A Generalization of Special Atom Spaces with Arbitrary Measure

by

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Abstract

A brief historical account of the development of special atom spaces is presented followed by the introduction of two new function spaces, $A_\phi(\mu, \alpha)$ and $B_\phi(\mu, \alpha)$, which are generalizations of previous special atom spaces utilizing arbitrary measures rather than Lebesgue measure of intervals. Known definitions relating to normed vector spaces are extended to apply to the new function spaces of arbitrary measure. The properties of the new function spaces are discussed including the relationship between the spaces as well as the relationship of the spaces with well known function spaces such as Lebesgue spaces, L_p, $Lip(\mu, \alpha)$ and $\Lambda(\mu, \alpha)$.

Major results include Hölder-type inequalities for both $A_\phi(\mu, \alpha)$ and $B_\phi(\mu, \alpha)$. In the case of $B_\phi(\mu, \alpha)$, the dual of $B_\phi(\mu, \alpha)$ is determined and a Representation Theorem for the weighted bounded linear functionals of $B_\phi(\mu, \alpha)$ is presented in detail. However, for $A_\phi(\mu, \alpha)$ we mention that the dual follows the same idea of the theorem for $B_\phi(\mu, \alpha)$. That is, that we only need to estimate $\|\chi_A\|_{A(\mu,\alpha)}$ for a μ-measurable set A. Indeed we show there is a positive constant M such that $\|\chi_A\|_{A(\mu,\alpha)} \leq M\mu^\alpha(A)$. The duality and representation theorems for $A_\phi(\mu, \alpha)$ follow easily. Interpolation of Operators Theorems are presented on sublinear operators which map $B(\mu, \frac{1}{p})$ into weak L_p and $A(\mu, \frac{1}{p})$ into weak L_p spaces. Finally, we present the multiplication operator on $A_\phi(\mu, \alpha)$ and $B_\phi(\mu, \alpha)$ for $\phi(t) = t$, and show under some conditions this operator is bounded on those spaces.

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Chapter 1

Introduction and Historical Background

This dissertation will introduce and explore the properties and applications of two new function spaces, denoted as $A_\phi(\mu,\alpha)$ and $B_\phi(\mu,\alpha)$. This dissertation begins with historical background motivating these spaces, followed by several definitions which streamline the introduction of the new function spaces. We will be presenting several theorems regarding the properties of the new function spaces as well as the relationship with other known function spaces.

1.1 A Major Question in the 20th Century

In 1923, Frigyes Riesz introduced a new function space he named after G.H. Hardy following a paper written by Hardy in 1915, see [32],[26]. The space is defined as:

Definition 1.1 (Hardy's Space $H^p(\mathbb{D}), 0 < p < \infty$) *$F \in H^p(\mathbb{D}) \Leftrightarrow F$ is analytic in the complex unit disc, $\mathbb{D}$ and $||F||_{H^p} = \sup\limits_{0<r<1}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|F(re^{i\theta})|^p d\theta\right)^{1/p} < \infty$.*

Hardy's Space, in turn, is related to another function space, namely Lebesgue space which Riesz had introduced previously in 1910, see [33]. The definition of Lebesgue Space follows:

Definition 1.2 (Lebesgue Spaces $L^p, 0 < p < \infty$) *$f \in L^p[-\pi,\pi] \Leftrightarrow f$ is a Lebesgue-measurable function and $\|f\|_{L^p} = \left(\int_{-\pi}^{\pi}|f(t)|^p d\mu(t)\right)^{1/p} < \infty$, where μ is a measure on $[-\pi,\pi]$.*

Recall that if $A = \{x \in [-\pi,\pi] \mid f(x) \neq g(x)\}$ and $\mu(A) = 0$, then $\|f\|_{L^p} = \|g\|_{L^p}$. Although both the Hardy's Spaces and Lebesgue Spaces are defined for $p > 0$, we are only concerned with the case $p \geq 1$.

During the first half of the twentieth century, a well known fact was that for $1 < p < \infty$, $H^p(\mathbb{D}) \cong L^p[-\pi,\pi]$. This follows from the fact that the Hilbert Transform of f, which is denoted by $\tilde{f}$, is invariant on L^p, and $\|\tilde{f}\|_{L^p} \leq M\|f\|_{L^p}$ for $1 < p < \infty$, M a constant. Thus, since the dual spaces of L^p were well known, the dual spaces of $H^p(\mathbb{D})$ were known for $1 < p < \infty$. However, for $p = 1$, $H^1(\mathbb{D})$ is not equivalent to $L^1[-\pi,\pi]$ since we have functions f in $L^1[-\pi,\pi]$ such that $\tilde{f}$ does not belong to $L^1[-\pi,\pi]$. Thus, the dual space of $H^1(\mathbb{D})$ became a major question in harmonic analysis in the middle of the 20th century. The search for a solution for the dual space of $H^1(\mathbb{D})$ led to several new areas of study, including the origin of special atom spaces.

Charley Fefferman first solved the question of the dual space of $H^1(\mathbb{D})$ in 1971, see [24]. Fefferman solved the problem by using the space of functions of Bounded Mean Oscillation (BMO), introduced by F. John and L. Nirenberg in 1961, see [28]. Here is the definition of BMO for reference:

Definition 1.3 (Functions of Bounded Mean Oscillation (BMO))
$g \in BMO[-\pi,\pi] \Leftrightarrow g$ is periodic and $||g||_{BMO} = \sup_I \frac{1}{|I|}\int_I |g(t) - g_I|dt < \infty$, where $g_I = \frac{1}{|I|}\int_I g(t)dt$.

Fefferman demonstrated that BMO was equivalent to the dual of $H^1(\mathbb{D})$ with equivalent norms, see [24]. Following this discovery, in 1974 R.R. Coifman provided a new characterization of $H^1(\mathbb{D})$, what is now referred to as the "Atomic Decomposition of $H^1(\mathbb{D})$." Coifman's characterization of $H^1(\mathbb{D})$ introduced the concept of atoms in his definition of ReH^1 which has come to be known as the Space of Atoms. In his paper, Coifman proved that $H^1(\mathbb{D}) \cong ReH^1$ with equivalent norms, hence the dual of ReH^1 is equivalent with BMO, see [1]. The definition of atom follows along with the definition of ReH^1, the Space of Atoms, as these definitions led directly to the introduction of Special Atom Spaces.

Definition 1.4 (Atom) *An atom is a function which is either $a(t) = \frac{1}{2\pi}$ or $a : [-\pi,\pi] \to \mathbb{R}$ satisfying:*

- *supp* $a \subset I \subset [-\pi, \pi]$
- $|a(t)| \leq \frac{1}{|I|}, \forall t \in [-\pi, \pi]$
- $\int\limits_I a(t)dt = 0$

where supp a is the support of the function a, and I is an interval.

Definition 1.5 (Space of Atoms ReH^1) $ReH^1 = \{f : [-\pi, \pi] \to \mathbb{R}, periodic; f(t) = \sum_{m=1}^{\infty} c_m a_m(t), \sum_{m=1}^{\infty} |c_m| < \infty\}$, $c_m \in \mathbb{R}$, $||f||_{ReH^1} = inf \sum_{m=1}^{\infty} |c_m|$ *where the infimum is taken over all possible representations of* f, *and the* a_m*'s are atoms supported on intervals* $I_n \subseteq [-\pi, \pi]$.

Notice that in Definition 1.4, the concept of atom is in general terms. The next logical question is what specific function satisfies all the requirements to be an atom? Specifically, can a real-valued function be found on $[-\pi, \pi]$ that is supported in an interval in $[-\pi, \pi]$, which is bounded by one over the Lebesgue measure of the interval, and when integrated over the interval results in zero? The answer to this question was the beginning of Special Atom Spaces.

1.2 Special Atom Space

G. S. de Souza was struck with this question when he was presented with the definitions of atoms and ReH^1. G.S. de Souza provided the answer to this question when he defined Special Atoms as part of his PhD dissertation, see [15].

Definition 1.6 (Special Atoms - de Souza - 1980) *Special atoms are defined as either* $b(t) = \frac{1}{2\pi}$ *or* $b : [-\pi, \pi] \to \mathbb{R}$ *given by* $b(t) = \frac{1}{|I|}[\chi_L(t) - \chi_R(t)], I = L \cup R,\ L,\ R$ *are halves of intervals* $I \subset [-\pi, \pi]$

Utilizing the above definition, de Souza introduced the first Special Atom Space, B_1, in 1980. B_1 was the beginning of a series of spaces which eventually led to the new spaces introduced in this dissertation. In his 1992 work on wavelets, Yves Meyer referred to B_1 as de Souza's Space and the name has subsequently remained, see [31]. The definition of de Souza's Space is:

Definition 1.7 (Special Atom Space - de Souza's Space B_1 - 1980) $B_1 = \{f : [-\pi, \pi] \to \mathbb{R}, periodic; f(t) = \sum_{m=1}^{\infty} c_m b_m(t), \sum_{m=1}^{\infty} |c_m| < \infty\}$ *where* b_m *are special atoms and* $c_m \in \mathbb{R}$ *for all* $m \in \mathbb{N}$*. The norm is given by* $||f||_{B_1} = inf \sum_{m=1}^{\infty} |c_m|$*, where the infimum is taken over all possible representations of* f*.*

Note here that the norm $||f||_{B_1}$ makes de Souza's Space a Banach space. One useful result after the discovery of B_1 was the analytic characterization of B_1. In fact, even before de Souza was working on his space, mathematicians were working to solve another problem. In order to formulate this problem, define the following function space J:

Definition 1.8 (Function Space J) $F \in J \Leftrightarrow F : \mathbb{D} \to \mathbb{C}$ *with* $\|F\|_J = \int_0^1 \int_{-\pi}^{\pi} |F'(re^{i\theta})| d\theta dr < \infty$*, where* $\mathbb{D}$ *is the complex unit disc and* F' *is the derivative of* F*.*

The open problem was how to characterize the boundary value of J. That is, for $F \in J$, what is $\lim_{r \to 1} ReF(re^{i\theta})$? In 1983 de Souza and Gary Sampson answered the question with the following theorem, see [8]:

Theorem 1.1 (Analytic Characterization of B_1) $f \in B_1[-\pi, \pi] \Leftrightarrow f(\theta) = \lim_{r \to 1} ReF(re^{i\theta})$ *with* $\int_0^1 \int_{-\pi}^{\pi} |F'(re^{i\theta})| d\theta dr < \infty$*. Moreover,* $||f||_{B_1} \cong \int_0^1 \int_{-\pi}^{\pi} |F'(re^{i\theta})| d\theta dr = \|F\|_J$*. That is* $||f||_{B_1} \cong \|F\|_J$*.*

Following this discovery, de Souza's space was further generalized in several iterations discussed in the next section.

1.3 Generalizations of Special Atom Space

The first generalization of de Souza's space was with the introduction of a weight to the bound of the special atom. The result was a generalization of B_1 to B_p given in the next definition.

Definition 1.9 (Generalization B_1 to $B_p, 1/2 < p < \infty$)
$f \in B_p \Leftrightarrow f(t) = \sum_{m=1}^{\infty} c_m b_m(t)$ with $\sum_{m=1}^{\infty} |c_m| < \infty$
where $b_1(t) = \frac{1}{2\pi}$, $b_m(t) = \frac{1}{|I_m|^{1/p}}[\chi_{L_m}(t) - \chi_{R_m}(t)], I_m = L_m \cup R_m,\ m \neq 1$
A norm for B_p is defined as $||f||_{B_p} = inf \sum_{m=1}^{\infty} |c_m|$, where the infimum is taken over all possible representations of f.

Note in the above definition for $p = 1$, B_p reduces to the original B_1. B_p was then generalized to the space C_p^q:

Definition 1.10 (Generalization to $C_p^q, 1/2 < p < \infty, 0 < q \leq 1$)
$f \in C_p^q \Leftrightarrow f(t) = \sum_{m=1}^{\infty} c_m b_m(t)$ with $\sum_{m=1}^{\infty} |c_m|^q < \infty$ where $b_1(t) = \frac{1}{2\pi}$,
$b_m(t) = \frac{1}{|I_m|^{1/p}}[\chi_{L_m}(t) - \chi_{R_m}(t)], I_m = L_m \cup R_m,\ m \neq 1$. A norm for C_p^q is defined as $||f||_{C_p^q} = inf \sum_{m=1}^{\infty} |c_m|^q$ where the infimum is taken over all possible representations of f.

Note that for $0 < q < 1$, $||f||_{C_p^q}$ is not a norm in the usual sense. Again, one can easily see the generalization that for $q = 1$, $C_p^1 = B^p$. The spaces, B_p and C_p^q, are very important spaces in harmonic analysis since they are the boundary characterizations of spaces of analytic functions in the disc called Besov spaces. While B_p is still a Banach space, C_p^q is a complete metric space for $0 < q < 1$. Also, the duals of these spaces are Lipschitz spaces. For these results, and a more complete summary of the above spaces, see [18],[19],[5],[16],[13],[17], and [36].

Work continued around B_p and C_p^q, resulting in weighted special atom spaces B_ρ. The weighted special atom spaces are formed by replacing the bound in the special atoms, $|I|$

with a weight function $\rho(|I|)$ where ρ satisfies certain conditions, see [4], [3]. Note that if $\rho(t) = t^{\frac{1}{p}}$ then B_ρ reduces to B_p. In 2006, de Souza utilized arbitrary measure in his latest iteration. He defined the spaces $A(\mu, \alpha)$ and $B(\mu, \alpha)$ using a finite measure μ in place of the original Lebesgue measure of intervals, see [21]. The definitions of $A(\mu, \alpha)$ and $B(\mu, \alpha)$ are included below as these spaces are integral to the new function spaces at the heart of this dissertation.

Definition 1.11 ($A(\mu, \alpha)$) *Given a finite measure space* $([-\pi, \pi], \mathcal{A}, \mu)$, *for* $n \in \mathbb{N}$ *and* $\alpha \in (0, 1]$, *let* A_n, B_n, *and* X_n *be* $\mu-$*measurable sets in* $\mathcal{A}$ *such that* $A_n \bigcup B_n = X_n$, $A_n \bigcap B_n = \emptyset$ *and* $\mu(A_n) = \mu(B_n)$. *Define the space* $A(\mu, \alpha)$ *as:*

$$A(\mu, \alpha) = \left\{ f : [-\pi, \pi] \to \mathbb{R} \mid f(t) = \sum_{n=1}^{\infty} c_n b_n(t), \sum_{n=1}^{\infty} |c_n| < \infty \right\}$$

Where $c_n \in \mathbb{R}$, $b_n(t)$, $b_1(t) = \frac{1}{\mu([-\pi,\pi])}$, *and* $b_n(t) = \frac{1}{\mu^\alpha(X_n)}[\chi_{A_n}(t) - \chi_{B_n}(t)]$, $n \neq 1$. *For* $f \in A(\mu, \alpha)$ *define a norm as*

$$\|f\|_{A(\mu,\alpha)} = \inf \sum_{n=1}^{\infty} |c_n|$$

where the infimum is taken over all possible representations of f.

Definition 1.12 ($B(\mu, \alpha)$) *Given a finite measure space* $([-\pi, \pi], \mathcal{A}, \mu)$, *for* $n \in \mathbb{N}$ *and* $\alpha \in (0, 1]$, *let* B_n *be* $\mu-$*measurable sets in* $\mathcal{A}$. *Define the space* $B(\mu, \alpha)$ *as:*

$$B(\mu, \alpha) = \left\{ f : [-\pi, \pi] \to \mathbb{R} \mid f(t) = \sum_{n=1}^{\infty} c_n b_n(t), \sum_{n=1}^{\infty} |c_n| < \infty \right\}$$

Where $c_n \in \mathbb{R}$, $b_1(t) = \frac{1}{\mu([-\pi,\pi])}$, *and* $b_n(t) = \frac{1}{\mu^\alpha(B_n)}[\chi_{B_n}(t)], n \neq 1$. *For* $f \in B(\mu, \alpha)$ *define a norm as*

$$\|f\|_{B(\mu,\alpha)} = \inf \sum_{n=1}^{\infty} |c_n|$$

where the infimum is taken over all possible representations of f.

Note that $\|f\|_{A(\mu,\alpha)}$ and $\|f\|_{B(\mu,\alpha)}$ are norms in the usual sense. As one can see, the space $A(\mu, \alpha)$ is a generalization of previous special atom spaces using arbitrary measure and arbitrary measurable sets rather than Lebesgue measure of intervals. The space $B(\mu, \alpha)$ is a slight variation and not a clear generalization since the "atoms"in $B(\mu, \alpha)$ consist of the characteristic function of one μ-measurable set. If one recalls the definition of atom, Definition 1.4, $B(\mu, \alpha)$ is not a linear combination of true atoms since the integral of b_n in the definition of $B(\mu, \alpha)$ will not necessarily be zero. However, one interesting result is that $A(\mu, \alpha) \subseteq B(\mu, \alpha)$. In fact, in 2009, de Souza and Miguel Pozo proved that $A(\mu, \alpha)$ and $B(\mu, \alpha)$ are equivalent as Banach spaces with equivalent norms, see [6]. The spaces $A(\mu, \alpha)$ and $B(\mu, \alpha)$ became much more intriguing later that year when de Souza showed that both of these spaces are characterizations of the Lorentz spaces $L(p, 1)$ for $p > 1$. Note $f \in L(p, 1)$ if $\|f\|_{L(p,1)} = \int_0^{2\pi} f^*(t)t^{\frac{1}{p}-1}dt < \infty$, where f^* is the decreasing rearrangement of f, see comments following Definition 2.10. Indeed, de Souza showed that for $p > 1$, $A(\mu, 1/p) \cong B(\mu, 1/p) \cong L(p, 1)$ with equivalent norms, see [9]. The main result in this dissertation is a further generalization of $A(\mu, \alpha)$ and $B(\mu, \alpha)$, utilizing different "norms"which are defined as weighted metrics. The remainder of this dissertation will define and explore $A_\phi(\mu, \alpha)$ and $B_\phi(\mu, \alpha)$, beginning with several useful definitions, including the definitions of $A_\phi(\mu, \alpha)$ and $B_\phi(\mu, \alpha)$.

Chapter 2

Definitions and Comments

In order to concisely define our new function spaces, we will introduce several new definitions in this section. Many of the following definitions are extensions of known definitions and are noted as such. We will include examples and comments for clarification as appropriate. For the remainder of this dissertation, we shall assume that any function denoted by the symbol ξ is defined and finite for real numbers in its given domain.

2.1 Basic Definitions

The first definition provided below defines a class of functions which we will utilize throughout this dissertation.

Definition 2.1 (Class C_ϕ functions) *We define C_ϕ to be a class of functions $\phi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:*

$$\phi(0) = 0, \quad \phi \textit{ is strictly increasing and continuous} \tag{2.1}$$

$$\phi(\lambda \cdot x) \leq \xi_\phi(\lambda)\phi(x) \textit{ for some function } \xi_\phi : \mathbb{R}^+ \mapsto \mathbb{R}^+ \tag{2.2}$$

$$\phi(x+y) \leq k_\phi(\phi(x) + \phi(y)) \textit{ for some constant } k_\phi \geq 1 \tag{2.3}$$

$$\phi(x) \to \infty \textit{ as } x \to \infty \tag{2.4}$$

In order to illustrate Class C_ϕ functions we present the following Lemma.

Lemma 2.1 (Class C_ϕ is not empty) *For $\alpha \in (0,1]$ the real functions $\phi_1(t)$ and $\phi_2(t)$ defined by $\phi_1(t) = t^\alpha$ and $\phi_2(t) = \ln^\alpha(t+1)$ on $[0,\infty)$ are in the Class C_ϕ functions.*

Proof. Let $\alpha \in (0,1]$. First, consider ϕ_1. $\phi_1(0) = 0$ and ϕ_1 is clearly continuous. Now $\phi_1'(t) = \alpha t^{\alpha-1} > 0$ so ϕ_1 is strictly increasing and property (2.1) is satisfied. For property (2.2) of Class C_ϕ functions let $\lambda \in \mathbb{R}^+$, then $\phi_1(\lambda t) = (\lambda t)^\alpha = \lambda^\alpha t^\alpha = \lambda^\alpha \phi_1(t)$ and we have $\xi_{\phi_1}(\lambda) = \lambda^\alpha$. Property (2.3) follows directly from the inequality $(t+s)^\alpha \leq t^\alpha + s^\alpha$, since $\phi_1(t+s) = (t+s)^\alpha \leq t^\alpha + s^\alpha = \phi_1(t) + \phi_1(s)$. This inequality is simple to prove:

$$(t+s)^\alpha = (t+s)(t+s)^{\alpha-1} = t(t+s)^{\alpha-1} + s(t+s)^{\alpha-1} \leq t \cdot t^{\alpha-1} + s \cdot s^{\alpha-1}$$

since $(t+s)^{\alpha-1} \leq t^{\alpha-1}$ and $(t+s)^{\alpha-1} \leq s^{\alpha-1}$. Thus, $(t+s)^\alpha \leq t^\alpha + s^\alpha$. The final property is clearly true and $\phi_1 \in C_\phi$.

The proof for ϕ_2 is slightly more involved. Property (2.1) is clear since natural log is strictly increasing and continuous with $\phi_2(0) = \ln^\alpha(1) = 0$. In order to prove property (2.2) We break λ into two cases with $\alpha = 1$. First consider the case $\lambda > 1$. Let $g(t) = \lambda \ln(t+1) - \ln(\lambda t + 1)$, then $g'(t) = \frac{\lambda}{t+1} - \frac{\lambda}{\lambda t+1}$ and $g'(t) = 0$ occurs only at $t = 0$. Since $g'(t) > 0$, $g(t)$ is strictly increasing and $g(0)$ is a minimum we conclude $g(0) < g(t)$ for all $t > 0$. Thus, we have $g(0) = 0 < g(t) = \lambda \ln(t+1) - \ln(t+1) \Rightarrow \ln(\lambda t+1) \leq \lambda \ln(t+1)$. Second we consider the case $\lambda < 1$. For $t \geq 0$ we have $\lambda t + 1 \leq (\lambda+1)t + 1$ and from the previous case we then have $\ln(\lambda t+1) \leq \ln((\lambda+1)t+1) \leq (\lambda+1)\ln(t+1)$ since $\lambda + 1 > 1$. Combining the two cases and the trivial case $\lambda = 1$ we conclude $\ln(\lambda t+1) \leq (\lambda+1)\ln(t+1)$ for all $\lambda \geq 0$, that is $\xi_{\phi_2}(\lambda) = \lambda + 1$. Finally consider $\alpha \neq 1$. We now have $\phi_2(\lambda t) = \ln^\alpha(t+1) \leq ((\lambda+1)\ln(\lambda t+1))^\alpha = (\lambda+1)^\alpha \ln^\alpha(t+1)$. Setting $\xi_{\phi_2}(\lambda) = (\lambda+1)^\alpha$ property (2.2) is proved. Since property (2.4) is clearly true, all that remains is to prove property (2.3). Since $\phi_2(t+s) = \ln^\alpha(t+s+1)$, letting $\alpha = 1$ we have $\phi_2(t+s) = \ln(t+s+1)$. Now utilizing logarithmic properties we see $\phi_2(t) + \phi_2(s) = \ln(t+1) + \ln(s+1) = \ln((t+1)(s+1)) = \ln(ts+t+s+1) \geq ln(t+s+1) = \phi_2(t+s)$ since $ts \geq 0$. Letting $\alpha \neq 1$, $\phi_2(t+s) = \ln^\alpha(t+s+1) \leq (\ln(t+1) + \ln(s+1))^\alpha \leq \ln^\alpha(t+1) + \ln^\alpha(s+1) = \phi_2(t) + \phi_2(s)$

by the third property proved above for $\phi_1(t) = t^\alpha$.$\square$

The following definition is an extension of a normed vector space which is applicable to the spaces introduced in this dissertation.

Definition 2.2 (Weighted Metric Space) *Let X be a vector space. X is said to be a weighted metric space if there is a given real valued function $\|\cdot\|_X$ called a weighted metric on X satisfying:*

$$\|x\|_X > 0 \text{ if } x \neq 0 \tag{2.5}$$

$$\|x\|_X = 0 \text{ if and only if } x = 0 \tag{2.6}$$

$$\|\lambda x\|_X \leq \xi_w(|\lambda|)\|x\|_X \text{ for all scalars } \lambda,\ \xi_w \text{ a function on } \mathbb{R}^+ \tag{2.7}$$

$$\|x+y\|_X \leq k_w\left(\|x\|_X + \|y\|_X\right) \text{ for all } x, y \in X,\ k_w \geq 1, \text{ a scalar} \tag{2.8}$$

Note if one replaces (2.7) and (2.8) by (2.7)': $\|\lambda x\|_X = |\lambda|\|x\|_X$ and (2.8)': $\|x+y\|_X \leq \|x\|_X + \|y\|_X$ for all $x, y \in X$, then $\|\cdot\|_X$ is a norm in the usual sense.

Note: for convenience, we will use the typical symbol for norm throughout this dissertation although in many cases our representation is not actually a norm. This distinction will be pointed out where appropriate. Wherever a metric satisfies the four above restrictions, we will refer to this metric as a weighted metric. The next definition defines a useful relationship between weighted metric spaces.

Definition 2.3 (Weighted Continuous Space Inclusion) *Let X and Y be two weighted metric spaces, one says that X is weighted continuously contained in Y if:*

1. $X \subseteq Y$ and

2. There are constants $M, k > 0$ and a function $\xi : [0,\infty) \to [0,\infty), \xi(0) = 0$ such that $\|f\|_Y \leq M\xi\left(k\|f\|_X\right)$.

Note if $\xi(t) = t$, then the above definition is the usual definition for continuously contained normed spaces. For convenience, we will now define Weighted Generalized Special Atoms. This definition provides us a type of shorthand notation which we utilize throughout the remainder of this dissertation.

Definition 2.4 (Weighted Generalized Special Atoms for arbitrary measure) *Given a finite measure space $([-\pi, \pi], \mathcal{A}, \mu)$, for $n \in \mathbb{N}$, let A_n, B_n, X_n, and E be $\mu-$measurable sets in $[-\pi, \pi]$ such that $A_n \bigcup B_n = X_n$, $A_n \bigcap B_n = \emptyset$ and $\mu(A_n) = \mu(B_n)$. For $\alpha \in (0, 1]$, we define the weighted generalized special atoms of Type* I *and* II, $b_n^{\mathtt{I},\mathtt{II}} : [-\pi, \pi] \to \mathbb{R}$ *as follows:*

$$b_1^{\mathtt{I},\mathtt{II}}(t) = \frac{1}{\mu([-\pi, \pi])}, \quad b_n^{\mathtt{I}}(t) = \frac{1}{\mu^\alpha(X_n)}[\chi_{A_n}(t) - \chi_{B_n}(t)], b_n^{\mathtt{II}}(t) = \frac{1}{\mu^\alpha(B_n)}[\chi_{B_n}(t)], \quad n \neq 1$$

where χ_E is the characteristic function of E.

Definition 2.5 (Completeness) *A Weighted Metric Space X is said to be complete if and only if every Cauchy sequence in X converges to an element in X.*

2.2 Definitions of $A_\phi(\mu, \alpha)$ and $B_\phi(\mu, \alpha)$

Armed now with the above definitions, we can now define the two new spaces $A_\phi(\mu, \alpha)$ and $B_\phi(\mu, \alpha)$, which are the foundation of this dissertation.

Definition 2.6 ($A_\phi(\mu, \alpha)$) *For $\alpha \in (0, 1]$, let $(b_n^{\mathtt{I}})_{n \geq 1}$ be weighted generalized special atoms of Type* I *and $(c_n)_{n \geq 1}$ be a sequence of real numbers, μ a finite measure on sets in a $\sigma-$algebra of $[-\pi, \pi]$ and $\phi \in C_\phi$. We define the space $A_\phi(\mu, \alpha)$ as:*

$$A_\phi(\mu, \alpha) = \left\{ f : [-\pi, \pi] \to \mathbb{R} \mid f(t) = \sum_{n=1}^{\infty} c_n b_n^{\mathtt{I}}(t), \sum_{n=1}^{\infty} \phi(|c_n|) < \infty \right\}$$

For $f \in A_\phi(\mu, \alpha)$ we define a "norm" as

$$\|f\|_{A_\phi(\mu, \alpha)} = \inf \sum_{n=1}^{\infty} \phi(|c_n|)$$

where the infimum is taken over all possible representations of f.

Definition 2.7 **($B_\phi(\mu,\alpha)$)** *For $\alpha \in (0,1]$, let $(b_n^{\mathtt{II}})_{n\geq 1}$ be weighted generalized special atoms of Type* II *and $(c_n)_{n\geq 1}$ be a sequence of real numbers, μ a finite measure on sets in a $\sigma-$algebra of $[-\pi,\pi]$ and $\phi \in C_\phi$. We define the space $B_\phi(\mu,\alpha)$ as*

$$B_\phi(\mu,\alpha) = \left\{ f : [-\pi,\pi] \to \mathbb{R} \mid f(t) = \sum_{n=1}^{\infty} c_n b_n^{\mathtt{II}}(t), \sum_{n=1}^{\infty} \phi(|c_n|) < \infty \right\}$$

For $f \in B_\phi(\mu,\alpha)$ we define a "norm" as

$$\|f\|_{B_\phi(\mu,\alpha)} = \inf \sum_{n=1}^{\infty} \phi(|c_n|)$$

where the infimum is taken over all possible representations of f.

The word "norm" in the previous two definitions is in quotations since in many instances, $\|\cdot\|_{A_\phi(\mu,\alpha)}$ and $\|\cdot\|_{B_\phi(\mu,\alpha)}$ will not turn out to be norms. However, we will show that this "norm" is, in fact, at all times at least a weighted metric.

We point out here for $\phi(t) = t$, $A_\phi(\mu,\alpha)$ and $B_\phi(\mu,\alpha)$ reduce to the spaces $A(\mu,\alpha)$ and $B(\mu,\alpha)$, which de Souza introduced in 2006, see [9]. Thus, considering the previous discussion, we see that $A_\phi(\mu,\alpha)$ and $B_\phi(\mu,\alpha)$ are indeed generalizations of all the spaces derived from the original de Souza space. The next definitions further generalize linear functionals and operator norms to meet the needs of subsequent proofs.

2.3 Functional and Operator type Definitions

A few operator-type definitions need extension for the purposes of this dissertation. Namely, we extend the definitions of bounded linear operators , operator norms, and dual spaces to suit our purposes.

Definition 2.8 (Weighted Bounded Linear Functional) *Let X be a weighted metric space, we say that ψ is a weighted bounded linear functional on X if $\psi : X \to \mathbb{R}$ such that:*

1. *ψ is linear*

2. *There are constants $M, k > 0$ and a continuous function $\xi : [0, \infty) \to [0, \infty), \xi(0) = 0$ such that for all $f \in X$, $|\psi(f)| \leq M\xi\left(k\|f\|_X\right)$.*

Note: If $\xi(t) = t$ then the above definition is the usual definition for bounded linear functionals.

Definition 2.9 (Weighted Operator Norm) *Let X be either $A_\phi(\mu, \alpha)$ or $B_\phi(\mu, \alpha)$ and φ be a weighted bounded linear functional on X. We define the weighted operator norm, $\|\varphi\|$ as follows for $f \in X$:*

$$\|\varphi\| = \sup_{f \neq 0} \frac{|\varphi(f)|}{\phi^{-1}(k_\phi \|f\|_X)}$$

Again, note that if $\phi(t) = t$ and $k_\phi = 1$, the weighted operator norm reduces to the usual definition of operator norms (or equivalent characterization of operator norms.) We now will extend the traditional definition of the dual space of a normed space to the weighted metric space equivalent.

Definition 2.10 (Dual Space of a Weighted Metric Space) *The space of all weighted bounded linear functionals on a weighted metric space X is called the dual of X and is denoted by X^*.*

The final definition of this section is a special case of a definition given by de Souza and Bloom in [3] relating to operators. In order to formulate this definition we must first define the decreasing rearrangement of a real valued, measurable function f. Let f be a real valued measurable function on T, then for $y > 0$ let

$$m(f, y) = m(|f|, y) = |\{x \in T, |f(x)| > y\}|$$

where $|\cdot|$ denotes the Lebesgue measure on T. $m(f,y)$ is known as the distribution function of f. Now we can define the decreasing arrangement of f, denoted by f^*, as

$$f^*(t) = inf\{y|m(f,y) \leq t\}.$$

Armed with the previous definition we can now define restricted weak type r operators.

Definition 2.11 (Restricted Weak Type r Operators) *Let μ be a finite measure. We say that an operator T is restricted weak type r if for any μ-measurable set A in $[-\pi,\pi]$*

$$t^{\frac{1}{r}} (T\chi_A)^* (t) \leq M\mu^{\frac{1}{r}}(A)$$

where M is an absolute constant and $*$ is the decreasing rearrangement of $T\chi_A(t)$.

2.4 Definitions of $Lip(\mu,\alpha)$, $\Lambda(\mu,\alpha)$, Lorentz spaces and some results

Here are the definitions of three well known spaces for later reference. We will see that $Lip(\mu,\alpha)$ and $\Lambda(\mu,\alpha)$ are not only related to each other but have relationships with $A_\phi(\mu,\alpha)$ and $B_\phi(\mu,\alpha)$ as well as their duals. The Lorentz spaces are useful in interpolation of operator theorems.

Definition 2.12 ($Lip(\mu,\alpha)$) *For $\alpha \in (0,1]$ and μ a finite measure on sets of $[-\pi,\pi]$ the space $Lip(\mu,\alpha)$ is defined as*

$$Lip(\mu,\alpha) = \left\{ g : [-\pi,\pi] \to \mathbb{R} \mid \frac{1}{\mu^\alpha(B)} \left| \int_B g(t) d\mu(t) \right| < M \right\}$$

where B is a $\mu-$measurable set in $[-\pi,\pi]$. A norm is defined on $Lip(\mu,\alpha)$ as

$$||g||_{Lip(\mu,\alpha)} = \sup_B \frac{1}{\mu^\alpha(B)} \left| \int_B g(t) d\mu(t) \right|$$

Note: the space $Lip(\mu,\alpha)$ is a generalization of the traditional Lipschitz α spaces usually denoted by Lip_α. Lip_α is the set of all continuous functions f on $[-\pi,\pi]$ such that for $\alpha \in (0,1]$, $\left|\frac{f(x+h)-f(x)}{h^\alpha}\right| < C$ where C is a constant. To see the generalization to $Lip(\mu,\alpha)$, take μ as the Lebesgue measure on $[-\pi,\pi]$, $X=[x,x+h]$, and f a differentiable function, thus we have:

$$\frac{1}{\mu^\alpha(X)}\left|\int_X f'(t)d\mu(t)\right| = \frac{1}{\mu^\alpha([x,x+h])}\left|\int_x^{x+h} f'(t)d\mu(t)\right| = \frac{1}{h^\alpha}\left|\int_x^{x+h} f'(t)d(t)\right| = \left|\frac{f(x+h)-f(x)}{h^\alpha}\right|.$$

If the function f above is not differentiable, take $G(x)=\int_{-\pi}^{x} f(t)dt$. Then $G(x+h) = \int_{-\pi}^{x+h} f(t)dt \Rightarrow |G(x+h)-G(x)| = \int_x^{x+h} f(t)dt$ and we see:

$$\frac{1}{\mu^\alpha(X)}\left|\int_X f(t)d\mu(t)\right| = \frac{1}{\mu^\alpha([x,x+h])}\left|\int_x^{x+h} f(t)d\mu(t)\right| = \frac{1}{h^\alpha}\left|\int_x^{x+h} f(t)d(t)\right| = \left|\frac{G(x+h)-G(x)}{h^\alpha}\right|.$$

G.G. Lorentz originally introduced this space in 1950, see [30],[29]. The next space we introduce, $\Lambda(\mu,\alpha)$, is a natural generalization of the traditional second difference Lipschitz α space usually denoted by Λ_α.

Definition 2.13 **($\Lambda(\mu,\alpha)$)** *For $\alpha \in (0,1]$ and μ a finite measure on sets of $[-\pi,\pi]$ the space $\Lambda(\mu,\alpha)$ is defined as*

$$\Lambda(\mu,\alpha) = \left\{ g:[-\pi,\pi]\to\mathbb{R} \middle| \frac{1}{\mu^\alpha(X)}\left|\int_A g(t)d\mu(t) - \int_B g(t)d\mu(t)\right| < M \right\}$$

where A,B and X are a $\mu-$measurable sets in $[-\pi,\pi]$ such that $A\bigcup B = X$, $A\bigcap B=\emptyset$. A norm is defined on $\Lambda(\mu,\alpha)$ as

$$||g||_{\Lambda(\mu,\alpha)} = \sup_{A\bigcup B=X, A\bigcap B=\emptyset} \frac{1}{\mu^\alpha(X)}\left|\int_A g(t)d\mu(t) - \int_B g(t)d\mu(t)\right|$$

Λ_α is defined as the set of all continuous functions f on $[-\pi,\pi]$ such that for $\alpha\in(0,1]$, $\left|\frac{f(x+h)+f(x-h)-f(x)}{(2h)^\alpha}\right| < C$ where C is a constant. To see the generalization to $\Lambda(\mu,\alpha)$, take μ

as the Lebesgue measure on $[-\pi, \pi]$, $X = [x-h, x+h]$, and f a differentiable function. Let $A = [x-h, x]$ and $B = (x, x+h]$ then:

$$\frac{1}{\mu^\alpha(X)} \left| \int_A f'(t) d\mu(t) - \int_B f'(t) d\mu(t) \right| = \frac{1}{\mu^\alpha([x-h, x+h])} \left| \int_{x-h}^{x} f'(t) d\mu(t) - \int_{x}^{x+h} f'(t) d\mu(t) \right|$$

$$= \frac{1}{(2h)^\alpha} \left| f(x) - f(x-h) - f(x+h) + f(x) \right| = \left| \frac{f(x+h) + f(x-h) - 2f(x)}{(2h)^\alpha} \right|.$$

If the function f above is not differentiable, one can make a similar argument as in the $Lip(\mu, \alpha)$ case. de Souza and M. Pozo introduced the space $\Lambda(\mu, \alpha)$ in earlier works, see [6].

The spaces $Lip(\mu, \alpha)$ and $\Lambda(\mu, \alpha)$ endowed with their respective norms are Banach Spaces. The proofs of these facts are simple using standard techniques. In fact, $\Lambda(\mu, \alpha) \cong Lip(\mu, \alpha)$ for $\alpha \in (0, 1)$, see [6].

Definition 2.14 (Lorentz Spaces) *A measurable function f belongs to the Lorentz space $L(p, q)$ if*

$$\|f\|_{pq} = \left(\frac{q}{p} \int_0^\infty \left(f^*(t) t^{\frac{1}{p}} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty$$

for $p \in (0, \infty)$,$q \in (0, \infty)$ where f^* is the decreasing arrangement of f.

For $q = \infty$, the space $L(p, \infty)$ is weak L_p space and $L(p, p)$ is the usual Lebesgue space L_p. Depending on the choices of p and q, $\|f\|_{pq}$ may not be a norm since the triangle inequality may fail. However, under certain restrictions on p and q, $\|f\|_{pq}$ is a norm and in this case $L(p, q)$ is a Banach space, see [3].

Chapter 3

Properties of $A_\phi(\mu, \alpha)$ and $B_\phi(\mu, \alpha)$

In this chapter, we will examine the spaces $A_\phi(\mu, \alpha)$ and $B_\phi(\mu, \alpha)$ in more detail including the relationship these spaces have to each other, other known spaces, as well relationships within the spaces themselves. The first few theorems present basic properties of these spaces. The first theorem is the fact that both of these new spaces are indeed vector spaces. Subsequent theorems present facts regarding the completeness of $A_\phi(\mu, \alpha)$ and $B_\phi(\mu, \alpha)$ and their relationship to L_p spaces. The final subsection demonstrates the relation of the new spaces among their own variants.

3.1 Basic Properties

The first theorem illustrates the simple fact that $A_\phi(\mu, \alpha)$ and $B_\phi(\mu, \alpha)$ are indeed vector spaces.

Theorem 3.1 *$A_\phi(\mu, \alpha)$ and $B_\phi(\mu, \alpha)$ are vector spaces with the usual definitions of addition and scalar multiplication for reals.*

A complete proof of Theorem 3.1 is relatively straightforward and is in Appendix A. Proof of closure under addition for $B_\phi(\mu, \alpha)$ is provided below, using a technique which will be repeated in further proofs.

Proof. Let $u, v \in B_\phi(\mu, \alpha)$ then for all $n \in \mathbb{N}$ there are real numbers c_{u_n}, c_{v_n} such that:

$$u = \sum_{n=1}^{\infty} c_{u_n} b^{\mathtt{II}}_{u_n}(t), \quad \sum_{n=1}^{\infty} \phi(|c_{u_n}|) < \infty$$
$$v = \sum_{n=1}^{\infty} c_{v_n} b^{\mathtt{II}}_{v_n}(t), \quad \sum_{n=1}^{\infty} \phi(|c_{v_n}|) < \infty$$

Now consider $u + v$:

$$u + v = \sum_{n=1}^{\infty} \left(c_{u_n} b^{\texttt{II}}_{u_n}(t) + c_{v_n} b^{\texttt{II}}_{v_n}(t) \right)$$

We introduce new variables and generalized special atoms of type II to re-index the above equation:

$$d_m = \begin{cases} c_{u_{\frac{m}{2}}}, & m \text{ even} \\ c_{v_{\frac{m+1}{2}}}, & m \text{ odd} \end{cases} \qquad b^{\texttt{II}}_m(t) = \begin{cases} b^{\texttt{II}}_{u_{\frac{m}{2}}}(t), & m \text{ even} \\ b^{\texttt{II}}_{v_{\frac{m+1}{2}}}(t), & m \text{ odd} \end{cases}$$

Then $u + v = \sum_{m=1}^{\infty} d_m b^{\texttt{II}}_m(t)$. Clearly all that remains to show is that $\sum_{m=1}^{\infty} \phi(|d_m|) < \infty$. This is clear since $\sum_{m=1}^{\infty} \phi(|d_m|) = \sum_{n=1}^{\infty} \phi(|c_{u_n}|) + \sum_{n=1}^{\infty} \phi(|c_{v_n}|) < \infty$. Thus $u+v \in B_\phi(\mu, \alpha)$ and $B_\phi(\mu, \alpha)$ is closed under addition.□

Our next theorem states the property mentioned earlier that the "norms" of the spaces are weighted metrics.

Theorem 3.2 (Weighted Metric) *The functions* $\|\cdot\|_{B_\phi(\mu,\alpha)}$ *and* $\|\cdot\|_{A_\phi(\mu,\alpha)}$ *are weighted metrics on* $B_\phi(\mu, \alpha)$ *and* $A_\phi(\mu, \alpha)$*, respectively.*

We provide the proof for $B_\phi(\mu, \alpha)$. The proof for $A_\phi(\mu, \alpha)$ is analogous.

Proof. Proof of (2.6) ($\|x\|_X = 0$ if and only if $x = 0$):

($\Leftarrow$) Since $\phi \in C_\phi$ we know ϕ is non-negative which implies $\|f\|_{B_\phi(\mu,\alpha)} \geq 0$. Let $f \in B_\phi(\mu, \alpha)$ such that $f = 0$. Then one possible representation of f is letting all coefficients c_n be zero. Hence, $\phi(|c_n|) = 0$ for all c_n's so $\sum_{n=1}^{\infty} \phi(|c_n|) = 0$. This implies $\|f\|_{B_\phi(\mu,\alpha)} = 0$.

($\Rightarrow$) Let $f \in B_\phi(\mu, \alpha)$ such that $\|f\|_{B_\phi(\mu,\alpha)} = 0$. Let f_k be a representation of f, then $f_k(t) = \sum_{n=1}^{\infty} c_{n_k} b^{\texttt{II}}_{n_k}(t)$. Define $d_k = \sum_{n=1}^{\infty} \phi(|c_{n_k}|)$. Now either $f = 0$ and the proof is complete or there exists a sequence of representations of f, $\{f_k\}_{k=1}^{\infty}$ such that $d_k \to 0$ as $k \to \infty$. Consider the second case where $f \neq 0$ and such a sequence of representations of f exists. Then for all $n \in \mathbb{N}$, $|c_{n_k}| \to 0$ as $k \to \infty$: Else, assume for some $m \in \mathbb{N}$, $|c_{m_k}|$ is bounded below by some $\delta > 0$ and $\phi(|c_{m_k}|) > 0$. Let $\epsilon = \frac{\phi(|c_{m_k}|}{2}$, then $\sum_{n=1}^{\infty} \phi(|c_{n_k}|) \geq \phi(|c_{m_k}|) > \epsilon > 0$ for all k. This contradicts our original assumption that $\|f\|_{B_\phi(\mu,\alpha)} = 0$. Thus the coefficients in the sequence of representations of f all converge to zero and hence $f = 0$.

Proof of (2.5) ($\|x\|_X > 0$ if $x \neq 0$):

This follows directly from (2.6) and the fact that $\|f\|_{B_\phi(\mu,\alpha)} \geq 0$ for all $f \in B_\phi(\mu, \alpha)$.

Proof of (2.7) ($\|\lambda x\|_X \leq \xi_w(|\lambda|)\|x\|_X$ for all scalars λ, ξ_w *real-valued function*):

Let $\lambda \in \mathbb{R}$, and $f \in B_\phi(\mu, \alpha)$ then $f(t) = \sum_{n=1}^{\infty} c_n b_n^{\mathtt{II}}(t)$ and

$$\lambda f(t) = \lambda \sum_{n=1}^{\infty} c_n b_n^{\mathtt{II}}(t) = \sum_{n=1}^{\infty} \lambda c_n b_n^{\mathtt{II}}(t)$$

so

$$\|\lambda f\|_{B_\phi(\mu,\alpha)} = \left\| \sum_{n=1}^{\infty} \lambda c_n b_n^{\mathtt{II}} \right\|_{B_\phi(\mu,\alpha)} \leq \sum_{n=1}^{\infty} \phi(|\lambda c_n|)$$

applying property (2.2) of ϕ we have

$$\sum_{n=1}^{\infty} \phi(|\lambda c_n|) = \sum_{n=1}^{\infty} \xi_\phi(|\lambda|)\phi(|c_n|) = \xi_\phi(|\lambda|) \sum_{n=1}^{\infty} \phi(|c_n|).$$

Thus, taking the infimum over all representations of f in the above inequality we conclude

$$\|\lambda f\|_{B_\phi(\mu,\alpha)} \leq \xi_\phi(|\lambda|) \|f\|_{B_\phi(\mu,\alpha)}.$$

Proof of (2.8) ($\|x + y\|_X \leq k_w (\|x\|_X + \|y\|_X)$ for all $x, y \in X$, $k_w \geq 1$, a scalar):

Let $f, g \in B_\phi(\mu, \alpha)$. Then $(f + g)(t) = \sum_{n=1}^{\infty} c_{(f+g)_n} b_{(f+g)_n}^{\mathtt{II}}(t)$ where $c_{(f+g)_n} \in \mathbb{R}$. Now let $\epsilon > 0$. By definition of $\|\cdot\|_{B_\phi(\mu,\alpha)}$ there are sequences of real numbers c_{f_n}, c_{g_n} and atoms b_{f_n}, b_{g_n} such that

$$f(t) = \sum_{n=1}^{\infty} c_{f_n} b_{f_n}^{\mathtt{II}}(t) \text{ and } g(t) = \sum_{n=1}^{\infty} c_{g_n} b_{g_n}^{\mathtt{II}}(t)$$

and for the constant k_ϕ given in (2.3) we have

$$\sum_{n=1}^{\infty} \phi(|c_{f_n}|) \leq \|f\|_{B_\phi(\mu,\alpha)} + \frac{\epsilon}{2k_\phi} \text{ and } \sum_{n=1}^{\infty} \phi(|c_{g_n}|) \leq \|g\|_{B_\phi(\mu,\alpha)} + \frac{\epsilon}{2k_\phi}.$$

Combining the above equations and multiplying by k_ϕ:

$$k_\phi \left(\sum_{n=1}^{\infty} \phi(|c_{f_n}|) + \sum_{n=1}^{\infty} \phi(|c_{g_n}|) \right) \leq k_\phi \left(\|f\|_{B_\phi(\mu,\alpha)} + \|g\|_{B_\phi(\mu,\alpha)} \right) + \epsilon.$$

Notice now that $(f+g)(t) = \sum_{n=1}^{\infty} c_{f_n} b^{\mathtt{II}}_{f_n}(t) + \sum_{n=1}^{\infty} c_{g_n} b^{\mathtt{II}}_{g_n}(t)$ which can be re-written as

$$(f+g)(t) = \sum_{n=1}^{\infty} \begin{cases} c_{f_{\frac{n}{2}}} b^{\mathtt{II}}_{f_{\frac{n}{2}}}(t), & \text{if } n \text{ even} \\ c_{g_{\frac{n+1}{2}}} b^{\mathtt{II}}_{g_{\frac{n+1}{2}}}(t), & \text{if } n \text{ odd.} \end{cases}$$

Therefore, taking the infimum below over all representations of $f+g$ we have

$$\|f+g\|_{B_\phi(\mu,\alpha)} = \inf \sum_{n=1}^{\infty} \phi\left(|c_{(f+g)_n}|\right) \leq \sum_{n=1}^{\infty} \begin{cases} \phi\left(\left|c_{f_{\frac{n}{2}}}\right|\right), & n \text{ even} \\ \phi\left(\left|c_{g_{\frac{n+1}{2}}}\right|\right), & n \text{ odd} \end{cases} = \sum_{n=1}^{\infty} \phi(|c_{f_n}|) + \sum_{n=1}^{\infty} \phi(|c_{g_n}|).$$

Applying property (2.3) of ϕ and continuing above

$$\sum_{n=1}^{\infty} \phi(|c_{f_n}|) + \sum_{n=1}^{\infty} \phi(|c_{g_n}|) \leq k_\phi \left(\sum_{n=1}^{\infty} \phi(|c_{f_n}|) + \sum_{n=1}^{\infty} \phi(|c_{g_n}|) \right) \leq k_\phi \left(\|f\|_{B_\phi(\mu,\alpha)} + \|g\|_{B_\phi(\mu,\alpha)} \right) + \epsilon.$$

Since ϵ was arbitrary we conclude

$$\|f+g\|_{B_\phi(\mu,\alpha)} \leq k_\phi \left(\|f\|_{B_\phi(\mu,\alpha)} + \|g\|_{B_\phi(\mu,\alpha)} \right). \square$$

The next theorem introduces the interesting result $A_\phi(\mu,\alpha) \subseteq B_\phi(\mu,\alpha)$.

Theorem 3.3 (Inclusion) *For all $\phi \in C_\phi$, $A_\phi(\mu,\alpha) \subseteq B_\phi(\mu,\alpha)$. Moreover, $||f||_{B_\phi(\mu,\alpha)} \leq 2\xi_\phi\left(\frac{1}{2^\alpha}\right) ||f||_{A_\phi(\mu,\alpha)}$.*

Proof. Let $f \in A_\phi(\mu,\alpha)$, then for all $n \in \mathbb{N}$ there are $c_n \in \mathbb{R}$ such that

$$f(t) = \sum_{n=1}^{\infty} c_n b^{\mathtt{I}}_n(t), \quad \text{where} \quad \sum_{n=1}^{\infty} \phi(|c_n|) < \infty$$

Re-writing $f(t)$:

$$\begin{aligned} f(t) &= \sum_{n=1}^{\infty} \frac{c_n}{\mu^\alpha(X_n)} \left(\chi_{A_n}(t) - \chi_{B_n}(t)\right) \\ &= \sum_{n=1}^{\infty} \frac{c_n \chi_{A_n}(t)}{\mu^\alpha(X_n)} - \sum_{n=1}^{\infty} \frac{c_n \chi_{B_n}(t)}{\mu^\alpha(X_n)}. \end{aligned}$$

Now, since $A_n \cap B_n = \emptyset$ and $\mu(A_n) = \mu(B_n)$ we have $\mu(X_n) = \mu(A_n \bigcup B_n) = \mu(A_n) + \mu(B_n) = 2\mu(A_n) = 2\mu(B_n)$. Substituting into previous equation we have:

$$\begin{aligned} f(t) &= \sum_{n=1}^{\infty} \frac{c_n \chi_{A_n}(t)}{(2\mu(A_n))^\alpha} - \sum_{n=1}^{\infty} \frac{c_n \chi_{B_n}(t)}{(2\mu(B_n))^\alpha} \\ &= \frac{1}{2^\alpha} \left(\sum_{n=1}^{\infty} \frac{c_n \chi_{A_n}(t)}{(\mu(A_n))^\alpha} - \sum_{n=1}^{\infty} \frac{c_n \chi_{B_n}(t)}{(\mu(B_n))^\alpha} \right). \end{aligned}$$

We introduce two new variables to re-index the above equation:

$$D_m = \begin{cases} B_{\frac{m}{2}}, & m \text{ even} \\ A_{\frac{m+1}{2}}, & m \text{ odd} \end{cases} \qquad p_m = \begin{cases} -c_{\frac{m}{2}}, & m \text{ even} \\ c_{\frac{m+1}{2}}, & m \text{ odd} \end{cases}$$

Substituting and noting that D_m are $\mu-$measurable sets we have:

$$f(t) = \sum_{m=1}^{\infty} \frac{p_m \chi_{D_m}(t)}{2^\alpha \mu^\alpha(D_m)} = \sum_{m=1}^{\infty} \frac{p_m}{2^\alpha} b_m^{\mathtt{II}}(t)$$

Finally, we have

$$\sum_{m=1}^{\infty} \phi(\left|\frac{p_m}{2^\alpha}\right|) = \sum_{n=1}^{\infty} \phi(\left|\frac{c_n}{2^\alpha}\right|) + \sum_{n=1}^{\infty} \phi(\left|\frac{c_n}{2^\alpha}\right|) = 2\sum_{n=1}^{\infty} \phi(\left|\frac{c_n}{2^\alpha}\right|)$$

Applying property (2.2) of C_ϕ functions

$$2\sum_{n=1}^{\infty}\phi(\left|\frac{c_n}{2^\alpha}\right|) \le 2\sum_{n=1}^{\infty}\xi_\phi\left(\frac{1}{2^\alpha}\right)\phi(|c_n|) = 2\xi_\phi\left(\frac{1}{2^\alpha}\right)\sum_{n=1}^{\infty}\phi(|c_n|) < \infty$$

So, we have $f(t) \in B_\phi(\mu,\alpha)$ and $A_\phi(\mu,\alpha) \subseteq B_\phi(\mu,\alpha)$.

Taking the infimum over all representations of f of both sides of the above inequality gives $||f||_{B_\phi(\mu,\alpha)} \le 2\xi_\phi\left(\frac{1}{2^\alpha}\right)||f||_{A_\phi(\mu,\alpha)}$.□

3.2 Completeness and L_p Inclusion

In this section, we will provide a proof that Weighted Metric Spaces are complete in the sense of Definition 2.5 if and only if every absolutely summable series in the space is summable. This property of Weighted Metric Spaces is very useful in several proofs presented. The relation of the spaces $A_\phi(\mu,\alpha)$ and $B_\phi(\mu,\alpha)$ to the Lebesgue spaces, L_p is then examined. This relationship with L_p spaces is vital to the last theorem presented in this section which demonstrates the completeness of $A_\phi(\mu,\alpha)$ and $B_\phi(\mu,\alpha)$.

Theorem 3.4 (Completeness of Weighted Metric Spaces) *Let X be a weighted metric space. Then X is complete if and only if every absolutely summable series is summable.*

The proof given below is an adaptation of a proof given by Royden, see [34].

Proof. ($\Rightarrow$) Let X be complete and $\langle f_i\rangle$ be an absolutely summable series of elements of X. Then $\sum_{i=1}^{\infty}\|f_i\|_X = M < \infty$. Let $\epsilon > 0$ then there is an $N \in \mathbb{N}$ such that $\sum_{i=N}^{\infty}\|f_i\|_X < \frac{\epsilon}{k_w}$ where k_w is the constant from Definition 2.2. Let $s_n = \sum_{i=1}^{n}\|f_i\|_X$ be the partial sum of $\langle f_i\rangle$, then for $n \ge m \ge N$, we have

$$\|s_n - s_m\|_X = \left\|\sum_{i=m}^{n} f_i\right\|_X \le k_w\sum_{i=m}^{\infty}\|f_i\|_X \le k_w\sum_{i=N}^{\infty}\|f_i\|_X < \epsilon$$

and hence $\langle s_i \rangle$ is a Cauchy sequence in X and must converge to an element f in X since X is complete. Therefore, $\lim\limits_{n\to\infty} \sum_{i=1}^{n} f_i = f$ in X.

($\Leftarrow$) Let $\langle f_n \rangle$ be a Cauchy sequence in X. Then for each integer k there is an integer n_k such that $\|f_n - f_m\|_X < 2^{-k}$ for all $n, m > n_k$. Hence, we choose $n_{k's}$ such that $n_{k+1} > n_k$ and obtain the subsequence $\langle f_{n_k} \rangle$ of $\langle f_n \rangle$. Let $g_1 = f_{n_1}$ and $g_k = f_{n_k} - f_{n_{k-1}}$, then for $k > 1$ we have a series $\langle g_k \rangle$ whose k^{th} partial sum is f_{n_k} but $\|g_k\|_X \leq 2^{-k+1}$. Therefore, we have

$$\sum_{k=1}^{\infty} \|g_k\|_X \leq \|g_1\|_X + \sum_{k=2}^{\infty} 2^{-k+1} = \|g_1\|_X + 1$$

and $\langle g_k \rangle$ is absolutely summable. Thus, there exists $f \in X$ such that $\langle f_{n_k} \rangle \to f$ in X. Finally, we show $\lim\limits_{n\to\infty} f_n = f$. Since $\langle f_n \rangle$ is a Cauchy sequence, for $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that $\|f_n - f_m\|_X \leq \frac{\epsilon}{2k_w}$ for all $n, m > N$. Since $f_{n_k} \to f$, there is a $K \in \mathbb{N}$ such that for all $k \geq K$ we have $\|f_{n_k} - f\|_X \leq \frac{\epsilon}{2k_w}$. Take k so large that $k > K$ and $n_k > N$, then

$$\|f_n - f\|_X \leq \|f_n - f_{n_k} + f_{n_k} - f\|_X \leq k_w(\|f_n - f_{n_k}\|_X + \|f_{n_k} - f\|_X) \leq k_w(\frac{\epsilon}{2k_w} + \frac{\epsilon}{2k_w}) = \epsilon.$$

Since ϵ was arbitrary, we have $\lim\limits_{n\to\infty} f_n = f$ and X is complete based on Definition 2.5 .□

The Lemma below will be used in several subsequent proofs.

Lemma 3.1 *For $n \in \mathbb{N}$, let c_n be real numbers and $\phi \in C_\phi$ such that $\sum_{n=1}^{\infty} \phi(|c_n|) < \infty$, and let k_ϕ be the constant from property (2.3) of C_ϕ functions, then*

$$\sum_{n=1}^{\infty} |c_n| \leq \phi^{-1}\left(k_\phi \sum_{n=1}^{\infty} \phi(|c_n|)\right)$$

where ϕ^{-1} is the inverse of ϕ.

Proof. Let n, c_n, ϕ, and k_ϕ be as above in the statement of the lemma. Choose $N \in \mathbb{N}$. Then, since ϕ is continuous, well defined, finite, strictly increasing, and $\phi(0) = 0$, we have

$$\sum_{n=1}^{N} |c_n| = \phi^{-1}\left(\phi\left(\sum_{n=1}^{N} |c_n|\right)\right) \leq \phi^{-1}\left(k_\phi \sum_{n=1}^{N} \phi(|c_n|)\right).$$

Taking the limit as $N \to \infty$ of the above inequality gives the desired result.□

It is interesting to note that if one chooses $\phi(t) = t^p$ for $p \in (0,1)$, then $\phi^{-1}(t) = t^{\frac{1}{p}}$ and $k_\phi = 1$ (see Lemma 2.1), applying Lemma 3.1 gives a well known inequality

$$\sum_{n=1}^{\infty} |c_n| \leq \left(\sum_{n=1}^{\infty} |c_n|^p\right)^{\frac{1}{p}}$$

The following theorem presents the aforementioned relationships between $A_\phi(\mu, \alpha)$, $B_\phi(\mu, \alpha)$, and L_p spaces.

Theorem 3.5 (L_p Inclusion) *For $p \geq 1$ and $\alpha < \frac{1}{p}$, $A_\phi(\mu, \alpha)$ and $B_\phi(\mu, \alpha)$ are weighted continuously contained in $L_p[-\pi, \pi]$.*

Proof. First, consider one weighted generalized special atom of Type `II` (where B is a $\mu-$measurable set in $[-\pi, \pi]$):

$$||b^{\mathtt{II}}||_{L_p}^p = \int_{-\pi}^{\pi} \left|\frac{\chi_B(t)}{\mu^\alpha(B)}\right|^p d\mu(t) = \frac{1}{\mu^{p\alpha}(B)} \int_B \chi_B(t)^p d\mu(t)$$

$$= \frac{\mu(B)}{\mu^{p\alpha}(B)} = \mu^{1-p\alpha}(B) \leq \mu^{1-p\alpha}([-\pi, \pi]).$$

Let $M_{\alpha,p} = (\mu^{1-p\alpha}([-\pi, \pi]))^{1/p}$, then for any weighted generalized special atom of Type `II` we have $||b^{\mathtt{II}}||_{L_p} \leq M_{\alpha,p}$.

Now let $f \in B_\phi(\mu, \alpha)$, then for $n \in \mathbb{N}$ there are $c_n \in \mathbb{R}$ such that

$$f(t) = \sum_{n=1}^{\infty} c_n b_n^{\mathtt{II}}(t), \quad \sum_{n=1}^{\infty} \phi(|c_n|) < \infty.$$

Taking the L_p norm of f we have:

$$||f||_{L_p} = \left\| \sum_{n=1}^{\infty} c_n b_n^{\mathrm{II}} \right\|_{L_p} \leq \sum_{n=1}^{\infty} ||c_n b_n^{\mathrm{II}}||_{L_p} = \sum_{n=1}^{\infty} |c_n| \|b_n^{\mathrm{II}}\|_{L_p}$$

$$\leq \sum_{n=1}^{\infty} |c_n| M_{\alpha,p} = M_{\alpha,p} \sum_{n=1}^{\infty} |c_n|.$$

Now, by Lemma 3.1 we know

$$\sum_{n=1}^{\infty} |c_n| \leq \phi^{-1} \left(k_\phi \sum_{n=1}^{\infty} \phi(|c_n|) \right).$$

Combining this with our previous inequality,

$$||f||_{L_p} \leq M_{\alpha,p} \sum_{n=1}^{\infty} |c_n| \Rightarrow ||f||_{L_p} \leq M_{\alpha,p} \phi^{-1} \left(k_\phi \sum_{n=1}^{\infty} \phi(|c_n|) \right).$$

Finally taking the infimum of the above inequality over all representations of f we have:

$$||f||_{L_p} \leq M_{\alpha,p} \phi^{-1} \left(k_\phi \|f\|_{B_\phi(\mu,\alpha)} \right).$$

In other words, $B_\phi(\mu, \alpha)$ is weighted continuously contained in $L_p[-\pi, \pi]$. Now consider $g \in A_\phi(\mu, \alpha)$. Using Theorem 3.3, $g \in B_\phi(\mu, \alpha)$ such that $||g||_{B_\phi(\mu,\alpha)} \leq 2\xi_\phi \left(\frac{1}{2^\alpha}\right) ||g||_{A_\phi(\mu,\alpha)}$, and we have $A_\phi(\mu, \alpha)$ weighted continuously contained in $L_p[-\pi, \pi]$. Indeed, there is a constant $M_{\alpha,p}$ such that $||g||_{L_p} \leq M_{\alpha,p} \phi^{-1} \left(k_\phi \|g\|_{B_\phi(\mu,\alpha)} \right)$ and we have the following relation between L_p and $A_\phi(\mu, \alpha)$:

$$||g||_{L_p} \leq M_{\alpha,p} \phi^{-1} \left(k_\phi \|g\|_{B_\phi(\mu,\alpha)} \right) \leq M_{\alpha,p} \phi^{-1} \left(k_\phi 2\xi_\phi \left(\frac{1}{2^\alpha} \right) ||g||_{A_\phi(\mu,\alpha)} \right). \square$$

We now have all of the tools needed to demonstrate that $A_\phi(\mu, \alpha)$ and $B_\phi(\mu, \alpha)$ are complete weighted metric spaces. We will provide the proof for the $B_\phi(\mu, \alpha)$ case since the $A_\phi(\mu, \alpha)$ case is similar.

Theorem 3.6 (Completeness of $A_\phi(\mu,\alpha)$ and $B_\phi(\mu,\alpha)$) *For $\alpha \in (0,1)$ the spaces $A_\phi(\mu,\alpha)$ and $B_\phi(\mu,\alpha)$ are complete.*

Proof. In order to demonstrate the completeness of $B_\phi(\mu,\alpha)$, we apply Theorem 3.4. To this end, let $\langle f_n \rangle$ be an absolutely summable sequence in $B_\phi(\mu,\alpha)$, then $\sum_{n=1}^{\infty} \|f_n\|_{B_\phi(\mu,\alpha)} < \infty$. Let $c_{n_m} \in \mathbb{R}$ such that $f_n(t) = \sum_{m=1}^{\infty} c_{n_m} b_{n_m}^{\mathtt{II}}(t)$ with $\sum_{m=1}^{\infty} \phi(|c_{n_m}|) < \infty$ for each $n \in \mathbb{N}$. We must show that $\sum_{n=1}^{\infty} f_n$ converges in $B_\phi(\mu,\alpha)$. By definition of $\|\|_{B_\phi(\mu,\alpha)}$, for all $\epsilon > 0$ there is a representation of f_n such that

$$\sum_{m=1}^{\infty} \phi(|c_{n_m}|) \leq \|f_n\|_{B_\phi(\mu,\alpha)} + \frac{\epsilon}{2^n}$$

for each n. Let $f(t) = \sum_{n=1}^{\infty} f_n(t)$, then $f(t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_{n_m} b_{n_m}^{\mathtt{II}}(t)$ and

$$\|f\|_{B_\phi(\mu,\alpha)} \leq \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \phi\left(|c_{n_m}|\right) \leq \sum_{n=1}^{\infty} \|f_n\|_{B_\phi(\mu,\alpha)} + \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \sum_{n=1}^{\infty} \|f_n\|_{B_\phi(\mu,\alpha)} + \epsilon < \infty.$$

Applying Theorem 3.4, we conclude that $B_\phi(\mu,\alpha)$ is complete.□

3.3 Relationships within $B_\phi(\mu,\alpha)$ and $A_\phi(\mu,\alpha)$

The following discussion presents two theorems regarding the relation of these spaces for different measures, alpha values and base functions ϕ. Theorems for both spaces are presented below while the proof is given for only the $B_\phi(\mu,\alpha)$ case. The proof for the $A_\phi(\mu,\alpha)$ is again analogous.

Theorem 3.7 (Relationship of A_ϕ spaces) *Given $A_{\phi_1}(\mu_1,\alpha_1)$ and $A_{\phi_2}(\mu_2,\alpha_2)$, if $\alpha_2 \leq \alpha_1$ and there exists $k_1, k_2 \in \mathbb{R}^+$ such that $\phi_1(t) \leq k_1\phi_2(t)$, and $\mu_1(E) \leq k_2\mu_2(E)$ for all μ_1, μ_2-measurable sets $E \subseteq [-\pi,\pi]$ then $A_{\phi_2}(\mu_2,\alpha_2) \subseteq A_{\phi_1}(\mu_1,\alpha_1)$. Moreover, there exists a constant $M \in \mathbb{R}^+$ such that $\|f\|_{A_{\phi_1}(\mu_1,\alpha_1)} \leq M\|f\|_{A_{\phi_2}(\mu_2,\alpha_2)}$.*

Theorem 3.8 (Relationship of B_ϕ spaces) *Given $B_{\phi_1}(\mu_1, \alpha_1)$ and $B_{\phi_2}(\mu_2, \alpha_2)$, if $\alpha_2 \leq \alpha_1$ and there exists $k_1, k_2 \in \mathbb{R}^+$ such that $\phi_1(t) \leq k_1\phi_2(t)$, and $\mu_1(E) \leq k_2\mu_2(E)$ for all μ_1, μ_2-measurable sets $E \subseteq [-\pi, \pi]$ then $B_{\phi_2}(\mu_2, \alpha_2) \subseteq B_{\phi_1}(\mu_1, \alpha_1)$. Moreover, there exists a constant $M \in \mathbb{R}^+$ such that $\|f\|_{B_{\phi_1}(\mu_1,\alpha_1)} \leq M\|f\|_{B_{\phi_2}(\mu_2,\alpha_2)}$*

Proof of Theorem 3.8.

Let $B_{\phi_1}(\mu_1, \alpha_1)$, $B_{\phi_2}(\mu_2, \alpha_2)$ and $k_1, k_2 \in \mathbb{R}^+$ be given such that $\alpha_2 \leq \alpha_1$, $\phi_1(t) \leq k_1\phi_2(t)$, and $\mu_1(E) \leq k_2\mu_2(E)$. Let $f \in B_{\phi_2}(\mu_2, \alpha_2)$ then

$$f(t) = \sum_{n=1}^{\infty} c_n b_n^{\mathtt{II}}(t) \text{ and } \sum_{n=1}^{\infty} \phi_2(|c_n|) < \infty$$

Substituting the definition of weighted generalized special atom of Type II we have:

$$\begin{aligned} f(t) &= \sum_{n=1}^{\infty} \frac{c_n \chi_{B_n}(t)}{\mu_2^{\alpha_2}(B_n)} \\ &= \sum_{n=1}^{\infty} c_n \left(\frac{\mu_1^{\alpha_1}(B_n)}{\mu_2^{\alpha_2}(B_n)} \right) \frac{\chi_{B_n}(t)}{\mu_1^{\alpha_1}(B_n)}. \end{aligned}$$

Now, we have:

$$\frac{\mu_1^{\alpha_1}(B_n)}{\mu_2^{\alpha_2}(B_n)} \leq \frac{k_2^{\alpha_1}\mu_2^{\alpha_1}(B_n)}{\mu_2^{\alpha_2}(B_n)} = k_2^{\alpha_1}\mu_2^{(\alpha_1-\alpha_2)}(B_n) \leq k_2^{\alpha_1}(\mu_2)^{(\alpha_1-\alpha_2)}([-\pi, \pi]).$$

Let $M_1 = k_2^{\alpha_1}(\mu_2)^{(\alpha_1-\alpha_2)}([-\pi, \pi])$.

Utilizing conditions (2.1) and (2.2) on C_ϕ:

$$\sum_{n=1}^{\infty} \phi_1(\left| c_n \frac{\mu_1^{\alpha_1}(B_n)}{\mu_2^{\alpha_2}(B_n)} \right|) = \sum_{n=1}^{\infty} \phi_1(|c_n| \frac{\mu_1^{\alpha_1}(B_n)}{\mu_2^{\alpha_2}(B_n)}) \leq \sum_{n=1}^{\infty} \phi_1(M_1|c_n|)$$

$$\leq \sum_{n=1}^{\infty} \xi_{\phi_1}(M_1)\phi_1(|c_n|).$$

Continuing the above inequality and using the fact that $\phi_1 \le k_1\phi_2$,

$$\sum_{n=1}^{\infty} \xi_{\phi_1}(M_1)\phi_1(|c_n|) = \xi_{\phi_1}(M_1)\sum_{n=1}^{\infty} \phi_1(|c_n|) \le \xi_{\phi_1}(M_1)k_1\sum_{n=1}^{\infty} \phi_2(|c_n|).$$

So,

$$\sum_{n=1}^{\infty} \phi_1(\left|c_n\frac{\mu_1^{\alpha_1}(B_n)}{\mu_2^{\alpha_2}(B_n)}\right|) < \infty$$

Thus, $f \in B_{\phi_1}(\mu_1, \alpha_1)$ and $B_{\phi_2}(\mu_2, \alpha_2) \subseteq B_{\phi_1}(\mu_1, \alpha_1)$ Finally, letting $M = k_1\xi_{\phi_1}(M_1)$ and taking the infimum of both sides of the above inequality over all representations of f we conclude $\|f\|_{B_{\phi_1}(\mu_1,\alpha_1)} \le M\|f\|_{B_{\phi_2}(\mu_2,\alpha_2)}$.$\square$

Chapter 4

Major Results

Armed with the basic properties of $A_\phi(\mu,\alpha)$ and $B_\phi(\mu,\alpha)$ we am now able to prove deeper results. This chapter discusses the duality of the new spaces as well as interpolation of operator theorems.

4.1 Hölder-Type Inequalities

To find the dual spaces of $A_\phi(\mu,\alpha)$ and $B_\phi(\mu,\alpha)$, a first step is to find a Hölder-type inequality for each space. Essentially, we desire to find a function space X associated with $B_\phi(\mu,\alpha)$ such that for $g \in X$ and $f \in B_\phi(\mu,\alpha)$ we have a result similar to:

$$\left|\int_{-\pi}^{\pi} g(t)f(t)d\mu(t)\right| \leq \|g\|_X \|f\|_{B_\phi(\mu,\alpha)}$$

Similarly, we would like to find a function space Y to couple with $A_\phi(\mu,\alpha)$ for a Hölder-type inequality.

Consider the case for $B_\phi(\mu,\alpha)$. As in previous proofs, we first consider one weighted generalized special atom of Type $\mathtt{II}$. For g in our arbitrary function space X we then have:

$$\left|\int_{-\pi}^{\pi} g(t)b^{\mathtt{II}}(t)d\mu(t)\right| = \left|\int_{-\pi}^{\pi} g(t)\frac{\chi_B(t)}{\mu^\alpha(B)}d\mu(t)\right| = \frac{1}{\mu^\alpha(B)}\left|\int_B g(t)d\mu(t)\right|.$$

Thus, in order to establish our desired inequality we would like a function space X for g which will provide the bound above. Recalling the definition of $Lip(\mu,\alpha)$ given in Definition 2.12, we see that $Lip(\mu,\alpha)$ is the desired candidate space to couple with $B_\phi(\mu,\alpha)$. Similarly we find that $\Lambda(\mu,\alpha)$ given in Definition 2.13 is the desired candidate space to couple with $A_\phi(\mu,\alpha)$. The next two theorems and proofs validate the above choices for the desired inequalities.

Theorem 4.1 (Hölder's-type inequality for $B_\phi(\mu,\alpha)$) *For $f \in B_\phi(\mu,\alpha)$ and $g \in Lip(\mu,\alpha)$ the following inequality holds for $\alpha \in (0,1)$:*

$$\left|\int_{-\pi}^{\pi} g(t)f(t)d\mu(t)\right| \leq ||g||_{Lip(\mu,\alpha)} \cdot \phi^{-1}\left(k_\phi ||f||_{B_\phi(\mu,\alpha)}\right)$$

where k_ϕ is the constant given in condition (2.3) of Class C_ϕ functions.

Proof. Let $f \in B_\phi(\mu,\alpha)$ be a weighted generalized special atom of Type $\mathtt{II}$, that is $f(t) = b^{\mathtt{II}}(t)$, and let $g \in Lip(\mu,\alpha)$. Then, by the argument in the previous paragraph and the definition of $Lip(\mu,\alpha)$, we have:

$$\left|\int_{-\pi}^{\pi} g(t)f(t)d\mu(t)\right| = \left|\int_{-\pi}^{\pi} g(t)b^{\mathtt{II}}(t)d\mu(t)\right| = \frac{1}{\mu^\alpha(B)}\left|\int_B g(t)d\mu(t)\right| \leq ||g||_{Lip(\mu,\alpha)}.$$

Now, let $N \in \mathbb{N}$ and f_N be a finite combination of Type $\mathtt{II}$ weighted generalized special atoms. Then, $f_N(t) = \sum_{n=1}^{N} c_n b_n^{\mathtt{II}}(t)$ and we have:

$$\left|\int_{-\pi}^{\pi} g(t)f_N(t)d\mu(t)\right| = \left|\int_{-\pi}^{\pi} g(t)\sum_{n=1}^{N} c_n b_n^{\mathtt{II}}(t)d\mu(t)\right| = \left|\sum_{n=1}^{N} c_n \int_{-\pi}^{\pi} g(t)b_n^{\mathtt{II}}(t)d\mu(t)\right| \leq$$

$$\sum_{n=1}^{N}|c_n|\left|\int_{-\pi}^{\pi} g(t)b_n^{\mathtt{II}}(t)d\mu(t)\right| \leq \sum_{n=1}^{N}|c_n|\,||g||_{Lip(\mu,\alpha)} = ||g||_{Lip(\mu,\alpha)}\sum_{n=1}^{N}|c_n|\,.$$

Applying Lemma 3.1 gives the result:

$$\left|\int_{-\pi}^{\pi} g(t)f_N(t)d\mu(t)\right| \leq ||g||_{Lip(\mu,\alpha)} \cdot \phi^{-1}\left(k_\phi \sum_{n=1}^{N} \phi\left(|c_n|\right)\right).$$

Taking the infimum of the above inequality over all representations of f_N, we conclude

$$\left|\int_{-\pi}^{\pi} g(t)f_N(t)d\mu(t)\right| \leq ||g||_{Lip(\mu,\alpha)} \cdot \phi^{-1}\left(k_\phi \left\|f_N\right\|_{B_\phi(\mu,\alpha)}\right). \tag{4.1}$$

In order to extend equation (4.1) as $N \to \infty$ take $f \in B_\phi(\mu, \alpha)$, then there are real coefficients c_n such that $f(t) = \sum_{n=1}^{\infty} c_n b_n^{\mathtt{II}}(t)$ with $\sum_{n=1}^{\infty} \phi(|c_n|) \leq \infty$. Now for $N \in \mathbb{N}$ define $f_N(t) = \sum_{n=1}^{N} c_n b_n^{\mathtt{II}}(t)$, then $f_N \in B_\phi(\mu, \alpha)$ and it is clear that $\lim_{n\to\infty} f_N(t) = f(t)$.
Let $r_N = \int_{-\pi}^{\pi} g(t) f_N(t) d\mu(t)$, then $r_N \in \mathbb{R}$ since from equation (4.1) we know

$$|r_N| \leq ||g||_{Lip(\mu,\alpha)} \cdot \phi^{-1}\left(k_\phi \left\|f_N\right\|_{B_\phi(\mu,\alpha)}\right).$$

Consider the sequence $\langle r_N \rangle$. Let $M \in \mathbb{N}$ such that $N > M$, then

$$r_N - r_M = \int_{-\pi}^{\pi} \left(f_N(t) - f_M(t)\right) g(t) d\mu(t)$$

and

$$|r_N - r_M| \leq ||g||_{Lip(\mu,\alpha)} \cdot \phi^{-1}\left(k_\phi \left\|f_N - f_M\right\|_{B_\phi(\mu,\alpha)}\right).$$

Also,

$$\left\|f_N - f_M\right\|_{B_\phi(\mu,\alpha)} = \left\|\sum_{n=M+1}^{N} c_n b_n^{\mathtt{II}}\right\|_{B_\phi(\mu,\alpha)} \leq \sum_{n=M+1}^{N} \phi(|c_n|)$$

Since $\sum_{n=1}^{\infty} \phi(|c_n|) < \infty$, we know $\sum_{n=M+1}^{N} \phi(|c_n|) \to 0$ as $N, M \to \infty$, and hence $\left\|f_N - f_M\right\|_{B_\phi(\mu,\alpha)} \to 0$ as $N, M \to \infty$. Having $\phi \in C_\phi$, we know $\phi(0) = 0$ and ϕ is strictly increasing and it follows that:

$$|r_N - r_M| \leq ||g||_{Lip(\mu,\alpha)} \cdot \phi^{-1}\left(k_\phi \left\|f_N - f_M\right\|_{B_\phi(\mu,\alpha)}\right) \to 0 \text{ as } N, M \to \infty.$$

Thus, $\langle r_N \rangle$ is a Cauchy sequence in $\mathbb{R}$ and convergent in $\mathbb{R}$ since $\mathbb{R}$ is complete. In other words, $\lim_{N\to\infty} r_N = \lim_{N\to\infty} \int_{-\pi}^{\pi} g(t) f_N(t) d\mu(t)$ exists. We now demonstrate that this limit is independent of the sequence $\langle f_N \rangle$ converging to f. To this end, let $\langle h_N \rangle$ be a sequence of functions in $B_\phi(\mu, \alpha)$ that converges to f. Then we must show:

$$\lim_{N\to\infty} \int_{-\pi}^{\pi} g(t) h_N(t) d\mu(t) = \lim_{N\to\infty} \int_{-\pi}^{\pi} g(t) f_N(t) d\mu(t).$$

Consider then,

$$\lim_{N\to\infty}\left|\int_{-\pi}^{\pi}\left(f_N(t)-h_N(t)\right)g(t)d\mu(t)\right|\leq \lim_{N\to\infty}||g||_{Lip(\mu,\alpha)}\cdot\phi^{-1}\left(k_\phi\left\|f_N-h_N\right\|_{B_\phi(\mu,\alpha)}\right).$$

We next examine $\|f_N-h_N\|_{B_\phi(\mu,\alpha)}$:

$$\|f_N-h_N\|_{B_\phi(\mu,\alpha)}=\|f_N-f+f-h_N\|_{B_\phi(\mu,\alpha)}\leq k_w\left(\|f_N-f\|_{B_\phi(\mu,\alpha)}+\|f-h_N\|_{B_\phi(\mu,\alpha)}\right).$$

Since $\|f_N-f\|_{B_\phi(\mu,\alpha)}\to 0$ and $\|f-h_N\|_{B_\phi(\mu,\alpha)}\to 0$ as $N\to\infty$, we have

$$\lim_{N\to\infty}\left|\int_{-\pi}^{\pi}\left(f_N(t)-h_N(t)\right)g(t)d\mu(t)\right|\to 0 \text{ as } N\to\infty.$$

and we conclude that the limit as $N\to\infty$ is independent of the sequence used to converge to f in $B_\phi(\mu,\alpha)$. Hence, we define

$$\lim_{N\to\infty}\int_{-\pi}^{\pi}f_N(t)g(t)d\mu(t)\doteq\int_{-\pi}^{\pi}f(t)g(t)d\mu(t).$$

With this result, we can now take the limit as $N\to\infty$ of equation (4.1) to obtain our final result:

$$\lim_{N\to\infty}\left|\int_{-\pi}^{\pi}f_N(t)g(t)d\mu(t)\right|=\left|\int_{-\pi}^{\pi}f(t)g(t)d\mu(t)\right|\leq$$

$$\lim_{N\to\infty}||g||_{Lip(\mu,\alpha)}\cdot\phi^{-1}\left(k_\phi\left\|f_N\right\|_{B_\phi(\mu,\alpha)}\right)=||g||_{Lip(\mu,\alpha)}\cdot\phi^{-1}\left(k_\phi\left\|f\right\|_{B_\phi(\mu,\alpha)}\right).\square$$

Theorem 4.2 (Hölder's-type inequality for $A_\phi(\mu,\alpha)$) *For $f\in A_\phi(\mu,\alpha)$ and $g\in\Lambda(\mu,\alpha)$ the following inequality holds for $\alpha\in(0,1)$:*

$$\left|\int_{-\pi}^{\pi}g(t)f(t)d\mu(t)\right|\leq||g||_{\Lambda(\mu,\alpha)}\cdot\phi^{-1}\left(k_\phi||f||_{A_\phi(\mu,\alpha)}\right)$$

where k_ϕ is the constant given in condition (2.3) of Class C_ϕ functions.

Proof. Let $f \in A_\phi(\mu,\alpha)$ be a weighted generalized special atom of Type $\mathtt{I}$, that is $f(t) = b^{\mathtt{I}}(t)$, and let $g \in \Lambda(\mu,\alpha)$. Then there are μ-measurable sets $X, A, B \in [-\pi,\pi]$ such that $X = A \bigcup B$, $A \bigcap B = \emptyset$, $\mu(A) = \mu(B)$, and we have

$$\left|\int_{-\pi}^{\pi} g(t)f(t)d\mu(t)\right| = \left|\int_{-\pi}^{\pi} g(t)b^{\mathtt{I}}(t)d\mu(t)\right| = \left|\int_{-\pi}^{\pi} \frac{1}{\mu^\alpha(X_n)}[\chi_{A_n}(t) - \chi_{B_n}(t)]g(t)d\mu(t)\right| =$$

$$\frac{1}{\mu^\alpha(X)}\left|\int_A g(t)d\mu(t) - \int_B g(t)d\mu(t)\right| \leq ||g||_{\Lambda(\mu,\alpha)}.$$

Now, let $N \in \mathbb{N}$ and f_N be a finite combination of Type $\mathtt{I}$ weighted generalized special atoms. Then $f_N(t) = \sum_{n=1}^N c_n b_n^{\mathtt{I}}(t)$, and we have:

$$\left|\int_{-\pi}^{\pi} g(t)f_N(t)d\mu(t)\right| = \left|\int_{-\pi}^{\pi} g(t)\sum_{n=1}^N c_n b_n^{\mathtt{I}}(t)d\mu(t)\right| = \left|\sum_{n=1}^N c_n \int_{-\pi}^{\pi} g(t)b_n^{\mathtt{I}}(t)d\mu(t)\right| \leq$$

$$\sum_{n=1}^N |c_n| \left|\int_{-\pi}^{\pi} g(t)b_n^{\mathtt{I}}(t)d\mu(t)\right| \leq \sum_{n=1}^N |c_n|\, ||g||_{\Lambda(\mu,\alpha)} = ||g||_{\Lambda(\mu,\alpha)} \sum_{n=1}^N |c_n|\,.$$

Applying Lemma 3.1 and taking the infimum over all representations of f_N gives the result

$$\left|\int_{-\pi}^{\pi} g(t)f_N(t)d\mu(t)\right| \leq ||g||_{\Lambda(\mu,\alpha)} \cdot \phi^{-1}\left(k_\phi \|f_N\|_{B_\phi(\mu,\alpha)}\right). \tag{4.2}$$

In a similar fashion to the previous proof, we can show that if a sequence $\langle f_N \rangle$ converges to f in $A_\phi(\mu,\alpha)$, then $\lim\limits_{N\to\infty} \int_{-\pi}^{\pi} g(t)f_N(t)d\mu(t)$ exists and is independent of the choice of the sequence used to converge to f in $A_\phi(\mu,\alpha)$. Thus, for such a sequence $\langle f_N \rangle$ and $g \in \Lambda(\mu,\alpha)$ we define:

$$\lim_{N\to\infty} \int_{-\pi}^{\pi} f_N(t)g(t)d\mu(t) \doteq \int_{-\pi}^{\pi} f(t)g(t)d\mu(t).$$

Taking the limit as $N \to \infty$ in equation (4.2) above provides our desired result:

$$\left|\int_{-\pi}^{\pi} g(t)f(t)d\mu(t)\right| \leq ||g||_{\Lambda(\mu,\alpha)} \cdot \phi^{-1}\left(k_\phi ||f||_{A_\phi(\mu,\alpha)}\right). \square$$

The following Lemma is useful in applying the Hölder's-type inequality in Theorem 4.1 in the search for the dual of $B_\phi(\mu,\alpha)$.

Lemma 4.1 *Let $b^{\mathtt{II}}(t)$ be a weighted generalized special atom of Type* II*, then for $\alpha \in (0,1)$*

$$\left\|b^{\mathtt{II}}\right\|_{B_\phi(\mu,\alpha)} \cong \phi(1).$$

Proof. Let $\alpha \in (0,1)$ and $p = \frac{1}{\alpha}$, then for $f \in B_\phi(\mu,\alpha)$ by Theorem 3.5, $f \in L_p$ with $||f||_{L_p} \leq \phi^{-1}\left(k_\phi \|f\|_{B_\phi(\mu,\alpha)}\right)$ since $M_{\alpha,p} = 1$ in this case. Now let f be a weighted generalized special atom of Type II, then

$$\|f\|_{L_p}^p = \left\|b^{\mathtt{II}}\right\|_{L_{\frac{1}{\alpha}}}^{\frac{1}{\alpha}} = \int_{-\pi}^{\pi}\left(\frac{\chi_B(t)}{\mu^\alpha(B)}\right)^{\frac{1}{\alpha}} d\mu(t) = \frac{1}{\mu(B)}\int_B d\mu(t) = 1.$$

By the definition of $\|\|_{B_\phi(\mu,\alpha)}$ we have

$$1 = \left\|b^{\mathtt{II}}\right\|_{L_{\frac{1}{\alpha}}} \leq \phi^{-1}\left(k_\phi \left\|b^{\mathtt{II}}\right\|_{B_\phi(\mu,\alpha)}\right) \leq \phi^{-1}\left(k_\phi \phi(1)\right).$$

Taking ϕ of both sides of the above inequality we obtain

$$\phi(1) \leq k_\phi \left\|b^{\mathtt{II}}\right\|_{B_\phi(\mu,\alpha)} \leq k_\phi \phi(1) \Rightarrow \frac{\phi(1)}{k_\phi} \leq \left\|b^{\mathtt{II}}\right\|_{B_\phi(\mu,\alpha)} \leq \phi(1)$$

In other words, $\|b^{\mathtt{II}}\|_{B_\phi(\mu,\alpha)} \cong \phi(1)$.□

We can now examine the duality of $A_\phi(\mu,\alpha)$ and $B_\phi(\mu,\alpha)$.

4.2 Duality and Representation

Perhaps the most significant result of this dissertation is the following theorem regarding the dual of $B_\phi(\mu,\alpha)$. In fact, the dual of $B_\phi(\mu,\alpha)$ is a characterization of $Lip(\mu,\alpha)$.

Theorem 4.3 (Duality and Representation for $B_\phi(\mu,\alpha)$) *φ is a weighted bounded linear functional on $B_\phi(\mu,\alpha)$ if and only if there is a unique $g \in Lip(\mu,\alpha)$ so that for all $f \in B_\phi(\mu,\alpha)$ we have $\varphi(f) = \int_{-\pi}^{\pi} f(t)g(t)d\mu(t)$. Moreover, $\|\varphi\|$ is equivalent to $\|g\|_{Lip(\mu,\alpha)}$. That is there are absolute real constants c_1 and c_2 such that $c_2\|g\|_{Lip(\mu,\alpha)} \le \|\varphi\| \le c_1\|g\|_{Lip(\mu,\alpha)}$. In other words, the dual space of $B_\phi(\mu,\alpha)$ is equivalent to $Lip(\mu,\alpha)$. That is $B_\phi^*(\mu,\alpha) \cong Lip(\mu,\alpha)$.*

Proof. Let $g \in Lip(\mu,\alpha)$. Define $\varphi_g : B_\phi(\mu,\alpha) \to \mathbb{R}$ by $\varphi_g(f) \doteq \int_{-\pi}^{\pi} f(t)g(t)d\mu(t)$ for $f \in B_\phi(\mu,\alpha)$. By Theorem 4.1, φ_g is a weighted bounded linear functional since $|\varphi_g| \le ||g||_{Lip(\mu,\alpha)} \cdot \phi^{-1}\left(k_\phi ||f||_{B_\phi(\mu,\alpha)}\right)$ and the integral operator is linear. Now, define ψ as follows:

$$\psi : Lip(\mu,\alpha) \to B_\phi^*(\mu,\alpha), \qquad g \mapsto \psi(g) = \varphi_g.$$

Then ψ is one-to-one:

Let $g_1, g_2 \in Lip(\mu,\alpha)$ such that $\psi(g_1) = \psi(g_2)$, then for $f \in B_\phi(\mu,\alpha)$ we have

$$\int_{-\pi}^{\pi} f(t)g_1(t)d\mu(t) = \int_{-\pi}^{\pi} f(t)g_2(t)d\mu(t) \Rightarrow \int_{-\pi}^{\pi} f(t)\left(g_1(t) - g_2(t)\right)d\mu(t) = 0$$

Hence, we conclude that $g_1(t) = g_2(t)$ almost everywhere which is sufficient for our purposes by the definition of $\|\|_{Lip(\mu,\alpha)}$.

It remains to show that ψ is onto:

Let $\varphi \in B_\phi^*(\mu,\alpha)$, then we must show that there is a unique $g \in Lip(\mu,\alpha)$ such that $\psi(g) = \varphi$ or in other words, $\psi = \varphi_g$. For any μ-measurable set E in $[-\pi,\pi]$, define $\lambda(E) \doteq \varphi(\chi_E)$. Note that by this definition λ is a signed measure, see [34]. We show that $\lambda << \mu$. That is, we show that if $\mu(E) = 0$ then $\lambda(E) = 0$. By the definition of weighted bounded linear functionals, Definition 2.8, there are constants $M, k > 0$ and a continuous real valued function ξ with $\xi(0) = 0$ such that

$$|\lambda(E)| = |\varphi(\chi_E)| \le M\xi\left(k\left\|\chi_E\right\|_{B_\phi(\mu,\alpha)}\right).$$

Since $\chi_E(t) = \mu^\alpha(E)\frac{\chi_E(t)}{\mu^\alpha(E)}$, we have $\|\chi_E\|_{B_\phi(\mu,\alpha)} \leq \phi(\mu^\alpha(E))$. Now due to the continuity of ϕ and ξ if $\mu(E) \to 0$ then $k\phi(\mu^\alpha(E)) \to 0$ which combined with the above inequality implies

$$|\lambda(E)| \leq M\xi\left(k\|\chi_E\|_{B_\phi(\mu,\alpha)}\right) \leq M\xi\left(k\phi\left(\mu^\alpha(E)\right)\right) \to 0 \text{ as } \mu(E) \to 0.$$

Therefore, if $\mu(E) = 0$ then $\lambda(E) = 0$ and we have $\lambda << \mu$. By the Radon-Nikodym Theorem, there is a μ-measurable function g such that $\lambda(E) = \int_E g(t)d\mu(t)$. Thus,

$$\varphi(\chi_E) = \int_E g(t)d\mu(t) = \int_{-\pi}^{\pi} \chi_E(t)g(t)d\mu(t) \text{ and } \varphi\left(\frac{\chi_E}{\mu^\alpha(E)}\right) = \int_{-\pi}^{\pi} \frac{\chi_E(t)}{\mu^\alpha(E)}g(t)d\mu(t)$$

by the linearity of φ. Hence, $\varphi\left(b^{\texttt{II}}\right) = \int_{-\pi}^{\pi} b^{\texttt{II}}(t)g(t)d\mu(t)$ for any weighted generalized special atom of Type II. Now let f_N be a finite linear combination of Type II atoms, then there are real coefficients c_n such that $f_N(t) = \sum_{n=1}^{N} c_n b_n^{\texttt{II}}(t)$ and we have

$$\varphi(f_N) = \varphi\left(\sum_{n=1}^{N} c_n b_n^{\texttt{II}}(t)\right) = \sum_{n=1}^{N} c_n \varphi\left(b_n^{\texttt{II}}(t)\right) = \sum_{n=1}^{N} c_n \int_{-\pi}^{\pi} b_n^{\texttt{II}}(t)g(t)d\mu(t)$$

$$= \int_{-\pi}^{\pi} \sum_{n=1}^{N} c_n b_n^{\texttt{II}}(t)g(t)d\mu(t) = \int_{-\pi}^{\pi} f_N(t)g(t)d\mu(t)$$

so

$$\lim_{N\to\infty} \varphi(f_N) = \lim_{N\to\infty} \int_{-\pi}^{\pi} f_N(t)g(t)d\mu(t).$$

Now, let $f \in B_\phi(\mu,\alpha)$. Then $f(t) = \sum_{n=1}^{\infty} c_n b_n^{\texttt{II}}(t)$ and similar to previous proofs, we define the sequence $\langle f_N \rangle$ by $f_N(t) = \sum_{n=1}^{N} c_n b_n^{\texttt{II}}(t)$. It is clear that $\langle f_N \rangle \to f$ as $N \to \infty$. Since φ is a weighted bounded linear functional, we have $\lim\limits_{N\to\infty} \varphi(f_N) = \varphi(f)$. So, by a similar argument to the proof of Theorem 4.1 we conclude:

$$\varphi(f) = \lim_{N\to\infty} \int_{-\pi}^{\pi} f_N(t)g(t)d\mu(t) = \int_{-\pi}^{\pi} f(t)g(t)d\mu(t).$$

Thus, all that remains to show is that $g \in Lip(\mu, \alpha)$. That is, we must show $\frac{1}{\mu^\alpha(E)} \left| \int_E g(t) d\mu t \right| <$ K for some constant K. Consider then

$$\frac{1}{\mu^\alpha(E)} \left| \int_E g(t) d\mu(t) \right| = \frac{1}{\mu^\alpha(E)} \left| \int_{-\pi}^{\pi} \chi_E(t) g(t) d\mu(t) \right| = \left| \int_{-\pi}^{\pi} \frac{\chi_E(t)}{\mu^\alpha(E)} g(t) d\mu(t) \right| = \varphi \left(\frac{\chi_E(t)}{\mu^\alpha(E)} \right).$$

Again, using the definition of weighted bounded linear functional, we can continue the above equation with constants $M, k > 0$ such that

$$\varphi \left(\frac{\chi_E(t)}{\mu^\alpha(E)} \right) \leq M\xi \left(k \left\| \frac{\chi_E(t)}{\mu^\alpha(E)} \right\|_{B_\phi(\mu,\alpha)} \right).$$

Finally, applying Lemma 4.1 we have

$$\frac{1}{\mu^\alpha(E)} \left| \int_E g(t) d\mu(t) \right| \leq M\xi \left(k \left\| \frac{\chi_E(t)}{\mu^\alpha(E)} \right\|_{B_\phi(\mu,\alpha)} \right) \leq M\xi \left(k_2 \phi(1) \right) < K$$

where K and k_2 are positive constants. Thus, $g \in Lip(\mu, \alpha)$ and $\varphi = \varphi_g$. All that remains to show is $\|\varphi\| = \|\varphi_g\| \cong \|g\|_{Lip(\mu,\alpha)}$. By Definition 2.9, for $g \in Lip(\mu, \alpha)$ and $f \in B_\phi(\mu, \alpha)$ we have

$$\|\varphi_g\| = \sup_{f \neq 0} \frac{|\varphi_g(f)|}{\phi^{-1}(k_\phi \|f\|_X)}.$$

By Theorem 4.1

$$|\varphi_g(f)| \leq ||g||_{Lip(\mu,\alpha)} \cdot \phi^{-1} \left(k_\phi ||f||_{B_\phi(\mu,\alpha)} \right) \Rightarrow \frac{|\varphi_g(f)|}{\phi^{-1} \left(k_\phi ||f||_{B_\phi(\mu,\alpha)} \right)} \leq ||g||_{Lip(\mu,\alpha)}$$

and taking the supremum above over all $f \in B_\phi(\mu, \alpha)$ such that $f \neq 0$:

$$\sup_{f \neq 0} \frac{|\varphi_g(f)|}{\phi^{-1}(k_\phi \|f\|_{B_\phi(\mu,\alpha)})} \leq ||g||_{Lip(\mu,\alpha)}. \tag{4.3}$$

Now let E be a μ-measurable set in $[-\pi, \pi]$ and let $h(t) = \frac{\phi^{-1}\left(\frac{\phi(1)}{k_\phi}\right)}{\mu^\alpha(E)}\chi_E(t)$. Then $h(t) \in B_\phi(\mu, \alpha)$ and

$$\|h\|_{B_\phi(\mu,\alpha)} = \phi\left(\phi^{-1}\left(\frac{\phi(1)}{k_\phi}\right)\right) = \frac{\phi(1)}{k_\phi}$$

(see Lemma 4.1). Thus, we have the following:

$$\frac{|\varphi_g(h)|}{\phi^{-1}\left(k_\phi ||h||_{B_\phi(\mu,\alpha)}\right)} = \left|\int_{-\pi}^{\pi} \frac{\phi^{-1}\left(\frac{\phi(1)}{k_\phi}\right)\chi_E(t)g(t)d\mu(t)}{\mu^\alpha(E)}\right| \cdot \frac{1}{\phi^{-1}\left(k_\phi ||h||_{B_\phi(\mu,\alpha)}\right)} =$$

$$\phi^{-1}\left(\frac{\phi(1)}{k_\phi}\right)\frac{1}{\mu^\alpha(E)}\left|\int_E g(t)d\mu(t)\right| \cdot \frac{1}{\phi^{-1}\left(\frac{k_\phi \phi(1)}{k_\phi}\right)} = \phi^{-1}\left(\frac{\phi(1)}{k_\phi}\right)\frac{1}{\mu^\alpha(E)}\left|\int_E g(t)d\mu(t)\right|$$

Let $K_1 = \phi^{-1}\left(\frac{\phi(1)}{k_\phi}\right)$ and take the sup above over all $h \in B_\phi(\mu, \alpha)$ such that $h \neq 0$ gives the result

$$\sup_{h\neq 0}\frac{|\varphi_g(h)|}{\phi^{-1}(k_\phi\|h\|_{B_\phi(\mu,\alpha)})} \geq \frac{K_1}{\mu^\alpha(E)}\left|\int_E g(t)d\mu(t)\right|.$$

Taking the sup over all μ-measurable sets in $[-\pi, \pi]$ gives us $\|\varphi_g\| \geq K_1||g||_{Lip(\mu,\alpha)}$ and combining this equation with equation 4.3 we obtain our final inequality:

$$K_1||g||_{Lip(\mu,\alpha)} \leq \|\varphi_g\| \leq ||g||_{Lip(\mu,\alpha)}$$

and we have $\|\varphi_g\| \cong ||g||_{Lip(\mu,\alpha)}$.$\square$

4.3 Interpolation of Operators

The final theorems we will present are interpolation of operator theorems which extend sublinear operators from $B(\mu, \frac{1}{p})$ into weak L_p spaces to sublinear operators from $B_\phi(\mu, \alpha)$ to Lorentz spaces and similarly from $A(\mu, \frac{1}{p})$ into weak L_p to sublinear operators from $A_\phi(\mu, \alpha)$ into Lorentz.

Theorem 4.4 (Interpolation of Operators for $B_\phi(\mu,\alpha)$) *If $1 \le p_2 < p < p_1$ and T a sublinear operator such that:*

$$T : B(\mu, \frac{1}{p_1}) \to L(p_1, \infty) \text{ and } T : B(\mu, \frac{1}{p_2}) \to L(p_2, \infty)$$

and T is both restricted weak type p_1 with constant M_1 and weak type p_2 with constant M_2 then for $q \ge 1$ and $t = \frac{p_1(p_2-p)}{p(p_2-p_1)}$:

$$T : B_\phi(\mu, \frac{1}{p}) \to L(p,q) \text{ with } \|Tf\|_{pq} \le C M_1^t M_2^{1-t} \phi^{-1}\left(k_\phi \|f\|_{B_\phi(\mu,\frac{1}{p})}\right)$$

where C, k_ϕ are constants with k_ϕ the constant given in condition (2.3) of Class C_ϕ functions.

Proof. Let T be a sublinear operator that meets the conditions stated in the theorem above and let f be a special atom in $B_\phi(\mu, \frac{1}{p})$, then there is a μ-measurable set B in $[-\pi, \pi]$ such that

$$f(t) = \frac{\chi_B(t)}{\mu^{\frac{1}{p}}(B)} = \left(\frac{\mu^{\frac{1}{p_1}}(B)}{\mu^{\frac{1}{p}}(B)}\right)\left(\frac{\chi_B(t)}{\mu^{\frac{1}{p_1}}(B)}\right).$$

Therefore, $f \in B(\mu, \frac{1}{p_1})$ with $\|f\|_{B(\mu,\frac{1}{p_1})} \le \mu^{\frac{1}{p_1}-\frac{1}{p}}(B)$. Similarly, $f \in B(\mu, \frac{1}{p_2})$ with $\|f\|_{B(\mu,\frac{1}{p_2})} \le \mu^{\frac{1}{p_2}-\frac{1}{p}}(B)$. Also, from the conditions of the theorem we have:

$$(Tf)^*(t) = \left(T\left(\frac{\chi_B}{\mu^{\frac{1}{p}}(B)}\right)\right)^*(t) \le \frac{1}{\mu^{\frac{1}{p}}(B)}\left(T\chi_B\right)^*(t) \le \frac{M_1 \mu^{\frac{1}{p_1}-\frac{1}{p}}(B)}{t^{\frac{1}{p_1}}}$$

and similarly,

$$(Tf)^*(t) \le \frac{M_2 \mu^{\frac{1}{p_2}-\frac{1}{p}}(B)}{t^{\frac{1}{p_2}}}$$

Now for $q \ge 1$ by Definition 2.14:

$$\frac{p}{q}\|Tf\|_{pq}^q = \int_0^\infty \left((Tf)^*(t) t^{\frac{1}{p}}\right)^q \frac{dt}{t}.$$

We split this integral for some $\sigma \in (0, \infty)$ to be determined later. So we get the series of inequalities:

$$\frac{p}{q}\|Tf\|_{pq}^q = \int_0^\sigma \left((Tf)^*(t)t^{\frac{1}{p}}\right)^q \frac{dt}{t} + \int_\sigma^\infty \left((Tf)^*(t)t^{\frac{1}{p}}\right)^q \frac{dt}{t} \leq$$

$$\int_0^\sigma \left(\frac{M_1 \mu^{\frac{1}{p_1}-\frac{1}{p}}(B)t^{\frac{1}{p}}}{t^{\frac{1}{p_1}}}\right)^q \frac{dt}{t} + \int_\sigma^\infty \left(\frac{M_2 \mu^{\frac{1}{p_2}-\frac{1}{p}}(B)t^{\frac{1}{p}}}{t^{\frac{1}{p_2}}}\right)^q \frac{dt}{t} \leq$$

$$M_1^q \int_0^\sigma \left(\mu^{\frac{1}{p_1}-\frac{1}{p}}(B)t^{\frac{1}{p}-\frac{1}{p_1}}\right)^q \frac{dt}{t} + M_2^q \int_\sigma^\infty \left(\mu^{\frac{1}{p_2}-\frac{1}{p}}(B)t^{\frac{1}{p}-\frac{1}{p_2}}\right)^q \frac{dt}{t} =$$

$$M_1^q \mu^{\frac{q}{p_1}-\frac{q}{p}}(B) \int_0^\sigma t^{\frac{q}{p}-\frac{q}{p_1}-1} dt + M_2^q \mu^{\frac{q}{p_2}-\frac{q}{p}}(B) \int_\sigma^\infty t^{\frac{q}{p}-\frac{q}{p_2}-1} dt.$$

Continuing the algebra above:

$$= M_1^q \mu^{\left(\frac{q(p-p_1)}{pp_1}\right)}(B) \frac{pp_1}{q(p_1-p)} \sigma^{\frac{q(p_1-p)}{pp_1}} + M_2^q \mu^{\left(\frac{q(p-p_2)}{pp_2}\right)}(B) \frac{pp_2}{q(p_2-p)} \cdot \lim_{\lambda\to\infty} \frac{pp_2}{q(p_2-p)} t^{\frac{q(p_2-p)}{pp_2}} \Big|_\sigma^\infty$$

$$= M_1^q \mu^{\left(\frac{q(p-p_1)}{pp_1}\right)}(B) \frac{pp_1}{q(p_1-p)} \sigma^{\frac{q(p_1-p)}{pp_1}} + M_2^q \mu^{\left(\frac{q(p-p_2)}{pp_2}\right)}(B) \frac{pp_2}{q(p_2-p)} \frac{pp_2}{q(p_2-p)} \left(-\sigma^{\frac{q(p_2-p)}{pp_2}}\right).$$

since $p_2 < p$. So to summarize above inequalities up to this point

$$\frac{p}{q}\|Tf\|_{pq}^q \leq M_1^q \mu^{\left(\frac{q(p-p_1)}{pp_1}\right)}(B) \frac{pp_1}{q(p_1-p)} \sigma^{\frac{q(p_1-p)}{pp_1}} + M_2^q \mu^{\left(\frac{q(p-p_2)}{pp_2}\right)}(B) \frac{pp_2}{q(p-p_2)} \sigma^{\frac{q(p_2-p)}{pp_2}}.$$

Due to the fact that σ was arbitrary we can now let $\sigma = C\mu(B)$ where C is a constant to be determined. We now define the function $g(C)$ below in order to find a minimum for the above bound:

$$g(C) = M_1^q \mu^{\left(\frac{q(p-p_1)}{pp_1}\right)}(B) \frac{pp_1}{q(p_1-p)} (C\mu(B))^{\frac{q(p_1-p)}{pp_1}} + M_2^q \mu^{\left(\frac{q(p-p_2)}{pp_2}\right)}(B) \frac{pp_2}{q(p-p_2)} (C\mu(B))^{\frac{q(p_2-p)}{pp_2}}.$$

For simplification purposes let $A_1 = \frac{q(p-p_1)}{pp_1}$ and $A_2 = \frac{pp_2}{q(p-p_2)}$ we continuing above:

$$\frac{p}{q}\|Tf\|_{pq}^q \leq g(C) = M_1^q \mu^{\left(\frac{q(p-p_1)+q(p_1-p)}{pp_1}\right)}(B) A_1 C^{\frac{q(p_1-p)}{pp_1}} + M_2^q \mu^{\left(\frac{q(p-p_2)+q(p_2-p)}{pp_2}\right)}(B) A_2 C^{\frac{q(p_2-p)}{pp_2}}$$

Notice that the exponents in the $\mu(B)$ terms above are zero so

$$\frac{p}{q}\|Tf\|_{pq}^q \leq g(C) = M_1^q A_1 C^{\frac{q(p_1-p)}{pp_1}} + M_2^q A_2 C^{\frac{q(p_2-p)}{pp_2}}$$

It can be shown (see Appendix B) that $C = \left(\frac{M_1}{M_2}\right)^{\frac{p_2 p_1}{p_2-p_1}}$ minimizes $g(C)$. Thus, plugging this value for C in above:

$$\frac{p}{q}\|Tf\|_{pq}^q \leq M_1^q A_1 \left(\left(\frac{M_1}{M_2}\right)^{\frac{p_2 p_1}{p_2-p_1}}\right)^{\frac{q(p_1-p)}{pp_1}} + M_2^q A_2 \left(\left(\frac{M_1}{M_2}\right)^{\frac{p_2 p_1}{p_2-p_1}}\right)^{\frac{q(p_2-p)}{pp_2}}$$

We note here that $\frac{p_2(p_1-p)}{p(p_1-p_2)} - \frac{p_1(p_2-p)}{p(p_1-p_2)} = 1$ and if we let $t = \frac{p_1(p_2-p)}{p(p_2-p_1)}$ then $1-t = \frac{p_2(p_1-p)}{p(p_1-p_2)}$. Thus expanding the previous inequality and substituting t we have:

$$\frac{p}{q}\|Tf\|_{pq}^q \leq A_1 M_1^{q\left(1+\frac{p_2(p_1-p)}{p(p_2-p_1)}\right)} M_2^{q\left(\frac{p_2(p-p_1)}{p(p_2-p_1)}\right)} + A_2 M_1^{q\left(\frac{p_1(p_2-p)}{p(p_2-p_1)}\right)} M_2^{q\left(1+\frac{p_1(p-p_2)}{p(p_2-p)}\right)} =$$

$$A_1 M_1^{q(1+t-1)} M_2^{q(1-t)} + A_2 M_1^{qt} M_2^{q(1-t)}.$$

Since $q \geq 1$ we now have

$$\|Tf\|_{pq}^q \leq \frac{q}{p}(A_1 + A_2) M_1^{qt} M_2^{q(1-t)} \Rightarrow \|Tf\|_{pq} \leq \left(\frac{q}{p}(A_1 + A_2)\right)^{\frac{1}{q}} M_1^t M_2^{(1-t)}$$

Now let $C = \left(\frac{q}{p}(A_1 + A_2)\right)^{\frac{1}{q}}$ then $\|Tf\|_{pq} \leq C M_1^t M_2^{(1-t)}$ where C, M_1, M_2, t are all constants. Now for $1 \leq q \leq p < \infty$, $\|\|_{pq}$ is a true norm. Let $h \in B_\phi(\mu, \frac{1}{p})$, then there are real coefficients c_n such that $h(t) = \sum_{n=1}^\infty c_n b_n^{\mathtt{II}}(t)$ with $\sum_{n=1}^\infty \phi(|c_n|) < \infty$. Now utilizing the sublinearity of

T and the properties of norm:

$$\|Th\|_{pq} = \left\| T\sum_{n=1}^{\infty} c_n b_n^{\mathtt{II}} \right\|_{pq} \leq \left\| \sum_{n=1}^{\infty} c_n T b_n^{\mathtt{II}} \right\|_{pq} \leq \sum_{n=1}^{\infty} \left\| c_n T b_n^{\mathtt{II}} \right\|_{pq} = \sum_{n=1}^{\infty} |c_n| \left\| T b_n^{\mathtt{II}} \right\|_{pq} \leq$$

$$\sum_{n=1}^{\infty} |c_n| C M_1^t M_2^{(1-t)} = C M_1^t M_2^{(1-t)} \sum_{n=1}^{\infty} |c_n| \leq C M_1^t M_2^{(1-t)} \phi^{-1} \left(k_\phi \sum_{n=1}^{\infty} \phi(|c_n|) \right)$$

by Lemma 3.1. Taking the infimum above over all representations of f we have

$$\|Th\|_{pq} \leq C M_1^t M_2^{(1-t)} \phi^{-1} \left(k_\phi \, \|f\|_{B_\phi(\mu, \frac{1}{p})} \right).$$

and we conclude $T : B_\phi(\mu, \frac{1}{p}) \to L(p,q)$ as desired.$\square$ The next theorem is the interpolation theorem for $A_\phi(\mu, \alpha)$.

Theorem 4.5 (Interpolation of Operators for $A_\phi(\mu, \alpha)$) *If $1 \leq p_2 < p < p_1$ and T a sublinear operator such that:*

$$T : A(\mu, \frac{1}{p_1}) \to L(p_1, \infty) \text{ and } T : A(\mu, \frac{1}{p_2}) \to L(p_2, \infty)$$

and T is both restricted weak type p_1 with constant M_1 and weak type p_2 with constant M_2 then for $q \geq 1$ and $t = \frac{p_1(p_2-p)}{p(p_2-p_1)}$:

$$T : A_\phi(\mu, \frac{1}{p}) \to L(p,q) \text{ with } \|Tf\|_{pq} \leq C_p M_1^t M_2^{1-t} \phi^{-1} \left(k_\phi \|f\|_{B_\phi(\mu, \frac{1}{p})} \right)$$

where C_p, k_ϕ are constants with k_ϕ the constant given in condition (2.3) of Class C_ϕ functions.

Proof. Let T be a sublinear operator that meets the conditions stated in the theorem above and let f be a special atom in $A_\phi(\mu, \frac{1}{p})$, then there are μ-measurable sets X, A, B in $[-\pi, \pi]$ such that $A \bigcup B = X$, $A \bigcap B = \emptyset$, and $\mu(A) = \mu(B)$ with

$$f(t) = \frac{1}{\mu^{\frac{1}{p}}(X)} \left(\chi_A(t) - \chi_B(t) \right)$$

Then since $A(\mu, \frac{1}{p_i}) \subseteq B(\mu, \frac{1}{p_i})$ for $i = 1, 2$ we have

$$\|Tf\|_{pq} = \left\| T\left(\frac{1}{\mu^{\frac{1}{p}}(X)}\left(\chi_A(t) - \chi_B(t)\right)\right)\right\|_{pq} \leq \left\| T\left(\frac{\chi_A(t)}{\mu^{\frac{1}{p}}(X)}\right)\right\|_{pq} + \left\| T\left(\frac{\chi_B(t)}{\mu^{\frac{1}{p}}(X)}\right)\right\|_{pq} =$$

$$\left\| T\left(\frac{\mu^{\frac{1}{p}}(A)}{\mu^{\frac{1}{p}}(X)} \cdot \frac{\chi_A(t)}{\mu^{\frac{1}{p}}(A)}\right)\right\|_{pq} + \left\| T\left(\frac{\mu^{\frac{1}{p}}(B)}{\mu^{\frac{1}{p}}(X)} \cdot \frac{\chi_B(t)}{\mu^{\frac{1}{p}}(B)}\right)\right\|_{pq} \leq$$

$$\left(\frac{\mu(A)}{\mu(X)}\right)^{\frac{1}{p}} \left\| T\left(\frac{\chi_A(t)}{\mu^{\frac{1}{p}}(A)}\right)\right\|_{pq} + \left(\frac{\mu(B)}{\mu(X)}\right)^{\frac{1}{p}} \left\| T\left(\frac{\chi_B(t)}{\mu^{\frac{1}{p}}(B)}\right)\right\|_{pq}.$$

Now since $\mu(X) = 2\mu(A) = 2\mu(B)$ by Theorem 4.4 we have:

$$\|Tf\|_{pq} \leq \left(\frac{1}{2}\right)^{\frac{1}{p}} CM_1^t M_2^{1-t} + \left(\frac{1}{2}\right)^{\frac{1}{p}} CM_1^t M_2^{1-t} = C_p M_1^t M_2^{1-t}$$

where $C_p = 2^{1-\frac{1}{p}}C$. The remainder of the proof is identical to the proof of Theorem 4.4.$\square$

We now take a slight detour to discuss an interesting side result.

4.4 Multiplication Operator on $L(p, 1)$

One interesting result of the research for this dissertation involves a special case of the Lorentz Spaces (see Definition 2.14), with $q = 1$. Recall in Chapter 1 that de Souza showed that the Lorentz space $L(p, 1)$ is equivalent to $B(\mu, \frac{1}{p})$ for $p > 1$ with equivalent norms, see [9]. We use this fact to characterize all functions g so that the multiplication operator T_g defined by $T_g f \doteq g \cdot f$ maps $L(p, 1)$ into $L(p, 1)$ and is bounded.

We first review the motivation for this result. Given the linear multiplication operator T_g defined above, we would like to characterize all functions g so that for $f \in L(p, 1)$, $\|T_g f\|_{L(p,1)} \leq C\|f\|_{L(p,1)}$ where C is a constant. To this end, recall that for $f \in L(p, 1)$, then there are constants c_n and μ-measurable sets B_n in $[-\pi, \pi]$ such that $f(t) = \sum_{n=1}^{\infty} c_n \chi_{B_n}(t)$ with $\sum_{n=1}^{\infty} |c_n| \, \mu^{\frac{1}{p}}(B_n) < \infty$ since $L(p, 1)$ is equivalent to $B(\mu, \frac{1}{p})$. Thus we consider a generic

μ-measurable set B in $[-\pi, \pi]$. We note first that for a function g we have:

$$\chi_B(t)g(t) = \begin{cases} g(t), & \text{if } t \in B \\ 0, & \text{if } t \text{ is not in } B \end{cases} \qquad \text{and thus} \qquad (\chi_B(t)g(t))^* = \begin{cases} g^*(t), & \text{if } t \in [0, \mu(B)] \\ 0, & \text{otherwise} \end{cases}$$

where $*$ denotes the decreasing rearrangement of a function. As a result, for $T_g\chi_B = \chi_B \cdot g$ we have:

$$\int_{-\pi}^{\pi} (T_g\chi_B(t))^* \, t^{\frac{1}{p}-1} d\mu(t) = \int_{-\pi}^{\pi} (\chi_B(t)g(t))^* \, t^{\frac{1}{p}-1} d\mu(t) = \int_0^{\mu(B)} (g(t))^* \, t^{\frac{1}{p}-1} d\mu(t) =$$

$$\mu^{\frac{1}{p}}(B) \cdot \frac{1}{\mu^{\frac{1}{p}}(B)} \int_0^{\mu(B)} (g(t))^* \, t^{\frac{1}{p}-1} d\mu(t).$$

The above equation leads to our next definition which provides the means to solve the characterization required.

Definition 4.1 (X^p) *Given a finite measure space $([-\pi, \pi], \mathcal{A}, \mu)$, for $B \in \mathcal{A}$, define the space X^p as:*

$$X^p = \left\{ g : [-\pi, \pi] \to \mathbb{R} \mid \frac{1}{\mu^{\frac{1}{p}}(B)} \int_0^{\mu(B)} (g(t))^* \, t^{\frac{1}{p}-1} d\mu(t) < M \right\}$$

where M is an absolute constant. For $g \in X^p$ define a "norm" as

$$\|g\|_{X^p} = \sup_{\mu(B) \neq 0} \frac{1}{\mu^{\frac{1}{p}}(B)} \int_0^{\mu(B)} (g(t))^* \, t^{\frac{1}{p}-1} d\mu(t).$$

Utilizing this new space we now present our theorem.

Theorem 4.6 (Characterization of Multiplication Operator on $L(p, 1)$)
For $1 < p < \infty$, $T_g : L(p,1) \to L(p,1)$ defined by $T_g f \doteq g \cdot f$ is bounded if and only if $g \in X^p$. Moreover, $\|T_g\| \cong \|g\|_{X^p}$, and $X^p \subseteq L(p,1)$ with $\|g\|_{L(p,1)} \leq (2\pi)^{\frac{1}{p}} \|g\|_{X^p}$.

Proof. (⇒) Assume T_g is bounded so there is a constant C such that for $f \in L(p,1)$, $\|T_g f\|_{L(p,1)} \leq C\|f\|_{L(p,1)}$. Let $B \in \mathcal{A}$ and $f(t) = \chi_B(t)$. Then

$$\int_{-\pi}^{\pi} (\chi_B(t)g(t))^* \, t^{\frac{1}{p}-1} d\mu(t) \leq C \int_{-\pi}^{\pi} (\chi_B(t))^* \, t^{\frac{1}{p}-1} d\mu(t) = C \int_{-\pi}^{\pi} \chi_{[0,\mu(B)]}(t) t^{\frac{1}{p}-1} d\mu(t) =$$

$$C \int_0^{\mu(B)} t^{\frac{1}{p}-1} d\mu(t) = pC\mu^{\frac{1}{p}}(B).$$

So

$$\int_{-\pi}^{\pi} (\chi_B(t)g(t))^* \, t^{\frac{1}{p}-1} d\mu(t) = \int_0^{\mu(B)} g(t)^* t^{\frac{1}{p}-1} d\mu(t) \leq pC\mu^{\frac{1}{p}}(B).$$

Thus,

$$\frac{1}{\mu^{\frac{1}{p}}(B)} \int_0^{\mu(B)} g(t)^* t^{\frac{1}{p}-1} d\mu(t) \leq pC$$

and taking the supremum above we conclude $\|g\|_{X^p} < pC$ and thus $g \in X^p$.

(⇐) Now let $g \in X^p$. First consider $\chi_B(t)$ where $B \in \mathcal{A}$. Then

$$\|T_g \chi_B\|_{L(p,1)} = \int_{-\pi}^{\pi} (T_g \chi_B(t))^* \, t^{\frac{1}{p}-1} d\mu(t) = \int_{-\pi}^{\pi} (\chi_B(t)g(t))^* \, t^{\frac{1}{p}-1} d\mu(t) = \int_0^{\mu(B)} g^*(t) t^{\frac{1}{p}-1} d\mu(t)$$

$$= \mu^{\frac{1}{p}}(B) \cdot \frac{1}{\mu^{\frac{1}{p}}(B)} \int_0^{\mu(B)} g^*(t) t^{\frac{1}{p}-1} d\mu(t) \leq \mu^{\frac{1}{p}}(B) \sup_{\mu(B) \neq 0} \frac{1}{\mu^{\frac{1}{p}}(B)} \int_0^{\mu(B)} g^*(t) t^{\frac{1}{p}-1} d\mu(t).$$

We conclude

$$\|T_g \chi_B\|_{L(p,1)} \leq \|g\|_{X^p} \mu^{\frac{1}{p}}(B).$$

Now let $f \in L(p,1)$ then there are constant c_n and sets B_n in $\mathcal{A}$ such that $f(t) = \sum_{n=1}^{\infty} c_n \chi_{B_n}(t)$ with $\sum_{n=1}^{\infty} |c_n| \, \mu^{\frac{1}{p}}(B_n) < \infty$, then

$$\|T_g f\|_{L(p,1)} = \|T_g \sum_{n=1}^{\infty} c_n \chi_{B_n}(t)\|_{L(p,1)} \leq \sum_{n=1}^{\infty} |c_n| \, \|T_g \chi_{B_n}(t)\|_{L(p,1)} \leq \sum_{n=1}^{\infty} |c_n| \, \|g\|_{X^p} \mu^{\frac{1}{p}}(B) =$$

$$\|g\|_{X^p} \sum_{n=1}^{\infty} |c_n| \, \mu^{\frac{1}{p}}(B) \leq \|g\|_{X^p} \|f\|_{B(\mu,\frac{1}{p})}$$

where we obtained the last inequality above by taking the infimum over all representations of f in $B(\mu, \frac{1}{p})$. Thus since $B(\mu, \frac{1}{p})$ is equivalent to $L(p, 1)$ with equivalent norms we conclude that T_g is bounded. Note that since $\chi_{[-\pi,\pi]}(t) \in L(p, 1)$ and $\|\chi_{[-\pi,\pi]}(t)\|_{L(p,1)} = (2\pi)^{\frac{1}{p}}$, then $b(t) = \frac{1}{(2\pi)^{\frac{1}{p}}}\chi_{[-\pi,\pi]}(t) \in L(p, 1)$ with $\|b\|_{L(p,1)} = 1$. Thus for $g \in X^p$, $T_g\chi_{[-\pi,\pi]} = \chi_{[-\pi,\pi]}(t) \cdot g(t) = g(t) \in L(p, 1)$ and $X^p \subset L(p, 1)$. If we take $f(t) = 1$ on $[-\pi, \pi]$ then $\|f\|_{L(p,1)} = (2\pi)^{\frac{1}{p}}$ and $\|gf\|_{L(p,1)} = \|g\|_{L(p,1)} \leq (2\pi)^{\frac{1}{p}}\|g\|_{X^p}$. Thus, $X^p \subseteq L(p, 1)$ with $\|g\|_{L(p,1)} \leq (2\pi)^{\frac{1}{p}}\|g\|_{X^p}\Box$.

Chapter 5

Comments on the dual of $A_\phi(\mu, \alpha)$

This chapter examines the possible dual of $A_\phi(\mu, \alpha)$. As we have seen in Theorem 4.3, $Lip(\mu, \alpha)$ is equivalent to the dual of $B_\phi(\mu, \alpha)$. While we have not proved that the dual of $A_\phi(\mu, \alpha)$ is $\Lambda(\mu, \alpha)$, we will show that this is most likely the case due to several factors. We start by formalizing the relationship between $Lip(\mu, \alpha)$ and $\Lambda(\mu, \alpha)$.

5.1 Relationship between $Lip(\mu, \alpha)$ and $\Lambda(\mu, \alpha)$

The following theorem demonstrates the connection between $Lip(\mu, \alpha)$ and $\Lambda(\mu, \alpha)$ and is due to de Souza and Pozo, see [6].

Theorem 5.1 (Equivalence of $Lip(\mu, \alpha)$ and $\Lambda(\mu, \alpha)$) *For $\alpha \in (0, 1)$ the spaces $Lip(\mu, \alpha)$ and $\Lambda(\mu, \alpha)$ are equivalent as Banach Spaces. That is, there are positive constants M, N such that*

$$M\|g\|_{Lip(\mu,\alpha)} < \|g\|_{\Lambda(\mu,\alpha)} < N\|g\|_{Lip(\mu,\alpha)}.$$

In order to prove Theorem 5.1 we provide another Theorem also by de Souza and Pozo, see [6].

Theorem 5.2 (de Souza and Pozo) *Let $(\Omega, \mathcal{A}, \mu)$ be a finite positive, nontrivial measure space and $\alpha \in (0, 1)$ with $f : \Omega \to \mathbb{R}$ such that $f \in L^1(\Omega, \mathcal{A}, \mu)$, then*

$$\sup_{C,D\in\mathcal{A},\mu(C\triangle D)\neq 0} \frac{\left|\int_C f(t)d\mu(t) - \int_D f(t)d\mu(t)\right|}{\mu^\alpha(C\triangle D)} \cong \sup_{A\in\mathcal{A},\mu(A)\neq 0} \frac{\left|\int_A f(t)d\mu(t)\right|}{\mu^\alpha(A)} \cong \sup_{A\in\mathcal{A},\mu(A)\neq 0} \frac{\int_A |f(t)|d\mu(t)}{\mu^\alpha(A)}.$$

We now prove Theorem 5.1. Although one can use Theorem 5.2 for the complete proof, we will provide the argument for one side of the inequality in Theorem 5.1.

Proof. Let $g \in Lip(\mu, \alpha)$ and A, B, X be μ-measurable sets such that $A \bigcap B = \emptyset$ and $X = A \bigcup B$. Consider then:

$$\frac{1}{\mu^\alpha(X)} \left| \int_A g(t) d\mu(t) - \int_B g(t) d\mu(t) \right| \leq \frac{1}{\mu^\alpha(X)} \left(\left| \int_A g(t) d\mu(t) \right| + \left| \int_B g(t) d\mu(t) \right| \right) =$$

$$\frac{\mu^\alpha(A)}{\mu^\alpha(X)} \left(\frac{1}{\mu^\alpha(A)} \left| \int_A g(t) d\mu(t) \right| \right) + \frac{\mu^\alpha(B)}{\mu^\alpha(X)} \left(\frac{1}{\mu^\alpha(B)} \left| \int_B g(t) d\mu(t) \right| \right) \leq 2\|g\|_{Lip(\mu,\alpha)}.$$

Taking the sup of the above inequality over all μ-measurable sets A, B, X such that $A \bigcap B = \emptyset$ and $X = A \bigcup B$ gives the result

$$\|g\|_{\Lambda(\mu,\alpha)} < 2\|g\|_{Lip(\mu,\alpha)}.$$

For the other side of the inequality, apply Theorem 5.2 with $C \bigcap D = \emptyset$.□

5.2 Closing Arguments

Given the Hölder's-type inequality for $A_\phi(\mu, \alpha)$ in Theorem 4.2, the dual of $A_\phi(\mu, \alpha)$ is in fact $\Lambda(\mu, \alpha)$. In order to see that we need an estimate for $\|\chi_A\|_{A(\mu,\alpha)}$ where A is a μ-measurable set in $[-\pi, \pi]$. Indeed using Theorem 1 by F.F. Bonsall along with the Closed Range Theorem for quasi Banach spaces one can show that $\|\chi_A\|_{A(\mu,\alpha)} \leq M\mu^\alpha(A)$, see [37]. Thus, along with the fact that $\Lambda(\mu, \alpha)$ and $Lip(\mu, \alpha)$ are equivalent as Banach spaces, it seems reasonable that the spaces $A_\phi(\mu, \alpha)$ and $B_\phi(\mu, \alpha)$ are equivalent (recall the dual of $B_\phi(\mu, \alpha)$ is $Lip(\mu, \alpha)$). Recall also that de Souza showed that the spaces $A(\mu, \alpha)$ and $B(\mu, \alpha)$ are equivalent so the extension to $A_\phi(\mu, \alpha)$ and $B_\phi(\mu, \alpha)$ as equivalent spaces is logical.

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Appendices

Appendix A

Vector Space proof

This appendix provides the remainder of the proof of Theorem 3.1 which states $A_\phi(\mu,\alpha)$ and $B_\phi(\mu,\alpha)$ are vector spaces. We will provide the proof for $B_\phi(\mu,\alpha)$. The proof in the $A_\phi(\mu,\alpha)$ is analogous. Closure under addition was proved following the statement of Theorem 3.1.

Proof. Let $u, v, w \in B_\phi(\mu,\alpha)$ and a, b be real scalars. Then by definition, u, v, w are all linear combinations of weighted general atoms of Type $\mathtt{II}$. Thus, associative and commutative properties are true by properties of summations. Also, the identity element for addition is zero which is in $B_\phi(\mu,\alpha)$ by setting all coefficients to zero. Clearly $av \in B_\phi(\mu,\alpha)$. Since $v \in B_\phi(\mu,\alpha)$, there are real coefficients c_{v_n} such that

$$v = \sum_{n=1}^{\infty} c_{v_n} b_{v_n}^{\mathtt{II}}(t), \quad \sum_{n=1}^{\infty} \phi(|c_{v_n}|) < \infty.$$

Hence,

$$av = \sum_{n=1}^{\infty} ac_{v_n} b_{v_n}^{\mathtt{II}}(t) \quad \text{and} \quad \sum_{n=1}^{\infty} \phi(|ac_{v_n}|) \le \sum_{n=1}^{\infty} \xi_\phi(|a|)\phi(|c_{v_n}|) = \xi_\phi(|a|) \sum_{n=1}^{\infty} \phi(|c_{v_n}|) < \infty$$

by property (2.2) of C_ϕ functions and therefore $av \in B_\phi(\mu,\alpha)$. The distributive properties are clear due to properties of summations: for any scalars a, b, we have $a(v+w) = av + aw$ and $(a+b)v = av + bv$ since we are dealing with the summation operator which is linear. Now the identity element for scalar multiplication is 1. Again, it is clear that $1 \in B_\phi(\mu,\alpha)$, since

$$\mu([-\pi,\pi]) b_1^{\mathtt{II}(t)} = \frac{\mu([-\pi,\pi])}{\mu([-\pi,\pi])} = 1.$$

Finally, we show that for $v \in B_\phi(\mu,\alpha)$, there exists a $w \in B_\phi(\mu,\alpha)$ such that $v + w = 0$. Let $v \in B_\phi(\mu,\alpha)$ then similar to above we have real coefficients c_{v_n} such that

$$v = \sum_{n=1}^{\infty} c_{v_n} b_{v_n}^{\mathtt{II}}(t), \quad \sum_{n=1}^{\infty} \phi(|c_{v_n}|) < \infty.$$

Now, let

$$w = \sum_{n=1}^{\infty} -c_{v_n} b_{v_n}^{\mathtt{II}}(t)$$

then $w \in B_\phi(\mu, \alpha)$ and

$$v + w = \sum_{n=1}^{\infty} c_{v_n} b^{\mathtt{II}}_{v_n}(t) + \sum_{n=1}^{\infty} -c_{v_n} b^{\mathtt{II}}_{v_n}(t) = \sum_{n=1}^{\infty} (c_{v_n} - c_{v_n})\, b^{\mathtt{II}}_{v_n}(t) = 0.$$

Therefore, we conclude that $B_\phi(\mu, \alpha)$ is indeed a vector space over $\mathbb{R}$.$\square$

Appendix B

Verification of minimum in proof of Theorem 4.4

This work will prove the assertion in the proof of Theorem 4.4 that $C = \left(\frac{M_1}{M_2}\right)^{\frac{p_1 p_2}{p_2 - p_1}}$ is a minimum of $g(C)$. Recall that

$$g(C) = M_1^q A_1 C^{\frac{q(p_1-p)}{pp_1}} + M_2^q A_2 C^{\frac{q(p_2-p)}{pp_2}}$$

with

$$A_1 = \frac{pp_1}{q(p_1 - p)} \quad \text{and} \quad A_2 = \frac{pp_2}{q(p - p_2)}.$$

Note that $g(C)$ can be re-written as

$$g(C) = M_1^q A_1 C^{\frac{1}{A_1}} + M_2^q A_2 C^{-\frac{1}{A_2}}.$$

Taking the derivative of $g(C)$

$$g'(C) = M_1^q A_1 \frac{1}{A_1} C^{\frac{1}{A_1}-1} + M_2^q A_2 \left(-\frac{1}{A_2}\right) C^{-\frac{1}{A_2}-1}.$$

Setting $g'(C) = 0$ leads to

$$M_1^q C^{\frac{1}{A_1}-1} = M_2^q C^{-\frac{1}{A_2}-1} \Rightarrow \left(\frac{M_1}{M_2}\right)^q = C^{-\frac{1}{A_2}-1} C^{1-\frac{1}{A_1}} = C^{-\frac{1}{A_2}-\frac{1}{A_1}}.$$

Consider $-\frac{1}{A_2} - \frac{1}{A_1}$:

$$-\frac{1}{A_2} - \frac{1}{A_1} = \frac{q(p_2 - p)}{pp_2} + \frac{q(p - p_1)}{pp_1} = \frac{q(p_1(p_2 - p) + p_2(p - p_1))}{pp_1p_2} = \frac{q(p_2 - p_1)}{p_1p_2}.$$

Substituting above

$$\left(\frac{M_1}{M_2}\right)^q = C^{\frac{q(p_2-p_1)}{p_1p_2}} \Rightarrow C = \left(\frac{M_1}{M_2}\right)^{\frac{p_1p_2}{p_2-p_1}}.$$

It remains to show that this value for C is indeed a minimum. We now take the second derivative of $g(C)$:

$$g''(C) = M_1^q \left(\frac{1}{A_1} - 1\right) C^{\frac{1}{A_1}-2} - M_2^q \left(-\frac{1}{A_2} - 1\right) C^{-\frac{1}{A_2}-2}.$$

Now if $g''(C) > 0$ at $C = \left(\frac{M_1}{M_2}\right)^{\frac{p_1 p_2}{p_2 - p_1}}$ as desired we would have

$$M_1^q \left(\frac{1}{A_1} - 1\right) C^{\frac{1}{A_1} - 2} > M_2^q \left(-\frac{1}{A_2} - 1\right) C^{-\frac{1}{A_2} - 2} \Rightarrow$$

$$\left(\frac{M_1}{M_2}\right)^q \left(\frac{1}{A_1} - 1\right) > \left(-\frac{1}{A_2} - 1\right) C^{-\frac{1}{A_2} - 2 + 2 - \frac{1}{A_1}} = \left(-\frac{1}{A_2} - 1\right) C^{-\frac{1}{A_2} - \frac{1}{A_1}} = \left(-\frac{1}{A_2} - 1\right) \left(\frac{M_1}{M_2}\right)^q.$$

So we must show $\frac{1}{A_1} - 1 > -\frac{1}{A_2} - 1$ or $\frac{1}{A_1} > -\frac{1}{A_2}$. If we show this inequality then one can work backwards through the above inequalities showing $g''(C) > 0$ for $C = \left(\frac{M_1}{M_2}\right)^{\frac{p_1 p_2}{p_2 - p_1}}$ which guarantees this value of C is the minimum of $g(C)$. Now $\frac{p_1 - p}{p_1} > \frac{p_2 - p}{p_2}$ since $p_2 < p < p_1$, which implies that $\frac{q(p_1 - p)}{pp_1} > \frac{q(p_2 - p)}{pp_2}$ which is equivalent to $\frac{1}{A_1} > -\frac{1}{A_2}$. Therefore $C = \left(\frac{M_1}{M_2}\right)^{\frac{p_1 p_2}{p_2 - p_1}}$ is the minimum of $g(C)$.$\square$