Phenomenal Three-Dimensional Objects

by

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Abstract

Original thesis project: *By studying the literature, collect and write a survey paper on special three-dimensional polyhedra and bodies. Read and understand the results, to which these polyhedra and bodies are related.*

Although most of the famous examples are constructed in a very clever way, it was relatively easy to understand them. The challenge lied in the second half of the project, namely at studying the theory behind the models. I started to learn things related to Schönhardt's polyhedron, Császár's polyhedron, and a Meissner body. During Fall 2010 I was a frequent visitor of Dr. Bezdek's 2nd studio class at the Industrial Design Department where some of these models coincidently were fabricated. A change in the thesis project came while reading a paper of R. Guy about a polyhedron which was stable only if placed on one of its faces. It turned out that for sake of brevity many details were omitted in the paper, and that gave room for a substantial amount of independent work. As a result, the focus of the thesis project changed.

Modified thesis project: *By studying the literature, write a survey paper on results concerning stable polyhedra.*

The following is the outcome of the thesis:

1. A collection of elementary facts/theorems/proofs concerning stable tetrahedra.

2. A description of a double tipping tetrahedron constructed by A. Heppes.

3. A description of a 19 faceted polyhedron of R. Guy, which has only one stable face.

4. Description of the Gömböc, a recently discovered mono-monostatic 3D body.

Acknowledgments

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Chapter 1

Introduction and outline of the thesis

Since tetrahedra and other special polyhedra occur so often in physics, in chemistry and in crystallography, it's particularly useful to understand their geometry. In Chapters 2-5 of this thesis we will study the stability properties of such bodies. Intuitively speaking we want to know how many stable positions these bodies can have and those specific positions. Recall that in planar geometry, the centroid, geometric center, or barycenter of a plane figure refers to the intersection of all straight lines that divide the figure into two parts of equal moment about the line (equal moment means balanced position). This definition extends to any object in 3-dimensional space. The centroid of an object is the intersection of all planes that divide the body into two parts of equal moment.

In physics, the mass center or the center of gravity refers to the point where the object is supported while maintaining a balanced position. Informally, the center of mass (and center of gravity in a uniform gravitational field) is the average of all points, weighted by the local density or specific weight. This gives rise for using integral calculus at computing the coordinates of the mass center. For special shapes and special density distributions this computation is simple, otherwise it can be quite involved.

If a physical object has uniform density, then its center of mass is the same as the centroid of its shape. In the subsequent sections, if we do not say otherwise, polyhedra are assumed to be solids with uniform mass density. We will call a face of given polyhedra *stable* if the polyhedra will rest on that particular face in a stable position on top of a horizontal table.

The following is an outline of the thesis:

In Chapter 2 we study the geometry of tetrahedra. We will list and prove some basic properties concerning the mass center of tetrahedra and also describe some special tetrahedra to give an insight on what makes a face stable.

Chapter 3 contains the proof of Conway's theorem, which claims that every tetrahedron has at least two stable faces.

In Chapter 4 we explain a construction of Guy [7], which shows that there are polyhedra with exactly one stable face. This is our main section where most of our independent work was done. It turned out that a crucial step of the proof in Guy's paper was given without explanation. This step was about finding a closed formula for a quite elaborate integral. Pages 26-36 contain these computations. Moreover Guy's paper ended by saying that if certain distance is sufficiently large then the constructed model has the right stability properties. We computed the numerical value of this measurement and then 3D printed Guy's model. The printed model will lead other students to a better understanding of Guy's construction.

Chapter 5 contains a brief description of a recently discovered body called the Gömböc. The Gömböc is a body with exactly one stable and exactly one unstable position. In this section we include the proof of planar theorem on the existence of stable and unstable equilibriums. This result shows the difference both in difficulty and in nature of the 2D and the 3D versions of our stability questions.

Finally, the Appendix contains Chapter's 6-9 which are further reading on phenomenal three-dimensional objects. They are not contained in the main part of the thesis because they are not closely related, but much time was spent on this before the discovery of the monostatic polyhedron [7].

Chapter 2

Geometry of tetrahedra

2.1 Things to know about the mass center of tetrahedra

In this section we assume that the tetrahedra have uniform mass density. The mass center of such a tetrahedron always coincides with its centroid.

1. Simple coordinate calculus shows that the centroid of a tetrahedron with vertices $(x_i, y_i, z_i), i = 1, \ldots 4$ is the point $\left(\frac{\sum_1^4 x_i}{4}, \frac{\sum_1^4 y_i}{4}, \frac{\sum_1^4 z_i}{4}\right)$.

2. It follows from (1) that the centroid of a solid tetrahedron is located on the line segment that connects the apex to the centroid of the base, moreover the centroid is at 1/4 of the distance from the base to the apex.

3. It follows from (1) that the centroid of a solid tetrahedron is also the midpoint of three segments each connecting the midpoints of the opposite edges.

4. Assume that the base ABC of tetrahedron ABCD lies on a horizontal surface. (3) implies that this tetrahedron will tip over the edge BC if and only if the perpendicular projection D' of the vertex D onto the plane of ABC lies further from the edge BC than the vertex A. Thus, an obtuse dihedral angle (angle between two neighboring faces) is necessary for a tip, moreover a tetrahedron can tip from a face to another face only if the later has larger area. This simple observation will play a very important role in most of the proofs later.

5. The following is a natural question: At most how many obtuse dihedral angles can a tetrahedron have? It turns out that the maximum is three. Although none of the

proofs of our subsequent results will use this bound, we present a proof here for sake of completeness. The solution is by M. Klamkin [9]. There is at least one vertex of the tetrahedron such that its corresponding face angles are all acute. This is necessary because otherwise the sum of all face angles would be greater than 4π (the sum of two face angles at a vertex is greater than the third face angle). Since there are four faces, the sum must be equal to 4π.

The following two lemmas are for the face angles a, b, c, and the opposite dihedral angles of a trihedral angle.

Lemma 2.1. *If $\frac{\pi}{2} > a \geq b \geq c$, then B and C are acute.*

Lemma 2.1 follows from the sign analysis of the spherical law of cosine: $\sin b \sin c \sin A = \cos a - \cos b \cos c$, *etc.*

Lemma 2.2. *If $A \geq B \geq \frac{\pi}{2} > C$, then a, $b \geq \frac{\pi}{2} > c$.*

Lemma 2.2 follows from the sign analysis of the spherical law of cosine: $\sin B \sin C \cos a = \cos A + \cos B \cos C$, *etc.*

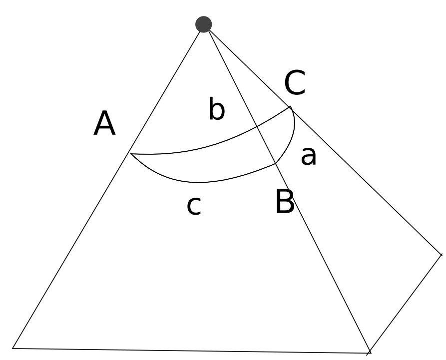


Figure 2.1: All tetrahedra have no more than three obtuse dihedral

Choose P as the vertex with all acute face angles in tetrahedron $PQRS$. By Lemma 1, we can take PQ and PS as the edges of the two acute dihedral angles. Now assume these are the only acute dihedral angles and obtain a contradiction. By the application of Lemma 2 to trihedral angles at vertices Q and S, it follows that angles PQS and PSQ are both non-acute. This is impossible, so there are always at least three acute dihedral angles in any tetrahedron.

6. Every tetrahedron has at least one stable face. To see this notice that a face of a given tetrahedron is stable iff it contains the perpendicular projection of its mass center. A simple indirect reasoning shows that the face nearest to the mass center has the later property.

7. As in (6) it follows that every tetrahedron with all acute dihedral angles has four stable faces.

8. Figure 2.2 shows a tetrahedron which has only two stable faces.

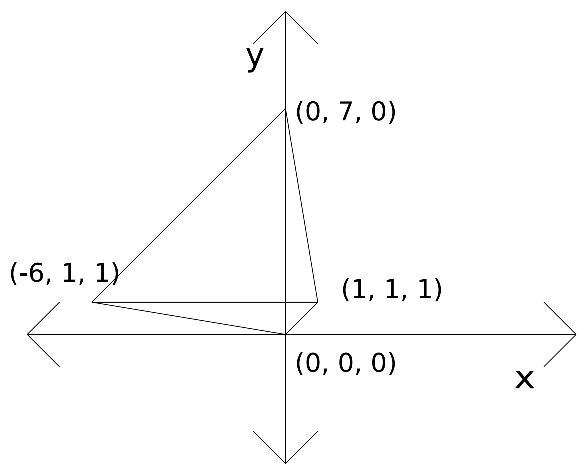


Figure 2.2: Top view of tetrahedron with only two stable faces

9. Heppes [8] discovered a "two-tip tetrahedron" which Guy [1] realized with edges 41, 26, 24, 20, 17 and 4 (amature carpenters may like to construct this). Section 2.2 contains Heppes's construction.

10. Dawson [2] includes an argument of Conway which proves that every tetrahedron has at least two stable faces. Section 3.1 contains Conway's proof.

2.2 A double tipping tetrahedron

Aladár Heppes posed the following problem in *SIAM Review* [8]. *Design a homogeneous tetrahedron which, when placed lying on one of its faces on the top of a horizontal table, will tip over to another face and then tip over again, finally coming to rest on a third face.* Heppes presents a solution and called his tetrahedron the double tipping tetrahedron.

Let A, B, C, D denote the vertices of a tetrahedron T, and let G' denote the orthogonal projection of its centroid G, on the "horizontal" coordinate plane z=o. We compute the x coordinate of G' by averaging the x coordinates of the vertices of T. The z coordinate of G' is obviously 0. We will be using the mass center's projection to see if the solid will tip over, and if it does tip over, the side with which it will rotate over. Heppes presents the following coordinates for the tetrahedron. The initial position of the tetrahedron is described by the following coordinates where $0 < \epsilon < 1$.

$$\begin{array}{llll} A & (-7, & -8(1-\epsilon), & 0), \\ B & (-1, & 0, & 0), \\ C & (1, & 0, & 0), \\ D & (7, & 8, & 8). \end{array}$$

For simplicity we choose $\epsilon = \frac{1}{2}$

$$\begin{array}{llll} A & (-7, & -4, & 0), \\ B & (-1, & 0, & 0), \\ C & (1, & 0, & 0), \\ D & (7, & 8, & 8). \end{array}$$

We have $G' = (0, 1, 0)$. Notice that G' falls outside of the triangle ABC on the y-axis therefore the tetrahedron will tip over side BC.

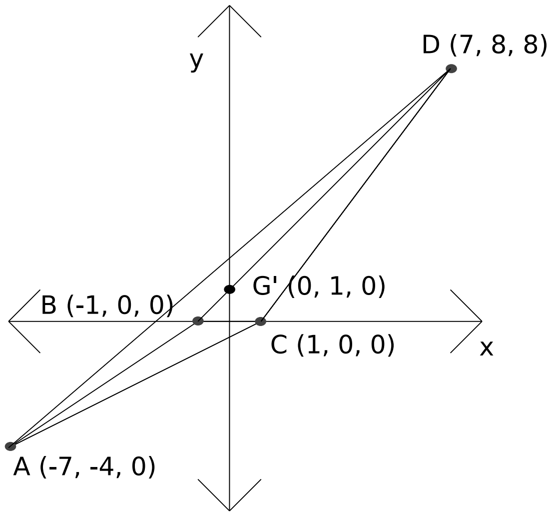


Figure 2.3: Top view of the double tipping tetrahedron first position

In order to determine coordinates of T after the tip, we need to evaluate each coordinate individually. The tetrahedron is tipping over side BC, so the coordinates of B and C are not going to change. Since the tetrahedron is flipping straight over the x-axis, all x^{th} coordinates of the vertices will remain the same. Both y and z coordinates of A and D will be determined through the right triangles with legs x,y. In the case of both vertices these form $45°, 45°, 90°$ triangles. These right triangles are known to have a side ratio of 1:1:$\sqrt{2}$, with $\sqrt{2}$ being the hypotenuse. The triangle associated with A has a hypotenuse of 4. With the given ratios, we know that the new y and z coordinates will have a length of $4(\frac{1}{\sqrt{2}})= \frac{4\sqrt{2}}{2} = 2\sqrt{2}$. The triangle associated with D has side lengths 8 and 8. With the given ratios, we know that the new y coordinate will have a length of $8(\sqrt{2})= 8\sqrt{2}$. The tetrahedron is tipping onto triangle BCD, so the z coordinate of D will become 0.

This gives us the following coordinates for the vertices of T after the first tip.

$$
\begin{array}{llll}
A & (-7, & -2\sqrt{2}, & 2\sqrt{2}), \\
B & (-1, & 0, & 0), \\
C & (1, & 0, & 0), \\
D & (7, & 8\sqrt{2}, & 0).
\end{array}
$$

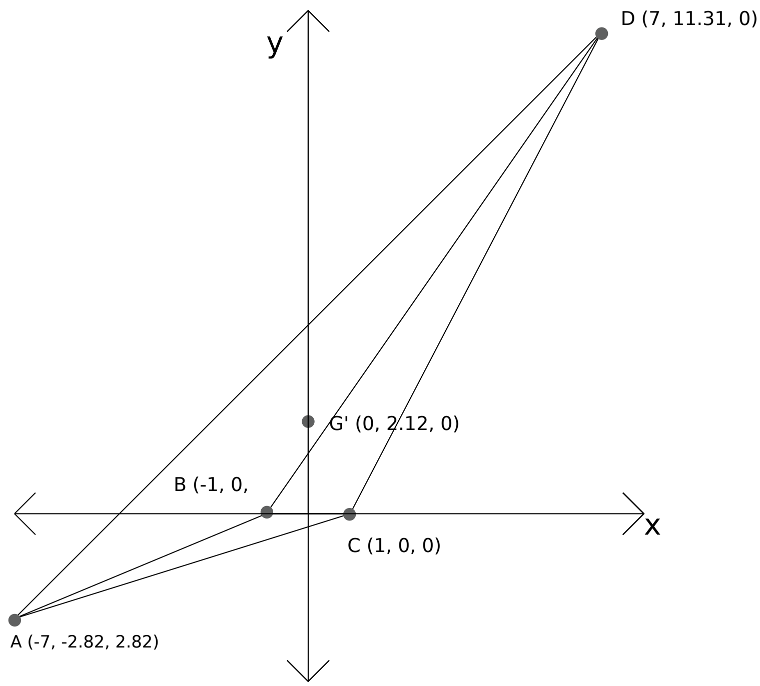


Figure 2.4: Top view of the double tipping tetrahedron second position

The projection for the mass center is $G' = (0, \frac{3\sqrt{2}}{2}, 0)$. This is also outside of triangle BCD, and our tetrahedron will tip over the side BD and come to rest on the base ABD.

Heppes found a tetrahedron with only two stable faces that tips over twice when placed on a horizontal surface. Richard Guy [1] modified this example and found that with edges 41, 26, 24, 20, 17, and 4, and with opposite pairs summing to 45, 43, and 44, an amateur carpenter could construct such a tetrahedron! In the next chapter we will discuss if there are tetrahedrons with only one stable face.

Chapter 3

All tetrahedra have at least two stable faces

J. Conway and R. Guy presented the following problem in the *SIAM Review*[7]: *Show that any homogeneous tetrahedron will rest in stable position when lying on any one of at least two of its faces.*

Michael Goldberg [7] published an incorrect solution to prove that all tetrahedra have at least two stable faces. Goldberg said that a tetrahedron is always stable when resting on the face nearest to the center of gravity. He stated that if you project the apex and the edges onto the stable base, then the projection of the center of gravity will fall onto one of the projected triangles or on the projected edges. If it lies within a projected triangle, then a perpendicular from the center of gravity to that corresponding face will meet within the face, thus proving that it is another stable face. If it projects onto an edge, then both corresponding faces are stable faces.

Apparently there are two problems concerning the above underlined sentence. A) Conway constructed a tetrahedron with non-uniform density which is stable only on one of the faces. Goldberg's solution makes no explicit use of the position of the mass center of a tetrahedron, thus in view of Conway's example, it cannot be complete. B) We constructed a tetrahedron, which is stable on two faces, but not on those which Goldberg claims, thus his proof cannot be completed. It was not until 1985 that R. J. Dawson [2] included Conway's proof in his paper. For the sake of completeness we also include Conways proof here.

Definition 3.1. *Bodies with just one stable equilibrium are called monostatic.*

Theorem 3.1. *No tetrahedron is monostatic.*

Proof (Conway): A tetrahedron will not tip about an edge unless the dihedral angle at that edge is obtuse. Thus, an obtuse angle is necessary for a tip. In view of (2) in 2.1 and the volume formula of the tetrahedron, the closer the center of mass is to a tetrahedron's face, the larger the area of the face. Thus a tetrahedron can only tip from a smaller face to a larger one.

Suppose there exists some tetrahedron which is monostatic. Let A and B be faces with the largest and second-largest areas. There are two ways for the tetrahedron to roll onto face A. Either it rolls from B onto A and from C onto A, or it rolls from another face C, onto B, and then from B onto A. Either way, one of the two largest area faces has two obtuse dihedral angles (call these sides with dihedral angles e and f). Either e or f is a side shared with the other of the two largest faces.

Since sides e and f are both obtuse dihedral angles, the vertex v must be outside the intersection of the extended lines of e and f. The projection of D shows that the face is larger than A or B, thus leading us to a contradiction.

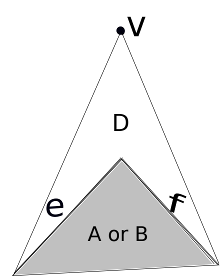


Figure 3.1: No monostatic tetrahedron exist

Chapter 4

A polyhedron, which has only one stable face

R. K. Guy [7] approached this problem in an attempt to define how to develop a mono-static body with only 1 stable equilibrium. His goal was to find the polyhedron which is stable on only one face with the *least* number of faces. In view of chapter 3, the minimum number of faces is between 4 and 19. The exact value is still unknown today.

Think about taking a long cylinder and diagonally cut off one end, then at the opposite angle chop off the other end. This truncated cylinder can lie stable on the table on it's longest "side", but in no other position. It only has one stable equilibrium. Intuitively we can think about how to prove that it only has one stable equilibrium. Think about a normal cylinder; the mass center of the object goes through the center of the circle. After we cut the edges diagonally, there is more mass below the center of the circle. Thus the mass center is below the center of the circular center of the truncated cylinder. This shows that the bottom of our cylinder is a stable equilibrium. If we place the cylinder down anywhere except the bottom of the cylinder then it will roll because the perpendicular from the mass center to the side will fall outside of the cylinder. Thus, it only has one stable equilibrium. We are going to evaluate a figure that is similar to the truncated cylinder, however it is multi-faceted and much more complicated.

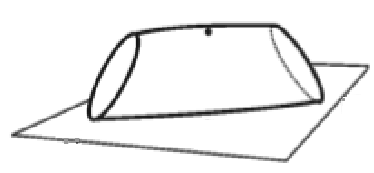


Figure 4.1: Truncated Cylinder

Guy's solid is a 17-sided prism. The first view (see Figure 4.2) is only half of the side view because they are symmetrical. Basically you are looking into the 17-sided prism from the 2 ends. The second view (see Figure 4.3) is looking at the prism from the front, and it is also symmetrical so we are only showing half of it once again.

In general, we are taking a $(2m-1)$ sided prism. The section is made up of $2m$ similar right triangles with angle $\beta = \frac{\pi}{m}$ at point O. This means that our m is 9, and we have 18 right triangles around the prism. The reasons we have 1 less side face is due to the two triangles at the bottom that meet to form just 1 side face. Our angle from O is $\beta = \frac{\pi}{9}$.

When on it's stable equilibrium, the longest hypotenuse is $r = r_o$ going from O to the top. This is where we start "counting," or naming off everything. We will name the rest of the hypotenuses $r_n = rcos^n\beta$, $0 < n < m$. Our side length $r_m = s$, so it is collinear with $r = r_0$. We can conclude from this that the measure of the height of this prism is $r + s = r_0 + r_m$. ($r_m = rcos^m\beta$ respectively).

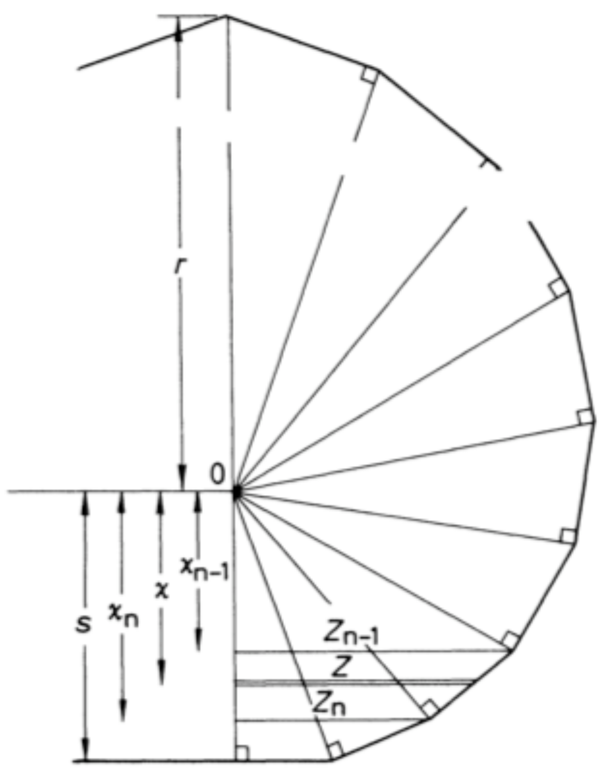


Figure 4.2: View 1

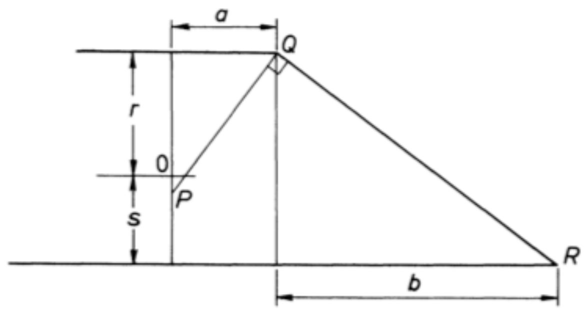


Figure 4.3: View 2

The vertices other than O (on the boundary) lie on two symmetrical equiangular spiral S. So, we start with r, then make the next hypotenuse (r_1) $\frac{\pi}{9}$ from r. Then connect the two, so that there is a right angle where the side and r_1 meets. Continue repeating until the prism is complete.

Our "view 2" (see Figure 4.3) of the cylinder is called the half length. It is represented as $y = fx + g$ where x is measured vertically downwards from O and where f and g denote the following numbers

$$f = \frac{b}{r+s}$$

$$g = a + \frac{br}{r+s}$$

Notice that when x moves below O it's positive, and above it is negative.

We want to choose b sufficiently large compared to a because we want the mass center below O and above P. For the definition of P see Figure 4.3. P is the point where the perpendicular line through Q to R intersects the vertical line containing O. P moves based off of the length of b because as it increases or decreases the right angle must remain. Since the angle PQR must remain a right angle, adjusting b moves P up or down.

We describe the half width (looking into the prism from the side) with depth x as well. We are again measuring x below O. It is going to be represented in terms of z as $z = px + q$.

We will represent p and q in the following way as well:

$$p = \frac{z_n - z_{n-1}}{x_n - x_{n-1}}$$

$$q = \frac{z_{n-1}x_n - z_n x_{n-1}}{x_n - x_{n-1}}$$

We will represent x_n and z_n in terms of r in the following way:

$$x_n = -r_n cosn\beta = -r(cos\beta)^n cosn\beta$$

$$z_n = r_n sinn\beta = r(cos\beta)^n sinn\beta$$

So, looking at Figure 4.2, our $-x_0$ is all the way at the top with z_o. Then $-x_1$ (since x is measured downward from O) goes up to where r_1 forms a right angle with the edge of the prism. The distance from the point $(-x_1, 0)$ to the point where r_1 forms a right angle is z_1. Continue this process until $n = m - 1$, in which case we have x_n and z_n where $x_n = x_{m-1}$ and $z_n = z_{m-1}$. Then we reach x_m and z_m where r_m meets the bottom of our prism and $r_m = s$.

It seems that the reason Guy constructed the prism this way is so that one could easily compute the coordinates of the mass center. The reason that each of the figures only shows half of the actual views is that they are symmetrical. To check to see where there is a stable face, drop a perpendicular from the mass center. If it is on the face, then it stays there when sitting on the face. If the dropped perpendicular falls outside of the face, then it tips over. For example, with a normal cube sitting on one of its faces we can drop a perpendicular, and it will fall inside the face (meaning the face will not tip over). In our other possible example the cube is stretched and crooked to where if we drop a perpendicular from the mass center, it would fall outside of the face. This means that after modification it will not stand on that face, and it will tip over. This way of thinking was most likely how Guy began forming this prism.

They knew the body would be multi-faceted, however, it needed to be somewhat simple so that they could easily compute the mass center. They want the perpendicular from the mass center to only fall on one face and none of the others. After all the computation is carried out we know that in the z plane we are alright with the 18 faces because when we drop a perpendicular from the mass center it falls outside of the faces. We also need to take care of the other 2 faces which we evaluated in the y plane. We need the mass center to be below O, but above P. P will take care of the other points when we drop the perpendicular because all of the points above it will not make a perpendicular with the two side faces, thus they are not stable. It has to be in between the two for it to work. That is when picking the correct proportion of a and b comes into play.

4.1 (1) Setting up the integral

So we know that we have constructed the prism symmetrically so that the y and z coordinates of the mass center are equal to 0. They are trivial (there is only one face where we drop a perpendicular from the mass center and land on a face). We are only going to have to set up the integral to find the first moment relative to the yz plane. Our goal is to show that the first moment is positive so that the mass center lies below 0. We are going to slice the prism parallel to the yz coordinate plane. Since they are both symmetrical, we are going to have 2 of each, giving us 4 on the outside of the integral. My slices are going to go from the very top of the prism where x_n is x_1, all the way down to the bottom of the prism where x_n is x_m. The product of xyz, as a function of x, depends on the subinterval $[x_{i-1}, x_i]$, thus our slices are going to be from x_{n-1} to x_n. Since the integral only takes care of one of the slices, we will need to sum from $n = 1$ to m. Thus giving us the following integral to compute the first moment (4.1) below O of the material solid:

$$4 \sum_{n=1}^{m} \int_{x_{n-1}}^{x_n} xyz \mathrm{d}x \tag{4.1}$$

The following excerpt (Figure 4.4) is from Guy's paper [7]. It shows how he evaluates 1. (the labeling 1... 5 is by us). As we go through 2-5 it turns out that a substantial amount of algebra had to be redone.

1
$$4\sum_{n=1}^{m}\int_{x_{n-1}}^{x_n} xyz\,dx,$$

and we shall choose m (and b) so that this is positive. The integral in (2) has value

2
$$\tfrac{1}{4}fp(x_n^4 - x_{n-1}^4) + \tfrac{1}{3}(fq + gp)(x_n^3 - x_{n-1}^3) + \tfrac{1}{2}gq(x_n^2 - x_{n-1}^2).$$

Algebraic and trigonometric manipulation reduces this to

3
$$\begin{aligned}&\tfrac{1}{4}f(x_n^3 z_n - x_{n-1}^3 z_{n-1}) + \tfrac{1}{3}g(x_n^2 z_n - x_{n-1}^2 z_{n-1})\\ &+ \tfrac{1}{12}f(x_n^3 z_{n-1} + x_n^2 x_{n-1} z_{n-1} + x_n x_{n-1}^2 z_{n-1} - x_n^2 x_{n-1} z_n - x_n x_{n-1}^2 z_n - x_{n-1}^3 z_n)\\ &+ \tfrac{1}{6}g(x_n^2 z_{n-1} + x_n x_{n-1} z_{n-1} - x_n x_{n-1} z_n - x_{n-1}^2 z_n).\end{aligned}$$

If we sum from $n = 1$ to m, the first two terms yield $\frac{1}{4}f(x_m^3 z_m - x_0^3 z_0)$ and $\frac{1}{3}g(x_m^2 z_m - x_0^2 z_0)$, which vanish, since $z_0 = z_m = 0$. It remains to sum the other two terms, which, on writing $\cos\beta = k$, may be thrown into the form

4
$$\begin{aligned}\frac{r^3\sin\beta}{24k^3}\{&fr[(k^4)^n + 2k^2(k^4)^n + k^2(k^4)^n\cos 2n\beta\\ &+ k(k^4)^n\cos(2n-1)\beta + (k^4)^n\cos(2n-2)\beta]\\ &- 4gk[k(k^3)^n\cos n\beta + (k^3)^n\cos(n-1)\beta]\},\end{aligned}$$

whose sum is

5
$$\begin{aligned}\frac{r^3k}{24\sin\beta}\Bigg\{&fr(1-k^{4m})\left[\frac{1+2k^2}{1+k^2} + \frac{1+3k^4-4k^6}{1+k^2+3k^4-k^6}\right]\\ &- 4g(1+k^{3m})\frac{(1-k^2)(1+2k^2)}{1+k^2-k^4}\Bigg\}.\end{aligned}$$

Figure 4.4: Guy's solution

4.2 (2) Evaluating the integral

The following completes Part 1 of the solution in Figure 4.4. We substitute the equations $y = fx + g$ and $z = px + q$ so that the integral is in terms of x.

$$\int_{x_{n-1}}^{x_n} xyz\mathrm{d}x = \int_{x_{n-1}}^{x_n} x(fx+g)(px+q)\mathrm{d}x$$

$$= \int_{x_{n-1}}^{x_n} x(fpx^2 + fqx + gpx + gq)\mathrm{d}x = \int_{x_{n-1}}^{x_n} (fpx^3 + fqx^2 + gpx^2 + gpx)\mathrm{d}x \tag{4.2}$$

When we evaluate the integral we get:

$$fp\frac{x^4}{4} + fq\frac{x^3}{3} + gp\frac{x^3}{3} + gq\frac{x^2}{2}\Big|_{x_{n-1}}^{x_n}$$

$$= \frac{1}{4}fp(x_n^4 - x_{n-1}^4) + \frac{1}{3}fq(x_n^3 - x_{n-1}^3) + \frac{1}{3}gp(x_n^3 - x_{n-1}^3) + \frac{1}{2}gq(x_n^2 - x_{n-1}^2) \tag{4.3}$$

So the integral in (4.2) has value:

$$= \frac{1}{4}fp(x_n^4 - x_{n-1}^4) + \frac{1}{3}(fq + gp)(x_n^3 - x_{n-1}^3) + \frac{1}{2}gq(x_n^2 - x_{n-1}^2) \tag{4.4}$$

The above equation is given in Part 2 of the solution in Figure 4.4.

4.3 (3) Algebraic/ trigonometric manipulation

Then when we substitute $p = \frac{z_n - z_{n-1}}{x_n - x_{n-1}}$ and $q = \frac{z_{n-1}x_n - z_n x_{n-1}}{x_n - x_{n-1}}$ into our equation we get the following (The equation is broken up into three parts so that we can see how each form is manipulated):

$$= \frac{1}{4}f\Big(\frac{z_n - z_{n-1}}{x_n - x_{n-1}}\Big)(x_n^4 - x_{n-1}^4) \tag{4.5}$$

$$+\frac{1}{3}\Big[f\Big(\frac{z_{n-1}x_n - z_n x_{n-1}}{x_n - x_{n-1}}\Big) + g\Big(\frac{z_n - z_{n-1}}{x_n - x_{n-1}}\Big)\Big](x_n^3 - x_{n-1}^3) \tag{4.6}$$

$$+\frac{1}{2}g\Big(\frac{z_{n-1}x_n - z_n x_{n-1}}{x_n - x_{n-1}}\Big)(x_n^2 - x_{n-1}^2) \tag{4.7}$$

We will start by evaluating 4.5:

$$\frac{1}{4}f\Big(\frac{z_n - z_{n-1}}{x_n - x_{n-1}}\Big)(x_n^2 - x_{n-1}^2)(x_n^2 + x_{n-1}^2)$$

$$= \frac{1}{4}f\Big(\frac{z_n - z_{n-1}}{x_n - x_{n-1}}\Big)(x_n - x_{n-1})(x_n + x_{n-1})(x_n^2 + x_{n-1}^2)$$

$$= \frac{1}{4}f(z_n - z_{n-1})(x_n + x_{n-1})(x_n^2 + x_{n-1}^2)$$

$$= \frac{1}{4}f(z_n - z_{n-1})(x_n^3 + x_n x_{n-1}^2 + x_n^2 x_{n-1} + x_{n-1}^3)$$

$$= \frac{1}{4}f\big((x_n^3 z_n)+(x_n x_{n-1}^2 z_n)+(x_n^2 x_{n-1} z_n)+(x_{n-1}^3 z_n)-(x_n^3 z_{n-1})-(x_n x_{n-1}^2 z_{n-1})-(x_n^2 x_{n-1} z_{n-1})-(x_{n-1}^3 z_{n-1})\big)$$

For future manipulation of the equation, we will break 4.5 up into:

$$= \frac{1}{4}f\big((x_n^3 z_n) - (x_{n-1}^3 z_{n-1})\big)$$

$$+\frac{1}{4}f\big((x_n x_{n-1}^2 z_n) + (x_n^2 x_{n-1} z_n) + (x_{n-1}^3 z_n) - (x_n^3 z_{n-1}) - (x_n x_{n-1}^2 z_{n-1}) - (x_n^2 x_{n-1} z_{n-1})\big)$$

Now we will evalute 4.6:

$$\frac{1}{3}\Big[f\Big(\frac{x_n z_{n-1} - x_{n-1} z_n}{x_n - x_{n-1}}\Big) + g\Big(\frac{z_n - z_{n-1}}{x_n - x_{n-1}}\Big)\Big](x_n - x_{n-1})(x_n^2 + x_n x_{n-1} + x_{n-1}^2)$$

$$= \frac{1}{3}\Big[\frac{f(x_n z_{n-1}) - f(x_{n-1} z_n) + g(z_n - z_{n-1})}{(x_n - x_{n-1})}\Big](x_n - x_{n-1})(x_n^2 + x_n x_{n-1} + x_{n-1}^2)$$

$$= \frac{1}{3}[f(x_n z_{n-1}) - f(x_{n-1} z_n) + g(z_n) - g(z_{n-1})](x_n^2 + x_n x_{n-1} + x_{n-1}^2)$$

$$= \frac{1}{3}\Big[f\big((x_n^3 z_{n-1}) + (x_n^2 x_{n-1} z_{n-1}) + (x_n x_{n-1}^2 z_{n-1}) - (x_n^2 x_{n-1} z_n) - (x_n x_{n-1}^2 z_n) - (x_{n-1}^3 z_n)\big)$$
$$+g\big((x_n^2 z_n) + (x_n x_{n-1} z_n) + (x_{n-1}^2 z_n) - (x_n^2 z_{n-1}) - (x_n x_{n-1} z_{n-1}) - (x_{n-1}^2 z_{n-1})\big)\Big]$$

For future manipulation of the equation, we will break 4.6 up into:

$$\frac{1}{3}g\big((x_n^2 z_n) - (x_{n-1}^2 z_{n-1})\big)$$

$$+\frac{1}{3}\Big[f\big((x_n^3 z_{n-1}) + (x_n^2 x_{n-1} z_{n-1}) + (x_n x_{n-1}^2 z_{n-1}) - (x_n^2 x_{n-1} z_n) - (x_n x_{n-1}^2 z_n) - (x_{n-1}^3 z_n)\big)$$
$$+g\big((x_n x_{n-1} z_n) + (x_{n-1}^2 z_n) - (x_n^2 z_{n-1}) - (x_n x_{n-1} z_{n-1})\big)\Big]$$

Now we will evaluate 4.7:

$$\frac{1}{2}g\Big(\frac{(x_n z_{n-1}) - (x_{n-1} z_n)}{(x_n - x_{n-1})}\Big)(x_n - x_{n-1})(x_n + x_{n-1})$$

$$= \frac{1}{2}g\big((x_n z_{n-1}) - (x_{n-1} z_n)\big)(x_n + x_{n-1})$$

$$= \frac{1}{2}g\big((x_n^2 z_{n-1}) + (x_n x_{n-1} z_{n-1}) - (x_n x_{n-1} z_n) - (x_{n-1}^2 z_n)\big)$$

Combining 4.5, 4.6, and 4.7 to simplify further we have:

$$\frac{1}{4}f\big((x_n^3 z_n) - (x_{n-1}^3 z_{n-1})\big)$$

$$+\frac{1}{3}g\big((x_n^2 z_n) - (x_{n-1}^2 z_{n-1})\big)$$

$$+\frac{1}{4}f\big((x_n x_{n-1}^2 z_n) + (x_n^2 x_{n-1} z_n) + (x_{n-1}^3 z_n) - (x_n^3 z_{n-1}) - (x_n x_{n-1}^2 z_{n-1}) - (x_n^2 x_{n-1} z_{n-1})\big)$$

$$+\frac{1}{3}f\big((x_n^3 z_{n-1}) + (x_n^2 x_{n-1} z_{n-1}) + (x_n x_{n-1}^2 z_{n-1}) - (x_n^2 x_{n-1} z_n) - (x_n x_{n-1}^2 z_n) - (x_{n-1}^3 z_n)\big)$$

$$+\frac{1}{3}g\big((x_n x_{n-1} z_n) + (x_{n-1}^2 z_n) - (x_n^2 z_{n-1}) - (x_n x_{n-1} z_{n-1})\big)$$

$$+\frac{1}{2}g\big((x_n^2 z_{n-1}) + (x_n x_{n-1} z_{n-1}) - (x_n x_{n-1} z_n) - (x_{n-1}^2 z_n)\big)$$

When we add the third and fourth rows

$$\frac{1}{4}f\big((x_n x_{n-1}^2 z_n) + (x_n^2 x_{n-1} z_n) + (x_{n-1}^3 z_n) - (x_n^3 z_{n-1}) - (x_n x_{n-1}^2 z_{n-1}) - (x_n^2 x_{n-1} z_{n-1})\big)$$

$$+\frac{1}{3}f\big((x_n^3 z_{n-1}) + (x_n^2 x_{n-1} z_{n-1}) + (x_n x_{n-1}^2 z_{n-1}) - (x_n^2 x_{n-1} z_n) - (x_n x_{n-1}^2 z_n) - (x_{n-1}^3 z_n)\big),$$

they are combined to get:

$$\frac{1}{12}f\big(- (x_n x_{n-1}^2 z_n) - (x_n^2 x_{n-1} z_n) - (x_{n-1}^3 z_n) + (x_n^3 z_{n-1}) + (x_n x_{n-1}^2 z_{n-1}) + (x_n^2 x_{n-1} z_{n-1})\big)$$

When we add the fifth and sixth rows

$$\frac{1}{3}g\big((x_n x_{n-1} z_n) + (x_{n-1}^2 z_n) - (x_n^2 z_{n-1}) - (x_n x_{n-1} z_{n-1})\big)$$

$$+\frac{1}{2}g((x_n^2 z_{n-1}) + (x_n x_{n-1} z_{n-1}) - (x_n x_{n-1} z_n) - (x_{n-1}^2 z_n)),$$

they are combined to get:

$$\frac{1}{6}g\big((x_n^2 z_{n-1}) + (x_n x_{n-1} z_{n-1}) - (x_n x_{n-1} z_n) - (x_{n-1}^2 z_n)\big)$$

Combining 4.5, 4.6, 4.7, and the simplifications we get:

$$\frac{1}{4}f\big((x_n^3 z_n) - (x_{n-1}^3 z_{n-1})\big) \tag{4.8}$$

$$+\frac{1}{3}g\big((x_n^2 z_n) - (x_{n-1}^2 z_{n-1})\big)$$

$$+\frac{1}{12}f\big(-(x_n x_{n-1}^2 z_n) - (x_n^2 x_{n-1} z_n) - (x_{n-1}^3 z_n) + (x_n^3 z_{n-1}) + (x_n x_{n-1}^2 z_{n-1}) + (x_n^2 x_{n-1} z_{n-1})\big)$$

$$+\frac{1}{6}g\big((x_n^2 z_{n-1}) + (x_n x_{n-1} z_{n-1}) - (x_n x_{n-1} z_n) - (x_{n-1}^2 z_n)\big)$$

The above equation is given in Part 3 of the solution in Figure 4.4.

4.4 (4) Sum from $n = 1$ to m

Applying the sum to the first and second row, $\frac{1}{4}f\big((x_n^3 z_n) - (x_{n-1}^3 z_{n-1})\big) + \frac{1}{3}g\big((x_n^2 z_n) - (x_{n-1}^2 z_{n-1})\big)$, means summing from $n = 1$ to m. These two terms yield $\frac{1}{4}f\big((x_m^3 z_m) - (x_0 z_0)\big)$ and $\frac{1}{3}g\big((x_m^2 z_m - x_0^2 z_0)\big)$. These both equal zero because $z_0 = z_m = 0$. In order to sum the third and fourth lines, we will recall the following formulas: $r = r_0$, $r_n = r\cos^n\beta, 0 < n < m$, $x_n = -r_n \cos n\beta \rightarrow x_n = -r\cos^n\beta\cos n\beta$, $z_n = r_n \sin n\beta \rightarrow z_n = r\cos^n\beta\sin n\beta$. When we write $k = \cos\beta$ we get:

$$x_n = -rk^n \cos n\beta$$

$$z_n = rk^n \sin n\beta$$

We will deal with line three and four individually and then combine them. When we plug x_n, z_n, and $k = \cos\beta$, into

$$\frac{1}{12}f\big(-(x_n x_{n-1}^2 z_n) - (x_n^2 x_{n-1} z_n) - (x_{n-1}^3 z_n) + (x_n^3 z_{n-1}) + (x_n x_{n-1}^2 z_{n-1}) + (x_n^2 x_{n-1} z_{n-1})\big)$$

we get:

$$\frac{1}{12}f\Big[(-rk^n \cos n\beta)^3 (rk^{n-1} \sin(n-1)\beta)$$

$$+(-rk^n \cos n\beta)^2(-rk^{n-1}\cos(n-1)\beta)(rk^{n-1}\sin(n-1)\beta)$$

$$+(-rk^n \cos n\beta)(-rk^{n-1}\cos(n-1)\beta)^2(rk^{n-1}\sin(n-1)\beta)$$

$$-(-rk^n \cos n\beta)^2(-rk^{n-1}\cos(n-1)\beta)(rk^n \sin n\beta)$$

$$-(-rk^n \cos n\beta)(-rk^{n-1}\cos(n-1)\beta)^2(rk^n \sin n\beta)$$

$$-(-rk^{n-1}\cos(n-1)\beta)^3(rk^n \sin n\beta)\Big].$$

Which can be reduced to the following form:

$$\frac{r^3}{24k^3}fr(k^4)^n\Big[-(2\cos^3 n\beta)(1)(\sin(n-1)\beta)(k^2)$$

$$-(2\cos^2 n\beta)(\cos(n-1)\beta)(\sin(n-1)\beta)(k)$$

$$-(2\cos n\beta)(\cos^2(n-1)\beta)(\sin(n-1)\beta)(1)$$

$$+(2\cos^2 n\beta)(\cos(n-1)\beta)(\sin n\beta)(k^2)$$

$$+(2\cos n\beta)(\cos^2(n-1)\beta)(\sin n\beta)(k)$$

$$+(2)(1)(\cos^3(n-1)\beta)(\sin n\beta)(1)\Big]$$

When combining the rows according to their last term (1, k, and k^2) we get:

$$\frac{r^3}{24k^3}fr(k^4)^n\Big[(2\cos n\beta)(\cos(n-1)\beta)\big[(-\cos n\beta)(\sin(n-1)\beta)+(\sin n\beta)(\cos(n-1)\beta)\big](k)$$

$$+(2\cos^2 n\beta)\big[-(\cos n\beta)(\sin(n-1)\beta)+(\sin n\beta)(\cos(n-1)\beta)\big](k^2)$$

$$+(2\cos^2(n-1)\beta)\big[-(\cos n\beta)(\sin(n-1)\beta)+(\sin n\beta)(\cos(n-1)\beta)\big](1)\Big]$$

From the "sum and difference formula" which says $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$, we know that $(-\cos n\beta)(\sin(n-1)\beta)+(\sin n\beta)(\cos(n-1)\beta)=\sin\beta$. So factor out $\sin\beta$ to have:

$$\frac{r^3}{24k^3}fr(k^4)^n\sin\beta\Big[2(\cos n\beta)(\cos(n-1)\beta)(k)$$

$$+2(\cos^2 n\beta)(k^2)+2(\cos^2(n-1)\beta)(1)\Big]$$

Using the "product to sum formula" which says $\cos\alpha\cos\beta=\frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$, and the "half angle formula" which says $\cos^2\alpha=\frac{1}{2}(1+\cos(2\alpha))$ the above yields the following:

$$\frac{r^3}{24k^3}fr(k^4)^n\sin\beta\Big[(\cos\beta+\cos(2n-1)\beta)(k)$$

$$+(1+\cos 2n\beta)(k^2)+(1+\cos(2n-2)\beta)(1)\Big]$$

Factoring in k, and k^2 yields (remember $k=\cos\beta$):

$$\frac{r^3}{24k^3}fr(k^4)^n\sin\beta\Big[(k^2+\cos(2n-1)\beta k)$$

$$+(k^2+\cos 2n\beta k^2)+(1+\cos(2n-2)\beta)\Big]$$

Combining the above to see the manipulation of the "f" part of 4.8 produces

$$\frac{r^3}{24k^3}fr(k^4)^n\sin\beta\Big[1+2k^2+k^2\cos 2n\beta+k\cos(2n-1)\beta+\cos(2n-2)\beta\Big]$$

Now we will look at the fourth line of 4.8 which is

$$\frac{1}{6}g\big((x_n^2 z_{n-1}) + (x_n x_{n-1} z_{n-1}) - (x_n x_{n-1} z_n) - (x_{n-1}^2 z_n)\big),$$

and becomes the following when using $x_n = -rk^n \cos n\beta$, $z_n = rk^n \sin n\beta$, and $k = \cos\beta$.

$$\frac{1}{6}g\Big[(-rk^n \cos n\beta)^2(rk^{n-1}\sin(n-1)\beta)$$

$$+(-rk^n \cos n\beta)(-rk^{n-1}\cos(n-1)\beta)(rk^{n-1}\sin(n-1)\beta)$$

$$-(-rk^n \cos n\beta)(-rk^{n-1}cos(n-1)\beta)(rk^n \sin n\beta)$$

$$-(-rk^{n-1}\cos(n-1)\beta)^2(rk^n \sin n\beta)\Big]$$

Which can be reduced to the following form:

$$\frac{r^3}{24k^3}(-4g)(k^3)^n k\Big[(-k)(\cos^2 n\beta)(\sin(n-1)\beta$$

$$+(-1)(\cos n\beta)(\cos(n-1)\beta)(\sin(n-1)\beta)$$

$$+(k)(\cos n\beta)(\cos(n-1)\beta)(\sin n\beta)$$

$$+(1)(\cos^2(n-1)\beta)(\sin n\beta)\Big]$$

When combining the rows according to their last term (1 and k) we get:

$$\frac{r^3}{24k^3}(-4g)(k^3)^n k\Big[k\big((\cos n\beta)(\cos(n-1)\beta)(\sin n\beta) - (\cos^2 n\beta)(\sin(n-1)\beta)\big)$$

$$+(\cos^2(n-1)\beta)(\sin n\beta) - (\cos n\beta)(\cos(n-1)\beta)(\sin(n-1)\beta)\Big]$$

Factor out $\cos n\beta$ from the first and $\cos(n-1)\beta$ from the second and get:

$$\frac{r^3}{24k^3}(-4g)(k^3)^n k\Big[k\cos n\beta\big((cos(n-1)\beta)(\sin n\beta)-(\cos n\beta)(\sin(n-1)\beta)\big)$$

$$+\cos(n-1)\beta\big((\cos(n-1)\beta)(\sin n\beta)-(\cos n\beta)(\sin(n-1)\beta)\big)\Big]$$

We know that we know that $(-\cos n\beta)(\sin(n-1)\beta)+(\sin n\beta)(\cos(n-1)\beta)=\sin\beta$, so factor out $\sin\beta$ and get:

$$\frac{r^3}{24k^3}(-4g)(k^3)^n k\sin\beta\Big[k\cos n\beta+\cos(n-1)\beta\Big].$$

When we combine our manipulations $\frac{r^3}{24k^3}fr(k^4)^n\sin\beta\Big[1+2k^2+k^2\cos 2n\beta+k\cos(2n-1)\beta+\cos(2n-2)\beta\Big]$ and $\frac{r^3}{24k^3}(-4g)(k^3)^n k\sin\beta\Big[k\cos n\beta+\cos(n-1)\beta\Big]$, factor in $(k^4)^n$ and $(k^3)^n$ for computation and get:

$$\frac{r^3\sin\beta}{24k^3}\Big[fr\big((k^4)^n+2k^2(k^4)^n+k^2(k^4)^n\cos 2n\beta+k(k^4)^n\cos(2n-1)\beta+(k^4)^n\cos(2n-2)\beta\big)$$

$$-4gk\big(k(k^3)^n\cos n\beta+(k^3)^n\cos(n-1)\beta\big)\Big] \tag{4.9}$$

The above equation is given in Part 4 of the given solution in Figure 4.4.

4.5 (5) Evaluating the sum

In order to take the sum of 4.9 we need to put it in the following form:

$$\frac{r^3\sin^2\beta k}{24\sin\beta k^4}\Big[fr\big((k^4)^n+2k^2(k^4)^n+k^2(k^4)^n\cos 2n\beta+k(k^4)^n\cos(2n-1)\beta+(k^4)^n\cos(2n-2)\beta\big)$$

$$-4gk\big(k(k^3)^n\cos n\beta+(k^3)^n\cos(n-1)\beta\big)\Big]$$

which is equal to

$$\frac{r^3k}{24\sin\beta}\Big[fr\big((k^4)^{n-1}\sin^2\beta+2k^2(k^4)^{n-1}\sin^2\beta+k^2(k^4)^{n-1}\cos 2n\beta\sin^2\beta \tag{4.10}$$

$$+k(k^4)^{n-1}\cos(2n-1)\beta\sin^2\beta+(k^4)^{n-1}\cos(2n-2)\beta\sin^2\beta\big)$$

$$-4g\big(k(k^3)^{n-1}\cos n\beta\sin^2\beta+(k^3)^{n-1}\cos(n-1)\beta\sin^2\beta\big)\Big]$$

We will look at the "f" and "g" part of 4.10 separately.

Starting with the "f" part we will be looking at

$$(k^4)^{n-1}\sin^2\beta+2k^2(k^4)^{n-1}\sin^2\beta+k^2(k^4)^{n-1}\cos 2n\beta\sin^2\beta$$

$$+k(k^4)^{n-1}\cos(2n-1)\beta\sin^2\beta+(k^4)^{n-1}\cos(2n-2)\beta\sin^2\beta$$

We need to remember the following for computations: $\sin^2\beta=(1-k^2), \beta=\frac{\pi}{m}, \cos 2\beta = \cos(2(m+1))\beta=2k^2-1, \cos 2m\beta=1$.

We will label it accordingly and work with each separately, and take the sum from $n=1$ to m:

$$a_n=(k^4)^{n-1}\sin^2\beta+2k^2(k^4)^{n-1}\sin^2\beta$$

$$b_n=k^2(k^4)^{n-1}\cos 2n\beta\sin^2\beta$$

$$c_n=k(k^4)^{n-1}\cos(2n-1)\beta\sin^2\beta$$

$$d_n=(k^4)^{n-1}\cos(2n-2)\beta\sin^2\beta$$

First,

$$S_a=\sum_{n=1}^{m}a_n=(1+2k^2)\sin^2\beta\Big(\sum_{n=1}^{m}(k^4)^{n-1}\Big)=(1+2k^2)(1-k^2)\frac{1-k^{4m}}{1-k^4}$$

$$= (1+2k^2)(1-k^2)\frac{1-k^{4m}}{(1-k^2)(1+k^2)} = (1+2k^2)\frac{1-k^{4m}}{1+k^2}$$

Second,

$$S_b = \sum_{n=1}^{m} b_n = \sum_{n=1}^{m} k^2(k^4)^{n-1}\cos 2n\beta \sin^2\beta$$

Notice:

$$k^2\cos 2n\beta = k\cos 2n\beta(\cos\beta) = k\frac{1}{2}(\cos(2n+1)\beta + \cos(2n-1)\beta)$$

$$= \frac{1}{2}\big(\cos(2n+1)\beta\cos\beta + \cos(2n-1)\beta\cos\beta\big)$$

$$= \frac{1}{2}\Big[\frac{1}{2}\Big(\cos(2n+2)\beta + \cos 2n\beta + \cos 2n\beta + \cos(2n-2)\beta\Big)\Big]$$

$$= \frac{1}{4}\cos(2n+2)\beta + \frac{1}{2}\cos 2n\beta + \frac{1}{4}\cos(2n-2)\beta$$

$$S_b = \sum_{n=1}^{m}\frac{1}{4}(k^4)^{n-1}\cos(2n+2)\beta\sin^2\beta + \sum_{n=1}^{m}\frac{1}{2}(k^4)^{n-1}\cos 2n\beta\sin^2\beta + \sum_{n=1}^{m}\frac{1}{4}(k^4)^{n-1}\cos(2n-2)\beta\sin^2\beta$$

$$= \frac{1}{4k^2}\sum_{n=1}^{m}k^2(k^4)^{n-1}\cos(2n+2)\beta\sin^2\beta + \frac{1}{2k^2}\sum_{n=1}^{m}k^2(k^4)^{n-1}\cos 2n\beta\sin^2\beta$$

$$+\frac{1}{4k^2}\sum_{n=1}^{m}k^2(k^4)^{n-1}\cos(2n-2)\beta\sin^2\beta$$

Evaluate the first (1), second (2), and third (3) place of the equation:

1.

$$\frac{1}{4k^2}\sum_{n=1}^{m}k^2(k^4)^{n-1}\cos(2n+2)\beta\sin^2\beta$$

$$= \frac{1}{4k^2}\Big[\frac{S_b}{k^4} - k^2\frac{1}{k^4}(k^4)^0\cos 2\beta\sin^\beta + k^2(k^4)^{m-1}\cos(2(m+1))\beta\sin^2\beta\Big]$$

$$= \frac{1}{4k^2}\Big[\frac{S_b}{k^4} - \frac{k^2}{k^4}\cos 2\beta\sin^2\beta + \frac{k^2}{k^4}(k^4)^m\cos(2(m+1))\beta\sin^2\beta\Big]$$

$$= \frac{1}{4k^6}S_b - \frac{(2k^2-1)(1-k^2)}{4k^4} + \frac{(k^4)^m(2k^2-1)(1-k^2)}{4k^4}$$

2.

$$\frac{1}{2k^2}\sum_{n=1}^{m} k^2(k^4)^{n-1}\cos 2n\beta \sin^2\beta$$

$$= \frac{1}{2k^2}S_b$$

3.

$$\frac{1}{4k^2}\sum_{n=1}^{m} k^2(k^4)^{n-1}\cos{(2n-2)\beta}\sin^2\beta$$

$$= \frac{1}{4k^2}\Big[S_b k^4 - k^2(k^4)^m\cos 2m\beta\sin^2\beta + k^2(k^4)^0\cos{(2(0))\beta}\sin^2\beta\Big]$$

$$= \frac{k^2}{4}S_b - \frac{(k^4)^m(1-k^2)}{4} + \frac{(1-k^2)}{4}$$

We get S_b by adding 1+2+3:

$$S_b = \frac{1}{4k^6}S_b - \frac{(2k^2-1)(1-k^2)}{4k^4} + \frac{(k^4)^m(2k^2-1)(1-k^2)}{4k^4} + \frac{1}{2k^2}S_b + \frac{k^2}{4}S_b - \frac{(k^4)^m(1-k^2)}{4} + \frac{(1-k^2)}{4}$$

$$S_b(1-\frac{1}{4k^6}-\frac{1}{2k^2}-\frac{k^2}{4}) = -\frac{(2k^2-1)(1-k^2)+k^{4m}(2k^2-1)(1-k^2)}{4k^4} + \frac{-k^{4m}(1-k^2)+(1-k^2)}{4}$$

$$S_b(1-\frac{1}{4k^6}-\frac{1}{2k^2}-\frac{k^2}{4}) = \frac{(k^{4m}-1)(2k^2-1)(1-k^2)}{4k^4} + \frac{k^4(1-k^{4m})(1-k^2)}{4k^4}$$

$$S_b(\frac{4k^6-1-2k^4-k^8}{4k^6}) = \frac{(1-k^{4m})}{4k^4}\big[(1-2k^2)(1-k^2)+(k^4-k^6)\big]$$

$$S_b = \Big(\frac{4k^6}{-k^8+4k^6-2k^4-1}\Big)\frac{(1-k^{4m})}{4k^4}\Big[-k^6+3k^4-3k^2+1\Big]$$

$$S_b = \Big(\frac{4k^6}{-k^8+4k^6-2k^4-1}\Big)\frac{(1-k^{4m})}{4k^6}\Big[k^2(-k^6+3k^4-3k^2+1)\Big]$$

Third,

$$S_c = \sum_{n=1}^{m} c_n = \sum_{n=1}^{m} k(k^4)^{n-1}\cos(2n-1)\beta\sin^2\beta$$

Notice:

$$k\cos{(2n-1)\beta} = \cos\beta\cos{(2n-1)\beta} = \frac{1}{2}(\cos{(2n-2)\beta}+\cos 2n\beta)$$

$$= \frac{k}{2k}(\cos(2n-2)\beta + \cos 2n\beta) = \frac{1}{2k}(\cos(2n-2)\beta\cos\beta + \cos 2n\beta\cos\beta)$$

$$= \frac{1}{2k}\Big[\frac{\cos(2n-3)\beta + \cos(2n-1)\beta}{2} + \frac{\cos(2n-1)\beta + \cos(2n+1)\beta}{2}\Big]$$

$$= \frac{1}{4k}\cos(2n+1)\beta + \frac{1}{2k}\cos(2n-1)\beta + \frac{1}{4k}\cos(2n-3)\beta$$

$$S_c = \sum_{n=1}^{m}\frac{1}{4k}(k^4)^{n-1}\cos(2n+1)\beta\sin^2\beta + \sum_{n=1}^{m}\frac{1}{2k}(k^4)^{n-1}\cos(2n-1)\beta\sin^2\beta$$

$$+ \sum_{n=1}^{m}\frac{1}{4k}(k^4)^{n-1}\cos(2n-3)\beta\sin^2\beta$$

$$S_c = \frac{1}{4k^2}\sum_{n=1}^{m}k(k^4)^{n-1}\cos(2n+1)\beta\sin^2\beta + \frac{1}{2k^2}\sum_{n=1}^{m}k(k^4)^{n-1}\cos(2n-1)\beta\sin^2\beta$$

$$+ \frac{1}{4k^2}\sum_{n=1}^{m}k(k^4)^{n-1}\cos(2n-3)\beta\sin^2\beta$$

Evaluate the first (1), second (2), and third (3) place of the equation:

1.

$$\frac{1}{4k^2}\sum_{n=1}^{m}k(k^4)^{n-1}\cos(2n+1)\beta\sin^2\beta$$

$$= \frac{1}{4k^2}\Big[\frac{S_c}{k^4} - k\frac{1}{k^4}\cos 1\beta\sin^2\beta + \frac{k}{k^4}(k^4)^m\cos(2m+1)\beta\sin^2\beta\Big]$$

$$= \frac{1}{4k^6}S_c - \frac{(1-k^2)}{4k^4} + \frac{k^{4m}(1-k^2)}{4k^4}$$

2.

$$\frac{1}{2k^2}\sum_{n=1}^{m}k(k^4)^{n-1}\cos(2n-1)\beta\sin^2\beta$$

$$= \frac{1}{2k^2}S_c$$

3.

$$\frac{1}{4k^2}\sum_{n=1}^{m} k(k^4)^{n-1}\cos{(2n-3)\beta}\sin^2\beta$$

$$=\frac{1}{4k^2}\Big[S_c k^4 - k(k^4)^m\cos{(2m-1)\beta}\sin^2\beta + k(k^4)^0\cos(-1)\beta$$

$$=\frac{k^2}{4}S_c - \frac{k^{4m}(1-k^2)}{4} + \frac{(1-k^2)}{4}$$

$$=\frac{k^2}{4}S_c + \frac{(1-k^2)(1-k^{4m})}{4}$$

We get S_c by adding 1+2+3:

$$S_c = \frac{1}{4k^6}S_c - \frac{(1-k^2)}{4k^4} + \frac{k^{4m}(1-k^2)}{4k^4} + \frac{1}{2k^2}S_c + \frac{k^2}{4}S_c + \frac{(1-k^2)(1-k^{4m})}{4}$$

$$S_c(1-\frac{1}{4k^6}-\frac{1}{2k^2}-\frac{k^2}{4}) = \frac{-(1-k^2)+k^{4m}(1-k^2)}{4k^4} + \frac{(1-k^2)(1-k^{4m})}{4}$$

$$S_c(1-\frac{1}{4k^6}-\frac{1}{2k^2}-\frac{k^2}{4}) = \frac{(k^{4m}-1)(1-k^2)}{4k^4} + \frac{(1-k^2)(1-k^{4m})}{4}$$

$$S_c = \Big(\frac{4k^6}{-k^8+4k^6-2k^4-1}\Big)\Big[\frac{(1-k^{4m})(k^2-1)}{4k^4} + \frac{(1-k^{4m})(1-k^2)}{4}\Big]$$

$$S_c = \Big(\frac{4k^6}{-k^8+4k^6-2k^4-1}\Big)\frac{1-k^{4m}}{4k^6}\Big(k^2(k^2-1)+k^6(1-k^2)\Big)$$

Fourthly,

$$S_d = \sum_{n=1}^{m} d_n = \sum_{n=1}^{m}(k^4)^{n-1}\cos(2n-2)\beta\sin^2\beta$$

Notice:

$$\frac{1}{k}[\cos{(2n-2)\beta}\cos\beta]$$

$$=\frac{1}{2k}(\cos{(2n-3)\beta}+\cos{(2n-1)\beta})$$

$$=\frac{1}{2k^2}(\cos{(2n-3)\beta}\cos\beta+\cos{(2n-1)\beta}\cos\beta)$$

$$=\frac{1}{2k^2}\Big[\frac{(\cos{(2n-4)\beta}+\cos{(2n-2)\beta}}{2} + \frac{\cos{(2n-2)\beta}+\cos{2n\beta}}{2}\Big]$$

$$= \frac{1}{4k^2}\cos(2n-4)\beta + \frac{1}{2k^2}\cos(2n-2)\beta + \frac{1}{4k^2}\cos 2n\beta$$

$$S_d = \sum_{n=1}^{m} \frac{1}{4k^2}(k^4)^{n-1}\cos(2n-4)\beta\sin^2\beta$$

$$+\sum_{n=1}^{m} \frac{1}{2k^2}(k^4)^{n-1}\cos(2n-2)\beta\sin^2\beta + \sum_{n=1}^{m} \frac{1}{4k^2}(k^4)^{n-1}\cos 2n\beta\sin^2\beta$$

$$S_d = \frac{1}{4k^2}\sum_{n=1}^{m}(k^4)^{n-1}\cos(2n-4)\beta\sin^2\beta$$

$$+\frac{1}{2k^2}\sum_{n=1}^{m}(k^4)^{n-1}\cos(2n-2)\beta\sin^2\beta + \frac{1}{4k^2}\sum_{n=1}^{m}(k^4)^{n-1}\cos 2n\beta\sin^2\beta$$

Evaluate the first (1), second (2), and third (3) place of the equation:

1.

$$\frac{1}{4k^2}\sum_{n=1}^{m}(k^4)^{n-1}\cos(2n-4)\beta\sin^2\beta$$

$$= \frac{1}{4k^2}(S_d k^4 - (k^4)^m\cos(2m-2)\beta\sin^2\beta + (k^4)^0\cos -2\beta\sin^2\beta)$$

$$= \frac{k^2}{4}S_d - \frac{k^{4m}(2k^2-1)(1-k^2)}{4k^2} + \frac{(2k^2-1)(1-k^2)}{4k^2}$$

$$= \frac{k^2}{4}S_d + \frac{(1-k^{4m})(2k^2-1)(1-k^2)}{4k^2}$$

2.

$$\frac{1}{2k^2}\sum_{n=1}^{m}(k^4)^{n-1}\cos(2n-2)\beta\sin^2\beta$$

$$= \frac{1}{2k^2}S_d$$

3.

$$\frac{1}{4k^2}\sum_{n=1}^{m}(k^4)^{n-1}\cos 2n\beta\sin^2\beta$$

$$= \frac{1}{4k^2}(\frac{S_d}{k^4} - \frac{1}{k^4}\cos 0\beta\sin^2\beta + \frac{k^{4m}}{k^4}\cos 2m\beta\sin^2\beta)$$

$$= \frac{S_d}{4k^6} - \frac{(1-k^2)}{4k^6} + \frac{k^{4m}(1-k^2)}{4k^6}$$

$$= \frac{S_d}{4k^6} + \frac{(1-k^2)(k^{4m}-1)}{4k^6}$$

$$= \frac{S_d}{4k^6} + \frac{(1-k^{4m})(k^2-1)}{4k^6}$$

We get S_d by adding 1+2+3:

$$S_d = \frac{k^2}{4}S_d + \frac{(1-k^{4m})(2k^2-1)(1-k^2)}{4k^2} + \frac{1}{2k^2}S_d + \frac{S_d}{4k^6} + \frac{(1-k^{4m})(k^2-1)}{4k^6}$$

$$S_d(1 - \frac{k^2}{4} - \frac{1}{2k^2} - \frac{1}{4k^6}) = \frac{(1-k^{4m})(2k^2-1)(1-k^2)}{4k^2} + \frac{(1-k^{4m})(k^2-1)}{4k^6}$$

$$S_d(1 - \frac{k^2}{4} - \frac{1}{2k^2} - \frac{1}{4k^6}) = \frac{k^4(1-k^{4m})(2k^2-1)(1-k^2)}{4k^6} + \frac{(1-k^{4m})(k^2-1)}{4k^6}$$

$$S_d(1 - \frac{k^2}{4} - \frac{1}{2k^2} - \frac{1}{4k^6}) = \frac{1-k^{4m}}{4k^6}\Big(k^4(2k^2-1)(1-k^2) + (k^2-1)\Big)$$

$$S_d = \Big(\frac{4k^6}{-k^8+4k^6-2k^4-1}\Big)\Big[\frac{1-k^{4m}}{4k^6}\Big(k^4(2k^2-1)(1-k^2) + (k^2-1)\Big)\Big]$$

To get the remaining part of "f" we will add $S_b + S_c + S_d$.

$$S_b = \Big(\frac{4k^6}{-k^8+4k^6-2k^4-1}\Big)\frac{1-k^{4m}}{4k^6}\Big[k^2(-k^6+3k^4-3k^2+1)\Big]$$

$$S_c = \Big(\frac{4k^6}{-k^8+4k^6-2k^4-1}\Big)\frac{1-k^{4m}}{4k^6}\Big(k^2(k^2-1) + k^6(1-k^2)\Big)$$

$$S_d = \Big(\frac{4k^6}{-k^8+4k^6-2k^4-1}\Big)\frac{1-k^{4m}}{4k^6}\Big(k^4(2k^2-1)(1-k^2) + (k^2-1)\Big)$$

$$S_b + S_c + S_d =$$

$$= \Big(\frac{4k^6}{-k^8+4k^6-2k^4-1}\Big)\Big(\frac{1-k^{4m}}{4k^6}\Big)\Big[k^2(-k^6+3k^4-3k^2+1) + k^2(k^2-1)$$

$$+k^6(1-k^2) + k^4(2k^2-1)(1-k^2) + (k^2-1)\Big]$$

$$= \frac{1-k^{4m}}{-k^8+4k^6-2k^4-1}\Big(-k^8+3k^6-3k^4+k^2+k^4-k^2+k^6-k^8+3k^6-2k^8-k^4+k^2-1\Big)$$

$$= \frac{1-k^{4m}}{-k^8+4k^6-2k^4-1}\Big(-4k^8+7k^6-3k^4+k^2-1\Big)$$

$$= (1-k^{4m})\Big[\frac{-4k^8+7k^6-3k^4+k^2-1}{-k^8+4k^6-2k^4-1}\Big]$$

$$= (1-k^{4m})\Big[\frac{(k^2-1)(1+3k^4-4k^6)}{(k^2-1)(1+k^2+3k^4-k^6)}\Big]$$

$$S_b+S_c+S_d = (1-k^{4m})\Big[\frac{1+3k4-4k^6}{1+k^2+3k^4-k^6}\Big]$$

Adding in what we got for S_a, our "f" part is the following:

$$\frac{r^3k}{24\sin\beta}\Big[fr(1-k^{4m})\Big(\frac{1+2k^2}{1+k^2}+\frac{1+3k^4-4k^6}{1+k^2+3k^4-k^6}\Big)\Big]$$

Taking the sum of the "g" part of 4.10. We will evaluate the equation by naming parts of the equation in the following way:

$$A = \sum_{n=1}^{m} k(k^3)^{n-1}\cos n\beta\sin^2\beta$$

$$B = \sum_{n=1}^{m}(k^3)^{n-1}\cos(n-1)\beta\sin^2\beta$$

We will begin by evaluating A:

$$A = \sum_{n=1}^{m}(k^3)^{n-1}\cos n\beta\cos\beta\sin^2\beta$$

From $\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$,

$$\cos n\beta\cos\beta\sin^2\beta = \frac{cos(n+1)\beta+\cos(n-1)\beta}{2}\sin^2\beta$$

$$A = \frac{1}{2k}\Big[\sum_{n=1}^{m}k(k^3)^{n-1}\cos(n+1)\beta\sin^2\beta + \sum_{n=1}^{m}k(k^3)^{n-1}\cos(n-1)\beta\sin^2\beta\Big]$$

$$= \frac{1}{2k}\Big[\frac{A}{k^3} - \frac{k(k^3)^0}{k^3}(\cos 1\beta)(\sin^2\beta) + k(k^3)^{m-1}\cos(m+1)\beta\sin^2\beta$$

$$+k^3A + k(k^3)^0\cos 0\beta\sin^2\beta - k(k^3)^m\cos m\beta\sin^2\beta\Big]$$

We need to remember the following for computations: $\sin^2\beta = (1-k^2)$, $\beta = \frac{\pi}{m}$, $\cos m\beta = \cos\pi = -1$, and $\cos(m+1)\beta = \cos(m+1)\frac{\pi}{m} = -\cos\beta = -k$.

$$= \frac{1}{2k}\Big[\frac{A}{k^3} - \frac{1-k^2}{k} - \frac{k^{3m}}{k}(1-k^2) + k^3A + k(1-k^2) + k(k^3)^m(1-k^2)\Big]$$

$$A = \frac{A}{2k^4} - \frac{1-k^2}{2k^2} - \frac{k^{3m}}{2k^2}(1-k^2) + \frac{k^2A}{2} + \frac{1-k^2}{2} + \frac{(k^3)^m(1-k^2)}{2}$$

$$A\Big(1 - \frac{1}{2k^4} - \frac{k^2}{2}\Big) = (1-k^2)(-\frac{1}{2k^2} - \frac{k^{3m}}{2k^2} + \frac{1}{2} + \frac{k^{3m}}{2})$$

$$A\Big(\frac{2k^4 - 1 - k^6}{2k^4}\Big) = (1-k^2)\Big(\frac{1 - k^{3m} + k^2 + k^{2+3m}}{2k^2}\Big)$$

$$A\Big(\frac{(1-k^2)(k^4-k^2-1)}{2k^4}\Big) = \frac{(1-k^2)}{2k^2}(-1 - k^{3m} + k^2 + k^{2+3m})$$

$$A\Big(\frac{(k^4-k^2-1)}{k^2}\Big) = (1 - k^{3m} + k^2 + k^{2+3m})$$

$$A = \frac{(1 - k^{3m} + k^2 + k^{2+3m})k^2}{k^4 - k^2 - 1}$$

$$A = \frac{(k^2-1)(1+k^{3m})k^2}{k^4-k^2-1}$$

$$A = \frac{(1-k^2)(1+k^{3m})k^2}{1+k^2-k^4}$$

Now we will evaluate B with A:

$$B = \sum_{n=1}^{m}(k^3)^{n-1}\cos(n-1)\beta\sin^2\beta$$

$$= \frac{1}{k}\sum_{n=1}^{m}k(k^3)^{n-1}\cos(n-1)\beta\sin^2\beta$$

$$= \frac{1}{k}\Big(k^3 A + k(k^3)^0 \cos 0\beta \sin^2\beta - k(k^3)^m \cos m\beta \sin^2\beta\Big)$$

$$= \frac{1}{k}\Big(k^3 A + k(1-k^2) + k(k^{3m}(1-k^2)\Big)$$

$$B = k^2 A + (1+k^{3m})(1-k^2)$$

Now evaluate A+B

$$A + B = \frac{(1-k^2)(1+k^{3m})}{1+k^2-k^4}k^2 + \frac{(1-k^2)(1+k^{3m})}{1+k^2-k^4}k^2(k^2) + (1+k^{3m})(1-k^2)$$

$$A + B = \frac{(1-k^2)(1+k^{3m})}{1+k^2-k^4}k^2(1+k^2) + (1+k^{3m})(1-k^2)$$

$$= \frac{(1-k^2)(1+k^{3m})}{1+k^2-k^4}k^2(1+k^2) + (1+k^{3m})(1-k^2)$$

$$= \frac{(1-k^2)(1+3^{3m})}{1+k^2-k^4}\Big(k^2(1+k^2) + 1 + k^2 - k^4\Big)$$

$$= \frac{(1-k^2)(1+k^{3m})}{1+k^2-k^4}(1+2k^2)$$

So now we know that the "g" part of our equation results in the following:

$$= 4g\frac{(1-k^2)(1+k^{3m})}{1+k^2-k^4}(1+2k^2)$$

Our final equation results in:

$$\frac{r^3 k}{24\sin\beta}\Big[fr(1-k^{4m})\Big(\frac{1+2k^2}{1+k^2} + \frac{1+3k^4-4k^6}{1+k^2+3k^4-k^6}\Big) - 4g(1+k^{3m})\frac{(1-k^2)(1+2k^2)}{1+k^2-k^4}\Big] \quad (4.11)$$

The above equation is given in Part 5 of the given solution in Figure 4.4.

4.6 Evaluating m

Working just with the inner brackets of 4.11, we will substitute our equations for $f = \frac{b}{r+s}$ and $g = a + \frac{br}{r+s}$ as follows

$$= \frac{br}{r+s}(1-k^{4m})\frac{2+4k^2+8k^4+4k^6-6k^8}{(1+k^2)(1+k^2+3k^4-k^6)}$$

$$-4(a+\frac{br}{r+s})(1+k^{3m})\frac{(1-k^2)(1+2k^2)}{1+k^2-k^4}$$

$$= -4a(1+k^{3m})\frac{(1-k^2)(1+2k^2)}{1+k^2-k^4} - \frac{4br}{r+s}(1+k^{3m})\frac{(1-k^2)(1+2k^2)}{1+k^2-k^4}$$

$$+\frac{2br}{r+s}(1-k^{4m})\frac{1+2k^2+4k^4+2k^6-3k^8}{(1+k^2)(1+k^2+3k^4-k^6)}$$

$$= -4a(1+k^{3m})\frac{(1-k^2)(1+2k^2)}{1+k^2-k^4} \tag{4.12}$$

$$+\frac{2br}{r+s}\left[(1-k^{4m})\frac{1+2k^2+4k^4+2k^6-3k^8}{(1+k^2)(1+k^2+3k^4-k^6)} - 2(1+k^{3m})\frac{(1-k^2)(1+2k^2)}{1+k^2-k^4}\right]$$

This entire equation needs to be positive. First, the equation in the brackets for the b part must yield a positive number. In order to determine the first integer in which m is positive, we will run a program in MATLAB. Figure 4.5 is the program that was run to determine our value for m.

The results in Table 4.1 yield that the brackets will be positive for all $m \geq 9$.

Now that we know our brackets are positive for all $m \geq 9$, we will find a b sufficiently large compared to a so that "P" will intersect the center on or below the bottom our solid. In order to find b we will set Equation 4.12 > 0.

To determine the value for b based off of given values for a, β, r, and s I wrote a MATLAB program (See Figure 4.7). This made it easier to find values of b based off of a given a. Remember that r is equal to length from the origin to the top of our solid, $s = r\cos^9\beta = rk^9$, and $\beta = \pi/9$.

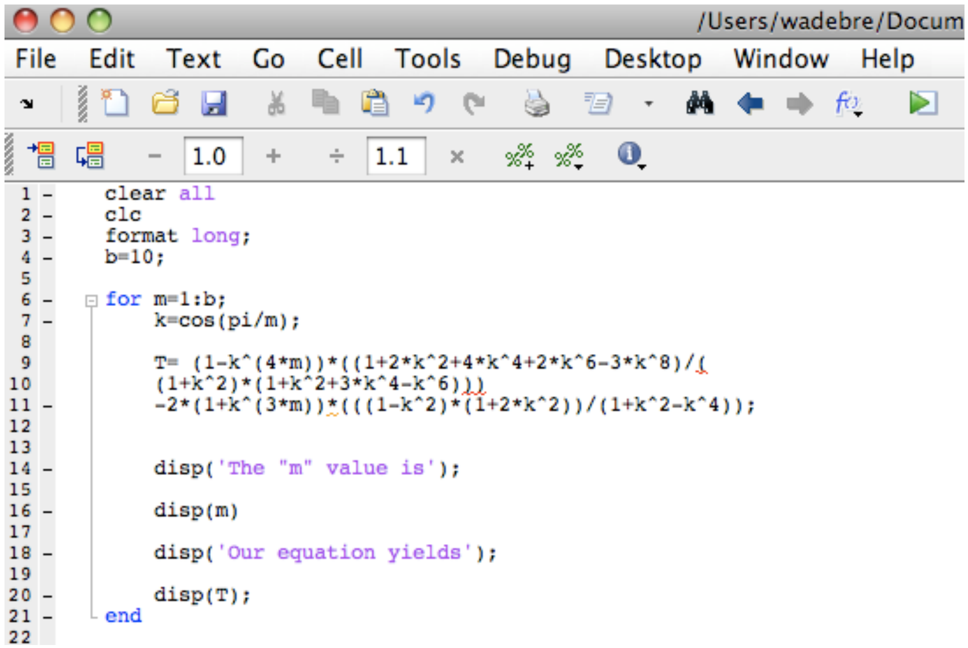


Figure 4.5: The "m" file that is run in MATLAB to determine our value for m

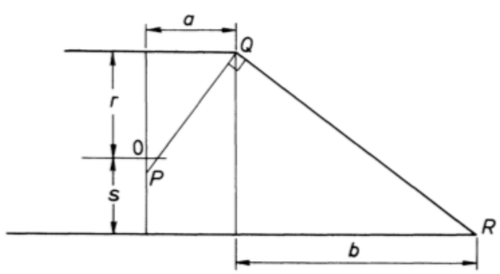


Figure 4.6: Side view of the solid

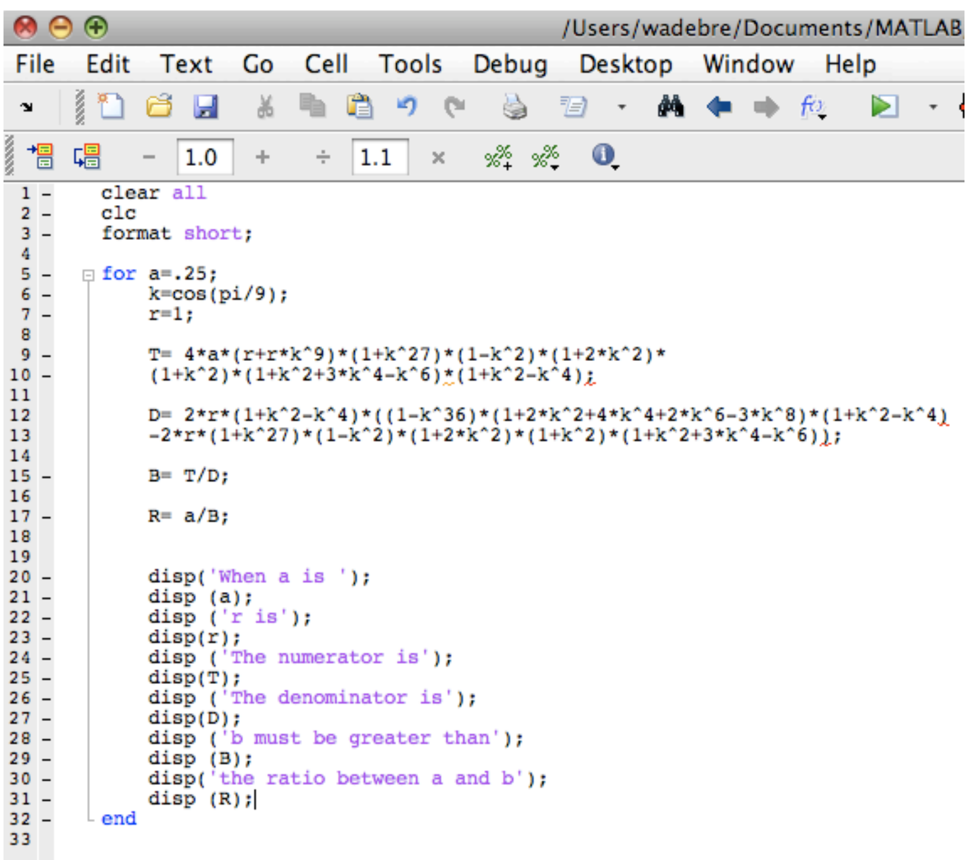
For the results when $a = 0.25$ and $r = 1$, b must be greater than 7.9762. For the given r value, the ratio between a and b is 0.0313.

In order to ensure that P is below the mass center, we want b long enough compared to a so that the right angle Q will cause $\overline{PQ}$ to intersect the center of the solid below the bottom of the solid.

Since $a = 0.25$, and the height of the solid is $r + s = 1 + 1\cos^9\beta \approx 1.57$, we know that the diagonal from Q to the center and bottom of the figure is ≈ 1.59. Using similar triangles, we find that any $b \geq 9.8624$ and $a = 0.25$ ensures that P is below the the mass center.

Table 4.1: Results for m

m value	equation result
1	0
2	-1
3	-0.903076
4	-0.667969
5	-0.452051
6	-0.279457
7	-0.146033
8	-0.043859
9	0.034274
10	0.094100




```
clear all
clc
format short;

for a=.25;
    k=cos(pi/9);
    r=1;

    T= 4*a*(r+r*k^9)*(1+k^27)*(1-k^2)*(1+2*k^2)*
    (1+k^2)*(1+k^2+3*k^4-k^6)*(1+k^2-k^4);

    D= 2*r*(1+k^2-k^4)*((1-k^36)*(1+2*k^2+4*k^4+2*k^6-3*k^8)*(1+k^2-k^4)
    -2*r*(1+k^27)*(1-k^2)*(1+2*k^2)*(1+k^2)*(1+k^2+3*k^4-k^6));

    B= T/D;

    R= a/B;


    disp('When a is ');
    disp (a);
    disp ('r is');
    disp(r);
    disp ('The numerator is');
    disp(T);
    disp ('The denominator is');
    disp(D);
    disp ('b must be greater than');
    disp (B);
    disp('the ratio between a and b');
    disp (R);
end
```

Figure 4.7: The "m" file to determine a value for b.

Chapter 5

Bodies which have exactly one stable and one unstable position.

Guy's construction of a polyhedron with just one stable equilibrium is fascinating and important because of the fact that it is a polyhedron with the *least* known number of faces and just one stable equilibrium (it is still an open question). Guy's construction had 1 stable equilibrium, and 3 unstable equilibrium. There are bodies with less than 4 equilibrium. These bodies are called mono-monostatic bodies. Gábor Domokos' interest was to find a three-dimensional body with just one stable and one unstable equilibrium. The following provides a proof that no monostatic bodies exist in two-dimensions, and then the construction of a three-dimensional mono-monostatic body.

5.1 All two-dimensional objects have at least two stable equilibrium.

Gábor Domokos was a Civil Engineering teacher in Hungary in the 1980's. He enjoys the mathematical side of things, and often times would discuss interesting problems with Andy Ruina. Ruina had a friend named Jim Papadopoulous and both were working on a simple conjecture. Papadopoulous started working with two-dimensional (2D) problems with plywood to see if he could find shapes with only one stable equilibrium.

We are considering these bodies as resting on a horizontal surface in the presence of uniform gravity. Monostatic bodies are possible to construct; for example, the popular children's toy called a "Comeback Kid" is a monostatic body. Looking at homogeneous, convex monostatic bodies in the 2D case, one can prove that:

Theorem 5.1. *Among planar (slab-like) objects rolling along their circumference no monostatic bodies exist.*

In 2D Papadopoulous found that a square, for example, had four positions for which it was stable. He looked at an ellipse, which is stable when horizontal on one of the two flatter parts. It is unstable when balanced on either end. Imagine balancing an upright egg. Papadopoulous' conjecture was that no matter what convex shape you draw and cut out, it has at least two orientations where it is stable.

The following ideas are published by G. Domokos and P. Várkonyi [4]. Consider a convex, homogeneous planar "body" B and a polar coordinate system with origin at the center of gravity of B. Let the continuous function $R(\varphi)$ denote the boundary of B. It is easy to see that non-degenerate stable/unstable equilibria of the body correspond to local minima/ maxima of $R(\varphi)$.

Assume that $R(\varphi)$ has only one local maximum and one local minimum. Meaning that there is only one point on the boundary that is closest to the origin, and one point of the boundary that is furthest away from the origin. Finally a simple continuity argument shows that there exists exactly one value $\varphi = \varphi_0$ for which $R(\varphi_0) = R(\varphi_0 + \pi)$. Rotate a directed chord which passes through O clockwise around O. Assume initially the chord points toward the closest boundary point. Notice that initially O is closest to the head of the directed chord, while at the end O is closest to the tail of the directed chord.

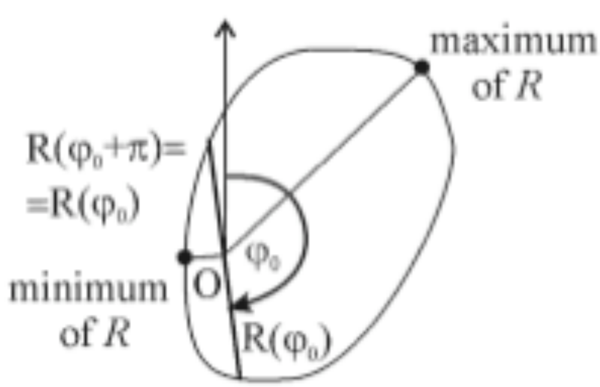


Figure 5.1: Two-Dimensional case

From this we know $R(\varphi) > R(\varphi_0)$ if $\pi > \varphi - \varphi_0 > 0$, and $R(\varphi) < R(\varphi_0)$ if $-\pi < \varphi - \varphi_0 < 0$. The straight line discussed above $\varphi = \varphi_0$ is the same as $\varphi = \varphi_0 + \pi$. It is passing through the origin O and cuts B into a "thin" ($R(\varphi) < R(\varphi_0)$), and a "thick" ($R(\varphi) > R(\varphi_0)$) part. This implies that O can not be the center of gravity, which contradicts our initial assumption.

Definition 5.1. *Mono-monostatic bodies are convex, homogeneous bodies with fewer than four equilibria.*

5.2 The Gömböc

In 1995 Domokos attended an International Congress on Industrial and Applied Mathematics in Hamburg. There were many lectures, but the one that everyone attended was Vladimir Igorevich Arnold's lecture. A returning theme of that particular talk was the recurrence of the integer 4 in many seemingly unrelated results. His lecture reminded Domokos of the 2D objects which always have at least four equilibrium (two unstable and two stable). He proposed a counter example to Arnold, saying that he has found a three-dimensional object with just one stable equilibrium. Arnold immediately explained that it was not a counter example because the figure has one stable equilibrium, but three unstable. Arnold did, however, conjecture that "convex homogeneous bodies with fewer than four equilibria (mono-monostatic bodies) may exist."

Domokos moved on to construct a three-dimensional solid which has less than four equilibrium. About ten years later, Domokos constructed the Gömböc to prove that Arnolds conjecture is indeed correct. The problem was solved in 2006 [4]. It is called the Gömböc, and is a convex three-dimensional homogeneous body which, when resting on a flat surface, has just one stable and one unstable point of equilibrium.

The shape is not unique, has countless varieties, and most of which are very close to a sphere with very strict shape tolerance. These bodies are hard to visualize, describe, or identify. They are neither "flat" nor "thin." The solution has curved edges and resembles a

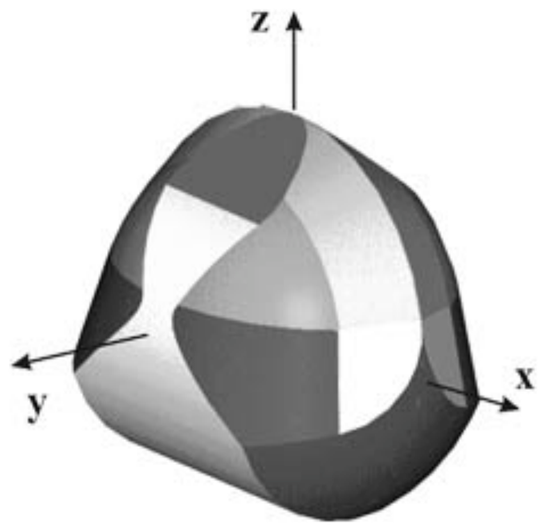


Figure 5.2: Gömböc

sphere with a squashed top. It rests in its stable equilibtirum, and its unstable equilibrium position is obtained by rotating the figure 180° about the horizontal axis. Theoretically it will rest on this point, but the smallest perturbation will bring it back to the stable point. There is not a specific description of how to create the Gömböc available at this time.

It has been proven that there exists a body that is convex with its center of gravity at the origin. Unfortunately, from the numerical proof it suggests that d must be very small ($d < 5 * 10^{-5}$) in order to satisfy convexity together with other restrictions. The created object with these properties is extremely similar to a sphere and nearly impossible to visually see the difference in the construction.

Appendix: For general appreciation and also to help me remember what was done during the project, the description of Schönhardt's polyhedron, Császár's polyhedron, and a Meissner body are included here.

Chapter 6

Császár's polyhedron

Theorem 6.1. *The tetrahedron and Császár's polyhedron are the only two known polyhedra without any diagonals.*

Definition 6.1. *In a polyhedron, a line between two nonadjacent vertices is called a diagonal.*

The Császár polyhedron is named after Ákos Császár, who discovered it in 1949. Ákos Császár is a Hungarian mathematician specializing in general topology and real analysis.

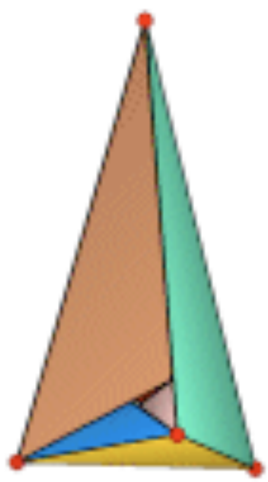


Figure 6.1: Császár's Polyhedron

It is very difficult to visualize the Császár Polyhedron because of its complexity. In an attempt, to help you understand what it looks like, there is a flat pattern of the polyhedron. It would look like this if it was completely unfolded (See Figure 6.2).

In geometry, the Császár polyhedron is a nonconvex polyhedron, topologically a torus, with 14 triangular faces. When every pair of vertices is connected by an edge, the polyhedron has thus no diagnols. Therefore, the Császár polyhedron has no diagonals because every

Figure 6.2: Flat Pattern of Császár's Polyhedron

pair of vertices is connected by an edge. The graph is isomorphic with the skeleton of a six-dimensional simplex, the 6-space analogue of the tetrahedron.

It may be easier to visualize the polyhedron if you understand how to construct it. It is not difficult to make a paper model of the Császár polyhedron. Copy the two patterns below and cut the copies out. Color the seven shaded triangles on both sides. Crease the paper to make "mountain folds" along each M line, and the "valley folds" along each v line. [6]

1. With the pattern for the base, fold the two largest triangles to the center and tape the A edges to each other. Turn the paper over. Fold the two smaller triangles to the center and tape the B edges together to obtain a completed base.

2. The six-faced conical top is formed by taping the C edges together. Place it on the base as shown in the drawing of the completed model. It will fit in two ways. Choose the fit that joins white to shaded triangles, then tape each of its six edges to the corresponding six edges of the base.

The polyhedron has 7 polyhedron vertices, 14 faces, and 21 polyhedron edges.

The Császár polyhedron and tetrahedron are the only two known polyhedra without any diagonals for which there are no interior diagonals.

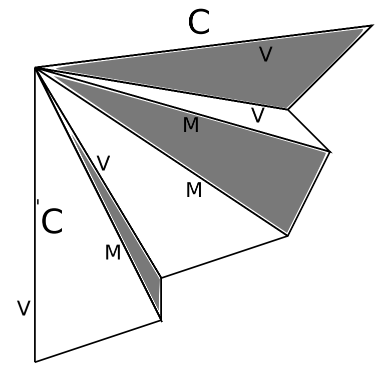


Figure 6.3: Pattern for top of Császár's polyhedron

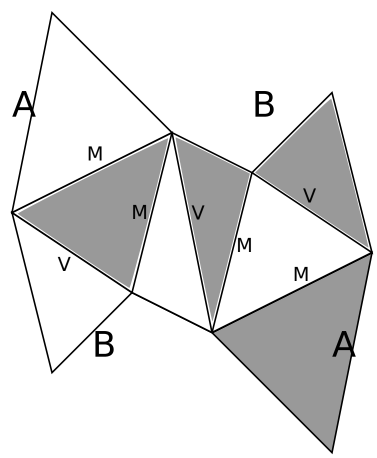


Figure 6.4: Pattern for base of Császár's polyhedron

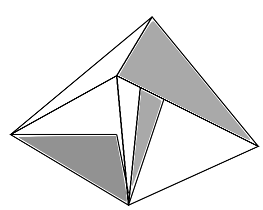


Figure 6.5: Completed Base of Császár's polyhedron

Chapter 7

Szilassi's polyhedron

Theorem 7.1. *The tetrahedron and Szilassi polyhedron are the only two known polyhedra in three-dimensions in which each face shares an edge with another face.*

The Szilassi polyhedron, named after Hungarian mathematics Lajos Szilassi, who discovered it in 1977, is the dual to the Császár polyhedron with 7 faces, 14 vertices, and 21 edges. While the Császár polyhedron has the fewest possible vertices of any toroidal polyhedron, the Szilassi polyhedron has the fewest possible faces of any toroidal polyhedron.

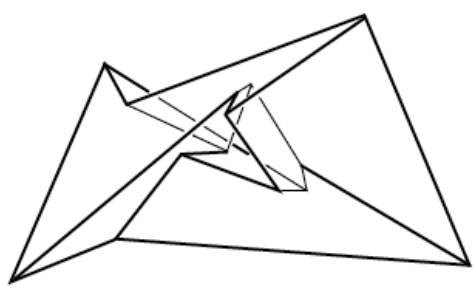


Figure 7.1: Transparent Szilassi's Polyhedron

Definition 7.1. *In geometry, polyhedra are associated into pairs called duals, where the edges correspond to the faces of the other. The dual of the dual is the original polyhedron. The dual of a polyhedron with equivalent vertices is one with equivalent faces, and of one with equivalent edges is another with equivalent edges.*

Duality means that every face of the Szilassi heptahedron has an edge in common with each of the other 6 faces.

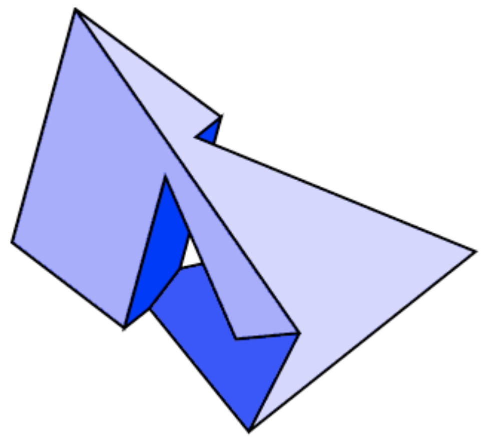


Figure 7.2: Szilassi's Polyhedron

Theorem 7.2. *The four color theorem states that given any separation of a plane into contiguous (adjacent) regions, no more than four colors are required to color the regions of the map so that no two adjacent regions have the same color.*

Definition 7.2. *Two regions are called adjacent if they share a border segment, not just a point.*

The Four Color Problem dates back to 1852 when Francis Guthrie, while trying to color the map of the counties of England, notices that four colors sufficed. He asked his brother Frederick if it was true that any map can be colored using four colors in such a way that adjacent regions receive different colors. Frederick Guthrie then communicated the conjecture to DeMorgan. A year later the first proof by Kempe appeared, and its incorrectness was pointed out 11 year later by Heawood. Another failed proof was presented by Tait in 1880. Although they both failed, Kempe discovered what became known as Kempe chains, and Tait found an equivalent formulation of the Four Color Theorem in terms of 3-edge-coloring.

Chapter 8

Schönhardt Polyhedron

Definition 8.1. *A tetrahedralization is a partition of the input domain, point set or polyhedron, into a collection of tetrahedra, that meet only at shared faces (vertices, edges, and triangles).*

The analogue of triangulation in three dimensions is called tetrahedralization.

If additional vertices (Steiner points) are allowed when constructing the tetrahedralization, then all polyhedra are tetrahedralizable.

Tetrahedralization turns out to be significantly more complicated than triangulation.

Theorem 8.1. *Any polyhedra can be triangulated with $O(n^2)$ Steiner points and $P(n^2)$ tetrahedra.*

Proof: Extend a vertical "wall" from each edge of the polyhedron boundary, up and down from that edge until is reaches some other part of the boundary. These walls divide the polygon into generalized cylinders. Triangulating the top and bottom faces of the cylinders partitions the polygon into $O(n^2)$ triangular prisms. Each vertical prism side is crossed at most once by a polyhedron edge, so the prisms are polyhedra with at most twelve vertices. Triangulate the faces of these polyhedra, making sure that tetrahedra from different prisms will meet face to face, and then triangulate each prism with at most 20 tetrahedra incident to the single interior Steiner point (Du [5]).

In geometry, the Schönhardt Polyhedron is the smallest non-convex polyhedron that cannot be triangulated into tetrahedra without adding new vertices.

It can be formed by two congruent equilateral triangles in two parallel planes so that the line through the centers of the triangles is perpendicular to the planes. Consider an equilateral triangular prism, where the base triangle ABC lies directly below the upper triangle $A'B'C'$.

Suppose that the edges of the prisms are constructed from wire and that the edges AB', BC', and CA' have been added with extra wire. If we slowly rotate the upper triangle in one direction, then the rectangular faces of the prism bend outwards along the extra edges AB', BC', and CA'. If we rotate in the other direction, however, the rectangular faces of the prism bend inward, yielding a polyhedron which is not convex.

When we have rotated a full 60°, then the three edges AB', BC', and CA' intersect at the center of the figure. Schönhardt's Polyhedron is obtained when the rotation is by some intermediate value, for example 30°. In this instance the line segments AB', BC', and CA' lie outside the polyhedron([3]).

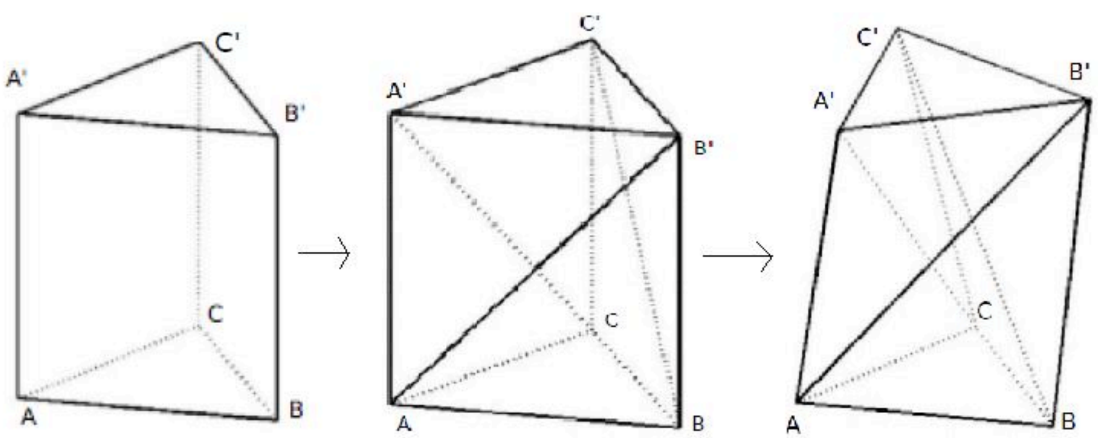


Figure 8.1: Schönhardt's Polyhedron

We take the three vertices on the bottom triangle, and we need to connect it to a fourth additional vertex to form a tetrahedron. If we connect it to any of the above three vertices, then the edge is outside of the polyhedron because every diagonal that is not a boundary edge lies completely in the exterior. It implies that there can be no triangulation of it without new vertices because there is simply no interior tetrahedron (all possible tetrahedra spanned by four of its six vertices would introduce new edges).

Schönhardt proved that this is the smallest example of an untetrahedralizable polyhedron. Bagemihl extended this example to construct a polyhedron of n vertices with the same properties for every $n \geq 6$.

Chapter 9

A Meissner body

Definition 9.1. *A three-dimensional body is of* constant width *if it has the following property: Whichever way it is clamped between two parallel plates, both plates are always at exactly the same distance from each other.*

Take a regular tetrahedron, which has four points in space that are arranged in such a way that they are equidistant from each other. Place spheres of equal size at each of the four tetrahedron vertices. The spheres expand at the same rate until they touch and have to penetrate each other. The surface of each sphere intersects the three tetrahedron vertices opposite the center of the sphere. This forms a Reuleaux tetrahedron, which is the intersection of four spheres at the vertices of a regular tetrahedron. The result is a body looking like a bulging tetrahedron. The Reuleaux tetrahedron has the same face structure as a regular tetrahedron, but with curved faces: four vertices, and four curved faces, connected by six circular-arc edges. Basically if a flat plate is placed on three equal Reuleaux tetrahedrons and pushed back and forth, it will move almost parallel to the surface of the table. In other words, it will move as though it is placed on three spherical balls with equal size, without wobbling. As noted above, the Reuleaux tetrahedron is almost constant width.

Since a Reuleaux tetrahedron is only of almost constant width, mostly the body touches two plates between which it is clamped with one of its vertices and one point on the opposite surface of the solid. In cases like this, the two plates are at the same distance from each other, but the two possible points of contact may be on two opposite edges of the solid. At this place on the solid, the width is greater. For two edge midpoints, it is maximally $\sqrt{3}$ - $\frac{\sqrt{2}}{2} \approx 1.0249$ times larger than the smallest width.

It is possible, to make the Reauleaux tetrahedron into a body of constant width by rounding three of its edges. First, pick a face of the original tetrahedron. Extend the chosen face in all three directions so that it intersects the Reuleaux tetrahedron. Then, extend the adjacent triangles so that they intersect with the original triangle. Remove the part of the surface of this body which is located between the extensions of two adjacent lateral surfaces of the tetrahedron. In the new space, replace it with a sphere so that there are three smooth and three original "edges." The resulting figure is called a Meissner body (Weber [10]).

Definition 9.2. *When three edges of a Reuleaux tetrahedron that meet at a common vertex are rounded as described, the resultant solid is a body of constant width, called a Meissner body.*

Bibliography

[1] H. T. Croft, K. J. Falconer and R. Guy, Unsolved problems in geometry, **Problem B12**, Springer, (1991), 61.

[2] R. Dawson, Monostatic simplexes, *The American Mathematical Monthly*, **92**, (8), (1985), 541-546.

[3] N. Do, Art gallery theorems, *Mathellaneous*, 288-294. http://www.austms.org.au/Gazette/2004/Nov04/mathellaneous.pdf

[4] G. Domokos and P. L. Várkonyi, Mono-monostatic bodies, *The Mathematical Intelligencer*, **28**, (2006), 34-38.

[5] D. Du, Computing in Euclidean geometry, World Scientific Publishing Co., (1992), 68-71.

[6] M. Gardner, The Császár polyhedron, Time Travel and Other Mathematical Bewilderments, W. H. Freeman and Company, (1988), 140-152.

[7] M. Goldberg and R. Guy, Problems and solutions, *SIAM Review*, **11**, (1), (1969), 78-82.

[8] A. Heppes, A double tipping tetrahdreon, *SIAM Review*, **9**, (1967), 599-600.

[9] M. Klamkin, Acute dihedral angles, *Mathematics Magazine*, **61**, (5), (1988), 320.

[10] C. Weber, What does this solid have to do with a ball?, (2009). http://www.swisseduc.ch/mathematik/material/gleichdick/docs/meissner_en.pdf