# Gregarious Path Decompositions of Some Graphs

by

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Abstract

Let $G$ be a simple graph and $f(v)$ a positive integer for each vertex $v$ of $G$. Form $G^f$ by replacing each $v$ by a set $F(v)$ of $f(v)$ vertices, and each edge $uv$ by complete bipartite graph on bipartition $(F(u), F(v))$. Can we partition $G^f$ into paths of length 2 which are gregarious, that is, meet three different $F(u)$'s?

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## List of Abbreviations

PBBG Parity Balanced Bipartite Graph

BGwFD Bipartite Graph with Four Degrees

L(G) Line Graph of G

GDDs Group Divisible Designs

NASCs Necessary and Sufficient Conditions

TFAE The following are equivalent

# Chapter 1

# Introduction

Let $G = (V,\, E)$ be a simple graph and $f : V \to \mathbb{N}$, where $f(v)$ is a positive integer for each vertex $v$ of $G$. Form the graph $G^f$ by replacing each $v$ by a set $F(v)$ of $f(v)$ vertices, and each edge $uv$ by a complete bipartite graph on bipartition $(F(u), F(v))$. Our question is: "Can we partition $G^f$ into paths of length 2, $P_3$, which are *gregarious*, that is, each vertex of $P_3$ is in a different $F(u)$?"

**Example 1.1.** Let $G = (V, E)$ be a graph where $|V| = 5$ and $f : V(G) \to \mathbb{P}$ such that $f : (v_1, v_2, v_3, v_4) \to (2, 3, 2, 2)$.

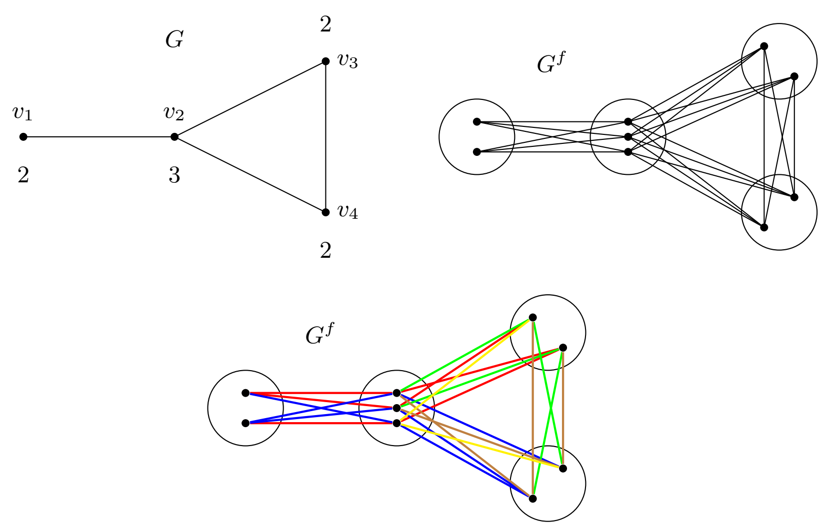


Figure 1.1: Example of Gregarious Path Decomposition

Each color represents a different class of $P_3$'s.

## 1.1 History of the problem

**Non-Gregarious Case:**

In a recent paper [3], the complete solution is given for a decomposition of any complete multipartite graph into paths of lengths 3 and 4. Dr. Hoffman & Dr. Billington introduce the problem "what if we try to solve the same problem with **gregarious** paths?"

**Previous work in Gregarious Decompositions:**

1. In [5], Dean G. Hoffman & Elizabeth Billington give necessary and sufficient conditions to decompose a complete tripartite graph into gregarious 4-cycles. They use the notion of gregarious decompositions as "a cycle is said to be gregarious if its vertices occur in as many different parts of the multipartite graph as possible".

2. In [6], Dean G. Hoffman & Elizabeth Billington give the necessary and sufficient conditions for gregarious 4-cycle decompositions of the complete equipartite graph $K_{n(m)}$ (with $n \geqslant 4$ parts of size $m$) whenever a 4-cycle decomposition (gregarious or not) is possible, and also of a complete multipartite graph in which all parts but one have the same size.

3. In [7], Benjamin R. Smith give necessary and sufficient conditions for the existence of a gregarious 5-cycle decomposition of the complete equipartite graph $K_{m(n)}$.

4. In [8], Elizabeth J. Billington, Benjamin R. Smith & D.G. Hoffman give necessary and sufficient conditions for gregarious cycle decomposition of the complete equipartite graph $K_{n(m)}$ (with n parts, $n \geqslant 6$ or $n \geqslant 8$, of size $m$) into both 6-cycles and 8-cycles.

5. In [9] Jung R. Cho & Ronald J. Gould give necessary and sufficient conditions for the existence of decompositions of the complete multipartite graph $K_{n(2t)}$ into gregarious 6-cycles if $n \equiv 0, 1, 3$ or $4 \pmod 6$ . They used the method of a complete set of differences in $\mathbb{Z}_n$.

6. In [10], Jung R. Cho gives another proof of the problem of decomposing the complete multipartite graph $K_{n(2t)}$ into gregarious 6-cycles for the case of $n \equiv 0$ or $3 \pmod 6$.

7. In [11], Benjamin R. Smith gives necessary and sufficient conditions for the existence of gregarious $k-$ cycle decomposition of a complete equipartite graph, having $n$ parts of size $m$, and either $n \equiv 0, 1 \pmod k$, or $k$ is odd and $m \equiv 0 \pmod k$.

8. In [12], Elizabeth J. Billington , Dean G. Hoffman & Chris A. Rodger give necessary and sufficient conditions for decomposing a complete equipartite graph $K_{n(m)}$ with $n$ parts of size $m$ into n-cycles in such a way that each cycle meets each part of $K_{n(m)}$; that is, each cycle is said to be gregarious. Furthermore, they give gregarious decompositions which are also resolvable.

9. In [13], Saad I. El-Zanati, Narong Punnim & Chris A. Rodger give necessary and sufficent conditions for the existence of Gregarious GDDs with Two Associate Classes having block size 3.

**Definition 1.2.** Let $G = (V, E)$ be simple graph and $h : E \to \mathbb{P}$. Define $G^{[h]}$ on vertex set $V$ as follows: if $u, v \in V$ and $uv = e \in E$ then put $h(e)$ edges in between $u$ and $v$ in $G^{[h]}$.

**Example 1.3.** Let's use the example 1.1 and define $h(v_iv_j) := f(v_i)f(v_j)$.

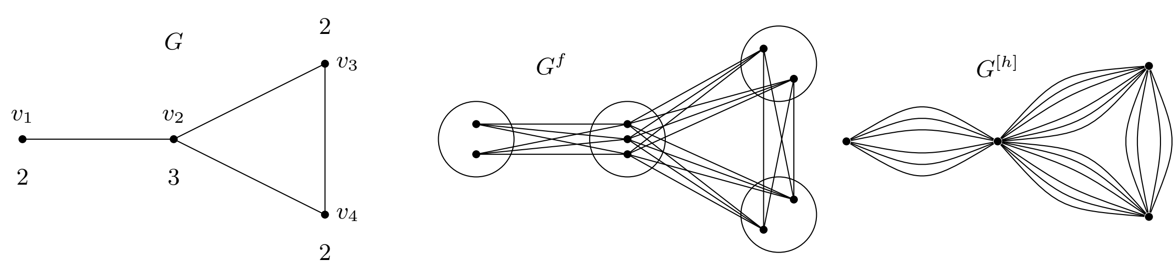


Figure 1.2: Example of $G^f$ and $G^{[h]}$

In $G^{[h]}$, any $P_3$ will be a gregarious path of length two.

Chapter 2

Tutte's f-Factor Theorem

## 2.1 Tutte's $f-$Factor Theorem

**Definition 2.1.** Let $G = (E, V)$ be a graph. G is called a $k$-regular graph if for every $v \in V$, $deg(v) = k$ for some $k \in \mathbb{N}$.

**Definition 2.2.** Let $G = (V, E)$ be a graph, then

1. A **factor** of a graph $G$ is a spanning subgraph of $G$.

2. A $k-$**factor** is a spanning $k-$regular subgraph.

3. Given a function $f : V(G) \to \mathbb{Z}$, an $f-$**factor** of a graph $G$ is a spanning subgraph $H$ such that $d_H(v) = f(v)$ for all $v \in V$.

**Example 2.3.** Let $G = (V, E)$ be a graph with $f : (v_1, v_2, v_3, v_4, v_5, v_6) \to (3, 1, 1, 2, 3, 2)$, then we can get the $f - factor$:

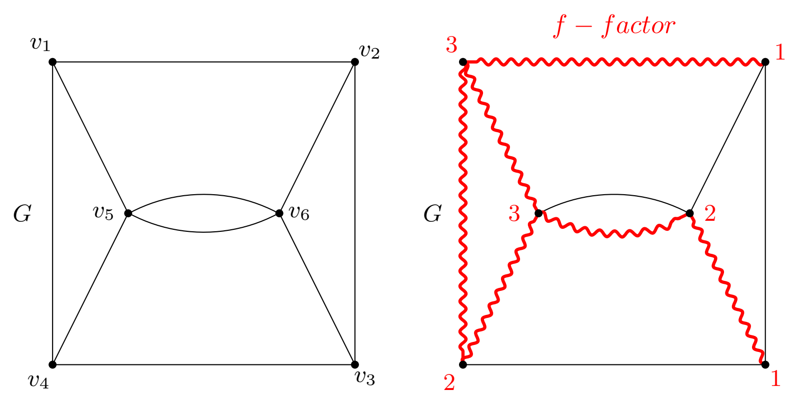


Figure 2.1: Example of an f-factor

**Definition 2.4.** If $f : V(G) \to \mathbb{Z}$, define $\bar{f} : V(G) \to \mathbb{Z}$ by $\bar{f}(v) = deg_G(v) - f(v)$.

**Example 2.5.** From the previous example $\bar{f} : (v_1, v_2, v_3, v_4, v_5, v_6) \to (0, 2, 2, 1, 1, 2)$

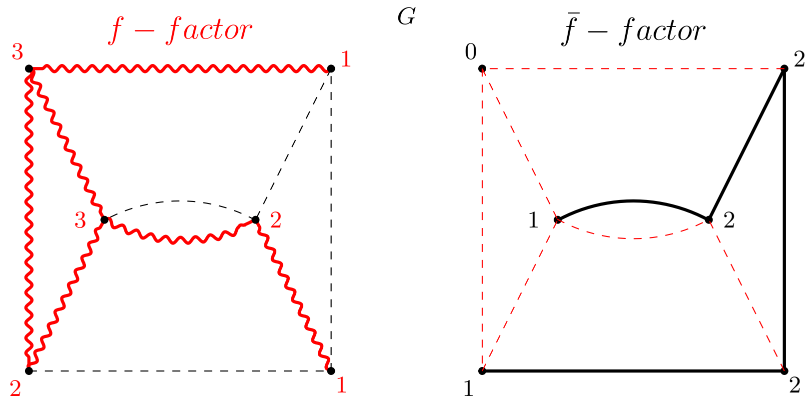


Figure 2.2: Example of an $\bar{f}$-factor

**Definition 2.6.** Let $S, T \subseteq V(G)$, with $S \cap T = \emptyset$, then $\lambda(S, T) =$ the set of edges with one end in $S$ and the other end in $T$.

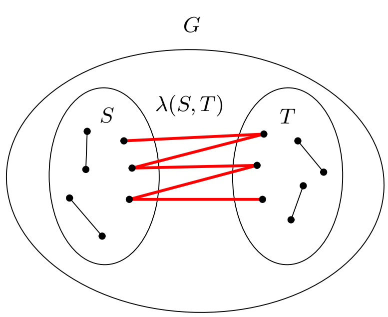


Figure 2.3: Definition of $\lambda(S, T)$

**Definition 2.7.** $B = (S, T, U)$ is a $G - triple$ if

1. $S, T, U \subseteq V(G)$,

2. $S \cup T \cup U = V(G)$

3. $S \cap T = S \cap U = T \cap U = \emptyset$

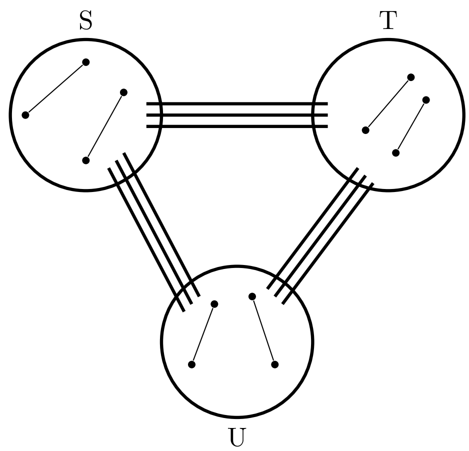


Figure 2.4: Example of a G-triple

**Definition 2.8.** Let $G = (V, E)$ be a graph and $f : V \to \mathbb{P}$ be a function. Then define $f(S) = \sum_{s \in S} f(s)$ for any $S \subseteq V$.

**Definition 2.9.** By a *component* of $B$, we mean a component of $G \setminus (S \cup T)$.

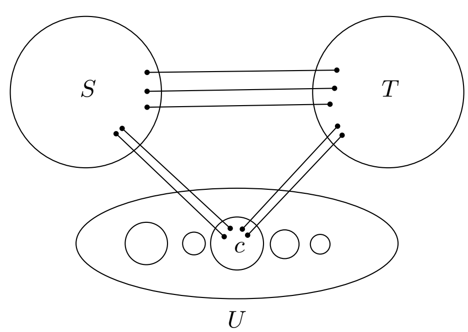


Figure 2.5: Component of $B$

1. If $c$ is a component of $B$, let $J(B, f, c) = f(c) + \lambda(c, T)$.

2. $c$ is called **odd** or **even** according to if $J(B, f, c)$ is odd or even.

3. $k(B, f) = \#$ of odd components of $B$.

**Theorem 2.10.** [4] *$G$ has an $f-factor$, iff for each $G-triple$ $B=(S,T,U)$*

$$k(B,f)+\lambda(S,T) \quad \leqslant \quad f(S)+\bar{f}(T)$$

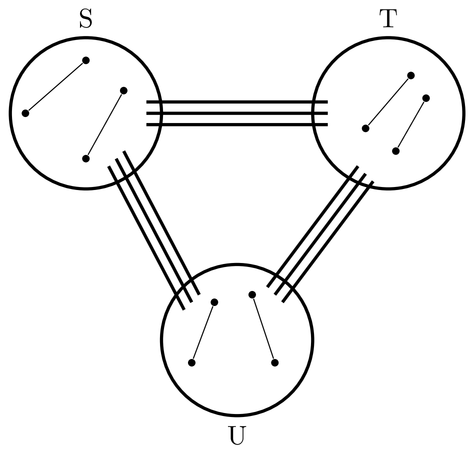


Figure 2.6: Tutte's f-factor Theorem

## 2.2 Applying Tutte's f-Factor Theorem

**Definition 2.11.** The *line graph* of a graph $G$, written $L(G)$, is the graph whose vertices are the edges of $G$, with $ef \in E(L(G))$ whenever $e$ and $f$ are different edges of $G$ having at least one vertex of $G$ in common.

Let $G=(V,\,E)$ be a simple graph and $f: V \to \mathbb{N}$, where $f(v)$ is a positive integer for each vertex $v$ of $G$. Then, let $h: E \to \mathbb{P}$ where $h(v_iv_j) := f(v_i)f(v_j)$. In this way we can get $G^f$ and $G^{[h]}$ with given $G$ and $f$.

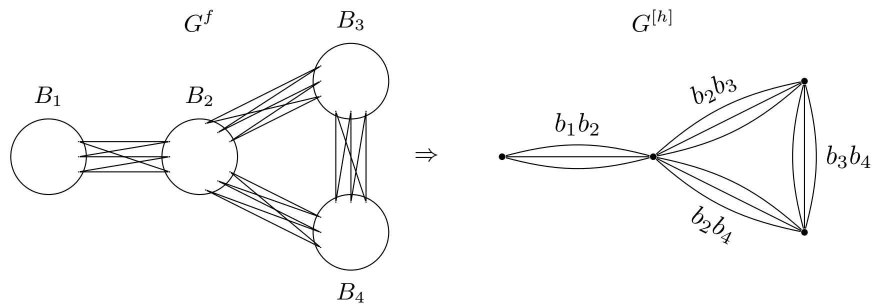


Figure 2.7: $G^f$ and $G^{[h]}$

Now, if we have a gregarious decomposition of $G^f$ into $P_3$'s (denoted by $G^f \overset{g}{\hookrightarrow} P_3$), then getting decomposition of $G^{[h]}$ into $P_3$'s (denoted by $G^{[h]} \hookrightarrow P_3$) is trivial.

So $G^f \overset{g}{\hookrightarrow} P_3 \Rightarrow G^{[h]} \hookrightarrow P_3$.

The opposite direction is not true, but it will still give us some of the necessary conditions for $G^f \overset{g}{\hookrightarrow} P_3$. We can apply Tutte's f-factor theorem to solve $G^{[h]} \hookrightarrow P_3$.

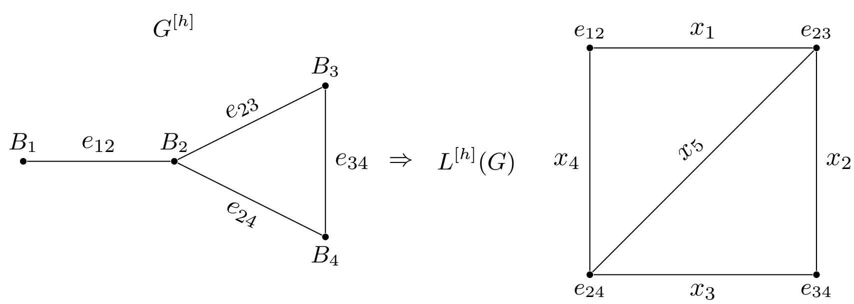


Figure 2.8: $G^{[h]}$ and $L^{[h]}(G)$

In $G^{[h]}$, assume that there are $e_{ij} = b_i b_j$ edges in between any pair of sets of vertices $B_i$ and $B_j$ where $|B_i| = b_i$ for any $i$. Firstly, get the line graph of $G$, $L(G)$, then blow the edges of $L(G)$ in such a way that for each vertex $e_{ij} \in V(L(G))$, $deg(e_{ij})$ is $b_i b_j$ in the new graph, $L^{[h]}(G)$. In $L^{[h]}(G)$, each edge will represent a $P_3$ in $G^{[h]}$. For example, in figure 2.8, $x_1$ represents the number of $P_3$'s passing through sets $B_1 \to B_2 \to B_3$ in $G^{[h]}$.

Therefore, if we find each $x_i$, we can find the $P_3$ decomposition of $G^{[h]}$. To find the $x_i$'s, we will use Tutte's f-factor theorem. Start with $G$, get $L(G)$, let $M$ be a sufficiently large number, then get $L^M(G)$ by replacing every edge of $L(G)$ with $M$ edges (fig. 2.9). So if we find an $h-$factor of $L^M(G)$ such that $h(e_{ij}) = b_i b_j$, then we can get $G^{[h]} \hookrightarrow P_3$.

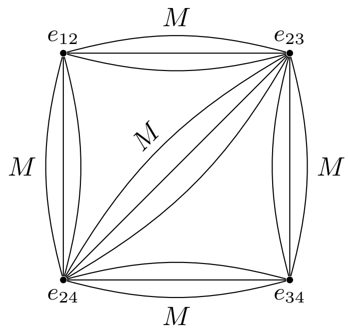


Figure 2.9: $L^M(G)$

**Theorem 2.12.** *$L^M(G)$ has an $h - factor$ for all sufficiently large $M$ , iff for each $L(G) - triple$ $B = (S, T, U)$ where $T$ is independent and $\lambda(T, U) = 0$,*

$$k(B, h) + h(T) \leqslant h(S).$$

*Proof.* Let $M$ be a sufficiently large number. From Tutte's f-factor theorem, for all $L(G)$-triples $B = (S, T, U)$:

$$\begin{aligned} k(B, h) + \lambda_{L^M}(S, T) &\leqslant h(S) + \bar{h}(T) \\ k(B, h) + M\lambda_L(S, T) &\leqslant h(S) + M deg_L(T) - h(T) \\ k(B, h) + h(T) - h(S) &\leqslant M(deg_L(T) - \lambda_L(S, T)) \end{aligned}$$

Here, $deg_L(T) \geqslant \lambda_L(S, T)$ for any $L(G)-$triple since $deg_L(T) = \sum_{t \in T} deg(t)$ and $\lambda_L(S, T)$ = number of edges in between $S$ & $T$. In addition, when $deg_L(T) > \lambda_L(S, T)$ the inequality holds since we can choose $M$ sufficiently large and the left hand side of the inequality doesn't depend on $M$. So the only problem is when $deg_L(T) = \lambda_L(S,T)$. Which means $T$ is independent and there is no edge in between $T$ & $U$ ($\lambda_L(T,U) = 0$). So the condition we need to check for each $L(G)-$triple reduces to:

$$k(B,h) + h(T) \leqslant h(S)$$

when $T$ is independent and $\lambda_L(T,U) = 0$. □

**Example 2.13.** Let $G$ be the underlying graph in Figure 2.8. Let's find the necessary conditions for $G^{[h]}$ to have a gregarious $P_3$ decomposition. We need to check
$k(B,h) + h(T) \leqslant h(S)$
for all $L(G)$ triples $B = (S,T,U)$ where $T$ is independent and $\lambda_L(T,U) = 0$.

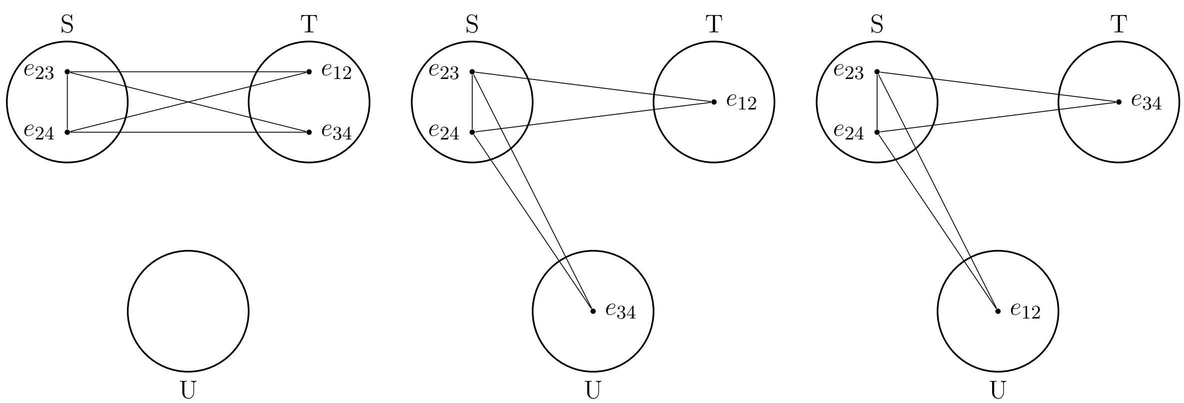


Figure 2.10: Example of how to apply Tutte's f-Factor Theorem

where $deg(e_{ij}) = b_i b_j$. So we can get conditions:

1. $b_1b_2 + b_3b_4 \leqslant b_2b_3 + b_2b_4$

2. $b_1b_2 \leqslant b_2b_3 + b_2b_4$

3. $b_3b_4 \leqslant b_2b_3 + b_2b_4$

Therefore the necessary condition to decompose $G^{[h]}$ into gregarious $P_3$'s is:

$b_1b_2 + b_3b_4 \leqslant b_2b_3 + b_2b_4$

since 1 is stronger than 2 and 3.

In summary, if we have a gregarious decomposition of $G^f$ into $P_3$'s ($G^f \overset{g}{\hookrightarrow} P_3$), then getting a decomposition of $G^{[h]}$ into $P_3$'s is trivial ($G^{[h]} \hookrightarrow P_3$). So $G^f \overset{g}{\hookrightarrow} P_3 \Rightarrow G^{[h]} \hookrightarrow P_3$.

The opposite direction is not true ($G^{[h]} \hookrightarrow P_3 \nRightarrow G^f \overset{g}{\hookrightarrow} P_3$, see the following counter example), but it will still give us some of the necessary conditions for $G^f \overset{g}{\hookrightarrow} P_3$.

**Example 2.14.** Let $G = S_3$ be a tristar and define $f : V(G) \to \mathbb{P}$ by

$f : (a, v_1, v_2, v_3) \to (2, 1, 1, 1)$.

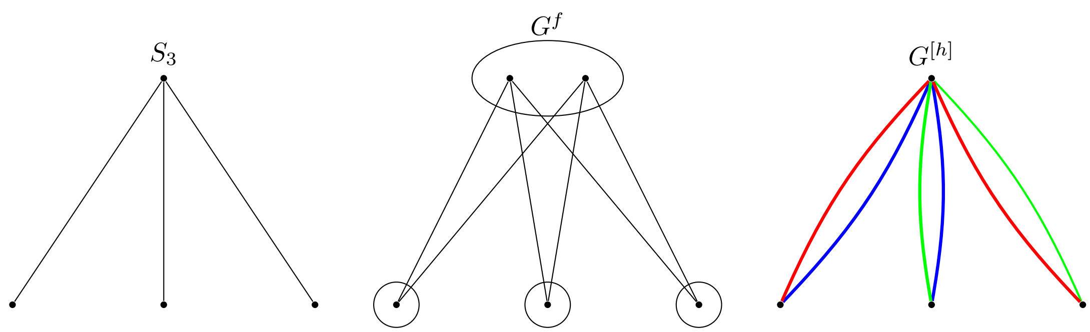


Figure 2.11: Counter Example

$G^f$ doesn't have a gregarious $P_3$ decomposition since both vertices in the root have odd degree. On the other side, $G^{[h]}$ has a gregarious $P_3$ decomposition, each color gives a different gregarious $P_3$.

Chapter 3

Parity Balanced Bipartite Graphs

Let $a$, $b \in \mathbb{P}$ and $e \in \mathbb{N}$ , and let $\epsilon_a$, $\epsilon_b \in \{0, 1\}$. We say the simple bipartite graph $G$ on bipartition $(A, B)$, where $|A| = a$ and $|B| = b$, with $e$ edges, is *parity balanced* with parameters $(a,\, b,\, e,\, \epsilon_a,\, \epsilon_b)$ if

$\forall u \in A$, $deg(u) \equiv \epsilon_a \pmod 2$, and further $\forall v \in A$, $|deg(u) - deg(v)| \leqslant 2$,

$\forall u \in B$, $deg(u) \equiv \epsilon_b \pmod 2$, and further $\forall v \in B$, $|deg(u) - deg(v)| \leqslant 2$.

We will give necessary and sufficient conditions on the parameters $(a,\, b,\, e,\, \epsilon_a,\, \epsilon_b)$ for the existence of such graphs.

## 3.1 Introduction

All the graphs are *simple,* i.e., they have no loops or multiple edges. Let $\mathbb{P}$ be the set of positive integers and $\mathbb{N}$ be the set of non-negative integers.

**Definition 3.1.** The integer vector $(x_1,\, x_2,\, \dots,\, x_t)$ is said to be *balanced* if $|x_i - x_j| \leqslant 1$ for all $1 \leqslant i,\, j \leqslant t$. Two vectors are *equivalent* if one can be obtained from the other by permuting the entries.

**Definition 3.2.** Let $G$ be a bipartite graph on bipartition $(A,\, B)$. If for all $v \in A$, $deg_G(v) = d_1$ and for all $w \in B$, $deg_G(w) = e_1$, then we will call $G$ a $(d_1,\, e_1) - \textit{regular}$ bipartite graph.

The following lemmas are proved in [1], p. 399.

**Lemma 3.3.** Let $v$ and $w$ be balanced vectors with the same number of coordinates. Then, for some vector $w'$ equivalent to $w$, $v + w'$ is balanced.

**Lemma 3.4.** Let $a$, $b \in \mathbb{P}$, and let $e \leqslant ab$ be a non-negative integer. Then there is a bipartite graph $G$ on bipartition $(A,\, B)$ with both $(deg_G(v) \,|\, v \in A)$ and $(deg_G(y) \,|\, y \in B)$ balanced.

### 3.2 Bipartite Graphs with Four Degrees

The theorems we will be proving here can be proven by using the Ryser-Gale theorem ([2], p. 185), but the proof is much harder.

**Theorem 3.5.** *Let $a_1$, $a_2$, $b_1$, $b_2$, $d_1$, $d_2$, $e_1$, $e_2$ be non-negative integers. Then:*
*There is a simple bipartite graph on bipartition $(A, B)$, where $A$ consists of $a_1$ vertices of degree $d_1$ and $a_2$ vertices of degree $d_2$, and $B$ consists of $b_1$ vertices of degree $e_1$ and $b_2$ vertices of degree $e_2$, if and only if*

$$(*)\, a_1d_1 + a_2d_2 = b_1e_1 + b_2e_2$$

1. $a_1d_1 \leqslant a_1b_1 + b_2e_2$, or, equivalently, $b_1e_1 \leqslant a_1b_1 + a_2d_2$
2. $a_1d_1 \leqslant a_1b_2 + b_1e_1$, or, equivalently, $b_2e_2 \leqslant a_1b_2 + a_2d_2$
3. $b_1e_1 \leqslant a_2b_1 + a_1d_1$, or, equivalently, $a_2d_2 \leqslant a_2b_1 + b_2e_2$
4. $b_2e_2 \leqslant a_2b_2 + a_1d_1$, or, equivalently, $a_2d_2 \leqslant a_2b_2 + b_1e_1$
5. either $a_1 = 0$, or $d_1 \leqslant b_1 + b_2$
6. either $a_2 = 0$, or $d_2 \leqslant b_1 + b_2$
7. either $b_1 = 0$, or $e_1 \leqslant a_1 + a_2$
8. either $b_2 = 0$, or $e_2 \leqslant a_1 + a_2$

**Necessity:**

*Proof.* Each side of $(*)$ counts the total number of edges, hence they must be equal. Conditions 5 - 8 come from the fact that maximum degree of any vertex is less than the number of vertices in the other part. Now for conditions 1 - 4:

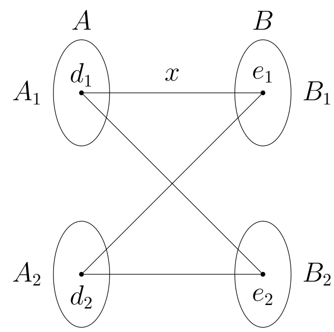


Figure 3.1: Bipartite Graph with Four Degrees

For $i = 1, 2$, let $A_i$ (resp. $B_i$) be the vertices of degree $d_i$ (resp. $e_i$) in $A_i$ (resp. $B_i$). In addition, let's assume that there are $x$ edges in between vertices of $A_1$ and $B_1$. If we look at the other pairs, we will get:

| | $B_1$ | $B_2$ |
|---|---|---|
| $A_1$ | $x$ | $a_1d_1 - x$ |
| $A_2$ | $b_1e_1 - x$ | $a_2d_2 - b_1e_1 + x$ $=$ $b_2e_2 - a_1d_1 + x$ |

Table 3.1: Distribution of the edges

Using table 3.1, we can get the following inequalities:

$$\begin{array}{lcl} 0 \leqslant & x & \leqslant a_1b_1 \\ 0 \leqslant & a_1d_1 - x & \leqslant a_1b_2 \\ 0 \leqslant & b_1e_1 - x & \leqslant a_2b_1 \\ 0 \leqslant & a_2d_2 - b_1e_1 + x & \leqslant a_2b_2 \end{array}$$

So, we can get:

$$
\begin{aligned}
0 &\leqslant x \leqslant & a_1 b_1 \\
a_1 d_1 - a_1 b_2 &\leqslant x \leqslant & a_1 d_1 \\
b_1 e_1 - a_2 b_1 &\leqslant x \leqslant & b_1 e_1 \\
b_1 e_1 - a_2 d_2 &\leqslant x \leqslant & a_2 b_2 - a_2 d_2 + b_1 e_1
\end{aligned}
$$

We can get sixteen inequalities on the variables ($a_1$, $a_2$, $b_1$, $b_2$, $d_1$, $d_2$, $e_1$, $e_2$) from above since we have $x$ in the middle of all of the four inequalities. If we use the left side of the first inequality and right sides of the all them, then we can get:

$$
\begin{aligned}
0 \leqslant & \quad a_1 b_1 \\
0 \leqslant & \quad a_1 d_1 \\
0 \leqslant & \quad b_1 e_1 \\
0 \leqslant & \quad a_2 b_2 - a_2 d_2 + b_1 e_1 \quad \Rightarrow a_2 d_2 \leqslant a_2 b_2 + b_1 e_1 \,, \text{cond. } 4 \checkmark
\end{aligned}
$$

From the second one:

$$
\begin{aligned}
a_1 d_1 - a_1 b_2 \leqslant & \quad a_1 b_1 & \Rightarrow d_1 \leqslant b_1 + b_2 \\
a_1 d_1 - a_1 b_2 \leqslant & \quad a_1 d_1 & \Rightarrow 0 \leqslant a_1 b_1 \\
a_1 d_1 - a_1 b_2 \leqslant & \quad b_1 e_1 & \Rightarrow a_1 d_1 \leqslant a_1 b_2 + b_1 e_1 \,, \text{cond. } 2 \checkmark \\
a_1 d_1 - a_1 b_2 \leqslant & \quad a_2 b_2 - a_2 d_2 + b_1 e_1 & \Rightarrow a_1 d_1 + a_2 d_2 \leqslant a_2 b_2 + b_1 e_1 + a_1 b_2
\end{aligned}
$$

If we use $(*)$, we will get $a_1 d_1 + a_2 d_2 = b_1 e_1 + b_2 e_2 \leqslant a_2 b_2 + b_1 e_1 + a_1 b_2$, so this reduces to $e_1 \leqslant a_1 + a_2$.

From the third one:

$$\begin{array}{lcl}
b_1e_1 - a_2b_1 \leqslant & a_1b_1 & \Rightarrow e_1 \leqslant a_1 + a_2 \\
b_1e_1 - a_2b_1 \leqslant & a_1d_1 & \Rightarrow b_1e_1 \leqslant a_2b_1 + a_1d_1 \text{ , cond. 3} \checkmark \\
b_1e_1 - a_2b_1 \leqslant & b_1e_1 & \Rightarrow 0 \leqslant a_2b_1 \\
b_1e_1 - a_2b_1 \leqslant & a_2b_2 - a_2d_2 + b_1e_1 & \Rightarrow d_2 \leqslant b_1 + b_2
\end{array}$$

From the fourth one:

$$\begin{array}{lcl}
b_1e_1 - a_2d_2 \leqslant & a_1b_1 & \Rightarrow b_1e_1 \leqslant a_1b_1 + a_2d_2 \text{ , cond. 1} \checkmark \\
b_1e_1 - a_2d_2 \leqslant & a_1d_1 & \Rightarrow b_1e_1 \leqslant a_1d_1 + a_2d_2 \\
b_1e_1 - a_2d_2 \leqslant & b_1e_1 & \Rightarrow 0 \leqslant a_2d_2 \\
b_1e_1 - a_2d_2 \leqslant & a_2b_2 - a_2d_2 + b_1e_1 & \Rightarrow 0 \leqslant a_2b_2
\end{array}$$

In the second equation, if we use $a_1d_1 + a_2d_2 = b_1e_1 + b_2e_2$, then we will get $b_1e_1 \leqslant b_1e_1 + b_2e_2 \Rightarrow 0 \leqslant b_2e_2$. $\square$

**Sufficiency**:

*Proof.* Assume there is bipartite graph $(A, B)$ satisfying the necessary conditions and there are $x$ edges in between the vertices of $A_1$ and $B_1$ like in figure 3.1. Therefore, we can find the number of edges in between other vertices using the remaining edges like in table 3.1.

Now using the construction in [1] on pg. 399,

1. distribute $x$ edges on $a_1$ vertices with balanced degrees.
2. distribute $a_1d_1 - x$ on $a_1$ vertices with balanced degrees.

So we will get two balanced vectors with the same number of entries.

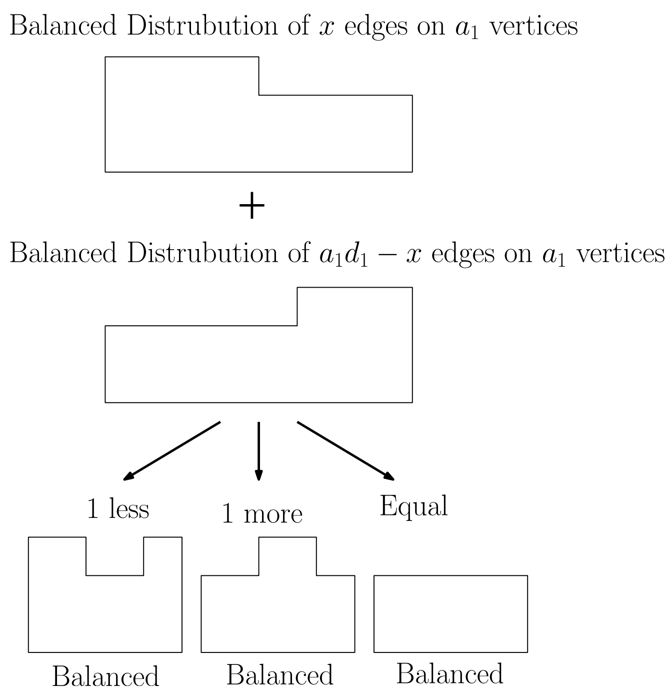


Figure 3.2: Adding balanced distributions

At the end, we will have one of these three cases from figure 3.2 and all of them will still have balanced distributions since both distributions were balanced to begin with. However, the first two cases are impossible, since we have a balanced distribution of $a_1d_1$ edges on $a_1$ vertices, this means that each vertex will be incident with $d_1$ edges. In the same manner we can prove that we can distribute the remaining edges with the desired degrees. □

### 3.3 Parity balanced Bipartite Graphs

**Definition 3.6.** Let $a,\, b \in \mathbb{P}$ and $e \in \mathbb{N}$ , and let $\epsilon_a,\, \epsilon_b \in \{0, 1\}$. We say the bipartite graph $G$ with $e$ edges on bipartition $(A, B)$, with $|A| = a$ and $|B| = b$, is *parity balanced* with parameters $(a,\, b,\, e,\, \epsilon_a,\, \epsilon_b)$ *if*

1. $\forall\, u \in A$, $deg(u) \equiv \epsilon_a \pmod 2$ and further $\forall\, v \in A$, $|deg(u) - deg(v)| \leqslant 2$.

2. $\forall\, u \in B$, $deg(u) \equiv \epsilon_a \pmod 2$ and further $\forall\, v \in B$, $|deg(u) - deg(v)| \leqslant 2$.

**Example 3.7.** Let $|A| = a = 6$, $|B| = b = 5$, $e = 14$, $\epsilon_a = 1$ and $\epsilon_b = 0$.

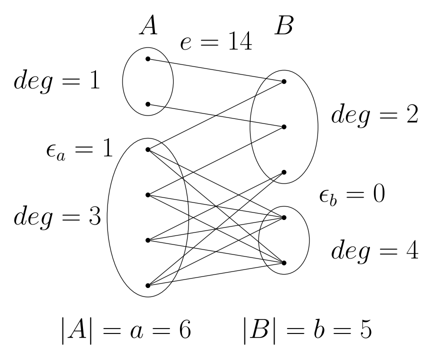


Figure 3.3: Example of a parity balanced bipartite graph

**Definition 3.8.** If $A$ and $B$ are disjoint sets, we denote $K_{A,B}$ to be the complete bipartite graph on bipartition $(A,\, B)$.

**Definition 3.9.** Let $K_{A,B}$ be a complete bipartite graph on bipartition $(A, B)$. A *bipartite complement* of a bipartite graph $G$ on bipartition $(A,\, B)$ with edge set $E$ is the bipartite graph $G'$ on bipartition $(A,\, B)$ with the edge set $E'$ where $E' = E(K_{A,B})\backslash E$.

**Fact 3.10.** If G is a parity balanced bipartite graph with parameters $(a,\, b,\, e,\, \epsilon_a\,, \epsilon_b)$, then $G'$ is a parity balanced bipartite graph with parameters $(a,\, b,\, e' = ab - e,\, \epsilon'_a\,, \epsilon'_b)$ where $\epsilon_a + \epsilon'_a \equiv b \pmod 2$ and $\epsilon_b + \epsilon'_b \equiv a \pmod 2$

**Example 3.11.** The bipartite complement of $G$ is $G'$ with parameters ($a = 6$, $b = 5$, $e' = ab - 14 = 30 - 14 = 16$, $\epsilon'_a = 0$, $\epsilon'_b = 0$):

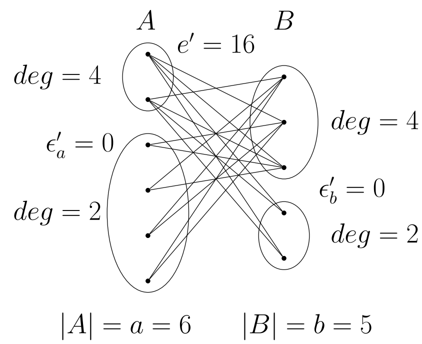


Figure 3.4: Example of a bipartite complement

where $\epsilon_a + \epsilon'_a = 1 + 0 = 1 \equiv 5 \pmod 2$ and $\epsilon_b + \epsilon'_b = 0 + 0 \equiv 6 \pmod 2$.

**Theorem 3.12.** Let $a$, $b \in \mathbb{P}$, $e \in \mathbb{N}$, $\epsilon_a$, $\epsilon_b$, $\epsilon^{'}_a$, $\epsilon^{'}_b \in \{0, 1\}$,
$\epsilon_a + \epsilon^{'}_a \equiv b \pmod 2$, $\epsilon_b + \epsilon^{'}_b \equiv a \pmod 2$.
Then, there is a parity balanced bipartite graph $G$ on bipartition ($A$, $B$) with parameters ($a$, $b$, $e$, $\epsilon_a$, $\epsilon_b$) if and only if
$\epsilon_a a \leqslant e \leqslant ab - \epsilon^{'}_a a$ , $\epsilon_b b \leqslant e \leqslant ab - \epsilon^{'}_b b$, and all of these are congruent $\pmod 2$, with the following exceptions:

<table>
<tr><th>$e$</th><th>$a$</th><th>$b$</th><th>$\epsilon_a$</th><th>$\epsilon'_a$</th><th>$\epsilon_b$</th><th>$\epsilon'_b$</th></tr>
<tr><td rowspan="10">2</td><td>2</td><td>2</td><td>0</td><td>0</td><td>0</td><td>0</td></tr>
<tr><td>2</td><td>3</td><td>0</td><td>1</td><td>0</td><td>0</td></tr>
<tr><td>3</td><td>2</td><td>0</td><td>0</td><td>0</td><td>1</td></tr>
<tr><td>odd $\geqslant 3$</td><td>odd $\geqslant 3$</td><td>0</td><td>1</td><td>0</td><td>1</td></tr>
<tr><td rowspan="2">odd $\geqslant 3$</td><td rowspan="2">even $\geqslant 3$</td><td>0</td><td>0</td><td>0</td><td>1</td></tr>
<tr><td>0</td><td>0</td><td>1</td><td>0</td></tr>
<tr><td rowspan="2">even $\geqslant 3$</td><td rowspan="2">odd $\geqslant 3$</td><td>0</td><td>1</td><td>0</td><td>0</td></tr>
<tr><td>1</td><td>0</td><td>0</td><td>0</td></tr>
<tr><td rowspan="2">even $\geqslant 3$</td><td rowspan="2">even $\geqslant 3$</td><td>1</td><td>1</td><td>1</td><td>1</td></tr>
<tr><td>0</td><td>0</td><td>0</td><td>0</td></tr>
<tr><td rowspan="10">$ab-2$</td><td>2</td><td>2</td><td>0</td><td>0</td><td>0</td><td>0</td></tr>
<tr><td>2</td><td>3</td><td>1</td><td>0</td><td>0</td><td>0</td></tr>
<tr><td>3</td><td>2</td><td>0</td><td>0</td><td>1</td><td>0</td></tr>
<tr><td>odd $\geqslant 3$</td><td>odd $\geqslant 3$</td><td>1</td><td>0</td><td>1</td><td>0</td></tr>
<tr><td rowspan="2">odd $\geqslant 3$</td><td rowspan="2">even $\geqslant 3$</td><td>0</td><td>0</td><td>1</td><td>0</td></tr>
<tr><td>0</td><td>0</td><td>0</td><td>1</td></tr>
<tr><td rowspan="2">even $\geqslant 3$</td><td rowspan="2">odd $\geqslant 3$</td><td>1</td><td>0</td><td>0</td><td>0</td></tr>
<tr><td>0</td><td>1</td><td>0</td><td>0</td></tr>
<tr><td rowspan="2">even $\geqslant 3$</td><td rowspan="2">even $\geqslant 3$</td><td>1</td><td>1</td><td>1</td><td>1</td></tr>
<tr><td>0</td><td>0</td><td>0</td><td>0</td></tr>
</table>

Table 3.2: Exceptions for Theorem 3.12

**Necessity:**

*Proof.* For any $u \in A$ we have $deg_G(u) + deg_{G'}(u) = b$ so $deg_G(u) + deg_{G'}(u) \equiv b \pmod 2$ where $deg_G(u) \equiv \epsilon_a \pmod 2$ and $deg_{G'}(u) \equiv \epsilon_a^{'} \pmod 2$ by definition, and $\epsilon_a + \epsilon_a^{'} \equiv b \pmod 2$ follows. In the same way, we can get $\epsilon_b + \epsilon_b^{'} \equiv a \pmod 2$.

To get $\epsilon_a a \leqslant e \leqslant ab - \epsilon_a^{'} a$ and $\epsilon_b b \leqslant e \leqslant ab - \epsilon_b^{'} b$, if one of $\epsilon_a$, $\epsilon_a^{'}$, $\epsilon_b$, $\epsilon_b^{'}$ is 1, then we have to have enough edges in either $G$ or $G'$.

Finally, to get $\epsilon_a a$, $e$, $ab - \epsilon_a^{'} a$, $\epsilon_b b$, $ab - \epsilon_b^{'} b$ all congruent $\pmod 2$;

$e = \sum_{u \in A} deg_G(u) \equiv a\epsilon_a \pmod 2$,and

$\epsilon_a + \epsilon_a^{'} \equiv b \pmod 2 \Rightarrow a\epsilon_a + a\epsilon_a^{'} \equiv ab \pmod 2 \Rightarrow a\epsilon_a \equiv ab - a\epsilon_a^{'} \pmod 2$.

In the same way we can get the other conditions. For the exceptions, it is easy to prove that there is no parity balanced bipartite graph with parameters given in table 4.1. Figure 3.6 shows all possible parity balanced bipartite graphs with 2 edges and and the ones with $ab - 2$ edges will be bipartite complement of these graphs. □

**Sufficiency:**

*Proof.* We can use theorem 3.5 for this proof. Define $n$, $m$, $q_a$, $r_a$, $q_b$, $r_b \in \mathbb{N}$ by,

$e = 2n + \epsilon_a a = 2m + \epsilon_b b$

$n = aq_a + r_a$, $m = bq_b + r_b$

$0 \leqslant r_a \leqslant a - 1$, $0 \leqslant r_b \leqslant b - 1$.

So $e = 2aq_a + 2r_a + \epsilon_a a = 2bq_b + 2r_b + \epsilon_b b$.

$q_a = \dfrac{e - \epsilon_a a - 2r_a}{2a} = \dfrac{e - \epsilon_a a}{2a} - \dfrac{r_a}{a} = \left\lfloor \dfrac{e - \epsilon_a a}{2a} \right\rfloor$. In the same way, $q_b = \left\lfloor \dfrac{e - \epsilon_b b}{2b} \right\rfloor$.

Now let's translate this problem into"bipartite graphs with four degrees" since we already know NASCs for those graphs (figure 3.5).

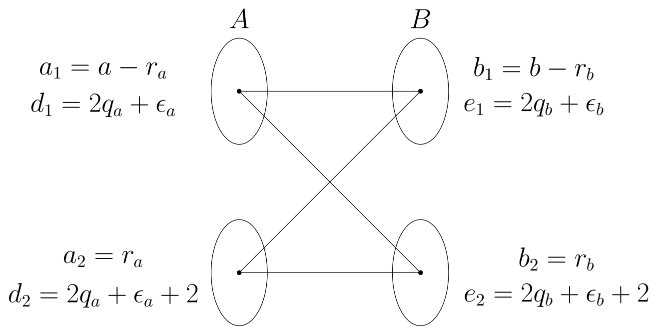


Figure 3.5: Translating PBBG to a BGwFDs

Note that $(*)$ holds since:

$(a - r_a)(2q_a + \epsilon_a) + r_a(2q_a + \epsilon_a) = (b - r_b)(2q_b + \epsilon_b) + r_b(2q_b + \epsilon_b)$

**Case 1**: Assume $a_2 = 0 = b_2$, then we will get a $(d_1, e_1)-$regular bipartite graph. Since $a_2 = 0$, and $b_2 = 0$, we only need to prove 1, 5 and 7 in theorem 3.5. Let's start proving conditions 5 and 7 which say:

$$\begin{aligned} d_1 &\leqslant b_1 + b_2 \\ 2q_a + \epsilon_a &\leqslant b - r_b + r_b = b \end{aligned}$$

and

$$\begin{aligned} e_1 &\leqslant a_1 + a_2 \\ 2q_b + \epsilon_b &\leqslant a - r_a + r_a = a \end{aligned}$$

So for 5 if we prove $2q_a + \epsilon_a \leqslant b$, we are done. Using $2q_a = \dfrac{e - \epsilon_a a - 2r_a}{a} = \dfrac{e}{a} - \epsilon_a - \dfrac{2r_a}{a}$; we can get $2q_a + \epsilon_a = \dfrac{e}{a} - \epsilon_a - \dfrac{2r_a}{a} + \epsilon_a = \dfrac{e}{a} - \dfrac{2r_a}{a} \leqslant b - \dfrac{2r_a}{a} < b$. We can prove 7 in the same way.

Now, let's prove 1:

$$\begin{aligned} a_1 d_1 &\leqslant a_1 b_1 + b_2 e_2 \\ a_1 d_1 &\leqslant a_1 b_1 \;\; \text{since } b_2 = 0 \\ d_1 &\leqslant b_1 \\ 2q_a + \epsilon_a &\leqslant b \;\; \text{and we just proved this in 5} \end{aligned}$$

**Case 2:** Assume $a_2 = 0$ and $b_2 \neq 0$ ($a_2 \neq 0$ and $b_2 = 0$ is just the symmetric case). Since $a_2 = 0$, we only need to prove 1, 2, 5, 7 and 8 in theorem 3.5. 5 and 7 are the same as in Case 1. 1 and 2 will reduce to 7 and 8, respectively since $a_2 = 0$. So just proving 8 is enough which says:

$$\begin{aligned} e_2 &\leqslant a_1 + a_2 \\ 2q_b + \epsilon_b + 2 &\leqslant a_1 = a \;\; \text{since } a_2 = 0 \end{aligned}$$

So $2q_b = \dfrac{e}{b} - \epsilon_b - \dfrac{2r_b}{b}$, using this:
$2q_b + \epsilon_b + 2 = \dfrac{e}{b} - \epsilon_b - \dfrac{2r_b}{b} + \epsilon_b + 2 = \dfrac{e}{b} - \dfrac{2r_b}{b} + 2 \leqslant a - \epsilon_b^{'} - \dfrac{2r_b}{b} + 2 < a + 2.$
The only problem is when $a = 2q_b + \epsilon_b + 1$, then $\epsilon_b + \epsilon_b^{'} \equiv a = 2q_b + \epsilon_b + 1 \pmod 2$.
So $\epsilon_b^{'} = 1$. In this case:
$2q_b + \epsilon_b + 2 = \dfrac{e}{b} - \epsilon_b - \dfrac{2r_b}{b} + \epsilon_b + 2 = \dfrac{e}{b} - \dfrac{2r_b}{b} + 2 \leqslant a - \epsilon_b^{'} - \dfrac{2r_b}{b} + 2 < a + 1.$
**Case 3:** We can assume $a_2 \neq 0 \neq b_2$. Let's start proving conditions 5 through 8 in theorem 3.5. So 5 and 6 say:

$$\begin{aligned} d_1 &\leqslant b_1 + b_2 \\ 2q_a + \epsilon_a &\leqslant b - r_b + r_b = b \end{aligned}$$

and

$$\begin{aligned}
d_2 &\leqslant b_1 + b_2 \\
2q_a + \epsilon_a + 2 &\leqslant b - r_b + r_b = b
\end{aligned}$$

If we prove $2q_a + \epsilon_a + 2 \leqslant b$, this will cover both cases. However, this is the same as (8) in Case 2, just switch $a$ and $b$. We can prove (7) and (8) in the same way.

Now let's prove conditions 1 through 4 of theorem 3.5.

For 1, we need to prove:

$$\begin{aligned}
a_1 d_1 &\leqslant a_1 b_1 + b_2 e_2 \\
(a - r_a)(2q_a + \epsilon_a) &\leqslant (a - r_a)(b - r_b) + r_b(2q_b + \epsilon_b + 2)
\end{aligned}$$

First, suppose $r_b \leqslant b - 2q_a - \epsilon_a$, then we get:

$$\begin{aligned}
(a - r_a)(2q_a + \epsilon_a) &\leqslant (a - r_a)(b - r_b) + r_b(2q_b + \epsilon_b + 2) \\
(a - r_a)(2q_a + \epsilon_a - b + r_b) &\leqslant r_b(2q_b + \epsilon_b + 2) \\
(a - r_a)(r_b - (b - 2q_a - \epsilon_a)) &\leqslant r_b(2q_b + \epsilon_b + 2)
\end{aligned}$$

So $r_b - (b - 2q_a - \epsilon_a) \leqslant 0$ and the inequality is automatically satisfied since $a - r_a > 0$, $r_b > 0$, $2q_b + \epsilon_b + 2 > 0$.

So we can assume $r_b \geqslant b - 2q_a - \epsilon_a + 1$. In addition, recall that $e = 2n + \epsilon_a a = 2aq_a + 2r_a + \epsilon_a a$.

Need to prove:

$$
\begin{aligned}
a_1 d_1 &\leqslant a_1 b_1 + b_2 e_2 \\
(a - r_a)(2q_a + \epsilon_a) &\leqslant (a - r_a)(b - r_b) + r_b(2q_b + \epsilon_b + 2) \\
2aq_a + a\epsilon_a - r_a(2q_a + \epsilon_a) &\leqslant ab - ar_b - br_a + r_a r_b + r_b(2q_b + \epsilon_b + 2) \\
2aq_a + a\epsilon_a - r_a(2q_a + \epsilon_a) + 2r_a - 2r_a &\leqslant ab - ar_b - br_a + r_a r_b + r_b(2q_b + \epsilon_b + 2) \\
(2aq_a + a\epsilon_a + 2r_a) - r_a(2q_a + \epsilon_a + 2) &\leqslant ab - ar_b - br_a + r_a r_b + r_b(2q_b + \epsilon_b + 2) \\
e + r_a(b - (2q_a + \epsilon_a + 2)) + r_b(a - (2q_b + \epsilon_b + 2)) - r_a r_b &\leqslant ab
\end{aligned}
$$

So if we show $e + r_a(b - (2q_a + \epsilon_a + 2)) + r_b(a - (2q_b + \epsilon_b + 2)) - r_a r_b \leqslant ab$ , we are done.
Using $r_b \geqslant b - 2q_a - \epsilon_a + 1$, we can get $r_a(b - (2q_a + \epsilon_a + 2)) < r_a r_b$. So

$$
\begin{aligned}
e + r_a(b - (2q_a + \epsilon_a + 2)) + r_b(a - (2q_b + \epsilon_b + 2)) - r_a r_b &< e + r_a r_b + r_b(a - (2q_b + \epsilon_b + 2)) - r_a r_b \\
&= e + r_b(a - (2q_b + \epsilon_b + 2))
\end{aligned}
$$

where we can use, $e = (b - r_b)(2q_b + \epsilon_b) + r_b(2q_b + \epsilon_b + 2)$.

$$
\begin{aligned}
&= e + r_b(a - (2q_b + \epsilon_b + 2)) \\
&= (b - r_b)(2q_b + \epsilon_b) + r_b(2q_b + \epsilon_b + 2) + r_b(a - (2q_b + \epsilon_b + 2)) \\
&= (b - r_b)(2q_b + \epsilon_b) + ar_b
\end{aligned}
$$

here using the fact that $2q_b + \epsilon_b \leqslant a$ (which we just proved in 7), we will get:

$$
\begin{aligned}
&= (b - r_b)(2q_b + \epsilon_b) + ar_b \\
&< (b - r_b)a + ar_b \quad = ab.
\end{aligned}
$$

Now let's prove 2:

$$\begin{aligned} a_1 d_1 &\leqslant a_1 b_2 + b_1 e_1 \\ (a-r_a)(2q_a+\epsilon_a) &\leqslant (a-r_a)r_b + (b-r_b)(2q_b+\epsilon_b) \end{aligned}$$

First suppose $2q_a+\epsilon_a \leqslant r_b$, then we get:

$$\begin{aligned} (a-r_a)(2q_a+\epsilon_a) &\leqslant (a-r_a)r_b + (b-r_b)(2q_b+\epsilon_b) \\ (a-r_a)(2q_a+\epsilon_a-r_b) &\leqslant (b-r_b)(2q_b+\epsilon_b). \end{aligned}$$

So $2q_a+\epsilon_a-r_b \leqslant 0$ and the inequality is automatically satisfied since $a-r_a > 0$, $(b-r_b) > 0$, $2q_b+\epsilon_b \geq 0$.

We can assume $2q_a+\epsilon_a+1 \geqslant r_b$. We need to prove:

$$\begin{aligned} a_1 d_1 &\leqslant a_1 b_2 + b_1 e_1 \\ (a-r_a)(2q_a+\epsilon_a) &\leqslant (a-r_a)r_b + (b-r_b)(2q_b+\epsilon_b) \\ 2aq_a + a\epsilon_a - r_a(2q_a+\epsilon_a) &\leqslant ar_b - r_a r_b + 2bq_b + b\epsilon_b - r_b(2q_b+\epsilon_b) \\ 2aq_a + a\epsilon_a - r_a(2q_a+\epsilon_a) + 2r_a - 2r_a &\leqslant ar_b - r_a r_b + 2bq_b + b\epsilon_b - r_b(2q_b+\epsilon_b) + 2r_b - 2r_b \\ (2aq_a + a\epsilon_a + 2r_a) - r_a(2q_a+\epsilon_a+2) &\leqslant ar_b - r_a r_b + (2bq_b + b\epsilon_b + 2r_b) - r_b(2q_b+\epsilon_b+2) \\ e - r_a(2q_a+\epsilon_a+2) &\leqslant ar_b - r_a r_b + e - r_b(2q_b+\epsilon_b+2) \\ r_a r_b + r_b(2q_b+\epsilon_b+2) &\leqslant ar_b + r_a(2q_a+\epsilon_a+2) \end{aligned}$$

If we show $r_a r_b + r_b(2q_b+\epsilon_b+2) \leqslant ar_b + r_a(2q_a+\epsilon_a+2)$, then we have shown (2). Using $2q_a+\epsilon_a+1 \geqslant r_b$ we can get $r_a r_b < r_a(2q_a+\epsilon_a+2)$. In addition, we can use the previously proved fact in 8 that $2q_b + \epsilon_b + 2 \leqslant a$. So

$$\begin{aligned} r_a r_b + r_b(2q_b + \epsilon_b + 2) &\leqslant r_a(2q_a + \epsilon_a + 2) + r_b(2q_b + \epsilon_b + 2) \\ &\leqslant r_a(2q_a + \epsilon_a + 2) + r_b a \\ &= a r_b + r_a(2q_a + \epsilon_a + 2) \end{aligned}$$

The proof of 3 is exactly the same as the proof of 2, if we switch parts $A \longleftrightarrow B$.

Now, let's prove the last condition, 4.

We need to prove $b_2e_2 \leqslant a_2b_2 + a_1d_1$, or, equivalently, $a_2d_2 \leq a_2b_2 + b_1e_1$.

$$\begin{aligned} b_2e_2 &\leqslant a_2b_2 + a_1d_1 \\ r_b(2q_b + \epsilon_b + 2) &\leqslant r_a r_b + (a - r_a)(2q_a + \epsilon_a) \\ r_b(2q_b + \epsilon_b + 2 - r_a) &\leqslant (a - r_a)(2q_a + \epsilon_a) \end{aligned}$$

which is equivalent to

$$\begin{aligned} a_2d_2 &\leqslant a_2b_2 + b_1e_1 \\ r_a(2q_a + \epsilon_a + 2) &\leqslant r_a r_b + (b - r_b)(2q_b + \epsilon_b) \\ r_a(2q_a + \epsilon_a + 2 - r_b) &\leqslant (b - r_b)(2q_b + \epsilon_b). \end{aligned}$$

First, assume $2q_b + \epsilon_b + 2 \leqslant r_a$ or $2q_a + \epsilon_a + 2 \leqslant r_b$.

Then $b_2e_2 \leqslant a_2b_2 + a_1d_1$ or $a_2d_2 \leq a_2b_2 + b_1e_1$ will be automatically satisfied.

So we can assume $2q_b + \epsilon_b + 2 > r_a$ and $2q_a + \epsilon_a + 2 > r_b \Rightarrow 2q_a + \epsilon_a + 1 \geqslant r_b$.

If we turn back to the problem and use the fact, which follows from 8, that $2q_b + \epsilon_b + 2 \leqslant a$, then:

$$\begin{aligned} r_b(2q_b+\epsilon_b+2) &\leqslant r_b a \\ &= r_a r_b + (a-r_a)r_b \\ &\leqslant r_a r_b + (a-r_a)(2q_a+\epsilon_a+1) \end{aligned}$$

So the only problem is when $r_b = 2q_a+\epsilon_a+1$. Similarly, we can assume $r_a = 2q_b+\epsilon_b+1$. Need to show:

$$\begin{aligned} b_2e_2 &\leqslant a_2b_2 + a_1d_1 \\ r_b(2q_b+\epsilon_b+2) &\leqslant r_ar_b + (a-r_a)(2q_a+\epsilon_a) \\ (2q_a+\epsilon_a+1)(2q_b+\epsilon_b+2) &\leqslant (2q_a+\epsilon_a+1)(2q_b+\epsilon_b+1) + (a-r_a)(2q_a+\epsilon_a) \\ 2q_a+\epsilon_a+1 &\leqslant (a-r_a)(2q_a+\epsilon_a) \\ 1 &\leqslant (a-r_a-1)(2q_a+\epsilon_a) \\ 1 &\leqslant (a-r_a-1)(r_b-1) \end{aligned}$$

Here $r_b \geqslant 1$ since $r_a \neq 0 \neq r_b$. In this case both are positive and we can assume $a - r_a \geqslant 1$ since $0 \leqslant r_a < a$. Therefore, we only need to prove $r_b \neq 1$ or $a - r_a \neq 1$. First, suppose $r_b = 1$, then $r_b = 2q_a+\epsilon_a+1 = 1$ so $q_a = 0$ and $\epsilon_a = 0$.

$$\begin{aligned} e = 2aq_a + 2ra + a\epsilon_a &= 2bq_b + 2r_b + b\epsilon_b \\ 2r_a &= 2bq_b + 2 + b\epsilon_b \\ 2(2q_b+\epsilon_b+1) &= b(2q_b+\epsilon_b)+2 \\ 2(2q_b+\epsilon_b) &= b(2q_b+\epsilon_b) \\ 0 &= (b-2)(2q_b+\epsilon_b) \\ 0 &= (b-2)(r_a-1) \end{aligned}$$

Therefore, in this case, either $b = 2$ or $r_a = 1$. If $r_a = 1$, then $e = 2r_a = 2$ and this is not possible since when $e = 2$ there are only two bipartite graphs with 2 edges (see figure 3.6) and both of them have either $b_2 = 0$ or $a_2 = 0 = b_2$, which contradicts the assumption $a_2 \neq 0 \neq b_2$.

We can get the exceptions in table 4.1 with parameters $(a,\ b,\ e = 2,\ \epsilon_a,\ \epsilon'_a,\ \epsilon_b,\ \epsilon'_b)$ easily since no other bipartite graphs exist with 2 edges but the ones in figure 3.6.

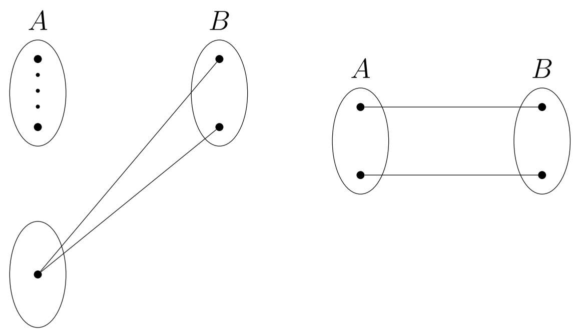


Figure 3.6: Bipartite graphs with 2 edges

Now, assume $b = 2$ where $r_b = 1$ and $r_a \geqslant 2$.

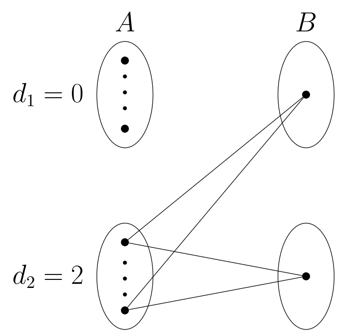


Figure 3.7: Exception for $b = 2$

There are only two vertices in $B$ and $d_2 = 2$ which means every vertex in $a_2$ will be adjacent to the vertices in $B$. This implies $b_1 = 2$, and $b_2 = 0$, and contradicts the assumption $b_2 \neq 0$. So we finished proving the case where $r_b \neq 1$. Therefore we can assume $r_b \geq 2$.

Now assume $a = r_a + 1$:

If $r_a = 1$, then $a = 2$ and it will be the same case as $b = 2$. Assume $r_a \geqslant 2$, so $a \geqslant 3$ where $r_a = 2q_b + \epsilon_b + 1 = a - 1$.

$$
\begin{aligned}
e &= 2bq_b + 2r_b + b\epsilon_b \\
&= b(2q_b + \epsilon_b) + 2r_b \\
&= b(a-2) + 2r_b \\
&= ab - 2(b - r_b)
\end{aligned}
$$

On the other side,

$$
\begin{aligned}
e = 2bq_b + 2r_b + b\epsilon_b &= 2aq_a + 2r_a + a\epsilon_a \\
(b - r_b)(2q_b + \epsilon_b) + r_b(2q_b + \epsilon_b + 2) &= 2aq_a + 2(a-1) + a\epsilon_a \\
(b - r_b)(a-2) + ar_b &= a(2q_a + \epsilon_a + 2) - 2 \\
(b - r_b)(a-2) + ar_b &= a(r_b + 1) - 2 \\
(b - r_b)(a-2) + ar_b &= ar_b + a - 2 \\
(b - r_b)(a-2) &= a - 2 \\
(b - r_b - 1)(a-2) &= 0
\end{aligned}
$$

We know $a \geqslant 3$, $b = r_b + 1$, which means $e = ab - 2(b - r_b) = ab - 2$, which is the bipartite complement of the exception $e = 2$. So we proved $r_b \neq 1$ or $a - r_a \neq 1$. This completes the proof. $\square$

Chapter 4

Results

## 4.1 Complete Multipartite Graphs

### 4.1.1 Complete Tripartite Graph

**Theorem 4.1.** *For a complete tripartite graph $K(A, B, C)$ with $|A| = a$, $|B| = b$ and $|C| = c$, assume $a \geqslant b \geqslant c$, then the NASCs are:*

1. $2 \mid (ab + ac + bc)$
2. $ab \leqslant ac + bc$

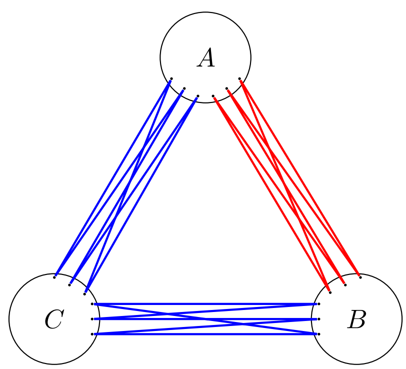


Figure 4.1: $K(A, B, C)$

**Necessity:**

*Proof.* $2 \mid (ab+ac+bc)$ comes from the fact that the total number of edges must be divisible by 2 since there are two edges in $P_3$. For $ab \leqslant ac + bc$:

We have three kinds of paths, let $x$, $y$, $z$ be the number of the paths $C \to A \to B$, $A \to B \to C$ and $A \to C \to B$ respectively, then

$$\begin{aligned} x + y &= ac \\ x + y &= ab \\ y + z &= bc \end{aligned}$$

So

$$\begin{aligned} x &= \frac{1}{2}(ac + ab - bc) \Rightarrow bc \leqslant ac + ab \\ y &= \frac{1}{2}(ab + bc - ac) \Rightarrow ac \leqslant ab + bc \\ z &= \frac{1}{2}(ac + bc - ab) \Rightarrow ab \leqslant ac + cb \end{aligned}$$

So if we have $ab \leqslant ac + cb$, the other two follow easily since $a \geqslant b \geqslant c$. □

**Sufficiency:**

*Proof.* Let $A$, $B$, $C$ be sets of size $a$, $b$, $c$ respectively. Assume the necessary conditions are satisfied, then we can find proper $x$, $y$, $z$. Find subgraphs $S_1$ of $K(C, A)$ and $S_2$ of $K(A, B)$ with $x$ edges, as in Lemma 3.4, so that their degrees agree on $A$ (thus $S_1 \cup S_2$ is a union of $x$ gregarious paths). Do the same for $y$ paths in $K(A, B) \cup K(B, C)$ and $z$ paths in $K(A, C) \cup K(C, B)$. Now we take the union of these three collections of paths, taking care to rename vertices as in Lemma 3.3. Thus the resulting graph will be the required complete tripartite graph. □

## 4.2 Star Multipartite Graphs

**Definition 4.2.** A *star* is a tree consisting of one vertex (called the root) adjacent to the all others. So a star multipartite graph $S = (A; B_1, B_2, \ldots, B_n)$ has $|A| = a$ non-adjacent vertices in the root which are adjacent to all the other sets of vertices $(B_1, B_2, \ldots, B_n)$ where $|B_i| = b_i$ for any $i$.

**Theorem 4.3.** *Let $S = (A; B_1, B_2, \ldots, B_n)$ be a star multipartite graph and assume $b_1 \geqslant b_2 \geqslant \cdots \geqslant b_n$. The NASCs are:*

1. $2 \,|\, (b_1 + b_2 + \cdots + b_n)$

2. $b_1 \leqslant b_2 + b_3 + \cdots + b_n$

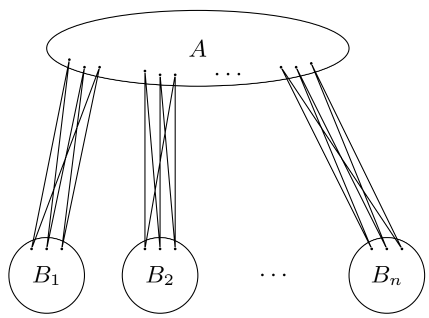


Figure 4.2: Multipartite Star $S(A; B_1, B_2, ..., B_n)$

**Necessity:**

*Proof.* Let $v$ be a vertex in $A$. So all the gregarious paths passing through $v$ have both ends in $B_1 \cup B_2 \cup \cdots \cup B_n$. So $2 \,|\, (b_1 + b_2 + \cdots + b_n)$. For the second condition, the number of the vertices in any $b_i$ should be less than the number of remaining vertices, because if you fix a vertex, say $v$ in $a$, then the gregarious paths passing through $v$ gives a one-to-one matching in between vertices. So the number of vertices in any part, $b_i$, should be less than the sum of the number of vertices in the remaining parts. So for any $1 \leqslant i \leqslant n$, $b_i \leqslant b_2 + \cdots + b_{i-1} + b_{i+1} + \cdots + b_n$. Therefore, if $b_1 \leqslant b_2 + b_3 + \cdots + b_n$ is true, then for any $1 \leqslant i \leqslant n$, $b_i \leqslant b_2 + \cdots + b_{i-1} + b_{i+1} + \cdots + b_n$ is also true since $b_1 \geqslant b_2 \geqslant \cdots \geqslant b_n$. $\square$

**Sufficiency:**

*Proof.* First, take any vertex $v$ in $a$, then find the gregarious decomposition of $(v;\, b_1,\, b_2\,,\, ...,\, b_n)$. Afterwards, we put the copies of this decomposition on the remaining vertices in $A$ (every vertex in $a$ has the same degree). To find the gregarious decomposition of $(v;\, b_1,\, b_2\,,\, ...,\, b_n)$:

1. take a $P_3$ between the first (biggest) two parts.

2. reorder $(b_1 - 1,\, b_2 - 1,\, ...,\, b_n)$ so it is non-increasing.

3. repeat steps 1 and 2 until there are no edges left.

Now we need to prove that in each step the graph we get still satisfies the necessary conditions. The proof of the first condition is easy since we start with an even number of vertices and in each step we just remove two vertices, so in the next step we should still have an even number of vertices.

Now we need prove that in each step we preserve the second condition. We will use induction. Assume that in the $k^{th}$ step we have $(b_1^{(k)},\, b_2^{(k)}, \ldots, b_n^{(k)})$. For $k = 1$ the second condition holds since $(b_1^{(1)},\, b_2^{(1)}, \ldots, b_n^{(1)}) = (b_1, b_2, \ldots, b_n)$ and
for any $1 \leqslant i \leqslant n$, $b_i \leqslant b_2 + \cdots + b_{i-1} + b_{i+1} + \cdots + b_n$ .
To use induction, assume the condition holds for $k$:
for any $1 \leqslant i \leqslant n$, $b_i^{(k)} \leqslant b_2^{(k)} + \cdots + b_{i-1}^{(k)} + b_{i+1}^{(k)} + \cdots + b_n^{(k)}$.
So, we need to prove it holds for $k+1$:
for any $1 \leqslant i \leqslant n$, $b_i^{(k+1)} \leqslant b_2^{(k+1)} + \cdots + b_{i-1}^{(k+1)} + b_{i+1}^{(k+1)} + \cdots + b_n^{(k+1)}$
Fix $i$, $1 \leqslant i \leqslant n$.
**Case 1:** $b_i^{(k+1)} = b_i^{(k)} - 1$:
If we remove one vertex from $b_i^k$, there exists an $m$ with $1 \leqslant m \leqslant n$ such that $b_m^{(k+1)} = b_m^k - 1$. In addition, $b_j^{(k+1)} = b_j^k$ for any $j$ except $j = m,\, i$. So,

$$
\begin{aligned}
b_i^{(k+1)} = b_i^{(k)} - 1 &\leqslant b_1^{(k)} + \cdots + b_m^{(k)} - 1 + \cdots + b_n^{(k)} \\
&\leqslant b_1^{(k+1)} + \cdots + b_m^{(k+1)} + \cdots + b_n^{(k+1)}
\end{aligned}
$$

**Case 2:** $b_i^{(k+1)} = b_i^{(k)}$:

Since we removed two vertices in each step, there exist $b_p^{(k)}$ and $b_q^{(k)}$ such that $b_p^{(k)} \geqslant b_q^{(k)} \geqslant b_i^{(k)}$.
If $b_i^{(k)} \geqslant 2$, then $b_i^{(k+1)} = 2 \leqslant b_p^{(k+1)} + b_q^{(k+1)} + \cdots$.
If $b_i^{(k)} = 1$ and $b_p^{(k)} = b_q^{(k)} = 1$, then we should have at least one more $b_w^{(k)} = 1$ since we have an even number of vertices in each step. So

$$\begin{aligned} b_i^{(k+1)} = 1 &\leqslant b_p^{(k+1)} + b_q^{(k+1)} + b_w^{(k+1)} + \cdots \\ &\leqslant b_p^{(k)} - 1 + b_q^{(k)} - 1 + b_w^{(k)} + \cdots \\ &\leqslant 1 - 1 + 1 - 1 + 1 + \cdots \\ &\leqslant 1 + \cdots \end{aligned}$$

□

Note that the following are equivalent:

1. There is a gregarious $P_3$ decomposition of $S(A; B_1, B_2, \ldots, B_n)$.

2. There is a loopless multigraph with degree sequence $(b_1, b_2, \ldots, b_n)$.

3. The complete multigraph $K(B_1, B_2, \ldots, B_n)$ has a perfect matching.

## 4.3 Cycle Multipartite Graphs

### 4.3.1 Even Cycles

**Theorem 4.4.** *For an even cycle multipartite graph $C(B_1, \ldots, B_{2n})$, the NASCs are:*

1. $b_1b_2 + b_3b_4 + \cdots + b_{2n-1}b_{2n} = b_2b_3 + b_4b_5 + \cdots + b_{2n}b_1$

2. *for any* $1 \leqslant i \leqslant 2n$, $b_ib_{i+1} \leqslant b_{i-1}b_i + b_{i+1}b_{i+2}$

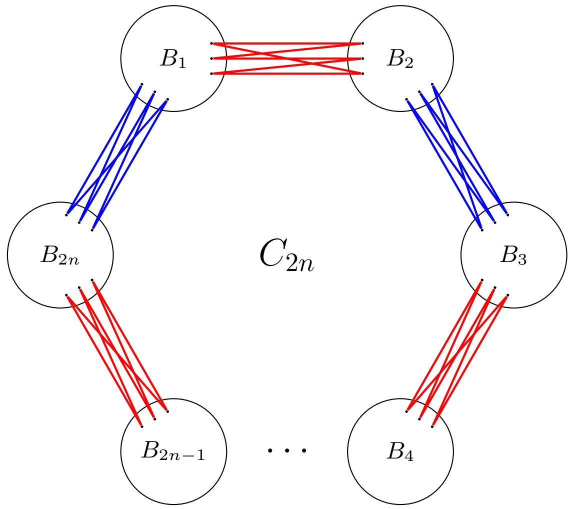


Figure 4.3: Multipartite Even Cycle $C_{2n}$

**Necessity:**

*Proof.* For any $1 \leqslant i \leqslant 2n$ let $x_i$ be the number of gregarious paths that have their middle vertex in $B_i$. Then,

$$\begin{aligned} x_1 + x_2 &= b_1 b_2 \\ x_2 + x_3 &= b_2 b_3 \\ &\vdots \\ x_{2n-1} + x_{2n} &= b_{2n-1} b_{2n} \\ x_{2n} + x_1 &= b_{2n} b_1 \end{aligned}$$

If we add the first, third, fifth, ..., and $(2n-1)^{th}$ equations, we get,

$$x_1 + x_2 + \cdots + x_{2n} = b_1 b_2 + b_3 b_4 + \cdots + b_{2n-1} b_{2n}$$

and if we add the second, fourth, sixth, ..., and $(2n)^{th}$ equations and rearrange the $x_i$'s, we get:

$$x_1 + x_2 + \cdots + x_{2n} \;=\; b_2b_3 + b_4b_5 + \cdots + b_{2n}b_1$$

So these two equations give the first condition. For the second condition, let $x_1 = x$ and $x \geqslant 0$, then:

$$\begin{aligned}
x_2 &= b_1b_2 - x \\
x_3 &= b_2b_3 - x_2 \\
x_4 &= b_3b_4 - x_3 \\
&\vdots \\
x_{2n} &= b_{2n-1}b_{2n} - x_{2n-1}
\end{aligned}$$

and if we get all equations in terms of $x$,

$$\begin{aligned}
x_2 &= b_1b_2 - x \\
x_3 &= b_2b_3 - b_1b_2 + x \\
x_4 &= b_3b_4 - b_2b_3 + b_1b_2 - x \\
&\vdots \\
x_{2n} &= b_{2n-1}b_{2n} - b_{2n-2}b_{2n-1} + \cdots + x
\end{aligned}$$

If we use $x \geqslant 0$ and the equations we have above, then we get:
$b_ib_{i+1} \leqslant b_{i-1}b_i + b_{i+1}b_{i+2}$ for any $1 \leqslant i \leqslant 2n$. $\square$

**Sufficiency:**

*Proof.* If the necessary conditions are satisfied, we can find all the $x_i$'s for $1 \leqslant i \leqslant 2n$, then we use the same technique that we used in the proof of Theorem 4.1 to find gregarious a $P_3$ decomposition. □

### 4.3.2 Odd Cycles

**Theorem 4.5.** *For an odd cycle multipartite graph $C(B_1, \ldots, B_{2n+1})$, the NASCs are:*

1. $2 \mid \sum_{i=1}^{i=2n} b_i b_{i+1}$ *(# of the edges)*
2. *for any* $1 \leqslant i \leqslant 2n+1$, $b_i b_{i+1} \leqslant b_{i-1} b_i + b_{i+1} b_{i+2}$
3. *for any* $1 \leqslant i \leqslant 2n+1$,

   $b_{i+1}b_{i+2} + b_{i+3}b_{i+4} + \cdots + b_{i+2n-2}b_{(i+1)+2n-2} \leqslant b_i b_{i+1} + b_{i+2}b_{i+3} + \cdots + b_{i+2n}b_{(i+1)+2n}$

   *where the subscripts of the b's are taken* $\pmod{2n+1}$.

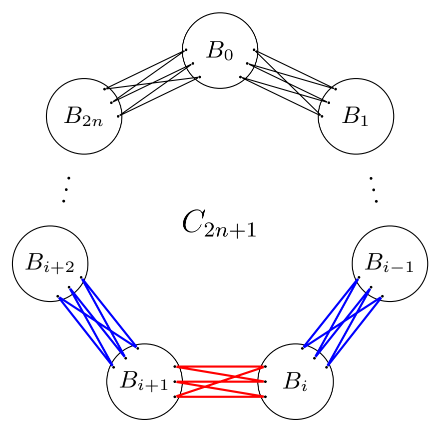


Figure 4.4: Multipartite Odd Cycle $C_{2n+1}$

**Necessity:**

*Proof.* The first condition comes from the fact that the total number of edges is divisible by 2. To get the second condition, for any $1 \leqslant i \leqslant 2n+1$ let $x_i$ be the number of gregarious paths that have their middle vertex in $B_i$ . Then

$$\begin{aligned} x_1 + x_2 &= b_1 b_2 \\ x_2 + x_3 &= b_2 b_3 \\ &\vdots \\ x_{2n} + x_{2n+1} &= b_{2n} b_{2n+1} \\ x_{2n+1} + x_1 &= b_{2n+1} b_1 \end{aligned}$$

To find $x_1$;

$$\begin{aligned} (+) x_1 + x_2 &= b_1 b_2 \\ (-) x_2 + x_3 &= b_2 b_3 \\ &\vdots \\ (-) x_{2n} + x_{2n+1} &= b_{2n} b_{2n+1} \\ (+) x_{2n+1} + x_1 &= b_{2n+1} b_1 \end{aligned}$$

then we get

$$\begin{aligned} x_1 &= \frac{b_1 b_2 - b_2 b_3 + b_3 b_4 - \cdots - b_{2n} b_{2n+1} + b_{2n+1} b_1}{2} \\ &= \frac{b_1 b_2 + b_3 b_4 + \cdots + b_{2n+1} b_1 - (b_2 b_3 + \cdots + b_{2n} b_{2n+1})}{2} \end{aligned}$$

Using the same technique we can get all the $x_i$'s along with condition 3 since each $x_i \geqslant 0$. Condition 2 is the same as the even cycle case. $\square$

**Sufficiency:**

*Proof.* After finding $x_i$, constructing the gregarious $P_3$ decomposition is the same as for the even cycle case. $\square$

## 4.4 Path Multipartite Graphs

**Theorem 4.6.** *For a path multipartite graph $P(B_1, B_2, \ldots, B_n)$, the NASCs are:*

1. $b_3 \geqslant b_1$ *and* $b_{n-2} \geqslant b_n$

2. $b_1b_2 + b_3b_4 \cdots + b_{k-1}b_k = b_2b_3 + b_4b_5 + \cdots + b_{l-1}b_l$

3. *for any* $2 \leqslant i \leqslant n-2$, $b_ib_{i+1} \leqslant b_{i-1}b_i + b_{i+1}b_{i+2}$

*where $k$ is the largest **even** number such that $k \leqslant n$ and $l$ is the largest **odd** number such that $l \leqslant n$.*

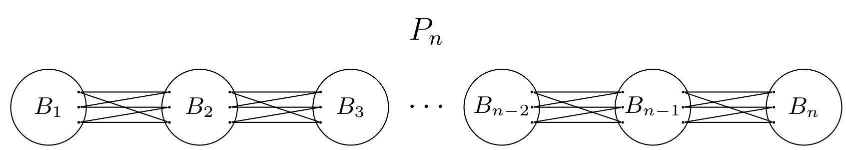


Figure 4.5: Multipartite Path $P_n$

**Necessity:**

*Proof.* For any $2 \leqslant i \leqslant n-1$ let $x_i$ be the number of gregarious paths that have the middle vertex in $B_i$ .

$$\begin{aligned} x_2 &= b_1 b_2 \\ x_2 + x_3 &= b_2 b_3 \\ x_3 + x_4 &= b_3 b_4 \\ &\vdots \\ x_{n-2} + x_{n-1} &= b_{n-2} b_{n-1} \\ x_{n-1} &= b_{n-1} b_n \end{aligned}$$

The first condition comes from the fact that $x_3 = b_2b_3 - x_2 = b_2b_3 - b_1b_2 = b_2(b_3 - b_1)$. So we get $b_3 \geqslant b_1$ since $x_2 \geqslant 0$. We can get $b_n \geqslant b_{n-2}$ in the same way. We can find the remaining $x_i$'s easily.

If we add the first, third, fifth,... , and $(k-1)^{th}$ equations, we get,

$$x_2 + x_3 \cdots + x_{n-1} = b_1b_2 + b_3b_4 + \cdots + b_{k-1}b_k$$

and if we add the second, fourth, sixth,... , and $(l-1)^{th}$ equations, we get:

$$x_2 + x_3 + \cdots + x_{n-1} = b_2b_3 + b_4b_5 + \cdots + b_{l-1}b_l$$

So these two equations give the second condition. The third condition is the same as the condition in the cycle case. $\square$

**Sufficiency:**

*Proof.* If the necessary conditions are satisfied, we can find all the $x_i$'s for $2 \leqslant i \leqslant n-1$, then we use the same technique that we used in the proof of Theorem 4.1 to find gregarious a $P_3$ decomposition. $\square$

### 4.5 Some Tree Multipartite Graphs

**Definition 4.7.** Let $T(C_1, \ldots C_m; A_1, A_2; B_1, \ldots, B_n)$ be a multipartite graph such that two multipartite stars $S(A_1; C_1, \ldots, C_m)$ and $S(A_2; B_1, \ldots, B_n)$ are attached to each other via putting a complete bipartite graph on bipartition $(A_1, A_2)$. See figure 4.6.

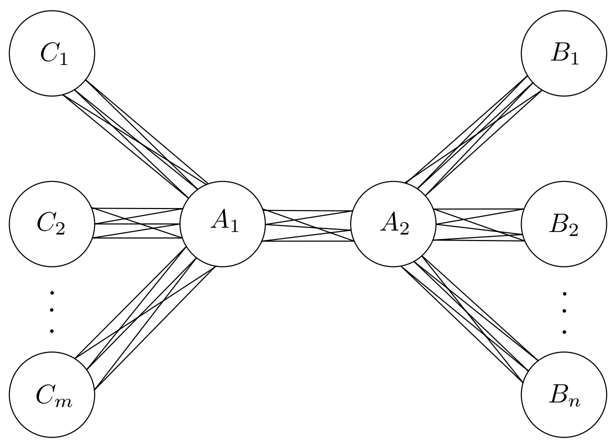


Figure 4.6: $T(C_1, \ldots C_m; A_1, A_2; B_1, \ldots, B_n)$

**Definition 4.8.** Define $T(A_1, A_2, A_3; B_1, \ldots, B_n)$ by using definition 4.7 as $T(A_1; A_2, A_3; B_1, \ldots, B_n)$. See figure 4.7.

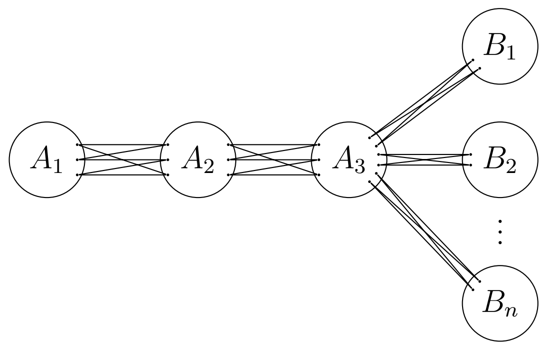


Figure 4.7: $T(A_1, A_2, A_3; B_1, \dots, B_n)$

**Lemma 4.9.** Let $G = (E, V)$ be a graph. There is an orientation of $G$ such that for all $v \in V$, $|out(v) - in(v)| \leqslant 1$.

*Proof.* We can assume that $G$ is connected.

**Case 1:** If all vertices have even degree, then there exists an Euler trail, we can orient the graph this way.

**Case 2:** If $G$ has some vertices with odd degree, make an extra vertex $u$ and connect all those vertices to $u$, then find an Euler trail on $G \cup \{u\}$ and remove the edges at the end. For all $v \in V$, we still have $|out(v) - in(v)| \leqslant 1$ since we remove one edge from each vertex with odd degree .

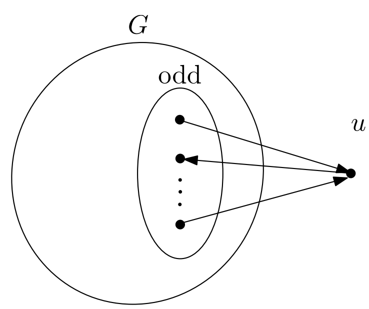


Figure 4.8: Orientation of $G$

□

**Theorem 4.10.** [14] *Let $A, B, I$ be finite non-empty sets, let $f : B \times I \to \mathbb{N}$ be such that for all $t \in B$, $\sum_{i \in I} f(t,i) = |A|$. Then the edges of $K(A,B)$ can be partitioned into spanning subgraphs $G_i$, $i \in I$, such that for each $i \in I$, $G_i$ is balanced on $A$, and for each $t \in B$, the degree of $t$ in $G_i$ is $f(t,i)$.*

*Proof.* If $|A| = 1$, then the proof is trivial.
Now suppose $|A| = 2$. Let $A = \{s_1, s_2\}$. Form a graph $H$ on vertex set $I$ as follows:
For each $t \in B$, $H$ has an edge $e_t$: If $f(t,i) = 2$, (and so $f(t,j) = 0$ for all other $j \in I$) then $e_t$ is a loop at vertex $i$ of $H$. If $f(t,i) = 1 = f(t,j)$, $i \neq j$, ($f(t,k) = 0$ for the other $k \in I$), then $e_t$ joins the vertices $i$ and $j$ in $H$.

Orient $H$ so that at each vertex of $H$ the indegree and outdegree differ by at most 1 using Lemma 4.9. For each $t \in B$, if $e_t$ is directed from $i$ to $j$ in the oriented $H$, place the edge between $t$ and $s_1$ in $G_i$, and the edge between $t$ and $s_2$ in $G_j$ (see example 4.11).

If $|A| \geqslant 3$, then partition the edges of $K(A,B)$ into spanning subgraphs $G_i$ whose degrees on $B$ are given by $f$ (this is certainly possible by the sum condition on $f$). If everything is balanced on $|A|$, then we are done. Otherwise degrees in some $G_i$ differ by 2 or more. Fix $i$, and let $s_1, s_2$ be two vertices in $A$, whose degrees differ by 2 or more in $G_i$. So use the previous case where $|A| = 2$ on this graph to find the balanced distribution. Using this method repeatedly for each unbalanced pair of vertices of $G_i$ in $A$, finally we can get the balanced distribution on $A$. Afterwards, we can repeat the same process for the other $G_j$ for each $j \in I$.

Now we need to prove that this process will stop after finitely many steps. Let $V_1 = (a_1, \ldots, a_i, \ldots, a_j, \ldots, a_n)$ be a integer vector with fixed sum $\sum a_i = a$. So the shortest integer vector with respect to the Euclidean metric with the fixed sum of the entries is the balanced one. To see this assume $a_j \geqslant a_i + 2$, then if we balance $a_i$ and $a_j$, we get

$V_2 = (a_1, \ldots, a_i + 1, \ldots, a_j - 1, \ldots, a_n)$ and $|V_2|^2 \leqslant |V_1|^2 - 2$ since,

$$\begin{aligned}
|V_2|^2 &= a_1^2 + \cdots + (a_i+1)^2 + \cdots + (a_j-1)^2 + \cdots + a_n^2 \\
&= a_1^2 + \cdots + a_i^2 + 2a_i + 1 + \cdots + a_j^2 - 2a_j + 1 + \cdots + a_n^2 \\
&= (a_1^2 + \cdots + a_i^2 + \cdots + a_j^2 + \cdots + a_n^2) + 2(a_i - a_j) + 2 \\
&= |V_1|^2 + 2(a_i - a_j) + 2 \\
&\leqslant |V_1|^2 + 2(-2) + 2 \\
&\leqslant |V_1|^2 - 2
\end{aligned}$$

This means that when we balance a pair of entries in the vector at a time, the vector gets shorter, and after finitely many steps we will find the shortest one. This completes the proof. □

**Example 4.11.** Let $G$ be a bipartite graph on bipartition $(A, B)$ where $A = \{s_1, s_2\}$ and $B = \{v_1, v_2, v_3, v_4\}$. Let $I$ ={green, blue, red}. We want to get green and blue balanced on $A$ without changing the color census on $B$ (see the first picture in Figure 4.9). Then using the method defined in Theorem 4.10 build a graph $H$ (see the second picture in Figure 4.9) and orient $H$ so that $|in(w) - out(w)| \leqslant 1$ for every vertex $w$ in $H$. Finally, we can swap edges of $G$ with respect to the orientation on $H$ to get a balanced coloring on $A$.

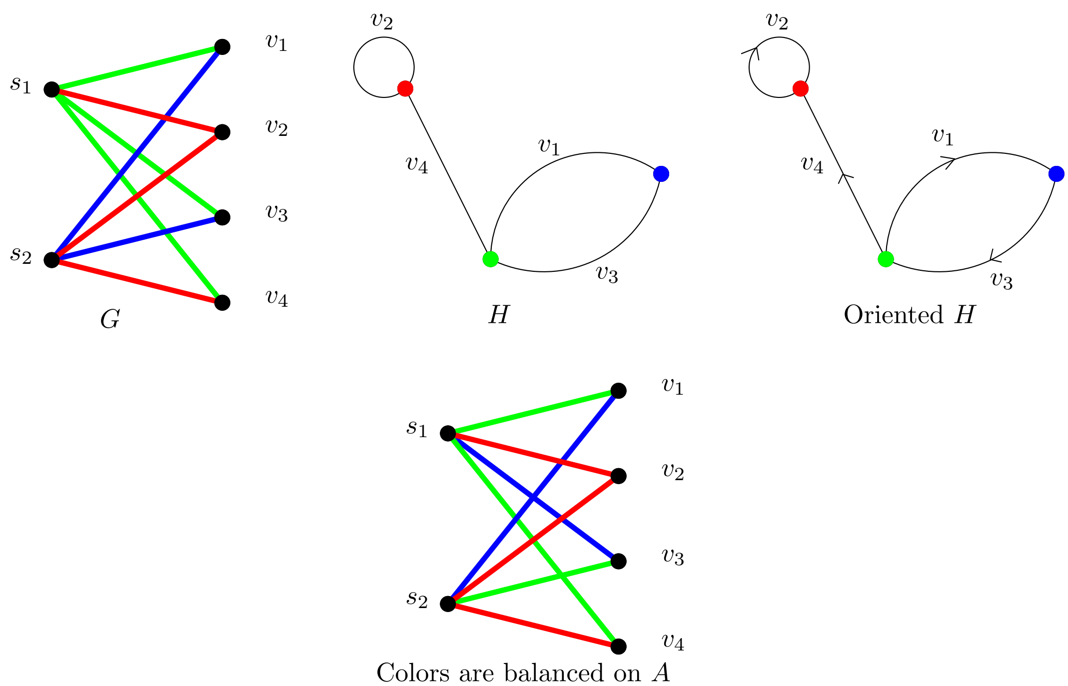


Figure 4.9: An Example for Theorem 4.10

### 4.5.1 Necessary And Sufficient Conditions for $T(A_1, A_2, A_3; B_1, \ldots, B_n)$

**Theorem 4.12.** *For a graph $T(A_1, A_2, A_3; B_1, \ldots, B_n)$ assume $b_n \leqslant \cdots \leqslant b_2 \leqslant b_1$, the NASCs are:*

1. $2 \,|\, [a_2(a_1 + a_3) + a_3(b_1 + b_2 + \cdots + b_n)]$

2. $a_1 \leqslant a_3$

3. $a_1a_2 + b_1a_3 \leqslant a_3(a_2 + b_2 + b_3 + \cdots + b_n)$

4. $a_2a_3 \leqslant a_1a_2 + a_3(b_1 + b_2 + \cdots + b_n)$

5. *If*

   - $a_2 + (b_1 + \cdots + b_n)$ *is even, then* $a_1a_2$ *is even.*

   - $a_2 + (b_1 + \cdots + b_n)$ *is odd, then* $a_1a_2 - a_3$ *is even and non-negative.*

**Necessity:**

*Proof.* Let $G = T(A_1, A_2, A_3; B_1, \ldots, B_n)$. Condition 1 comes from the fact that the number of edges is even. We can get conditions 2 - 4 using Tutte's f-factor Theorem on $L(G)$. $L(G)$ is union of a complete graph on $n+1$ vertices and an edge attached at the vertex $A_2A_3$. If we check all the possible $L(G)$ triples $B = (S, T, U)$ where $T$ is independent and $\lambda(T, U) = 0$ from theorem 2.12, all will reduce to the the following three cases in figure 4.10. We need to check $k(B, h) + h(T) \leqslant h(S)$ for each case. From the first picture in the figure 4.10, we will get $a_2a_3 \leqslant a_1a_2$ which gives condition 2. From the second picture, we get $a_1a_2 + b_1a_3 \leqslant a_2a_3 + b_2a_3 + \cdots + b_na_3$ which gives condition 3. From the last picture, we get $a_2a_3 \leqslant a_1a_2 + a_3b_1 + a_3b_2 + \cdots + a_3b_n$ which gives condition 4.

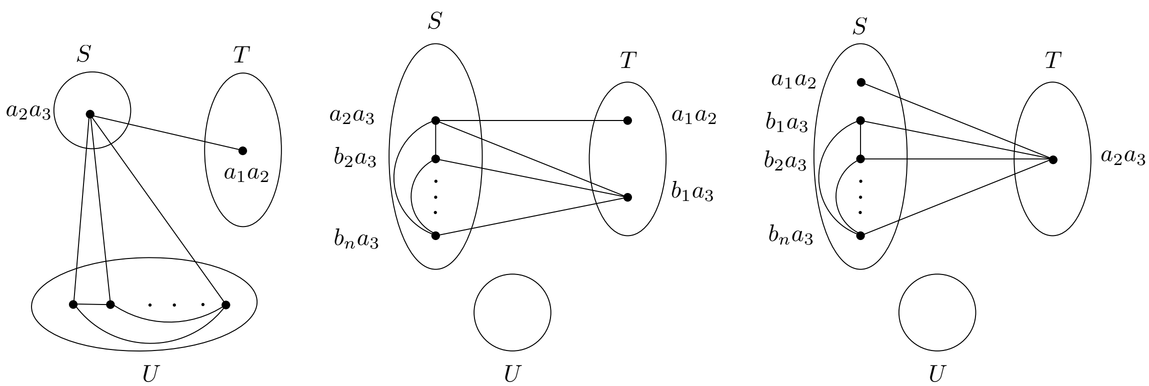


Figure 4.10: How to get conditions 2 - 4

For condition 5, we need to consider all the types of paths we have and the degree of any vertex in $A_3$. Firstly, the degree of any vertex $v$ in $A_3$ is $deg(v) = a_2 + b_1 + \cdots + b_n$. Let $x_1$ be the number of paths passing through the sets of vertices $A_1 \to A_2 \to A_3$ so $x_1 = a_1a_2$. In the same way, $y_i : A_2 \to A_3 \to B_i$ for any $1 \leqslant i \leqslant n$, and $w_{ij} : B_i \to A_3 \to B_j$ for any $1 \leqslant i \leqslant j \leqslant n$. Here, both $y_i$ and $w_{ij}$ have their middle vertices in $A_3$, so if $deg(v)$ is even, then $x_1$ must be even. If $deg(v)$ is odd, then we should have enough $x_1$ type paths which means $a_1a_2 \geqslant a_3$. That gives $a_1a_2 - a_3$ non-negative. To see that $a_1a_2 - a_3$ is even, consider the vertices in $A_3$ and distribution of $a_1a_2$ edges on $A_3$. There are $\alpha_i$'s for

$1 \leqslant i \leqslant a_3$ such that $\sum_{i=1}^{a_3} \alpha_i = a_1a_2$ where each $\alpha_i = 2\beta_i + 1$, an odd number. $\beta_i = \frac{\alpha_i - 1}{2}$. $\sum_{i=1}^{a_3} \beta_i = \sum_{i=1}^{a_3} \frac{\alpha_i - 1}{2} = \frac{1}{2}(a_1a_2 - a_3)$. Therefore $a_1a_2 - a_3$ is an even number. □

**Sufficiency:**

*Proof.* If the necessary conditions are satisfied we can find proper $x_1$, $y_i$'s and $w_{ij}$'s. In between pairs of sets $(A_1, A_2)$ and $(A_3, B_i)$ for any $i$, we can find balanced edge distributions with the required numbers as we did for the sufficiency case of the Theorem 4.1. The only problem is finding a construction for $(A_2, A_3)$ since we need to find a parity balanced distribution of $x_1$ edges on $A_3$ with respect to the parity of $a_2 + (b_1 + \cdots + b_n)$ (see condition 5 in Theorem 4.12). We also need to find a balanced distribution for the remaining $y_i$'s. To be able to find this special distribution we can use theorem 4.10 and choose the degrees on $A_3$ to get balanced degrees on $A_2$. □

### 4.5.2 Necessary And Sufficient Conditions for $T(C_1, \ldots, C_m; A_1, A_2; B_1, \ldots, B_n)$

**Theorem 4.13.** *For a graph $T(C_1, \ldots, C_m; A_1, A_2; B_1, \ldots, B_n)$ assume $c_m \leqslant \cdots \leqslant c_2 \leqslant c_1$ and $b_n \leqslant \cdots \leqslant b_2 \leqslant b_1$, and let $C = C_1 \cup C_2 \cup \cdots \cup C_m$ and $|C| = c$, $B = B_1 \cup B_2 \cup \cdots \cup B_n$, $|B| = b$ and $d_1 = c + a_2$, $d_2 = b + a_1$. The NASCs are:*

1. $2 \,|\, [a_1(c_1 + c_2 + \cdots + c_m) + a_1a_2 + a_2(b_1 + b_2 + \cdots + b_n)]$
2. $c_1 \leqslant a_2 + (c_2 + \cdots + c_m)$ *and* $b_1 \leqslant a_1 + (b_2 + \cdots + b_n)$
3. $a_1c_1 + a_2b_1 \leqslant a_1(c_2 + \cdots + c_m) + a_1a_2 + a_2(b_2 + \cdots + b_n)$
4. $a_1a_2 \leqslant a_1(c_1 + c_2 + \cdots + c_m) + a_2(b_1 + b_2 + \cdots + b_n)$
5. *If*
   - *$d_1$ and $d_2$ are even, then $a_1a_2$ is even.*

- *$d_1$ is even and $d_2$ is odd, then either both $a_1$ and $a_2$ are odd, both even or $a_1$ is odd and $a_2$ even. In addition, $ca_1 - a_2 \geqslant 0$.*

- *$d_1$ is odd and $d_2$ is even, then either both $a_1$ and $a_2$ are odd, both even or $a_1$ is even and $a_2$ odd. In addition, $ba_2 - a_1 \geqslant 0$.*

- *$d_1$ and $d_2$ are odd, then both $a_1$ and $a_2$ are even. In addition, $ca_1 - a_2 \geqslant 0$ and $ba_2 - a_1 \geqslant 0$.*

*with the following exceptions:*

| $d_1$ | $d_2$ | $a_1$ | $a_2$ | $c_1$ & $b_1$ |
|---|---|---|---|---|
| even | even | 2 | 2 | any $c_1$ with $1 + (b_2 + \cdots + b_n) \leqslant b_1 \leqslant 2 + (b_2 + \cdots + b_n)$ |
| | | even | 1 | any $c_1$ with $1 + (b_2 + \cdots + b_n) \leqslant b_1 \leqslant a_1 + (b_2 + \cdots + b_n)$ |
| | | 1 | even | any $b_1$ with $1 + (c_2 + \cdots + c_m) \leqslant c_1 \leqslant a_2 + (c_2 + \cdots + c_m)$ |
| even | odd | odd | 1 | any $c_1$ with $1 + (b_2 + \cdots + b_n) \leqslant b_1 \leqslant a_1 + (b_2 + \cdots + b_n)$ |
| | | 1 | even | any $c_1$ with $b_1 = 1 + (b_2 + \cdots + b_n)$ |
| odd | even | 1 | odd | any $b_1$ with $1 + (c_2 + \cdots + c_m) \leqslant c_1 \leqslant a_2 + (c_2 + \cdots + c_m)$ |
| | | even | 1 | any $b_1$ with $c_1 = 1 + (c_2 + \cdots + c_m)$ |

Table 4.1: Exceptions for Theorem 4.13

**Necessity:**

*Proof.* Let $G = T(C_1, \ldots, C_m; A_1, A_2; B_1, \ldots, B_n)$. Condition 1 comes from the fact that the number of edges is even. We can get conditions 2 - 4 using Tutte's f-factor Theorem on $L(G)$. $L(G)$ is union of two complete graphs on $m$ and $n$ vertices attached at the vertex $A_1A_2$. If we check all the possible $L(G)$ triples $B = (S, T, U)$ where $T$ is independent and $\lambda(T, U) = 0$ from theorem 2.12, all will reduce to the the following three cases in figure 4.11. We need to check $k(B, h) + h(T) \leqslant h(S)$ for each case. From the first picture in the figure 4.11, we will get $c_1 \leqslant a_2 + (c_2 + \cdots + c_m)$ and in the same picture if we replace $B$'s with $C$'s and $C$'s with $B$'s then we get $b_1 \leqslant a_1 + (b_2 + \cdots + b_n)$ which gives condition 2. From the second picture, we get $a_1c_1 + a_2b_1 \leqslant a_1(c_2 + \cdots + c_m) + a_1a_2 + a_2(b_2 + \cdots + b_n)$ which gives condition 3. From the last picture, we get $a_1a_2 \leqslant a_1(c_1 + c_2 + \cdots + c_m) + a_2(b_1 + b_2 + \cdots + b_n)$ which gives condition 4.

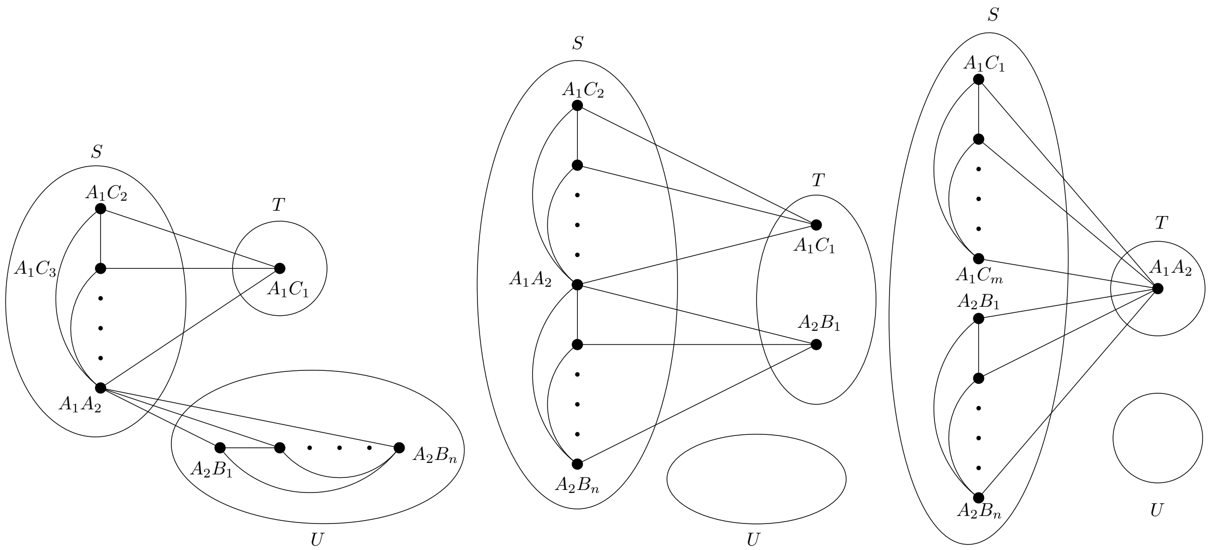


Figure 4.11: How to get conditions 2 - 4

For condition 5, we need to consider all the types of paths we have and the degree of any vertex in $A_1$ and $A_2$ . Firstly, the degree of any vertex $v_1$ in $A_1$ is:

$d_1 = deg(v_1) = a_2 + c_1 + \cdots + c_m = a_2 + c$

and the degree of any vertex $v_2$ in $A_2$ is:

$d_2 = deg(v_2) = a_1 + b_1 + \cdots + b_n = a_1 + b.$

Let $x_i$ be the number of paths passing through the sets of vertices $C_i \to A_1 \to A_2$ for any $1 \leqslant i \leqslant m$ and let $x = \sum_{i=1}^{m} x_i$. In the same way, $y_j : B_j \to A_2 \to A_1$ for any $1 \leqslant j \leqslant n$ and $y = \sum_{j=1}^{n} y_j$. So $x + y = a_1a_2$. In addition, we have $w_{ij} : C_i \to A_1 \to C_j$ for any $1 \leqslant i \leqslant j \leqslant m$ and $z_{kl} : B_k \to A_2 \to B_l$ for any $1 \leqslant k \leqslant l \leqslant n$. Here $w_{ij}$'s have their middle vertex in $A_1$ and $z_{kl}$'s have their middle vertex in $A_2$, so we have four cases with respect to the parity of $d_1$ and $d_2$. So the parity of $x$ and $d_2$, and $y$ and $x_1$ must be consistent (see figure 4.12).

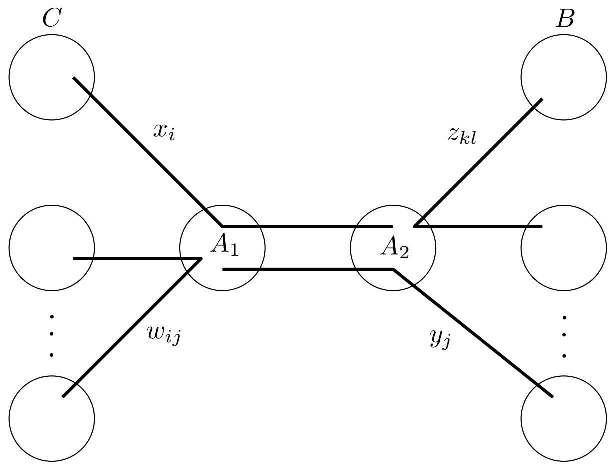


Figure 4.12: Types of paths

**Case 1:** If $d_1$ and $d_2$ are even, then $y$ and $x$ are even so $a_1a_2$ is even since $x + y = a_1a_2$.

**Case 2:** If $d_1$ is even and $d_2$ is odd, then $y$ is even and $x \equiv a_2 \pmod 2$ and $x \geqslant a_2$. So we can get $ca_1 - a_2 \geqslant 0$ since $ca_1 \geqslant x$. To get either both $a_1$ and $a_2$ odd, both even or or $a_1$ is odd and $a_2$ even, see:

$$\begin{aligned} x + y &= a_1a_2 \\ x + y &\equiv a_1a_2 \pmod 2 \\ x &\equiv a_1a_2 \pmod 2 \text{ since } y \text{ is even} \end{aligned}$$

**Case 3:** If $d_1$ is odd and $d_2$ is even, then this is the same as case 2, just replace $a_2$ with $a_1$.

**Case 4:** If $d_1$ is odd and $d_2$ is odd, then $y \equiv a_1 \pmod 2$ and $y \geqslant a_1$, and $x \equiv a_2 \pmod 2$ and $x \geqslant a_2$. We can get $ca_1 - a_2 \geqslant 0$ and $ba_2 - a_1 \geqslant 0$ in the same way as in case 2. To get both $a_1$ and $a_2$ even, see:

$$\begin{aligned} x + y &= a_1a_2 \\ x + y &\equiv a_1a_2 \pmod 2 \\ a_1 + a_2 &\equiv a_1a_2 \pmod 2 \text{ since } x \equiv a_2 \pmod 2 \text{ and } y \equiv a_1 \pmod 2 \end{aligned}$$

$a_1 + a_2 \equiv a_1 a_2 \pmod 2$ is only satisfied when both $a_1$ and $a_2$ are even.

Note that theorem 4.12 is a special case of theorem 4.13. In theorem 4.13, if we get $c_2 = c_3 = \cdots = c_m = 0$ and replace $c_1$ with $a_1$, $a_1$ with $a_2$ and $a_2$ with $a_3$ we will get exactly the same conditions as in theorem 4.12. $\square$

**Sufficiency:**

*Proof.* If the necessary conditions are satisfied we can find proper $x_i$'s, $y_j$'s, $w_{ij}$'s and $v_{kl}$'s. First we find a proper $x$ and $y$ then we will find $w_{ij}$'s and $v_{kl}$'s since we have more restriction on $x$ and $y$. In between pairs of sets $(C_i, A_1)$ for any $1 \leqslant i \leqslant m$, and $(A_2, B_j)$ for any $1 \leqslant j \leqslant n$, we can find balanced edge distributions with the required numbers as we did for the sufficiency case of theorem 4.1. The only problem is finding a construction for $(A_1, A_2)$ since we need to find a parity balanced distribution of $x+y$ edges on $A_1$ and $A_2$ with respect to the parity of $d_1$ and $d_2$ (see condition 5 in theorem 4.13). To find this parity balanced distribution, we will use theorem 3.12.

**Case 1:** Assume $d_1$ and $d_2$ are even, then $y$ and $x$ are even so $a_1a_2$ is even since $x+y = a_1a_2$. So there are three cases for $(a_1, a_2)$: (even, even), (even, odd) and (odd, even).

If $a_1$ and $a_2$ are both even, then we need to find a parity balanced bipartite graph (PBBG) with pararameters $(a = a_1, b = a_2, e = x, \epsilon_a = 0, \epsilon_b = 0)$ with bipartite complement $(a = a_1, b = a_2, e = ab - x = y, \epsilon'_a = 0, \epsilon'_b = 0)$ so that the distribution of $x$ on $A_2$ has even parity ($\epsilon_b = 0$) and the distribution of $y$ on $A_1$ has even parity ($\epsilon'_a = 0$). We can we can find such a PBBG since the necessary conditions of theorem 3.12 are satisfied.

$$\begin{aligned}
\epsilon_a + \epsilon'_a &= 0 + 0 = 0 \equiv b \pmod 2 \text{ and } \epsilon_b + \epsilon'_b = 0 + 0 = 0 \equiv a \pmod 2 \\
\epsilon_a a &\leqslant e \leqslant ab - \epsilon'_a a \Rightarrow 0 \leqslant x \leqslant a_1 a_2 \\
\epsilon_b b &\leqslant e \leqslant ab - \epsilon'_b b \Rightarrow 0 \leqslant x \leqslant a_1 a_2 \\
\epsilon_a a &\equiv ab - \epsilon'_a a \equiv e = x \equiv \epsilon_b b \equiv ab - \epsilon'_b b \equiv 0 \pmod 2
\end{aligned}$$

For the exceptions in table 4.1 the only problem concerning this case is when $a = a_1 =$even, $b = a_2 = even$, $e = x = 2$, $\epsilon_a = 0$, $\epsilon'_a = 0$, $\epsilon_b = 0$,$\epsilon'_b = 0$. We can solve this problem by choosing $x \geqslant 4$ since $a_1 \geqslant 2$ and $a_2 \geqslant 2$ in the exceptions. In the case of $a_1 = 2 = a_2$, we will not have any $y$'s which means we need to put a gregarious $P_3$ decomposition of star multipartite graph on $S = (A_2; B_2, B_2, \ldots, B_n)$. The first condition of theorem 4.3 is satisfied since $2 \mid b = b_1 + b_2 + \ldots + b_n$ ($d_1 =$even$= a_1 + b$ and $a_1$ is even so $b$ is even). For the second condition of theorem 4.3, we can get $b_1 \leqslant 2 + b_2 + \ldots + b_n$ from condtion 2 of theorem 4.13. From here we get two exceptions: $b_1 = 1 + b_2 + \cdots + b_n$ and $b_1 = 2 + b_2 + \cdots + b_n$. So $T(C_1, \ldots C_m; 2, 2; B_1, \ldots, B_n)$ doesn't exist when either $b_1 = 1 + b_2 + \cdots + b_n$ or $b_1 = 2 + b_2 + \cdots + b_n$.

If $a_1$ is even and $a_2$ is odd, then we need to find a PBBG with pararameters ($a = a_1, b = a_2, e = x, \epsilon_a = 1, \epsilon_b = 0$) with bipartite complement ($a = a_1, b = a_2, e = ab - x = y, \epsilon'_a = 0, \epsilon'_b = 0$) so that the distribution of $x$ edges on $a_2$ has even parity ($\epsilon_b = 0$) and the distribution of $y$ edges on $a_1$ has even parity($\epsilon'_a = 0$). We can we can find such a PBBG since the necessary conditions of theorem 3.12 are satisfied.

$$\begin{aligned}
\epsilon_a + \epsilon'_a &= 1 + 0 = 1 \equiv b \pmod 2 \text{ and } \epsilon_b + \epsilon'_b = 0 + 0 = 0 \equiv a \pmod 2 \\
\epsilon_a a &\leqslant e \leqslant ab - \epsilon'_a a \Rightarrow a_1 \leqslant x \leqslant a_1 a_2 \\
\epsilon_b b &\leqslant e \leqslant ab - \epsilon'_b b \Rightarrow 0 \leqslant x \leqslant a_1 a_2 \\
\epsilon_a a &\equiv ab - \epsilon'_a a \equiv e = x \equiv \epsilon_b b \equiv ab - \epsilon'_b b \equiv 0 \pmod 2
\end{aligned}$$

If we check the exceptions in table 4.1, then we see ($a = even \geqslant 4, b = odd \geqslant 3, e = 2, \epsilon_a = 1, \epsilon'_a = 0, \epsilon_b = 0, \epsilon'_b = 0$). However, in this case $e = x \geqslant a_1$ and $a_1 \geqslant 4$ so we don't have any exception for $e = 2$. There are other exceptions coming from $x \geqslant a_1$. If $a_2 = 1$, then we don't have any $y$'s which means we need to put a gregarious $P_3$ decomposition of a star multipartite graph on $S = (A_2 = 1; B_2, B_2, \ldots, B_n)$. The first condition of theorem 4.3 is satisfied since $2 \mid b = b_1 + b_2 + \ldots + b_n$ ($d_2 =$even$= a_1 + b$ and $a_1$ is even so $b$ is even). For the second condition of theorem 4.3, we can get $b_1 \leqslant a_1 + (b_2 + \ldots + b_n)$ from condition 2 of theorem 4.13. From here we get exceptions when $1 + (b_2 + \cdots + b_n) \leqslant b_1 \leqslant a_1 + (b_2 + \cdots + b_n)$. So $T(C_1, \ldots C_m; A_1, 1; B_1, \ldots, B_n)$ doesn't exist when $1+(b_2+\cdots+b_n) \leqslant b_1 \leqslant a_1+(b_2+\cdots+b_n)$.

If $a_1$ is odd and $a_2$ is even, then this case is the same as the previous case, just switch $a_2$ with $a_1$.

**Case 2:** If $d_1$ is even and $d_2$ is odd, then $y$ is even and $x \equiv a_2 \pmod 2$ and $x \geqslant a_2$. So there are three cases for $(a_1, a_2)$: (even, even), (odd, odd) and (odd, even).

If $a_1$ and $a_2$ are both even, then $x$ and $y$ are both even. We need to find a PBBG with parameters $(a = a_1, b = a_2, e = x, \epsilon_a = 0, \epsilon_b = 1)$ with bipartite complement $(a = a_1, b = a_2, e = a_1a_2 - x = y, \epsilon'_a = 0, \epsilon'_b = 1)$ so that the distribution of $x$ on $A_2$ has odd parity $(\epsilon_b = 1)$ and the distribution of $y$ on $A_1$ has even parity $(\epsilon'_a = 0)$. We can we can find such a PBBG since the necessary conditions of theorem 3.12 are satisfied.

$$\begin{aligned}
\epsilon_a + \epsilon'_a &= 0 + 0 = 0 \equiv b \pmod 2 \text{ and } \epsilon_b + \epsilon'_b = 1 + 1 = 0 \equiv a \pmod 2 \\
\epsilon_a a &\leqslant e \leqslant ab - \epsilon'_a a \Rightarrow 0 \leqslant x \leqslant a_1 a_2 \\
\epsilon_b b &\leqslant e \leqslant ab - \epsilon'_b b \Rightarrow a_2 \leqslant x \leqslant a_1 a_2 - a_2 \\
\epsilon_a a &\equiv ab - \epsilon'_a a \equiv e = x \equiv \epsilon_b b \equiv ab - \epsilon'_b b \equiv 0 \pmod 2
\end{aligned}$$

If we check the exceptions in table 4.1, we see that we don't have any exception for this case.

If $a_1$ and $a_2$ are both odd, then $x$ is odd and $y$ is even. We need to find a PBBG with parameters $(a = a_1, b = a_2, e = x, \epsilon_a = 1, \epsilon_b = 1)$ with bipartite complement $(a = a_1, b = a_2, e = a_1a_2 - x = y, \epsilon'_a = 0, \epsilon'_b = 0)$ so that the distribution of $x$ on $A_2$ has odd parity $(\epsilon_b = 1)$ and the distribution of $y$ on $A_1$ has even parity $(\epsilon'_a = 0)$. We can we can find such a PBBG since the necessary conditions of theorem 3.12 are satisfied.

$$\begin{aligned}
\epsilon_a + \epsilon'_a &= 0+1=1 \equiv b \pmod 2 \text{ and } \epsilon_b + \epsilon'_b = 1+0=1 \equiv a \pmod 2 \\
\epsilon_a a &\leqslant e \leqslant ab - \epsilon'_a a \Rightarrow a_1 \leqslant x \leqslant a_1 a_2 \\
\epsilon_b b &\leqslant e \leqslant ab - \epsilon'_b b \Rightarrow a_2 \leqslant x \leqslant a_1 a_2 \\
\epsilon_a a &\equiv ab - \epsilon'_a a \equiv e = x \equiv \epsilon_b b \equiv ab - \epsilon'_b b \equiv 1 \pmod 2
\end{aligned}$$

If we check the exceptions in table 4.1, then we see ($a = odd \geqslant 3, b = odd \geqslant 3, e = 2, \epsilon_a = 1, \epsilon'_a = 0, \epsilon_b = 1, \epsilon'_b = 0$). However, in this case $e = x \geqslant max\{a_1, a_2\}$ and $a_1, a_2 \geqslant 3$ so we don't have any exception for $e = 2$. There are other exceptions coming from $x \geqslant max\{a_1, a_2\}$. If $a_2 = 1$, then $x = a_1 = a_1 a_2$ so we don't have any $y$'s which means we need to put a gregarious $P_3$ decomposition of a star multipartite graph on $S = (A_2 = 1; B_2, B_2, \ldots, B_n)$. The first condition of theorem 4.3 is satisfied since $2 \mid b = b_1 + b_2 + \ldots + b_n$ ($d_2$ =odd= $a_1 + b$ and $a_1$ is odd so $b$ is even). For the second condition of theorem 4.3, we can get $b_1 \leqslant a_1 + (b_2 + \ldots + b_n)$ from condition 2 of theorem 4.13. From here we get exceptions when $1 + (b_2 + \cdots + b_n) \leqslant b_1 \leqslant a_1 + (b_2 + \cdots + b_n)$. So $T(C_1, \ldots C_m; A_1, 1; B_1, \ldots, B_n)$ doesn't exist when $1 + (b_2 + \cdots + b_n) \leqslant b_1 \leqslant a_1 + (b_2 + \cdots + b_n)$.

If $a_1$ is odd and $a_2$ is even, then $x$ and $y$ are both even. We need to find a PBBG with parameters ($a = a_1, b = a_2, e = x, \epsilon_a = 0, \epsilon_b = 1$) with bipartite complement ($a = a_1, b = a_2, e = a_1 a_2 - x = y, \epsilon'_a = 0, \epsilon'_b = 0$) so that the distribution of $x$ on $A_2$ has odd parity ($\epsilon_b = 1$) and the distribution of $y$ on $A_1$ has even parity ($\epsilon'_a = 0$). We can we can find such a PBBG since the necessary conditions of theorem 3.12 are satisfied.

$$\begin{aligned}
\epsilon_a + \epsilon'_a &= 0+0=0 \equiv b \pmod 2 \text{ and } \epsilon_b + \epsilon'_b = 1+0=1 \equiv a \pmod 2 \\
\epsilon_a a &\leqslant e \leqslant ab - \epsilon'_a a \Rightarrow 0 \leqslant x \leqslant a_1 a_2 \\
\epsilon_b b &\leqslant e \leqslant ab - \epsilon'_b b \Rightarrow a_2 \leqslant x \leqslant a_1 a_2 \\
\epsilon_a a &\equiv ab - \epsilon'_a a \equiv e = x \equiv \epsilon_b b \equiv ab - \epsilon'_b b \equiv 0 \pmod 2
\end{aligned}$$

If we check the exceptions in table 4.1, then we see ($a = odd \geqslant 3, b = even \geqslant 3, e = 2, \epsilon_a = 0, \epsilon'_a = 0, \epsilon_b = 1, \epsilon'_b = 0$). However, in this case $e = x \geqslant a_2$ and $a_2 \geqslant 4$ so we don't have any exception for $e = 2$. There is an exception coming from $x \geqslant a_2$. If $a_1 = 1$, then $x = a_2 = a_1 a_2$ so we don't have any $y$'s which means we need to put a gregarious $P_3$ decomposition of a star multipartite graph on $S = (A_2; B_2, B_2, \ldots, B_n)$. The first condition of theorem 4.3 is satisfied since $2 \mid b = b_1 + b_2 + \ldots + b_n$ ($d_2$ =odd= $a_1 + b$ and $a_1$ is odd so $b$ is even). For the second condition of theorem 4.3, we can get $b_1 \leqslant a_1 + (b_2 + \ldots + b_n)$ from condition 2 of theorem 4.13. From here we get exceptions when $b_1 = 1 + (b_2 + \cdots + b_n)$. So $T(C_1, \ldots C_m; 1, A_2; B_1, \ldots, B_n)$ doesn't exist when $b_1 = 1 + (b_2 + \cdots + b_n)$.

**Case 3:** If $d_1$ is odd and $d_2$ is even, then this case is the same as case 2, just switch $a_1$ and $a_2$.

**Case 4:** If $d_1$ and $d_2$ are both odd, then $x, y, a_1, a_2$ are even and $y \geqslant a_1$, $x \geqslant a_2$. We need to find a PBBG with parameters ($a = a_1, b = a_2, e = x, \epsilon_a = 1, \epsilon_b = 1$) with bipartite complement ($a = a_1, b = a_2, e = a_1 a_2 - x = y, \epsilon'_a = 1, \epsilon'_b = 1$) so that the distribution of $x$ on $A_2$ has odd parity ($\epsilon_b = 1$) and the distribution of $y$ on $A_1$ has odd parity too ($\epsilon'_a = 1$). We can we can find such a PBBG since the necessary conditions of theorem 3.12 are satisfied.

$$\begin{aligned}
\epsilon_a + \epsilon'_a &= 1 + 1 = 0 \equiv b \pmod 2 \text{ and } \epsilon_b + \epsilon'_b = 1 + 1 = 0 \equiv a \pmod 2 \\
\epsilon_a a &\leqslant e \leqslant ab - \epsilon'_a a \Rightarrow a_1 \leqslant x \leqslant a_1 a_2 - a_1 \\
\epsilon_b b &\leqslant e \leqslant ab - \epsilon'_b b \Rightarrow a_2 \leqslant x \leqslant a_1 a_2 - a_2 \\
\epsilon_a a &\equiv ab - \epsilon'_a a \equiv e = x \equiv \epsilon_b b \equiv ab - \epsilon'_b b \equiv 0 \pmod 2
\end{aligned}$$

If we check the exceptions in table 4.1, we see that we don't have any exception for this case. □

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