Generalizing Clatworthy Group Divisible Designs

by

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Abstract

Clatworthy described the eleven group divisible designs with three groups, block size four, and replication number at most 10. Each of these can be generalized in natural ways. In this dissertation neat constructions are provided for these new families of group divisible designs. In a previous paper the existence of one such design was settled. Here we essentially settle the existence of generalizations of eight of the remaining ten Clatworthy designs. In each case (namely, $\lambda_1 = 4$ and $\lambda_2 = 5$, $\lambda_1 = 4$ and $\lambda_2 = 2$, $\lambda_1 = 8$ and $\lambda_2 = 4$, $\lambda_1 = 2$ and $\lambda_2 = 1$, $\lambda_1 = 10$ and $\lambda_2 = 5$, $\lambda_1 = 6$ and $\lambda_2 = 3$, $\lambda_1 = 3$ and $\lambda_2 = 1$, and $\lambda_1 = 6$ and $\lambda_2 = 2$), we have proved that the necessary conditions found are also sufficient for the existence of such GDD's with block size four and three groups, with one possible exception.

Acknowledgments

First and foremost, I need to give thanks to God for the many blessings He has given me. I would also like to thank my family and friends for their never-ending love and support. Last, but definitely not least, I need to thank my advisor, Dr. Chris Rodger, for his patience and guidance through the years. I know that I would never have reached this point without him.

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Chapter 1

Introduction

In this dissertation, a group divisible design $GDD(n, m, k; \lambda_1, \lambda_2)$ is defined to be an ordered pair (V, B), where V is a set of mn elements called symbols, together with a partition of V into m sets of size n, each element of which is called a group, and a collection B of k-subsets of V called blocks, such that each pair of symbols occurring in the same group appears together in precisely λ_1 blocks, while each pair of symbols occurring in different groups appears together in exactly λ_2 blocks. Symbols occuring in the same or different groups are known as first or second associates respectively. A restricted version of this original definition with $\lambda_1 = 0$ is more commonly used as the definition of GDD in the milieu of combinatorial designs; in this setting, a $GDD(n, m, k; 0, \lambda)$ is more commonly known as a (k, λ)-GDD of type n^m. The existence of a $GDD(n, m, 3; \lambda_1, \lambda_2)$ was completely settled by Fu, Rodger, and Sarvate [5, 6]. The most difficult and novel constructions were required when the number of groups, m, was less than k, namely when $m = 2$ [5]. The existence of $GDDs$ when $m < k$ is, in general, a difficult case to solve. Indeed, when $k = 4$, little is known about the existence of such $GDDs$. For example, when $k = 4$ Henson, Hurd, and Sarvate [9, 11, 12] reported existence results for $GDDs$ that are necessary and sufficient for small values of m and n, and were then used to construct some infinite families of $GDDs$. They also considered a restricted version of the problem in which the number of symbols in each block in any group had the same parity as in any other group. Hurd, Mishra, and Sarvate [10] reported some results when $k = 5$ and $m = 6$. Clatworthy's table from 1973 [2] lists eleven such designs (all with replication number at most 10). (See Table 1)

Recently Henson and Sarvate [8] generalized one of these 11 designs, namely $R127$, proving the following theorem.

name	n	m	k	λ_1	λ_2
S1	2	3	4	2	1
S2	2	3	4	4	2
S3	2	3	4	6	3
S4	2	3	4	8	4
S5	2	3	4	10	5
R96	2	3	4	4	5
R104	3	3	4	3	1
R105	3	3	4	6	2
R111	4	3	4	2	3
R117	5	3	4	1	2
R127	8	3	4	2	1

Table 1.1: Clatworthy's Table

Theorem 1.1. *There exists a* $GDD(n, 3, 4; 2, 1)$ *if and only if* $n \equiv 2 \ (mod\ 6)$.

In this dissertation we provide necessary and sufficient conditions for the existence of designs which generalize eight more designs in the Clatworthy table. The construction used for two such designs, namely, $R96$ and $S2$ (see Theorem 3.1 and Theorem 4.1), is a neat method that can also be used to provide another proof of Theorem 1.1. We also provide a similar generalization of $S4$ in the Clatworthy table (see Corollary 4.1.1), easily following from Theorem 4.1. Nestings are introduced and used in the generalizations of the rest of the designs, namely, $S1$, $S5$, $S3$, $R104$, and $R105$ (see Theorem 5.1, Corollary 5.1.1, Theorem 6.1, Theorem 7.1, and Theorem 8.1 respectively). These results were successfully submitted for review in two papers, one published in 2010 by the Journal of Statistical Planning and Inference [13], the other accepted for publication by the Journal of Combinatorial Mathematics and Combinatorial Computing [14].

Chapter 2

Preliminaries

The neat method mentioned in the previous section makes use of holey self orthogonal latin squares(HSOLSs). For the purposes of this dissertation, it is convenient to define

$$H(n) = \begin{cases} \{\{2i, 2i+1\} \times \{2i, 2i+1\} \mid i \in \mathbb{Z}_{n/2}\} & \text{if } n \text{ is even} \\ \{\{2i-1, 2i\} \times \{2i-1, 2i\} \mid i \in (\mathbb{Z}_{\lfloor n/2 \rfloor} \setminus \mathbb{Z}_3)\} \cup (\mathbb{Z}_5 \times \mathbb{Z}_5) & \text{if } n \text{ is odd} \\ & \text{and } n \geq 17 \end{cases}$$

Each element h of $H(n)$ is a subset of $\mathbb{Z}_n \times \mathbb{Z}_n$ called a hole. A HSOLS of order n and of type $2^{n/2}$ if n is even and of type $2^{(n-5)/2}5^1$ if n is odd, on the set of symbols $\mathbb{Z}_n$ with holes in $H(n)$, is an $n \times n$ array, L, in which:

1. Each cell (x, y) in L contains at most one symbol, containing no symbols if and only if $(x, y) \in h$ for some $h \in H(n)$;

2. Each symbol x occurs in row y of L if and only if $(x, y) \notin h$ for all $h \in H(n)$;

3. Each symbol x occurs in column y of L if and only if $(x, y) \notin h$ for all $h \in H(n)$; and

4. For each (x, y) in no hole of $H(n)$, there is exactly one ordered pair (k, l) such that in L, cell (k, l) contains x and cell (l, k) contains y.

Throughout this dissertation we will adopt quasigroup notation by denoting the symbol in cell (i, j) of a HSOLS by $i \circ j$. The set $H(n)$ will also be routinely used. From *Existence of frame SOLS of type a^nb^1* by Xu, Zhang, and Zhu [16] (or Theorem 5.18 in [3] on page 214) we obtain the following result.

Theorem 2.1. *Let $n \geq 6$. There exists a HSOLS of order n and*

1. *type $2^{n/2}$ if n is even; and*

2. *type $2^{(n-5)/2}5^1$ if and only if $n \geq 17$ and n is odd.*

We also use the following design constructed by Brouwer, Schrijver, and Hanani [1] (for a more general setting see also [17] and Theorem 4.6 in [7] on page 256).

Theorem 2.2. *Necessary and sufficient conditions for the existence of a $(4, \lambda)$-GDD of type m^u are:*

1. *$u \geq 4$,*

2. *$\lambda(u-1)m \equiv 0 \pmod 3$, and*

3. *$\lambda u(u-1)m^2 \equiv 0 \pmod{12}$,*

with the exception of $(m, u, \lambda) \in \{(2, 4, 1), (6, 4, 1)\}$, in which case no such GDD exists.

It is also fruitful to describe these designs as graph decompositions, each symbol being represented by a vertex. Let $G(n, 3; \lambda_1, \lambda_2)$ be the graph with vertex set $\mathbb{Z}_n \times \mathbb{Z}_3$ in which (u, i) is joined to (v, j) with

1. λ_1 edges if $i = j$, and

2. λ_2 edges if $i \neq j$.

Then a $GDD(n, 3, 4; \lambda_1, \lambda_2)$ is clearly equivalent to a partition of the edges of $G(n, 3; \lambda_1, \lambda_2)$ into sets of size 6, each of which induces a copy of K_4; for each $i \in \mathbb{Z}_3$, $\mathbb{Z}_n \times \{i\}$ is a group. These two notions will be used interchangeably throughout this dissertation.

To complete these designs we must first define the nesting of a GDD. A nesting of a $GDD(V, P, B)$ with associated graph $G(n, 3; \lambda_1, \lambda_2)$ is defined to be a function of $f : B \to V$

such that $\{\{x, f(b)\} \mid x \in b \in B\} = E(G(n, 3; \lambda_1, \lambda_2))$. More informally, a GDD with block size 3 is said to be nested if a fourth point can be added to each block such that the edges gained from the nesting cover precisely the same edges as the original GDD. So each pair $\{u, v\}$ of vertices occurs together in twice as many blocks of size 4 in the nested design as the number of triples containing $\{u, v\}$ in the original GDD. We will use the following theorem provided by Jin Hua Wang [15].

Theorem 2.3. *There exists a nesting of a $GDD(t, n, 3; \lambda_1 = 0, \lambda_2 = \lambda)$ if and only if $\lambda t(n-1) \equiv 0 \ (mod\ 6)$ and $n \geq 4$.*

Chapter 3

Generalizing Clatworthy Design $R96$

In this chapter we generalize $R96$-designs (see Figure 3.1), completely settling their existence. For completeness, we present a cyclic construction for the smallest $R96$-design.

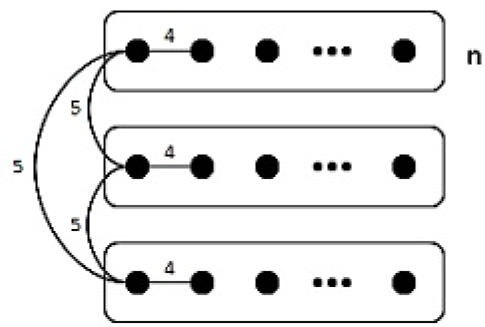

Figure 3.1: R96

Lemma 3.0.1. *There exists a $GDD(2,3,4;4,5)$.*

Proof. $(\mathbb{Z}_6, B)$ with $B = \{\{i, i+1, i+3, i+5\}, \{i, i+1, i+2, i+5\} \mid i \in \mathbb{Z}_6\}$ is such a design, where the groups are $(i, i+3)$ for each $i \in \mathbb{Z}_3$. (See Figure 3.2.)

□

We now use this small $R96$-design to obtain the following result.

Theorem 3.1. *There exists a $GDD(n,3,4;4,5)$ if and only if $n \equiv 2 \pmod 6$.*

Proof. We start by proving the necessity, so suppose there exists a $GDD(n,3,4;4,5)$. Let us begin by looking at the number of blocks. Since each block contains six edges, the number of blocks in any such design is

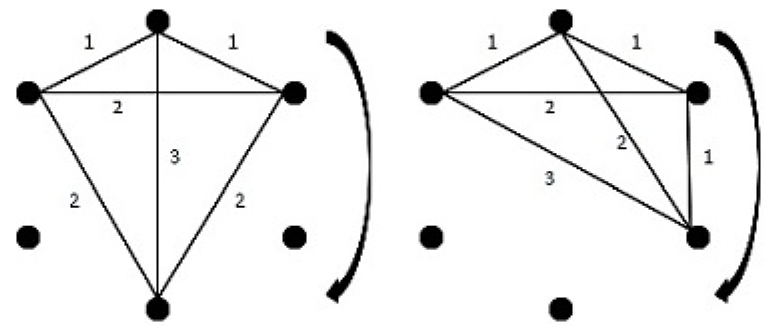

Figure 3.2: R96, $n = 3$

$$b = \frac{|E(G(n,3;4,5))|}{6} = \frac{3(\frac{4n(n-1)}{2} + 5n^2)}{6} = \frac{7n^2 - 2n}{2}.$$

Clearly the number of blocks is an integer, so n must be even.

Each time a vertex (u, i) is used in a block, 3 of its incident edges are used. So the number of blocks containing (u, i) is

$$d_{G(n,3;4,5)}(u, i) = \frac{4(n-1) + 10n}{3} = \frac{14}{3}n - \frac{4}{3},$$

which also must be an integer. So $n \equiv 2 \pmod 3$. So $n \equiv 2 \pmod 6$ is a necessary condition.

Next we prove the sufficiency, so suppose that $n \equiv 2 \pmod 6$. We will show there exists a $GDD(n, 3, 4; 4, 5)$, $(\mathbb{Z}_n \times \mathbb{Z}_3, B)$ with groups $\mathbb{Z}_n \times \{l\}$ for each $l \in \mathbb{Z}_3$. Since Lemma 3.0.1 produces a $GDD(2, 3, 4; 4, 5)$, we can assume that $n \geq 8$. The design will be described as a graph decomposition of the graph $G(n, 3; 4, 5)$.

For each $i \in \mathbb{Z}_{n/2}$ let $B(i)$ be a copy of $R96$ on the vertices in $C(i) = \{2i, 2i+1\} \times \mathbb{Z}_3$, where for each $l \in \mathbb{Z}_3$, $\{2i, 2i+1\} \times \{l\}$ is a group. Since $n \geq 8 > 6$, by Theorem 2.1 we can let S be a HSOLS of order n and of type $2^{n/2}$ on the set of symbols, $\mathbb{Z}_n$ with holes in $H(n)$. By Theorem 2.2 (since $n \geq 8$ and $u = n/2 \geq 4$), for each $l \in \mathbb{Z}_3$ let

$B'(l)$ be a copy of a $(4,2)$-GDD of type $2^{n/2}$ on the vertex set $\mathbb{Z}_n \times \{l\}$ with groups in $H'(l) = \{\{2i, 2i+1\} \times \{l\} \mid i \in \mathbb{Z}_{n/2}\}$. Then define the blocks in the design as follows:

$$\begin{aligned} B &= (\textstyle\bigcup_{i\in\mathbb{Z}_{n/2}} B(i)) \cup (\bigcup_{l\in\mathbb{Z}_3} B'(l)) \\ &\quad \cup \{\{(i,a),(j,a),(i\circ j, a+1),(j\circ i, a+2)\} \mid 0 \le i,j < n, (i,j) \notin h \text{ for each} \\ &\quad h \in H(n),\ a \in \mathbb{Z}_3\}, \end{aligned}$$

reducing the sum in the second component of each vertex modulo 3.

We first count the number blocks we get in the construction to see if it equals $b = (7n^2 - 2n)/2$ (calculated above when proving the necessity).

$$\begin{aligned} |B| &= |(\textstyle\bigcup_{i\in\mathbb{Z}_{n/2}} B(i))| + |(\bigcup_{l\in\mathbb{Z}_3} B'(l))| \\ &\quad +|(\{(i,a),(j,a),(i\circ j, a+1),(j\circ i, a+2) \mid 0 \le i,j < n, (i,j) \notin h \text{ for each} \\ &\quad h \in H(n),\ a \in \mathbb{Z}_3\})| \\ &= (12)(n/2) + 3(\tfrac{n(n-1)-n}{6}) + 3(2(\tbinom{n}{2}) - (n/2))) \\ &= 6n + \tfrac{n(n-1)-n}{2} + (3n)(n-1) - (3n) \\ &= \tfrac{7n^2-2n}{2} \end{aligned}$$

Since $|B| = b$ it suffices to check that each edge occurs in at least the correct number (that is, λ_1 or λ_2) of blocks in B. We consider each edge, $e = \{(x,a),(y,b)\}$, in turn.

1. Suppose e joins two vertices in $C(i)$ for some $i \in \mathbb{Z}_{n/2}$. Then clearly e occurs in $\lambda_1 = 4$ blocks in $B(i)$ if e joins two vertices in the same group and e occurs in $\lambda_2 = 5$ blocks in $B(i)$ if e joins two vertices in different groups, as required.

2. Next suppose that $e = \{(x,a),(y,a)\}$ for some $a \in \mathbb{Z}_3$ and $0 \le x,y < n$ where for each $i \in \mathbb{Z}_{n/2}$, e does not join two vertices in $C(i)$. The $\lambda_1 = 4$ blocks containing the edge $\{(x,a),(y,a)\}$ are as follows:
   $\{(x,a),(y,a),(x\circ y, a+1),(y\circ x, a+2)\}$, $\{(x,a),(y,a),(y\circ x, a+1),(x\circ y, a+2)\}$, and two blocks in $B'(a)$.

3. Finally suppose $e = \{(x,a),(y,b)\}$ where $a,b \in \mathbb{Z}_3$, $a \ne b$ and where for each $i \in \mathbb{Z}_{n/2}$, e does not join two vertices in $C(i)$. We can assume that $b \equiv a+1 \pmod 3$. By

properties (2) and (3) of a HSOLS, there exist unique symbols $z_1, ..., z_4$ in S such that $x \circ z_1 = y$, $z_2 \circ x = y$, $y \circ z_3 = x$, and $z_4 \circ y = x$. By property (4) of a HSOLS, there exists a unique pair z_5, z_6 in S such that $z_5 \circ z_6 = x$ and $z_6 \circ z_5 = y$. Therefore $e = \{(x, a), (y, b = a + 1)\}$ occurs in the following $\lambda_2 = 5$ blocks:
$\{(x, a), (z_1, a), (y = x \circ z_1, a + 1), (z_1 \circ x, a + 2)\}$, $\{(x, a), (z_2, a), (y = z_2 \circ x, a + 1), (x \circ z_2, a+2)\}$, $\{(y, a+1), (z_3, a+1), (z_3 \circ y, a+2), (x = y \circ z_3, a)\}$, $\{(z_4, a+1), (y, a+1), (y \circ z_4, a+2), (x = z_4 \circ y, a)\}$, and $\{(z_5, a+2), (z_6, a+2), (x = z_5 \circ z_6, a), (y = z_6 \circ z_5, a+1)\}$.

Thus, every edge is covered the correct number of times by the blocks, so $n \equiv 2 \pmod 6$ is a sufficient condition for a $GDD(n, 3, 4; 4, 5)$ to exist.

$\square$

Chapter 4

Generalizing Clatworthy Design $S2$

With one exception, the existence of generalized $S2$-designs (see Figure 4.1) is settled in this section. We use a similar construction that is used in the previous section. As before, we first find some small $S2$-designs.

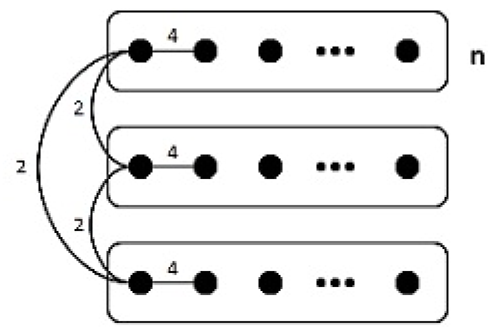

Figure 4.1: S2

Lemma 4.0.2. *There exists a* $GDD(2,3,4;4,2)$*, and a* $GDD(5,3,4;4,2)$.

Proof. To produce a $GDD(2,3,4;4,2)$, let $V = \mathbb{Z}_2 \times \mathbb{Z}_3$ and $B = \{\{(i,a),(i+1,a),(i,a+1),(i+1,a+1)\} \mid i \in \mathbb{Z}_2, a \in \mathbb{Z}_3\}$; for each $l \in \mathbb{Z}_3$, $\mathbb{Z}_2 \times \{l\}$ is a group. (See Figure 4.2.)

When $n = 5$, 3 base blocks are provided that can be rotated "vertically and horizontally" producing 45 blocks as required (see $(*)$ below). Formally, a $GDD(5,3,4;4,2)$ is produced by $(\mathbb{Z}_5 \times \mathbb{Z}_3, B)$, where

$B = \{\{(i,a),(i+1,a),(i+2,a),(i+3,a)\},\{(i,a),(i+2,a),(i+1,a+1),(i+1,a+2)\},\{(i,a),(i+1,a),(i+3,a+1),(i+3,a+2)\} \mid i \in \mathbb{Z}_5, a \in \mathbb{Z}_3\}$,

and where for each $l \in \mathbb{Z}_3$, $\mathbb{Z}_5 \times \{l\}$ is a group. (See Figure 4.3.)

□

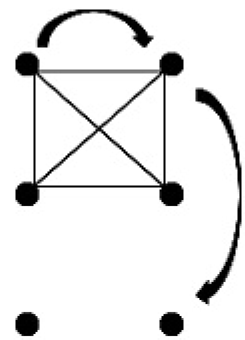

Figure 4.2: S2, $n = 2$

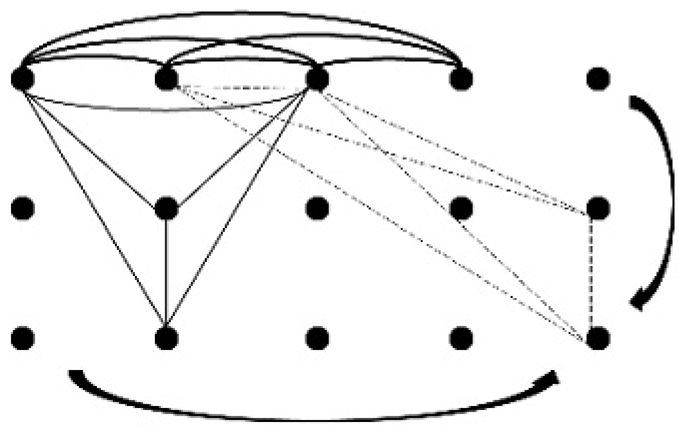

Figure 4.3: S2, $n = 5$

We now use these small $S2$-designs to obtain the following result.

Theorem 4.1. *There exists a* $GDD(n, 3, 4; 4, 2)$ *if and only if* $n \equiv 2$ $(mod\, 3)$*, except possibly if* $n = 11$*.*

Proof. Again, we start by proving the necessity, so suppose there exists a

$GDD(n, 3, 4; 4, 2)$. Each block containing a vertex (u, i) uses 3 of its incident edges. So the number of blocks containing (u, i) is

$$d_{G(n,3;4,2)}(u, i) = \frac{4(n-1) + 4n}{3} = \frac{8n-4}{3},$$

which must be an integer. Thus $n \equiv 2$ (mod 3) is a necessary condition.

To prove the sufficiency we assume that $n \equiv 2 \pmod 3$, $n \neq 11$, and show there exists a $GDD(n, 3, 4; 4, 2)$. We will consider two cases in turn: $n \equiv 2 \pmod 6$ and $n \equiv 5 \pmod 6$.

First, suppose $n \equiv 2 \pmod 6$. Since Lemma 4.0.2 produces a $GDD(2, 3, 4; 4, 2)$, we can assume that $n \geq 8$.

For each $i \in \mathbb{Z}_{n/2}$ let $B(i)$ be a copy of $S2$ on the vertices in $C(i) = \{2i, 2i+1\} \times \mathbb{Z}_3$, where for each $l \in \mathbb{Z}_3$, $\{2i, 2i+1\} \times \{l\}$ is a group. Since $n \geq 8 > 6$, by Theorem 2.1 we can let S be a HSOLS of order n and of type $2^{n/2}$ on the set of symbols, $\mathbb{Z}_n$, with holes in $H(n)$. By Theorem 2.2 (since $n \geq 8$ and $u = n/2 \geq 4$), for each $l \in \mathbb{Z}_3$ let $B'(l)$ be a copy of a $(4, 2)$-GDD of type $2^{n/2}$ on the vertex set $\mathbb{Z}_n \times \{l\}$ with groups in $H'(l) = \{\{2i, 2i+1\} \times \{l\} \mid i \in \mathbb{Z}_{n/2}\}$. Then define the blocks in the design as follows:

$$\begin{aligned} B \;=\; & (\textstyle\bigcup_{i \in \mathbb{Z}_{n/2}} B(i)) \cup (\bigcup_{l \in \mathbb{Z}_3} B'(l)) \\ & \cup \{\{(i, a), (j, a), (i \circ j, a+1), (j \circ i, a+1)\} \mid 0 \leq i < j < n, (i, j) \notin h \text{ for} \\ & \text{each } h \in H(n),\, a \in \mathbb{Z}_3\}, \end{aligned}$$

reducing the sum in the second component of each vertex modulo 3.

First we check to see the right number of blocks has been defined. Since each block contains six edges, the number of blocks in any such design is

$$b = \frac{|E(G(n, 3; 4, 2))|}{6} = \frac{3(\frac{4n(n-1)}{2} + 2n^2)}{6} = 2n^2 - n. \qquad (*)$$

We now count the number of blocks we get in our construction to see if it equals $b = 2n^2 - n$.

$$
\begin{aligned}
|B| &= |(\bigcup_{i\in\mathbb{Z}_{n/2}} B(i))| + |(\bigcup_{l\in\mathbb{Z}_3} B'(l))| \\
&\quad +|(\{(i,a),(j,a),(i\circ j,a+1),(j\circ i,a+1) \mid 0\le i<j<n, (i,j)\notin h \text{ for} \\
&\quad \text{each } h\in H(n),\ a\in\mathbb{Z}_3\})| \\
&= (6)(n/2) + 3(\tfrac{(\binom{n}{2}-\frac{n}{2})2}{6}) + 3(\binom{n}{2}-(n/2)) \\
&= 3n + \tfrac{n(n-1)-n}{2} + 3(\tfrac{n(n-1)-n}{2}) \\
&= 3n + 4(\tfrac{n(n-1)-n}{2}) \\
&= 3n + 2n^2 - 4n \\
&= 2n^2 - n
\end{aligned}
$$

Since $|B| = b$ it suffices to check that each edge occurs in at least the correct number (that is, λ_1 or λ_2) of blocks in B. We consider each edge, e, in turn.

1. Suppose e joins two vertices in $C(i)$ for some $i \in \mathbb{Z}_{n/2}$. Then clearly e occurs in $\lambda_1 = 4$ blocks in $B(i)$ if e joins two vertices in the same group and e occurs in $\lambda_2 = 2$ blocks in $B(i)$ if e joins two vertices in different groups, as required.

2. Next suppose that $e = \{(x,a),(y,a)\}$ for some $a \in \mathbb{Z}_3$ and $0 \le x < y < n$ where for each $i \in \mathbb{Z}_{n/2}$, e does not join two vertices in $C(i)$. By property (4) of a HSOLS there exists a unique pair of symbols z_1, z_2 in S such that $z_1 \circ z_2 = x$ and $z_2 \circ z_1 = y$. The $\lambda_1 = 4$ blocks containing the edge $\{(x,a),(y,a)\}$ are as follows:
$\{(x,a),(y,a),(x\circ y,a+1),(y\circ x,a+1)\}$, $\{(z_1,a+2),(z_2,a+2),(x=z_1\circ z_2,a),(y=z_2\circ z_1,a)\}$, and in two blocks in $B'(a)$.

3. Finally suppose $e = \{(x,a),(y,b)\}$ where $a,b \in \mathbb{Z}_3$, $a \ne b$ and where for each $i \in \mathbb{Z}_{n/2}$, e does not join two vertices in $C(i)$, say $b \equiv a+1 \pmod 3$. By properties (2) and (3) of a HSOLS, there exist unique symbols z_3, z_4 in S such that $x \circ z_3 = y$ and $z_4 \circ x = y$. Then edge $\{(x,a),(y,b=a+1)\}$ occurs in the following $\lambda_2 = 2$ blocks:
$\{(x,a),(z_3,a),(y=x\circ z_3,a+1),(z_3\circ x,a+1)\}$ and $\{(x,a),(z_4,a),(x\circ z_4,a+1),(y=z_4\circ x,a+1)\}$.

Thus, $n \equiv 2 \pmod 6$ is a sufficient condition for a $GDD(n,3,4;4,2)$ to exist.

Now suppose $n \equiv 5 \pmod 6$. Again, the smallest case when $n = 5$ is constructed in Lemma 4.0.2, and the case $n = 11$ is excluded in the theorem, so we can assume that $n \geq 17$.

For each $i \in (\mathbb{Z}_{\lfloor n/2 \rfloor} \setminus \mathbb{Z}_3)$ let $B(i)$ be a copy of $S2$ on the vertices in $C(i) = \{2i-1, 2i\} \times \mathbb{Z}_3$, where for each $l \in \mathbb{Z}_3$, $\{2i-1, 2i\} \times \{l\}$ is a group. Let $B(2)$ be the $GDD(5, 3, 4; 4, 2)$ from Lemma 4.0.2. Since $n \geq 17 > 6$, by Theorem 2.1 we can let S be a HSOLS of order n and of type $2^{(n-5)/2}5^1$ on the set of symbols $\mathbb{Z}_n$, with holes in $H(n)$. By Theorem 2.2 (since $n \geq 17$ and $u = (n-5)/2 \geq 6 > 4$), for each $l \in \mathbb{Z}_3$ let $B'(l)$ be a copy of a $(4, 2)$-GDD of type $2^{(n-5)/2}5^1$ on the vertex set $\mathbb{Z}_n \times \{l\}$ with groups in $H'(l) = \{\{\{2i-1, 2i\} \times \{l\} \mid i \in \mathbb{Z}_{\lfloor n/2 \rfloor} \setminus \mathbb{Z}_3\} \cup (\mathbb{Z}_5 \times \{l\})\}$. Then define the blocks in the design as follows:

$$\begin{aligned} B &= (\textstyle\bigcup_{i \in (\mathbb{Z}_{\lfloor n/2 \rfloor} \setminus \mathbb{Z}_2)} B(i)) \cup (\bigcup_{l \in \mathbb{Z}_3} B'(l)) \\ &\quad \cup \{\{(i, a), (j, a), (i \circ j, a+1), (j \circ i, a+1)\} \mid 0 \leq i < j < n, (i, j) \notin h \text{ for} \\ &\quad \text{each } h \in H(n),\ a \in \mathbb{Z}_3\}, \end{aligned}$$

reducing the sum in the second component of each vertex modulo 3.

We first count the number of blocks we get in our construction to see if it equals $b = 2n^2 - n$ (see $(*)$).

$$\begin{aligned} |B| &= |(\textstyle\bigcup_{i \in (\mathbb{Z}_{\lfloor n/2 \rfloor} \setminus \mathbb{Z}_2)} B(i))| + |(\bigcup_{l \in \mathbb{Z}_3} B'(l))| \\ &\quad + |(\{(i, a), (j, a), (i \circ j, a+1), (j \circ i, a+1) \mid 0 \leq i < j < n, (i, j) \notin h \text{ for} \\ &\quad \text{each } h \in H(n),\ a \in \mathbb{Z}_3\})| \\ &= 45 + (6)(\tfrac{n-5}{2}) + 3(\tfrac{(\binom{n}{2} - \frac{n-5}{2} - 10)2}{6}) + 3(\binom{n}{2} - \tfrac{n-5}{2} - 10) \\ &= 45 + 3n - 15 + (\tfrac{n(n-1) - n - 15}{2}) + 3(\tfrac{n(n-1) - n - 15}{2}) \\ &= 30 + 3n + 4(\tfrac{n^2 - 2n - 15}{2}) \\ &= 30 + 3n + 2n^2 - 4n - 30 \\ &= 2n^2 - n \end{aligned}$$

Since $|B| = b$ it suffices to check that each edge occurs in at least the correct number (that is, λ_1 or λ_2) of blocks in B; the proof is essentially identical to the previous case, so is left to the reader.

Thus, $n \equiv 5 \pmod 6$ is a sufficient condition for a $GDD(n, 3, 4; 4, 2)$ to exist.

□

4.1 A Corollary - Generalizing $S4$

It turns out that the necessary conditions for the existence of a $GDD(n, 3, 4; 8, 4)$, generalizing the Clatworthy design $S4$, are the same as for the existence of a

$GDD(n, 3, 4; 4, 2)$ (i.e. $S2$-design). So we immediately obtain the following corollary.

Corollary 4.1.1. *There exists a $GDD(n, 3, 4; 8, 4)$ if and only if $n \equiv 2 \pmod 3$, except possibly if $n = 11$.*

Proof. The necessity follows since the degree of each vertex, namely $(8(n-1) + 8n)/3$ must be an integer. The sufficiency follows by taking two copies of the $S2$-design constructed in Theorem 4.1. □

Chapter 5

Generalizing Clatworthy Design $S1$

We first find a small $S1$-design.

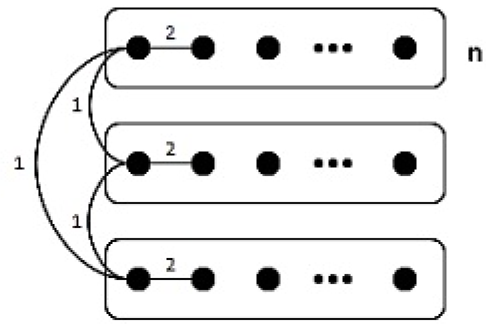

Figure 5.1: S1

Lemma 5.0.2. *There exists a* $GDD(2, 3, 4; 2, 1)$.

Proof. To produce a $GDD(2, 3, 4; 2, 1)$, let $V = \mathbb{Z}_2 \times \mathbb{Z}_3$ and $B = \{\{(0, a), (1, a), (0, a + 1), (1, a + 1)\} \mid a \in \mathbb{Z}_3\}$; for each $l \in \mathbb{Z}_3$, $\mathbb{Z}_2 \times \{l\}$ is a group. (See Figure 5.2.)

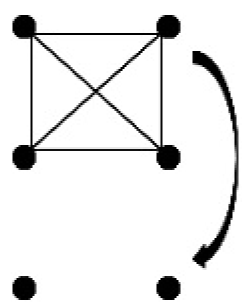

Figure 5.2: S1, $n = 2$

□

Theorem 5.1. *There exists a $GDD(n, 3, 4; 2, 1)$ if and only if $n \equiv 2$ (mod 6).*

Proof. We start by proving the necessity, so suppose there exists a

$GDD(n, 3, 4; 2, 1)$. Since each block contains six edges, the number of blocks in any such design is

$$b = \frac{|E(G(n,3;2,1))|}{6} = \frac{3(\frac{2n(n-1)}{2} + 3n^2)}{6} = n^2 - \frac{n}{2}.$$

Clearly the number of blocks is an integer, so n must be even.

For each block, each vertex contains 3 of its incident edges. So the number of blocks containing each vertex v is

$$d_{G(n,3;2,1)}(v) = \frac{2(n-1) + 2n}{3} = \frac{4n-2}{3},$$

which must be an integer. Thus $n \equiv 2$ (mod 3). Since n must also be even, $n \equiv 2$ (mod 6) is a necessary condition.

Next we prove the sufficiency, so suppose that $n \equiv 2$ (mod 6). We will show there exists a $GDD(n, 3, 4; 2, 1)$, $(\mathbb{Z}_n \times \mathbb{Z}_3, B)$ with groups $\mathbb{Z}_n \times \{l\}$ for each $l \in \mathbb{Z}_3$. Since Lemma 5.0.2 produces a $GDD(2, 3, 4; 2, 1)$, we can assume that $n \geq 8$. The design will be described as a graph decomposition of the graph $G(n, 3; 2, 1)$.

For each $i \in \mathbb{Z}_{n/2}$, let $B(i)$ be a copy of $S2$ on the vertices in $C(i) = \{2i, 2i+1\} \times \mathbb{Z}_3$, where for each $l \in \mathbb{Z}_3$, $\{2i, 2i+1\} \times \{l\}$ is a group. By Theorem 2.3, there exists a $(3,1)$-GDD, $(Z_n, \{\{2i, 2i+1\} \mid i \in \mathbb{Z}_{n/2}\}, B_1)$, that has nesting f of type $2^{n/2}$. Let $B_1(l) = \{\{(x,l),(y,l),(z,l),(f(b),l+1)\}, \{(x,l+1),(y,l+1),(z,l+1),(f(b),l)\} \mid l \in Z_3, \{x,y,z\} \in B_1\}$, reducing the sums in the second coordinate of each vertex modulo 3. Then define the blocks in the design as follows:

$$B = (\bigcup_{i \in \mathbb{Z}_{n/2}} B(i)) \cup (\bigcup_{l \in \mathbb{Z}_3} B_1(l))$$

□

We first count the number of blocks we get in the construction to see if it equals $b = n^2 - \frac{n}{2}$ (calculated above when proving the necessity).

$$\begin{aligned}
|B| &= |(\bigcup_{i\in\mathbb{Z}_{n/2}} B(i))| + |(\bigcup_{l\in\mathbb{Z}_3} B_1(l))| \\
&= (3)(n/2) + (2)(3)((\tfrac{n(n-1)-n}{2})/3) \\
&= \tfrac{3n}{2} + \tfrac{2(n(n-1)-n)}{2} \\
&= \tfrac{3n+2n^2-2n-2n}{2} \\
&= \tfrac{2n^2-n}{2} \\
&= n^2 - \tfrac{n}{2}
\end{aligned}$$

Since $|B| = b$ it suffices to check that each edge occurs in at least the correct number (that is, λ_1 or λ_2) of blocks in B. We consider each edge, $e = \{(x,a),(y,b)\}$, in turn.

1. Suppose e joins two vertices in $C(i)$ for some $i \in \mathbb{Z}_{n/2}$. Then clearly e occurs in $\lambda_1 = 2$ blocks in $B(i)$ if e joins two vertices in the same group and e occurs in $\lambda_2 = 1$ block in $B(i)$ if e joins two vertices in different groups, as required.

2. Next suppose that $e = \{(x,a),(y,a)\}$ for some $a \in \mathbb{Z}_3$ and $0 \le x, y < n$ where for each $i \in \mathbb{Z}_{n/2}$, e does not join two vertices in $C(i)$. Let $\{x, y, z_1\}$ be the triple in $B_1(a)$ that contains $\{x, y\}$, and suppose $f(\{x, y, z_1\}) = z_2$ is the vertex added to the triple by the nesting. Then the $\lambda_1 = 2$ blocks containing the edge $\{(x,a),(y,a)\}$ are as follows: $\{(x,a),(y,a),(z_1,a),(z_2,a+1)\}$, $\{(x,a),(y,a),(z_1,a),(z_2,a+2)\}$.

3. Finally suppose $e = \{(x,a),(y,b)\}$ where $a, b \in \mathbb{Z}_3$, $a \neq b$ and where for each $i \in \mathbb{Z}_{n/2}$, e does not join two vertices in $C(i)$. We can assume that $b \equiv a+1 \pmod 3$. Since $\{x, y\} \in G(n, 3; 2, 1)$, exactly one of the following occurs: either there exists a triple $t_1 = \{x, z_3, z_4\} \in B_1(a)$ such that $f(t_1) = y$ or there exists a triple $t_2 = \{y, z_3, z_4\} \in B_1(a)$ such that $f(t_2) = x$. Therefore $e = \{(x,a),(y,b = a+1)\}$ occurs in $\lambda_2 = 1$ of the following blocks:
$\{(x,a),(z_3,a),(z_4,a),(y,a+1)\}$ or $\{(x,a),(y,a+1),(z_3,a+1),(z_4,a+1)\}$.

Thus, every edge is covered the correct number of times by the blocks, so $n \equiv 2 \pmod 6$ is a sufficient condition for a $GDD(n, 3, 4; 2, 1)$ to exist.

5.1 A Corollary - Generalizing $S5$

It turns out that the necessary conditions for the existence of a $GDD(n, 3, 4; 10, 5)$, generalizing the Clatworthy design $S5$, are the same as for the existence of a

$GDD(n, 3, 4; 2, 1)$ (i.e. $S1$-design). So we immediately obtain the following corollary.

Corollary 5.1.1. *There exists a $GDD(n, 3, 4; 10, 5)$ if and only if $n \equiv 2$ (mod 6).*

Proof. The necessity follows since the degree of each vertex, namely $(20n - 10)/3$ must be an integer. The sufficiency follows by taking five copies of the $S1$-design constructed in Theorem 5.1. $\square$

Chapter 6

Generalizing Clatworthy Design $S3$

To complete this design we will use the nesting described in the Preliminaries section, as well as Theorem 2.3 provided by Jin Hua Wang [15].

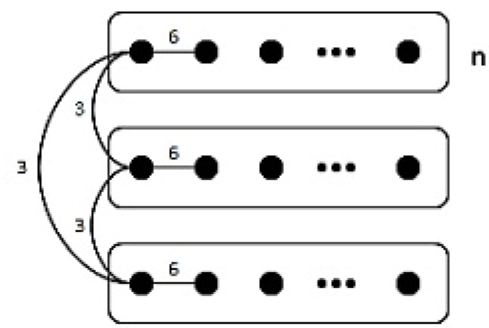

Figure 6.1: S3

We first find a small $S3$-design.

Lemma 6.0.2. *There exists a $GDD(2,3,4;6,3)$; and a $GDD(4,3,4;6,3)$.*

Proof. To produce a $GDD(2,3,4;6,3)$, let $V = \mathbb{Z}_2 \times \mathbb{Z}_3$ and each $l \in \mathbb{Z}_3$, $\mathbb{Z}_2 \times \{l\}$ is a group, and take three copies of each block produced in the small $S1$-design described in the proof of Lemma 5.0.2. (See Figure 6.2.)

When $n = 4$, four base blocks are provided that can be rotated "vertically and horizontally" producing 42 blocks as required. Formally, a $GDD(4,3,4;6,3)$ is produced by $(\mathbb{Z}_4 \times \mathbb{Z}_3, B)$, where

$B = \{\{(i,a),(i+1,a),(i+2,a),(i,a+1)\},\{(i,a),(i+1,a),(i+1,a+1),(i+2,a+1)\},\{(i,a),(i+2,a),(i,a+1),(i+3,a+2)\} \mid i \in \mathbb{Z}_4, a \in \mathbb{Z}_3\} \cup \{\{(i,a),(i+1,a),(i+$

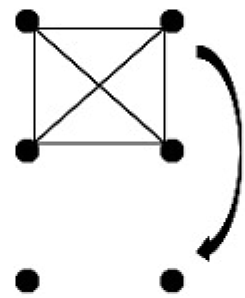

Figure 6.2: S3, $n = 2$ (Take 3 copies of each K_4)

$2, a), (i+3, a)\} \mid i \in \mathbb{Z}_2, a \in \mathbb{Z}_3\}$, and where for each $l \in \mathbb{Z}_3$, $\mathbb{Z}_4 \times \{l\}$ is a group. (See Figure 6.3.)

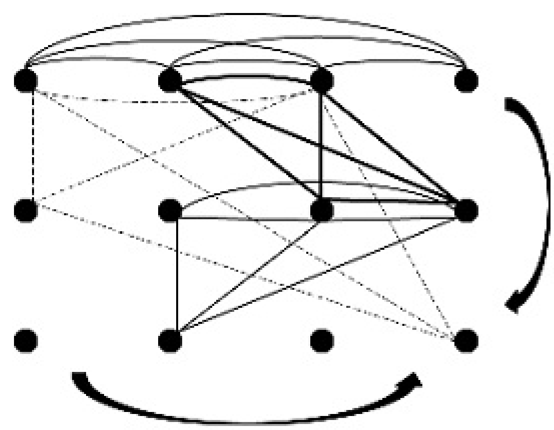

Figure 6.3: S3, $n = 4$ (The top K_4 only rotates halfway horizontally on each level)

□

Theorem 6.1. *There exists a $GDD(n, 3, 4; 6, 3)$ if and only if n is even, except possibly if $n = 6$.*

Proof. We start by proving the necessity, so suppose there exists a

$GDD(n, 3, 4; 6, 3)$. Since each block contains six edges, the number of blocks in any such design is

$$b = \frac{|E(G(n, 3; 6, 3))|}{6} = \frac{3(\frac{6n(n-1)}{2}) + 3(3n^2)}{6} = 3n^2 - \frac{3n}{2}.$$

Clearly the number of blocks is an integer, so n must be even.

For each block, each vertex contains 3 of its incident edges. So the number of blocks containing each vertex v is

$$d_{G(n,3;6,3)}(v) = \frac{6(n-1)+2(3n)}{3} = 4n-2,$$

which makes no restrictions on n. Thus n is even is a necessary condition.

To prove the sufficiency we assume that n is even, $n \neq 6$, and show there exists a $GDD(n,3,4;6,3)$. We will consider three cases in turn: $n \equiv 2 \pmod 6$, $n \equiv 4 \pmod 6$, and $n \equiv 0 \pmod 6$.

First suppose $n \equiv 2 \pmod 6$. Since $n \equiv 2 \pmod 6$ is the necessary condition for the existence of a $GDD(n,3,4;2,1)$ (i.e. $S1$-design), we immediately obtain the following corollary.

Corollary 6.0.3. *There exists a $GDD(n,3,4;6,3)$ if and only if $n \equiv 2$ (mod 6)*

Proof. The necessity follows since the number of blocks, namely $3n^2 - \frac{3n}{2}$ must be an integer. The sufficiency follows by taking three copies of the $S1$-design constructed in Theorem 5.1. □

Now suppose $n \equiv 4 \pmod 6$. Since Lemma 6.0.2 produces a $GDD(4,3,4;6,3)$, we can assume that $n \geq 10$. The design will be described as a graph decomposition of the graph $G(n,3;6,3)$.

For each $i \in \mathbb{Z}_{n/2}$, let $B(i)$ be a copy of $S3$ on the vertices in $C(i) = \{2i, 2i+1\} \times \mathbb{Z}_3$, where for each $l \in \mathbb{Z}_3$, $\{2i, 2i+1\} \times \{l\}$ is a group. By Theorem 2.3, there exists a $(3,3)$-GDD, $(Z_n, \{\{2i, 2i+1\} \mid i \in \mathbb{Z}_{n/2}\}, B_1)$, that has nesting f of type $2^{n/2}$. Let $B_1(l) = \{\{(x,l),(y,l),(z,l),(f(b),l+1)\}, \{(x,l+1),(y,l+1),(z,l+1),(f(b),l)\} \mid l \in Z_3, \{x,y,z\} \in B_1\}$, reducing the sums in the second coordinate of each vertex modulo 3. Then define the blocks in the design as follows:

$$B = (\bigcup_{i\in\mathbb{Z}_{n/2}} B(i)) \cup (\bigcup_{l\in\mathbb{Z}_3} B_1(l)) \qquad \square$$

We first count the number of blocks we get in the construction to see if it equals $b = 3n^2 - \frac{3n}{2}$ (calculated above when proving the necessity).

$$\begin{aligned}
|B| &= |(\bigcup_{i\in\mathbb{Z}_{n/2}} B(i))| + |(\bigcup_{l\in\mathbb{Z}_3} B_1(l))| \\
&= (9)(n/2) + (3)(2)(3)((\tfrac{n(n-1)-n}{2})/3) \\
&= \tfrac{9n}{2} + \tfrac{6(n(n-1)-n)}{2} \\
&= \tfrac{9n+6n^2-6n-6n}{2} \\
&= \tfrac{6n^2-3n}{2} \\
&= 3n^2 - \tfrac{3n}{2}
\end{aligned}$$

Since $|B| = b$ it suffices to check that each edge occurs in at least the correct number (that is, λ_1 or λ_2) of blocks in B. We consider each edge, $e = \{(x,a),(y,b)\}$, in turn.

1. Suppose e joins two vertices in $C(i)$ for some $i \in \mathbb{Z}_{n/2}$. Then clearly e occurs in $\lambda_1 = 6$ blocks in $B(i)$ if e joins two vertices in the same group and e occurs in $\lambda_2 = 3$ block in $B(i)$ if e joins two vertices in different groups, as required.

2. Next suppose that $e = \{(x,a),(y,a)\}$ for some $a \in \mathbb{Z}_3$ and $0 \le x, y < n$ where for each $i \in \mathbb{Z}_{n/2}$, e does not join two vertices in $C(i)$. Let $\{x,y,z_1\}, \{x,y,z_3\}$, and $\{x,y,z_5\}$ be the triples in $B_1(a)$ that contain $\{x,y\}$, and suppose $f(\{x,y,z_1\}) = z_2$, $f(\{x,y,z_3\}) = z_4$, and $f(\{x,y,z_5\}) = z_6$ are the vertices added to the triples by the nesting. Then the $\lambda_1 = 6$ blocks containing the edge $\{(x,a),(y,a)\}$ are as follows: $\{(x,a),(y,a),(z_1,a),(z_2,a+1)\}$, $\{(x,a),(y,a),(z_1,a),(z_2,a+2)\}$, $\{(x,a),(y,a),$ $(z_3,a),(z_4,a+1)\}$, $\{(x,a),(y,a),(z_3,a),(z_4,a+2)\}$, $\{(x,a),(y,a),(z_5,a),(z_6,a+1)$, and $\{(x,a),(y,a),(z_5,a),(z_6,a+2)$.

3. Finally suppose $e = \{(x,a),(y,b)\}$ where $a, b \in \mathbb{Z}_3$, $a \ne b$ and where for each $i \in \mathbb{Z}_{n/2}$, e does not join two vertices in $C(i)$. We can assume that $b \equiv a+1 \pmod 3$. Since $\{x,y\} \in G(n,3;6,3)$, exactly one of the following occurs for each of the $\lambda_2 = 3$ $\{x,y\}$

edges: either there exists a triple $t_1 = \{x, z_7, z_8\} \in B_1(a)$ such that $f(t_1) = y$ or there exists a triple $t_2 = \{y, z_7, z_8\} \in B_1(a)$ such that $f(t_2) = x$. The same argument can be made for the other two $\{x, y\}$ edges using $\{z_9, z_{10}, z_{11}, z_{12}\}$. Therefore $e = \{(x, a), (y, b = a + 1)\}$ occurs in $\lambda_2 = 3$ of the following blocks: $\{(x, a), (z_7, a), (z_8, a), (y, a+1)\}$ or $\{(x, a), (y, a+1), (z_7, a+1), (z_8, a+1)\}$, $\{(x, a), (z_9, a), (z_{10}, a), (y, a+1)\}$ or $\{(x, a), (y, a+1), (z_9, a+1), (z_{10}, a+1)\}$, and $\{(x, a), (z_{11}, a), (z_{12}, a), (y, a + 1)\}$ or $\{(x, a), (y, a + 1), (z_{11}, a + 1), (z_{12}, a + 1)\}$.

Thus, $n \equiv 4 \pmod 6$ is a sufficient condition for a $GDD(n, 3, 4; 6, 3)$ to exist.

Now suppose $n \equiv 0 \pmod 6$. Since $n = 6$ is the possible exception, we can assume that $n \geq 12$. Similarly to the $n \equiv 4 \pmod 6$ case, there exists a $(3, 3)$-GDD, $(Z_n, \{\{2i, 2i + 1\} \mid i \in \mathbb{Z}_{n/2}\}, B_1)$, that has nesting f of type $2^{n/2}$ by Theorem 2.3. Therefore, the arguments for the $n \equiv 0 \pmod 6$ case are essentially the same for the $n \equiv 4 \pmod 6$ case.

Thus, $n \equiv 0 \pmod 6$ is a sufficient condition for a $GDD(n, 3, 4; 6, 3)$ to exist.

Chapter 7

Generalizing Clatworthy Design $R104$

With two possible exceptions, the existence of a generalized $R104$-design (see Figure 7.1) is settled in this section. We use a similar construction that is used in the previous section. As before, we first find some small $R104$-designs.

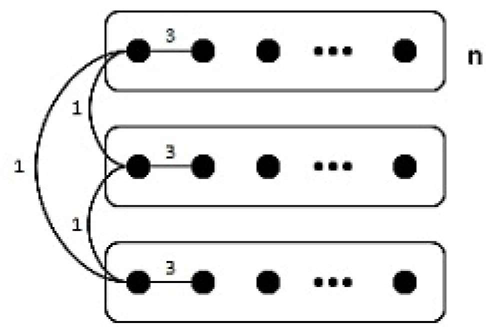

Figure 7.1: R104

Lemma 7.0.4. *There exists a* $GDD(3,3,4;3,1)$*, and a* $GDD(12,3,4;3,1)$*.*

Proof. To produce a $GDD(3,3,4;3,1)$, let $V = \mathbb{Z}_3 \times \mathbb{Z}_3$ and $B = \{\{(i,a),(i+1,a),(i+2,a),(i,a+1)\} \mid i \in \mathbb{Z}_3, a \in \mathbb{Z}_3\}$; for each $l \in \mathbb{Z}_3$, $\mathbb{Z}_3 \times \{l\}$ is a group. (See Figure 7.2.)

When $n = 12$, 4 base blocks are provided that can be rotated producing 144 blocks towards the required number of blocks. Let the 36 vertices be labeled $0, 1, 2, ..., 35$ and partitioned into three groups such that vertices with labels that are 0 (mod 3) are one group, 1 (mod 3) are the second group, and 2 (mod 3) are the third group. Then consider the following four blocks: $\{0,2,6,9\}, \{0,9,10,22\}, \{0,6,18,21\}$, and $\{0,8,20,25\}$. (See Figure 7.3; the first block is depicted.) When these four blocks are rotated they cover the edges of difference

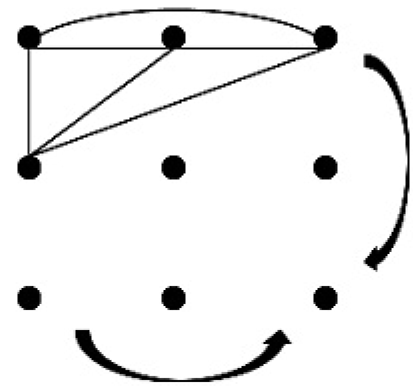

Figure 7.2: R104, $n = 3$

$3, 6, 9, 15, 18$ twice, difference 12 three times, and differences $1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17$ once. In other words, we have covered the mixed edges the required one time and the pure edges two of the required three times, with the exception of the pure edges of difference 12 which are completely covered. Finally, to cover the pure edges that are left we use Theorem 2.2; this exists since $n = 12$ and $u = n/3 \geq 4$, to put a $(4, 1) - GDD$ of type $3^{n/3}$ on each level with group G_i. The edges of difference 12 make 4 triangles so the vertices in each triangle form a group in the $(4, 1) - GDD$ of type 3^4. On each level, the $(4, 1) - GDD$ of type $3^{n/3}$ produces 9 blocks for a total of 27 blocks. Thus, the $144 + 27 = 171$ which is the required number of blocks (see $(*)$ below). □

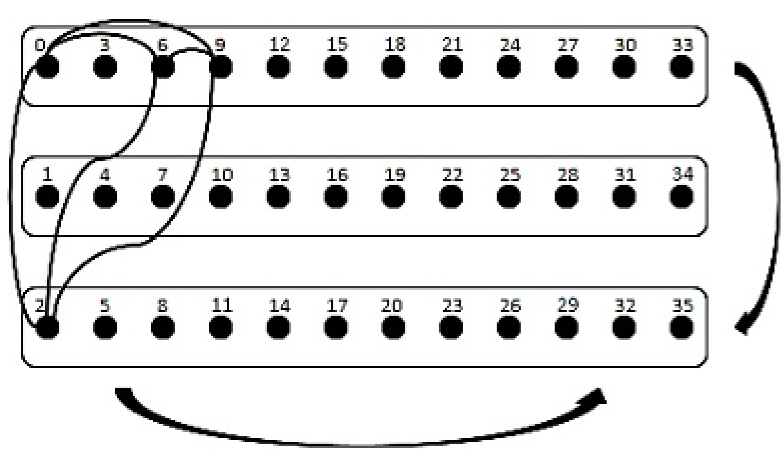

Figure 7.3: R104, $n = 12$; the first block of four base blocks is depicted

Theorem 7.1. *There exists a $GDD(n,3,4;3,1)$ if and only if $n \equiv 0, 3 \pmod{12}$, except possibly if $n = 24, 36$.*

Proof. We start by proving the necessity, so suppose there exists a

$GDD(n,3,4;3,1)$. Since each block contains six edges, the number of blocks in any such design is

$$b = \frac{|E(G(n,3;3,1))|}{6} = \frac{3(\frac{3n(n-1)}{2}) + 3(n^2)}{6} = \frac{5n^2 - 3n}{4}. \qquad (*)$$

Clearly the number of blocks is an integer, so $n \equiv 0, 3 \pmod{12}$.

For each block, each vertex contains 3 of its incident edges. So the number of blocks containing each vertex v is

$$d_{G(n,3;3,1)}(v) = \frac{3(n-1) + 2(n)}{3} = \frac{5}{3}n - 3,$$

which means $n \equiv 0 \pmod 3$. Thus $n \equiv 0, 3 \pmod{12}$ is a necessary condition.

To prove the sufficiency we assume that $n \equiv 0, 3 \pmod{12}$, $n \neq 24, 36$, and show there exists a $GDD(n,3,4;3,1)$. We will consider two cases in turn: $n \equiv 3 \pmod{12}$ and $n \equiv 0 \pmod{12}$, $n \geq 48$.

First suppose $n \equiv 3 \pmod{12}$. Since Lemma 7.0.4 produces a $GDD(3,3,4;3,1)$, we can assume that $n \geq 15$. The design will be described as a graph decomposition of the graph $G(n,3;3,1)$.

For each $i \in \mathbb{Z}_{n/3}$, let $B(i)$ be a copy of $R104$ on the vertices in $C(i) = \{3i, 3i+1, 3i+2\} \times \mathbb{Z}_3$, where for each $l \in \mathbb{Z}_3$, $\{3i, 3i+1, 3i+2\} \times \{l\}$ is a group. By Theorem 2.3, there exists a (3,1)-*GDD*, $(Z_n, \{\{3i, 3i+1, 3i+2\} \mid i \in \mathbb{Z}_{n/3}\}, B_1)$, that has nesting f of type $3^{n/3}$. Let $B_1(l) = \{\{(x,l),(y,l),(z,l),(f(b),l+1)\}, \{(x,l+1),(y,l+1),(z,l+1),(f(b),l)\} \mid l \in Z_3, \{x,y,z\} \in B_1\}$, reducing the sums in the second coordinate of each vertex modulo 3. By Theorem 2.2 (since $n \geq 15$ and $u = n/3 \geq 4$), for each $l \in Z_3$ let

$B'(l)$ be a copy of a $(4,1)-GDD$ of type $3^{n/3}$ on the vertex set $Z_n \times \{l\}$ with groups in $\{\{3i, 3i+1, 3i+2\} \times \{l\} \mid i \in Z_{n/3}\}$. Then define the blocks in the design as follows:

$$B \quad = \quad (\bigcup_{i\in\mathbb{Z}_3} B(i)) \cup (\bigcup_{l\in\mathbb{Z}_3} B_1(l)) \cup (\bigcup_{l\in\mathbb{Z}_3} B'(l))$$

We first count the number of blocks we get in the construction to see if it equals $b = \frac{5n^2-3n}{4}$ (calculated above when proving the necessity).

$$\begin{aligned}
|B| &= |(\bigcup_{i\in\mathbb{Z}_3} B(i))| + |(\bigcup_{l\in\mathbb{Z}_3} B_1(l))| + |(\bigcup_{l\in\mathbb{Z}_3} B'(l))| \\
&= (9)(n/3) + (2)(3)((\tfrac{n(n-1)-2n}{2})/3) + 3(\tfrac{(\binom{n}{2}-n)}{6}) \\
&= \tfrac{9n}{3} + n^2 - n - 2n + \tfrac{\frac{n(n-1)}{2}-n}{2} \\
&= 3n + n^2 - 3n + \tfrac{n^2-n}{4} - \tfrac{n}{2} \\
&= \tfrac{4n^2+n^2-n-2n}{4} \\
&= \tfrac{5n^2-3n}{4}
\end{aligned}$$

Since $|B| = b$ it suffices to check that each edge occurs in at least the correct number (that is, λ_3 or λ_1) of blocks in B. We consider each edge, $e = \{(x,a),(y,b)\}$, in turn.

1. Suppose e joins two vertices in $C(i)$ for some $i \in \mathbb{Z}_{n/3}$. Then clearly e occurs in $\lambda_1 = 3$ blocks in $B(i)$ if e joins two vertices in the same group and e occurs in $\lambda_2 = 1$ block in $B(i)$ if e joins two vertices in different groups, as required.

2. Next suppose that $e = \{(x,a),(y,a)\}$ for some $a \in \mathbb{Z}_3$ and $0 \le x, y < n$ where for each $i \in \mathbb{Z}_{n/2}$, e does not join two vertices in $C(i)$. Let $\{x, y, z_1\}$ be the triple in $B_1(a)$ that contains $\{x, y\}$, and suppose $f(\{x, y, z_1\}) = z_2$ is the vertex added to the triple by the nesting. Then the $\lambda_1 = 3$ blocks containing the edge $\{(x,a),(y,a)\}$ are as follows: $\{(x,a),(y,a),(z_1,a),(z_2,a+1)\}$, $\{(x,a),(y,a),(z_1,a),(z_2,a+2)\}$, and in one block in $B'(a)$.

3. Finally suppose $e = \{(x,a),(y,b)\}$ where $a, b \in \mathbb{Z}_3$, $a \neq b$ and where for each $i \in \mathbb{Z}_{n/2}$, e does not join two vertices in $C(i)$. We can assume that $b \equiv a + 1 \pmod 3$. Since $\{x, y\} \in G(n, 3; 3, 1)$, exactly one of the following occurs: either there exists a triple $t_1 = \{x, z_3, z_4\} \in B_1(a)$ such that $f(t_1) = y$ or there exists a triple $t_2 = \{y, z_3, z_4\} \in$

$B_1(a)$ such that $f(t_2) = x$. Therefore $e = \{(x,a),(y,b=a+1)\}$ occurs in $\lambda_2 = 1$ of the following blocks:

$\{(x,a),(z_3,a),(z_4,a),(y,a+1)\}$ or $\{(x,a),(y,a+1),(z_3,a+1),(z_4,a+1)\}$.

Thus, $n \equiv 3 \pmod{12}$ is a sufficient condition for a $GDD(n,3,4;3,1)$ to exist.

Now suppose $n \equiv 0 \pmod{12}$, $n \geq 48$.

For each $i \in \mathbb{Z}_{n/12}$, let $B(i)$ be a copy of the $GDD(12,3,4;3,1)$ created in Lemma 7.0.4 on the vertices in $C(i) = \{12i, 12i+1, 12i+2, 12i+3, 12i+4, 12i+5, 12i+6, 12i+7, 12i+8, 12i+9, 12i+10, 12i+11\} \times \mathbb{Z}_3$, where for each $l \in \mathbb{Z}_3$, $\{12i, 12i+1, 12i+2, 12i+3, 12i+4, 12i+5, 12i+6, 12i+7, 12i+8, 12i+9, 12i+10, 12i+11\} \times \{l\}$ is a group. By Theorem 2.3, there exists a $(3,1)-GDD$, $(\mathbb{Z}_n, \{\{12i, 12i+1, 12i+2, 12i+3, 12i+4, 12i+5, 12i+6, 12i+7, 12i+8, 12i+9, 12i+10, 12i+11\} \mid i \in \mathbb{Z}_{n/12}\}, B_1)$, that has nesting f of type $12^{n/12}$. Let $B_1(l) = \{\{(x,l),(y,l),(z,l),(f(b),l+1)\}, \{(x,l+1),(y,l+1),(z,l+1),(f(b),l)\} \mid l \in Z_3, \{x,y,z\} \in B_1\}$, reducing the sums in the second coordinate of each vertex modulo 3. By Theorem 2.2 (since $n \geq 48$ and $u = n/3 \geq 4$), for each $l \in Z_3$ let $B'(l)$ be a copy of a $(4,1)-GDD$ of type $12^{n/12}$ on the vertex set $Z_n \times \{l\}$ with groups in $\{\{12i, 12i+1, 12i+2, 12i+3, 12i+4, 12i+5, 12i+6, 12i+7, 12i+8, 12i+9, 12i+10, 12i+11\} \times \{l\} \mid i \in Z_{n/3}\}$. Then define the blocks in the design as follows:

$$B = (\textstyle\bigcup_{i \in \mathbb{Z}_{12}} B(i)) \cup (\bigcup_{l \in \mathbb{Z}_3} B_1(l)) \cup (\bigcup_{l \in \mathbb{Z}_3} B'(l))$$

We first count the number of blocks we get in the construction to see if it equals $b = \frac{5n^2-3n}{4}$ (calculated above when proving the necessity).

$$\begin{aligned}
|B| &= |(\textstyle\bigcup_{i \in \mathbb{Z}_{12}} B(i))| + |(\bigcup_{l \in \mathbb{Z}_3} B_1(l))| + |(\bigcup_{l \in \mathbb{Z}_3} B'(l))| \\
&= (171)(n/12) + (2)(3)((\tfrac{n(n-1)-11n}{2})/3) + 3(\tfrac{\binom{n}{2} - (\frac{(12)(11)}{2})(\frac{n}{12})}{6}) \\
&= \tfrac{171n}{12} + n^2 - n - 11n + \tfrac{\frac{n(n-1)}{2} - \frac{11n}{2}}{2} \\
&= \tfrac{57n}{4} + n^2 - 12n + \tfrac{n^2-n}{4} - \tfrac{11n}{4} \\
&= \tfrac{57n + 4n^2 - 48n + n^2 - n - 11n}{4} \\
&= \tfrac{5n^2-3n}{4}
\end{aligned}$$

Since $|B| = b$ it suffices to check that each edge occurs in at least the correct number (that is, λ_3 or λ_1) of blocks in B. We consider each edge, $e = \{(x,a),(y,b)\}$, in turn.

1. Suppose e joins two vertices in $C(i)$ for some $i \in \mathbb{Z}_{n/3}$. Then clearly e occurs in $\lambda_1 = 3$ blocks in $B(i)$ if e joins two vertices in the same group and e occurs in $\lambda_2 = 1$ block in $B(i)$ if e joins two vertices in different groups, as required.

2. Next suppose that $e = \{(x,a),(y,a)\}$ for some $a \in \mathbb{Z}_3$ and $0 \leq x, y < n$ where for each $i \in \mathbb{Z}_{n/2}$, e does not join two vertices in $C(i)$. Let $\{x, y, z_1\}$ be the triple in $B_1(a)$ that contains $\{x, y\}$, and suppose $f(\{x, y, z_1\}) = z_2$ is the vertex added to the triple by the nesting. Then the $\lambda_1 = 3$ blocks containing the edge $\{(x,a),(y,a)\}$ are as follows: $\{(x,a),(y,a),(z_1,a),(z_2,a+1)\}$, $\{(x,a),(y,a),(z_1,a),(z_2,a+2)\}$, and in one block in $B'(a)$.

3. Finally suppose $e = \{(x,a),(y,b)\}$ where $a, b \in \mathbb{Z}_3$, $a \neq b$ and where for each $i \in \mathbb{Z}_{n/2}$, e does not join two vertices in $C(i)$. We can assume that $b \equiv a+1 \pmod 3$. Since $\{x,y\} \in G(n,3;3,1)$, exactly one of the following occurs: either there exists a triple $t_1 = \{x, z_3, z_4\} \in B_1(a)$ such that $f(t_1) = y$ or there exists a triple $t_2 = \{y, z_3, z_4\} \in B_1(a)$ such that $f(t_2) = x$. Therefore $e = \{(x,a),(y,b=a+1)\}$ occurs in $\lambda_2 = 1$ of the following blocks:
$\{(x,a),(z_3,a),(z_4,a),(y,a+1)\}$ or $\{(x,a),(y,a+1),(z_3,a+1),(z_4,a+1)\}$.

Thus, $n \equiv 0 \pmod{12}$, $n \geq 48$, is a sufficient condition for a $GDD(n,3,4;3,1)$ to exist. $\square$

Chapter 8

Generalizing Clatworthy Design $R105$

With one possible exception, the existence of a generalized $R105$-design (see Figure 8.1) is settled in this chapter.

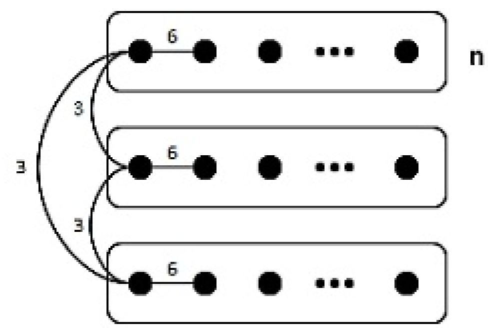

Figure 8.1: R105

Lemma 8.0.5. *There exists a $GDD(6, 3, 4; 6, 2)$ and a $GDD(9, 3, 4; 6, 2)$.*

Proof. To produce a $GDD(6, 3, 4; 6, 2)$, let $V = \mathbb{Z}_9 \times \mathbb{Z}_2$ where the groups are vertices 0 (mod 3) $\times \mathbb{Z}_2$, 1 (mod 3) $\times \mathbb{Z}_2$, and 2 (mod 3) $\times \mathbb{Z}_2$. Now, consider the following nine blocks $\{(0, 1), (1, 1), (3, 1), (4, 2)\}$, $\{(0, 1), (1, 1), (5, 1), (0, 2)\}$, $\{(0, 1), (3, 1), (6, 1), (0, 2)\}$, $\{(4, 1), (0, 2), (1, 2), (3, 2)\}$, $\{(3, 1), (0, 2), (1, 2), (5, 2)\}$, $\{(0, 1), (0, 2), (3, 2), (6, 2)\}$, $\{(0, 1), (2, 1), (2, 2), (5, 2)\}$, $\{(0, 1), (3, 1), (1, 2), (3, 2)\}$, and $\{(0, 1), (3, 1), (0, 2), (6, 2)\}$. (See Figure 8.2; the first block is depicted.) When rotated, these nine blocks cover the edges within a group the required six times and the edges between groups the required two times. Also, note that each block produces nine blocks giving 81 total blocks, which is the required number of blocks.

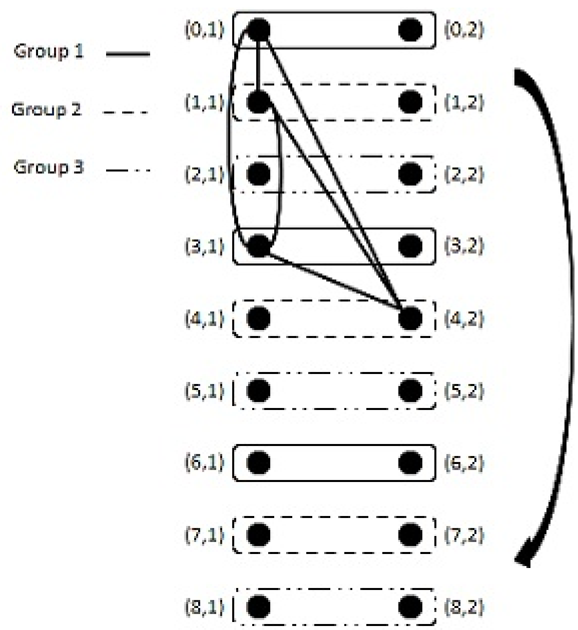

Figure 8.2: R105, $n = 6$; the first block of nine base blocks is depicted

To produce a $GDD(9, 3, 4; 6, 2)$, let $V = \mathbb{Z}_{27}$ where the groups are vertices 0 (mod 3), 1 (mod 3), and 2 (mod 3). Then consider the following seven blocks: $\{0, 1, 3, 9\}, \{0, 3, 9, 13\}, \{0, 3, 12, 16\}, \{0, 3, 15, 20\}, \{0, 6, 11, 12\}, \{0, 6, 8, 15\}, \{0, 3, 9, 12\}$. (See Figure 8.3; the first block is depicted.) When rotated, these seven blocks cover the edges of differences $3, 6, 9, 12$ six times and the edges of differences $1, 2, 4, 5, 7, 8, 10, 11, 13$ twice. In other words, we have covered the pure edges the required six times and the mixed edges the required two times. Also, note that each of these seven blocks produces 27 blocks giving 189 total blocks, which is the required number of blocks. □

Theorem 8.1. *There exists a $GDD(n, 3, 4; 6, 2)$ if and only if $n \equiv 0$ $(mod\ 3)$.*

Proof. We start by proving the necessity, so suppose there exists a

$GDD(n, 3, 4; 6, 2)$. Since each block contains six edges, the number of blocks in any such design is

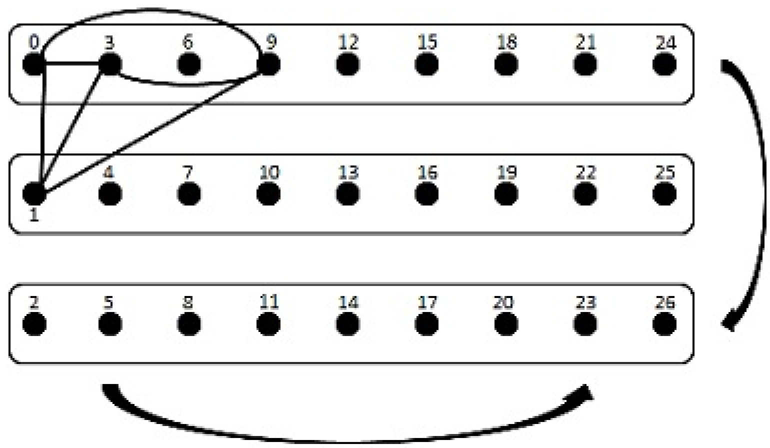

Figure 8.3: R105, $n = 9$; the first of seven base blocks is depicted

$$b = \frac{|E(G(n,3;6,2))|}{6} = \frac{3(\frac{6n(n-1)}{2}) + 3(2n^2)}{6} = \frac{5n^2 - 3n}{2} = \frac{n(5n-3)}{2}. \qquad (*)$$

Clearly the number of blocks is an integer, so there are no restrictions on n because either n or $5n - 3$ is even.

For each block, each vertex contains 3 of its incident edges. So the number of blocks containing each vertex v is

$$d_{G(n,3;6,2)}(v) = \frac{6(n-1) + 2(2n)}{3} = \frac{10}{3}n - 2,$$

which means $n \equiv 0 \pmod 3$ is a necessary condition.

To prove the sufficiency we assume that $n \equiv 0 \pmod 3$, $n \neq 9$, and show there exists a $GDD(n, 3, 4; 6, 2)$.

For each $i \in \mathbb{Z}_{n/3}$, let $B(i)$ be a copy of $R105$ on the vertices in $C(i) = \{3i, 3i + 1, 3i + 2\} \times \mathbb{Z}_3$, where for each $l \in \mathbb{Z}_3$, $\{3i, 3i + 1, 3i + 2\} \times \{l\}$ is a group. By Theorem 2.3, there exists a (3,2)-GDD, $(Z_n, \{\{3i, 3i + 1, 3i + 2\} \mid i \in \mathbb{Z}_{n/3}\}, B_1)$, that has nesting f of type $3^{n/3}$. Let $B_1(l) = \{\{(x, l), (y, l), (z, l), (f(b), l + 1)\}, \{(x, l + 1), (y, l + 1), (z, l +$

$1), (f(b), l)\} \mid l \in Z_3, \{x, y, z\} \in B_1\}$, reducing the sums in the second coordinate of each vertex modulo 3. By Theorem 2.2 (since $n \geq 12$ and $u = n/3 \geq 4$), for each $l \in Z_3$ let $B'(l)$ be a copy of a $(4,1) - GDD$ of type $3^{n/3}$ on the vertex set $Z_n \times \{l\}$ with groups in $\{\{3i, 3i+1, 3i+2\} \times \{l\} \mid i \in Z_{n/3}\}$. Then define the blocks in the design as follows:

$$B = (\textstyle\bigcup_{i \in \mathbb{Z}_3} B(i)) \cup (\bigcup_{l \in \mathbb{Z}_3} B_1(l)) \cup (\bigcup_{l \in \mathbb{Z}_3} B'(l))$$

We first count the number of blocks we get in the construction to see if it equals $b = \frac{5n^2-3n}{2}$ (calculated above when proving the necessity).

$$\begin{aligned}
|B| &= |(\textstyle\bigcup_{i \in \mathbb{Z}_3} B(i))| + |(\bigcup_{l \in \mathbb{Z}_3} B_1(l))| + |(\bigcup_{l \in \mathbb{Z}_3} B'(l))| \\
&= (18)(n/3) + (2)(2)(3)((\tfrac{n(n-1)-2n}{2})/3) + (2)(3)(\tfrac{(\binom{n}{2}-n)}{6}) \\
&= \tfrac{18n}{3} + 2n^2 - 2n - 4n + \tfrac{n(n-1)-2n}{2} \\
&= 6n + 2n^2 - 6n + \tfrac{n^2-3n}{2} \\
&= \tfrac{4n^2+n^2-3n}{2} \\
&= \tfrac{5n^2-3n}{2}
\end{aligned}$$

Since $|B| = b$ it suffices to check that each edge occurs in at least the correct number (that is, λ_3 or λ_1) of blocks in B. We consider each edge, $e = \{(x, a), (y, b)\}$, in turn.

1. Suppose e joins two vertices in $C(i)$ for some $i \in \mathbb{Z}_{n/3}$. Then clearly e occurs in $\lambda_1 = 6$ blocks in $B(i)$ if e joins two vertices in the same group and e occurs in $\lambda_2 = 2$ block in $B(i)$ if e joins two vertices in different groups, as required.

2. Next suppose that $e = \{(x, a), (y, a)\}$ for some $a \in \mathbb{Z}_3$ and $0 \leq x, y < n$ where for each $i \in \mathbb{Z}_{n/2}$, e does not join two vertices in $C(i)$. Let $\{x, y, z_1\}$ and $\{x, y, z_3\}$ be the triples in $B_1(a)$ that contain $\{x, y\}$, and suppose $f(\{x, y, z_1\}) = z_2$ and $f(\{x, y, z_3\}) = z_4$ are the vertices added to the triples by the nesting. Then the $\lambda_1 = 6$ blocks containing the edge $\{(x, a), (y, a)\}$ are as follows:
   $\{(x, a), (y, a), (z_1, a), (z_2, a+1)\}$, $\{(x, a), (y, a), (z_1, a), (z_2, a+2)\}$, $\{(x, a), (y, a), (z_3, a), (z_4, a+1)\}$, $\{(x, a), (y, a), (z_3, a), (z_4, a+2)\}$ and in two blocks in $B'(a)$.

3. Finally suppose $e = \{(x,a),(y,b)\}$ where $a,b \in \mathbb{Z}_3$, $a \neq b$ and where for each $i \in \mathbb{Z}_{n/2}$, e does not join two vertices in $C(i)$. We can assume that $b \equiv a+1 \pmod 3$. Since $\{x,y\} \in G(n,3;6,2)$, exactly one of the following occurs: either there exists a triple $t_1 = \{x,z_5,z_6\} \in B_1(a)$ such that $f(t_1) = y$ or there exists a triple $t_2 = \{y,z_5,z_6\} \in B_1(a)$ such that $f(t_2) = x$. The same argument can be made for the other $\{x,y\}$ edge using $\{z_7,z_8\}$. Therefore $e = \{(x,a),(y,b=a+1)\}$ occurs in $\lambda_2 = 2$ of the following blocks:

   $\{(x,a),(z_5,a),(z_6,a),(y,a+1)\}$ or $\{(x,a),(y,a+1),(z_5,a+1),(z_6,a+1)\}$, and $\{(x,a),(z_7,a),(z_8,a),(y,a+1)\}$ or $\{(x,a),(y,a+1),(z_7,a+1),(z_8,a+1)\}$.

Thus, $n \equiv 6,9 \pmod{12}$, $n \neq 6,9$, is a sufficient condition for a $GDD(n,3,4;6,2)$ to exist.

$\square$

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