Countably Compact, Countably Tight, Non-Compact Spaces

by

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Abstract

In this work, we will study several examples of countably compact, countably tight, non-compact spaces. After reviewing the important basic notions, we will examine a construction of several such spaces first given by Manes in “Monads in Topology” and will then detail how to construct such spaces using a more direct and explicit topological process. We will then use this new framework to describe several new spaces and to prove several propositions which are much more transparent from this new viewpoint.

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Chapter 1

Introduction

A vast body of literature is devoted to the study of compact Hausdorff spaces. The class of compact Hausdorff spaces is well behaved in many important ways, but there is at least one sense in which compact Hausdorff spaces may be quite poorly behaved: any metric space is sequential, i.e. the topology is entirely determined by convergent sequences in the space, but there are non-trivial compact Hausdorff spaces with no non-trivial convergent sequences. A natural question is: are there topological spaces satisfying a weaker form of compactness but which are to some degree determined by sequences in the space? In this work, we will single out countable compactness as the weaker form of compactness under consideration.

When considering countable compactness, there is a natural question: what can be said about spaces which are countably compact but not compact? For instance, no such space can be Lindelöf. The specific question "is every separable, countably compact, countably tight space compact?" was posed as a "classic problem" by Nyikos in [2]. A series of examples of countably compact, countably tight, non-compact spaces satisfying successively stronger separation properties has been constructed [3] [4], but most such constructions require the assumption of additional axioms beyond ZFC. The strongest such example currently known to exist in ZFC was constructed by Manes in [1] using the category-theoretical concept of a monad.

In this work, we will give a construction of the spaces first described by Manes using a new and more explicit topological approach. In Chapter 2, we review the basic background notions. In Chapter 3, we summarize Manes' construction using monads. In Chapter 4, we will detail the new topological construction of Manes' spaces and prove, using our description, that they give examples of countably compact, countably tight, non-compact spaces and discuss their other properties. We will also use this topological framework to construct a new larger class of spaces with many similar properties. Finally, in Chapter 5, we will discuss some applications of this new framework by providing several propositions and constructions that would be either much more opaque or outright impossible from the categorical perspective.

Chapter 2

Background Definitions and Theorems

The material in this section is standard and can be found in any introductory topology book, but definitions and proofs can be found in [5]. Basic set theory notions are assumed, and whenever unspecified, notation matches that used in [5].

The following terms are well known and will be used without further comment: *topology*, *topological space*, *subspace topology*, *finer topology*, *open set*, *neighborhood*, *closed set*, *limit point*, *closure*, *sequence*, *continuous function*, *homeomorphism*, *compact*, T_1, *Hausdorff*, *regular*, *completely regular*, *connected*, *dense*.

We will often consider two different topologies on the same space. For clarity, we make the following definition:

Definition 2.1. If τ is a topology on a set X and $A \subset X$, $cl_\tau(A)$ is the closure of A in (X, τ). More specifically:

$$cl_\tau(A) = \cap\{C \mid A \subset C, X \setminus C \in \tau\}$$

We may use $cl(A)$ or simply $\overline{A}$ if τ is clear from context.

Throughout this chapter, X will denote an arbitrary topological space. The following propositions are well known:

Proposition 2.2. *If $C \subset X$ is closed and X is compact, then C is compact.*

Proposition 2.3. *If $C \subset X$ is compact and X is Hausdorff then C is closed in X.*

Proposition 2.4. *If $C \subset X$ is compact and $f : X \to Y$ is continuous then $f(C) \subset Y$ is compact.*

Proposition 2.5. *If $x_n \to x$ in X and $f : X \to Y$ continuous then $f(x_n) \to f(x)$ in Y.*

Proposition 2.6. *If $D \subset X$ is dense, Y is Hausdorff, and $f, g : X \to Y$ are continuous with $f|_D = g|_D$, then $f = g$.*

The following definitions are less widely known, and so they are included for completeness:

Definition 2.7. X is *Urysohn* provided that for every pair of points $x, y \in X$ there are open sets $U_x, U_y \subset X$ with $x \in U_x$ and $y \in U_y$ with $\overline{U_x} \cap \overline{U_y} = \emptyset$.

Note that any Urysohn space is clearly Hausdorff.

Definition 2.8. X is *extremally disconnected* if for every open $O \subset X$, $\overline{O}$ is open.

The next several results outline the basic construction of and results about the Stone-Čech compactification. Proofs are omitted, but may be found in [7].

Definition 2.9. A *filter* $\mathcal{F}$ on a set S is a collection of nonempty subsets of S such that:

- If $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$.
- If $A \in \mathcal{F}$ and $C \supset A$ then $C \in \mathcal{F}$.

Definition 2.10. An *ultrafilter* $\mathfrak{u}$ on a set S is maximal filter on S.

Note that we must appeal to some form of the Axiom of Choice to show that ultrafilters exist. By using Zorn's Lemma, any filter can be extended to an ultrafilter.

Proposition 2.11. *Let $\mathcal{F}$ be a filter on a set S. The following are equivalent:*

1. *$\mathcal{F}$ is an ultrafilter.*
2. *For every $A \subset S$, either $A \in \mathcal{F}$ or $S \setminus A \in \mathcal{F}$.*
3. *If $A \cup B = S$, then either $A \in \mathcal{F}$ or $B \in \mathcal{F}$.*

4. If $A \subset S$ and $A \cap B \neq \emptyset$ for every $B \in \mathcal{F}$, then $A \in \mathcal{F}$.

Definition 2.12. $C \subset X$ is a *zero-set* in X if there is a continuous $f : X \to \mathbb{R}$ with $C = f^{-1}(0)$.

Definition 2.13. A z-filter is a filter $\mathcal{F}$ on X so that for every $C \in \mathcal{F}$, C is a zero-set in X.

Definition 2.14. A z-ultrafilter is a filter which is maximal among z-filters.

Definition 2.15. For $x \in X$, $prin(x) = \{B \subset X \mid B \text{ a zero-set}, x \in B\}$.

A z-ultrafilter of the form $prin(x)$ is called a *principal ultrafilter.*

Definition 2.16. βX is the set of all z-ultrafilters on X. The standard topology on βX is defined as follows: if $F \subset X$ is a zero-set, let $\overline{F} = \{\mathfrak{u} \in \beta X \mid F \in \mathfrak{u}\}$. A basis for the standard topology consists of all sets of the form $\beta X \setminus \overline{F}$ for F a zero-set in X.

We will primarily consider the case when X is discrete, in which case every subset is a zero-set and βX can be described more simply as the set of all ultrafilters on X with basic open sets of the form $\overline{O}$ for any $O \subset X$.

We will typically regard X as a subset of βX by identifying $x \in X$ with $prin(x) \in \beta X$. With this identification,

Proposition 2.17. *$X \subset \beta X$ is dense.*

Proposition 2.18. *βX is a compact Hausdorff space.*

If X is a completely regular, Hausdorff space, the inclusion map $X \hookrightarrow \beta X$ is an embedding and βX is often referred to as the *Stone-Čech compactification of X.*

Proposition 2.19. *If Y is any compact Hausdorff space and $f : X \to Y$ is a continuous function, there is a unique continuous function $F : \beta X \to Y$ so that $F|_X = f$. Moreover, this property uniquely characterizes βX.*

Proposition 2.20. *If $C, D \subset X$ are disjoint zero-sets, then $cl_{\beta X}(C)$ and $cl_{\beta X}(D)$ are disjoint.*

Definition 2.21. ω is the first infinite ordinal.

Proposition 2.22. $|\beta\omega| = 2^{2^{\omega}}$.

Proposition 2.23. *If X is discrete, βX contains no non-trivial convergent sequences. That is, any sequence that converges in βX is eventually constant.*

Definition 2.24. If $f : X \to Y$ and $\mathfrak{u} \in \beta X$, then $f(\mathfrak{u}) = \{B \subset Y \mid f^{-1}(B) \in \mathfrak{u}\} = \{C \subset Y \mid C \supset f(A) \text{ for some } A \in \mathfrak{u}\}$.

The notation $f\mathfrak{u}$ is often used in place of $f(\mathfrak{u})$.

Definition 2.25. $X^* = \beta X \setminus X$ with the subspace topology inherited from βX.

Thus an element of X^* is a non-principal ultrafilter. Such an ultrafilter is often called *free*.

Proposition 2.26. *Let $p \in \omega^*$. p contains no finite subsets of ω.*

Proof. Suppose there is some finite $F \in p$ of smallest cardinality. If $|F| > 1$ and $n \in F$, then either $\{n\} \in p$ or $F \setminus \{n\} \in p$ by 2.11.2, contradicting the minimality of $|F|$. Thus $F = \{n\}$ for some $n \in \omega$ and so for every $B \subset \omega$, with $n \in B$, $B \in p$ since p is a filter. Thus $p = prin(n)$ and so $p \notin \omega^*$. □

Definition 2.27. A set $S \subset X$ is a *weak P-set* in X if for every countable $C \subset X$ with $C \cap S = \emptyset$, $\overline{C} \cap S = \emptyset$.

Definition 2.28. A point $x \in X$ is a *weak P-point* in X if $\{x\} \subset X$ is a weak P-set in X.

An important fact which we will use frequently is:

Proposition 2.29. *There are weak P-points in ω^*.*

In fact, ω^* contains a dense set of weak P-points [7], but we will not have need of this stronger result.

A central notion in this work is that of a p-limit:

Definition 2.30. If $p \in \omega^*$ and (x_n) is a sequence in X, then a point $y \in X$ is the *p-limit of* (x_n), denoted $y = p\text{-}\lim x_n$ provided that for every open $O \subset X$ containing y, $\{n \in \omega \mid x_n \in O\} \in p$.

The definition of a p-limit depends essentially on the topology of X. We may use $p\text{-}\lim_X x_n$ or $p\text{-}\lim_\tau x_n$ to emphasize the space X or topology τ with respect to which the limit is taken.

Note that if $x_n \to x$ in the usual sense then for any open O about x, $\{n \mid x_n \in O\}$ is cofinite in ω and so this set is in *every* $p \in \omega^*$ by 2.26. Thus the notion of a p-limit is a weakening of the usual notion of a limit.

The importance of p-limits can be seen in the following proposition:

Proposition 2.31. *Let (x_n) be a sequence in X. $y \in \overline{\{x_n\}}$ if and only if $y = p\text{-}\lim x_n$ for some $p \in \beta\omega$.*

Proof. Suppose $y = p\text{-}\lim x_n$. Then for any open O containing y, $\{n \mid x_n \in O\} \in p$ and since every set in p is nonempty, $O \cap \{x_n\} \neq \emptyset$ and $y \in \overline{\{x_n\}}$.

Conversely, if $y \in \overline{\{x_n\}}$, let $\mathcal{F} = \{\{n \mid x_n \in O\} \mid O \text{ a neighborhood of } y\}$. For any $A, B \in \mathcal{F}$, $A \cap B \in \mathcal{F}$ and so $\mathcal{F}$ can be extended to an ultrafilter $p \in \beta\omega$. Then by definition, for any neighborhood O of y, $\{n \mid x_n \in O\} \in \mathcal{F} \subset p$ and so $y = p\text{-}\lim x_n$. $\square$

Let us record a few basic facts about p-limits:

Proposition 2.32. *Let (x_n) be a sequence in X and $p \in \omega^*$. If X is Hausdorff, then $p\text{-}\lim x_n$ is unique if it exists.*

Proof. Suppose x and y are both p-limits of (x_n). If X is Hausdorff, there are open U_x, U_y containing x and y respectively with $U_x \cap U_y = \emptyset$. Since x is a p-limit of (x_n), the set

$A_x = \{n \mid x_n \in U_x\} \in p$ and similarly $A_y = \{n \mid x_n \in U_y\} \in p$. But $A_x \cap A_y = \emptyset$ since $U_x \cap U_y = \emptyset$, which contradicts the fact that p is a filter. $\square$

Thus we are justified in referring to "the" p-limit of a sequence.

Proposition 2.33. *Let (x_n) be a sequence in X, $p \in \omega^*$ and $x = p\text{-}\lim x_n$. For any continuous $f : X \to Y$, $f(x) = p\text{-}\lim f(x_n)$.*

Proof. Let $f : X \to Y$ be continuous and $O \subset Y$ be an open neighborhood of $f(x)$. Then $x \in f^{-1}(O)$, which is open since f is continuous. Since $x = p\text{-}\lim x_n$, $\{n \mid x_n \in f^{-1}(O)\} = \{n \mid f(x_n) \in O\} \in p$ as required. $\square$

Proposition 2.34. *Let (x_n), (y_n) be sequences in X and $p \in \omega^*$. If $\{n \mid x_n = y_n\} \in p$ then $p\text{-}\lim x_n = p\text{-}\lim y_n$.*

Proof. Suppose $A = \{n \mid x_n = y_n\} \in p$. For any open neighborhood O of $p\text{-}\lim x_n$, $B = \{n \mid x_n \in O\} \in p$. Since p is a filter, $A \cap B \in p$ and for every $n \in A \cap B$, $y_n = x_n \in O$. Thus $A \cap B \subset \{n \mid y_n \in O\} \in p$ since p is a filter and so $p\text{-}\lim x_n = p\text{-}\lim y_n$. $\square$

Definition 2.35. For $p \in \omega^*$, X is *p-compact* if for every sequence (x_n) in X, $p\text{-}\lim x_n \in X$ (in particular, $p\text{-}\lim x_n$ exists).

p-compactness is a weakening of compactness:

Proposition 2.36. *If X is compact, then X is p-compact for every $p \in \omega^*$.*

Proof. Let (x_n) be a sequence in X. For $B \subset \omega$, let $x_B = \{x_n \mid n \in B\}$. Then for any $p \in \omega^*$, $\{\overline{x_B} \mid B \in p\}$ is a collection of closed subsets of X with the finite intersection property. Since X is compact, there is some $x \in \cap\{\overline{x_B} \mid B \in p\}$. Let O be an open neighborhood of x and $A = \{n \mid x_n \in O\}$. For every $B \in p$, $x \in \overline{x_B}$, so there is some $n \in B$ with $x_n \in O$. Thus $A \cap B \neq \emptyset$ for each $B \in p$ and since p is an ultrafilter, $A \in p$ by 2.11.4. Thus $x = p\text{-}\lim x_n$. $\square$

Definition 2.37. X is *ultracompact* if X is p-compact for every $p \in \omega^*$.

p-compactness can be used to define an important partial order on $\beta\omega$:

Definition 2.38. For $p, q \in \omega^*$, the *Comfort preorder* $\leq_C$ on $\beta\omega$ is defined by $p \leq_C q \iff$ every q-compact space is p-compact.

Definition 2.39. For $p, q \in \omega^*$, the *Rudin-Keisler preorder* $\leq_{RK}$ on $\beta\omega$ is defined by $p \leq_{RK} q \iff p = fq$ for some $f : \omega \to \omega$.

Definition 2.40. X is *countably compact* if every countable open cover has a finite subcover.

Proposition 2.41. *If X is p-compact and T_1, then X is countably compact.*

Proof. If X is p-compact, then every infinite set has a limit point by 2.31. Since X is T_1, it is countably compact by [5]. $\square$

Proposition 2.42. *If X is countably compact and $\{C_n \mid n \in \omega\}$ is a countable collection of nonempty closed sets such that $C_{n+1} \subset C_n$, then $\cap C_n \neq \emptyset$.*

Proof. Let $U_n = X \setminus C_n$. Each U_n is open. If $\cap C_n = \emptyset$ then $\cup U_n = X \setminus \cap C_n = X$ so $\{U_n\}$ is a cover. If X is countably compact, there is some $N \in \omega$ so that $\cap_{n \leq N} C_n = X \setminus \cup_{n \leq N} U_n = \emptyset$, contrary to the assumption that $\cap_{n \leq N} C_n = C_N \neq \emptyset$. $\square$

Definition 2.43. X is *sequential* if for every non-closed $A \subset X$, there is a sequence (x_n) in A with $x_n \to x \notin A$.

A sequential space is one in which the topology is entirely determined by convergent sequences. Using p-limits we can weaken this definition in the obvious way:

Definition 2.44. For $p \in \omega^*$, a space X is p-sequential if for every non-closed $A \subset X$, there is a sequence (x_n) in A with $p\text{-}\lim x_n \notin A$.

We can weaken this even further to get the following definition:

Definition 2.45. X is countably tight if for every set $A \subset X$ and $x \in \overline{A}$, there is some countable $C \subset A$ with $x \in \overline{C}$.

That this is a weaker notion will be shown shortly, but first let us introduce some notation to aid in the proof:

Definition 2.46. For $Y \subset X$ and $S \subset \omega^*$, define $A^X_\alpha(S,Y)$ for each $\alpha \leq \omega_1$ by transfinite induction as follows:

- $A^X_0(S,Y) = Y$
- $A^X_{\alpha+1}(S,Y) = \{x \in X \mid x = p\text{-}\lim x_n \text{ for some } p \in S \text{ and some sequence } (x_n) \text{ in } A^X_\alpha(S,Y)\}$ for successor ordinals
- $A^X_\alpha(S,Y) = \cup_{\beta<\alpha} A_\beta(S,Y)$ for limit ordinals

We may write $A_\alpha(S,Y)$ if X is clear from context.

Proposition 2.47. *Let $p \in \omega^*$. If X is p-sequential and $Y \subset X$, then $cl_X(Y) = A^X_{\omega_1}(\{p\},Y)$.*

Proof. Fix $Y \subset X$. Suppose that $A^X_{\omega_1}(\{p\},Y)$ is not closed. Then since X is p-sequential, there is some sequence (x_n) in $A^X_{\omega_1}(\{p\},Y)$ with $p\text{-}\lim x_n \notin A^X_{\omega_1}(\{p\},Y)$. But then by construction, there is some $\alpha < \omega_1$ with $\{x_n\} \subset A^X_\alpha(\{p\},Y)$ and so $p\text{-}\lim x_n \in A^X_{\alpha+1}(\{p\},Y) \subset A^X_{\omega_1}(\{p\},Y)$, a contradiction. Thus $A^X_{\omega_1}(\{p\},Y)$ is closed and so $cl_X(Y) \subset A^X_{\omega_1}(\{p\},Y)$.

Clearly $A_0(\{p\},Y) \subset cl_X(Y)$. Suppose $A_\beta(\{p\},Y) \subset cl_X(Y)$ for all $\beta < \alpha$. If α is a limit ordinal then $A_\alpha(\{p\},Y) \subset cl_X(Y)$ trivially. If not, then $\alpha = \gamma+1$ and for any $a \in A_\alpha(\{p\},Y)$, $a = p\text{-}\lim x_n$ for some sequence (x_n) in $A_\gamma(\{p\},Y)$. Thus $a \in \overline{A_\gamma(\{p\},Y)} \subset \overline{cl_X(Y)}$ by 2.31 and the induction hypothesis. Thus $A_{\omega_1}(\{p\},Y) \subset cl_X(Y)$ by transfinite induction. $\square$

Proposition 2.48. *If $S \subset \omega^*$ so that for any $Y \subset X$, $cl_X(Y) = A^X_{\omega_1}(S,Y)$, then X is countably tight.*

Proof. Suppose $S \subset \omega^*$ satisfies the hypotheses. We claim that for any $\alpha < \omega_1$ and $y \in A^X_\alpha(S,Y)$ there is a countable $B \subset Y$ with $y \in \overline{B}$. If $y \in A_1(S,Y)$ then this is clear from 2.31. Proceeding by transfinite induction, suppose $y \in A^X_\alpha(S,Y)$. If $\alpha = \gamma+1$, then $y = p\text{-}\lim y_n \in \overline{\{y_n\}}$ for some sequence (y_n) in $A^X_\gamma(S,Y)$ by 2.31. By induction, for each n there is

$B_n \subset Y$ with $y_n \in \overline{B_n}$. Then $y \in \overline{\cup B_n}$. If α is a limit ordinal, then $y \in A^X_\gamma(S, Y)$ for some $\gamma < \alpha$ already.

If $Y \subset X$ and $y \in cl_X(Y) = A^X_{\omega_1}(S, Y)$ then $y \in A^X_\alpha(S, Y)$ for some $\alpha < \omega_1$ and so $y \in \overline{B}$ for some countable $B \subset Y$. Thus X is countably tight. □

Corollary 2.49. *Every p-sequential space is countably tight.*

Definition 2.50. $C(X) = \{f : X \to \mathbb{R} \mid f \text{ is continuous}\}$.

Definition 2.51. $C(X, Y) = \{f : X \to Y \mid f \text{ is continuous}\}$.

Definition 2.52. A *category* $\mathcal{C}$ is a collection of objects $ob(\mathcal{C})$, morphisms $mor(\mathcal{C})$ and a composition operator $\circ$ so that for every collection of objects A, B, C and morphisms $f : A \to B$ and $g : B \to C$, $g \circ f : A \to C$ with the following properties:

- If f, g, h are morphisms, $(f \circ g) \circ h = f \circ (g \circ h)$, provided both expressions are defined.
- For each object A there is a unique identity morphism $id_A : A \to X$ such that for any morphism $f : A \to B$, $f \circ id_A = f = id_B \circ f$.

Definition 2.53. A *(covariant) functor* F is a function $F : \mathcal{C} \to \mathcal{D}$ between categories so that for every $A \in ob(\mathcal{C})$, $F(A) \in ob(\mathcal{D})$, for every $f : A \to B$ in $\mathcal{C}$, $F(f) : F(A) \to F(B)$ in $\mathcal{D}$ and $F(f \circ g) = F(f) \circ F(g)$ whenever the composition $f \circ g$ is defined.

For convenience, $F(A)$ and $F(f)$ may be denoted FA and Ff, respectively.

Definition 2.54. The *identity functor* $1_\mathcal{C} : \mathcal{C} \to \mathcal{C}$ is defined by $1_\mathcal{C}A = A$ and $1_\mathcal{C}f = f$ for every object A and morphism f.

Definition 2.55. If F, G are two functors from $\mathcal{C}$ to $\mathcal{D}$ a *natural transformation* η is a collection of morphisms η_A - called the *component functions* of η - one for each $A \in ob(\mathcal{C})$ so that for every $f : A \to B$ in $\mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd}
FA \arrow[r, "Ff"] \arrow[d, "\eta_A"'] & FB \arrow[d, "\eta_B"] \\
GA \arrow[r, "Gf"'] & GB
\end{tikzcd}$$

Definition 2.56. A *semigroup operation* on a set S is an associative binary operation, usually denoted $a * b$ for $a, b \in S$.

Chapter 3

Monads of Ultrafilters and Their Algebras

Definition 3.1. Given a category $\mathcal{C}$, a *monad* is a triple (T, η, μ) where T is a functor $T : \mathcal{C} \to \mathcal{C}$, η is a natural transformation $\eta : 1_{\mathcal{C}} \to T$ and μ is a natural transformation $\mu : T^2 \to T$ so that the following diagrams commute:

$$\begin{tikzcd} T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"'] & T \end{tikzcd}$$

$$\begin{tikzcd} T \arrow[r, "\eta T"] \arrow[d, "T\eta"'] \arrow[dr, "id"] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"'] & T \end{tikzcd}$$

Where $T\mu$ is the natural transformation with component functions $T(\mu_X) : T^3X \to T^2X$ and μT is the natural transformation with component functions $\mu_{TX} : T^3X \to T^2X$ for each object $X \in ob(\mathcal{C})$.

This is the standard definition of a monad, given, for instance, in [8]. In [1], Manes gives the following equivalent definition if $\mathcal{C} = \mathbf{Set}$:

Definition 3.2. A *monad over* **Set** is a triple $(T, \eta, (\cdot)^{\#})$ where T is a function $T : \mathbf{Set} \to \mathbf{Set}$, $\eta_X : X \to TX$ for every set X, and for every pair of sets X, Y and function $f : X \to TY$, an extension $f^{\#} : TX \to TY$ so that:

1. $f^{\#} \circ \eta_X = f$

2. $(\eta_X)^{\#} = id_{TX}$

3. $(g^\sharp \circ f)^\sharp = g^\sharp \circ f^\sharp$ for any $g : Y \to TZ$

That the two definitions of a monad are equivalent is given by the following:

Proposition 3.3. *A monad (T, η, μ) over* **Set** *is equivalent to a monad over* **Set** $(T, \eta, (\cdot)^\sharp)$.

Proof. First, suppose we are given a monad as $(T, \eta, (\cdot)^\sharp)$. Extend T to a functor by defining, for $f : X \to Y$, $Tf = (\eta_Y \circ f)^\sharp : TX \to TY$. T is a functor since if $g : Y \to Z$, $T(g \circ f) = (\eta_Z \circ g \circ f)^\sharp$ whereas $Tg \circ Tf = (\eta_Z \circ g)^\sharp \circ (\eta_Y \circ f)^\sharp = ((\eta_Z \circ g)^\sharp \circ \eta_Y \circ f)^\sharp = (\eta_Z \circ g \circ f)^\sharp$. For any such f, $Tf \circ \eta_X = (\eta_Y \circ f)^\sharp \circ \eta_X = \eta_Y \circ f$ so η is a natural transformation $\eta : 1_{\mathcal{C}} \to T$.

For each set X let $\mu_X = id^\sharp_{TX} : TTX \to TX$. $\mu : T^2 \to T$ is a natural transformation since for any $f : X \to Y$, $\mu_Y \circ TTf = id^\sharp_{TY} \circ (\eta_{TY} \circ Tf)^\sharp = (id^\sharp_{TY} \circ \eta_{TY} \circ Tf)^\sharp = (id_{TY} \circ Tf)^\sharp = (Tf)^\sharp$ and $Tf \circ \mu_X = (\eta_Y \circ f)^\sharp id^\sharp_{TX} = ((\eta_Y \circ f)^\sharp \circ id_{TX})^\sharp = (Tf)^\sharp$. The following computations show that the diagrams in 3.1 commute and so (T, η, μ) is a monad:

$\mu_X \circ \eta_{TX} = id^\sharp_{TX} \circ \eta_{TX} = id_{TX}$

$\mu_X \circ T(\eta_X) = id^\sharp_{TX} \circ (\eta_{TX} \circ \eta_X)^\sharp = (id^\sharp_{TX} \circ \eta_{TX} \circ \eta_X)^\sharp = (id_{TX} \circ \eta_{TX})^\sharp = (\eta_{TX})^\sharp = id_{TX}$

$\mu_X \circ \mu_{TX} = id^\sharp_{TX} \circ id^\sharp_{TTX} = (id^\sharp_{TX} \circ id_{TTX})^\sharp = id^{\sharp\sharp}_{TX}$ and $\mu_X \circ T(\mu_X) = id^\sharp_{TX} \circ T(id^\sharp_{TX}) = id^\sharp_{TX} \circ (\eta_{TX} \circ id^\sharp_{TX})^\sharp = (id^\sharp_{TX} \circ \eta_{TX} \circ id^\sharp_{TX})^\sharp = (id_{TX} \circ id^\sharp_{TX})^\sharp = id^{\sharp\sharp}_{TX}$

Conversely, suppose we are given (T, η, μ). If $f : X \to TY$, define $f^\sharp = \mu_Y \circ Tf : TX \to TY$. Then we can check that for any $f : X \to TY$ and $g : Y \to TZ$,

1. $f^\sharp \circ \eta_X = \mu_Y \circ Tf \circ \eta_X = \mu_Y \circ \eta_{TY} \circ f = f$

2. $(\eta_X)^\sharp = \mu_{TX} \circ T(\eta_X) = id_{TX}$

3. $(g^\sharp \circ f)^\sharp = \mu_Z \circ T(g^\sharp \circ f) = \mu_Z \circ T(g^\sharp) \circ Tf = \mu_Z \circ T(\mu_Z \circ Tg) \circ Tf = \mu_Z \circ T(\mu_Z) \circ TTg \circ Tf = \mu_Z \circ \mu_{TZ} \circ TTg \circ Tf = \mu_Z \circ Tg \circ \mu_Y \circ Tf = g^\sharp \circ f^\sharp$

Given $(\cdot)^\sharp$ and defining $\mu_X = id^\sharp_{TX}$, note that for any $f : X \to TY$, $\mu_Y \circ Tf = id^\sharp_{TY} \circ Tf = id^\sharp_{TY} \circ (\eta_{TY} \circ f)^\sharp = (id^\sharp_{TY} \circ \eta_{TY} \circ f)^\sharp = (id_{TY} \circ f)^\sharp = f^\sharp$ so we recover our original notion of $(\cdot)^\sharp$. Conversely, given μ and defining $f^\sharp = \mu_Y \circ Tf$ for $f : X \to TY$, we have $id_{TX}^\sharp = \mu_X \circ T(id_{TX}) = \mu_X \circ id_{TTX} = \mu_X$ and we recover our original notion of μ. Thus these two definitions are equivalent.

□

We will sometimes refer to T as a monad when η and μ (or $(\cdot)^\sharp$) are clear from context.

The prototypical example of a monad over **Set** is β, with $\eta_X : X \to \beta X$ by $x \mapsto prin(x)$ for each set X and $x \in X$ and $(\cdot)^\sharp$ defined by $f^\sharp(\mathfrak{u}) = \{B \subset Y \mid \{x \in X \mid B \in f(x)\} \in \mathfrak{u}\}$ for each $f : X \to \beta Y$ and $\mathfrak{u} \in \beta X$.

Given a monad T, there is an important classical notion of an algebra of T:

Definition 3.4. Given a monad T on a category $\mathcal{C}$, an *algebra of* T is an object $X \in \mathcal{C}$ with a morphism $\xi : TX \to X$ so that the following diagrams commute:

$$\begin{tikzcd} T^2X \arrow[r, "T\xi"] \arrow[d, "\mu_X"'] & TX \arrow[d, "\xi"] \\ TX \arrow[r, "\xi"'] & X \end{tikzcd}$$

$$\begin{tikzcd} TX \arrow[dr, "\xi"] & \\ X \arrow[u, "\eta_X"] \arrow[r, "id"'] & X \end{tikzcd}$$

Definition 3.5. If A is an algebra of the monad T over **Set**, $B \subset A$ is a *subalgebra* if there is a set map ξ_0 which makes the following diagram commute:

$$\begin{tikzcd} TB \arrow[r, "Ti"] \arrow[d, "\xi_0"'] & TA \arrow[d, "\xi"] \\ B \arrow[r, "i"'] & A \end{tikzcd}$$

where $i : B \to A$ is the inclusion map.

Note that if B is a subalgebra of (A, ξ) as above then (B, ξ_0) is itself an algebra.

3.1 $T_p\omega$

Throughout this section, p will denote an arbitrary free ultrafilter on ω. The following definitions are taken from [1].

Definition 3.6. $G_pX = \{r \in \beta X \mid r = fp \text{ for some } f : \omega \to X\}$

Definition 3.7. T_p is the smallest submonad of $(\beta, \eta, (\cdot)^\sharp)$ over **Set** such that $G_pX \subset T_pX$ for each set X.

The importance of this definition is established by the following:

Theorem 3.8. *Define a topology on $T_p\omega$ by declaring $A \subset T_p\omega$ closed if and only if A is a subalgebra of $(T_p\omega, id^\sharp_{TX})$. With this topology, $T_p\omega$ is a countably compact, countably tight, separable, Urysohn, non-compact, non-sequential space.*

In [1], Manes established this fact, giving the first ZFC construction of such a space. We will prove this result later by giving a topological construction of the space $T_p\omega$.

Chapter 4

Topological Descriptions and Properties

4.1 $\beta_p X$

Given any topological space X and $p \in \omega^*$, the set of p-compact subsets of X form the closed sets of a new topology on X. We'll examine some properties of this new topology and then look at this construction on a particular space to get the example we seek.

Throughout this section, let (X, τ) be an arbitrary completely regular, Hausdorff space and $p \in \omega^*$ a free ultrafilter.

Definition 4.1.

$$\tau_p = \{U \subset X \mid X \setminus U \text{ is } p\text{-compact in } (X, \tau)\} \cup \{\emptyset\}$$

Proposition 4.2. *τ_p is a topology on X.*

Proof. $\emptyset \in \tau_p$ by definition and $X \in \tau_p$ trivially. We show that the finite union and arbitrary intersection of p-compact subsets of X are p-compact.

First, let C_1 and C_2 be p-compact subsets of X and (x_n) a sequence in $C_1 \cup C_2$. Let $B_i = \{n \mid x_n \in C_i\}$. Since p is an ultrafilter and $B_1 \cup B_2 \in p$, by 2.11.3 we may assume without loss of generality that $B_1 \in p$. Fix an arbitrary $z \in C_1$ and let $x'_n = x_n$ for all $n \in B_1$ and $x'_n = z$ otherwise. Then (x'_n) is a sequence in C_1, so $p\text{-}\lim x'_n$ exists and is in C_1 by assumption and $p\text{-}\lim x_n = p\text{-}\lim x'_n$ by 2.34. Thus $p\text{-}\lim x_n \in C_1 \cup C_2$ and $C_1 \cup C_2$ is p-compact.

Next, let C_i be p-compact for each i in some index set. If (x_n) is a sequence in $\cap C_i$, then (x_n) is a sequence in C_i for each i so $p\text{-}\lim x_n \in C_i$ for each i and thus $\cap C_i$ is p-compact. □

As observed in the preceding proof, τ_p may defined by declaring $C \subset (X, \tau_p)$ closed if and only if $C \subset (X, \tau)$ is p-compact.

Proposition 4.3. *If X is p-compact and $C \subset X$ is closed, then C is p-compact.*

Proof. If (x_n) is a sequence in C then $x = p\text{-}\lim x_n$ exists since X is p-compact and $x \in \overline{C} = C$ by 2.31. Thus C is p-compact in the subspace topology inherited from X. $\square$

Corollary 4.4. *If X is p-compact, then τ_p is finer than τ.*

Note that even though τ_p is usually finer than τ in the examples we'll consider, p-limits are still the same when considered in τ or τ_p:

Proposition 4.5. *Let (x_n) be a sequence in X. If $p\text{-}\lim_\tau x_n$ exists, then $p\text{-}\lim_{\tau_p} x_n = p\text{-}\lim_\tau x_n$.*

Proof. Let $x = p\text{-}\lim_\tau x_n$. Suppose $x \neq p\text{-}\lim_{\tau_p} x_n$. Then there is some $O \in \tau_p$ such that $x \in O$ but $\{n \mid x_n \in O\} \notin p$. Thus $\{n \mid x_n \notin O\} \in p$ since p is a z-ultrafilter. Fix $z \in X \setminus O$ and define a new sequence (x'_n) by $x'_n = x_n$ if $x_n \notin O$ and $x'_n = z$ otherwise. Then (x'_n) is a sequence in $X \setminus O$ which is p-compact in (X, τ) and so $p\text{-}\lim_\tau x'_n \in X \setminus O$. But $x_n = x'_n$ for all $n \in \{n \mid x_n \notin O\}$, so by 2.34 $p\text{-}\lim_\tau x_n = p\text{-}\lim_\tau x'_n \notin O$, a contradiction. $\square$

Corollary 4.6. *If (X, τ) is p-compact, (X, τ_p) is p-compact.*

Proposition 4.7. *If (X, τ) is p-compact, then (X, τ_p) is p-sequential.*

Proof. Let $A \subset X$ be non-closed in τ_p. Then $A \subset X$ is not p-compact in τ. So by definition there is a sequence (x_n) in A so that $x = p\text{-}\lim x_n \in X \setminus A$. $\square$

Corollary 4.8. *If (X, τ) is p-compact, $cl_{\tau_p}(Y) = A_{\omega_1}(\{p\}, Y)$*

Proof. By 2.47. $\square$

Corollary 4.9. *If X is p-compact and $Y \subset X$, $|cl_{\tau_p}(Y)| \leq |Y|^\omega$.*

Mimicking the proof of 2.47, we can show that the two corollaries above are true even if (X, τ) is not p-compact to begin with, but we will not need this fact in what follows.

Now let's look at the specific example of interest:

Definition 4.10. Let θ be the standard topology on βX. $\beta_p X = cl_{\theta_p}(X)$ with the subspace topology inherited from $(\beta X, \theta_p)$.

Note that $\beta_p X = \cap\{Y \subset \beta X \mid X \subset Y \text{ and } Y \text{ is } p\text{-compact}\}$, so this notation is commensurate with the notation found in [9], although there they consider $\beta_p X$ to have the subspace topology inherited from βX.

We will show that $\beta_p \omega$ is a countably compact, countably tight, separable, Urysohn space that is not compact or sequential. We will then show that $T_p X = \beta_p X$ (regarding X as a set and a discrete space, respectively), giving a topological proof of 3.8.

Proposition 4.11. *$\beta_p \omega$ is p-compact and thus countably compact.*

Proof. $\beta\omega$ is compact and so p-compact by 2.36. This follows from 4.6 and 2.41. □

Proposition 4.12. *$\beta_p \omega$ is p-sequential and thus countably tight.*

Proof. By 4.7 and 2.49. □

Proposition 4.13. *$\beta_p \omega$ is separable.*

Proof. $\omega \subset \beta_p \omega$ is a countable dense set. □

Proposition 4.14. *$\beta_p \omega$ is Urysohn.*

Proof. By 4.4, if θ is the standard topology on $\beta\omega$, θ_p is finer than θ. Given $x, y \in \beta_p \omega$, since $\beta\omega$ is Urysohn, there are $U_x, U_y \in \theta$ containing x and y respectively with $cl_\theta(U_x) \cap cl_\theta(U_y) = \emptyset$. Since θ_p is finer than θ, $U_x, U_y \in \theta_p$ and $cl_{\theta_p}(U_x) \cap cl_{\theta_p}(U_y) \subset cl_\theta(U_x) \cap cl_\theta(U_y) = \emptyset$ so $(\beta\omega, \theta_p)$ is Urysohn. Since $\beta_p \omega$ is a subspace of $(\beta\omega, \theta_p)$, it is Urysohn as well. □

Proposition 4.15. *$\beta_p \omega$ is not compact.*

Proof. Let $i : \beta_p\omega \to \beta\omega$ denote the inclusion map. i is continuous since the topology on $\beta_p\omega$ is finer than the standard topology on $\beta\omega$ by 4.4. If $\beta_p\omega$ were compact, then $i(\beta_p\omega)$ would be a compact set (by 2.4) with $\omega \subset i(\beta_p\omega) \subset \beta\omega$, and since compact subsets of $\beta\omega$ are closed (by 2.18 and 2.3) and $\omega \subset \beta\omega$ is dense (by 2.17), $i(\beta_p\omega) = \beta\omega$. But $|\beta_p\omega| \leq 2^\omega$ by 4.9 and $|\beta\omega| = 2^{2^\omega}$ by 2.22, a contradiction. □

Proposition 4.16. *$\beta_p\omega$ contains no non-trivial convergent sequences.*

Proof. Since the topology on $\beta_p\omega$ is finer than the standard topology, the inclusion map $i : \beta_p\omega \to \beta\omega$ is continuous. Thus if $x_n \to x$ in $\beta_p\omega$, $i(x_n) \to i(x)$ in $\beta\omega$ by 2.5 and thus (x_n) is eventually constant by 2.23. □

Corollary 4.17. *$\beta_p\omega$ is not sequential.*

Proof. If $F \subset \beta_p\omega$ and (x_n) is a sequence in F with $x_n \to x$ then $x = x_m$ for some m so $x \in F$. Thus every subset of $\beta_p\omega$ is sequentially closed. But $\beta_p\omega$ is not discrete ($\omega \subset \beta_p\omega$ is not closed), so $\beta_p\omega$ is not sequential. □

We can now recover a proof of 3.8 by showing the following:

Theorem 4.18. *Let X be a set. The topological spaces T_pX and β_pX (considering X as a discrete space) are identical.*

The proof will follow from the next several propositions. For the remainder of this section, suppose that X is discrete. Recall the definitions related to T_p from 3.1. T_pX denotes the functor T_p applied to the underlying set of X.

Proposition 4.19. *For any $A \subset T_pX$ and $g : \omega \to A$, if $i : A \to T_pX$ is the inclusion map, $id^{\#}_{T_pX}\big((T_pi)(gp)\big) = p\text{-}\lim_{\beta X} g(n)$.*

Proof. Let A, g and i be as in the hypothesis. By definition,

$$id^{\#}_{T_pX}\big((T_pi)(gp)\big) = \{D \subset X \mid \{x \in T_pX \mid D \in x\} \in (T_pi)(gp)\}$$

and

$$\{x \in T_pX \mid D \in x\} \in (T_pi)(gp) \iff$$

$$\exists C \in gp(C \subset \{x \in T_pX \mid D \in x\}) \iff$$

$$\exists C \in gp(\forall y \in C(y \in \{x \in T_pX \mid D \in x\})) \iff$$

$$\exists C \in gp(\forall y \in C(D \in y)) \iff$$

$$\exists B \in p \land \exists C \supset g(B)(\forall y \in C(D \in y)) \iff$$

$$\exists B \in p(\forall y \in g(B)(D \in y))$$

The last equivalence follows from taking $C = g(B)$. Thus $D \in id^{\#}_{T_pX}\big((T_pi)(gp)\big) \iff \exists B \in p$ with $D \in \cap g(B)$, and so $id^{\#}_{T_pX}\big((T_pi)(gp)\big) = \cup_{B\in p} \cap g(B)$

Given any basic open $\overline{O} \subset \beta X$ containing $\cup_{B\in p} \cap g(B)$, $O \in \cup_{B\in p} \cap g(B)$ so there is some $B \in p$ so that $O \in x$ for every $x \in g(B)$. Thus $B \subset \{n \mid O \in g(n)\} = \{n \mid g(n) \in \overline{O}\}$ and so $\{n \mid g(n) \in \overline{O}\} \in p$ and $\cup_{B\in p} \cap g(B) = p\text{-}\lim g(n)$ as required. □

Corollary 4.20. *For any $g : \omega \to T_pX, id^{\#}_{T_pX}(gp) = p\text{-}\lim g(n)$.*

Proof. $T_p(id_{T_pX}) = id_{T_pT_pX}$. □

Proposition 4.21. *$T_pX = \beta_pX$ as sets.*

Proof. Since $(T_p, \eta, (\cdot)^\sharp)$ is a monad and $id_{T_pX} : T_pX \to T_pX$, $id^{\#}_{T_pX}(T_pT_pX) \subset T_pX$. For any sequence (x_n) in T_pX, let $g : \omega \to T_pX$ by $n \mapsto x_n$, so $gp \in T_pT_pX$ and $id^{\#}_{T_pX}(gp) = p\text{-}\lim x_n \in T_pX$. Thus T_pX is p-compact with $X \subset T_pX \subset \beta X$ and so $T_pX \supset \beta_pX$.

On the other hand, recalling definition 2.46, $A_0(\{p\}, X) = X \subset T_pX$ trivially. If $A_\beta(\{p\}, X) \subset T_pX$ for all $\beta < \alpha$, then if α is a limit ordinal, $A_\alpha(\{p\}, X) \subset T_pX$ trivially. If not, let $\alpha = \gamma + 1$ and $x \in A_\alpha(\{p\}, X)$. Then there is a sequence (x_n) in $A_\gamma(\{p\}, X)$ with $x = p\text{-}\lim x_n$. Let $g : \omega \to A_\gamma(\{p\}, X) \subset T_pX$ by $n \mapsto x_n$. Then $x = p\text{-}\lim x_n = id^{\#}_{T_pX}(gp) \in T_pX$. Thus $A_\alpha(\{p\}, X) \subset T_pX$ for all $\alpha \leq \omega_1$ and so $\beta_pX \subset T_pX$. □

Proposition 4.22. *The topologies on T_pX and β_pX coincide.*

Proof. Recall that $A \subset T_pX$ is closed provided there is a set map $\xi_0 : T_pA \to A$ with $id^{\#}_{T_pX} \circ T_pi = i \circ \xi_0$, where i is the inclusion map. Such a ξ_0 will exist if and only if $id^{\#}_{T_p\omega}\big((T_pi)(T_pA)\big) \subset A$, or $\forall gp \in T_pA, id^{\#}_{T_p\omega}\big((T_pi)(gp)\big) = p\text{-}\lim g(n) \in A$. Thus $A \subset T_pX \subset \beta X$ is closed in T_pX if and only if it is p-compact when considered as a subset of βX. □

4.2 UX

For different choices of p, each $\beta_p\omega$ captures only a small part of the structure of $\beta\omega$ (since $|\beta_p\omega| = 2^\omega$ and $|\beta\omega| = 2^{2^\omega}$). For this section, fix a completely regular, Hausdorff space X. In what follows, we'll describe a general way to glue together the various β_pX into a new topology on the entire set βX.

By [9], if $p \leq_C q$ then $\beta_pX \subset \beta_qX$.

Definition 4.23. For $p \leq_C q$, let $\iota_{pq} : \beta_pX \hookrightarrow \beta_qX$ denote the inclusion map.

Definition 4.24. For any $p \in \omega^*$, let $\iota_p : \beta_pX \hookrightarrow \beta X$ denote the inclusion map.

Proposition 4.25. *If $p \leq_C q$ then ι_{pq} is continuous.*

Proof. Suppose $p \leq_C q$. If $C \subset \beta_qX$ is closed, then C is q-compact. Since $p \leq_C q$, C is also p-compact. Thus $\iota_{pq}^{-1}(C) = C \cap \beta_pX$ is p-compact and so closed in β_pX. □

Definition 4.26. let $UX = \varinjlim\{\beta_pX, \iota_{pq}\}$ where $\iota_{pq} : \beta_pX \to \beta_qX$ is the inclusion map.

As a set, $UX = \{[r] \mid r \in \beta_p\omega \text{ for some } p\}$ with $[r] = [r']$ if $\iota_{pq}(r) = \iota_{p'q}(r')$. Since each ι is an inclusion map, we will identify UX with βX as a set. By definition, $O \subset UX$ is open if and only if $\iota_p^{-1}(O)$ is open for every p.

Proposition 4.27. *UX is ultracompact.*

Proof. Let (x_n) be a sequence in UX and $p \in \omega^*$. For each n, $x_n \in \beta_{p_n}X$ for some $p_n \in \omega^*$. By [9], there is an $r \in \omega^*$ with $p \leq_C r$ and $p_n \leq_C r$ for each n. Also by [9], $\beta_{p_n}X \subset \beta_rX$

and so $\{x_n\} \subset \beta_r X$. $\beta_r X$ is r-compact and since $p \leq_C r$, $\beta_r X$ is p-compact. Thus p-$\lim x_n \in \beta_r X \subset UX$. $\square$

We can characterize the topology on UX in several ways:

Proposition 4.28. *The following are equivalent:*

1. *$C \subset UX$ is closed.*

2. *C is ultracompact.*

3. *$C \cap \beta_p X$ is p-compact for every $p \in \omega^*$.*

4. *$UX \setminus C$ is a weak P-set in βX.*

Proof. (1) $\Rightarrow$ (2): Since UX is ultracompact and a closed subset of a p-compact space is p-compact, any closed $C \subset UX$ is ultracompact.

(2) $\Rightarrow$ (3): Given $p \in \omega^*$, if C is ultracompact, C is p-compact and thus $C \cap \beta_p X$ is p-compact.

(3) $\Rightarrow$ (1): By definition of the direct limit, $C \subset UX$ is closed if and only if $\iota_p^{-1}(C) = C \cap \beta_p X$ is closed for each p. But $C \cap \beta_p X$ is closed in $\beta_p X$ if and only if $C \cap \beta_p X$ is p-compact.

(2) $\iff$ (4): Suppose $C \subset UX$ is ultracompact. Let $O = UX \setminus C$. If $D \subset UX \setminus O = C$ is countable and y is a limit point of D, then by 2.31, $y = p$-$\lim x_n$ for some sequence (x_n) in $D \subset C$ and $p \in \omega^*$. Since C is ultracompact, $y \in C = UX \setminus O$. Thus O is a weak P-set. Conversely, suppose O is a weak P-set. Let $C = UX \setminus O$ and let (x_n) be a sequence in C. For any $p \in \omega^*$, p-$\lim x_n \in \overline{\{x_n\}}$, so p-$\lim x_n \notin O$ by assumption. Thus p-$\lim x_n \in C$ and C is ultracompact. $\square$

$U\omega$ gives us another (much larger) example of a countably compact, countably tight, separable, Urysohn, non-compact, non-sequential space, as the next several propositions demonstrate:

Proposition 4.29. *$U\omega$ is countably compact.*

Proof. $U\omega$ is ultracompact and thus p-compact for every p. Thus $U\omega$ is countably compact by 2.41. □

Proposition 4.30. *For any $Y \subset U\omega$, $cl_{U\omega}(Y) = A_{\omega_1}(\omega^*, Y)$.*

Proof. For any sequence (x_n) in $A_{\omega_1}(\omega^*, Y)$ and $p \in \omega^*$, $\{x_n\} \subset A_\alpha(\omega^*, Y)$ for some $\alpha < \omega_1$ and so, by construction, $p\text{-}\lim x_n \in A_{\alpha+1}(\omega^*, Y) \subset A_{\omega_1}(\omega^*, Y)$. Thus $A_{\omega_1}(\omega^*, Y)$ is ultracompact and so closed. Thus $cl_{U\omega}(Y) \subset A_{\omega_1}(\omega^*, Y)$

On the other hand, $A_0(\omega^*, Y) \subset cl_{U\omega}(Y)$ trivially. Suppose $A_\beta(\omega^*, Y) \subset cl_{U\omega}(Y)$ for all $\beta < \alpha$. If α is a limit ordinal, $A_\alpha(\omega^*, Y) \subset cl_{U\omega}(Y)$ trivially. If $\alpha = \gamma + 1$ and $x \in A_\alpha(\omega^*, Y)$ then $x = p\text{-}\lim x_n$ for some sequence (x_n) in $A_\gamma(\omega^*, Y)$ and $p \in \omega^*$. Since $\{x_n\} \subset cl_{U\omega}(Y)$ and $x \in \overline{\{x_n\}}$ by 2.31, $x \in cl_{U\omega}(Y)$. Thus, by transfinite induction, $A_{\omega_1}(\omega^*, Y) \subset cl_{U\omega}(Y)$. □

Corollary 4.31. *$U\omega$ is countably tight.*

Proof. By 2.48, taking $S = \omega^*$. □

Proposition 4.32. *The topology on $U\omega$ is strictly finer than that of $\beta\omega$.*

Proof. If $C \subset \beta\omega$ is closed in the usual topology, C is compact by 2.2 and thus p-compact for every $p \in \omega^*$ by 2.36. Thus C is closed in $U\omega$ as well. If $p \in \omega^*$ is a weak P-point in ω^* (which exist by 2.29) then $\omega^* \setminus \{p\}$ is ultracompact in $\beta\omega$, so $\{p\} \cup \omega$ is open in $U\omega$ but not open in $\beta\omega$. □

Corollary 4.33. *$U\omega$ is Urysohn and not compact.*

Proposition 4.34. *$U\omega$ is not p-sequential for any p.*

Proof. By 4.9, since $|U\omega| = |\beta\omega| = 2^{2^\omega}$. □

Unfortunately, $U\omega$ does not satisfy any stronger separation axioms.

Proposition 4.35. *$U\omega$ is not regular.*

Proof. If $p \in \omega^*$ is a weak P-point, then $\omega \cup \{p\}$ is open in $U\omega$. If $p \in V \subset U$ with V open, then $V \cap \omega$ must be infinite and so $\overline{V} \not\subset U$. $\square$

Though the topologies on $U\omega$ and $\beta\omega$ differ, continuous functions on each space behave very similarly.

Proposition 4.36. $C(U\omega) = C(\beta\omega)$

Proof. Since the topology on $U\omega$ is finer, any continuous $f : \beta\omega \to \mathbb{R}$ is also continuous when considered as a map $f : U\omega \to \mathbb{R}$. Thus $C(U\omega) \supset C(\beta\omega)$.

On the other hand, suppose $f : U\omega \to \mathbb{R}$ is continuous. If $im(f)$ were not compact, there would be a sequence $y_n \in im(f)$ with no limit point. Choose $x_n \in U\omega$ so that $f(x_n) = y_n$. Then since $U\omega$ is ultracompact, for any $p \in \omega^*$, $p\text{-}\lim x_n$ exists and $f(p\text{-}\lim x_n) = p\text{-}\lim y_n$ is a limit point of $\{y_n\}$, a contradiction. Thus $f : U\omega \to im(f)$ is a map into a compact set. By the universal property of $\beta\omega$ (2.19), $f|_\omega$ extends to a unique continuous $F : \beta\omega \to im(f) \subset \mathbb{R}$. F is still continuous when considered as a map from $U\omega$ into $\mathbb{R}$ and F and f agree on the dense set ω. Since $\mathbb{R}$ is Hausdorff, $F = f$ by 2.6 and so f is a continuous as a map from $\beta\omega$ into $\mathbb{R}$. $\square$

Proposition 4.37. $C(U\omega, U\omega) = C(\beta\omega, \beta\omega)$.

Proof. If $f : \beta\omega \to \beta\omega$ is continuous and C is closed in $U\omega$, then C is ultracompact in $\beta\omega$. Let (x_n) be a sequence in $f^{-1}(C)$ and $y_n = f(x_n) \in C$. Given $p \in \omega^*$, let $x = p\text{-}\lim x_n$. Then by 2.33, $f(x) = p\text{-}\lim f(x_n) = p\text{-}\lim y_n \in C$. Thus $x \in f^{-1}(C)$ and so $f^{-1}(C)$ is ultracompact and thus closed in $U\omega$.

Conversely, suppose $f : U\omega \to U\omega$ is continuous. Let $i : U\omega \hookrightarrow \beta\omega$ be the inclusion map. Then $i \circ f$ is continuous. By the universal property of $\beta\omega$, $i \circ f|_\omega$ extends to a unique continuous function $F : \beta\omega \to \beta\omega$. As above, F is still continuous when considered as a function from $U\omega$ to $\beta\omega$. F and $i \circ f$ agree on the dense subset ω and since $U\omega$ is Hausdorff, $F = i \circ f$, so f is continuous considered as a map from $\beta\omega$ to $\beta\omega$. $\square$

Proposition 4.38. $C(U\omega \setminus \omega) \neq C(\omega^*)$.

Proof. Let $p \in \omega^*$ be a weak P-point (2.29). Let $\chi_p : \omega^* \to \mathbb{R}$ by $p \mapsto 0$ and $q \mapsto 1$ for all $q \neq p$. $\chi_p \notin C(\omega^*)$ since p is not isolated in $\beta\omega$. But since p is a weak P-point, p is isolated in $U\omega \setminus \omega$ and thus $\chi_p \in C(U\omega \setminus \omega)$. $\square$

Chapter 5

Applications of the Topological Viewpoint

Historically, the spaces $T_p\omega$ were of interest primarily because they satisfied the strongest separation properties among ZFC examples of countably compact, countably tight, non-compact spaces. $T_p\omega$ is Urysohn for every $p \in \omega^*$ so a natural question is: could $T_p\omega$ be regular for any p? While this is not known in general, we do know

Proposition 5.1. *If p is a weak P-point in ω^*, $\beta_p\omega$ is not regular.*

Proof. Let p be a weak P-point in ω^*. First note that $p\text{-}\lim n = p$ since if $\overline{O} \subset \beta\omega$ is a basic open set about p then $O = \{n \mid n \in O\} \in p$. Thus $p \in \beta_p\omega$.

Since p is a weak P-point, p is not a limit point of any countable subset of $\omega^* \setminus \{p\}$ and so $\omega^* \setminus \{p\}$ is p-compact. Thus $\omega \cup \{p\} \subset \beta_p\omega$ is open.

Suppose $p \in U \subset \overline{U} \subset \omega \cup \{p\}$ for an open $U \subset \beta_p\omega$. $\{n \in \omega \mid n \in U\} \in p$ and so $\omega \cap U$ must be infinite. But then it follows that $|\overline{\omega \cap U}| > \omega$, so $|\overline{U}| > |\omega \cup \{p\}|$, a contradiction. $\square$

With what we have established so far about $T_p\omega$, we can answer a question posed by Manes in [1].

Definition 5.2. $p \in \omega^*$ is an m-point if $G_p\omega \subsetneq T_p\omega$.

Manes showed that there is an m-point if we assume the Continuum Hypothesis and further conjectured that every $p \in \omega^*$ is an m-point in ZFC. We will prove Manes' conjecture using some tools from [9]. The key idea lies in the following:

Definition 5.3. If $*$ is a semigroup operation on ω, we may extend $*$ to a semigroup operation on $\beta\omega$. The extension, also denoted $*$, may be defined as follows: for $m \in \omega$ and $q \in \omega^*$, let $m * q = q\text{-}\lim m * n$ as n ranges over ω; for $p, q \in \omega^*$, let $p * q = p\text{-}\lim m * q$ as m ranges over ω.

Proposition 5.4. *No $p \in \omega^*$ is an m-point.*

Proof. Pick any $p \in \omega^*$. By [9, 2.1], $T_p\omega = \beta_p\omega = \{q \in \beta\omega \mid q \leq_C p\}$ is always a sub-semigroup under $*$, while $G_p\omega = \{q \in \beta\omega \mid q = fp \text{ for some } f : \omega \to \omega\} = \{q \in \omega \mid q \leq_{RK} p\}$ is never a sub-semigroup [9, 2.15]. $\square$

We can also say a little bit more about the interplay between the order $\leq_C$, the semigroup operation $*$ and the topology on $\beta_p\omega$. The next result says, in effect, that for any $p \in \omega^*$ there is an $r \leq_C p$ that is "infinitely right-divisible" by p:

Proposition 5.5. *For any $s \in \omega$ and $p \in \omega^*$, there is some $q \leq_C p$ so that for every $n \in \omega$, $q = s^n * r * p^n$ for some $r \leq_C p$.*

Proof. Fix $s \in \omega$ and $p \in \omega^*$. As observed in [9], the map $f : \beta\omega \to \beta\omega$ by $r \mapsto s * r * p$ is continuous. By [9, 2.1], $\beta_p\omega \subset \beta\omega$ is a subsemigroup under $*$, so f restricts to a function $f : \beta_p\omega \to \beta_p\omega$. If $C \subset \beta_p\omega$ is closed and (x_n) is a sequence in $f^{-1}(C)$ then $f(p\text{-}\lim x_n) = p\text{-}\lim f(x_n) \in C$ since C is p-compact. Thus, since $\beta_p\omega$ is p-sequential, $f^{-1}(C)$ is closed and so f is still continuous in the new topology on $\beta_p\omega$. For each $m \in \omega$, $f^m(\beta_p\omega) = s^m * \beta_p\omega * p^m$ is p-compact and thus closed in $\beta_p\omega$ since for each $m \in \omega$, $p\text{-}\lim s^m * r_n * p^m = p\text{-}\lim f^m(r_n) = f^m(p\text{-}\lim r_n) \in f^m(\beta_p\omega)$. Thus

$$\beta_p\omega \supset s * \beta_p\omega * p \supset s^2 * \beta_p\omega * p^2 \supset \dots$$

is a countable descending chain of closed subsets of a countably compact space. Thus by 2.42 there is some $q \in \cap s^n * \beta_p\omega * p^n$. Such a q has the property given in the statement. $\square$

We've seen that T_pX and β_pX define the same topological spaces when X is a discrete space, but β_pX can be defined for any arbitrary space X. This gives us more flexibility and allows us to construct examples that could not be constructed using the monads T_p alone.

Fix a free ultrafilter $p \in \omega^*$.

Proposition 5.6. *For any set X, T_pX contains no non-trivial convergent sequences.*

Proof. Let X be a set. This proposition can be seen from the definition of T_pX, but in light of 4.18, it is easier to see that β_pX contains no non-trivial convergent sequences when X is considered as a discrete space: since the topology on β_pX is finer than the standard subspace topology, the inclusion map $i : \beta_pX \to \beta X$ is continuous. If a sequence (x_n) converges in β_pX, then $(i(x_n))$ converges in βX by 2.5 and so $i(x_n)$ is eventually constant by 2.23. $\square$

Proposition 5.7. *There is a space X so that $UX \setminus X$ contains a non-trivial convergent sequence.*

Proof. By [10], there is a completely regular space X so that X^* contains a non-trivial convergent sequence. Let $x_n \to x$ be that sequence. We claim that $x_n \to x$ in UX as well. For any $p \in \omega^*$, p-$\lim x_n$ exists because UX is p-compact. If p-$\lim x_n = y \neq x$ then let U_x and U_y be disjoint open sets in βX around x and y respectively. Since $x_n \to x$, $\{n \mid x_n \in U_y\}$ is finite and thus $\{n \mid x_n \in U_y\} \notin p$ by 2.26 since $p \in \omega^*$.

Suppose (x_n) does not converge to x in UX. Then there is some open $O \subset UX$ so that $A = \{n \mid x_n \notin O\}$ is infinite. Fix an ultrafilter $p \in \omega^*$ with $A \in p$. Fix $z \notin O$ and let $x'_n = x_n$ if $x_n \notin O$ and $x'_n = z$ otherwise. Then by 2.34, p-$\lim_{\beta X} x_n = p$-$\lim_{\beta X} x'_n \notin O$ since $\beta X \setminus O$ is ultracompact. In particular p-$\lim_{\beta X} x_n \neq x$, contradicting the fact that $x_n \to x$ in βX. $\square$

Proposition 5.8. *For any set X, T_pX is extremally disconnected.*

Proof. Let X be a discrete space. It suffices to show that β_pX is extremally disconnected. Let τ be the standard topology on βX. Suppose $B, C \subset X$ with $B \cap C = \emptyset$ and $B \cup C = X$. We claim that $cl_{\tau_p}(B) \cap cl_{\tau_p}(C) = \emptyset$ and $cl_{\tau_p}(B) \cup cl_{\tau_p}(C) = \beta_p\omega$.

First, since τ_p is finer than τ, $cl_{\tau_p}(B) \subset cl_\tau(B)$. Thus $cl_{\tau_p}(B) \cap cl_{\tau_p} \subset cl_\tau(B) \cap cl_\tau(C) = \emptyset$ by 2.20.

To prove the second assertion, recall that by 2.47 $\beta_pX = A^Y_{\omega_1}(\{p\}, X)$ where Y is the space $(\beta X, \tau_p)$. So it suffices to show that $A_\alpha(\{p\}, X) \subset A_\alpha(\{p\}, B) \cup A_\alpha(\{p\}, C)$ for all $\alpha < \omega_1$ (the reverse inclusion is clear). Note that $A_0(\{p\}, X) = A_0(\{p\}, B) \cup A_0(\{p\}, C)$. If $A_\beta(\{p\}, X) \subset A_\beta(\{p\}, X) \cup A_\beta(\{p\}, X)$ for all $\beta < \alpha$ and α is a limit ordinal, then

$A_\alpha(\{p\}, X) \subset A_\alpha(\{p\}, X) \cup A_\alpha(\{p\}, X)$ trivially. On the other hand, if $\alpha = \gamma + 1$ for some γ and if $a \in A_\alpha(\{p\}, X)$ then $a = p\text{-}\lim x_n$ for some sequence (x_n) in $A_\gamma(\{p\}, X)$. By the induction hypothesis, (x_n) is a sequence in $A_\gamma(\{p\}, B) \cup A_\gamma(\{p\}, C)$. Let $B' = \{n \mid x_n \in A_\gamma(\{p\}, B)\}$ and $C' = \{n \mid x_n \in A_\gamma(\{p\}, C)\}$. $B' \cup C' = X$, so by 2.11.3 we may assume without loss of generality that $B' \in p$. Fix $z \in A_\gamma(\{p\}, B)$ and define $x'_n = x_n$ if $n \in B'$ and $x'_n = z$ otherwise. Then (x'_n) is a sequence in $A_\gamma(\{p\}, B)$ and so $p\text{-}\lim x'_n \in A_\alpha(\{p\}, B)$. But since $\{n \mid x_n = x'_n\} \in p$, by 2.34 $a = p\text{-}\lim x_n = p\text{-}\lim x'_n \in A_\alpha(\{p\}, B)$ as required.

Now suppose $U \subset \beta_p X$ is open. Then since X is dense in $\beta_p X$, $cl_{\tau_p}(U) = cl_{\tau_p}(U \cap X)$. By the above, $cl_{\tau_p}(U \cap X) = \beta_p X \setminus cl_{\tau_p}(X \setminus U)$, so $cl_{\tau_p}(U)$ is open and $\beta_p X$ is extremally disconnected. □

Proposition 5.9. *For any connected, completely regular, Hausdorff space X, $\beta_p X$ is connected.*

Proof. Suppose X satisfies the hypotheses. Since X is connected, $cl_Y(X)$ is connected for any space $Y \supset X$ [5]. $\beta_p X = cl_Y(X)$ where Y is the space $(\beta X, \theta_p)$ for θ the standard topology on βX. □

Thus this new topological method allows us to describe a significantly larger class of examples.

Bibliography

[1] E. Manes, *Monads in topology*, Topology Appl. 157 (2010), 961-989.

[2] P. Nyikos, *Classic problems - 25 years later (part 2)*, Topology Proc. 27 (2003), 365-378.

[3] P. Nyikos and J. Vaughn, *The Scarborough-Stone problem for Hausdorff spaces*, Topology Appl. 44 (1992), 309-316.

[4] A. Dow, *A countably compact, countably tight, non-sequential space*, Proceedings of the 1988 Northeast Conference on General Topology and Applications (1990), 71-80.

[5] A. Munkres, *Topology*, Prentice Hall, 1975.

[6] A. Kombarov, *Compactness and sequentiallity with respect to a set of ultrafilters*, Moscow Univ. Math. Bull. 40 (1985), 15-18.

[7] J. van Mill, *An introduction to $\beta\omega$*, Handbook of Set-Theoretic Topology 1984, 503-567.

[8] S. Mac Lane, *Categories for the working mathematician*, Springer-Verlag, 1971.

[9] S. Garcia-Ferreira, N. Hindman, D. Strauss, *Orderings of the Stone-Čech remainder of a discrete semigroup*, Topology Appl. 97 (1999) 127-148.

[10] R. Frič and P. Vojtáš, *Convergent Sequences in βX*, Proceedings of the 11th Winter School on Abstract Analysis, Circolo Matematico di Palermo (1984), 133-137.