THE TRIANGLE INTERSECTION PROBLEM FOR HEXAGON TRIPLE SYSTEMS

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THE TRIANGLE INTERSECTION PROBLEM FOR HEXAGON TRIPLE SYSTEMS

Carl Stuart Pettis

A Dissertation

Submitted to

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Auburn, Alabama
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THE TRIANGLE INTERSECTION PROBLEM FOR HEXAGON TRIPLE SYSTEMS

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Vita

Carl Stuart Pettis was born on June 30, 1979 in Augusta, Georgia, the second of two sons of Charles S. Pettis and Jane H. Pettis. He was an honor graduate of Thomson High School in 1997. He chose Alabama State University in Montgomery, Alabama as his institution of higher learning to pursue his Bachelor of Science degree in Mathematics, which he earned in May, 2001 as a Magna Cum Laude graduate. He would later receive his Master of Science degree in Mathematics from Alabama State University in May, 2003. Auburn University became his home in August, 2003 where he began his pursuit of the Doctorate of Philosophy.

Dissertation Abstract

THE TRIANGLE INTERSECTION PROBLEM FOR HEXAGON TRIPLE SYSTEMS

Carl Stuart Pettis

Doctor of Philosophy, August 7, 2006
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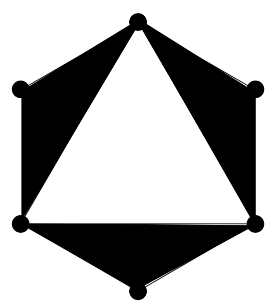
(Bachelor of Science May, 2001)

88 Typed Pages

Directed by Charles C. Lindner

A hexagon triple is the graph



and a hexagon system is an edge disjoint decomposition of $3k_n$ into hexagon triples. Note that a hexagon triple is the union of 3 triangles (= triples). The intersection problem for 3-fold triple systems has been solved for some time now. The purpose of this dissertation is to give a complete solution of the intersection problem for 3-fold triple systems each of which can be organized into hexagon triple systems.

Acknowledgments

The author would like to extend his deepest gratitude to all those who aided in this endeavor. To the members of his advisory committee and most especially Dr. Charles C. Lindner who contributed ideas and direction during the course of this research, the author would like to say thank you. The author would like for the faculty and staff of Alabama State University to know that they will always have a special place in his heart. To Mrs. Rosie Torbert, who did a wonderful job with the typing and editing of this dissertation, the author extends his warm thanks. To his family and friends for their undying support and encouragement, the author is forever indebted.

Style manual or journal used Journal of Approximation Theory (together with the style known as "aums"). Bibliograpy follows van Leunen's *A Handbook for Scholars*.

Computer software used The document preparation package TeX (specifically LaTeX) together with the departmental style-file `aums1.sty`.

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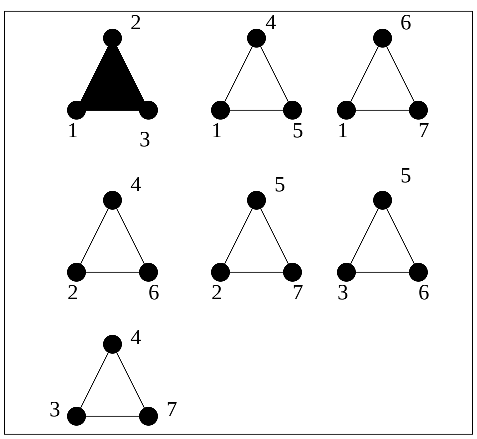
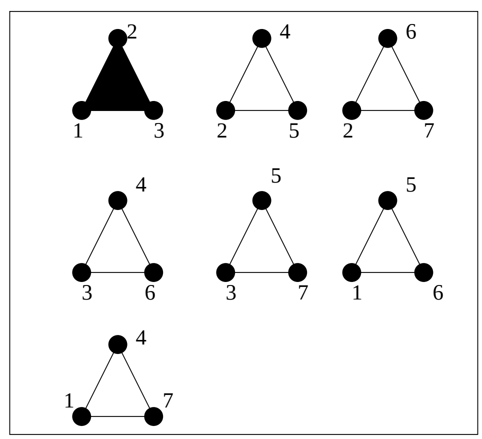
Chapter 1

Introduction

A Steiner triple system (more simply triple system) of order n is a pair (S, T), where T is a collection of edge disjoint triangles (or triples) which partitions the edge set of K_n (= the complete undirected graph on n vertices) with vertex set S. It is straightforward to see that $|T| = n(n-1)/6$. In 1846 T. P. Kirkman proved that the spectrum for triple systems (= the set of all n such that a triple system of order n exists) is precisely the set of all $n \neq 1$ or 3 (mod 6) [2].

Example 1.1 (Two triple systems of order 7)

$T_1 =$ 

$T_2 =$ 

Inspection shows that T_1 and T_2 have exactly one triple in common. In [3], C. C. Lindner and A. Rosa gave a complete solution of the intersection problem for triple systems. Let $I(n) = \{0, 1, 2, \ldots, n(n-1)/6 = x\} \backslash \{x-1, x-2, x-3, x-5\}$ and let $Int(n)$ be the set of all k such that there exists a pair of triple systems of order n having exactly k triples in common. Lindner and Rosa proved that $Int(n) = I(n)$ for all $n \equiv 1$ or 3 (mod 6), except for $n = 9$. In this case $Int(9) = I(9)\backslash\{5, 8\} = \{0, 1, 2, 3, 4, 6, 12\}$.

In what follows we will denote by λK_n the graph on n vertices with each pair of vertices connected by λ edges. A λ-fold triple system of order n is a pair (S, T), where T is a collection of triples which partitions the edge set of λK_n with vertex set S. It is easy to show that a necessary condition for the existence of a 3-fold triple system of order n is that n is odd and the number of triples is $\binom{n}{2}$. The construction of 3-fold triple systems showing that this condition is sufficient is quite easy and whoever did this for the first time is lost to history.

In what follows we will denote the triangle (triple) by $\{a, b.c\}$ or just abc.

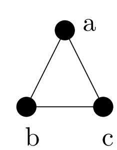


Figure 1.2: Triangle

1.3 The 3-fold Construction

Let $(Q, \circ)$ be an idempotent commutative quasigroup of order n (n must be odd). Let $T = \{\{a, b, a \circ b = b \circ a\} \mid a \neq b \in Q\}$. Then (Q, T) is a 3-fold triple system. This is quite easy to see. Let $a \neq b \in Q$, then the three triples $\{a, b, a \circ b\}, \{a, x, a \circ x = x \circ a = b\}$, and $\{b, y, b \circ y = y \circ b = a\} \in T$.

Example 1.4 (3-fold triple system of order 7)

$\circ$	1	2	3	4	5	6	7
1	1	5	2	6	3	7	4
2	5	2	6	3	7	4	1
3	2	6	3	7	4	1	5
4	6	3	7	4	1	5	2
5	3	7	4	1	5	2	6
6	7	4	1	5	2	6	3
7	4	1	5	2	6	3	7

$\{1,2,5\}$ $\{2,4,3\}$ $\{3,7,5\}$

$\{1,3,2\}$ $\{2,5,7\}$ $\{4,5,1\}$

$\{1,4,6\}$ $\{2,6,4\}$ $\{4,6,5\}$

$\{1,5,3\}$ $\{2,7,1\}$ $\{4,7,2\}$

$\{1,6,7\}$ $\{3,4,7\}$ $\{5,6,2\}$

$\{1,7,4\}$ $\{3,5,4\}$ $\{5,7,6\}$

$\{2,3,6\}$ $\{3,6,1\}$ $\{6,7,3\}$

Now just as we looked at the intersection problem for triple systems, we can look at the intersection problem for 3-fold triple systems. It is not difficult to see (an easy exercise) that if two 3-fold triple systems of order n have k triples in common, $k \in \{0, 1, 2, \ldots, \binom{n}{2} = x\} \backslash \{x-1, x-2, x-3, x-5\} = 3I(n)$. Several people have shown, except for $n = 5$, that this necessary condition is sufficient. (See [1] for example).

Example 1.5 (3-fold triple system of order 7)

$\{1,2,5\}$, $\{2,4,7\}$, $\{3,7,5\}$

$\{1,3,7\}$, $\{2,5,1\}$, $\{4,5,3\}$

$\{1,4,6\}$, $\{2,6,4\}$, $\{4,6,5\}$

$\{1,5,2\}$, $\{2,7,3\}$, $\{4,7,1\}$

$\{1,6,3\}$, $\{3,4,2\}$, $\{5,6,7\}$

$\{1,7,4\}$ $\{3,5,4\}$, $\{5,7,6\}$

$\{2,3,6\}$, $\{3,6,1\}$, $\{6,7,2\}$

A cursory check shows that the two 3-fold triple systems in Examples 1.4 and 1.5 have exactly 10 triples in common. They are

$\{1,2,5\},\{1,4,6\},\{1,4,7\},\{2,3,6\},\{2,4,6\},\{3,4,5\},$

$\{1,3,6\},\{3,5,7\},\{4,5,6\},\{5,6,7\}.$

The graph below is called a *hexagon triple* and a *hexagon triple system* of order n is a

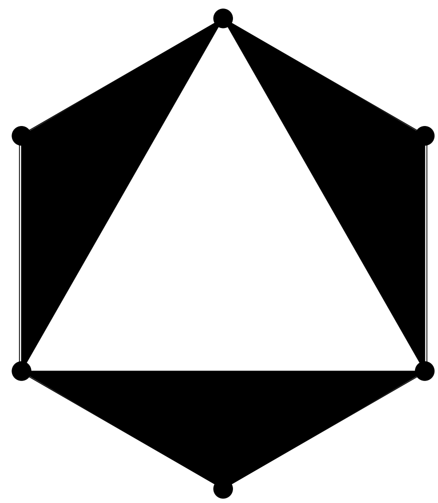
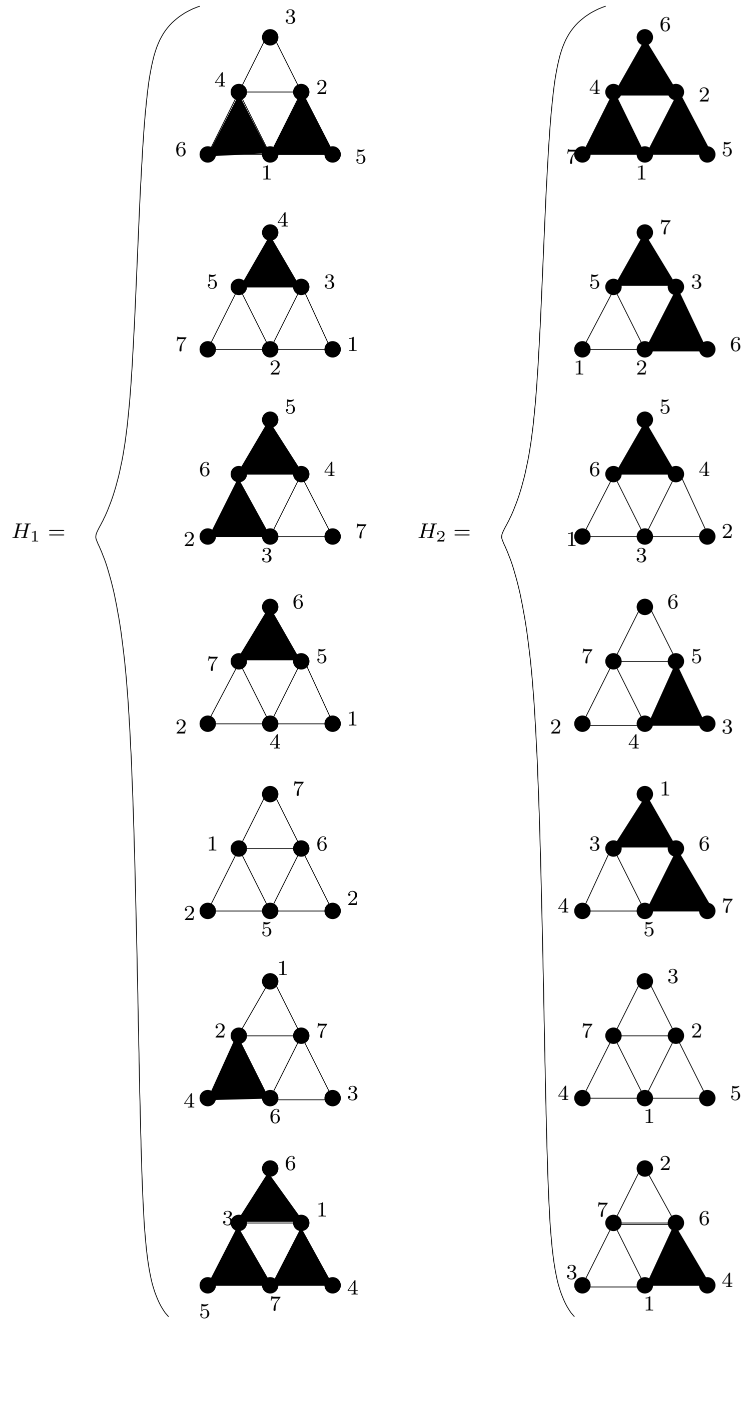


Figure 1.6: Hexagon triple

pair (X,H), where H is a collection of edge disjoint hexagon triples which partitions the edge set of $3K_n$. It is well-known (see [4] for example) that the spectrum for hexagon triple systems is precisely the set of all $n \equiv 1$ or $3 \pmod 6 \geq 7$ and that if (X,H) is a hexagon triple system of order n that $|H| = n(n-1)/6$. Note that a hexagon triple consists of 3 triples and so we can think of a hexagon triple system as the piecing together of the triples of a 3-fold triple system into hexagon triples. In the example below the two hexagon triple systems H_1 and H_2 of order 7 have been pieced together from the 3-fold triple systems in Examples 1.4 and 1.5.

Example 1.7 (Two hexagon triple systems of order 7 intersecting in 10 triples)



The object of this thesis is the solution of the following problem:

1.8 The Δ-intersection problem for hexagon triple systems:

For each $n \equiv 1$ or $3 \pmod 6 \geq 7$ and each $k \in 3I(n) = \{0, 1, 2, 3, \ldots, n(n-1)/2 = x\} \backslash \{x-1, x-2, x-3, x-5\}$ construct a pair of 3-fold triple systems of order n intersecting in k triples each of which can be organized into a hexagon triple system.

We give a complete solution of this problem, with a few possible exceptions for $n = 13$.

Chapter 2

Preliminaries

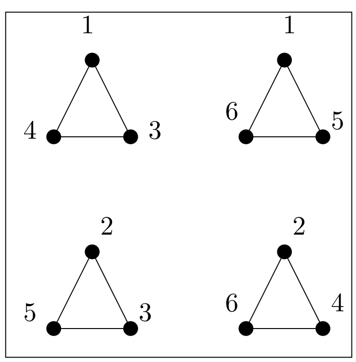
We collect together in this section some of the ideas and background material necessary to obtain the main results.

We begin with partial triple systems. A partial triple system of order n is a pair (X, P), where P is a collection of edge disjoint triples of the edge set of K_n. The difference between a partial triple system and a triple system is that the triples in a partial triple system (X, P) do not necessarily partition the edge set of K_n. We note that a triple system is also a partial triple system.

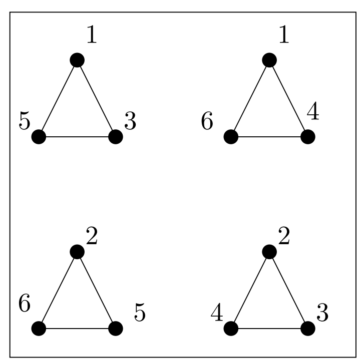
Two partial triple systems (X, P_1) and (X, P_2) are said to be balanced if the triples in P_1 and P_2 cover the same edges.

Example 2.1 (Balanced partial triple systems of order 6)

$P_1 =$



$P_2 =$

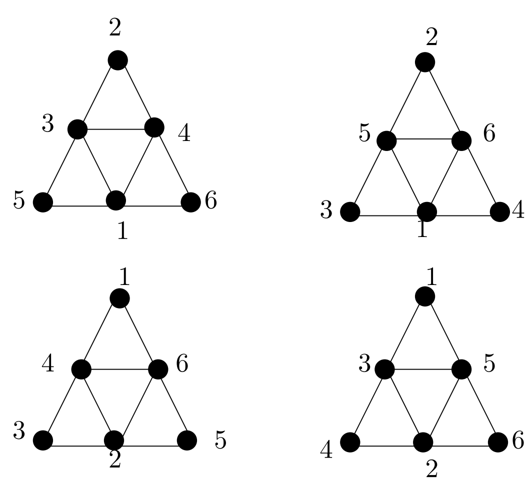


The following result is due to Lucia Grionfriddo (unpublished).

Lemma 2.2 *Let (X, P_1) and (X, P_2) be partial triple systems that are balanced and* disjoint (*having no triples in common*). *Then using each triple in P_2 three times, we can construct a partial hexagon triple system whose inside triples are P_1.*

Proof Let $\{a, b, c\} \in P_2$ and place $\{a, b, c\}$ on the triples in P_1 containing the edges $\{a, b\}, \{a, c\}$, and $\{b, c\}$. Since P_1 and P_2 are disjoint, the three triples in P_1 are disjoint. Since P_2 is a partial triple system, the three triples in P_2 placed on each triple in P_1 form a hexagon triple. □

Example 2.3 (Partial hexagon triple system constructed from Example 2.1)



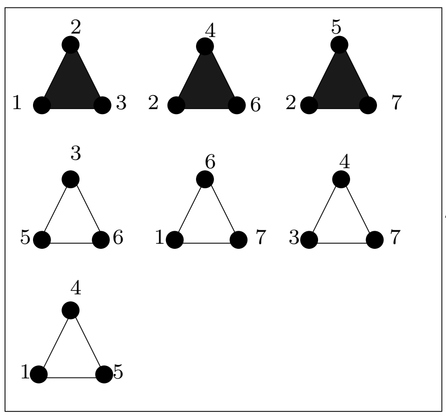
Corollary 2.4 *If $k \in Int(n)$ for Steiner triple systems, then $3k \in 3Int(n)$ for hexagon triple systems.*

Proof Let (S, T_1) and (S, T_2) be a pair of triple systems intersecting in k triples. Luc Teirlinck [7] has shown that every Steiner triple system has a disjoint mate. So let (S, T_1^*)

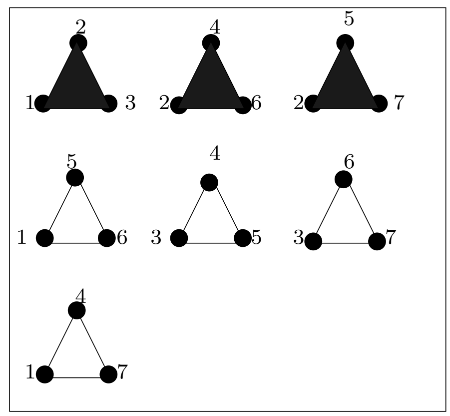
and (S, T_2^*) be triple systems such that $T_1 \cap T_1^* = \emptyset$ and $T_2 \cap T_2^* = \emptyset$. If we place the triples of $3T_1$ on T_1^* and the triples of $3T_2$ on T_2^* the resulting hexagon triple systems H_1 and H_2 have exactly $3k$ triples in common. $\square$

Example 2.5 (Hexagon triple systems of order 7 intersecting in 9 triples)

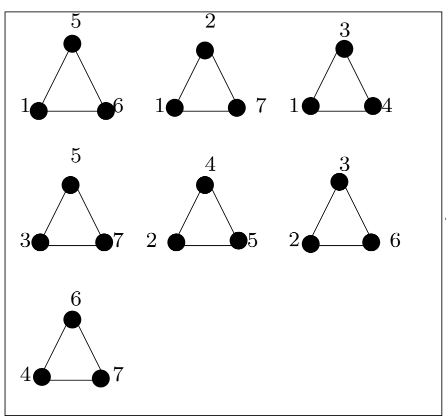
$T_1 =$



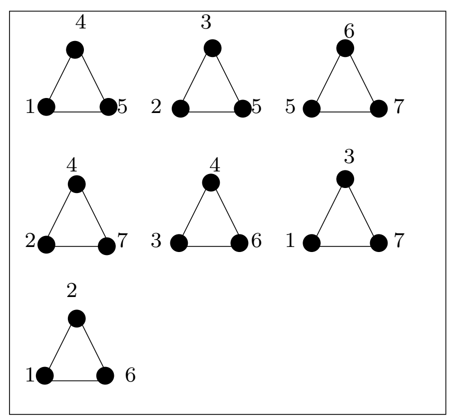
$T_2 =$

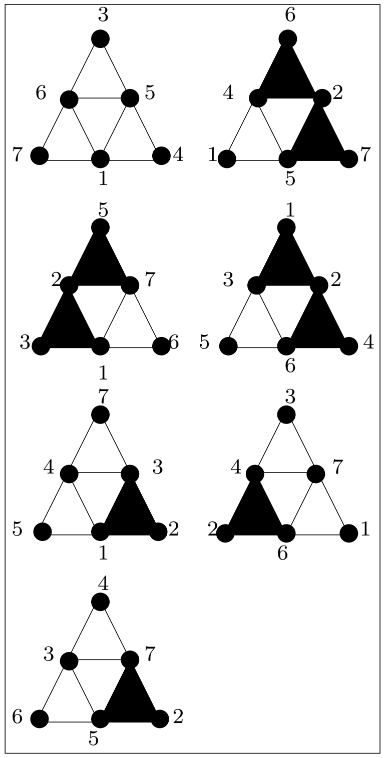
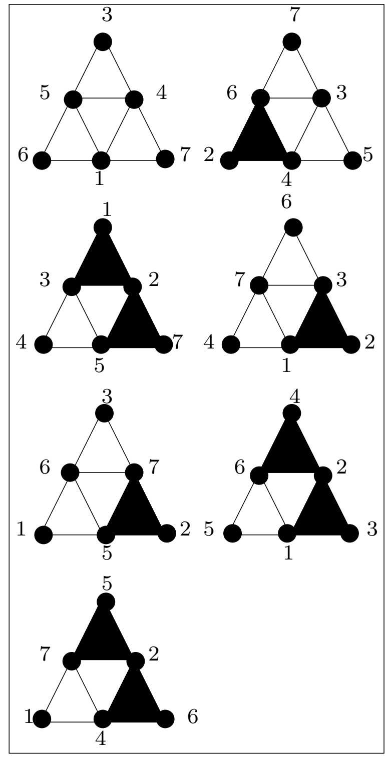


$T_1^* =$



$T_2^* =$



$H_1 =$  $H_2 =$ 

Finally, we will need both pairwise balanced designs and group divisible designs for the main constructions in this thesis.

A *pairwise balanced design* (PBD) of order n is a pair (X, B), where $|X| = n$, and B is a collection of subsets of X called *blocks* such that each pair of distinct elements of X belongs to *exactly* one block of B. The blocks in B are not necessarily the same size. In terms of graph theory, we can think of B as a collection of complete graphs (not necessarily of the same size) which partition the edge set of K_n with vertex set X. So, for example, a Steiner triple system is a pairwise balanced design in which every block has size 3.

A *group divisible design* (GDD) of order n is a triple (X, G, B), where $(X, G \cup B)$ is a PBD of order n and G is a parallel class of blocks, called *groups*, which partitions X.

Example 2.6 (**GDD of order 6 with groups of size 2 and blocks of size 3**)

$$\begin{cases} X & = \{1,2,3,4,5,6\} \\ G & = \{\{1,2\},\{3,4\},\{5,6\}\} \\ B & = \{\{1,3,5\},\{1,4,6\},\{2,4,5\},\{2,3,6\}\} \end{cases}$$

Example 2.7 (**GDD of order 10 with group sizes 4 and 2 and block size 3**)

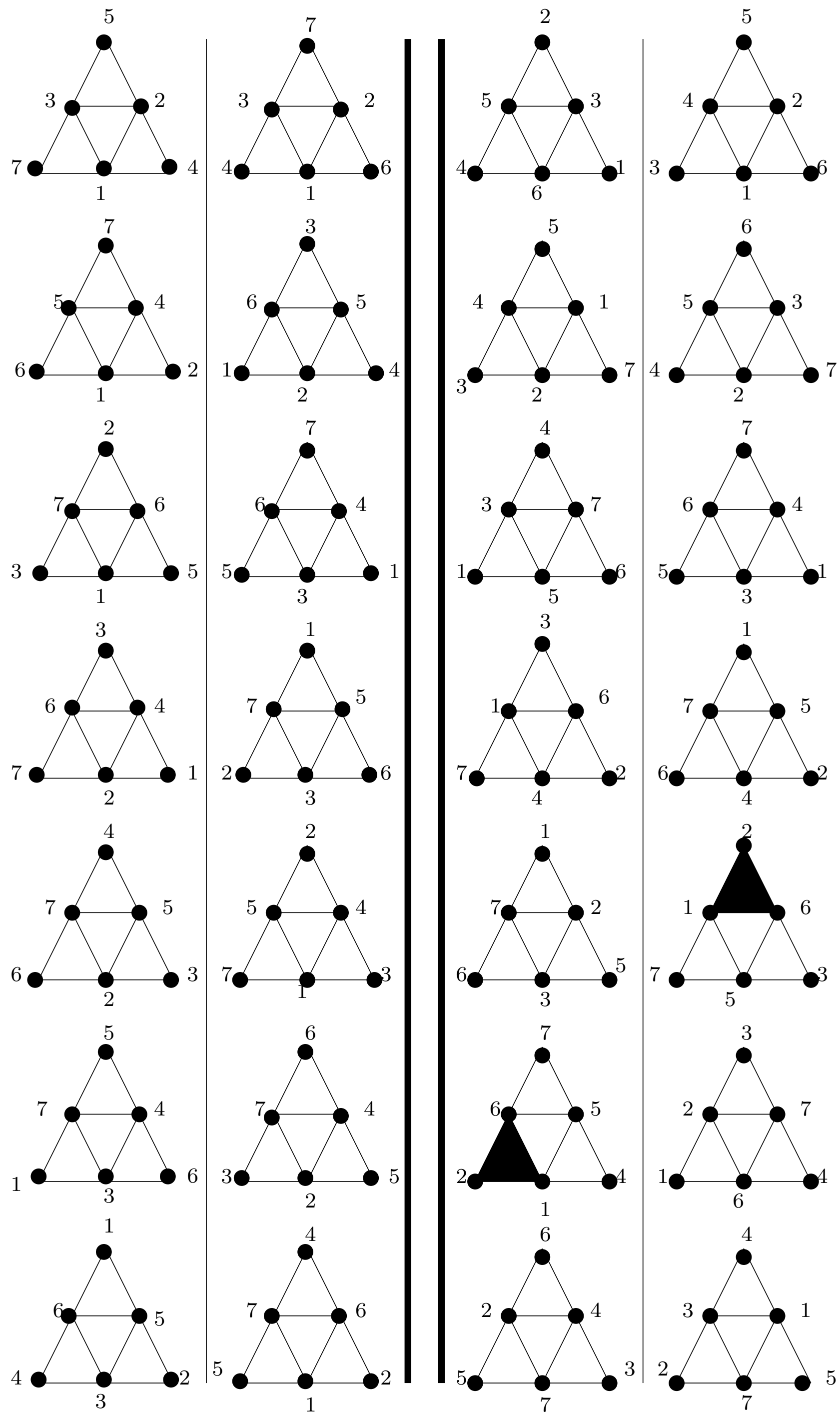
$$\begin{cases} X = & \{1,2,3,4,5,6,7,8,9,10\} \\ G = & \{\{1,2,3,4\},\{5,6\},\{7,8\},\{9,10\}\} \\ B = & \{\{1,5,9\},\{1,6,8\},\{1,7,10\},\{2,5,10\},\{2,6,7\},\{2,8,9\}, \\ & \{3,5,8\},\{3,7,9\},\{3,6,10\},\{4,6,9\},\{4,5,7\},\{4,8,10\}\} \end{cases}$$

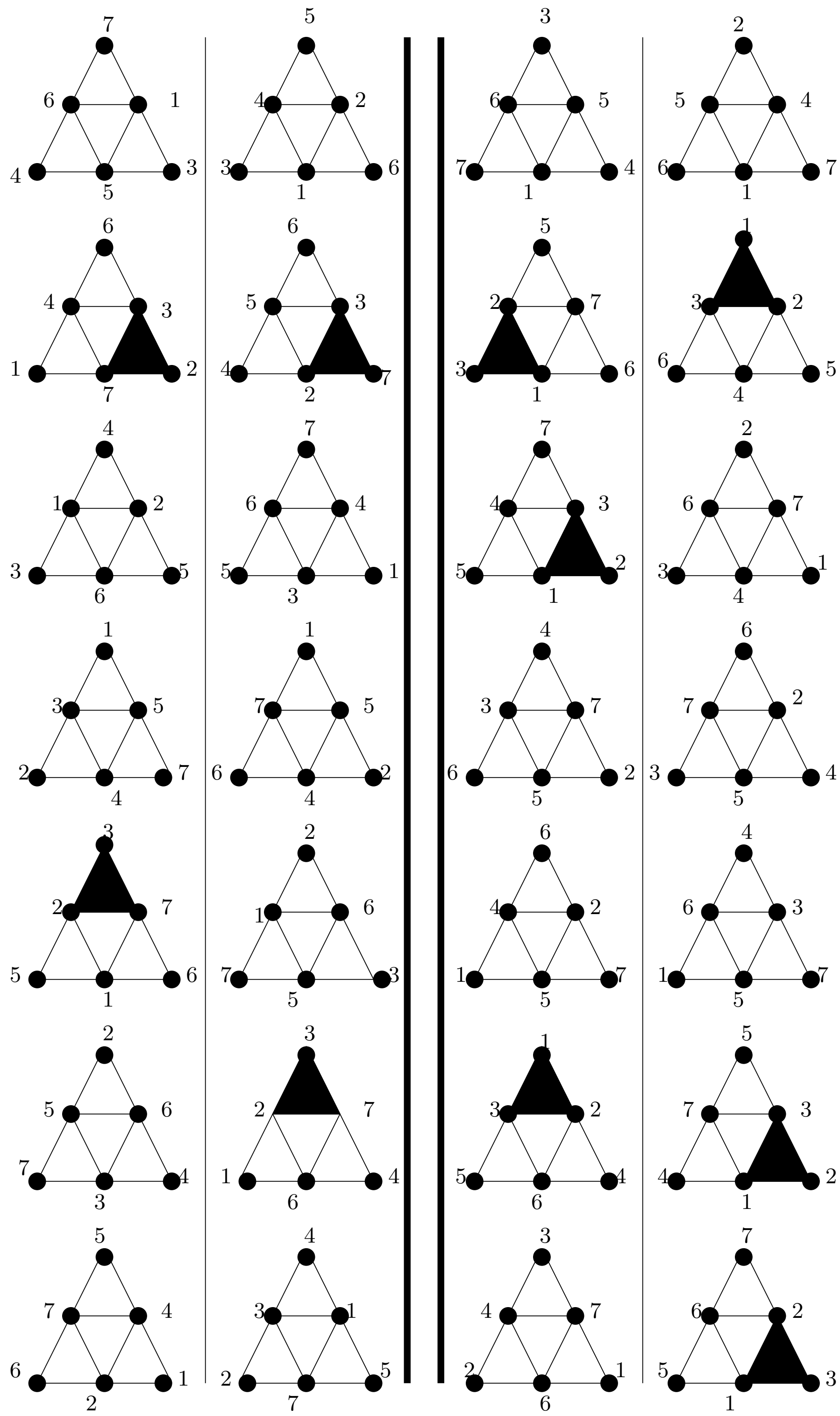
If $2n \equiv 0$ or $2 \pmod 6 \geq 6$, there is a GDD of order $2n$ with all groups of size 2 and blocks (= triples) of size 3. If $2n \equiv 4 \pmod 6 \geq 10$, there is a GDD of order $2n$ with *exactly one* group of size 4 and the remaining groups of size 2 and blocks of size 3. (See [6] for example.)

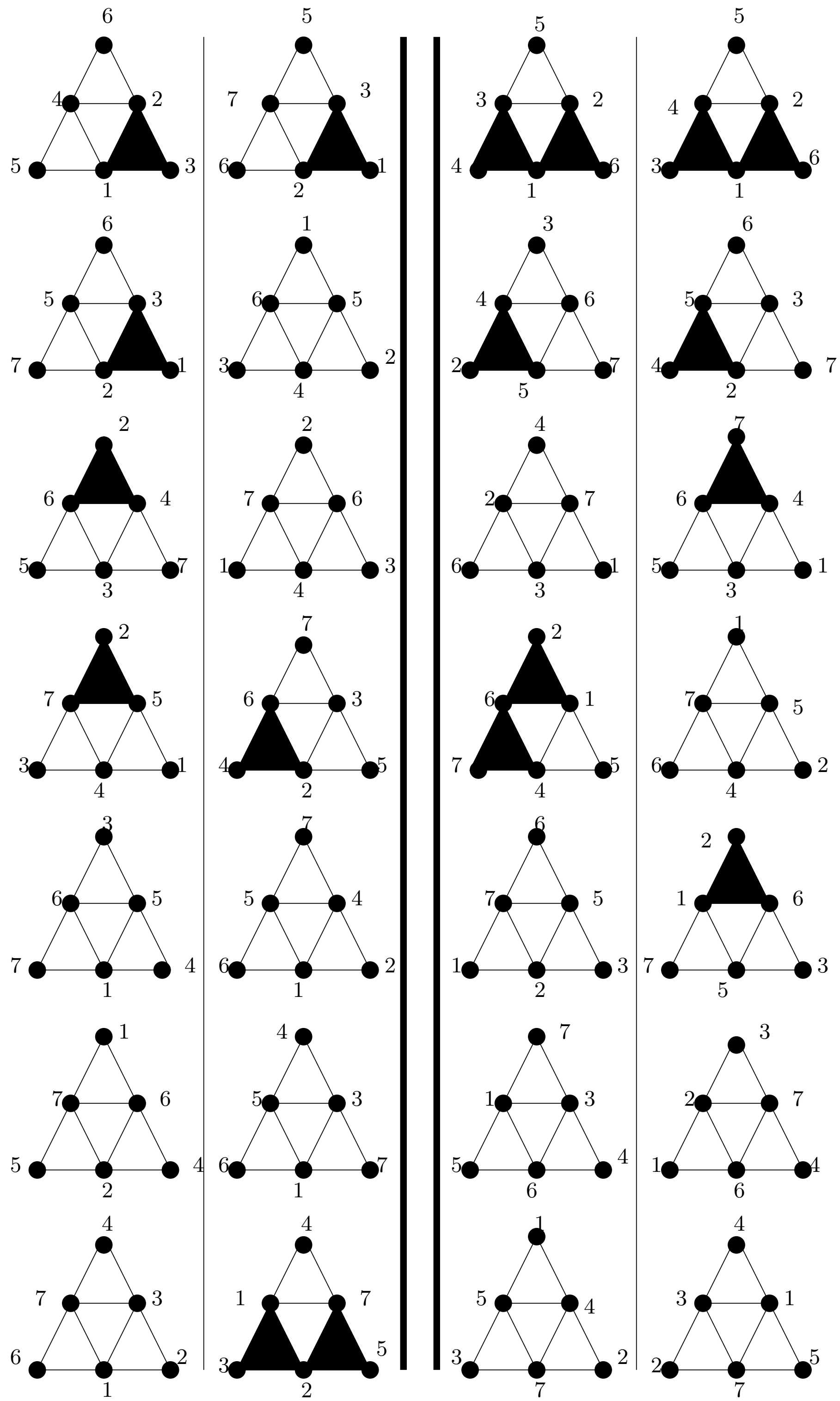
Chapter 3

$n = 7$

We give a complete solution of the Δ-intersection problem for $n = 7$ in this section. In what follows for each $x \in Int(7) = \{0, 1, 2, \ldots, 21\} \backslash \{20, 19, 18, 16\}$ we list the pair of hexagon triple systems having x triples in common. This gives a complete solution.



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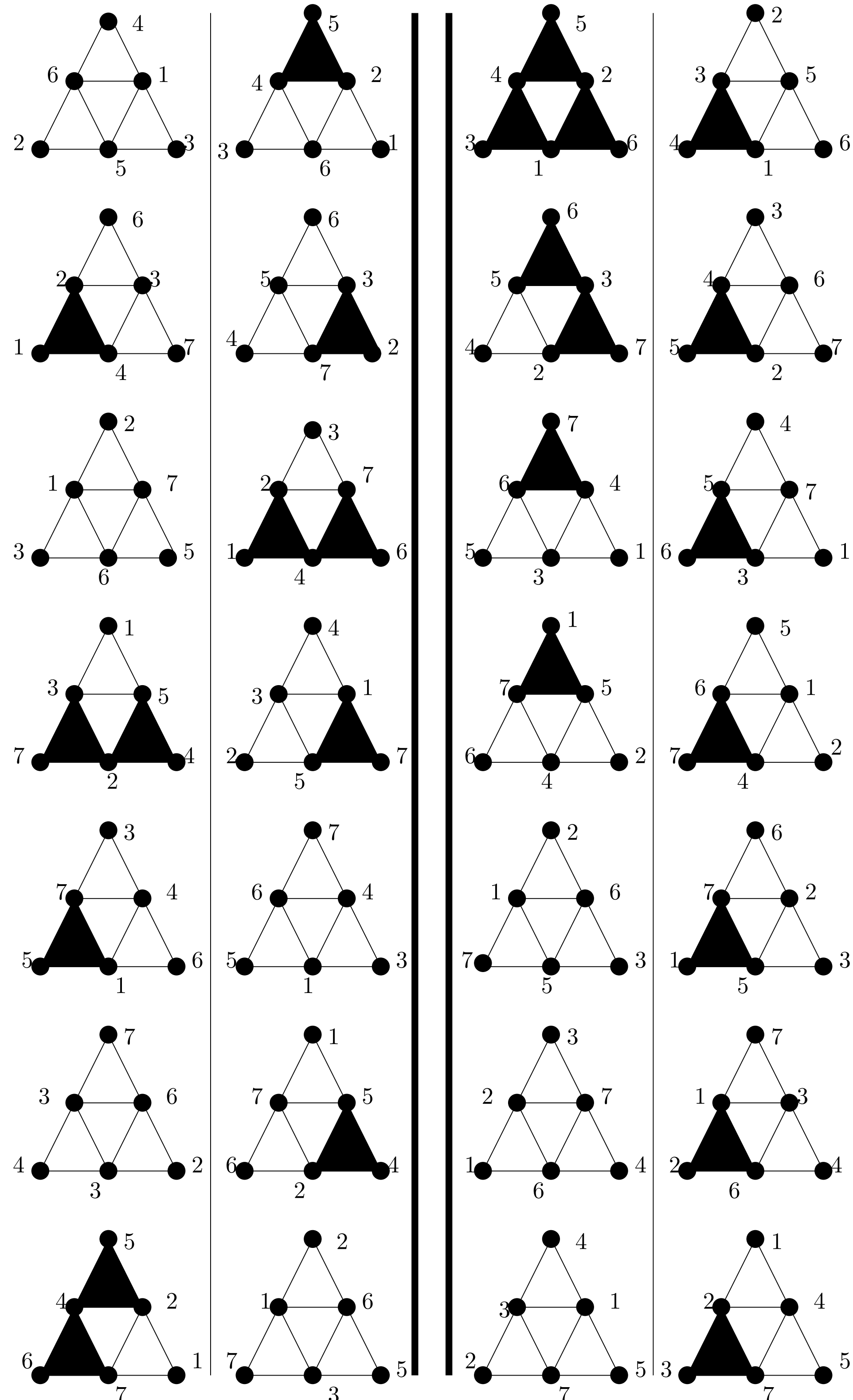
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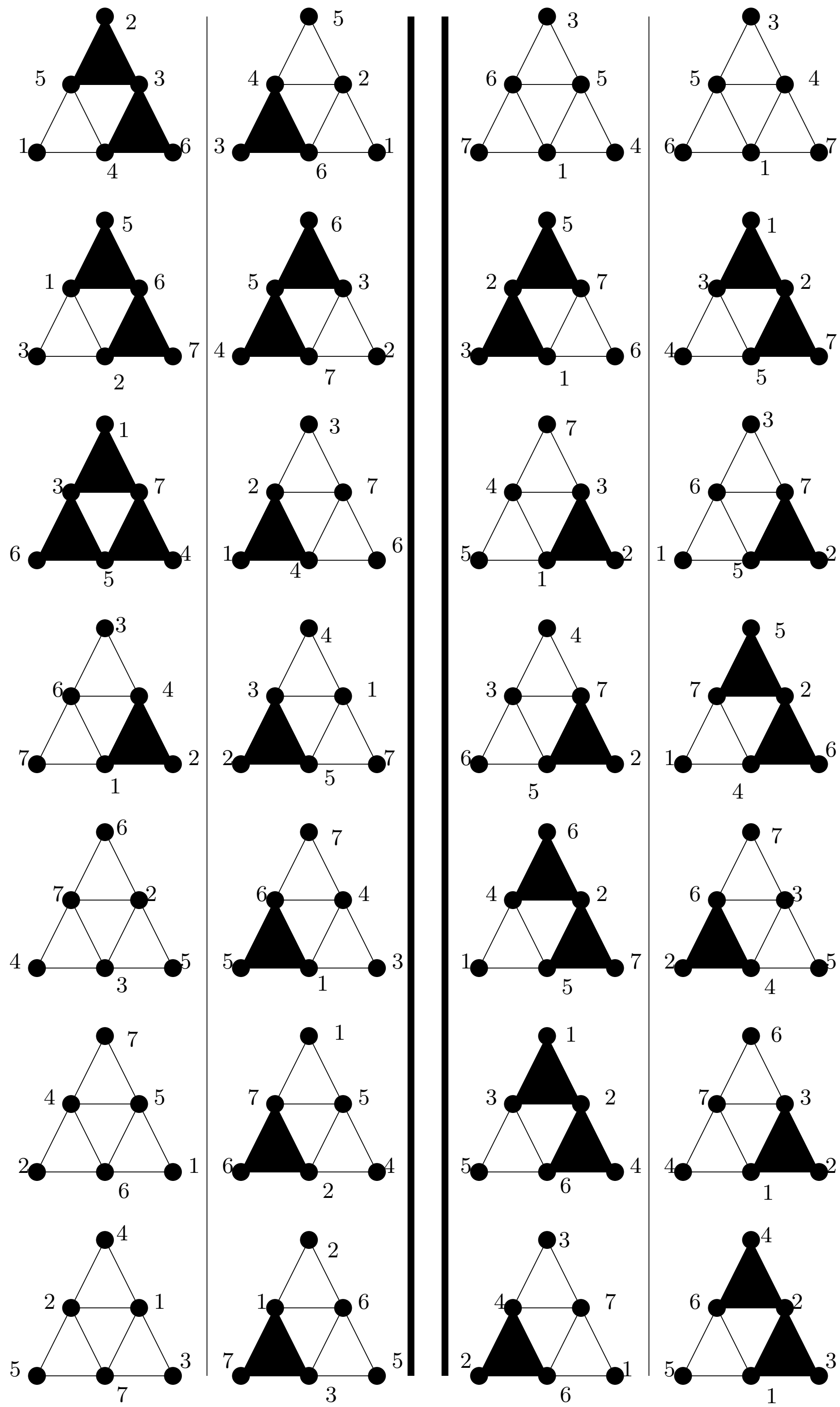
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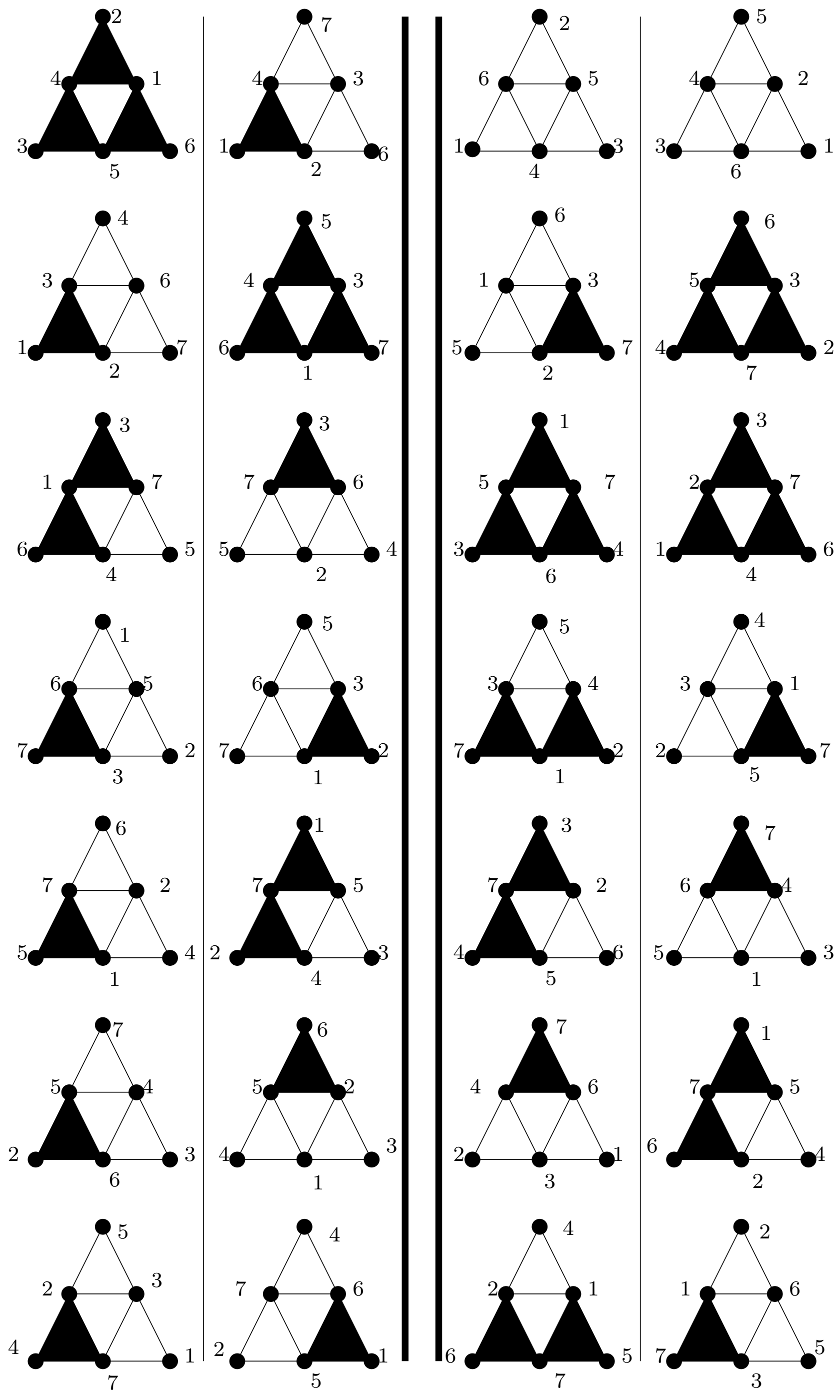

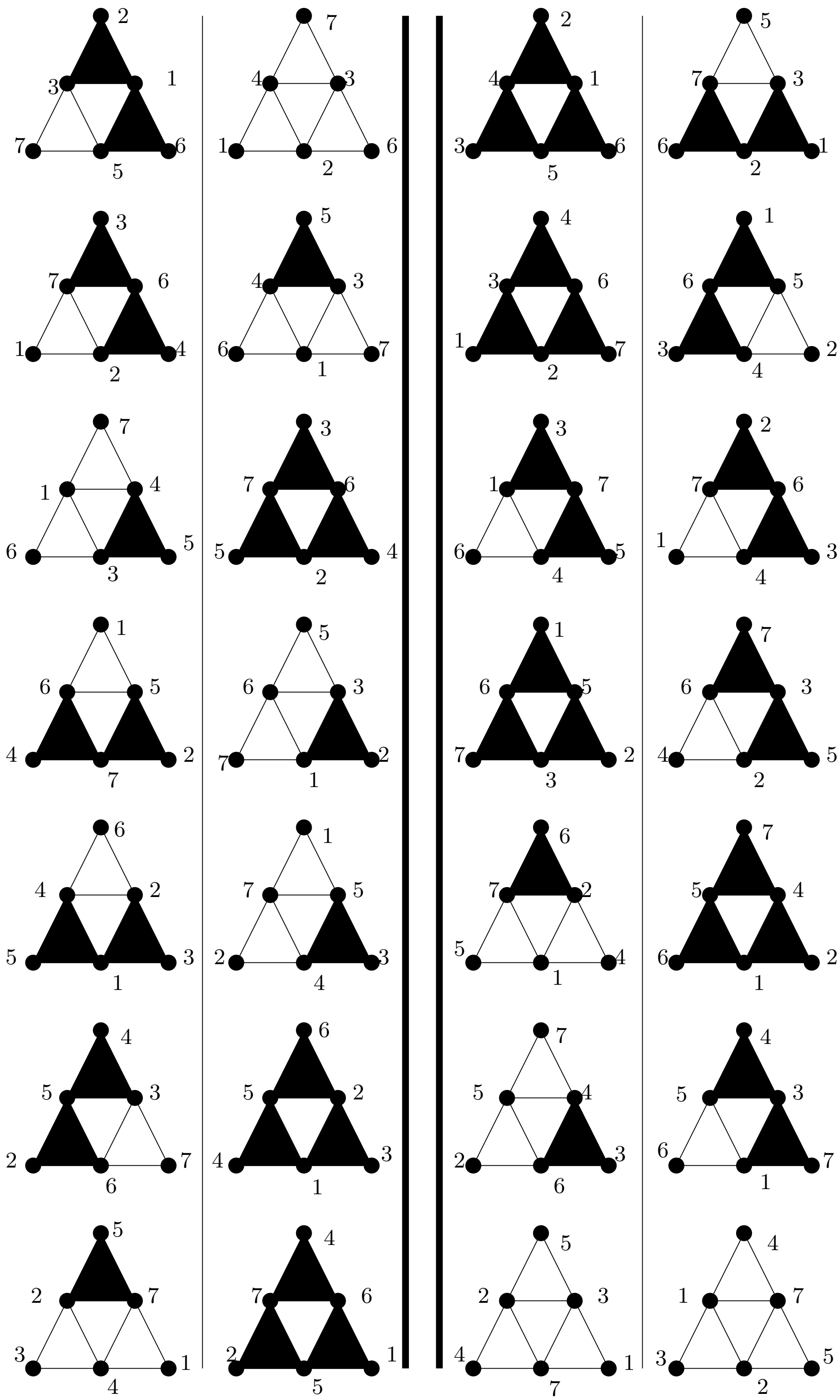

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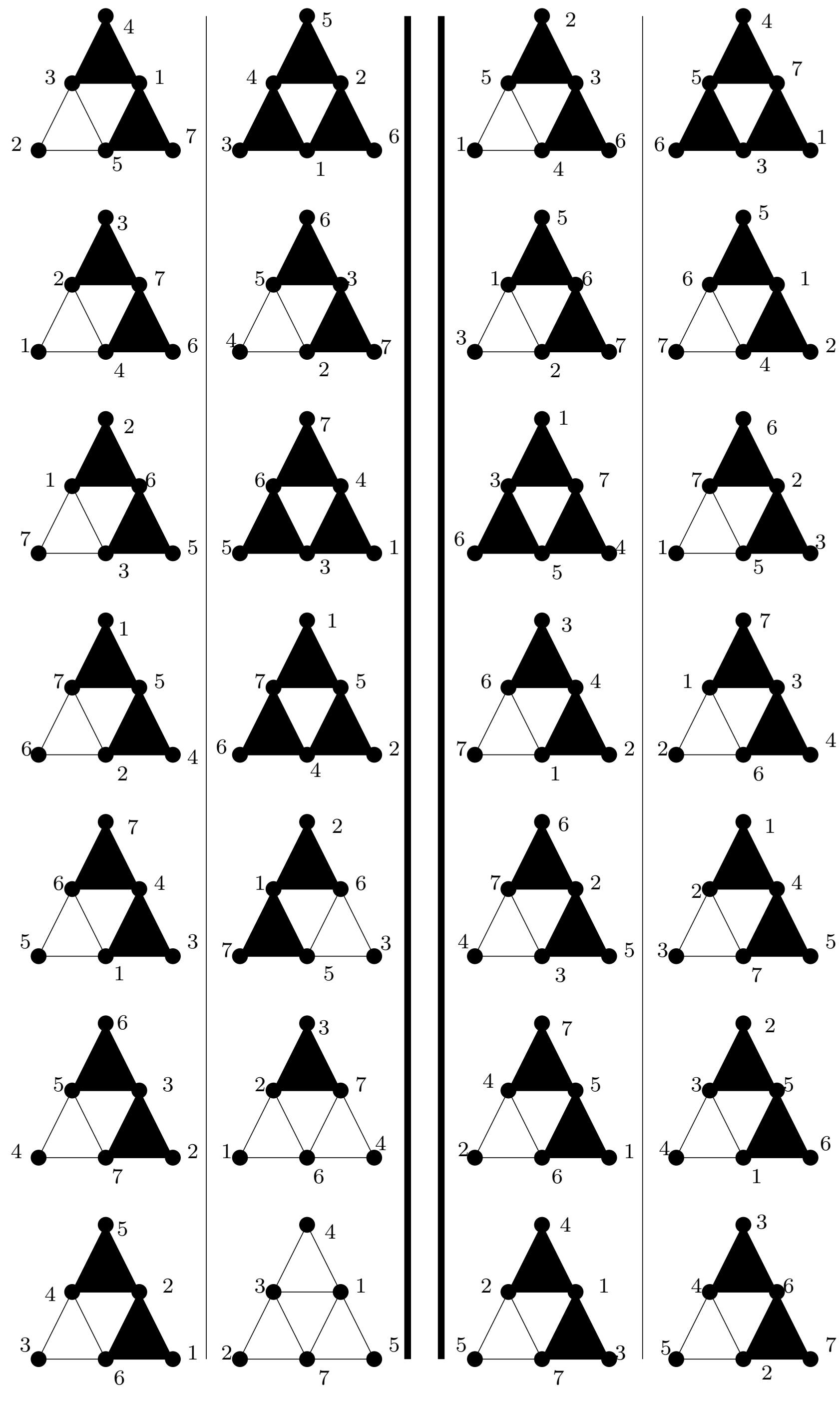
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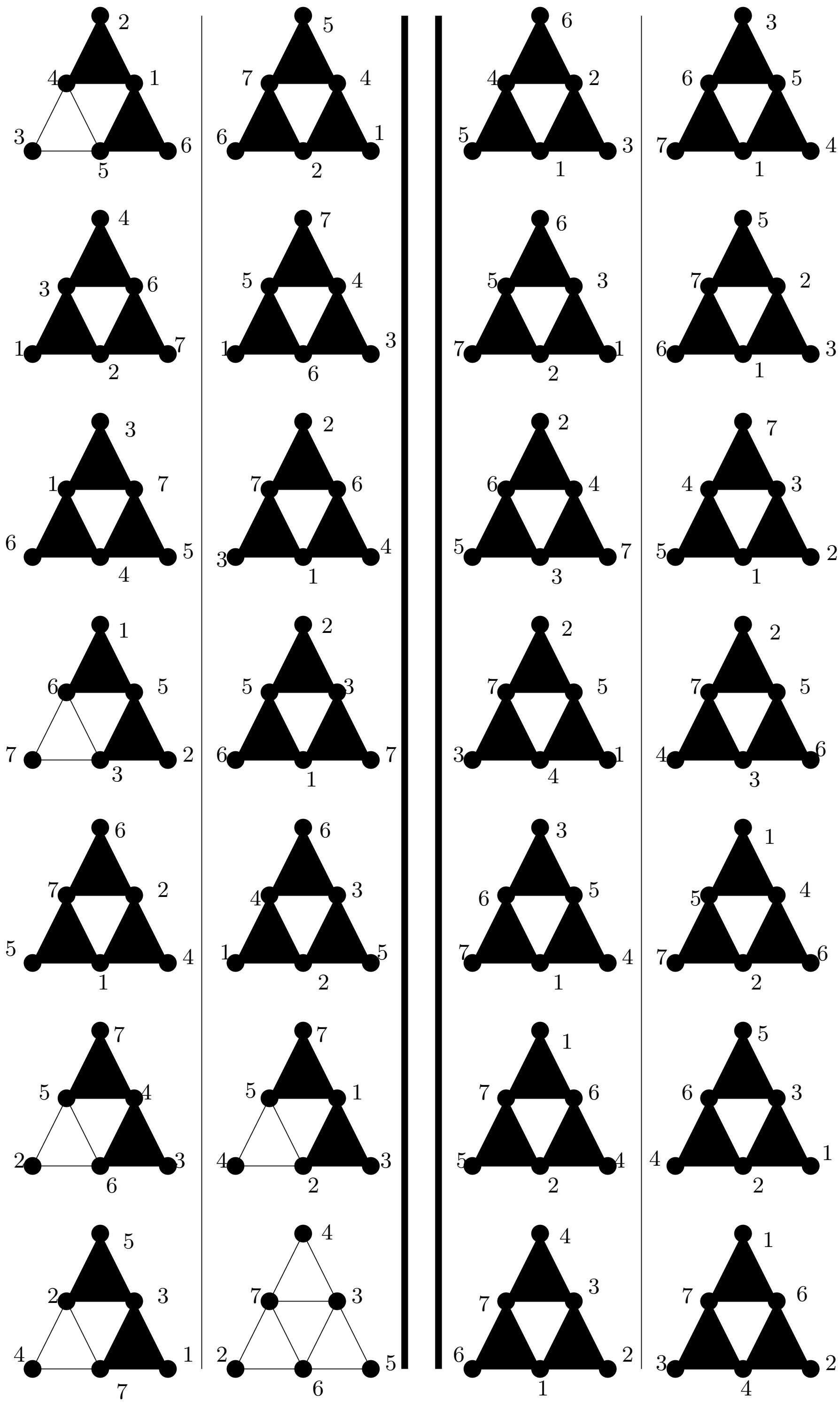


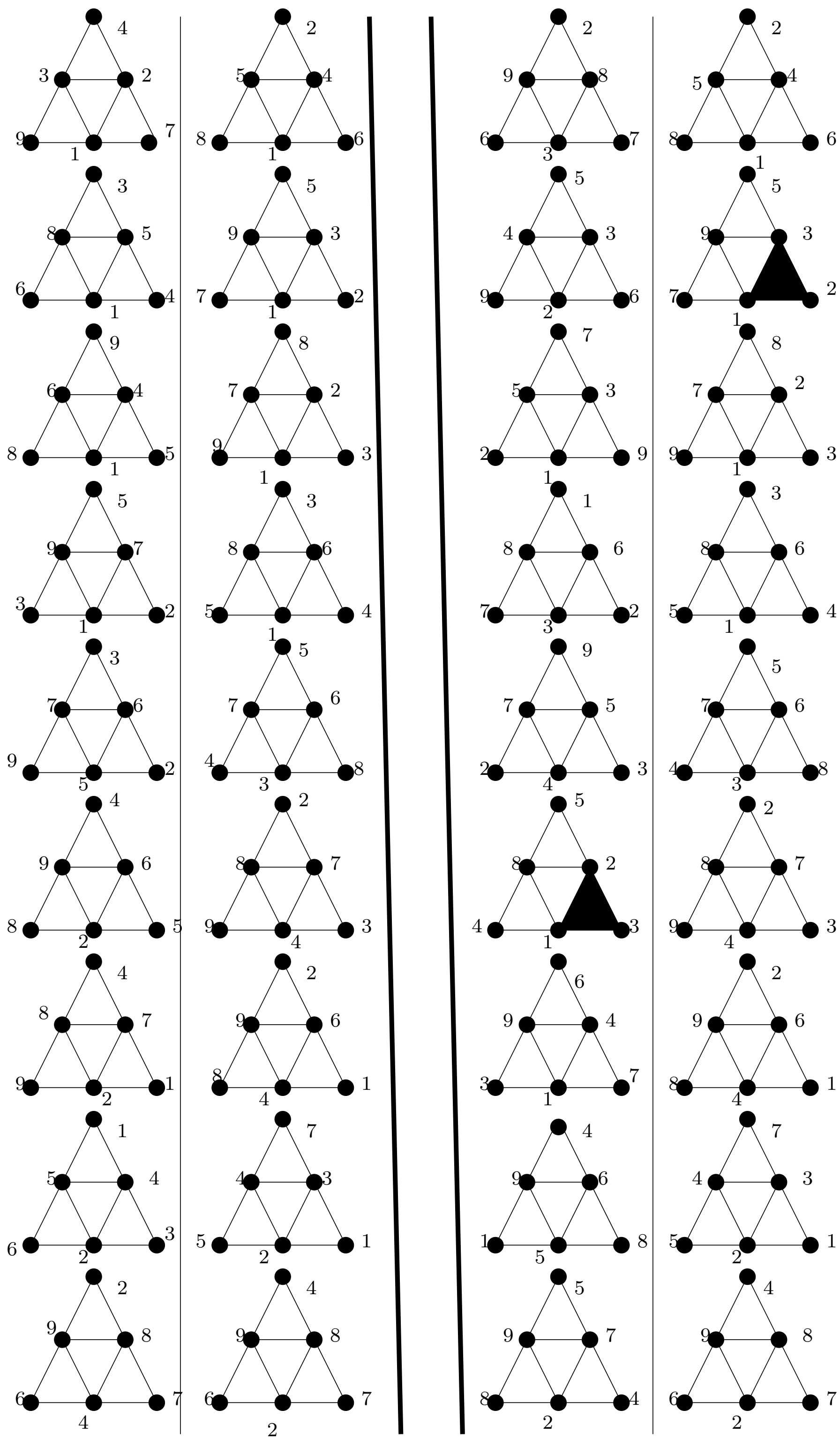
The preceding pages give a complete solution to the Δ-intersection problem for hexagon triple systems of order 7.

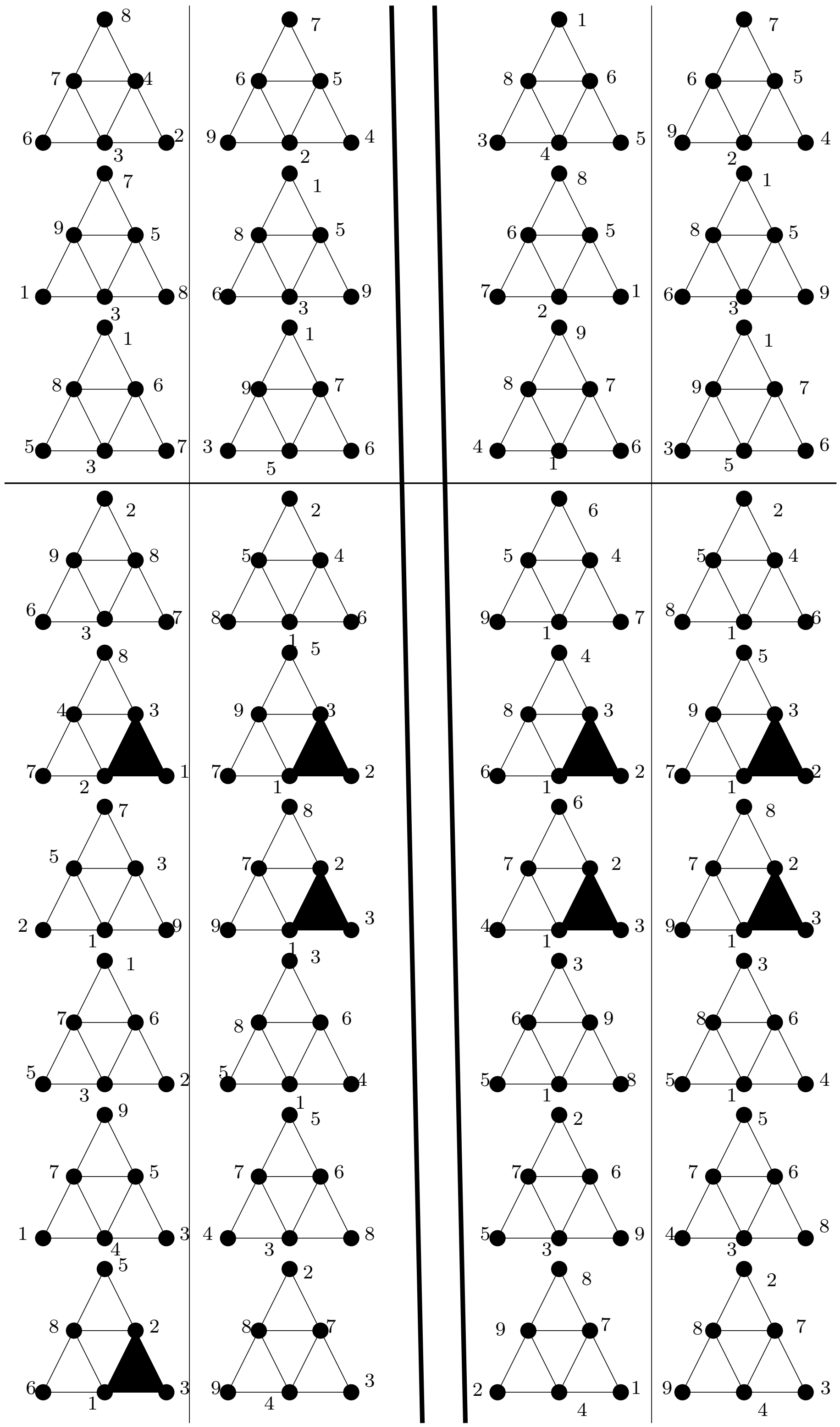
Lemma 3.1 $3Int(7) = \{0, 1, 2, \ldots, 21\} \backslash \{20, 19, 18, 16\}$. $\square$

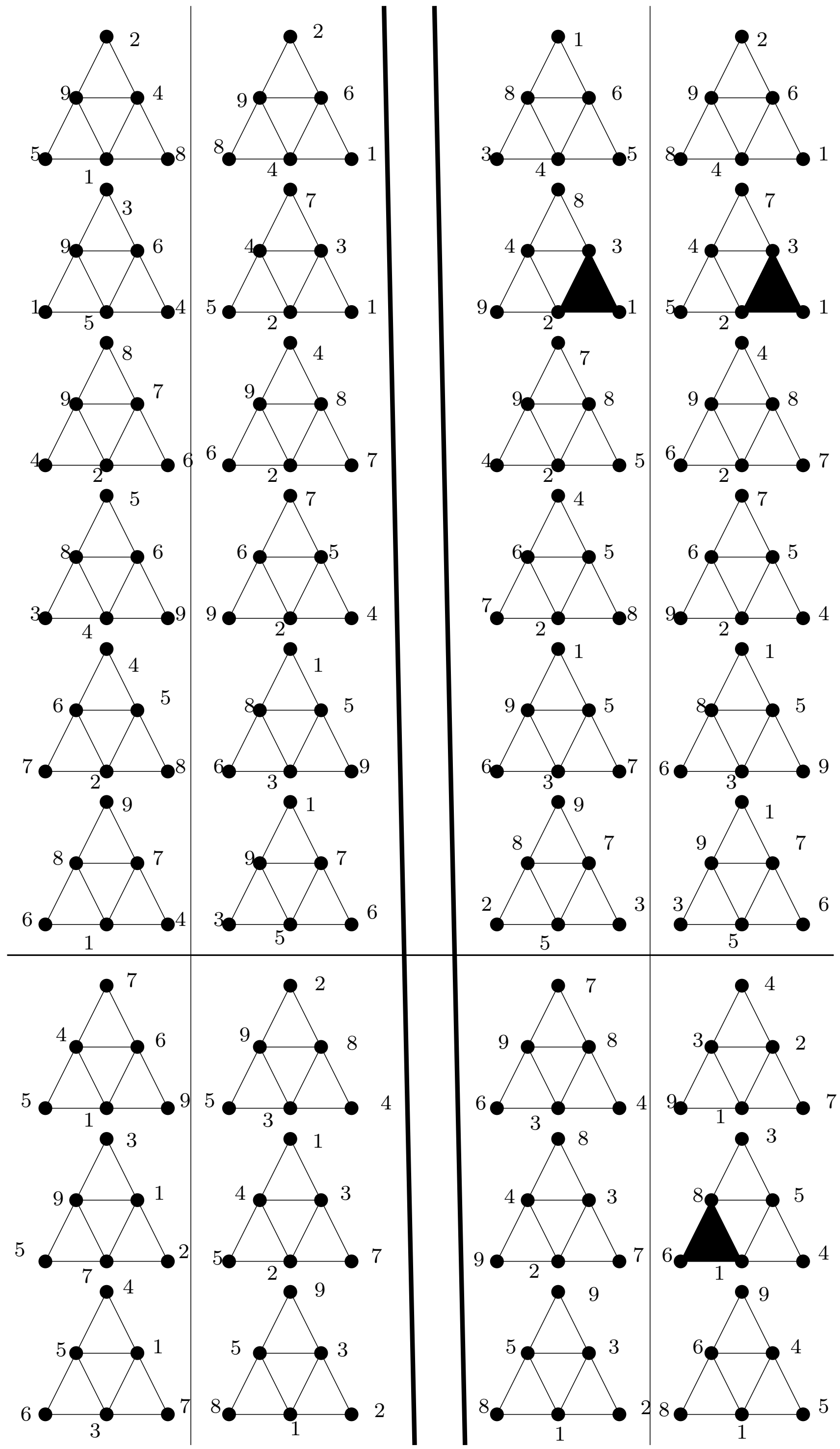
Chapter 4

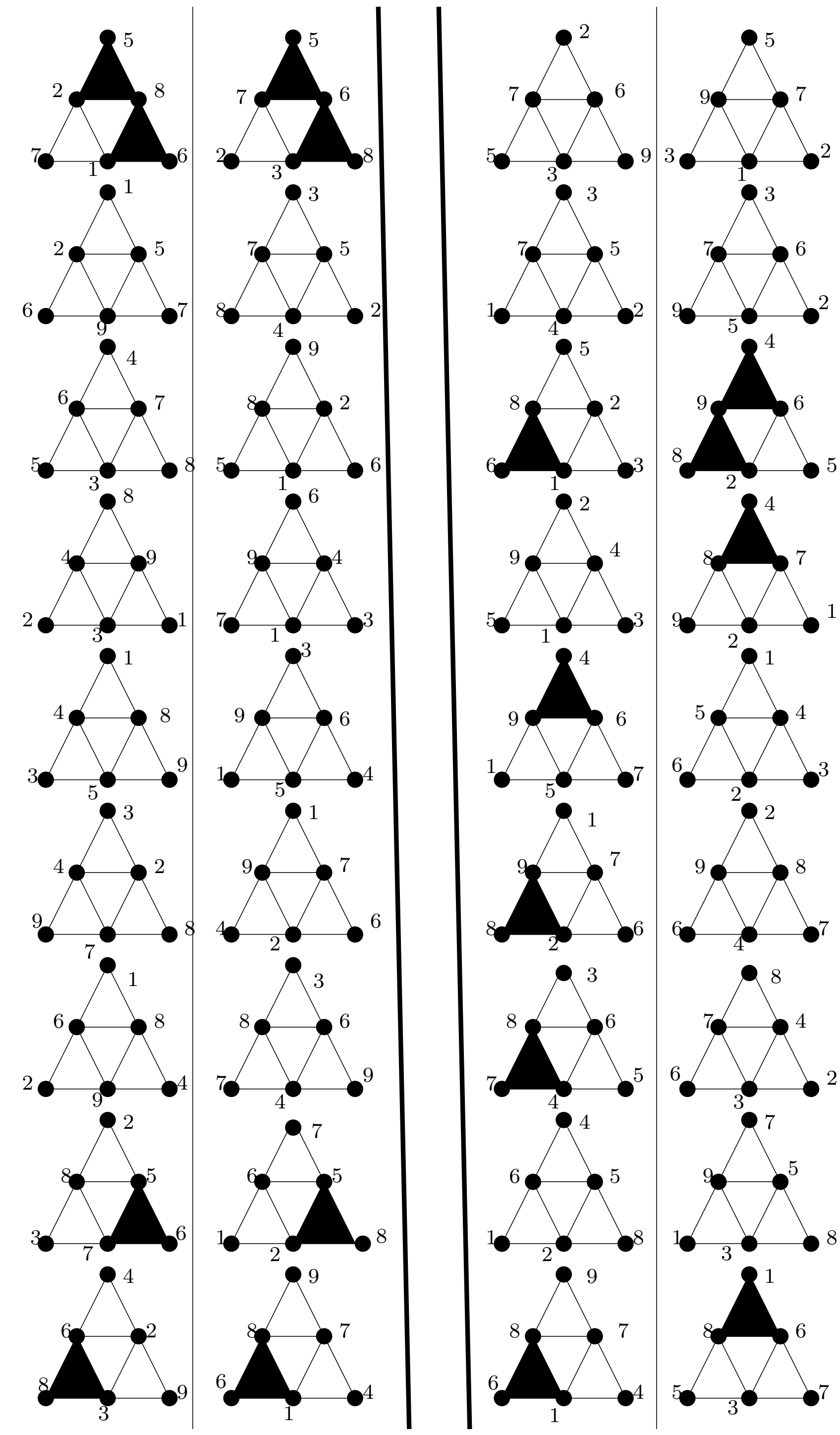
$n = 9$

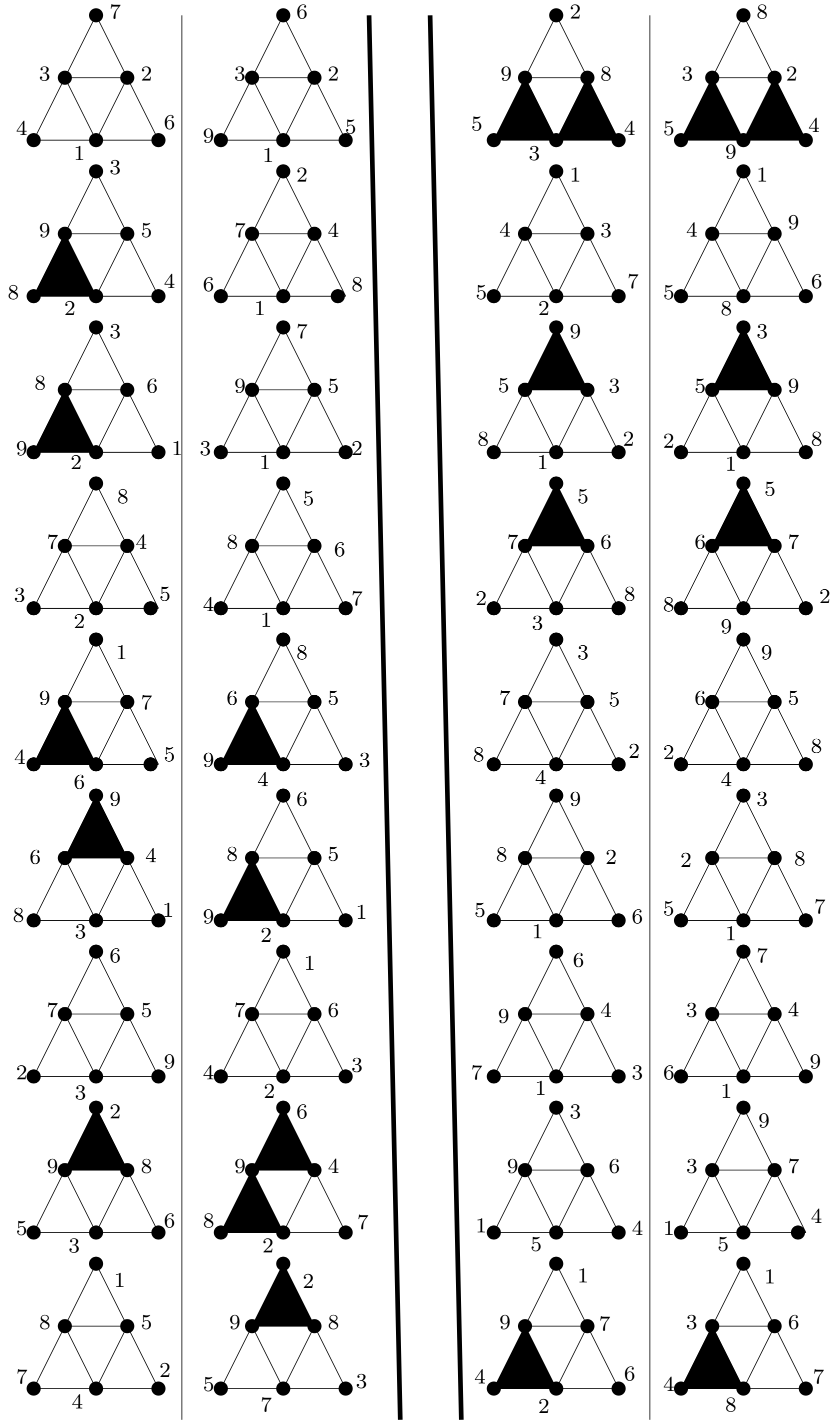
We give a complete solution of the Δ-intersection problem for $n = 9$ in this section. In what follows for each $x \in Int(9) = \{0, 1, 2, \ldots, 36\} \setminus \{35, 34, 33, 31\}$ we list the pair of hexagon triple systems having x triples in common. This gives a complete solution.

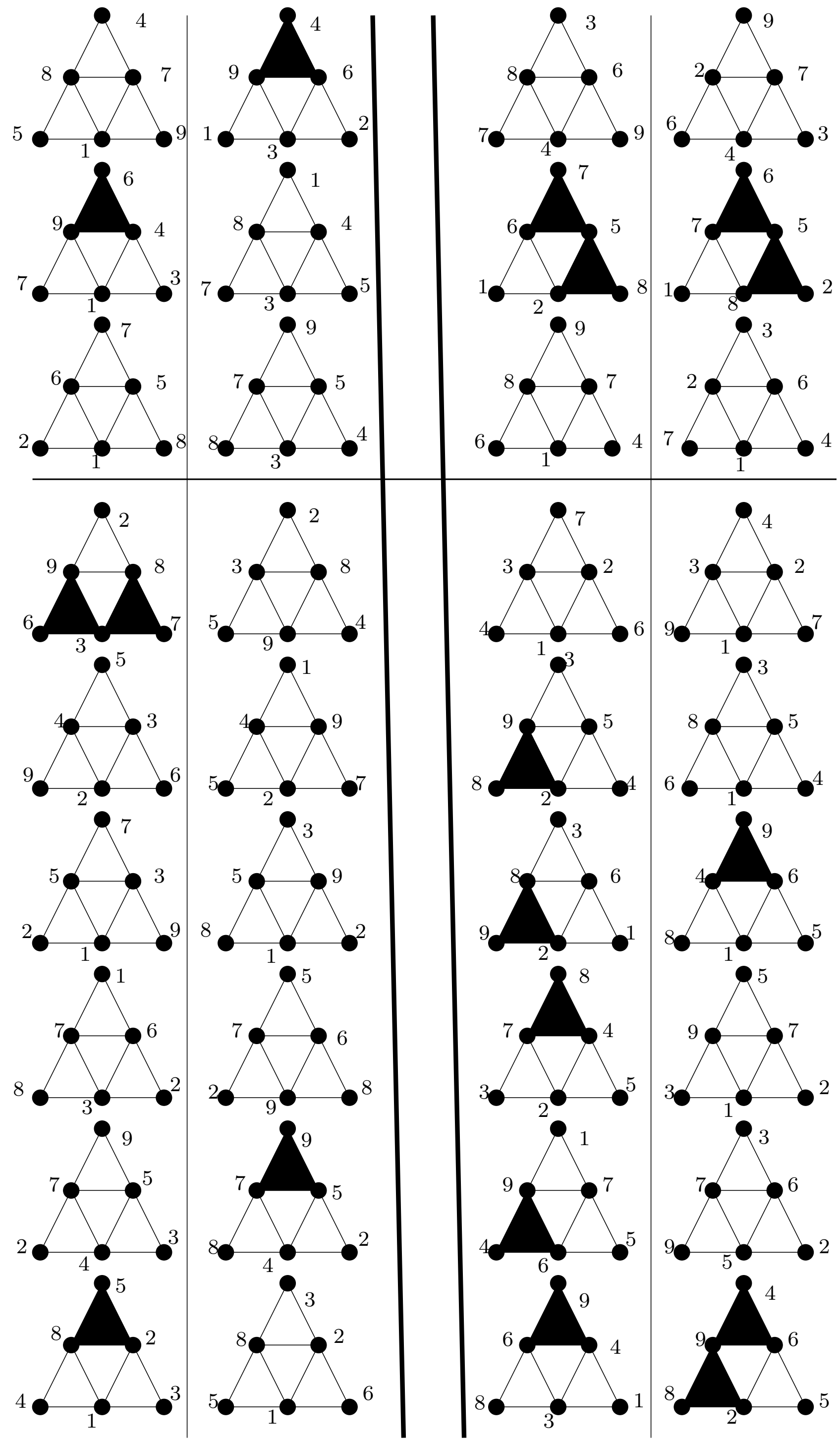


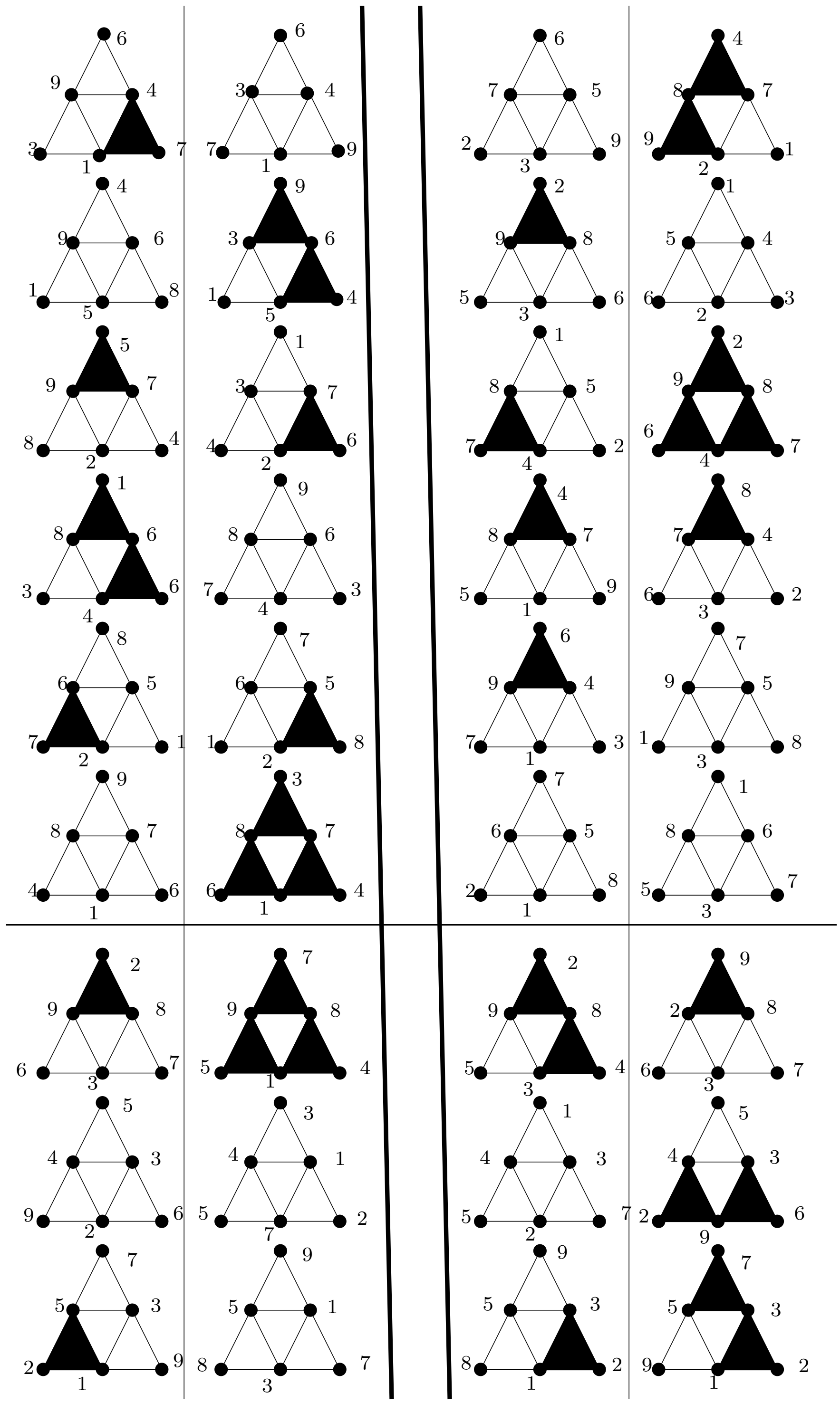


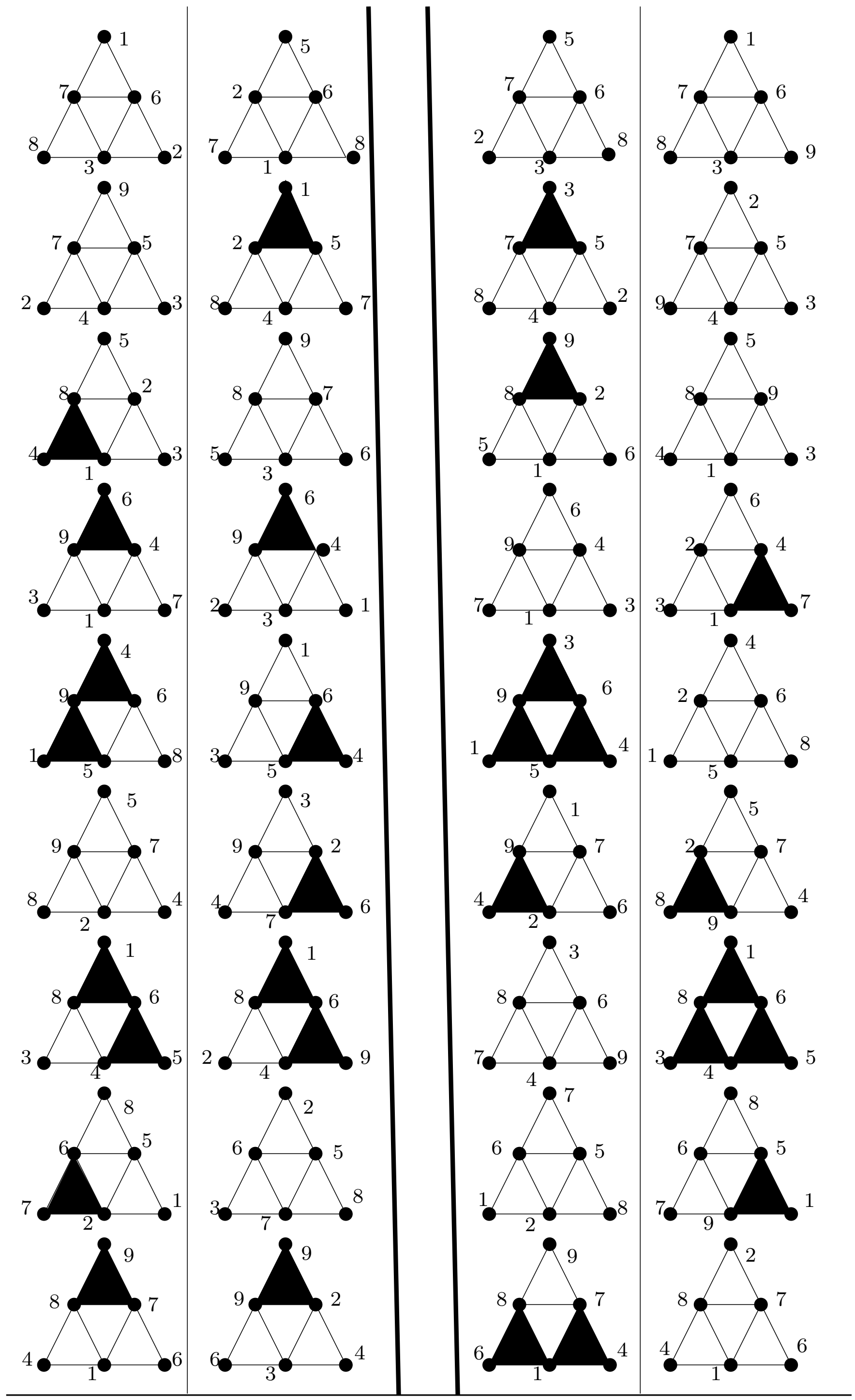


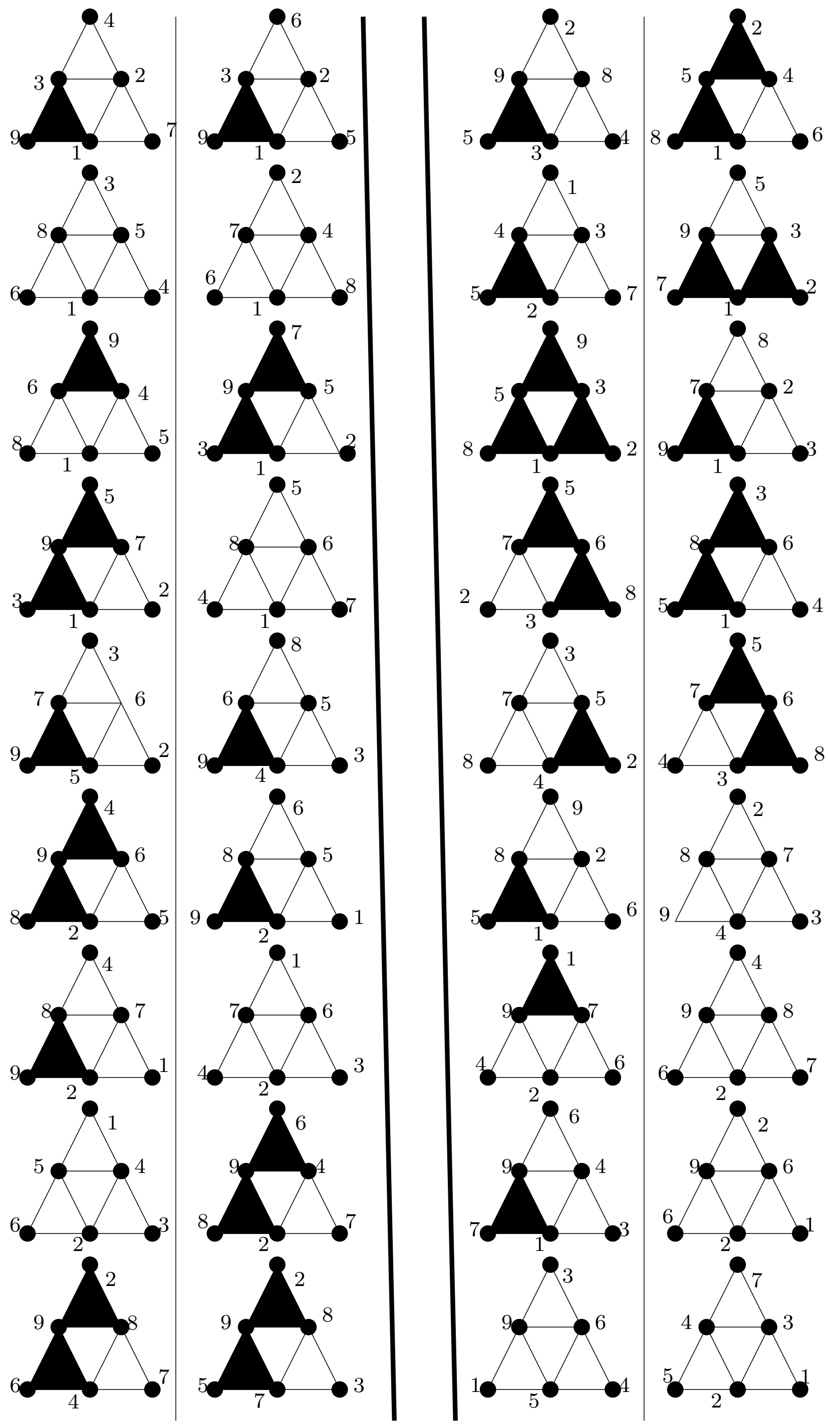


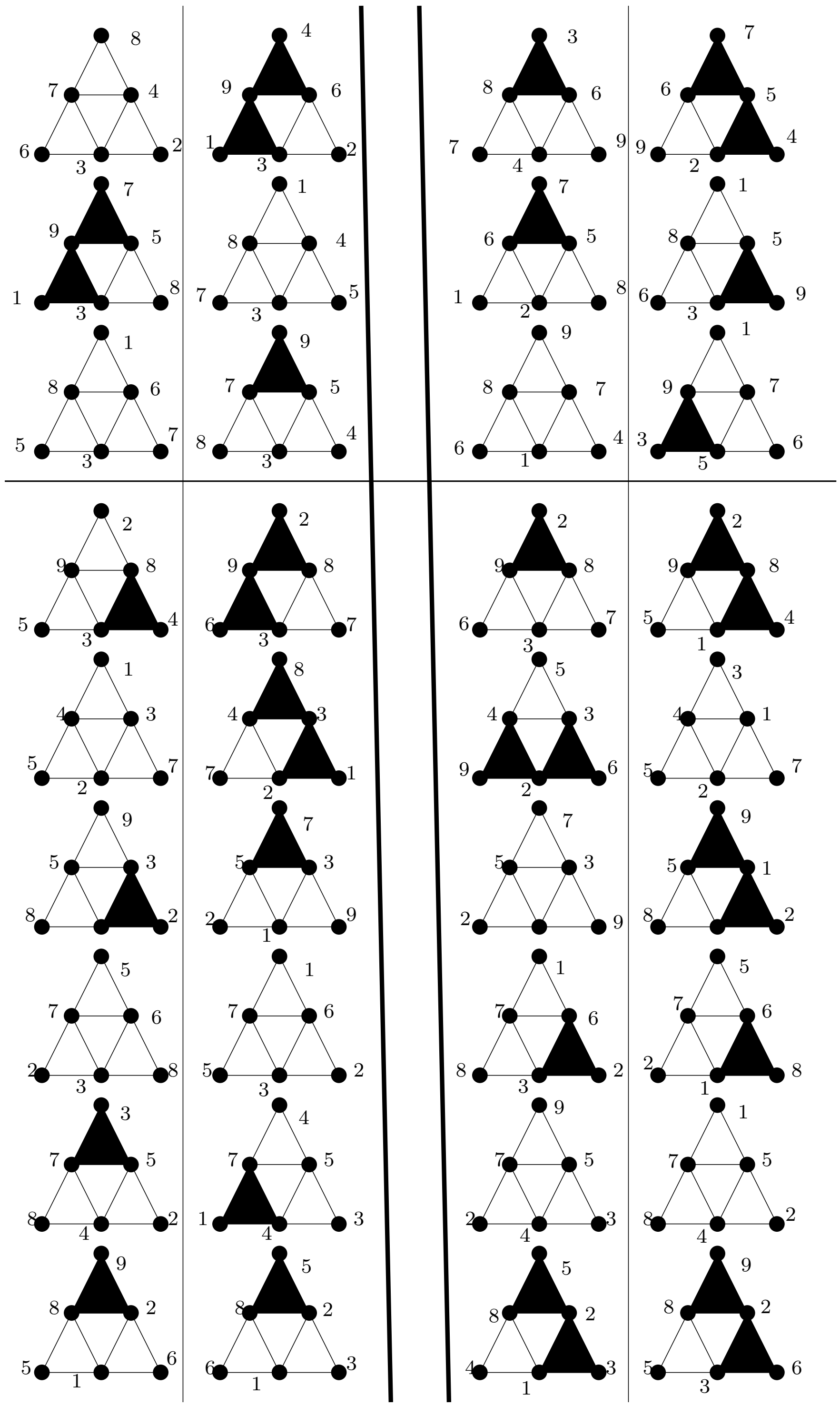


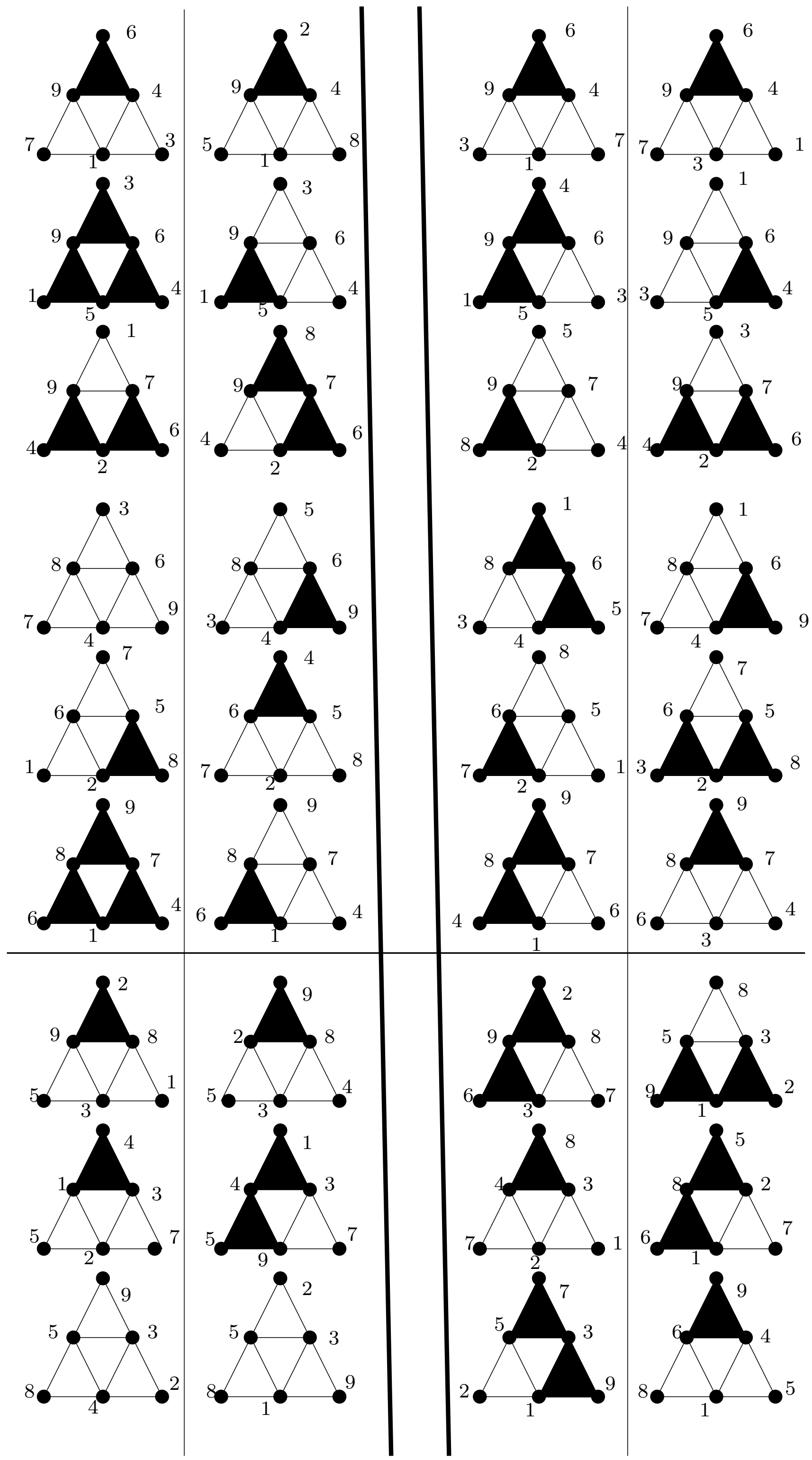


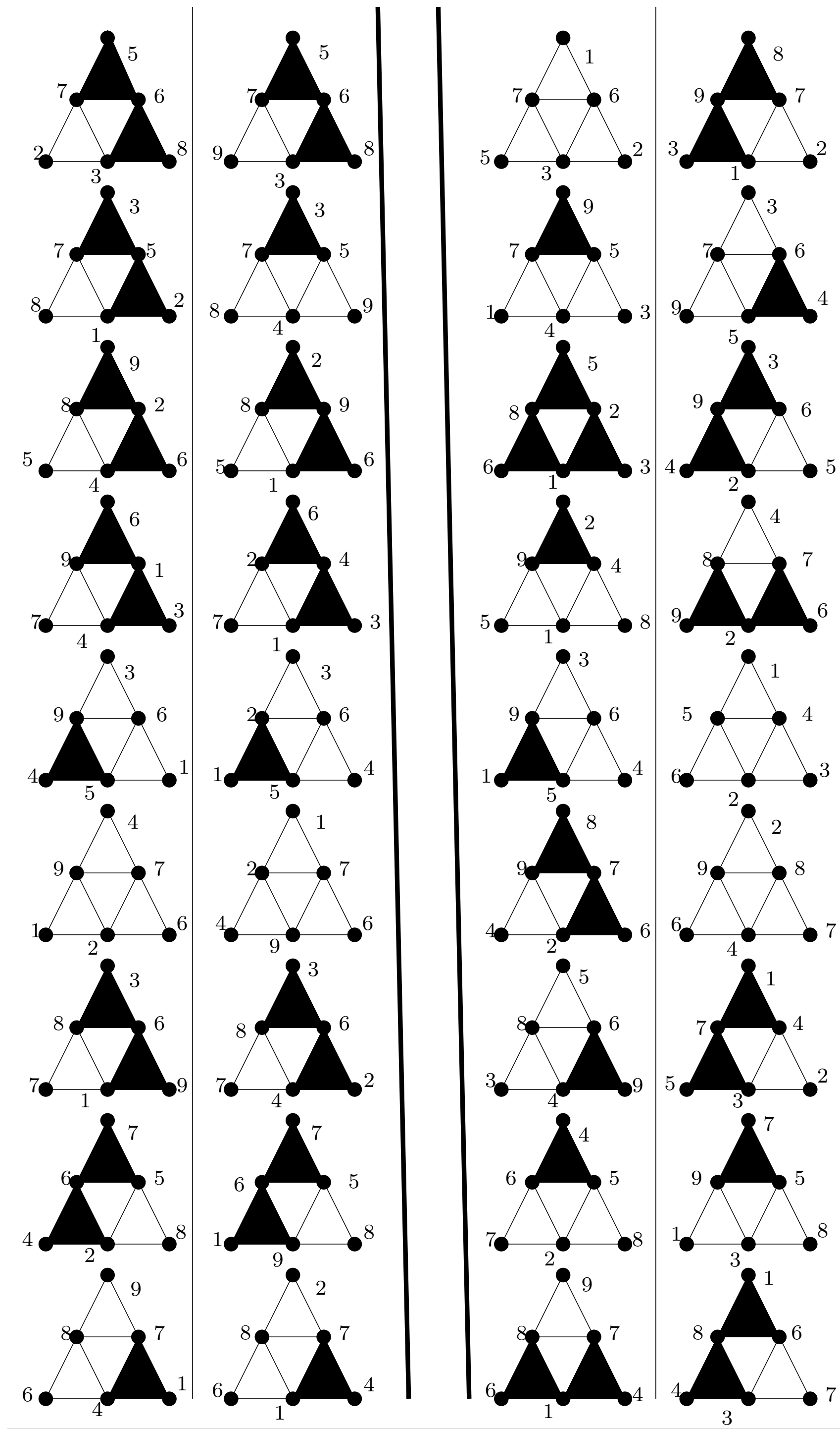


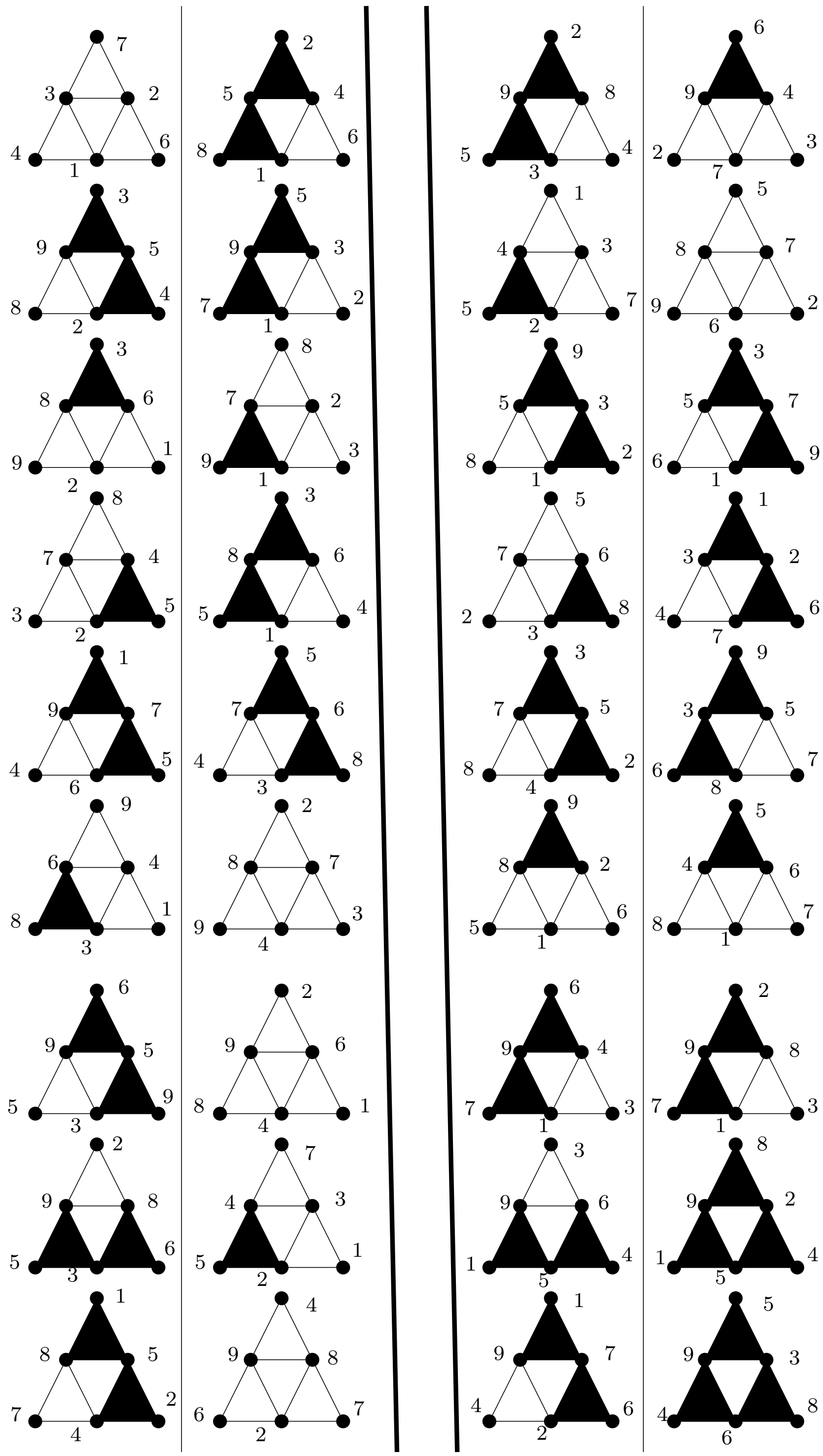








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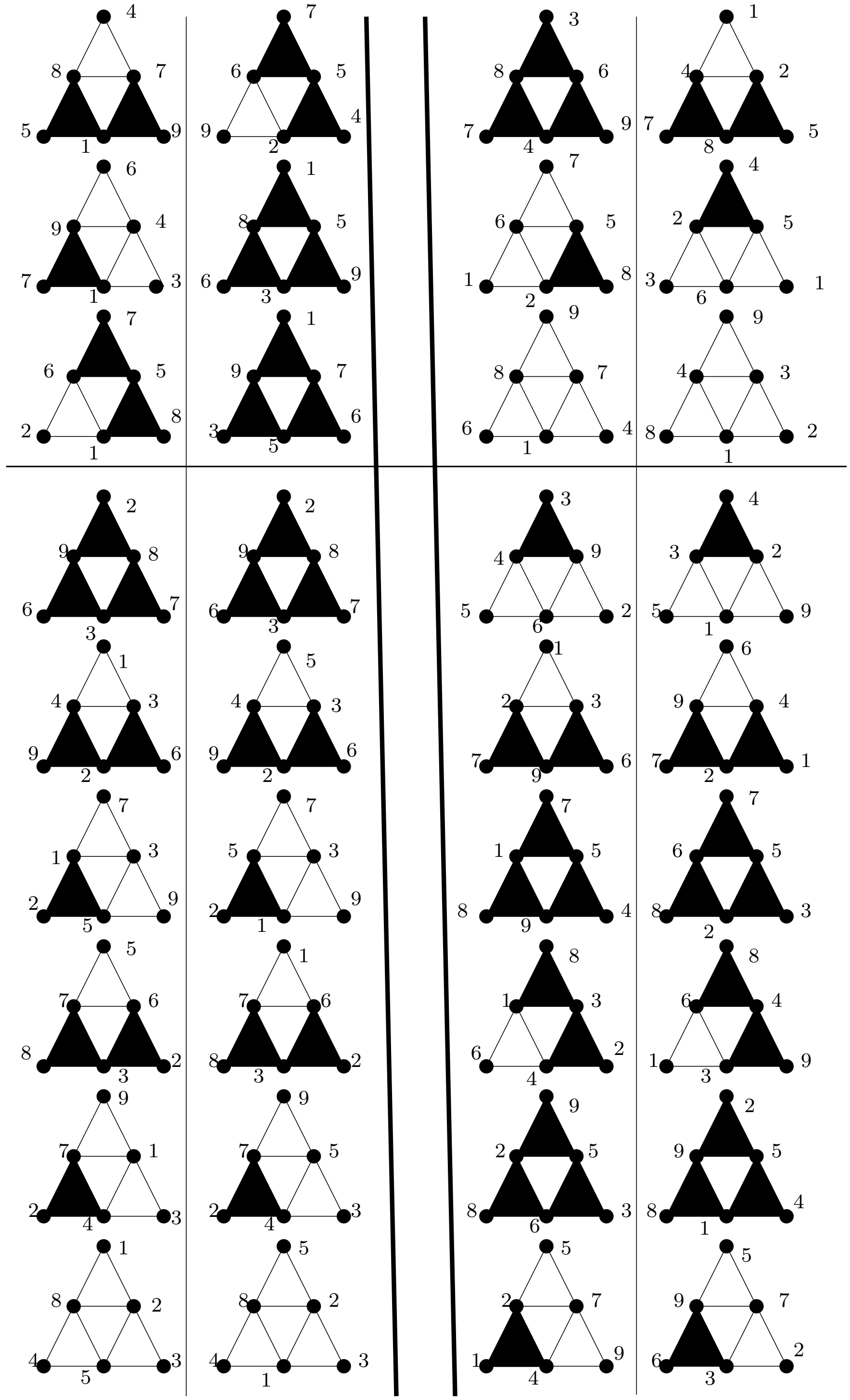

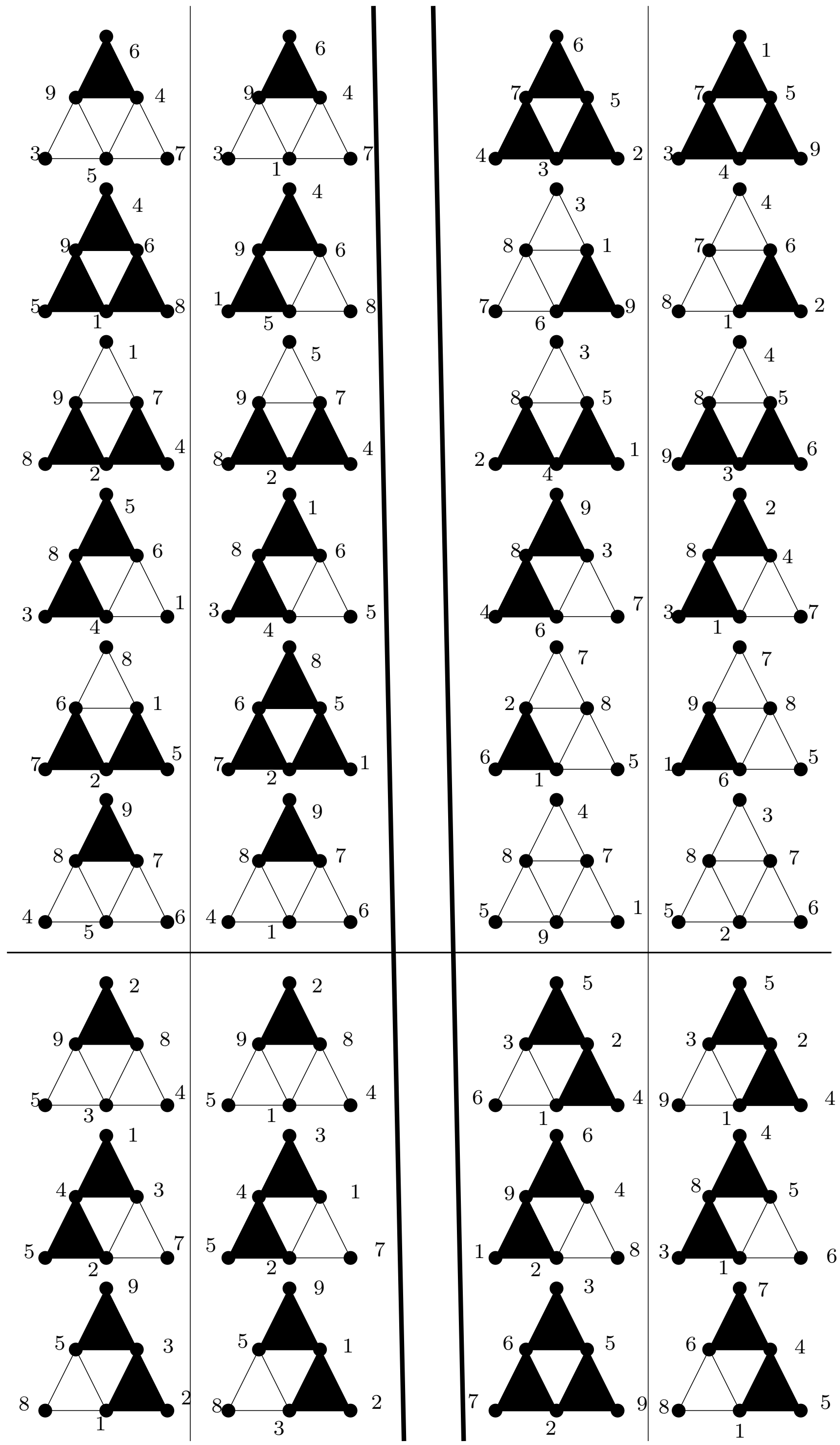

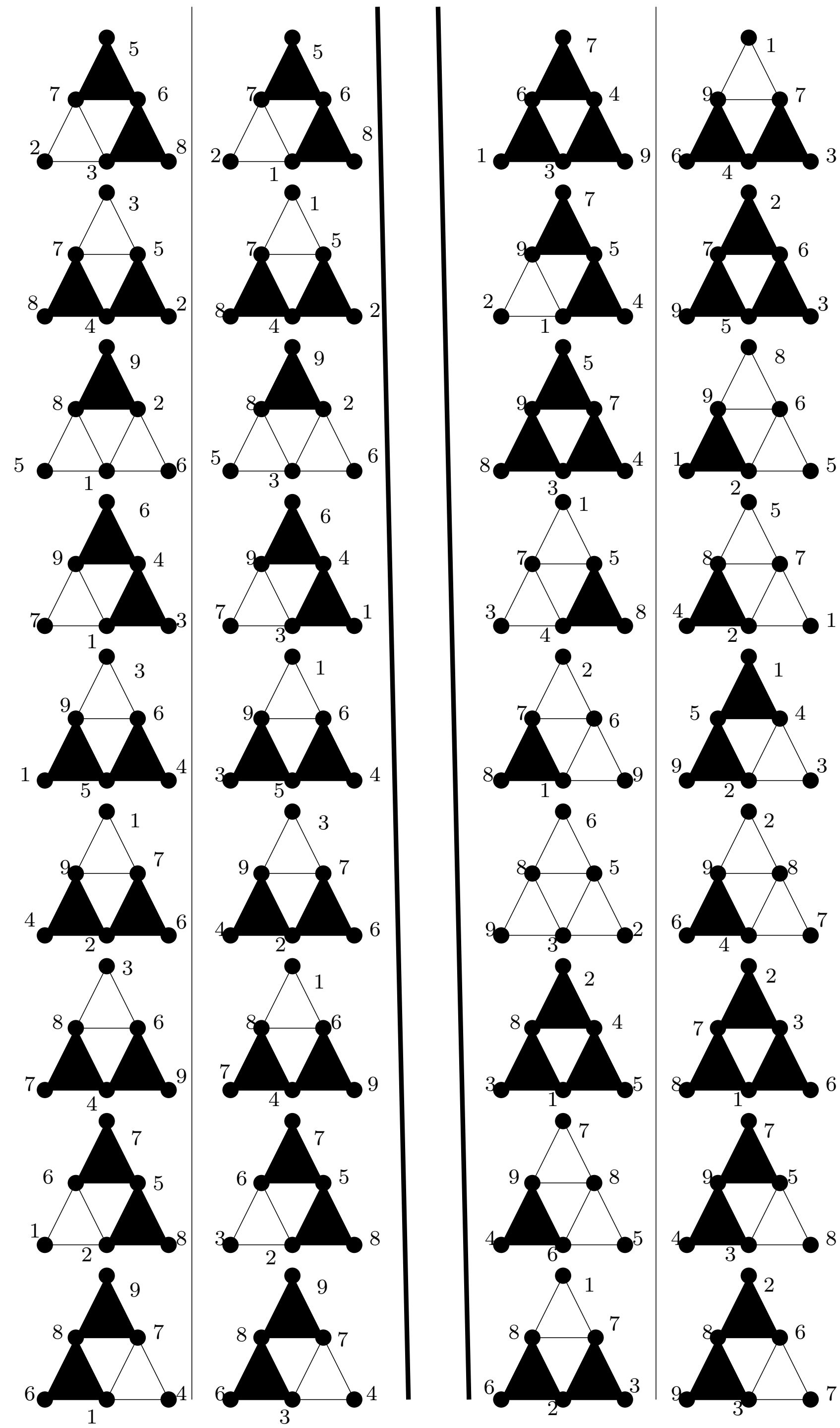

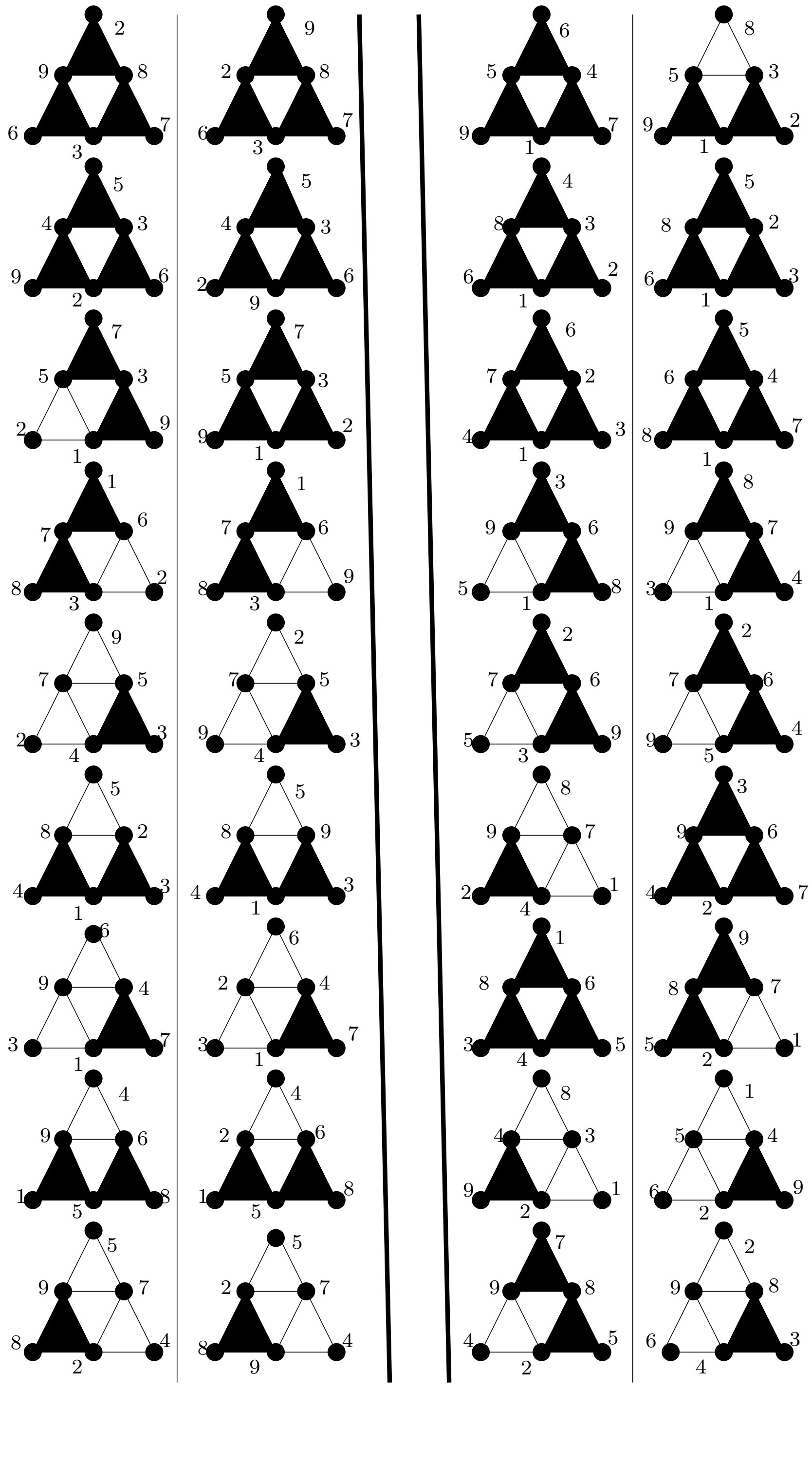
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5 8 9 4 3 1
6 2 4 3 7 1
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5 2 7 8 4 9
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3 9 6 5 8 1
2 7 6 5 9 3
8 9 7 2 1 4
1 8 6 3 5 4
8 4 3 9 1 2
7 9 8 4 5 2
8 5 3 9 2 1
5 8 2 6 3 1
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1 5 4 6 9 2
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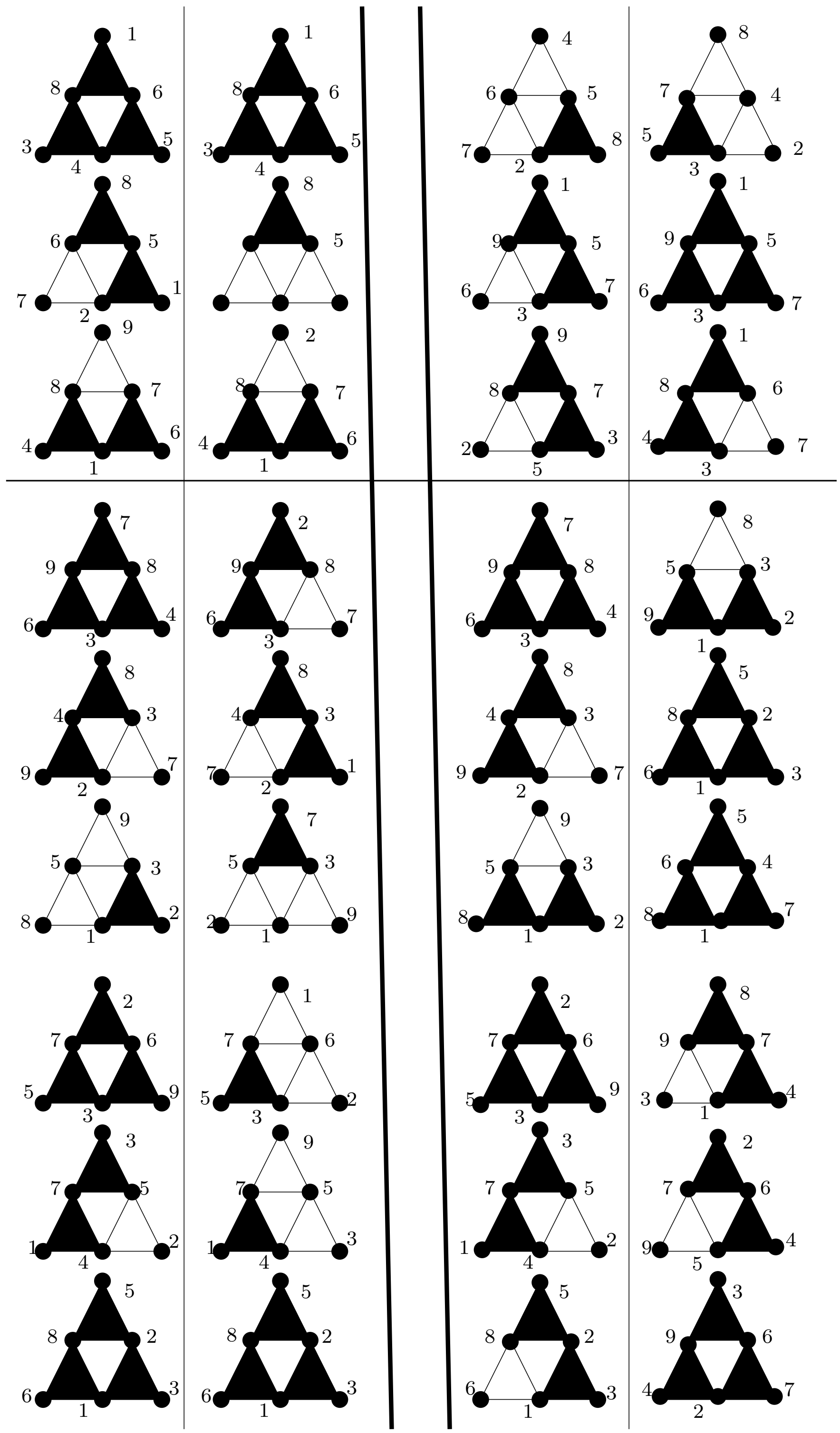
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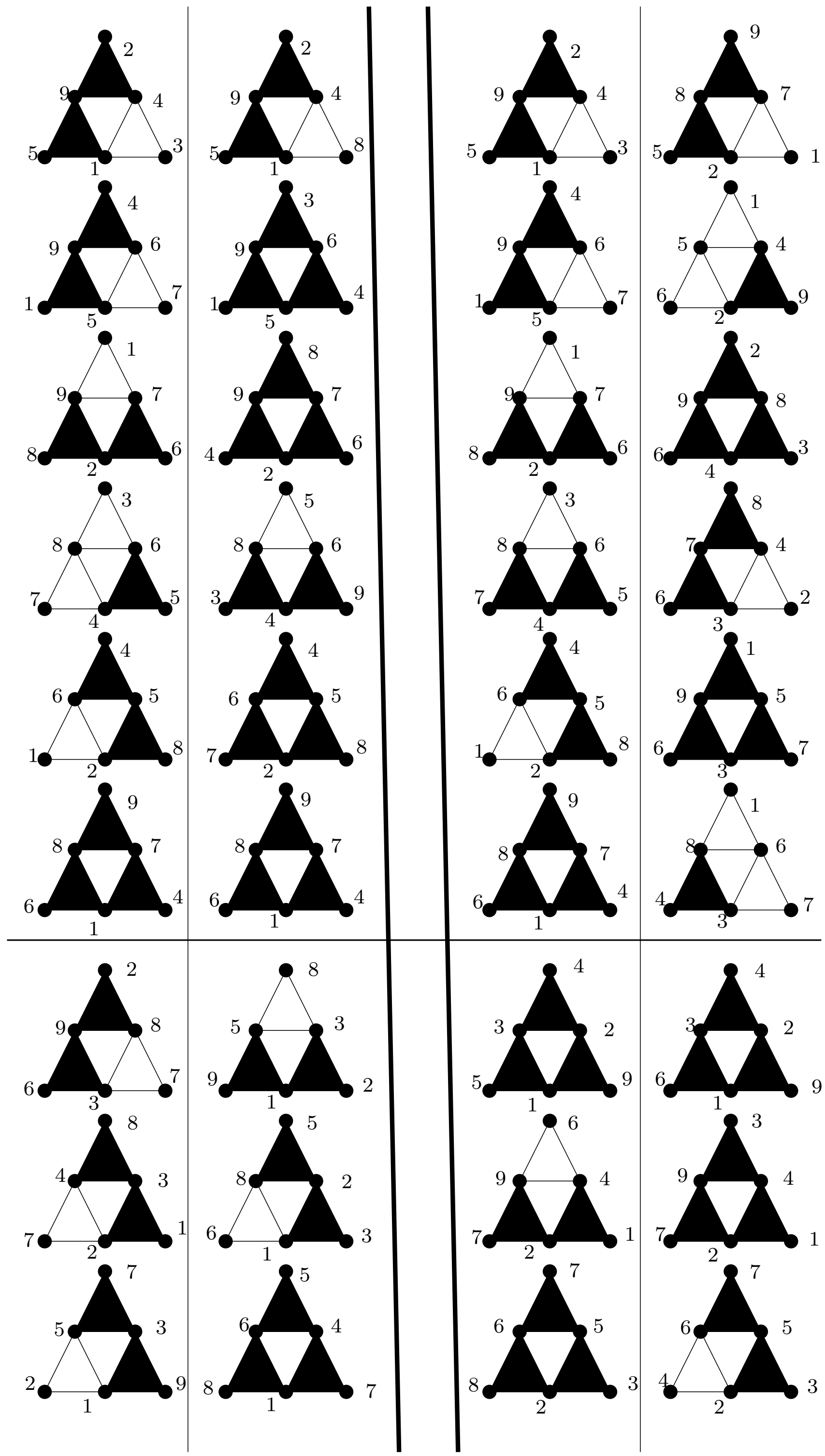

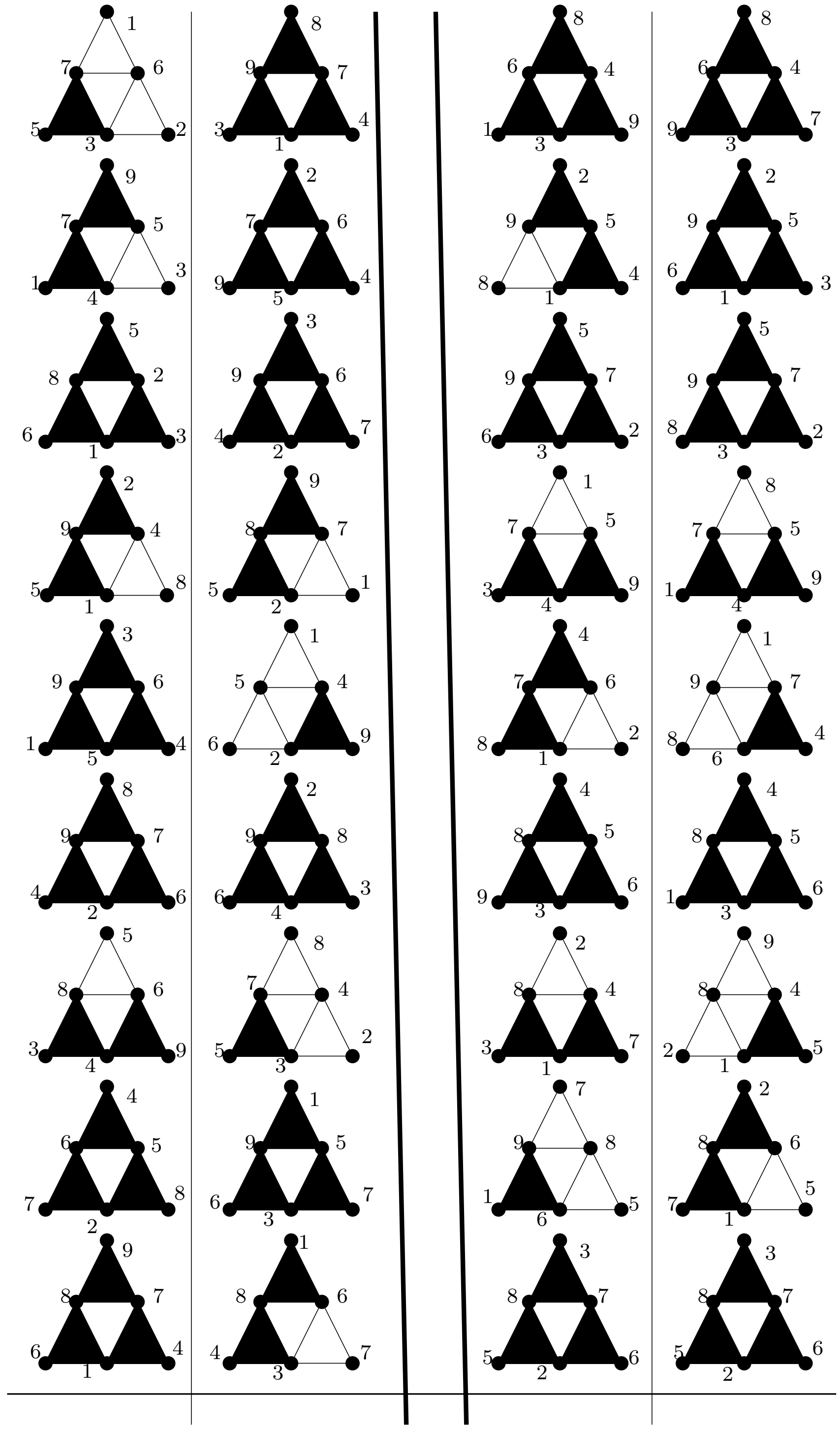

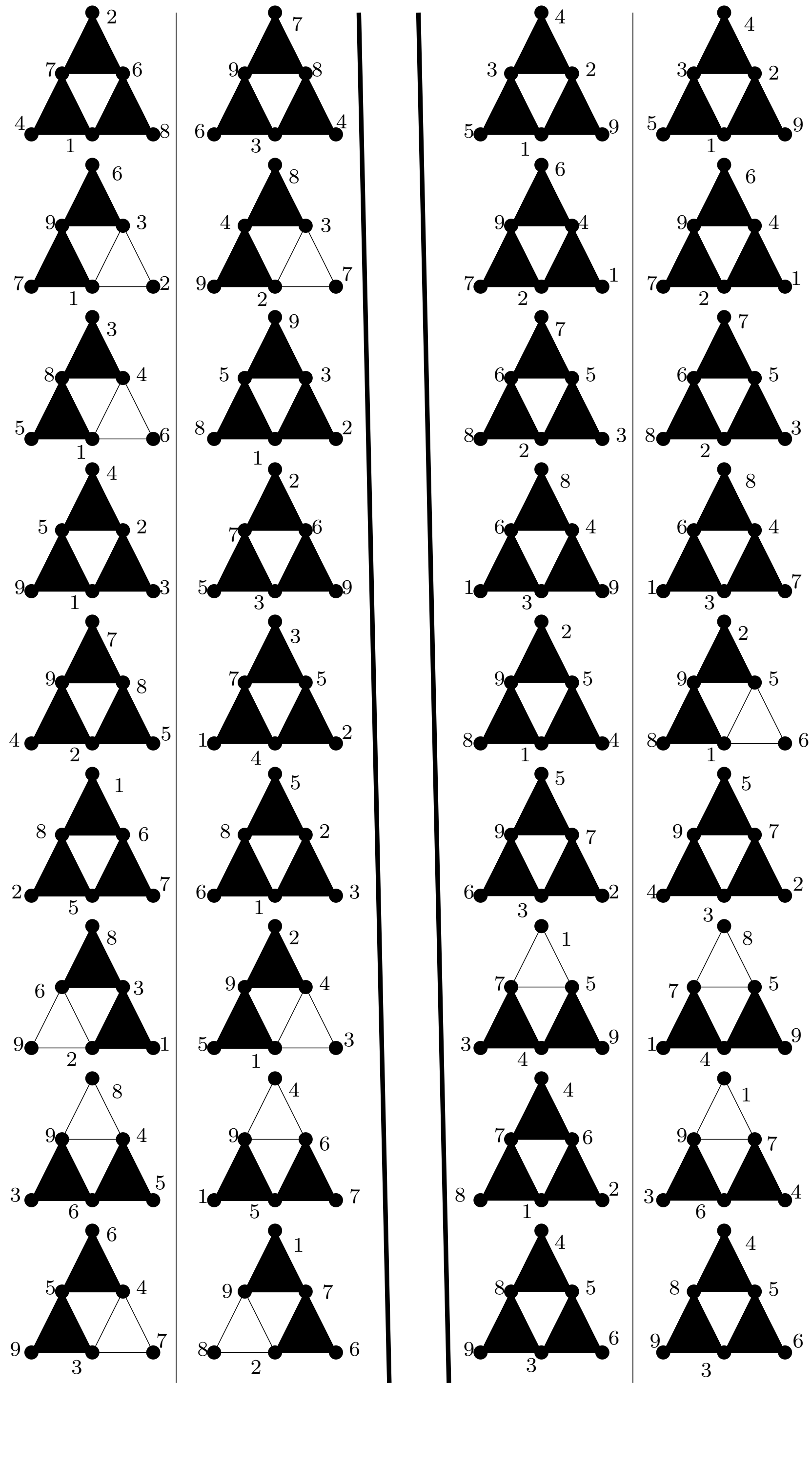

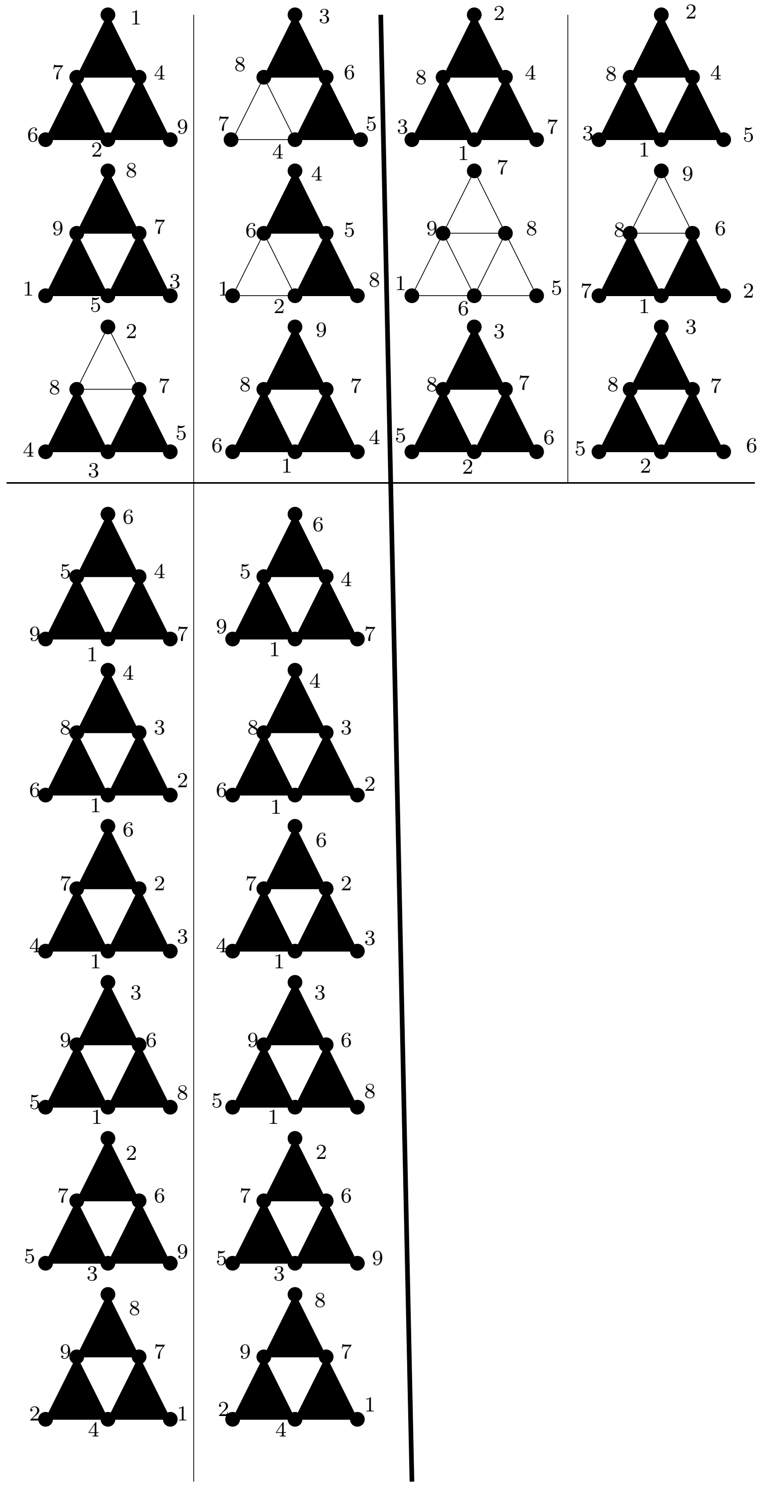
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2 8 4 3 1 5
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9 8 7 6 1 4
3 8 7 5 2 6
3 8 7 5 2 6
6 5 4 9 1 7
6 5 4 9 1 7
4 8 3 6 1 2
4 8 3 6 1 2
6 7 2 4 1 3
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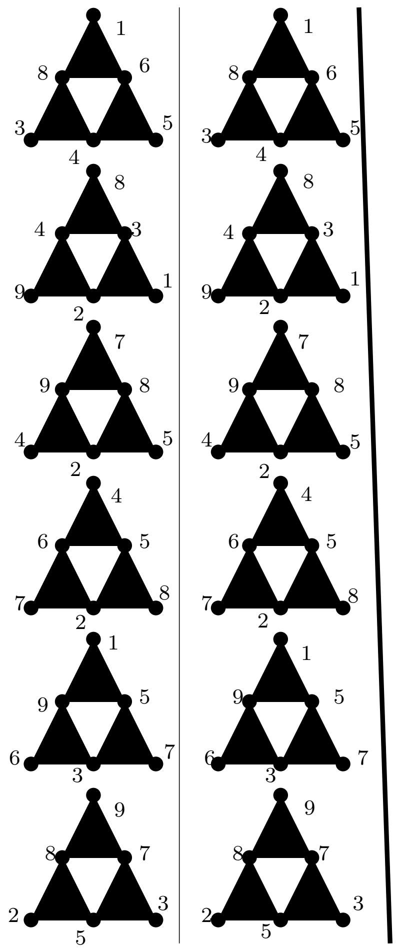
2 7 6 5 3 9
8 9 7 2 4 1
8 9 7 2 4 1



The preceding pages give a complete solution of the Δ-intersection problem for hexagon triple systems of order 9.

Lemma 4.1 $3Int(9) = \{0, 1, 2, \ldots, 36\} \backslash \{35, 34, 33, 31\}$. $\square$

Chapter 5

$n = 13$

$n = 13$ (A work in progress).

To begin with, to keep the pictures from getting out of hand we will denote the hexagon triple by $[x_1, x_2, x_3, x_4, x_5, x_6]$ or $[x_1, x_6, x_5, x_4, x_3, x_2]$ or any cyclic 2-shift. The

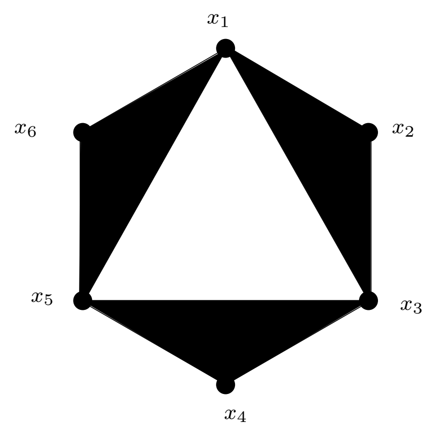


Figure 5.1: Hexagon triple

complete solution of the Δ-intersection problem for hexagon triple systems of order 13 remains elusive; it is a very difficult problem. To date we can show that 60 of the 75 intersection numbers are possible. Hopefully a complete solution can be obtained at a later date. Since $\{3k \mid k \in Int(13)\} \subseteq 3Int(13)$, we need only look at numbers in $3Int(13) \backslash \{3k \mid k \in Int(13)\}$. We will do this by listing 36 hexagon triple systems and 37 intersections between them.

To begin with $3Int(n) \supseteq \{3k \mid k \in Int(n)\}$ follows from Lemma 2.2. We will list the remaining solutions. This will be done by listing 36 hexagon triple system and then the various intersections between them for a total of 37.

[2, 10, 3, 13, 1, 7]	[2, 11, 3, 10, 1, 7]	[2, 11, 3, 10, 1, 12]
[2, 4, 6, 10, 11, 3]	[2, 9, 6, 8, 11, 3]	[2, 9, 6, 10, 11, 3]
[11, 9, 7, 5, 4, 3]	[11, 9, 6, 3, 4, 12]	[11, 9, 7, 3, 4, 12]
[4, 13, 8, 9, 3, 7]	[4, 6, 8, 12, 3, 7]	[4, 6, 8, 12, 3, 7]
[3, 8, 9, 2, 6, 5]	[3, 13, 9, 1, 6, 5]	[3, 13, 9, 2, 6, 5]
[6, 12, 10, 8, 7, 13]	[6, 11, 10, 8, 7, 12]	[6, 11, 10, 5, 7, 12]
[7, 4, 5, 1, 8, 10]	[7, 10, 5, 9, 8, 1]	[7, 10, 5, 9, 8, 1]
[8, 2, 12, 10, 9, 5]	[8, 2, 12, 10, 9, 3]	[8, 3, 12, 10, 9, 5]
[9, 2, 13, 11, 10, 5]	[9, 2, 13, 4, 10, 5]	[9, 3, 13, 4, 10, 12]
[10, 4, 1, 11, 5, 7]	[10, 4, 1, 11, 5, 7]	[10, 3, 1, 11, 5, 7]
[5, 11, 2, 1, 12, 13]	[5, 4, 2, 1, 12, 13]	[5, 4, 2, 8, 12, 13]
[12, 4, 11, 8, 13, 5]	[12, 1, 11, 10, 13, 5]	[12, 4, 11, 8, 13, 5]
[13, 10, 4, 9, 1, 3]	[13, 8, 4, 9, 1, 3]	[13, 10, 4, 9, 1, 11]
[1, 6, 9, 7, 11, 12]	[1, 4, 9, 7, 11, 5]	[1, 4, 9, 7, 11, 5]
[2, 3, 10, 1, 4, 6]	[2, 3, 10, 13, 4, 5]	[2, 1, 10, 13, 4, 5]
[3, 4, 11, 2, 5, 6]	[3, 4, 11, 2, 5, 6]	[3, 2, 11, 8, 5, 6]
[4, 9, 12, 7, 6, 8]	[4, 11, 12, 10, 6, 2]	[4, 11, 12, 7, 6, 1]
[3, 9, 13, 6, 7, 12]	[3, 9, 13, 6, 7, 12]	[3, 9, 13, 2, 7, 4]
[6, 9, 1, 5, 8, 11]	[6, 13, 1, 5, 8, 4]	[6, 13, 1, 7, 8, 4]
[7, 1, 2, 13, 9, 11]	[7, 13, 2, 6, 9, 11]	[7, 13, 2, 6, 9, 11]
[8, 6, 11, 13, 10, 2]	[8, 13, 11, 6, 10, 2]	[8, 13, 11, 6, 10, 2]
[9, 12, 4, 2, 5, 10]	[9, 12, 4, 7, 5, 8]	[9, 8, 4, 2, 5, 1]
[10, 1, 3, 8, 12, 6]	[10, 1, 3, 8, 12, 9]	[10, 8, 3, 1, 12, 9]
[5, 3, 6, 1, 13, 12]	[5, 3, 6, 1, 13, 1]	[5, 3, 6, 1, 13, 12]
[12, 3, 7, 8, 1, 11]	[12, 6, 7, 8, 1, 2]	[12, 6, 7, 8, 1, 2]
[13, 4, 8, 12, 1, 7]	[13, 11, 8, 10, 2, 7]	[13, 6, 8, 10, 2, 7]
H_1	H_2	H_3

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[4, 12, 10, 5, 1, 13]	[4, 12, 10, 13, 1, 5]	[7, 5, 11, 2, 6, 10]
[1, 4, 5, 9, 8, 6]	[1, 10, 5, 9, 8, 6]	[8, 6, 4, 1, 7, 3]
[5, 6, 2, 12, 11, 10]	[5, 12, 2, 6, 11, 10]	[9, 1, 5, 13, 8, 11]
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[10, 3, 7, 4, 8, 2]	[10, 6, 7, 5, 8, 2]	[3, 8, 7, 6, 10, 13]
[8, 3, 12, 5, 2, 10]	[8, 9, 12, 5, 2, 13]	[12, 10, 8, 7, 3, 1]
[2, 7, 9, 12, 13, 8]	[2, 7, 9, 10, 13, 8]	[13, 7, 9, 4, 12, 11]
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[7, 6, 11, 3, 9, 1]	[7, 4, 11, 1, 9, 2]	[11, 9, 8, 2, 1, 10]
[12, 3, 10, 9, 6, 1]	[12, 3, 10, 7, 6, 4]	[4, 12, 9, 10, 2, 13]
[3, 6, 9, 5, 8, 4]	[3, 11, 9, 12, 8, 4]	[5, 4, 10, 1, 11, 7]
[6, 11, 2, 9, 4, 12]	[6, 5, 2, 10, 4, 3]	[6, 9, 3, 11, 4, 8]
[3, 5, 13, 10, 1, 2]	[3, 7, 13, 4, 1, 2]	[7, 2, 12, 6, 5, 11]
[4, 11, 7, 8, 5, 1]	[4, 11, 7, 8, 5, 9]	[8, 5, 13, 1, 6, 4]
[1, 7, 12, 2, 11, 8]	[1, 7, 12, 2, 11, 8]	[9, 5, 1, 4, 7, 13]
[5, 4, 9, 13, 10, 11]	[5, 4, 9, 6, 10, 1]	[10, 9, 2, 1, 8, 12]
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[10, 7, 3, 1, 2, 4]	[10, 12, 3, 1, 2, 4]	[12, 9, 4, 5, 10, 8]
[8, 7, 4, 11, 13, 6]	[8, 7, 4, 1, 13, 6]	[13, 8, 5, 2, 3, 10]
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H_4	H_5	H_6

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[10, 7, 6, 1, 9, 2]	[9, 3, 13, 4, 10, 12]	[9, 3, 13, 4, 10, 12]
[1, 8, 7, 6, 10, 13]	[10, 2, 1, 9, 5, 7]	[10, 3, 1, 11, 5, 7]
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H_7	H_8	H_9

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[12, 2, 11, 4, 13, 9]

H_{10}

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H_{11}

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[12, 2, 5, 6, 13, 11]

H_{12}

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[3, 5, 11, 13, 6, 7]	[4, 5, 7, 9, 11, 3]	[11, 9, 7, 3, 4, 12]
[3, 1, 2, 9, 5, 11]	[3, 9, 8, 13, 4, 11]	[4, 9, 8, 10, 3, 7]
[3, 8, 13, 12, 4, 9]	[6, 1, 9, 8, 3, 5]	[3, 13, 9, 1, 6, 5]
[3, 6, 7, 10, 8, 13]	[7, 8, 10, 12, 6, 13]	[6, 11, 10, 5, 7, 12]
[12, 5, 8, 11, 9, 7]	[8, 1, 5, 4, 7, 10]	[7, 4, 5, 11, 8, 10]
[12, 13, 4, 5, 6, 1]	[9, 4, 12, 2, 8, 3]	[8, 2, 12, 10, 9, 3]
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[1, 10, 11, 3, 5, 7]	[1, 10, 4, 8, 13, 3]	[1, 5, 9, 7, 11, 12]
[1, 9, 13, 5, 10, 11]	[11, 7, 9, 6, 1, 12]	[2, 1, 10, 13, 4, 5]
[1, 3, 2, 6, 8, 4]	[4, 1, 10, 3, 2, 6]	[3, 4, 11, 8, 5, 6]
[9, 2, 5, 4, 6, 10]	[5, 2, 11, 4, 3, 6]	[4, 9, 12, 10, 6, 2]
[9, 12, 7, 4, 11, 8]	[6, 10, 12, 9, 4, 2]	[3, 9, 13, 2, 7, 12]
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[6, 8, 2, 4, 10, 9]	[8, 5, 1, 9, 6, 11]	[7, 1, 2, 6, 9, 11]
[6, 11, 13, 3, 8, 2]	[9, 13, 2, 1, 7, 11]	[8, 6, 11, 13, 10, 3]
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[11, 9, 8, 7, 10, 1]	[12, 7, 3, 2, 10, 6]	[5, 3, 6, 7, 13, 12]
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H_{13}	H_{14}	H_{15}

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[11, 9, 7, 5, 4, 3]	[11, 9, 7, 5, 4, 3]	[7, 9, 11, 5, 4, 3]
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[2, 11, 10, 8, 7, 13]	[6, 11, 10, 8, 7, 13]	[2, 7, 10, 8, 11, 13]
[7, 10, 5, 11, 8, 1]	[7, 10, 5, 11, 8, 1]	[11, 10, 5, 7, 8, 1]
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[9, 3, 13, 11, 10, 12]	[9, 3, 13, 11, 10, 12]	[9, 3, 13, 7, 10, 12]
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[12, 4, 11, 1, 13, 5]	[12, 4, 11, 1, 13, 5]	[12, 4, 7, 1, 13, 5]
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[1, 5, 9, 7, 11, 12]	[1, 5, 9, 7, 11, 12]	[1, 5, 9, 11, 7, 12]
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[3, 4, 11, 6, 5, 2]	[3, 4, 11, 2, 5, 6]	[3, 4, 7, 6, 5, 2]
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[3, 1, 13, 6, 7, 12]	[3, 1, 13, 2, 7, 12]	[3, 1, 13, 6, 11, 12]
[2, 9, 1, 5, 8, 13]	[6, 9, 1, 5, 8, 13]	[2, 9, 1, 5, 8, 13]
[7, 1, 6, 13, 9, 11]	[7, 1, 2, 13, 9, 11]	[11, 1, 6, 13, 9, 7]
[8, 2, 11, 13, 10, 7]	[8, 6, 11, 13, 10, 7]	[8, 2, 7, 13, 10, 11]
[9, 8, 4, 7, 5, 10]	[9, 8, 4, 7, 5, 10]	[9, 8, 4, 11, 5, 10]
[10, 6, 3, 7, 12, 2]	[10, 2, 3, 7, 12, 6]	[10, 6, 3, 11, 12, 2]
[5, 3, 2, 7, 13, 12]	[5, 3, 6, 7, 13, 12]	[5, 3, 2, 11, 13, 12]
[12, 2, 7, 6, 1, 11]	[12, 6, 7, 2, 1, 11]	[12, 2, 11, 6, 1, 7]
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H_{16}	H_{17}	H_{18}

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H_{19}	H_{20}	H_{21}

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H_{22}	H_{23}	H_{24}

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[5, 3, 6, 4, 2, 11]	[5, 1, 8, 4, 2, 11]	[5, 1, 8, 4, 2, 11]
[7, 9, 11, 8, 6, 13]	[7, 9, 11, 6, 8, 13]	[7, 9, 11, 6, 8, 13]
[7, 10, 8, 13, 4, 5]	[7, 10, 6, 13, 4, 5]	[7, 3, 6, 13, 4, 5]
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[12, 5, 13, 2, 9, 4]	[12, 5, 13, 2, 9, 4]	[12, 5, 13, 2, 9, 4]
[10, 11, 13, 3, 1, 4]	[10, 11, 13, 3, 1, 4]	[3, 11, 13, 10, 1, 4]
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[4, 12, 9, 13, 2, 6]	[4, 12, 9, 13, 2, 8]	[4, 12, 9, 13, 2, 8]
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[7, 3, 12, 13, 5, 4]	[7, 3, 12, 13, 5, 4]	[7, 10, 12, 13, 5, 4]
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H_{25}	H_{26}	H_{27}

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[5, 1, 6, 4, 7, 11]	[5, 3, 8, 4, 2, 11]	[5, 3, 8, 4, 2, 11]
[2, 9, 11, 8, 6, 13]	[7, 9, 11, 6, 8, 13]	[7, 9, 11, 6, 8, 13]
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[5, 12, 13, 1, 3, 8]	[5, 12, 13, 1, 3, 6]	[5, 12, 13, 1, 3, 8]
[6, 10, 12, 11, 1, 5]	[8, 10, 12, 11, 1, 5]	[8, 10, 12, 11, 1, 5]
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H_{28}	H_{29}	H_{30}

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H_{31}

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[1, 6, 11, 3, 12, 7]

H_{32}

[2, 12, 8, 4, 13, 9]
[1, 13, 3, 10, 2, 7]
[11, 8, 6, 4, 2, 5]
[4, 5, 7, 9, 11, 3]
[3, 9, 8, 13, 4, 11]
[6, 1, 9, 8, 3, 5]
[7, 3, 10, 12, 6, 13]
[8, 1, 5, 4, 7, 10]
[9, 4, 12, 2, 8, 3]
[10, 11, 13, 2, 9, 5]
[5, 8, 1, 4, 10, 9]
[12, 3, 2, 11, 5, 13]
[13, 10, 11, 1, 12, 5]
[1, 10, 4, 8, 13, 3]
[11, 7, 9, 6, 1, 12]
[4, 1, 10, 8, 2, 6]
[5, 2, 11, 4, 3, 6]
[6, 10, 12, 9, 4, 2]
[7, 6, 13, 1, 3, 12]
[8, 5, 1, 9, 6, 11]
[9, 13, 2, 1, 7, 11]
[10, 13, 11, 6, 8, 7]
[5, 7, 4, 12, 9, 10]
[12, 7, 3, 2, 10, 6]
[13, 7, 6, 3, 5, 12]
[1, 2, 7, 8, 12, 11]

H_{33}

[2, 12, 8, 4, 13, 9]	[2, 3, 10, 13, 1, 7]	[1, 13, 3, 12, 2, 7]
[1, 13, 3, 12, 2, 7]	[5, 1, 12, 6, 2, 11]	[2, 3, 12, 6, 10, 8]
[11, 8, 6, 4, 2, 5]	[5, 3, 13, 6, 4, 10]	[4, 8, 13, 11, 10, 1]
[4, 5, 7, 9, 11, 13]	[4, 7, 11, 8, 1, 3]	[1, 6, 5, 7, 4, 10]
[3, 9, 8, 13, 4, 11]	[11, 1, 8, 4, 2, 5]	[2, 4, 6, 1, 5, 11]
[6, 1, 9, 8, 3, 5]	[7, 9, 5, 6, 8, 13]	[6, 12, 10, 3, 7, 13]
[7, 3, 10, 12, 6, 13]	[7, 3, 6, 13, 4, 11]	[4, 13, 8, 12, 7, 5]
[8, 1, 5, 4, 7, 10]	[6, 1, 9, 3, 11, 10]	[5, 10, 9, 1, 8, 3]
[9, 4, 12, 7, 8, 3]	[9, 11, 3, 12, 8, 10]	[6, 8, 11, 7, 9, 3]
[10, 11, 13, 2, 9, 5]	[7, 6, 3, 2, 10, 12]	[3, 4, 11, 9, 7, 10]
[5, 8, 1, 4, 10, 8]	[6, 2, 12, 7, 10, 11]	[3, 2, 12, 7, 8, 5]
[12, 3, 2, 11, 5, 13]	[12, 11, 13, 2, 9, 4]	[9, 2, 13, 5, 12, 4]
[13, 10, 11, 1, 12, 5]	[3, 5, 13, 10, 1, 4]	[1, 3, 13, 10, 11, 12]
[1, 10, 4, 8, 13, 3]	[6, 8, 5, 12, 1, 9]	[1, 4, 10, 2, 8, 9]
[11, 7, 9, 6, 1, 12]	[4, 12, 9, 13, 2, 8]	[2, 13, 9, 12, 4, 6]
[4, 1, 10, 8, 2, 6]	[3, 13, 5, 2, 11, 9]	[5, 2, 11, 13, 10, 9]
[5, 2, 11, 4, 3, 6]	[4, 2, 8, 9, 10, 5]	[3, 9, 6, 2, 4, 11]
[6, 10, 12, 9, 4, 2]	[7, 10, 12, 13, 11, 4]	[5, 13, 12, 8, 7, 4]
[7, 6, 13, 1, 3, 10]	[6, 4, 13, 7, 8, 5]	[6, 7, 13, 4, 8, 11]
[8, 5, 1, 9, 6, 11]	[7, 5, 9, 6, 1, 2]	[1, 8, 9, 11, 7, 2]
[9, 13, 2, 1, 7, 11]	[6, 7, 3, 10, 2, 12]	[2, 5, 11, 6, 8, 10]
[10, 13, 11, 6, 8, 2]	[9, 7, 5, 4, 10, 8]	[3, 7, 10, 5, 9, 6]
[5, 7, 4, 12, 9, 10]	[3, 8, 12, 9, 4, 1]	[4, 9, 12, 1, 11, 3]
[12, 7, 3, 2, 10, 6]	[11, 12, 13, 1, 10, 6]	[3, 1, 13, 12, 5, 8]
[13, 7, 6, 3, 5, 12]	[8, 3, 12, 5, 1, 11]	[1, 11, 12, 10, 6, 5]
[1, 2, 7, 8, 12, 11]	[7, 8, 13, 9, 2, 1]	[2, 9, 13, 6, 7, 1]
H_{34}	H_{35}	H_{36}

$|H_{12} \cap H_{16}| = 1;$ $\{\{2, 8, 13\}\}$

$|H_{11} \cap H_{16}| = 2;$ $\{\{2, 7, 13\}, \{2, 7, 13\}\}$

$|H_4 \cap H_6| = 4;$ $\{\{2, 6, 11\}, \{3, 6, 9\}, \{6, 7, 10\}, \{11, 12, 13\}\}$

$|H_4 \cap H_7| = 5;$ $\{\{1, 10, 13\}, \{1, 10, 13\}, \{6, 7, 10\}, \{2, 6, 11\}, \{11, 12, 13\}\}$

$|H_{12} \cap H_{20}| = 7;$ $\{\{8, 9, 12\}, \{8, 9, 12\}, \{4, 6, 12\}, \{4, 6, 12\}, \{4, 6, 12\}, \{1, 7, 12\}, \{1, 7, 12\}\}$

$|H_6 \cap H_{18}| = 8;$ $\{\{1, 3, 12\}, \{4, 9, 12\}, \{4, 9, 12\}, \{1, 5, 9\}, \{3, 4, 11\}, \{2, 3, 5\}, \{2, 3, 5\}, \{2, 3, 5\}\}$

$|H_3 \cap H_{14}| = 10;$ $\{\{7, 9, 11\}, \{7, 9, 11\}, \{3, 5, 6\}, \{3, 5, 6\}, \{5, 12, 13\}, \{5, 12, 13\}, \{3, 5, 6\}, \{2, 8, 12\}, \{7, 9, 11\}, \{5, 12, 13\}\}$

$|H_{14} \cap H_{22}| = 11;$ $\{\{3, 5, 6\}, \{3, 5, 6\}, \{2, 8, 12\}, \{2, 8, 12\}, \{7, 9, 11\}, \{7, 9, 11\}, \{5, 12, 13\}, \{5, 12, 13\}, \{5, 12, 13\}, \{3, 5, 6\}, \{7, 9, 11\}\}$

$|H_3 \cap H_{19}| = 13;$ $\{\{2, 8, 12\}, \{3, 5, 6\}, \{3, 5, 6\}, \{3, 5, 6\}, \{3, 4, 7\}, \{3, 4, 7\}, \{3, 4, 7\}, \{7, 9, 11\}, \{7, 9, 11\}, \{7, 9, 11\}, \{5, 12, 13\}, \{5, 12, 13\}, \{5, 12, 13\}\}$

$|H_8 \cap H_{20}| = 14;$ $\{\{1, 13, 12\}, \{7, 9, 11\}, \{7, 9, 11\}, \{7, 9, 11\}, \{1, 5, 9\}, \{5, 8, 11\}, \{5, 8, 11\}, \{3, 9, 13\}, \{9, 10, 12\}, \{2, 6, 9\}, \{4, 8, 9\}, \{5, 12, 13\}, \{5, 12, 13\}, \{5, 12, 13\}\}$

$|H_8 \cap H_{18}| = 16;$ $\{\{1, 3, 12\}, \{7, 9, 11\}, \{7, 9, 11\}, \{7, 9, 11\}, \{1, 5, 9\}, \{4, 10, 13\}, \{3, 8, 10\}, \{3, 9, 13\}, \{3, 4, 7\}, \{3, 4, 7\}, \{9, 10, 12\}, \{2, 6, 9\}, \{4, 8, 9\}, \{5, 12, 13\}, \{5, 12, 13\}, \{4, 12, 13\}\}$

$|H_{18} \cap H_{21}| = 19;$ $\{\{5,12,13\},\{5,12,13\},\{4,8,13\},\{4,8,13\},\{5,9,10\},\{5,9,10\},$
$\{1,3,13\},\{1,3,13\},\{4,9,12\},\{4,9,12\},\{1,4,10\},\{1,4,10\},$
$\{7,9,11\},\{7,9,11\},\{2,4,6\},\{2,4,6\},\{7,9,11\},$
$\{3,4,11\},\{5,12,13\}\}$

$|H_9 \cap H_{17}| = 22;$ $\{\{7,9,11\},\{7,9,11\},\{3,5,6\},\{3,5,6\},\{5,12,13\},\{5,12,13\},$
$\{6,7,12\}\{7,9,11\},\{6,10,11\}\{4,10,13\},\{2,7,13\},\{3,9,13\},$
$\{3,4,7\},\{9,10,12\},\{1,7,8\},\{2,6,9\},\{3,5,6\},\{4,11,12\},$
$\{5,7,10\},\{2,3,11\},\{5,12,13\},\{2,4,5\}\}$

$|H_{14} \cap H_{18}| = 23;$ $\{\{4,9,12\},\{4,9,12\},\{3,4,11\},\{1,4,10\},\{1,4,10\},\{1,5,8\},$
$\{1,5,8\},\{7,9,11\},\{7,9,11\},\{3,8,9\},\{3,8,9\},\{2,4,6\},$
$\{2,4,6\},\{1,3,13\},\{1,3,13\},\{5,9,10\},\{5,9,10\},\{5,9,10\},$
$\{4,8,13\},\{4,8,13\},\{5,12,13\},\{5,12,13\},\{7,9,11\}\}$

$|H_{23} \cap H_{27}| = 25;$ $\{\{4,9,12\},\{1,5,8\},\{1,6,9\},\{2,5,11\},\{2,9,13\},\{7,9,11\},$
$\{4,5,7\},\{1,11,12\},\{6,8,11\},\{2,3,10\},\{1,2,7\},\{5,12,13\},$
$\{4,9,12\},\{1,5,8\},\{1,6,9\},\{2,5,11\},\{2,9,13\},\{7,9,11\},$
$\{4,5,7\},\{1,11,12\},\{6,8,11\},\{2,3,10\},\{1,2,7\},\{5,12,13\},$
$\{4,9,12\}\}$

$|H_2 \cap H_{21}| = 26;$ $\{\{3,7,12\},\{4,9,12\},\{3,4,11\},\{1,4,10\},\{2,5,11\},\{10,11,13\},$
$\{7,8,10\},\{2,8,12\},\{2,9,13\},\{6,7,13\},\{7,9,11\},\{6,10,12\},$
$\{2,4,6\},\{1,3,13\},\{4,5,7\},\{1,11,12\},\{5,9,10\},\{6,8,11\},$
$\{2,3,10\},\{4,8,13\},\{1,2,7\},\{5,12,13\},\{5,12,13\},\{5,12,13\},$
$\{7,9,11\},\{7,9,11\}\}$

$|H_5 \cap H_{11}| = 28;$ $\{\{1,2,3\},\{1,2,3\},\{1,2,3\},\{2,11,12\},\{4,11,13\},\{1,4,5\},$
$\{2,5,6\},\{6,7,11\},\{4,7,8\},\{5,8,9\},\{6,9,10\},\{3,7,10\},$
$\{3,8,12\},\{9,12,13\},\{1,10,13\},\{1,8,11\},\{2,4,9\},\{5,10,11\},$
$\{3,4,6\},\{5,7,12\},\{6,8,13\},\{1,7,9\},\{2,8,10\},\{3,9,11\},$
$\{4,10,12\},\{3,5,13\},\{1,6,12\},\{2,7,13\}\}$

$|H_{23} \cap H_{28}| = 29;$ $\{\{4,9,12\},\{3,4,11\},\{1,4,10\},\{10,11,13\},\{6,10,12\},\{1,3,13\},$
$\{1,11,12\},\{5,9,10\},\{6,8,11\},\{4,8,13\},\{1,2,7\},\{5,12,13\},$
$\{4,9,12\},\{3,4,11\},\{1,4,10\},\{10,11,13\},\{6,10,12\},\{1,3,13\},$
$\{1,11,12\},\{5,9,10\},\{6,8,11\},\{4,8,13\},\{1,2,7\},\{5,12,13\},$
$\{7,9,13\},\{3,6,9\},\{5,7,11\},\{4,9,12\},\{3,4,11\}\}$

$|H_{17} \cap H_{18}| = 32;$ $\{\{4,9,12\},\{4,9,12\},\{3,4,7\},\{1,4,10\},\{1,4,10\},\{1,5,8\},$
$\{1,5,8\},\{7,9,11\},\{7,9,11\},\{3,8,9\},\{3,8,9\},\{2,4,6\},$
$\{2,4,6\},\{1,3,13\},\{1,3,13\},\{5,9,10\},\{5,9,10\},\{4,8,13\},$
$\{4,8,13\},\{5,12,13\},\{5,12,13\},\{1,13,12\},\{7,9,11\},\{1,5,9\},$
$\{4,10,13\},\{3,8,10\},\{3,9,13\},\{3,4,11\},\{9,10,12\},\{2,6,9\},$
$\{4,8,9\},\{5,12,13\}\}$

$|H_{14} \cap H_{15}| = 34;$ $\{\{3,7,12\},\{4,9,12\},\{1,6,9\},\{3,5,6\},\{3,4,11\},\{1,4,10\},$
$\{1,5,8\},\{2,5,11\},\{10,11,13\},\{7,8,10\},\{2,8,12\},\{2,9,13\},$
$\{6,7,13\},\{7,9,11\},\{6,10,12\},\{3,8,9\},\{2,4,6\},\{1,3,13\},$
$\{4,5,7\},\{1,11,13\},\{5,9,10\},\{6,8,11\},\{2,3,10\},\{4,8,13\},$
$\{1,2,7\},\{5,12,13\},\{3,5,6\},\{3,5,6\},\{2,8,12\},\{2,8,12\},$
$\{7,9,11\},\{7,9,11\},\{5,12,13\},\{5,12,13\}\}$

$|H_{15} \cap H_{16}| = 35;$ $\{\{3,7,12\},\{4,9,12\},\{3,4,11\},\{1,4,10\},\{1,5,8\},\{10,11,13\},$
$\{7,8,10\},\{2,7,13\},\{2,7,13\},\{7,9,11\},\{7,9,11\},\{3,8,9\},$
$\{2,4,6\},\{1,3,13\},\{4,5,7\},\{1,11,12\},\{5,9,10\},\{4,8,13\},$
$\{5,12,13\},\{5,12,13\},\{1,3,12\},\{7,9,11\},\{1,5,9\},\{3,8,10\},$
$\{6,7,13\},\{3,9,13\},\{3,4,7\},\{9,10,12\},\{1,7,8\},$
$\{2,6,9\},\{1,11,13\},\{4,11,12\},\{5,7,10\},\{4,8,9\},\{5,12,13\}\}$

$$
\begin{aligned}
|H_{23} \cap H_{29}| = 40; \quad & \{\{3,7,12\},\{3,7,12\},\{4,9,12\},\{4,9,12\},\{3,8,9\},\{1,5,8\},\\
& \{3,4,11\},\{3,4,11\},\{1,4,10\},\{1,4,10\},\{1,6,9\},\{2,5,11\},\\
& \{2,5,11\},\{10,11,13\},\{10,11,13\},\{2,9,13\},\{2,9,13\},\{7,9,11\},\\
& \{7,9,11\},\{3,5,6\},\{1,3,13\},\{1,3,13\},\{4,5,7\},\{4,5,7\},\\
& \{1,11,12\},\{1,11,12\},\{5,9,10\},\{5,9,10\},\{6,8,11\},\{6,8,11\},\\
& \{2,3,10\},\{2,3,10\},\{1,2,7\},\{1,2,7\},\{5,12,13\},\{5,12,13\},\\
& \{4,9,12\},\{6,7,10\},\{3,4,1\},\{8,10,12\}\}\\
|H_{17} \cap H_{26}| = 43; \quad & \{\{3,7,12\},\{4,9,12\},\{3,8,9\},\{1,5,8\},\{3,4,11\},\{1,4,10\},\\
& \{1,6,9\},\{2,5,11\},\{10,11,13\},\{2,9,13\},\{7,9,11\},\{3,5,6\},\\
& \{1,3,13\},\{4,5,7\},\{1,11,12\},\{5,9,10\},\{6,8,11\},\{2,3,10\},\\
& \{1,2,7\},\{5,12,13\},\{3,7,12\},\{4,9,12\},\{3,8,9\},\{1,5,8\},\\
& \{3,4,11\},\{1,4,10\},\{1,6,9\},\{2,5,11\},\{10,11,13\},\\
& \{2,9,13\},\{7,9,11\},\{3,5,6\},\{1,3,13\},\{4,5,7\},\{1,11,12\},\\
& \{5,9,10\},\{6,8,11\},\{2,3,10\},\{1,2,7\},\{5,12,13\},\{7,9,11\},\\
& \{3,5,6\},\{5,12,13\}\}\\
|H_{23} \cap H_{26}| = 44; \quad & \{\{3,7,12\},\{4,9,12\},\{3,8,9\},\{1,5,8\},\{3,4,11\},\{1,4,10\},\\
& \{1,6,9\},\{2,5,11\},\{10,11,13\},\{2,9,13\},\{7,9,11\},\{3,5,6\},\\
& \{1,3,13\},\{4,5,7\},\{1,11,12\},\{5,9,10\},\{6,8,11\},\{2,3,10\},\\
& \{1,2,7\},\{5,12,13\},\{3,7,12\},\{4,9,12\},\{3,8,9\},\{1,5,8\},\\
& \{3,4,11\},\{1,4,10\},\{1,6,9\},\{2,5,11\},\{10,11,13\},\{2,9,13\},\\
& \{7,9,11\},\{3,5,6\},\{1,3,13\},\{4,5,7\},\{1,11,12\},\{5,9,10\},\\
& \{6,8,11\},\{2,3,10\},\{1,2,7\},\{5,12,13\},\{4,9,12\},\{6,7,10\},\\
& \{3,4,11\},\{8,10,12\}\}
\end{aligned}
$$

$|H_1 \cap H_{15}| = 46;$ $\{\{3,7,12\},\{4,9,12\},\{1,6,9\},\{3,5,6\},\{3,4,11\},\{1,4,10\},$
$\{1,5,8\},\{2,5,11\},\{10,11,13\},\{7,8,10\},\{2,8,12\},\{2,9,13\},$
$\{6,7,13\}\{7,9,11\},\{6,10,12\},\{3,8,9\},\{2,4,6\},\{1,3,13\},$
$\{4,5,7\},\{1,11,12\},\{5,9,10\},\{6,8,11\},\{2,3,10\},\{4,8,13\},$
$\{1,2,7\},\{5,12,13\},\{3,5,6\},\{2,8,12\},\{7,9,11\},\{5,12,13\},$
$\{6,7,12\},\{7,9,11\},\{6,10,11\},\{4,10,13\},\{2,7,13\},\{3,9,13\},$
$\{3,4,7\},\{9,10,12\},\{1,7,8\},\{2,6,9\},\{3,5,6\},\{4,11,12\},$
$\{5,7,10\},\{2,3,11\},\{5,12,13\},\{2,4,5\}\}$

$|H_{17} \cap H_{21}| = 47;$ $\{\{3,7,12\},\{4,9,12\},\{3,4,11\},\{1,4,10\},\{2,5,11\},\{10,11,13\},$
$\{7,8,10\},\{2,8,12\},\{2,9,13\},\{6,7,13\},\{7,9,11\},\{6,10,12\},$
$\{2,4,6\},\{1,3,13\},\{4,5,7\},\{1,11,12\},\{5,9,10\},\{6,8,11\},$
$\{2,3,10\},\{4,8,13\},\{1,2,7\},\{5,12,13\},\{3,7,12\},\{4,9,12\},$
$\{3,4,11\},\{1,4,10\},\{2,5,11\},\{10,11,13\},\{7,8,10\},\{2,8,12\},$
$\{2,9,13\},\{6,7,13\},\{7,9,11\},\{6,10,12\},\{2,4,6\},\{1,3,13\},$
$\{4,5,7\},\{1,11,12\},\{5,9,10\},\{6,8,11\},\{2,3,10\},\{4,8,13\},$
$\{1,2,7\},\{5,12,13\},\{5,12,13\},\{7,9,11\},\{8,8,12\}\}$

$|H_{16} \cap H_{18}| = 50;$ $\{\{4,9,12\},\{4,9,12\},\{1,2,9\},\{2,3,5\},\{2,3,5\},\{1,2,9\},\{3,4,7\},$
$\{1,9,10\},\{1,4,10\},\{1,5,8\},\{1,5,8\},\{6,8,12\},\{6,8,12\},$
$\{6,9,13\},\{6,9,13\},\{7,9,11\},\{7,9,11\},\{2,10,12\},\{2,10,12\},$
$\{3,8,,9\},\{3,8,9\},\{2,4,6\},\{2,4,6\},\{1,3,13\},\{1,3,13\},$
$\{5,9,10\},\{5,9,10\},\{3,6,10\},\{3,6,10\},\{4,8,13\},\{4,8,13\},$
$\{5,12,13\},\{5,12,13\},\{1,3,12\},\{7,9,11\},\{1,5,9\},\{1,2,4\},$
$\{4,5,6\},\{4,10,13\},\{3,8,10\},\{6,8,12\},\{3,9,13\},\{3,4,11\},$
$\{9,10,12\},\{2,6,9\},\{2,3,5\},\{4,8,9\},\{16,10\},\{2,8,13\},$
$\{5,12,13\}\}$

$|H_{14} \cap H_{30}| = 52;$ $\{\{3,7,12\},\{3,7,12\},\{3,7,12\},\{4,9,12\},\{4,9,12\},\{4,9,12\},$
$\{3,8,9\},\{1,5,8\},\{3,4,11\},\{3,4,11\},\{3,4,11\},\{1,4,10\},$
$\{1,4,10\},\{1,4,10\},\{1,6,9\},\{2,5,11\},\{2,5,11\},\{2,5,11\},$
$\{10,11,13\},\{10,11,13\},\{2,9,13\},\{2,9,13\},\{2,9,13\},\{7,9,11\},$
$\{7,9,11\},\{7,9,11\},\{3,5,6\},\{1,3,13\},\{1,3,13\},\{1,3,13\},$
$\{4,5,7\},\{4,5,7\},\{4,5,7\},\{1,11,12\},\{1,11,12\},\{1,11,12\},$
$\{5,9,10\},\{5,9,10\},\{5,9,10\},\{6,8,11\},\{6,8,11\},\{6,8,11\},$
$\{2,3,10\},\{2,3,10\},\{2,3,10\},\{1,2,7\},\{1,2,7\},\{1,2,7\},$
$\{5,12,13\},\{5,12,13\},\{5,12,13\},\{10,11,13\}\}$

$|H_{14} \cap H_{17}| = 56;$ $\{\{3,7,12\},\{4,9,12\},\{4,9,12\},\{1,6,9\},\{1,6,9\},\{3,5,6\},\{3,5,6\},$
$\{3,4,11\},\{3,4,11\},\{1,4,10\},\{1,4,10\},\{1,5,8\},\{1,5,8\},$
$\{2,5,11\},\{2,5,11\},\{10,11,13\},\{10,11,13\},\{7,8,10\},\{7,8,10\},$
$\{2,8,12\},\{2,8,12\},\{2,9,13\},\{2,9,13\}.\{6,7,13\},\{6,7,13\},$
$\{7,9,11\},\{7,9,11\},\{6,10,12\},\{6,10,12\},\{3,8,9\},\{3,8,9\},$
$\{2,4,6\},\{2,4,6\},\{1,3,13\},\{1,3,13\},\{4,5,7\},\{4,5,7\},$
$\{1,11,12\},\{1,11,12\},\{5,9,10\},\{5,9,10\},\{6,8,11\},\{6,8,11\},$
$\{2,3,10\},\{2,3,10\},\{4,8,13\},\{4,8,13\},\{1,2,7\},\{1,2,7\},$
$\{5,12,13\},\{5,12,13\},\{3,5,6\},\{2,8,12\},\{7,9,11\},\{5,12,13\},$
$\{3,7,12\}\}$

$|H_2 \cap H_{15}| = 58;$ $\{\{3,7,12\},\{4,9,12\},\{1,6,9\},\{3,5,6\},\{3,4,11\},\{1,4,10\},$
$\{1,5,8\},\{2,5,11\},\{10,11,13\},\{7,8,10\},\{2,8,12\},\{2,9,13\},$
$\{6,7,13\},\{7,9,11\},\{6,10,12\},\{3,8,9\},\{2,4,6\},\{1,3,13\},$
$\{4,5,7\},\{1,11,12\},\{5,9,10\},\{6,8,11\},\{2,3,10\},\{4,8,13\},$
$\{1,2,7\},\{5,12,13\},\{6,7,12\},\{6,7,12\},\{7,9,11\},\{7,9,11\},$
$\{6,10,11\},\{6,10,11\},\{4,10,13\},\{4,10,13\},\{2,7,13\},\{2,7,13\},$
$\{3,9,13\},\{3,4,7\},\{3,4,7\},\{3,9,13\},\{9,10,12\},\{9,10,12\},$
$\{1,7,8\},\{1,7,8\},\{2,6,9\},\{2,6,9\},\{3,5,6\},\{3,5,6\},$
$\{4,11,12\},\{4,11,12\},\{5,7,10\},\{5,7,10\},\{2,3,11\},\{2,3,11\},$
$\{5,12,13\},\{5,12,13\},\{2,4,5\},\{2,4,5\}\}$

$|H_{14} \cap H_{25}| = 70;$ $\{\{3,7,12\},\{3,7,12\},\{3,7,12\},\{4,9,12\},\{4,9,12\},\{4,9,12\},$
$\{1,6,9\},\{3,5,6\},\{3,4,11\},\{3,4,11\},\{3,4,11\},\{1,4,10\},$
$\{1,4,10\},\{1,4,10\},\{1,5,8\},\{2,5,11\},\{2,5,11\},\{2,5,11\},$
$\{10,11,13\},\{10,11,13\},\{10,11,13\},\{7,8,10\},\{7,8,10\},\{7,8,10\},$
$\{2,8,12\},\{2,8,12\},\{2,8,12\},\{2,9,13\},\{2,9,13\},\{2,9,13\},$
$\{6,7,13\},\{6,7,13\},\{6,7,13\},\{7,9,11\},\{7,9,11\},\{7,9,11\},$
$\{6,10,12\},\{6,10,12\},\{6,10,12\},\{3,8,9\},\{2,4,6\},\{2,4,6\},$
$\{2,4,6\},\{1,3,13\},\{1,3,13\},\{1,3,13\},\{4,5,7\},\{4,5,7\},$
$\{4,5,7\},\{1,11,12\},\{1,1,12\},\{1,11,12\},\{5,9,10\},\{5,9,10\},$
$\{5,9,10\},\{6,8,11\},\{6,8,11\},\{2,3,10\},\{2,3,10\},\{2,3,10\},$
$\{4,8,13\},\{4,8,13\},\{4,8,13\},\{1,2,7\},\{1,2,7\},\{1,2,7\},$
$\{5,12,13\},\{5,12,13\},\{5,12,13\},\{6,8,11\}\}$

$|H_{14} \cap H_{24}| = 74;$ $\{\{3,7,12\},\{3,7,12\},\{3,7,12\},\{4,9,12\},\{4,9,12\},\{4,9,12\},$
$\{1,6,9\},\{1,6,9\},\{3,5,6\},\{3,5,6\},\{3,4,11\},\{3,4,11\},$
$\{3,4,11\},\{1,4,10\},\{1,4,10\},\{1,4,10\},\{1,5,8\},\{1,5,8\},$
$\{2,5,11\},\{2,5,11\},\{2,5,11\},\{10,11,13\},\{10,11,13\},\{10,11,13\},$
$\{7,8,10\},\{7,8,10\},\{7,8,10\},\{2,8,12\},\{2,8,12\},\{2,8,12\},$
$\{2,9,13\},\{2,9,13\},\{2,9,13\},\{6,7,13\},\{6,7,13\},\{6,7,13\},$
$\{7,9,11\},\{7,9,11\},\{7,9,11\},\{6,10,12\},\{6,10,12\},\{6,10,12\},$
$\{3,8,9\},\{3,8,9\},\{2,4,6\},\{2,4,6\},\{2,4,6\},\{1,3,13\},$
$\{1,3,13\},\{1,3,13\},\{4,5,7\},\{4,5,7\},\{4,5,7\},\{1,11,12\},$
$\{1,11,12\},\{1,11,12\},\{5,9,10\},\{5,9,10\},\{5,9,10\},\{6,8,11\},$
$\{6,8,11\},\{6,8,11\},\{2,3,10\},\{2,3,10\},\{2,3,10\},\{4,8,13\},$
$\{4,8,13\},\{4,8,13\},\{1,2,7\},\{1,2,7\},\{1,2,7\},$
$\{5,12,13\},\{5,12,13\},\{5,12,13\}\}.$

$|H_{16} \cap H_{34}| = 31;$ $\{\{4, 9, 12\}, \{4, 9, 12\}, \{3, 4, 11\}, \{3, 4, 11\}, \{1, 4, 10\}, \{1, 5, 8\}$
$\{1, 5, 8\}, \{10, 11, 13\}, \{10, 11, 13\}, \{6, 7, 13\}, \{7, 9, 11\}, \{7, 9, 11\},$
$\{1, 4, 10\}, \{7, 9, 11\}, \{3, 8, 9\}, \{3, 8, 9\}, \{2, 4, 6\}, \{2, 4, 6\},$
$\{1, 3, 13\}, \{1, 3, 13\}\{4, 5, 7\}, \{4, 5, 7\}, \{1, 11, 12\}, \{1, 11, 12\},$
$\{5, 9, 10\}, \{5, 9, 10\}, \{4, 8, 13\}, \{4, 8, 13\}, \{5, 12, 13\}, \{5, 12, 13\},$
$\{5, 12, 13\}\}$

$|H_{33} \cap H_{35}| = 17;$ $\{\{4, 9, 12\}, \{4, 9, 12\}, \{4, 9, 12\}, \{1, 6, 9\}, \{1, 6, 9\}, \{1, 6, 9\},$
$\{2, 5, 11\}, \{2, 5, 11\}, \{2, 5, 11\}, \{2, 9, 13\}, \{2, 9, 13\}, \{2, 9, 13\},$
$\{1, 2, 7\}, \{1, 2, 7\}, \{1, 2, 7\}, \{2, 3, 10\}, \{2, 3, 10\}\}.$

$|H_{17} \cap H_{36}| = 38;$ $\{\{4, 9, 12\}, \{4, 9, 12\}, \{3, 4, 11\}, \{3, 4, 11\}, \{1, 4, 10\}, \{1, 4, 10\},$
$\{2, 5, 11\}, \{2, 5, 11\}, \{10, 11, 13\}, \{10, 11, 13\}, \{2, 9, 13\}, \{2, 9, 13\},$
$\{6, 7, 13\}, \{6, 7, 13\}, \{7, 9, 11\}, \{7, 9, 11\}, \{6, 10, 12\}, \{6, 10, 12\},$
$\{2, 4, 6\}, \{2, 4, 6\}, \{1, 3, 13\}, \{1, 3, 13\}, \{4, 5, 7\}, \{4, 5, 7\},$
$\{1, 11, 12\}, \{1, 11, 12\}, \{5, 9, 10, \}, \{5, 9, 10\}, \{6, 8, 11\}, \{6, 8, 11\},$
$\{4, 8, 13\}, \{4, 8, 13\}, \{1, 2, 7\}, \{1, 2, 7\}, \{5, 12, 13\}, \{5, 12, 13\},$
$\{5, 12, 13\}, \{7, 9, 11\}\}.$

$|H_{31} \cap H_{32}| = 20;$ $\{\{4, 9, 12\}, \{4, 9, 12\}, \{4, 9, 12\}, \{7, 9, 11\}, \{7, 9, 11\}, \{7, 9, 11\},$
$\{2, 4, 6\}, \{2, 4, 6\}, \{2, 4, 6\}, \{5, 9, 10\}, \{5, 9, 10\}, \{4, 8, 13\},$
$\{4, 8, 13\}, \{4, 8, 13\}, \{5, 12, 13\}, \{5, 12, 13\}, \{5, 12, 13\}, \{1, 4, 10\},$
$\{1, 3, 13\}, \{5, 9, 10\}\}$

$|H_{21} \cap H_{33}| = 62;$ $\{\{3,7,12\},\{3,7,12\},\{4,9,12\},\{4,9,12\},\{4,9,12\},\{3,4,11\},$
$\{3,4,11\},\{3,4,11\},\{1,4,10\},\{1,4,10\},\{1,4,10\},\{2,5,11\},$
$\{2,5,11\},\{2,5,11\},\{10,11,13\},\{10,11,13\},\{10,11,13\},\{7,8,10\},$
$\{7,8,10\},\{2,8,12\},\{2,8,12\},\{2,9,13\},\{2,9,13\},\{2,9,13\},$
$\{7,9,11\},\{7,9,11\},\{7,9,11\},\{6,10,12\},\{6,10,12\},\{6,10,12\},$
$\{2,4,6\},\{2,4,6\},\{2,4,6\},\{1,3,13\},\{1,3,13\},\{1,3,13\},$
$\{4,5,7\},\{4,5,7\},\{4,5,7\},\{1,11,12\},\{1,11,12\},\{1,11,12\},$
$\{5,9,10\},\{5,9,10\},\{5,9,10\},\{6,8,11\},\{6,8,11\},\{6,8,11\},$
$\{2,3,10\},\{2,3,10\},\{4,8,13\},\{4,8,13\},\{4,8,13\},\{1,2,7\},$
$\{1,2,7\},\{1,2,7\},\{5,12,13\},\{5,12,13\},\{5,12,13\},\{6,7,13\},$
$\{6,7,13\},\{6,7,13\}\}.$

Combining all of the above gives the following lemma.

Lemma 5.2 $3Int(13) \supseteq \{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,$
$20,21,22,23,24,25,26,27,28,29,30,31,32,33,34,35,36,38,39,40,42,43,44,45,$
$46,47,48,50,51,52,54,56,57,58,60,62,66,70,74,78\}.$

Chapter 6

$n = 15$

Let (X, F) and (X, G) be two 1-factorizations of K_{2n}, where $F = \{F_1, F_2, \ldots, F_{2n-1}\}$ and $G = \{G_1, G_2, \ldots G_{2n-1}\}$. We say that (X, F) and (X, G) have k edges in common provided $\sum_{i=1}^{2n-1} |F_i \cap G_i| = k$. The intersection problem for 1-factorization of K_{2n} was solved in 1982 by C. C. Lindner and W. D. Wallis [5]. In particular, for $2n = 8$ the intersection numbers are $\{0, 1, 2, \ldots, 28\} \backslash \{27, 26, 25, 23\}$.

We will need the following construction; the $2n + 1$ Construction specialized to $2n + 1 = 15$.

Let (S, T) be a 3-fold triple system of order 7 where $S = \{1, 2, 3, 4, 5, 6, 7\}$. Let (X, F) be 1-factorization of K_8 where $S \cap X = \emptyset$ and $F = \{F_1, F_2, F_3, F_4, F_5, F_6, F_7\}$. Define a collection of triples T^* as follows:

(1) $T \subseteq T^*$, and

(2) for each edge $\{x, y\} \in F_i$, place three copies of $\{i, x, y\}$ in T^*.

Then $(S \cup X, T^*)$ is a 3-fold triple system.

The following lemma is immediate.

Lemma 6.1 *Let (X, F) and (X, G) have k triples in common. Then the type (2) triples in (2) have $3k$-triples in common.* □

Lemma 6.2 *If (S, T) can be organized into hexagon triples, then $(S \cup X, T^*)$ can be organized into hexagon triples.*

Proof It is only necessary to organize the triples of type (2) into hexagon triples. This is quite easy. Let (X, G) be a 1-factorization of K_8 where $G \cap F = \emptyset$. Then by Corollary 2.4 we can place the triples of type (2) on the triples $\{i, x, y\} \in G_i \in G$ to obtain a collection of hexagon triples. □

Corollary 6.3 *If* $x \in 3Int(7)$ *and* $k \in \{0, 1, 2, \ldots, 28\} \backslash \{27, 26, 25, 23\}$*, then* $x + 3k \in 3Int(15)$.

Proof Let (S, T_1) and (S, T_2) be 3-fold triple system having k triples in common which can be organized into hexagon triples and let (X, F) and (X, G) be 1-factorizations of K_8 having k edges in common. Then the 3-fold triple systems $(S \cup X, T_1^*)$ and $(S \cup X, T_2^*)$ constructed using the $2n + 1$ Construction can be organized into hexagon triple systems having $x + 3k$ triples in common. □

Lemma 6.4 $3Int(15) = \{0, 1, 2, \ldots, 78\} \backslash \{77, 76, 75, 73\}$.

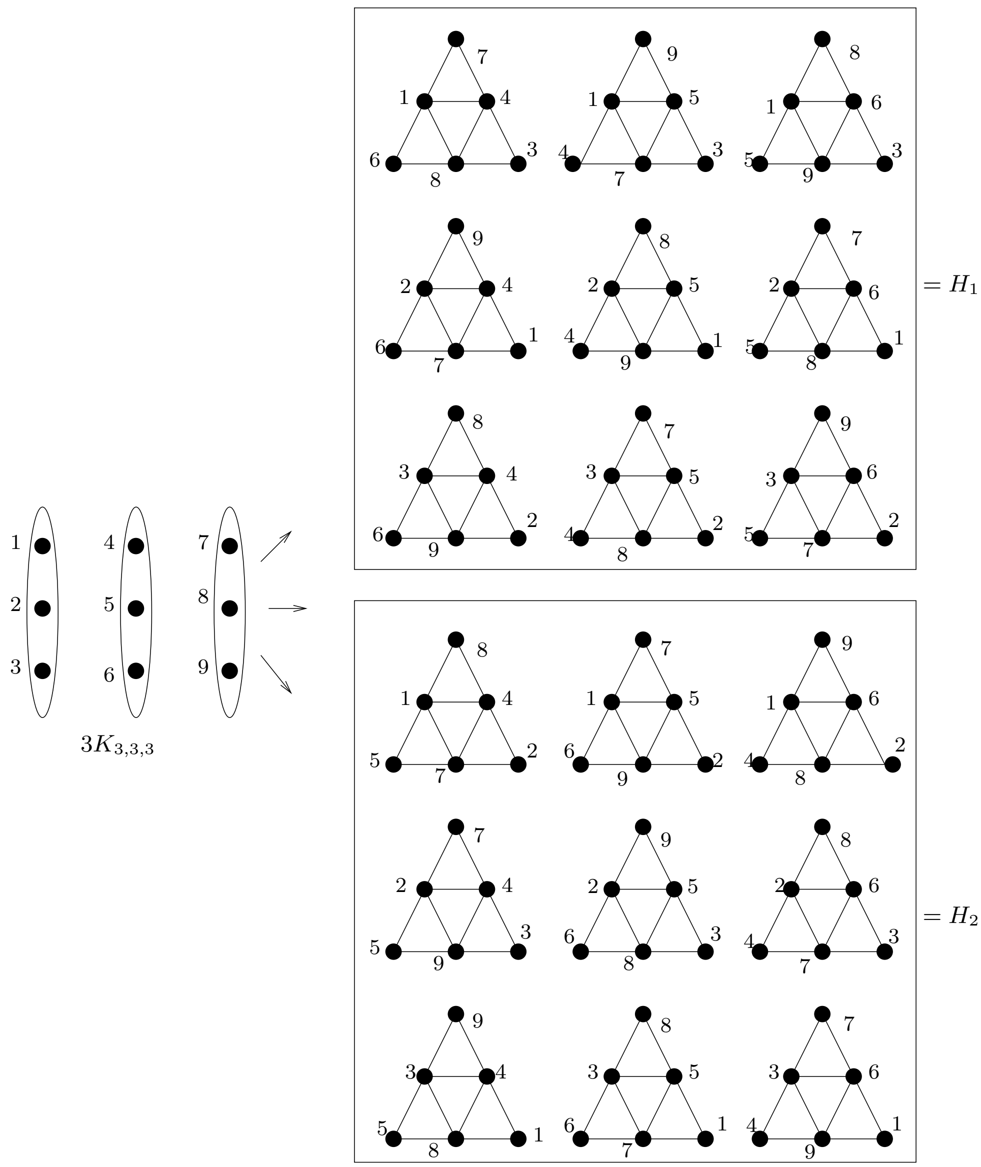
Proof Each $n \in 3Int(15)$ can be written in the form $n = x + 3k$, where $x \in 3Int(7)$ and $k \in \{0, 1, 2, \ldots, 28\} \backslash \{27, 26, 25, 23\}$. □

Chapter 7

The $6n + 1 \geq 19$ Construction

Since we have a solution for $7, 9, 13$ (modulo a few exceptions), and 15 we need consider only the cases $6n + 1 \geq 19$. Before giving the $6n + 1$ Construction, we need the following example.

Example 7.1 (Two decompositions of $3K_{3,3,3}$ into hexagon triples having no triples in common.)



Corollary 7.2 *There exists a pair of $3K_{3,3,3}$ hexagon triple systems having* 0 *or* 27 *triples in common.* $\square$

With this example in hand, we can proceed to the $6n+1 \geq 19$ Construction.

The $\mathbf{6n+1 \geq 19}$ Construction

Let (X, G, T) be a group divisible design (GDD) of order $2n \geq 6$ with at most one group of size 4 and the remaining groups of size 2. If $2n \equiv 0$ or 2 (mod 6) all groups are of size 2 and if $2n \equiv 4$ (mod 6) exactly one group is of size 4, the others of size 2. (See [6].) Let $S = \{\infty\} \cup (X \times \{1, 2\})$ and define a collection of hexagon triples H as follows:

(1) For each $g \in G$, let $(\{\infty\} \cup (g \times \{1, 2, 3\}), H(g))$ be a hexagon triple system of order 7 or 13, as the case may be, and place the hexagon triples of $H(g)$ in H.

(2) For each block $t = \{a, b, c\} \in T$, let $(3K_{3,3,3}, T(t))$ be a decomposition of $3K_{3,3,3}$ with parts $\{a\} \times \{1, 2, 3\}, \{b\} \times \{1, 2, 3\}$, and $\{c\} \times \{1, 2, 3\}$ into hexagon triples and place these hexagon triples in H.

Then (S, H) is a hexagon triple system.

Lemma 7.3 *There exists a pair of hexagon triple systems of order* $6n+1$ *(mod 6)* ≥ 19 *having* x *triples in common for all* $x \in 3Int(n)$.

Proof Let $x \in 3Int(n)$. If $2n \equiv 0$ or 2 (mod 6), we can write $x = \sum_{i=1}^{n} a_i + \sum_{i=1}^{|T|}\{0, 27\}$ where the a_i's belong to $3Int(7)$. If $2n \equiv 4$ (mod 6), we can write $x = a_1 + \sum_{i=1}^{n-2} a_i + \sum_{i=1}^{|T|}\{0, 27\}$, where $a_1 \in \{0, 78\}$ (see Lemma 6.4) and $a_2, a_3, \ldots, a_{n-2} \in 3Int(7)$. $\square$

Combining the results in Chapters 3, 4, 5, 6, and 7 giving the following theorem.

Theorem 7.4 $3Int(n) = 3I(n)$ *for all* $n \equiv 1$ *(mod 6), with possibly a few exceptions for* $n = 13$ *(see Section 5).* $\square$

Chapter 8

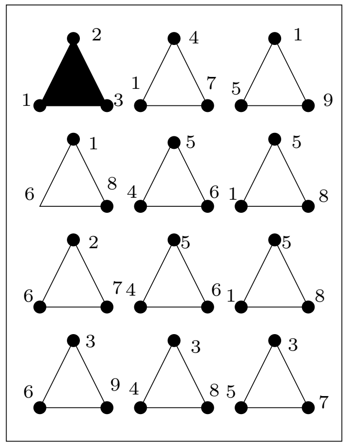
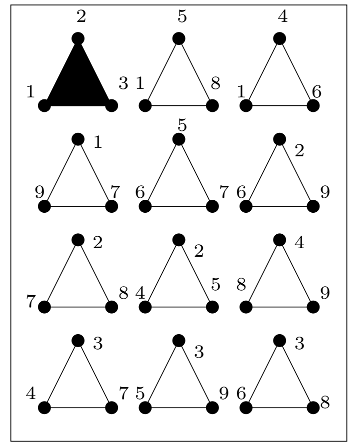
The $6n + 3$ Construction

Before giving the $6n + 3$ Construction we will need the following lemma.

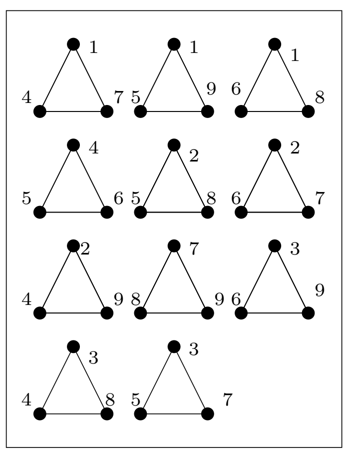
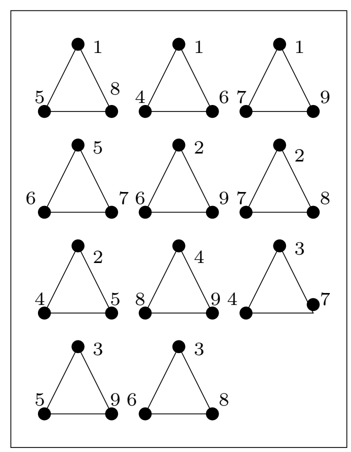
Lemma 8.1 *There exist a pair of partial hexagon triple systems of order* 9 *which are disjoint, balanced, and cover the edges of* $3K_9 \backslash 3K_3$.

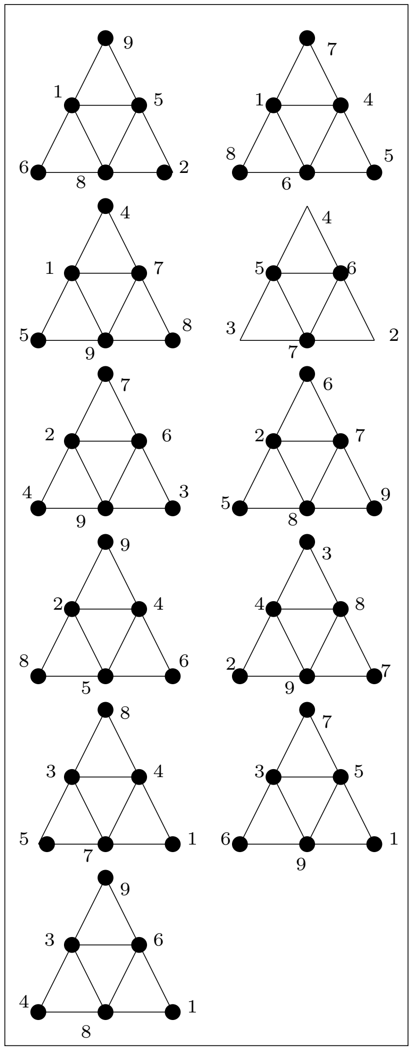
Proof Let (S, T_1) and (S, T_2) be a pair of triple systems of order 9 having *exactly* the one triple t^* in common. Then $(S, T_1 \backslash t^*)$ and $(S, T_2 \backslash t^*)$ are disjoint and balanced. Putting the triples of $T_1 \backslash t^*$ on the triples of $T_2 \backslash t^*$ and vice versa gives the desired pair of partial hexagon triple systems. □

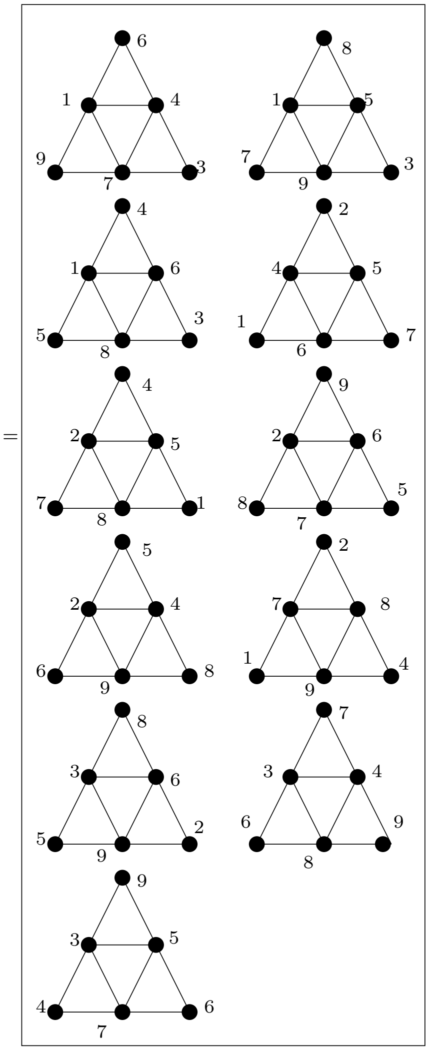
Example 8.2 (A pair of partial hexagon triple systems of order 9.)

$T_1 =$  $T_2 =$ 

$t^* = \{1, 2, 3\}$

$T_1 \backslash t^* =$  $T_2 \backslash t^* =$ 

$H_1 =$ 

$H_2 =$ 

We can now give the $6n+3$ Construction.

Let (X, G, B) be a GDD of order $2n$ with *at most* one group of size 4 and the remaining groups of size 2. Set $S = \{\infty_1, \infty_2, \infty_3\} \cup (X \times \{1,2,3\})$ and define a collection of hexagon triples H as follows:

Let $G = \{g_1^*, g_2, g_3, \ldots, g_{2n}\}$ be the groups of G with $|g_1^*| = 4$ if G contains a group of size 4.

(1) Place a hexagon triple system on $\{\infty_1, \infty_2, \infty_3\} \cup (g_1^* \times \{1,2,3\})$ and place these hexagon triples in H.

(2) For each $g_i \in G$, $i \geq 2$, place a partial hexagon triple system on $\{\infty_1, \infty_2, \infty_3\} \cup \{g_i \times \{1, 2, 3\})$ as in Lemma 8.1

(3) For each triple $\{a, b, c\} \in B$, place a hexagon triple system on $3K_{3,3,3}$ with parts $\{a\} \times \{1, 2, 3\}, \{b\} \times \{1, 2, 3\}$, and $\{c\} \times \{1, 2, 3\}$.

Then (S, H) is a hexagon triple system of order $6n + 3$.

Theorem 8.3 *The intersection numbers for hexagon triple systems of order* $n \equiv 3$ *(mod 6) are precisely* $\{0, 1, 2, \ldots, \binom{n}{2} = x\} \backslash \{x-1, x-2, x-3, x-5\}\}$.

Proof 9 and 15 are taken care of in Sections 4 and 6. So we need only concern ourselves here with $6n + 3 \geq 21$.

In case $|g_1^*| = 4$, any number in $\{0, 1, 2, \ldots \binom{n}{2} = x\} \backslash \{x-1, x-2, x-3, x-5\}$ can be written as a sum $a + \sum b_i + \sum\{0, 27\}$, where $a \in 3Int(15)$ and $b_i \in \{0, 33\}$. If $|g_i^*| = 2$ then any number in $\{0, 1, 2, \ldots, \binom{n}{2} = x\} \backslash \{x-1, x-2, x-3, x = 5\}$ can be written as a sum $a + \sum b_i + \sum\{0, 27\}$ where $a \in 3Int(9)$. □

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