On the Spectrum of Minimal Covers By Triples

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On the Spectrum of Minimal Covers By Triples

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Dissertation Abstract

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A Minimal Cover by Triples is an ordered pair (V, T) where V is a finite set and T is a collection of three element subsets of V with two properties. First, that every pair of elements of V appear together in at least one element of T. Second, if any element of T is removed the first property no longer holds. In this dissertation we explore the possible values $|T|$ can take on for a given $|V|$. In addition, construction techniques are given to construct a Minimal Cover by Triples for given values of $|V|$ and $|T|$.

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Style manual or journal used Journal of Approximation Theory (together with the style known as "aums"). Bibliograpy follows van Leunen's *A Handbook for Scholars.*

Computer software used The document preparation package TEX (specifically LATEX) together with the departmental style-file `aums.sty`.

Table of Contents

List of Figures

Chapter 1

Introduction

We will refer to the ordered pair (V, T) as a *Collection of Triples (CoT)* whenever V is a finite non–empty set and T is a collection of three element subsets of V. The elements of T will be referred to as *triples*. A *Steiner Triple System* is a CoT (V, T) with the additional property that every pair of elements of V appears together in exactly one triple of T. It is well known that a Steiner Triple System exists on V if and only if $|V| \equiv 1 \text{ or } 3 \mod 6$.

The natural next question of how close you can come is also well known in two forms. One can look at maximum packings or minimum covers. A *Packing of Triples* is a CoT (V, T) with the property that every pair of elements of V appears together in at most one triple of T. A Packing of Triples (V, T) is a *maximum packing* if for every other Packing of Triples (V, T') we have that $|T'| \leq |T|$. A *Cover by Triples* is a CoT (V, T) with the property that every pair of elements of V appears together in at least one triple of T. A Cover by Triples (V, T) is a *minimum cover* if for every Cover by Triples (V, T') we have that $|T| \leq |T'|$. More detailed information on maximum packings and minimum covers can be found in [1].

The purpose of this document is to take the idea of minimum covers one step further and look at minimal covers. A *Minimal Cover by Triples* (MCT) is a cover by triples, (V, T) , with the property that if you remove any triple from T you no longer have a cover by triples. If (V, T) is a MCT with $|V| = v$ and $|T| = b$ then we call it a (v, b) MCT

We will address specifically how many triples can be in a minimal cover by triples for a given value for v. An attempt, albeit unsuccessful, to do so (as well as for block size 4) is made in [2]. In chapter 5 of this document we will look at that particular attempt.

We state our main claim, which will be proved in this discourse, as follows:

Theorem 1.1 *There is a* (v, b) *MCT if and only if*

$$\left\lceil \frac{v}{3} \left\lceil \frac{v-1}{2} \right\rceil \right\rceil \leq b \leq \binom{v-1}{2} - 2, \text{ or } b = \binom{v-1}{2} \tag{1.1}$$

with the following exceptions:

1. *there is a* $(5, 5)$ *MCT*
2. *there is no* $(7, 8)$ *MCT*

Chapter 2

The Max, the Min and the Missing.

The lower bound of equation 1.1 is well known and can be obtained from information found in [1] where there is a section devoted to minimum covers with triples. The upper bound and missing value will be dealt with in this chapter.

2.1 The Upper Bound

It is often useful to consider this problem in graph theoretical terms. If we consider our minimal cover to be a cover of the complete graph with v vertices using copies of K_3 we can use this notion to help prove the upper bound. With this in mind we will refer to any edge that appears in exactly one triple of the cover to be a *unique edge* and any edge that appears in at least two triples of the cover will be called a *common edge*. In any minimal cover each triple will have at least one unique edge. Choose a unique edge from each triple and denote those edges as special edges.

Note that each triple then has one special edge and two non–special edges. We also consider the fact that each non–special edge can appear in at most $v-2$ triples. Thus we have b special edges and at least $\frac{2b}{v-2}$ non special edges giving us that

$$b + \frac{2b}{v-2} \leq \binom{v}{2}. \tag{2.1}$$

Applying some algebra to that gets us

$$b \leq \binom{v-1}{2} \tag{2.2}$$

To show that this bound is sharp, simply choose your favorite point of V, name it ∞, and let $T = \{\{\infty, u, v\} | u \neq v,\ u, v \in V \setminus \{\infty\}\}$.

2.2 Solving the Mystery of the Missing Value

Possibly the most interesting result that contributes to the main theorem is the following:

Theorem 2.1 *There exists a $(v, \binom{v-1}{2} - 1)$ MCT if and only if $v = 5$.*

If we let $V = \{0, 1, 2, 3, 4\}$ and $T = \{\{0, 1, 3\}, \{1, 2, 4\}, \{2, 3, 0\}, \{3, 4, 1\}, \{4, 0, 2\}\}$, then we have a $(5, 5)$ MCT. So it remains to be shown that one can't be obtained if $v \neq 5$.

Suppose (V, T) is a (v, b) MCT. As in section 2.1 we will choose one unique edge of each triple in T and label it as a special edge all other edges will be considered non–special edges. Let G be the subgraph of K_v induced by the set of non–special edges. Note that $|E(G)| = \binom{v}{2} - b$.

For each $x, y \in V$ with $x \neq y$ and $xy \notin E(G)$ there is a path P_{xy} of length 2 from x to y in G. Since xy is not in G it was a special edge so we will take the non–special edges from the triple it appears in to make up the path P_{xy}. If you define the path P_{xy} in such a way, it guarantees that each edge of G is in at least one P_{xy}.

With the existence of all the P_{xy}'s we have that every pair of vertices are either adjacent or distance 2 apart in G. Hence the diameter of G is 2 and it is connected. It also follows

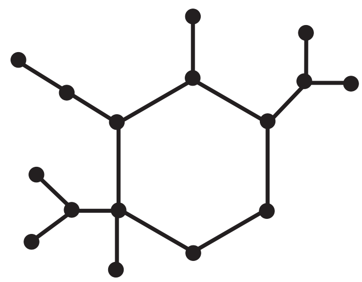

Figure 2.1: An example of a graph with exactly one cycle of length 6

that $v - 1 \leq |E(G)|$ thus $b \leq \binom{v}{2} - (v-1) = \binom{v-1}{2}$. (A fact that we have previously proved in section 2.1.)

Suppose that $b = \binom{v-1}{2} - 1$. Then we have that $|E(G)| = v$ and it follows that removing any edge of G either disconnects it or makes it acyclic. Hence G contains exactly one cycle of some length k. See figure 2.1 for an example of such a graph. If we were to remove the edges of the cycle from G, we would be left with a forest. We will refer to this forest as the fringe of G. So let's consider what possible values we can have for k.

If $k \geq 6$, then G would have diameter of at least 3; hence $k < 6$. Suppose $k = 5$ or 4, if there is a fringe edge then again the diameter will be at least 3 so in this case G can only be a cycle. If $k = 3$ and the fringe is anything other than a star then the diameter will again be at least 3. So the possible forms of G will be as in figure 2.2.

If $k = 3$ then there is no way for edge $y_1 y_2$ to be in any P_{xy} so $k \neq 3$. If $k = 4$ then we will have P_{xz} and P_{wy}. Without loss of generality, assume that P_{xz} is the path xyz. If P_{wy} is the path wxy then the edge wz is in no P_{xy} and if P_{wy} is the path wzy then the edge wx is in no P_{xy}. So it follows that $k \neq 4$. Thus if $b = \binom{v-1}{2} - 1$ then $v = 5$.

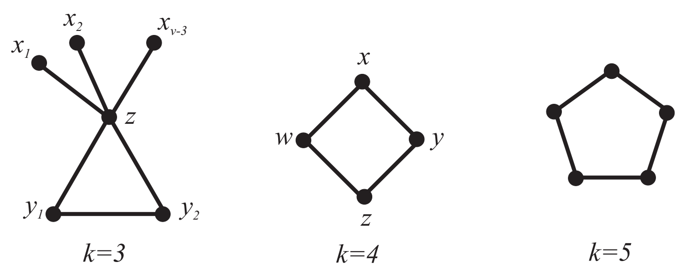

Figure 2.2: Narrowing down G when $b = \binom{v-1}{2} - 1$

Chapter 3

Needed Constructions

3.1 Minimum cover when v $\equiv$ 0 mod 6

We will make use of the "1–factor covering Construction" found in [1]. It will be used not only as the way to obtain the minimum number of triples in its respective case but as a starting point to obtain most of the other possible amounts of triples. The construction makes use of a pairwise balanced design with one block of size 5 and the rest of size 3. A *pairwise balanced design* (or PBD for short) is an ordered pair (X, B) were X is a finite nonempty set and B is a collection of subsets of X, called blocks, with the property that every pair of elements of X appears in exactly one block of B. How to construct the required PBD for the 1–factor covering construction is also found in [1].

We reproduce the 1–factor covering construction here with a few minor changes to how things are labeled.

Construction 3.1 (The 1–factor covering Construction)

Let $v = 6n$ and let (X, B) be a PBD of order $v - 1$ with one block $\{d, e, f, x_0, y_0\}$ of size 5 and the remaining blocks of size 3. Denote by T the collection of blocks of size 3. Let $S = \{\infty\} \cup X$ and let $\pi = \{\{x_1, y_1\}, \{x_2, y_2\}, \ldots, \{x_{3n-3}, y_{3n-3}\}\}$ be any partition of $X \setminus \{d, e, f, x_0, y_0\}$. Let $\pi(\infty) = \{\{\infty, x_1, y_1\}, \{\infty, x_2, y_2\}, \ldots, \{\infty, x_{3n-3}, y_{3n-3}\}\}$ and $F(\infty) = \{\{\infty, d, f\}, \{\infty, e, f\}, \{\infty, x_0, y_0\}, \{x_0, y_0, f\}, \{d, e, x_0\}, \{d, e, y_0\}\}$. Then (S, T^*) is a minimum covering of order v where $T^* = T \cup \pi(\infty) \cup F(\infty)$.

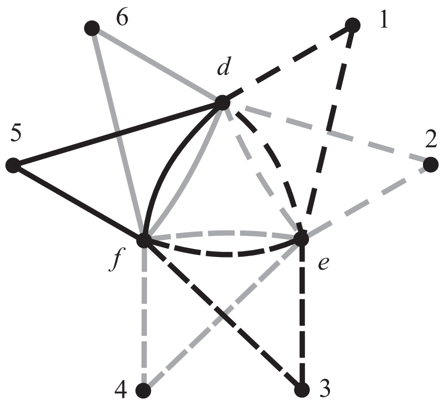

Figure 3.1: Handling the common edges

3.2 One more triple than a minimum cover when $\mathbf{v \equiv 1}$ or 3 mod 6

We will denote the multigraph obtained by taking all the edges of the triples of the cover as the *cover graph*. Every vertex of this graph will be of even degree. When $v \equiv 1$ or 3 mod 6 the cover graph for a minimum cover is the complete graph. So if we are interested in one more triple than there is in a minimum cover the associated cover graph will be a complete graph plus three extra edges. Since a complete graph with $v \equiv 1$ or $3 \mod 6$ has even degree it follows that the extra edges must form a K_3. The only way to do this and have a minimal cover is for no two of the three common edges to appear in a triple together. This forces us to have the structure in figure 3.1. Since this requires at least nine vertices it follows that we can not have a $(7, 8)$ MCT.

If we are looking to obtain a $(9, 13)$ MCT we can first add the triples $\{d, 3, 4\}$, $\{e, 5, 6\}$ and $\{f, 1, 2\}$ to what we have in figure 3.1. For future reference we will refer to this collection of triples as α. What we are then left with is the edges of a K_6 minus a matching (denoted

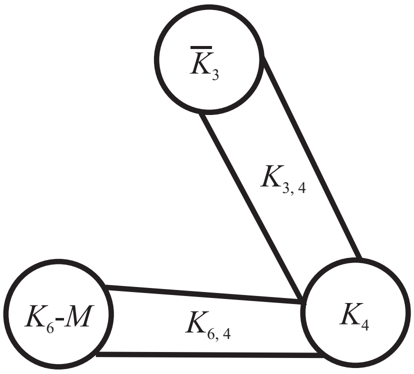

Figure 3.2: Building a $(13, 27)$ MCT

as $K_6 - M$ for brevity here on) which can easily be partitioned into triples. An example of a $(9, 13)$ MCT can be found in Appendix A.1.

In order to obtain a $(13, 27)$ MCT we again start with α. What remains is the graph we have in figure 3.2. We then 3 edge color the K_4 using the vertex labels of the $\overline{K_3}$ as colors and we 4 edge color the $K_6 - M$ using the vertex labels of the K_4 as colors. Then for each edge $\{u, w\}$ of either the K_4 or $K_6 - M$ we add the triple $\{u, w, c\}$ where c is the color of $\{u, w\}$. An example of a $(13, 27)$ MCT constructed in such a manner can be found in Appendix A.2.

To obtain a $(15, 36)$ MCT we again start with α after which we are left with the graph in figure 3.3. This time we will need to make use of an equitable edge coloring. For our purposes, an edge coloring is *equitable* if for any two colors i and j the number of edges colored i is equal to the number of edges colored j. We then 6 edge color the $K_6 - M$ equitably, using the vertex labels of the K_6 as colors, in such a way that the color pairs missing at each vertex form a 2–regular subgraph of the K_6 when taken as edges. We will call this 2–regular subgraph G. Next we 3 edge color the $K_6 - G$ using the labels of the $\overline{K_3}$

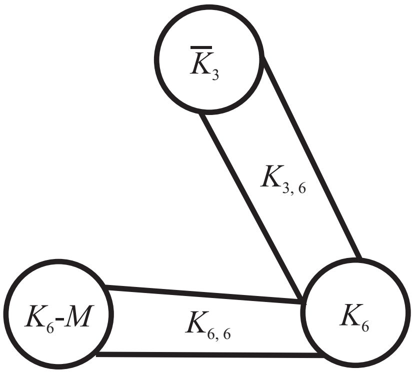

Figure 3.3: Building a $(15, 36)$ MCT

as colors. For each edge $\{u, w\}$ that has been colored we add the triple $\{u, w, c\}$ where c is the color of the edge. Finally for each edge $\{u, w\}$ of G if x is the vertex of the $K_6 - M$ that has no edge colored u or w then add the triple $\{u, w, x\}$. An example of a $(15, 36)$ MCT constructed in such a manner can be found in Appendix A.2.

For any $v \equiv 1$ or $3 \mod 6$ with $v > 18$ we can start by embedding any STS(9) in a STS(v) using the process found in [3]. Replacing the triples of the STS(9) with the triples of our $(9, 13)$ MCT will give us a $(v, \frac{v(v-1)}{6} + 1)$ MCT.

3.3 One more triple than a minimum cover when v $\equiv$ 2 or 4 mod 6

We are going to make use of a modified version of the "tripole covering Construction" a minimum cover for these orders found in [1]. Instead of starting with a Steiner triple system of order $v - 1$ we will start with the minimal cover (V', T') of order $v - 1$ with one extra triple constructed in section 3.2. We begin by renaming the elements of V'. Choose any triple of T' that does not contain any of the points $d, e, f, 2, 4$ or 6 and rename its points

u, s and w. Let $a = x_1$, $b = x_2$, $c = x_3$, $2 = y_3$, $4 = y_1$, $6 = y_2$. The remaining points rename in any fashion with the labels $x_4, \ldots, x_{\frac{1}{2}v-2}, y_4, \ldots, y_{\frac{1}{2}v-2}$.

Once we have renamed the points we let $V = V' \cup \{\infty\}$ construct T in the following manner. Let A be all the triples of $T' \setminus \{\{u, s, w\}\}$ (properly renamed); let $B = \{\{\infty, u, s\}, \{\infty, s, w\}, \{\infty, u, w\}\}$; let $C = \{\{\infty, x_i, y_i\} | 1 \leq i \leq \frac{1}{2}v - 2\}$. Finally $T = A \cup B \cup C$. We then have the desired MCT.

3.4 Minimum cover when $\mathbf{v \equiv 5 \bmod 6}$

We will be making use of the "double edge covering Construction" found in [1].

Construction 3.2 (The double edge covering Construction)

Let $v = 6n + 5$ and let (S, B) be a PBD of order v with one block $\{a, c, d, e, f\}$ of size 5 and the remaining blocks of size 3. Denote by T the collection of blocks of size 3, and let $T^* = \{\{a, f, c\}, \{a, f, d\}, \{a, f, e\}, \{c, d, e\}\}$. Then $(S, T \cup T^*)$ is a minimum cover by triples of order v.

Chapter 4

proof of Main Theorem

The techniques we will use to construct minimal covers for all possible values of b will start with a MCT with either the least possible number of triples or one more than the least possible. It will entail choosing one special point (akin to choosing the point in the construction of a MCT with the largest possible number of triples). Then in a strategic manner we will remove triples that don't contain the special point. For each unique edge from a triple that has been removed we will add a triple containing that edge and the special point. When we do this the only edges we cause to become duplicated are edges incident to the special point. The only danger in this is if there are triples whose unique edges are incident to the special point. In the cases where this is a problem we will carefully remove triples in a manner that will not cause those critical unique edges to be duplicated until after the common edge of the triple they are in has become a unique edge.

4.1 $\mathbf{v \equiv 0 \bmod 6}$

Let $v = 6n$, $n \geq 1$. The minimum value, $6n^2$, can be obtained using construction 3.1. To get the remaining values we will next separate the triples of this initial minimal cover into 4 classes. For notational purposes if we are referring to a vertex x_i or y_i and it is not important whether it is an x or a y but the subscript is important we will refer to it as z_i. We categorize the triples as follows:

Type 0 The triples $\{\infty, e, f\}$, $\{\infty, d, f\}$ and the triples of the form $\{\infty, x_i, y_i\}$ for $0 \leq i \leq 3n-3$.

Type 1 The triple $\{x_0, y_0, f\}$ and the triples of the form $\{x_i, y_i, v_i\}$ where $1 \le i \le 3n-3$ and v_i is the third element of the triple containing x_i and y_i but not ∞.

Type 2 Any triple of the form $\{z_i, z_j, z_k\}$ where $0 \le i < j < k \le 3n-3$ and any triple of the form $\{u, z_i, z_j\}$ where $u \in \{d, e, f\}$ and $1 \le i < j \le 3n-3$

Type 3 The triples $\{d, e, x_0\}$ and $\{d, e, y_0\}$

Obviously we have $3n$ Type 0, $3n-2$ Type 1 and 2 Type 3 triples; which leaves us $6n^2 - 6n$ Type 2 triples. We now proceed to show how to use this starting point as a way to construct a minimal cover with b triples for all b, $6n^2 < b < 18n^2 - 9n$.

We also want to take note that the Type 0 triples of the form $\{\infty, x_i, y_i\}$ have unique edges that are adjacent to infinity. Each one shares the edge $\{x_i, y_i\}$ with a Type 1 triple. So we need to take care how we remove triples until all the Type 1 triples are removed.

If $b \le 6n^2 + 3n - 2$ it can be obtained simply by removing $b - 6n^2$ type one triples and replacing them with new triples in the following manner:

1. Remove the triple $\{x_0, y_0, f\}$ and replace it with the triples $\{\infty, x_0, f\}$ and $\{\infty, y_0, f\}$.

2. Set $i = 1$.

3. Remove the Type 1 triple $\{x_i, y_i, v_i\}$ and replace it with the triples $\{\infty, x_i, v_i\}$ and $\{\infty, y_i, v_i\}$.

4. If $v_i = z_j$ for some j and the type 1 triple containing x_j and y_j has not been removed yet set $i = j$ otherwise set i to the lowest possible value such that $\{x_i, y_i, v_i\}$ has not been removed.

5. Go to step 3 unless we now have enough triples or there are no more Type 1 triples left.

When we remove the triple $\{x_i, y_i, v_i\}$ with $v_i = z_j$ and if $\{x_j, y_j, v_j\}$ has not yet been removed we are in danger of causing a triple to contain no unique edges. Assume without loss of generality that $v_i = x_j$. At this point in the Type 0 triple $\{\infty, x_j, y_j\}$ the only unique edge is $\{\infty, y_i\}$. If there is also a Type 1 triple $\{x_k, y_k, v_k\}$ where $v_k = y_j$ and we remove it and add the corresponding triples it will result in the triple $\{\infty, x_j, y_j\}$ no longer having a unique edge and hence we no longer have a MCT. If we first remove the triple $\{x_j, y_j, v_j\}$ we can avoid this problem. This is the reason for step 4 being as it is.

We need not worry about the removal of a single Type 1 triple causing a problem. The removal of $\{x_i, y_i, v_i\}$ with $v_i = z_j$ (without loss of generality assume $z_j = x_j$) can only cause the edges $\{\infty, x_i\}$, $\{\infty, y_i\}$, and $\{\infty, x_j\}$ to be come common edges. This isn't a problem because $\{x_i, y_i\}$ has just become a unique edge and if $\{\infty, y_j\}$ is common then the triple $\{x_j, y_j, v_j\}$ would have previously been removed so $\{x_j, y_j\}$ will be a unique edge.

If $b > 6n^2+3n-2$ we first remove and replace all the type 1 triples in the above described manner. We then choose $\lfloor \frac{b-(6n^2+3n-2)}{2} \rfloor$ type 2 triples. For each such triple $\{u, v, w\}$ chosen, we remove it and replace it with the triples $\{\infty, u, v\}$,$\{\infty, v, w\}$ and $\{\infty, u, w\}$. If we now have a system containing b triples we are done, otherwise we have one containing $b-1$ triples and then remove the triple $\{d, e, x_0\}$ and replace it with the triples $\{\infty, x_0, d\}$ and $\{\infty, x_0, e\}$.

Example 4.1 $(v = 18, b = 66)$

Suppose we wish to construct a $(18, 66)$ MCT. Lets start with $V = \{d, e, f, x_0, x_1, \ldots, x_6,$ $y_0, y_1, \ldots, y_6\}$ and T contain the following triples:

Type 0 $\{\infty, d, f\}$, $\{\infty, e, f\}$, and $\{\infty, x_i, y_i\}$ for all i, $0 \le i \le 6$;

Type 1 $\{x_0, y_0, f\}$, $\{x_1, y_1, x_3\}$, $\{x_2, y_2, y_3\}$, $\{x_3, y_3, y_6\}$, $\{x_4, y_4, x_5\}$, $\{x_5, y_5, y_2\}$, $\{x_6, y_6, x_1\}$;

Type 2 $\{x_0, x_3, x_5\}$, $\{x_0, y_3, x_1\}$, $\{x_0, y_5, y_1\}$, $\{x_0, x_4, x_6\}$, $\{x_0, y_4, x_2\}$, $\{x_0, y_6, y_2\}$, $\{y_0, x_5, x_1\}$, $\{y_0, y_3, y_5\}$, $\{y_0, x_4, y_1\}$, $\{y_0, x_6, x_2\}$, $\{y_0, y_4, y_6\}$, $\{y_0, x_3, y_2\}$, $\{d, x_3, x_6\}$, $\{d, y_3, x_5\}$, $\{d, x_4, y_6\}$, $\{d, y_4, y_5\}$, $\{d, x_1, y_2\}$, $\{d, y_1, x_2\}$, $\{e, x_5, x_2\}$, $\{e, y_5, x_1\}$, $\{e, x_6, y_2\}$, $\{e, y_6, y_1\}$, $\{e, x_3, y_4\}$, $\{e, y_3, x_4\}$, $\{f, x_1, y_4\}$, $\{f, y_1, y_3\}$, $\{f, x_2, x_3\}$, $\{f, y_2, x_4\}$, $\{f, x_5, y_6\}$, $\{f, y_5, x_6\}$, $\{x_3, x_4, y_5\}$, $\{x_5, x_6, y_1\}$, $\{x_4, x_1, x_2\}$, $\{y_3, y_4, x_6\}$, $\{y_5, y_6, x_2\}$, $\{y_4, y_1, y_2\}$;

Type 3 $\{d, e, x_0\}$ and $\{d, e, y_0\}$.

So we are starting with $b = 54$. First we must remove the Type 1 triples. So lets start by removing $\{x_0, y_0, f\}$ and replacing it with $\{\infty, x_0, f\}$ and $\{\infty, y_0, f\}$. Next we remove $\{x_1, y_1, x_3\}$ and replace it with $\{\infty, x_1, x_3\}$ and $\{\infty, y_1, x_3\}$. At this point we are supposed to set $i = 3$ but let us consider what happens if we don't bother to and just remove $\{x_2, y_2, y_3\}$ and replace it with $\{\infty, x_2, y_3\}$ and $\{\infty, y_2, y_3\}$. If we do that T now has as a subset $\{\{\infty, x_1, x_3\}, \{\infty, x_2, y_3\}, \{\infty, x_3, y_3\}, \{x_3, y_3, y_6\}\}$ which means that $\{\infty, x_3, y_3\}$ has no unique edge.

So let us proceed as we are supposed to and set $i = 3$ and then remove $\{x_3, y_3, y_6\}$ and replace it with $\{\infty, x_3, y_6\}$ and $\{\infty, y_3, y_6\}$. Next we need to set $i = 6$ and then remove $\{x_6, y_6, x_1\}$ and replace it with $\{\infty, x_6, x_1\}$ and $\{\infty, y_6, x_1\}$. This time, since we have already removed $\{x_1, y_1, x_3\}$, we set $i = 2$. We then remove $\{x_2, y_2, y_3\}$ and replace it with $\{\infty, x_2, y_3\}$ and $\{\infty, y_2, y_3\}$. Next we set $i = 4$ and remove $\{x_4, y_4, x_5\}$ and replace it with $\{\infty, x_4, x_5\}$ and $\{\infty, y_4, x_5\}$. Finally we set $i = 5$ and remove $\{x_5, y_5, y_2\}$ and replace it with $\{\infty, x_5, y_2\}$ and $\{\infty, y_5, y_2\}$.

At this point we now have $b = 61$ so we need to add five more triples. Dividing by two and rounding down tells us to pick two Type 2 triples for removal. So lets remove $\{x_0, x_3, x_5\}$ and $\{x_0, y_3, x_1\}$. We replace them with $\{\infty, x_0, x_3\}$, $\{\infty, x_3, x_5\}$, $\{\infty, x_0, x_5\}$, $\{\infty, x_0, y_3\}$, $\{\infty, y_3, x_1\}$, and $\{\infty, x_0, x_1\}$. At this point we now have $b = 65$, so finally we remove $\{d, e, x_0\}$ and replace it with $\{\infty, x_0, d\}$ and $\{\infty, x_0, e\}$. So following this process we have constructed an $(18, 66)$ MCT.

4.2 $\mathbf{v \equiv 1}$ or 3 mod 6

We will cover all possibilities with $v \geq 9$, the case when $v = 7$ will be handled separately in section B.2. Assume $v = 6n + 1$ with $n \geq 2$ or that $v = 6n + 3$ with $n \geq 1$. A Steiner Triple System of order v is a minimum cover by triples in this case [1]. We will obtain the rest of the values in the range by starting with the $(v, \frac{v(v-1)}{6} + 1)$ MCT we constructed in section 3.2 and building larger covers by replacing select triples with multiple replacement triples.

Noting that $V = \{d, e, f, 1, 2, \ldots, v - 3\}$ and $\alpha \subset T$, we will start by categorizing the triples in the following manner:

Type 0 All triples of T of the form $\{d, u, w\}$ where $u, w \in V - \{d\}$,

Type 1 The triples $\{e, f, 3\}$ and $\{e, f, 4\}$,

Type 2 All triples of T of the form $\{x, y, u\}$ where $x, y \in V - \{d, e, f\}$ and $u \in V - \{d\}$.

It follows that we have $\frac{v+1}{2}$ Type 0 triples and 2 Type 1 triples. Thus it leaves us either $6n^2-2n-2$ or $6n^2+2n-2$ Type 2 triples depending on if $v=6n+1$ or $6n+3$ respectively.

We see that in this case, none of the Type 0 triples are lacking a unique edge that is not incident to d so the construction process will be much less complicated. Suppose we want to construct a (v,b) MCT where $\frac{v(v-1)}{6}+1<b<\frac{(v-1)(v-2)}{2}-1$. First we choose $\left\lfloor \frac{b-\left(\frac{v(v-1)}{6}+1\right)}{2} \right\rfloor$ type 2 triples. For each such triple $\{x,y,u\}$ chosen we remove it and replace it with the triples $\{d,x,y\}$, $\{d,y,u\}$ and $\{d,u,x\}$. At this point our T either contains b or $b-1$ triples. In the first case we are done, otherwise we simply remove the triple $\{e,f,3\}$ and replace it with the triples $\{d,e,3\}$ and $\{d,f,3\}$.

Example 4.2 $(v=9, b=20)$

Suppose we wish to construct a $(9,20)$ MCT. Lets start with the $(9,13)$ MCT from section A.1. Categorizing the triples we have:

Type 0 $\{d,e,1\}$, $\{d,e,2\}$, $\{d,f,5\}$, $\{d,f,6\}$,and $\{d,3,4\}$;

Type 1 $\{e,f,3\}$ and $\{e,f,4\}$;

Type 2 $\{e,5,6\}$, $\{f,1,2\}$, $\{1,3,5\}$, $\{1,4,6\}$, $\{2,3,6\}$, and $\{2,4,5\}$.

We need to add 7 triples, so we start by removing 3 Type 2 triples. Let's remove $\{e,5,6\}$, $\{f,1,2\}$, and $\{1,3,5\}$. Next we replace them with the triples $\{d,e,5\}$,$\{d,e,6\}$, $\{d,5,6\}$, $\{d,f,1\}$, $\{d,f,2\}$, $\{d,1,2\}$, $\{d,1,3\}$, $\{d,1,5\}$, and $\{d,3,5\}$. At this point we have a $(9,19)$ MCT. All we need to do is remove $\{e,f,3\}$ and replace it with $\{d,e,3\}$ and $\{d,f,3\}$ giving us a $(9,20)$ MCT.

4.3 v $\equiv$ 2 or 4 mod 6

We will cover all possibilities with $v \geq 10$, the case when $v = 8$ will be handled separately in section B.3. When $v = 4$ the results are trivial as $\left\lceil \frac{4}{3} \left\lceil \frac{3}{2} \right\rceil \right\rceil = \binom{3}{2}$. Assume $v = 6n+2$ with $n \geq 2$ or that $v = 6n+4$ with $n \geq 1$. A construction for a minimum cover with $b = \frac{1}{3}\left(\frac{v^2}{2}+1\right)$ (which is $6n^2+4n+1$ or $6n^2+8n+3$ respectively) can be found in [1]. We will make use of the construction from section 3.3 that creates a MCT with one extra triple as a starting point.

We categorize the triples in the following manner:

Type 0 The triples $\{\infty, u, s\}$, $\{\infty, u, w\}$, $\{\infty, s, w\}$ and all triples of the form $\{\infty, x_i, y_i\}$ for all i, $1 \leq i \leq \frac{1}{2}v - 2$;

Type 1 the triples of the form $\{x_i, y_i, \alpha_i\}$ for all i, $1 \leq i \leq \frac{1}{2}v - 2$ where $\alpha_i \in V \setminus \{x_i, y_i, \infty\}$;

Type 2 the triples of the form $\{z_i, z_j, z_k\}$ or $\{z_i, z_k, \gamma\}$ where $1 \leq i < j < k \leq \frac{1}{2}v - 2$ and $\gamma \in \{u, s, w\}$ except for the triples described in the next type;

Type 3 The triples $\{x_1, x_2, y_3\}$, $\{x_1, x_2, \beta_1\}$, $\{x_2, x_3, y_1\}$, $\{x_2, x_3, \beta_3\}$, $\{x_1, x_3, y_2\}$ and $\{x_1, x_3, \beta_5\}$ where β_i is the point that i was renamed to in section 3.3.

We are starting with a total of $6n^2 + 4n + 2$ (or $6n^2 + 8n + 4$ if $v = 6n + 4$) triples. Clearly there are $3n + 2$ (or $3n + 3$) Type 0 , $3n - 1$ (or $3n$) Type 1 and 6 Type 3 triples; which leaves us with $6n^2 - 2n - 5$ (or $6n^2 + 2n - 5$) Type 2 triples. We now proceed to show how to use this starting point as a way to construct a minimal cover with b triples for all b, $6n^2 + 4n + 2 < b < 18n^2 + 3n - 1$ (or $6n^2 + 8n + 4 < b < 18n^2 + 15n + 2$).

If $b \leq 6n^2 + 7n + 1$ (or $b \leq 6n^2 + 11n + 4$) it can be obtained simply by removing $b - (6n^2 + 4n + 2)$ (or $b - 6n^2 + 8n + 4$) type one triples and replacing them with new triples in the following manner:

1. Set $i = 1$.

2. Remove the Type 1 triple $\{x_i, y_i, \alpha_i\}$ and replace it with the triples $\{\infty, x_i, \alpha_i\}$ and $\{\infty, y_i, \alpha_i\}$.

3. If $\alpha_i = z_j$ for some j and the type 1 triple containing x_j and y_j has not been removed yet set $i = j$ otherwise set i to the lowest possible value such that $\{x_i, y_i, \alpha_i\}$ has not been removed.

4. Go to step 2 unless we no longer need any additional triples or there are no more Type 1 triples to remove.

As in the construction in section 4.1 we have to take care while removing triples if there are still Type 1 triples. Thus we have taken similar precaution in our algorithm for their removal.

If $6n^2 + 7n + 1 < b \leq 18n^2 + 3n - 9$ (or $6n^2 + 11n + 4 < b \leq 18n^2 + 15n - 6$) we first remove and replace all the type 1 triples in the above described manner. We then choose $\lfloor \frac{b-(6n^2+7n+1)}{2} \rfloor$ (or $\lfloor \frac{b-(6n^2+11n+4)}{2} \rfloor$) type 2 triples. For each such triple $\{p, q, r\}$ chosen, we remove it and replace it with the triples $\{\infty, p, q\}$,$\{\infty, q, r\}$ and $\{\infty, p, r\}$. If we now have a system containing b triples we are done, otherwise we have one containing $b - 1$ triples and then remove the triple $\{x_1, x_2, y_3\}$ and replace it with the triples $\{\infty, y_3, x_1\}$ and $\{\infty, y_3, x_2\}$.

If $b > 18n^2 + 3n - 9$ (or $b > 18n^2 + 15n - 6$) we first remove all Type 1 triples followed by all Type two triples using the methods described above. At this point the number of triples we need to add is between 1 and 7, let this number be b'. We then add the remaining number of triples in the following manner:

if $\mathbf{b'} = \mathbf{1}$, remove $\{x_1, x_2, y_3\}$ and replace it with $\{\infty, y_3, x_1\}$ and $\{\infty, y_3, x_2\}$;

if $\mathbf{b'} = \mathbf{2}$, proceed as if $b' = 1$ and then remove $\{x_1, x_3, y_2\}$ and replace it with $\{\infty, y_2, x_1\}$ and $\{\infty, y_2, x_3\}$;

if $\mathbf{b'} = \mathbf{3}$, proceed as if $b' = 1$ and then remove $\{x_1, x_2, \beta_1\}$ and replace it with $\{\infty, x_1, x_2\}$, $\{\infty, x_1, \beta_1\}$ and $\{\infty, x_2, \beta_1\}$;

if $\mathbf{b'} = \mathbf{4}$, proceed as if $b' = 3$ and then remove $\{x_1, x_3, y_2\}$ and replace it with $\{\infty, y_2, x_1\}$ and $\{\infty, y_2, x_3\}$;

if $\mathbf{b'} = \mathbf{5}$, proceed as if $b' = 4$ and then remove $\{x_2, x_3, y_1\}$ and replace it with $\{\infty, y_1, x_2\}$ and $\{\infty, y_1, x_3\}$;

if $\mathbf{b'} = \mathbf{6}$, proceed as if $b' = 4$ and then remove $\{x_1, x_3, \beta_5\}$ and replace it with $\{\infty, x_1, x_3\}$, $\{\infty, x_1, \beta_5\}$ and $\{\infty, x_3, \beta_5\}$;

if $\mathbf{b'} = \mathbf{7}$, proceed as if $b' = 6$ and then remove $\{x_2, x_3, y_1\}$ and replace it with $\{\infty, y_1, x_2\}$ and $\{\infty, y_1, x_3\}$.

Example 4.3 $(v = 10, b = 30)$

Suppose we wish to construct a $(10, 30)$ MCT. We will first start with the $(10, 18)$ MCT constructed using the technique from section 3.3. We have $V = \{\infty, u, v, w, x_1, x_2, x_3, y_1, y_2, y_3\}$ and T had the following triples:

Type 0 $\{\infty u, s\}$, $\{\infty, u, w\}$, $\{\infty, s, w\}$, $\{\infty, x_1, y_1\}$, $\{\infty, x_2, y_2\}$, and $\{\infty, x_3, y_3\}$;

Type 1 $\{x_1, y_1, s\}$, $\{x_2, y_2, w\}$, and $\{x_3, y_3, u\}$;

Type 2 $\{u, y_1, y_2\}$, $\{y_3, s, y_2\}$, and $\{y_3, y_1, w\}$;

Type 3 $\{x_1, x_2, y_3\}$, $\{x_1, x_2, u\}$, $\{x_2, x_3, y_1\}$, $\{x_2, x_3, s\}$, $\{x_1, x_3, y_2\}$ and $\{x_1, x_3, w\}$

So initially we have $b = 18$. We start by removing $\{x_1, y_1, s\}$ and replacing it with $\{\infty, x_1, s\}$ and $\{\infty, y_1, s\}$. Next we set $i = 2$ and remove $\{x_2, y_2, w\}$ and replacing it with $\{\infty, x_2, w\}$ and $\{\infty, y_2, w\}$. Now we set $i = 3$ and remove $\{x_3, y_3, u\}$ and replacing it with $\{\infty, x_3, u\}$ and $\{\infty, y_3, u\}$. After removing all the Type 1 triples we now have $b = 21$.

We will need to remove all the Type 2 triples. After removing them we replace them with the triples $\{\infty, u, y_1\}$, $\{\infty, u, y_2\}$, $\{\infty, y_1, y_2\}$, $\{\infty, y_3, s\}$, $\{\infty, y_3, y_2\}$, $\{\infty, s, y_2\}$, $\{\infty, y_3, y_1\}$, $\{\infty, y_3, w\}$, and $\{\infty, y_1, w\}$. At this point we now have $b = 27$ this means that $b' = 3$.

So proceeding as directed we now remove $\{x_1, x_2, y_3\}$ and replace it with $\{\infty, y_3, x_1\}$ and $\{\infty, y_3, x_2\}$ making $b = 28$. Finally we remove $\{x_1, x_2, u\}$ and replace it with $\{\infty, x_1, x_2\}$, $\{\infty, x_1, u\}$ and $\{\infty, x_2, u\}$ giving us a $(10, 30)$ MCT.

4.4 v $\equiv$ 5 mod 6

We will handle all possibilities for $v \geq 11$, the case when $v = 5$ will be handled in section B.1. Assume that $v = 6n + 5$; the minimum value is obtained using Construction 3.2 and will be used as a starting point to obtain the remaining values needed. As done previously we will build a MCT with the desired value for b by carefully replacing select triples with multiple replacement triples.

We start by choosing any point of $V \setminus \{a, c, d, e, f\}$ and renaming it ∞. Noting that we start with $6n^2 + 9n + 4$ triples, we then categorize the triples in the following manner:

Type 0 All triples of T of the form $\{\infty, u, w\}$ where $u, w \in V - \{\infty\}$,

Type 1 The triples $\{a, f, c\}$, $\{a, f, d\}$ and $\{a, f, e\}$,

Type 2 All triples of T of the form $\{x, y, u\}$ where $x, y \in V - \{a, b, \infty\}$ and $u \in V - \{\infty\}$.

It follows that we have $3n + 2$ Type 0 triples and 3 Type 1 triples. Thus it leaves us $6n^2 + 6n - 1$ Type 2 triples. We now proceed to show how to use this to obtain a (v, b) MCT when $6n^2 + 9n + 4 < b < 18n^2 + 21n + 5$.

If $b \leq 18n^2 + 21n + 3$ we choose $\lfloor \frac{b-(6n^2+9n+4)}{2} \rfloor$ Type 2 triples. For each such triple $\{u, s, w\}$ we remove it and replace it with the triples $\{\infty, u, s\}$, $\{\infty, u, w\}$ and $\{\infty, s, w\}$. Our MCT now has either b or $b - 1$ triples, so either we are done or we remove $\{a, f, c\}$ and replace it with $\{\infty, a, c\}$ and $\{\infty, f, c\}$. If $b = 18n^2 + 21n + 4$ we replace all the Type 2 triples as above, and then remove $\{a, f, c\}$ and $\{a, f, d\}$ and replace them with $\{\infty, a, c\}$, $\{\infty, f, c\}$, $\{\infty, a, d\}$ and $\{\infty, f, d\}$.

Example 4.4 ($v = 11, b = 25$)

Suppose we wish to construct a $(11, 25)$ MCT. We will start with the $(11, 18)$ MCT we get from the construction in section 3.2. We have $V = \{a, c, d, e, f, 6, 7, 8, 9, 10, \infty\}$ and T is made up of:

Type 0 $\{\infty, a, 10\}$, $\{\infty, f, 7\}$, $\{\infty, c, 6\}$, $\{\infty, d, 8\}$, and $\{\infty, e, 9\}$;

Type 1 $\{a, f, c\}$, $\{a, f, d\}$, and $\{a, f, e\}$;

Type 2 $\{a, 6, 7\}$, $\{a, 8, 9\}$, $\{f, 6, 9\}$, $\{f, 8, 10\}$, $\{c, 7, 8\}$, $\{c, 9, 10\}$, $\{d, 6, 10\}$, $\{d, 7, 9\}$, $\{e, 6, 8\}$, and $\{e, 7, 10\}$.

We first need to remove three Type 2 triples, so lets remove $\{a, 6, 7\}$, $\{a, 8, 9\}$, and $\{f, 6, 9\}$. We then replace them with $\{\infty, a, 6\}$, $\{\infty, a, 7\}$, $\{\infty, 6, 7\}$, $\{\infty, a, 8\}$, $\{\infty, a, 9\}$, $\{\infty, 8, 9\}$, $\{\infty, f, 6\}$, $\{\infty, f, 9\}$, and $\{\infty, 6, 9\}$. This leaves us with $b = 24$ so we remove $\{a, f, c\}$ and replace it with $\{\infty, a, c\}$ and $\{\infty, f, c\}$. This gives us a $(11, 25)$ MCT.

Chapter 5

Imbrical Design Flaws

When I started this project I initially did what I thought was an exhaustive search for papers on the subject. After I had a large portion of the results worked out, my advisor contacted Charles J. Colburn in order to make sure that we had not missed any papers that may have been out there. It was then that he brought the paper on imbrical designs to our attention [2]. At that point Dr. Hoffman suggested that I compare my results to those in [2]

The authors define imbrical designs as follows:

"A pair (V, D) is an $ID(v, k, \lambda, b)$ if

(a) $|V| = v$;

(b) D is a collection of b k–subsets of V called blocks;

(c) every pair of elements of V is in at least λ blocks of D;

(d) for every $b \in D$ there is a pair $\{x, y\} \subset b$ so that xy is in exactly λ blocks ({x,y} ia called essential for b);

(d') equivalently, if $D' \subset D$, (V, D') is not an imbrical design."

If $k = 3$ and $\lambda = 1$ the associated imbrical design is a minimal cover by triples. What they refer to as an essential edge is then what I have referred to as a unique edge. They also define the spectrum as follows:

"Given v, k and λ the spectrum $Spec(v, k, \lambda) = \{b : \text{there exists } ID(v, k, \lambda, b)$."

Later, they make the claim that

“The results are that min $Spec(v,3,1) = \left\lceil \frac{v}{3} \left\lceil \frac{v-1}{2} \right\rceil \right\rceil$, max $Spec(v,3,1) = \binom{v-2}{2}$ and that $Specv, 3, 1$ is an interval except that $v = 3$, $b = 2$ and $v = 7$ and $b = 8$ do not exist.”

Their claim that the spectrum is an interval for most values of v is what first caught my attention as it is contradicted by theorem 2.1 of my work. The rest of this chapter explains why my results don’t agree with theirs.

5.1 The basic augmentation procedure

The authors of [2] make use of what they call augmentations. The augmentations are very similar to the techniques I used. They both involve the removal of one block followed by then replacing it with at most 3 new blocks. The main difference is in [2] they try to have one generalized process that works in all cases while I used different processes tailored to various situations.

There are two problems with their basic augmentation. The first problem involves being a little to liberal with the term *without loss of generality.* The authors write:

“Let (V, B) be an $ID(v, k, l, b)$. Let $\infty = \{\infty_0, \infty_1, \ldots, \infty_{k-1}\} \in B$ be fixed. Let a be a block of B, $a = \{a_0, a_1, \ldots, a_{k-1}\}$, with $\{a_0, a_{k-1}\}$ essential (i.e. in no other block). Thus not both $a_0, a_{k-1} \in \infty$. Assume without loss of generality that if one of them is in ∞ it is a_0 and $a_0 = \infty_{k-1}$.”

That last assumption garners no loss of generality if the augmentation is to be used a single time, but unfortunately the authors make use of multiple iterations of the augmentation. Since these are done with ∞ fixed, once the first augmentation is done its labels are fixed and any further augmentations are in danger of loss of generality.

The more severe of the two problems is the authors have not taken pains to ensure that there augmentation produces a cover that is still minimal. The steps the authors give us to take in the augmentation procedure are as follows:

"

1. Delete from B the block $\{a_0, \dots a_{k-1}\}$.

2. Add to B the block $\{\infty_i, a_0, a_1, \dots a_{k-2}\}$ where ∞_i has the least i such that $\infty_i \in \infty - a$, provided there is an essential edge in this new block.

3. Add to B the block $\{\infty_i, a_1, a_2, \dots a_{k-1}\}$ where ∞_i has the least i such that $\infty_i \in \infty - a$, provided there is an essential edge in this new block.

4. Add the block $\{\infty_0, \infty_1, \dots \infty_{k-3}, a_0, a_{k-1}\}$.

"

Here is an example of the augmentation procedure going wrong. Let $V = \{1, 2, \dots, 8\}$ and let $B = \{\{8, 1, 2\}, \{8, 1, 4\}, \{8, 2, 4\}\{8, 3, 5\}, \{8, 6, 7\}, \{2, 3, 5\}, \{3, 4, 6\}, \{5, 6, 1\}, \{6, 7, 2\}, \{7, 1, 3\}, \{8, 4, 5\}, \{8, 4, 7\}, \{8, 5, 7\}\}$. If we choose to let $\infty = \{8, 7, 6\}$ and $a = \{6, 4, 3\}$. Following the steps for the augmentation we first remove $\{6, 4, 3\}$ from B. Next we add $\{8, 6, 4\}$ since $\{4, 6\}$ is essential. Now we have a problem though because $\{8, 7, 6\}, \{6, 7, 2\}$, $\{8, 4, 7\}$ and $\{8, 6, 4\}$ are all in B hence ∞ no longer has an essential edge. Since no blocks will be removed during the rest of the augmentation process we will be left with a cover that is no longer minimal.

5.2 The Proof of the Main Theorem

Even if we found a way to tweak the augmentation so it would still work, there is a major flaw in the use of the second theorem of the paper to prove the authors claims about the spectrum of imbrical designs. They state:

"Let $k = 3$. Suppose that we have a block $\{a, b, c\}$ all distinct from ∞_0 with ab repeated, and ac and bc both essential. Then the augmentation to ∞ac, ∞bc increases the size by one, and no other augmentation by ∞ affects this."

The liberty taken with notation in this passage cause the authors claim to be imprecisely stated. In addition to that the authors provide no proof whatsoever of this claim. If one carefully considers their implication one can see that augmenting first by all blocks sharing $\{a, b\}$ with $\{a, b, c\}$ will leave $\{a, b, c\}$ with all its edges being essential. At this point, augmenting by ∞ on $\{a, b, c\}$ will add the blocks $\{\infty_0, a, b\}, \{\infty_0, a, c\}$ and $\{\infty_0, b, c\}$. So it no longer adds only one to the size.

A specific example of this happening can be obtained taking V and B to be as in the previous section and letting $\infty = \{6, 7, 8\}$ and $\{a, b, c\} = \{3, 5, 2\}$. In this case $\{3, 5\}$ is shared with the block $\{3, 5, 8\}$. If we first augment by ∞ on $\{3, 5, 8\}$ we will remove $\{3, 5, 8\}$ and add $\{3, 6, 8\}$ If we now augment by ∞ on $\{3, 5, 2\}$ we will remove $\{3, 5, 2\}$ and add $\{6, 3, 5\}$, $\{6, 5, 2\}$ and $\{6, 3, 2\}$.

So we see that the claim the authors make which is the key statement in proving their claims of the spectrum when $k = 3$ is quite false. This coupled with the inherent problems with the augmentation process itself discussed in the previous section unfortunately invalidates their results.

Bibliography

[1] C.C.Lindner and C.A.Rodger, "Design Theory," CRC press LLC, 1997.

[2] E.Mendelsohn and A.Assaf, *On the Spectrum of Imbrical Designs*, Annals of Discrete Mathematics 34 (1987), 363–369

[3] Jean Doyen and Richard M. Wilson, *Embeddings of Steiner triple systems*, Discrete Math, 5 (1973), 229–239.

Appendices

Appendix A

Some Specific MCT's

A.1 $(9, 13)$ MCT

$V = \{d, e, f, 1, 2, 3, 4, 5, 6\}$, $T = \{\{d, e, 1\}, \{d, e, 2\}, \{e, f, 3\}, \{e, f, 4\}, \{d, f, 5\}, \{d, f, 6\}, \{d, 3, 4\}, \{e, 5, 6\}, \{f, 1, 2\}, \{1, 3, 5\}, \{1, 4, 6\}, \{2, 3, 6\}, \{2, 4, 5\}\}$.

A.2 $(13, 27)$ MCT

$V = \{d, e, f, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $T = \{\{d, e, 1\}, \{d, e, 2\}, \{e, f, 3\}, \{e, f, 4\}, \{d, f, 5\}, \{d, f, 6\}, \{d, 3, 4\}, \{e, 5, 6\}, \{f, 1, 2\}, \{d, 7, 8\}, \{d, 9, 10\}, \{e, 7, 9\}, \{e, 8, 10\}, \{f, 7, 10\}, \{f, 8, 9\}, \{1, 3, 7\}, \{1, 4, 8\}, \{1, 5, 9\}, \{1, 6, 10\}, \{2, 3, 10\}, \{2, 4, 9\}, \{2, 5, 7\}, \{2, 6, 8\}, \{3, 5, 8\}, \{3, 6, 9\}, \{4, 5, 10\}, \{4, 6, 7\}\}$.

A.3 $(15, 36)$ MCT

$V = \{d, e, f, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}, T = \{\{d, e, 1\}, \{d, e, 2\}, \{e, f, 3\}, \{e, f, 4\}, \{d, f, 5\}, \{d, f, 6\}, \{d, 3, 4\}, \{e, 5, 6\}, \{f, 1, 2\}, \{d, 7, 10\}, \{d, 8, 12\}, \{d, 9, 11\}, \{e, 7, 11\}, \{e, 8, 10\}, \{e, 9, 12\}, \{f, 7, 12\}, \{f, 8, 11\}, \{f, 9, 10\}, \{1, 3, 10\}, \{1, 4, 9\}, \{1, 5, 12\}, \{1, 6, 11\}, \{1, 7, 8\}, \{2, 3, 12\}, \{2, 4, 7\}, \{2, 5, 8\}, \{2, 6, 9\}, \{2, 10, 11\}, \{3, 5, 11\}, \{3, 6, 7\}, \{3, 8, 9\}, \{4, 5, 10\}, \{4, 6, 8\}, \{4, 11, 12\}, \{5, 7, 9\}, \{6, 10, 12\}\}$.

Appendix B

$\mathbf{v = 5, 7}$ or $\mathbf{8}$

B.1 $\mathbf{v = 5}$

$V = \{a, c, d, e, f\}$

$\mathbf{b = 4}$: $T = \{\{a, f, c\}, \{a, f, d\}, \{a, f, e\}, \{c, d, e\}\}$

$\mathbf{b = 5}$: $T = \{\{a, f, c\}, \{f, c, d\}, \{c, d, e\}, \{d, e, a\}, \{e, a, f\}\}$

$\mathbf{b = 6}$: $T = \{\{a, f, c\}, \{a, f, d\}, \{a, f, e\}, \{a, c, d\}, \{a, c, e\}, \{a, d, e\}\}$

B.2 $\mathbf{v = 7}$

$V = \{1, 2, 3, 4, 5, 6, 7\}$

$\mathbf{b = 7}$: $T = \{\{1, 2, 4\}, \{1, 3, 7\}, \{1, 5, 6\}, \{2, 3, 5\}, \{2, 6, 7\}, \{3, 4, 6\}, \{4, 5, 7\}\}$

$\mathbf{b = 9}$: $T = \{\{1, 2, 4\}, \{1, 3, 7\}, \{1, 5, 6\}, \{1, 2, 3\}, \{1, 2, 5\}, \{1, 3, 5\}, \{2, 6, 7\}, \{3, 4, 6\}, \{4, 5, 7\}\}$

$\mathbf{b = 10}$: $T = \{\{1, 2, 5\}, \{1, 3, 6\}, \{1, 4, 7\}, \{2, 3, 5\}, \{2, 3, 6\}, \{2, 4, 5\}, \{2, 4, 7\}, \{3, 4, 6\}, \{3, 4, 7\}, \{5, 6, 7\}\}$

$\mathbf{b = 11}$: $T = \{\{1, 2, 4\}, \{1, 3, 7\}, \{1, 5, 6\}, \{1, 2, 3\}, \{1, 2, 5\}, \{1, 3, 5\}, \{1, 2, 6\}, \{1, 2, 7\}, \{1, 6, 7\}, \{3, 4, 6\}, \{4, 5, 7\}\}$

$\mathbf{b = 12}$: $T = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\}, \{1, 2, 6\}, \{1, 2, 7\}, \{2, 3, 5\}, \{2, 4, 5\}, \{2, 5, 6\}, \{2, 6, 7\}, \{2, 5, 7\}, \{3, 4, 6\}, \{3, 4, 7\}\}$

$\mathbf{b=13}$: $T = \{\{1,2,4\},\{1,3,7\},\{1,5,6\},\{1,2,3\},\{1,2,5\},\{1,3,5\},\{1,2,6\},\{1,2,7\},\{1,6,7\},$
$\{1,3,4\},\{1,3,6\},\{1,4,6\},\{4,5,7\}\}$

$\mathbf{b=15}$: $T = \{\{1,2,4\},\{1,3,7\},\{1,5,6\},\{1,2,3\},\{1,2,5\},\{1,3,5\},\{1,2,6\},\{1,2,7\},\{1,6,7\},$
$\{1,3,4\},\{1,3,6\},\{1,4,6\},\{1,4,5\},\{1,4,7\},\{1,5,7\}\}$

B.3 $\mathbf{v=8}$

$V = \{\infty,1,2,3,4,5,6,7\}$

$\mathbf{b=11}$: $T = \{\{\infty,1,2\},\{\infty,1,4\},\{\infty,2,4\},\{\infty,3,5\},\{\infty,6,7\},\{2,3,5\},\{3,4,6\},\{4,5,7\},$
$\{5,6,1\},\{6,7,2\},\{7,1,3\}\}$

$\mathbf{b=12}$: $T = \{\{\infty,1,2\},\{\infty,1,4\},\{\infty,2,4\},\{\infty,3,5\},\{\infty,6,7\},\{\infty,2,3\},\{\infty,2,5\},\{3,4,6\},$
$\{4,5,7\},\{5,6,1\},\{6,7,2\},\{7,1,3\}\}$

$\mathbf{b=13}$: $T = \{\{\infty,1,2\},\{\infty,1,4\},\{\infty,2,4\},\{\infty,3,5\},\{\infty,6,7\},\{\infty,3,4\},\{\infty,3,6\},\{\infty,4,6\},$
$\{2,3,5\},\{2,6,7\},\{4,5,7\},\{5,6,1\},\{7,1,3\}\}$

$\mathbf{b=14}$: $T = \{\{\infty,1,2\},\{\infty,1,4\},\{\infty,2,4\},\{\infty,3,5\},\{\infty,6,7\},\{\infty,3,4\},\{\infty,3,6\},\{\infty,4,6\},$
$\{\infty,2,3\},\{\infty,2,5\},\{2,6,7\},\{4,5,7\},\{5,6,1\},\{7,1,3\}\}$

$\mathbf{b=15}$: $T = \{\{\infty,1,2\},\{\infty,1,4\},\{\infty,2,4\},\{\infty,3,5\},\{\infty,6,7\},\{\infty,3,4\},\{\infty,3,6\},\{\infty,4,6\},$
$\{\infty,2,3\},\{\infty,2,5\},\{\infty,2,6\},\{\infty,2,7\},\{4,5,7\},\{5,6,1\},\{7,1,3\}\}$

$\mathbf{b=16}$: $T = \{\{\infty,1,2\},\{\infty,1,4\},\{\infty,2,4\},\{\infty,3,5\},\{\infty,6,7\},\{\infty,3,4\},\{\infty,3,6\},\{\infty,4,6\},$
$\{\infty,2,3\},\{\infty,2,5\},\{\infty,5,6\},\{\infty,5,1\},\{\infty,6,1\},\{2,6,7\},\{4,5,7\},\{7,1,3\}\}$

$\mathbf{b=17}:\ T=\{\{\infty,1,2\},\{\infty,1,4\},\{\infty,2,4\},\{\infty,3,5\},\{\infty,6,7\},\{\infty,3,4\},\{\infty,3,6\},\{\infty,4,6\},$
$\{\infty,2,3\},\{\infty,2,5\},\{\infty,5,6\},\{\infty,5,1\},\{\infty,6,1\},\{\infty,2,6\},\{\infty,2,7\},\{4,5,7\},$
$\{7,1,3\}\}$

$\mathbf{b=18}:\ T=\{\{\infty,1,4\},\{\infty,1,5\},\{\infty,1,6\},\{\infty,2,4\},\{\infty,2,5\},\{\infty,2,6\},\{\infty,2,7\},\{\infty,3,4\},$
$\{\infty,3,5\},\{\infty,3,6\},\{\infty,4,5\},\{\infty,4,6\}\{\infty,4,7\},\{\infty,5,6\},\{\infty,5,7\},\{\infty,6,7\},$
$\{1,3,7\},\{1,2,3\}\}$

$\mathbf{b=19}:\ T=\{\{\infty,1,2\},\{\infty,1,4\},\{\infty,1,5\},\{\infty,1,6\},\{\infty,2,3\},\{\infty,2,4\},\{\infty,2,5\},\{\infty,2,6\},$
$\{\infty,2,7\},\{\infty,3,4\},\{\infty,3,5\},\{\infty,3,6\},\{\infty,4,5\},\{\infty,4,6\}\{\infty,4,7\},\{\infty,5,6\},$
$\{\infty,5,7\},\{\infty,6,7\},\{1,3,7\}\}$

$\mathbf{b=21}:\ T=\{\{\infty,1,2\},\{\infty,1,3\},\{\infty,1,4\},\{\infty,1,5\},\{\infty,1,6\},\{\infty,1,7\},\{\infty,2,3\},\{\infty,2,4\},$
$\{\infty,2,5\},\{\infty,2,6\},\{\infty,2,7\},\{\infty,3,4\},\{\infty,3,5\},\{\infty,3,6\},\{\infty,3,7\},\{\infty,4,5\},$
$\{\infty,4,6\}\{\infty,4,7\},\{\infty,5,6\},\{\infty,5,7\},\{\infty,6,7\}\}$