The Intersection Problem for Steiner Triple Systems

by

Whitney Koetter

A thesis submitted to the Graduate Faculty of
Auburn University
in partial fulfillment of the
requirements for the Degree of
Master of Science

Auburn, Alabama
December 12, 2011

Keywords: Steiner triple system, intersection problem

Approved by

Charles Lindner, Chair, Professor of Mathematics
Peter Johnson, Professor of Mathematics
Dean Hoffman, Professor of Mathematics

Abstract

In this thesis we give a new solution to the intersection problem for Steiner triple systems, using results that were not available when the original solution was given. In particular we show for each pair (n, k), where $n \equiv 1$ or $3 \pmod 6 \geq 19$ and $k \in \{0, 1, 2, ..., x = \frac{n(n-1)}{6}\} \backslash \{x-1, x-2, x-3, x-5\}$, the existence of a pair of Steiner triple systems (S,T_1) and (S,T_2) of order n with the property that $|T_1 \cap T_2| = k$.

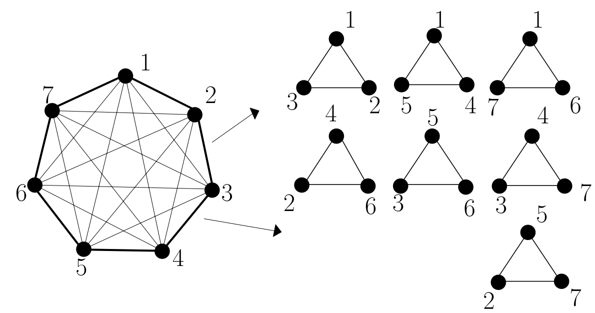
Table of Contents

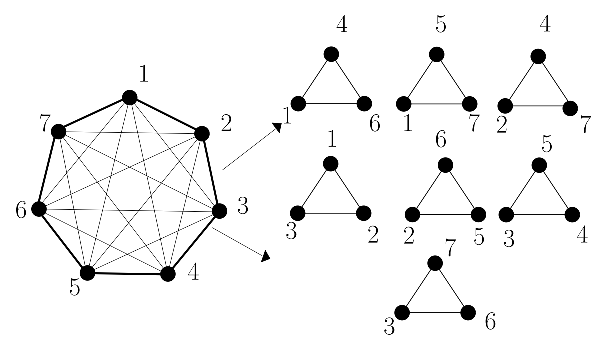
Chapter 1

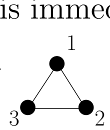
Introduction and outline of the thesis

A Steiner triple system (STS) of order n is a pair (X,T) where T is a collection of edge disjoint triangles (or triples) which partitions the edge set of K_n with vertex set X.

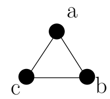
Example 1.1 (two triple systems of order 7).





It is immediate that the two triples in this example have exactly one triple in common, namely  which we will denote by $\{1, 2, 3\}$.

In general we will denote the triangle by $\{a, b, c\}$.



In this thesis, we will be looking at the intersection problem for Steiner triple systems. It is well-known that the spectrum for Steiner triple systems is precisely the set of all $n \equiv 1$ or 3 (mod 6) [3] and that if (X,T) is a triple system of order n, $|T| = \frac{n(n-1)}{6}$. Hence the following problem:

THE INTERSECTION PROBLEM: For each $n \equiv 1$ or 3 (mod 6), determine the set of all k such that there exists a pair of STS(n) having exactly k triples in common.

The two triple systems in Example 1.1 have exactly one triple in common; namely $\{1, 2, 3\}$.

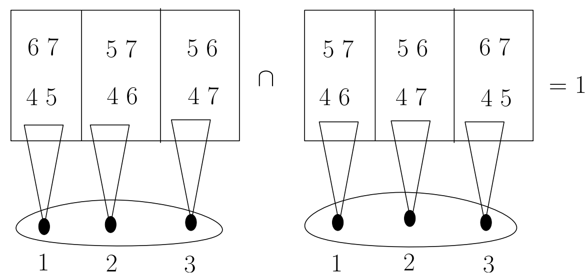
Now, it turns out that a necessary condition for a pair of STS(n) to have k triples in common is $k \in I(n) = \{0, 1, 2, ..., x = \frac{n(n-1)}{6}\} \backslash \{x-1, x-2, x-3, x-5\}$. This will be proved in Chapter 2. It also turns out that except for n = 9, this necessary condition is sufficient. Denote by $J(n) = \{k$ such that there exists two STS(n) having k triples in common$\}$. The following Theorem is due to C.C. Lindner and A. Rosa [5].

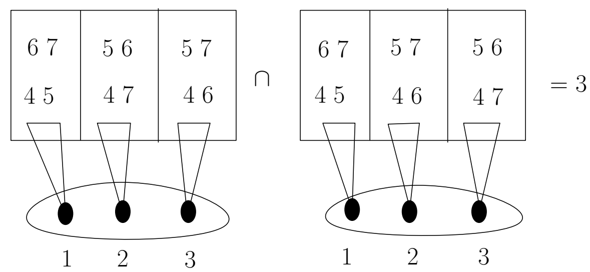
<u>Theorem 1.2</u> $J(n) = I(n)$ for all $n \equiv 1$ or 3 (mod 6), except for $J(9)$. In this case $J(9) = \{0, 1, 2, 3, 4, 6, 12\}$ [4]. $\square$

The following is an example showing $J(7) = I(7) = \{0, 1, 3, 7\}$.

Example 1.3 ($J(7) = I(7)$).

$$\boxed{\begin{array}{c} 1\,2\,4 \\ 2\,3\,5 \\ 3\,4\,6 \\ 4\,5\,7 \\ 5\,6\,1 \\ 6\,7\,2 \\ 7\,1\,3 \end{array}} \cap \boxed{\begin{array}{c} 1\,2\,6 \\ 2\,3\,7 \\ 3\,4\,1 \\ 4\,5\,2 \\ 5\,6\,3 \\ 6\,7\,4 \\ 7\,1\,5 \end{array}} = \emptyset$$





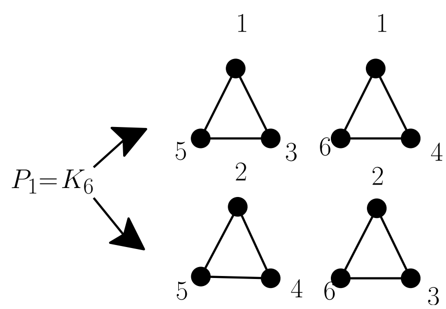
Any STS(7) intersects with itself in 7 triples.

Chapter 2

Necessary Conditions

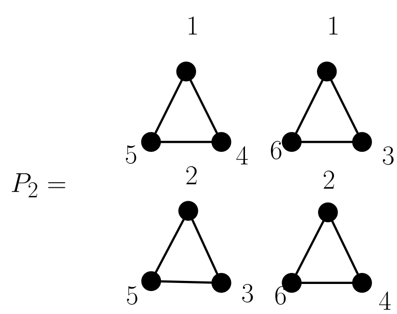
In this section we show that a necessary condition for a pair of triple systems (S,T_1) and (S,T_2) of order n to have k triples in common is for $k \in \{0, 1, 2, 3, 4, ..., x = \frac{n(n-1)}{6}\} \backslash \{x - 1, x - 2, x - 3, x - 5\}$. A partial triple system of order n is a pair (S,P) where P is a collection of edge disjoint triples of the edge set of K_n with vertex set S.

Example 2.1 (partial triple system of order 6)



Two partial triple systems (S,P_1) and (S,P_2) are said to be balanced provided P_1 and P_2 cover the same edges. Let P_1 be the collection of triples in Example 2.1 and P_2 be the following collection of triples.

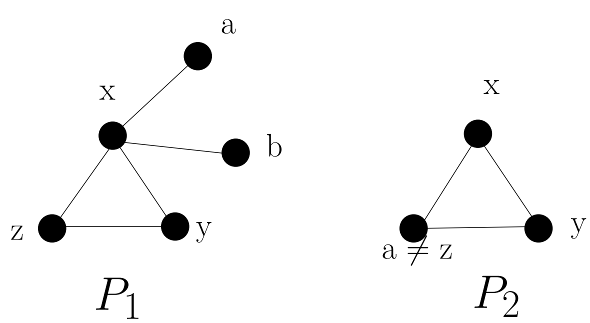
Example 2.2 (partial triple systems of order 6)



Then P_1 and P_2 are balanced. It is also the case that P_1 and P_2 are disjoint, that is, they have no triples in common.

Now let (S,T_1) and (S,T_2) be a pair of triple systems of order n. Then the partial triple systems (S,$T_1 \setminus (T_1 \cap T_2)$) and (S,$T_2 \setminus (T_1 \cap T_2)$) are balanced and disjoint. We will show that there does not exist a pair of balanced and disjoint partial triple systems containing 1,2,3, or 5 triples. It follows that $|T_1 \setminus (T_1 \cap T_2)| \notin \{1,2,3,5\}$ and so $|T_1 \cap T_2| \notin \{\frac{n(n-1)}{6} = x-1, x-2, x-3, x-5\}$. It follows that $|T_1 \cap T_2| \in \{0, 1, 2, ..., \frac{n(n-1)}{6} = x\} \setminus \{x-1, x-2, x-3, x-5\}$ is a necessary condition for a pair of triple systems to have x triples in common.

To begin, if (S, P_1) and (S, P_2) are balance and disjoint, every vertex must belong to at least 2 triples in both P_1 and P_2. Suppose $\{x, y, z\} \in P_1$ and is the only triple containing x. Then x has degree 2 in P_1. Now, in P_2 we must have a triple of the form $\{x, y, a\}$ since the edge $\{x, y\}$ has to be covered. However, if $a \neq z$, then the edge $\{x, a\}$ must be covered in P_1, so x has to have degee at least 4.



Now let (S, P_1) and (S, P_2) be a pair of partial balanced and disjoint triple systems. Since every vertex in P_1 must have degree at least 4 we cannot have $|P_1| = |P_2| \in \{1, 2, 3\}$.

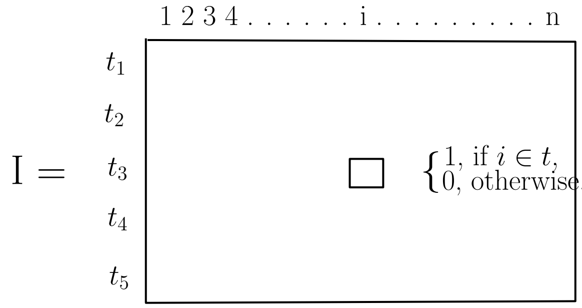
We now show that we cannot have

1. $|P_1 \cap P_2| = 5$, and

2. P_1 and P_2 are balanced

To begin, $|S|$ must be at least 6, otherwise we could not cover 15 edges. However, since a maximum partitioning of K_6 contains 4 triples, we cannot have $|S| = 6$.

Construct the following incidence matrix

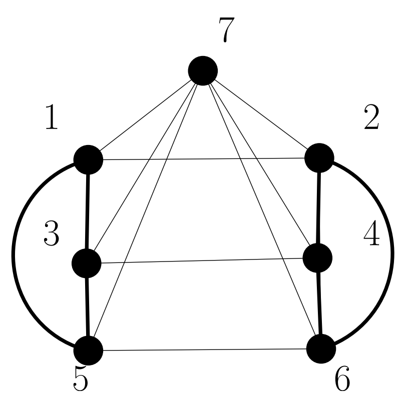


Then I contains 15 ones. Since each vertex belongs to at least 2 triples we must have $2n \leq 15$ so, $n \leq \frac{15}{2}$, and so $n = 7$.

It is now clear that I looks like

$$\begin{array}{c|ccccccc|} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline t_1 & & & & & 1 & & 1 \\ t_2 & 1 & 1 & & 1 & & & \\ t_3 & 1 & 1 & 1 & & & 1 & 1 \\ t_4 & & & 1 & & 1 & & 1 \\ t_5 & & & & 1 & & 1 & \\ \hline \end{array}$$

This is to say exactly one vertex belongs to 3 triples and the rest to 2 triples. It follows that P_1 looks like



These edges cannot be covered by 5 disjoint triples, and so, we have the following result:

Lemma 2.3 A necessary condition for a pair of triple systems (S,T_1) and (S,T_2) of order n to have k triples in common is for $k \in \{0, 1, 2, 3, ..., \frac{n(n-1)}{6} = x\} \backslash \{x-1, x-2, x-3, x-5\}$.

$\square$

In [5] this was shown to be sufficient for all $n \equiv 1$ or 3 (mod 6), except for $n = 9$. In this case the intersection numbers are {0,1,2,3,4,6,12} [4].

The object of this thesis is a different and much simpler proof of the intersection problem. We will first sketch a proof of the original solution. We then give a new construction using results which were not available when the original paper was written.

Chapter 3

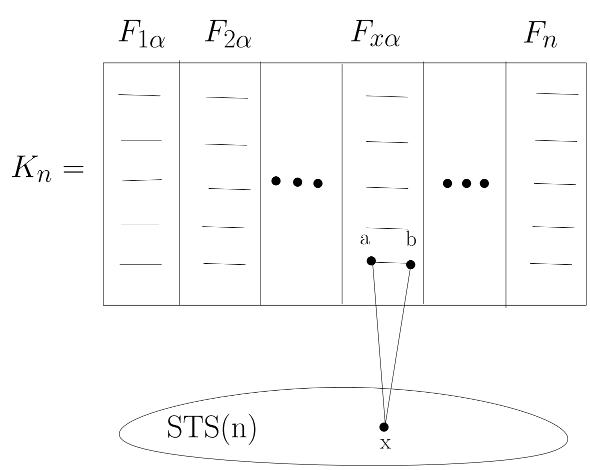
The Original Construction

This chapter will give a brief sketch of the original solution of the intersection problem [5]. The interested reader is referenced to the original paper for details. The original solution uses the following two constructions.

The $2n+1$ Construction: Let (S,T) be a STS(n) and (X,F), F = $\{F_1, F_2, ..., F_n\}$, a 1-factorization of K_{n+1} with vertex set X, where X ∩ S = ∅. Let S^* = S ∪ X and define a collection of triples T^* as follows

1. T ⊂ T^*, and
2. let α be any 1-1 mapping from S onto $\{1, 2, 3, ..., n\}$. For each $x \in$ S and each $\{a, b\} \in F_{x\alpha}$ place the triple $\{x, a, b\}$ in T^*.

Then (S^*, T^*) is a STS$(2n + 1)$

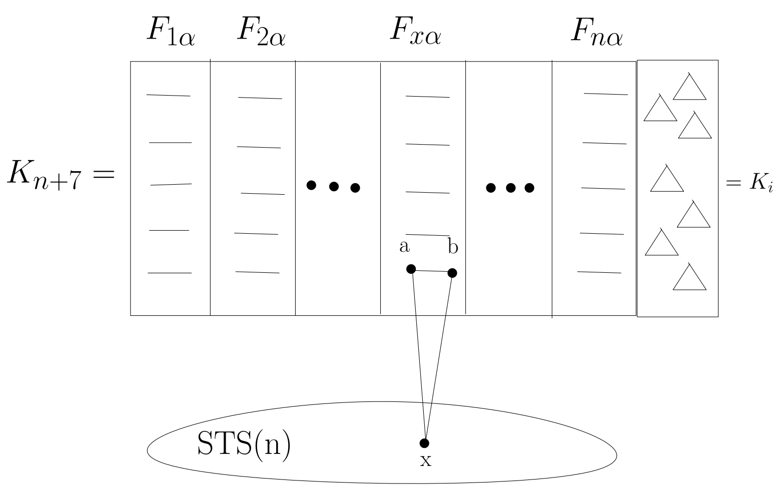


The $2n+7$ Construction: Let$F = \{F_1, F_2, ..., F_n\}$ be a collection of n 1-factors of K_{n+7} with vertex set X. Further let K = $K_{n+7} \backslash F$ and K_1 and K_2 two partitions of K into $n+7$ triples, where $K_1 \cap K_2 = \emptyset$. Now let (S,T) be a STS(n) with vertex set S, such that S $\cap$ X = $\emptyset$. Set S^* = S$\cup$ X and define a collection of triples T^* as follows:

1. T $\subset T^*$,

2. $K_i \subset T^*$, where $i = 1$ or 2 (but not both), and

3. let α be any 1-1 mapping from S onto $\{1, 2, 3, ..., n\}$. For each $x \in$ S and each $\{a, b\} \in F_{x\alpha}$ place the triple $\{x, a, b\}$ in T^*. Then (S^*, T^*) is a STS($2n+7$).



The Original Construction

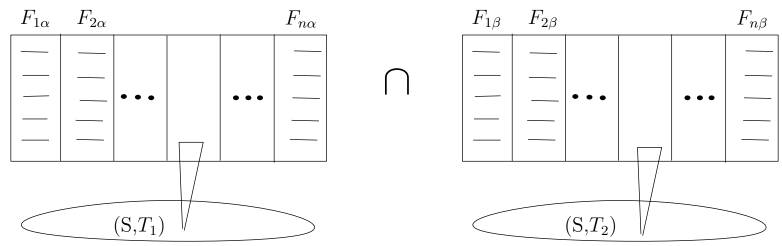
1. Ad hoc constructions are used to solve the problem for all $n \leq 33$. So we can assume $n \geq 37$.

2. Every $m \equiv 1$ or 3 (mod 6) can be written in the form $2n+1$ or $2n+7$, where $n \equiv 1$ or 3 (mod 6).

3. The proof uses induction. So assume we have solved the intersection problem for all $n \equiv 1$ or 3 (mod 6) ≤ 33.

4. If $n \equiv 1 (\text{mod} 6)$ we use the $2n+1$ Construction:

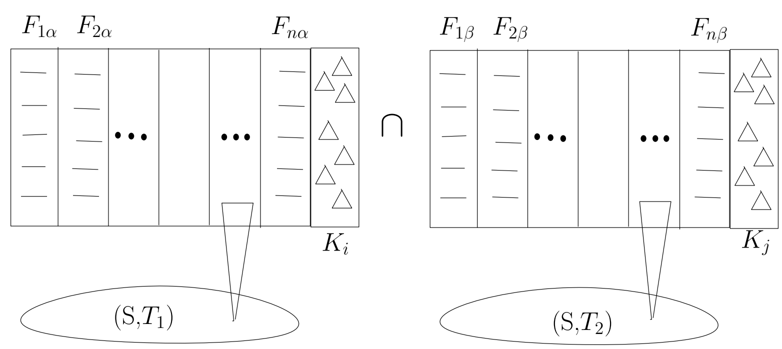
   Let (S,T_1) and (S,T_2) be any two STS(n), F a 1-factorization of K_{n+1}, and α and β 1-1 mappings from S onto $\{1, 2, 3, ..., n\}$. Then,

   $= |T_1 \cap T_2| + \sum |F_{i\alpha} \cap F_{i\beta}|$. A bit of reflection shows that $I(2n+1) = J(2n+1)$.

5. If $n \equiv 3$ (mod 6) use the $2n+7$ construction.

   Let (S,T_1) and (S,T_2) be any two STS(n), F a collection of n 1-factors of K_{n+7}, and α and β 1-1 mappings from S onto $\{1, 2, 3, ..., n\}$ Then,

   $=|T_1 \cap T_2| + \sum |F_{1\alpha} \cap F_{i\beta}| + |K_i \cap K_j|$. As with the $2n+1$ Construction it is quite easy to show that $I(2n+7) = J(2n+7)$.

   With the original construction out of the way we can now proceed to a completly different and new construction.

Chapter 4

The intersection of quasigroups

Two quasigroups (Q,$\circ_1$) and (Q,$\circ_2$) are said to intersect in k products provided their tables agree in exactly k cells.

Example 4.1 (Two quasigroups of order 4 intersecting in 6 products).

$\circ_1$	1	2	3	4
1	1	3	4	2
2	4	2	1	3
3	2	4	3	1
4	3	1	2	4

$\circ_2$	1	2	3	4
1	1	4	2	3
2	3	2	1	4
3	2	3	4	1
4	4	1	3	2

In [2] H.L Fu proved the following theorem.

<u>Theorem 4.2</u> (H.L. Fu[2]) If $n \geq 5$, there exists a pair of quasigroups having k products in common if and only if $k \in \{0, 1, 2, ..., n^2\} \backslash \{n^2 - 1, n^2 - 2, n^2 - 3, n^2 - 5\}$. $\square$

Let Q $= \{1, 2, ..., 2n\}$ and let H $= \{h_1, h_2, ..., h_n\}$ be a partition of Q into 2-element subsets (called holes of size 2).

Let (Q, $\circ$) be a quasigroup with the property that $(h_i, \circ)$ is a subgroup for every hole $h_i \in H$. Then (not too surprisingly) (Q, $\circ$) is said to be a quasigroup with holes H.

Two communitive quasigroups (Q,$\circ_1$) and (Q,$\circ_2$) with the same holes H are said to intersect in k products provided their tables agree in exactly k cells above the holes.

Example 4.3 (Two communitive quasigroups of order 6 with holes intersecting in 8 products)

$\circ_1$	1	2	3	4	5	6
1	1	2	5	6	4	3
2	2	1	6	5	3	4
3	5	6	3	4	2	1
4	6	5	4	3	1	2
5	4	3	2	1	5	6
6	3	4	1	2	6	5

$\circ_2$	1	2	3	4	5	6
1	1	2	6	5	4	3
2	2	1	5	6	3	4
3	6	5	3	4	2	1
4	5	6	4	3	1	2
5	4	3	1	2	5	6
6	3	4	2	1	6	5

The following theorem is due to C.M Fu [1].

Theorem 4.4 (C.M Fu [1]) If $2n \geq 10$, there exists a pair of commutative quasigroups of order $2n$ having the same holes intersecting in k products if and only if
k $\in \{0, 1, 2, ..., x = 2n(n-1)\} \backslash \{x-1, x-2, x-3, x-5\}$. $\square$

The results in Theorems 4.2 and 4.3 were obtained many years after the original solution of the intersection problem for Steiner triple systems. We can now use the results in these two theorems to give a new and much easier solution to this problem.

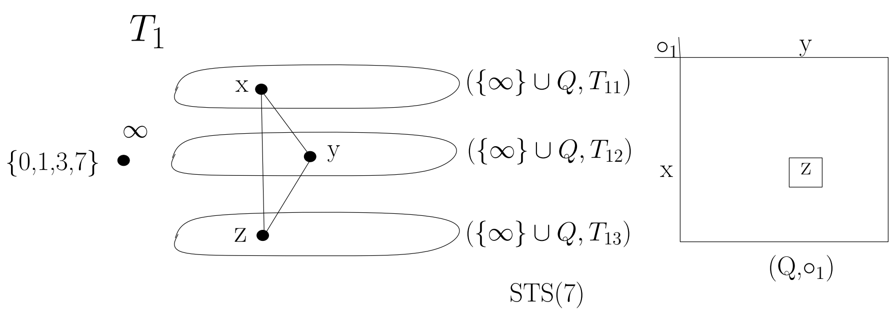
Chapter 5

The $6n + 1$ Construction

In this chapter we will give a $6n + 1$ Construction along with the results in Theorems 4.2 and 4.4 to give a new solution of the intersection problem for all $6n + 1 \geq 19$. We will give two examples before giving the general construction.

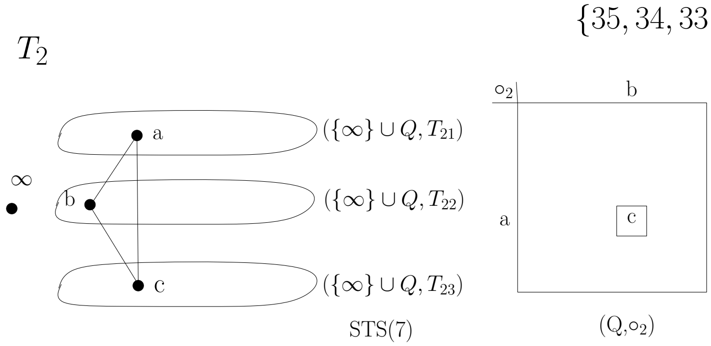
Example 5.1 ($n = 19$)

Write $n = 3 \bullet 6 + 1$. Let $|Q| = 6$ and set $S = \{\infty\} \cup (Q \times \{1, 2, 3\})$. Let (S,$T_1$) and (S,$T_2$) be two STS(19)s defined by:



and

$\cap\{0, 1, 2, ..., 36\}\backslash$
$\{35, 34, 33, 31\}$



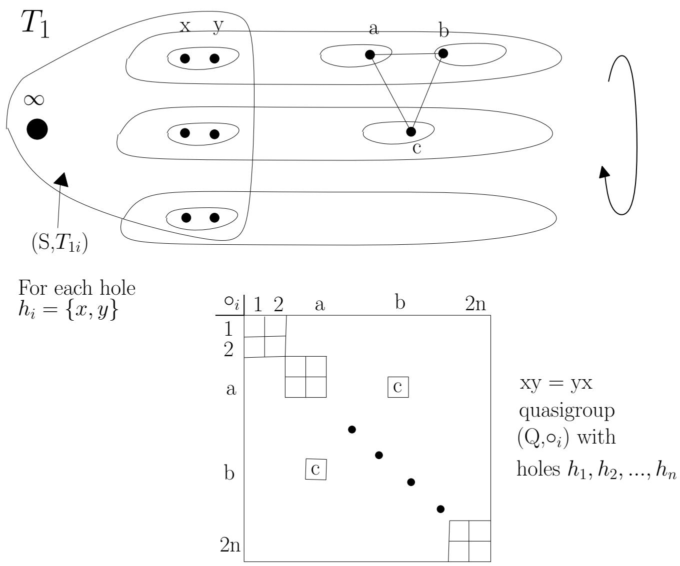
It is a straightforward computation to see that any number $n \in \{0, 1, 2, ..., 57\} \setminus \{56, 55, 54, 52\}$ can be written in the form $|T_{11} \cap T_{21}| + |T_{12} \cap T_{22}| + |T_{13} \cap T_{23}| + |(Q, \circ_1) \cap (Q, \circ_2)|$.

Example 5.2 ($n = 25$)

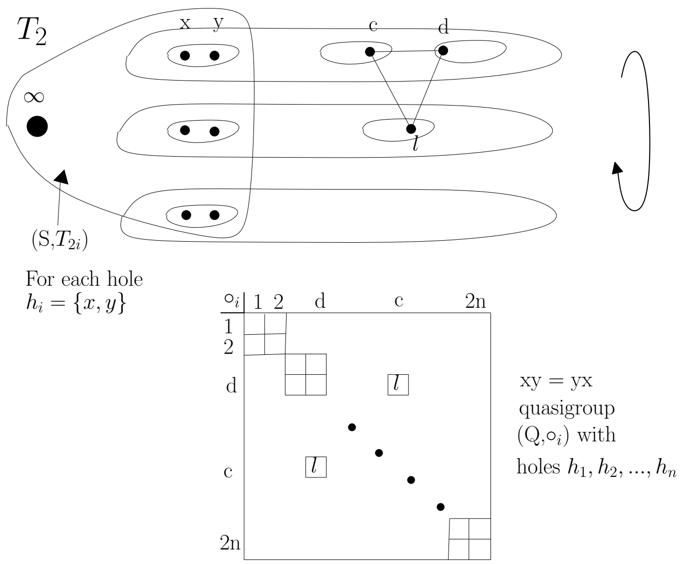
Write $n = 3 \bullet 8 + 1$. Let $|Q| = 8$, set $S = \{\infty\} \cup (Q \times \{1, 2, 3\})$, and proceed exactly as in Example 5.1

The solution for 6n + 1 ≥ 31

Write $6n + 1 = 3(2n) + 1(2n \geq 10)$, let $|Q| = 2n$, and set $S = \{\infty\} \cup (Q \times \{1, 2, 3\})$. Let (S, T_1) and (S, T_2) be STS$(6n + 1)$ defined as follows:



Where (Q,$\circ_1$) is used between levels 1 and 2, (Q,$\circ_2$) is used between levels 2 and 3, and (Q,$\circ_3$) is used between levels 3 and 1.



Where (Q,$\circ_1$) is used between levels 1 and 2, (Q,$\circ_2$) is used between levels 2 and 3, and (Q,$\circ_3$) is used between levels 3 and 1.

As in the above two examples it is easy to see that any number $m \in I(n)$ can be written in the form $\sum |T_{1i} \cap T_{2i}| + |(Q, \circ_1) \cap (Q, \circ_1)| + |(Q, \circ_2) \cap (Q, \circ_2)| + |(Q, \circ_3) \cap (Q, \circ_3)|$. We will illistrate this with an example.

Example 5.3 (Two STS(31)s intersecting in 87 triples).

1. Take $|T_{1i} \cap T_{2i}| = 7$, for $i = 1, 2, 3, 4, 5$;

2. $|(Q, \circ_1) \cap (Q, \circ_1)| = 40$ between $Q \times \{1\}$ and $Q \times \{2\}$; and

3. $|(Q, \circ_2) \cap (Q, \circ_2)| = 12$ between $Q \times \{2\}$ and $Q \times \{3\}$.

4. $|(Q, \circ_3) \cap (Q, \circ_3)| = 0$ between $Q \times \{3\}$ and $Q \times \{1\}$.

   Then $\sum |T_{1i} \cap T_{2i}| + |(Q, \circ_1) \cap (Q, \circ_1)| + |(Q, \circ_2) \cap (Q, \circ_2)| + |(Q, \circ_3) \cap (Q, \circ_3)| = 35 + 40 + 12 + 0 = 87$

Summary Examples 5.1, 5.2, and the $6n+1 \geq 31$ Construction gives a complete solution of the intersection problem for all $6n + 1 \geq 19$. As previously mentioned, the case of $6n + 1 = 13$ is handled by an ad hoc construction in the original paper[5].

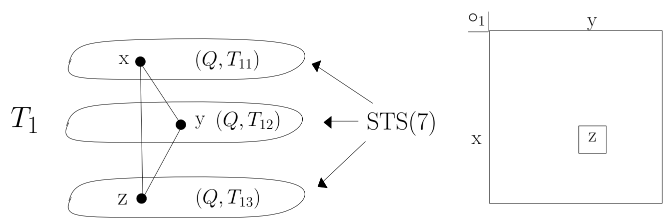
Chapter 6

The $6n+3$ Construction

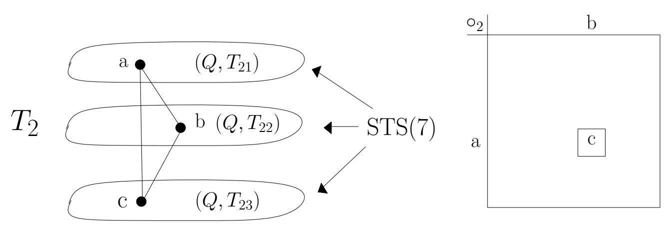
It is well known that $I(9) = \{0,1,2,3,4,6,12\}$ and $I(15) = J(15)$. So we will begin with examples for 21 and 27, which cannot be done in general with the $6n+3$ Construction.

Example 6.1 ($n = 21$)

Let Q be a set of size 7 and set S $=$ Q $\times\{1,2,3\}$. Define two STS(21)s (S,T_1) and (S,T_2) as follows:



(Q,$\circ_1$) a quasigroup of order 7.



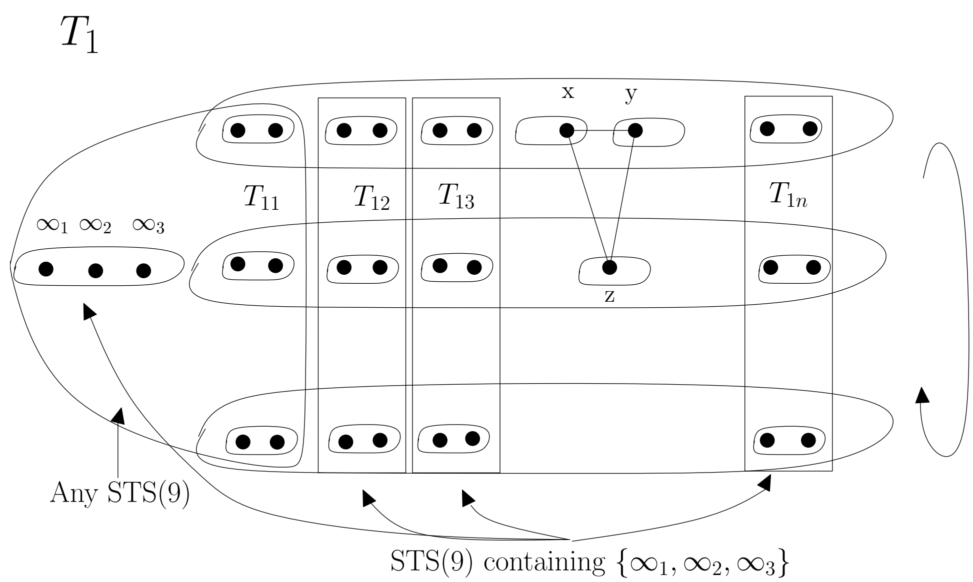
(Q,$\circ_2$) a quasigroup of order 7.

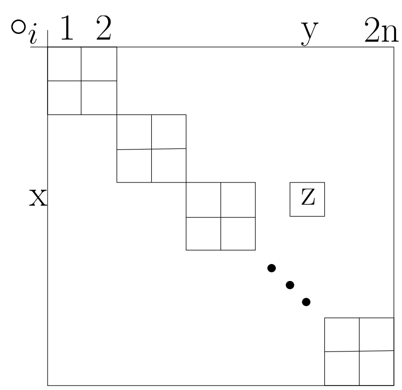
Then if $k \in \{0,1,2,...,70\}\backslash\{69,68,67,65\}, k = |T_{11} \cap T_{21}| + |T_{12} \cap T_{22}| + |T_{13} \cap T_{23}| + |(Q,\circ_1) \cap (Q,\circ_2)|$.

Example 6.2 ($n = 27$)

Let Q be a set of size 9 and set $S = Q \times \{1,2,3\}$. Define two STS(27) (S,T_1) and (S,T_2) and proceed as in Example 6.1

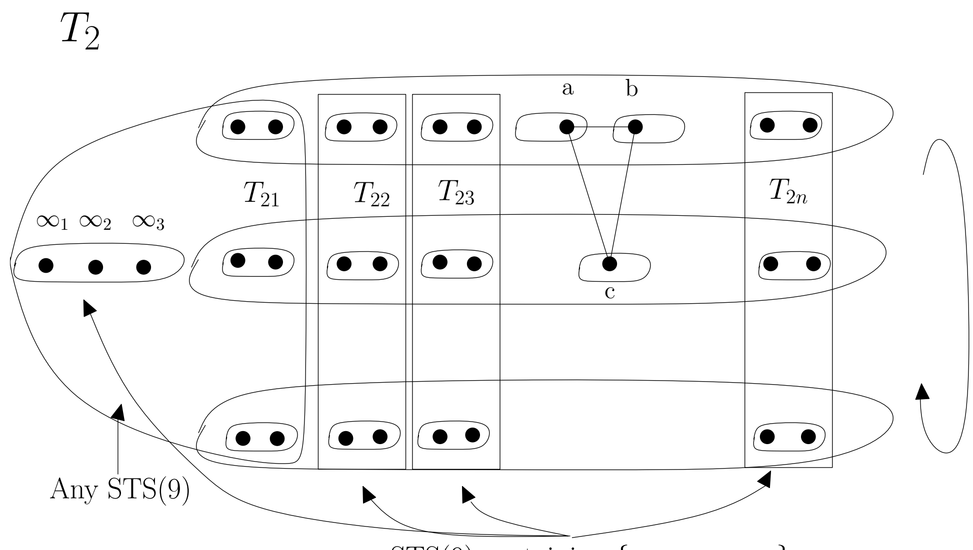
The solution for 6n+3 ≥ 33 Write $6n + 3 = 3 + 3 \bullet 2n$ and let Q be a set of size 2n with holes $H = \{h_1, h_2, ..., h_n\}$ of size 2. Set $S = \{\infty_1, \infty_2, \infty_3\} \cup (Q \times \{1, 2, 3\})$ and define two STS$(6n + 3)$s, (S,T_1) and (S,T_2) as follows:

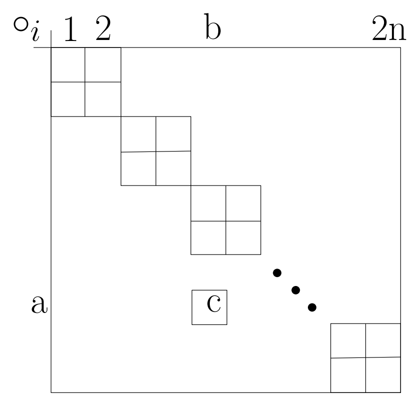




xy=yx quasigroup (Q,$\circ_i$)
with holes $H = \{h_1, h_2, ..., h_n\}$.

Where (Q,$\circ_1$) is used between levels 1 and 2, (Q,$\circ_2$) is used between levels 2 and 3, and (Q,$\circ_3$) is used between levels 3 and 1.





xy=yx quasigroup (Q,$\circ_i$).
with holes $H = \{h_1, h_2, ..., h_n\}$.

Where (Q,$\circ_1$) is used between levels 1 and 2, (Q,$\circ_2$) is used between levels 2 and 3, and (Q,$\circ_3$) is used between levels 3 and 1.

Now, let $m \in I(6n+3)$. Then $m = |T_{11} \cap T_{21}| + \sum |(T_{1i} \cap T_{2i}) \setminus \{\infty_1, \infty_2, \infty_3\}| + |(Q, \circ_1) \cap (Q, \circ_1)|$(between levels 1 and 2) $+|(Q, \circ_2) \cap (Q, \circ_2)|$(between levels 2 and 3)$+|(Q, \circ_3) \cap (Q, \circ_3)|$(between levels 3 and 1).

We illustrate this construction for $6n + 3 = 33$.

Example 6.3 (A pair of STS(33) intersecting in 102 triples).

Take $|T_{11} \cap T_{21}| = 0$ each $|(T_{1i} \backslash \{\infty_1, \infty_2, \infty_3\}| = 0$ (this is simply a pair of triple systems of order 9 having just the triple $\{\infty_1, \infty_2, \infty_3\}$ in common, $|(Q, \circ_1) \cap (Q, \circ_1)| = 40$ between levels 1 and 2 and 2 and 3, and $|(Q, \circ_3) \cap (Q, \circ_3)| = 22$ between levels 3 and 1.

Summary In this chapter we have given a complete solution of the intersection problem for all $n \equiv 3 (\text{mod } 6) \geq 21$.

Chapter 7

Concluding remarks

In this thesis we have given a new solution to the intersection problem for Steiner triple systems using results that were not available when the original solution was obtained. In particular we have given a new solution for all $n \equiv 1$ or $3 \pmod 6 \geq 19$. Our solution is much simpler than the original solution and has the added benefit of not using induction.

Bibliography

[1] C.M. Fu, The intersection problem for pentagon systems, Ph D thesis, Auburn University, (1987).

[2] H.L. Fu, On the construction of certain types of latin squares with prescribed intersections, Ph D thesis, Auburn University, (1980)

[3] T.P. Kirkman, On a problem in combinations, Cambridge and Dublin Math, J.,2(1847), 191-204.

[4] E.S. Kramer and D.M. Messner, Intersections among Steiner systems, J. Combinatorial Theory A, 16(1974), 273-285.

[5] C.C. Lindner and A. Rosa, Steiner triple systems having a prescribed number of tripples in common, Carnad J. Math., 27(1975), 1166-1175.