

Cycle Systems

by

Nidhi Sehgal

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Approved by

Dr. C.A. Rodger, Chair, C. Harry Knowles Professor, Assoc. Dean for Research and
Graduate Studies

Dr. Curt Lindner, Distinguished University Professor

Dr. Dean Hoffman, Professor

Dr. Pete Johnson, Professor

Abstract

In this dissertation the author has found the necessary and sufficient conditions for obtaining a 6-cycle system of the Cartesian product of two complete graphs covering 2-paths in the corresponding bipartite graph. She has found the maximum *fair* 6-cycle system as well as 6-cycle system of the Cartesian product of two complete graphs. As a part of this dissertation, the author has also found the necessary and sufficient conditions required to obtain a 4-cycle system of complete graph on n vertices with a nearly 2-regular leave. Finally the author has worked on the problem of finding a 4-cycle system of the line graph of a complete multipartite graph.

Acknowledgments

My Back Pages

Nidhi was born in a small town called Jalandhar in the northern state of Punjab in India. She grew up in Mumbai on the western coast of India. She completed her Bachelor of Science from St. Xavier's College in Mumbai. She completed her Master in Science from Auburn University under the guidance of Dr. Chis Rodger. After completing her MSc. in 2008, she continued to work under the guidance of Dr. Chris Rodger for doctoral research. Nidhi worked on the problem of finding the necessary and sufficient conditions for obtaining the 4-cycle system of the Line graph of a complete multipartite graph during her Masters. She then continued to work on this problem for her doctoral research. And went on to work on four other problems too for her doctoral research. She would like to thank her Major professor Dr. Rodger for his continued support and guidance. He is a great mentor and friend. She would like to thank her other committee members: Dr. Lindner, Dr. Hoffman and Dr. Johnson for their support. she would also like to express her thanks to all the members/colleagues of the Dept. of Mathematics and Statistics for their support. Nidhi would also like to express her heartfelt gratitude to her family and friends for their continued support throughout the five years.

“Ah, but I was so much older then

Im younger than that now”

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Chapter 1

Introduction

1.1 Introduction

A *graph* G , consists of a set of vertices or points, and a set of edges. G is denoted as an ordered pair, $(V(G), E(G))$, where $V(G)$ denotes the set of vertices or points, and $E(G)$ denotes the set of edges or line segments $\{u, v\}$ (or more simply uv), such that $u, v \in V(G)$. The number of vertices in G is said to be the *order* of G and the number of edges in G is said to be the *size* of G . Edges (e) are unordered pairs of vertices ($\{u, v\}$) that can be depicted as line segments that connect the vertices, and the two vertices (u and v) that are connected by the edge (e) are said to be its *endpoints*, and the edge (e) is said to be *incident* to the two vertices (u and v). Any two vertices (u and v) are said to be *adjacent* if they form the endpoints of an edge (e). The degree of a vertex v in $V(G)$ is the number of vertices adjacent to the vertex v in G . The *degree* of a vertex is also denoted by $deg(v)$. A loop is an edge that connects a vertex to itself. A graph is said to have multiple edges if at least one pair of vertices are connected by more than one edge. A *simple graph* is a graph that contains no loops nor multiple edges. A graph G' is said to be a *subgraph* of a graph G if $V'(G) \subseteq V(G)$ and $E'(G) \subseteq E(G)$. And G' is said to be an *induced subgraph* of a graph G if it is formed by the set of vertices $V'(G)$, where $V'(G) \subseteq V(G)$, and the edge set of G' contains the edges in $E(G)$ that join the vertices in $V'(G)$.

A graph on n vertices in which each vertex is adjacent to every other vertex is said to be a *complete graph*. It is denoted by K_n , where n is the order of the graph. The numbers of edges in K_n is given by $|E(G)| = (n)(n-1)/2$. λK_n denotes the graph formed by joining each pair of adjacent vertices in K_n by λ edges.

A *complete multipartite graph* is denoted by $G = K(a_1, \dots, a_p)$; the vertex set of G can be expressed as the disjoint union, $V(G) = V(K_{a_1}) \cup V(K_{a_2}) \cup \dots \cup V(K_{a_p})$, and $\{u, v\} \in E(G)$ if and only if $u \in V(K_{a_i}), v \in V(K_{a_j})$ such that $i \neq j$. The case when $p = 2$ is defined to be the *bipartite graph* denoted by $K(m, n)$ where $a_1 = m$ and $a_2 = n$.

An *m-path* is a sequence of $m + 1$ vertices such that each vertex is connected to the next vertex and two vertices have degree one, rest have degree two. It is also denoted by $P_{m+1} = \{v_1, v_2, \dots, v_{m+1}\}$. An *m-cycle* is an *m-path* with the property that $v_1 = v_{m+1}$. It is also denoted by $C_m = (v_1, v_2, \dots, v_m)$. Any connected graph with no cycles is said to be a *tree*. A *forest* is a disjoint union of trees. A *hamilton path* is a path that spans the vertex set of a graph G . Similarly we also define a *hamilton cycle* which spans the vertex set of a graph G . An *m-cycle system* of a graph G is denoted by $(V(G), C)$ where the set C contains cycles of length m whose edges form a partition of the edge set of G .

The *line graph* of a graph G , $L(G)$ is defined as follows: Every edge $\{u, v\} \in E(G)$ is a vertex in $L(G)$ and two vertices are adjacent in $L(G)$ if the corresponding edges in G have a common end point. A *clique* of a graph G is a subset of the vertex set of G such that any two vertices in the clique are adjacent to one another.

The history presented below is in chronological order.

In 1847, Kirkman [62] posed one of the earliest problems of graph theoretical importance given below. Historically the problem is stated as follows:

“If Q_x denotes the greatest number of triads that can be formed with x symbols, so that no duad shall be twice employed, then

$$3Q_x = c(x - 1)/2 - V_x$$

if for V_x we put $6k + 4$, when $x = 6n - 1$; $x/2 + 3k + 1$, when $x = 6n - 2$; 0 when $x = 6n + 1$ or $6n + 3$; and $x/2$, when $x = 6n$ or $6n + 2$: where $2^m(2k + 1) = n$; x, n, m, k , being all integers ≥ 0 ". Notice that V_x is the number of duads excluded from Q_x .

He then gave solutions for Q_3, Q_7, Q_{15} and Q_{25} . In graph theory terminology, this problem asks for a 3-cycle system of as big a subgraph of K_x as possible. Kirkman gave the solution for the 3-cycle systems of K_3, K_7, K_{13} and K_{25} .

Since then, there has been a lot of interest in problems pertaining to m -cycle systems of G , especially the case where $G = K_n$. In 1965, Kötzig [63] gave several results regarding even cycle systems of K_n . He first showed that there exists a $4k$ -cycle system of K_n when $n \equiv 1 \pmod{8k}$ (sufficient condition). For the case where n and k are relatively prime he gave necessary and sufficient conditions to obtain a $4k$ -cycle system of K_n where $n \equiv 1 \pmod{8k}$. Finally he showed that K_n can be decomposed into 2^p -cycles, where p is prime if and only if $n \equiv 1 \pmod{2^{p+1}}$. In 1966, Rosa [80] completed Kötzig's result regarding even cycle systems of K_n by giving a solution for the case where $m \equiv 2 \pmod{4}$. He showed that there exists a cyclic decomposition of K_n into p -cycles where $p \equiv 2 \pmod{4}$ and $n = 2kp + 1$ for any k and p . In 1966 Rosa [81], solved the problem further by giving partial solutions to the case where m is odd, $n \equiv 1 \pmod{2m}$, and n is an odd prime number satisfying the condition, $n \equiv m \pmod{2m}$.

There are results in literature about various other type of H decompositions of K_n too, where H is a subgraph of K_n . In 1979, Tarsi [88] gave necessary and sufficient conditions for obtaining the star decomposition, S_n of λK_n . A *star* S_n is defined to be the bipartite graph $K_{1,n}$. A few years later in 1983, Tarsi [89] gave a result regarding path decompositions of λK_n . He showed that the obvious necessary condition for obtaining an

m -path decomposition of λK_n , $n(n-1) \equiv 0 \pmod{2m}$ and $n \geq m+1$, is also sufficient. In 1980, Alspach and Varma [4] gave necessary and sufficient conditions for obtaining a $2p^e$ -cycle system of K_n , where p is prime and e is any positive integer. They showed that there exists a $2p^e$ -cycle system of K_n if and only if n is odd, $n \geq 2p^e$ and $2p^e$ divides $\binom{n}{2}$. In 1981, D. Sotteau [84] proved a prominent result on cycle systems of the complete bipartite graphs, $K(m, n)$. She showed that the complete bipartite graph $K(m, n)$ can be decomposed into cycles of length $2k$ if and only if m and n are even integers not less than k and $2k$ divides mn . She also gave the conditions for obtaining a directed $2k$ -cycle system of the digraph $K(m, n)^*$. In 1988, Jackson [51] gave a result for obtaining odd cycle systems of K_n . He proved that there exists an m -cycle system of K_n , where $n \equiv 1$ or $m \pmod{2m}$ and m is odd. Finally in 1989, Hoffman, Rodger, Lindner [45] gave necessary conditions for finding the m -cycle system of K_n , when m is odd. They showed that these necessary conditions are also sufficient if and only if there is an m -cycle system of all orders n satisfying the necessary conditions with $m \leq n < 3m$. In this paper the authors also gave an important result about complete graphs with a hole of size v , where v is odd. They showed that for odd $v = ql + r$, where $1 \leq r \leq l$, if $q \leq m + 2r - 1$, there exists an m -cycle system of K_{2m+v} with a hole of size v . The reader is referred to the paper by Rodger [79] in 1990 for a complete survey on the above mentioned results. By 1992, many researchers had solved the problem of finding an m -cycle system of K_n for values of $m \leq 37$. In 1993, Saad Zanati and Rodger [28] solved the problem of obtaining the H decomposition of K_n , where H is a simple connected subgraph of K_n containing at most 5 edges, which had the additional property of being able to 2-color the vertices so that no copy of H is monochromatic.

The existence of m -cycle systems of $L(K_n)$ was settled when $m \in \{4, 6\}$ in [16, 21, 20]. In 1993, Colby and Rodger [21] showed that there exists a 4-cycle system of λK_n if and only if n and λ satisfy (a) n is even, or (b) $n \equiv 1 \pmod{4}$ and $\lambda \equiv 0 \pmod{2}$, or (c) $n \equiv 3 \pmod{4}$ and $\lambda \equiv 0 \pmod{4}$, or (d) $n \equiv 1 \pmod{8}$ and λ is odd. Continuing the work on

m -cycle systems of $L(K_n)$, Cox and Rodger [20] in 1996 showed that such cycle systems exist for $n \equiv 1 \pmod{2m}$ and for all m, n with $m \equiv 0 \pmod{4}$ and $n \equiv 0$ or $2 \pmod{m}$. Also, there have been some results for obtaining m -cycle systems of $K(a_1, a_2, \dots, a_p)$, for example being settled when all parts have the same size in [28] where m is even and when p is small and then: There is a companion result for obtaining 4-cycle systems [78] of $L(K(a_1, a_2, \dots, a_n))$, but much remains to be done in this area. Some new results regarding this problem are discussed in Chapter 6.

By that time, a lot of research had been done on finding necessary and sufficient conditions for obtaining m -cycle systems of K_n in the case where n is odd. However, when n is even, the vertices of K_n have odd degree and therefore it is not possible to obtain an m -cycle system of K_n in this case. Thus, the natural question to ask was what is the maximum number of m -cycles that can be obtained in the case when n is even? Due to the odd degree of each vertex in this case, we will have to remove some edges from K_n in order to be able to construct an m -cycle system. The set of edges removed from the graph G is said to be the *leave*, (F). The problem of finding necessary and sufficient conditions to obtain m -cycle systems of K_n and $K_n - F$, where F is a leave, has generated a lot of interest.

A *k-regular graph* is a graph in which every vertex has degree k . A *k-factor* of a graph G is a spanning k -regular subgraph of the graph G . In 1986, Colbourn and Rosa [19] used difference methods to show that there exists a 3-cycle system of $K_n - F$ where F is any 2-regular leave. The problem of obtaining an m -cycle system of K_n minus a 1-factor was looked at by Alspach and Marshall in 1994 [3]. In their paper they gave necessary conditions for obtaining an m -cycle system of $K_n - I$ where m and n are both even and I is a 1-factor. In 1996, Buchanan in his PhD dissertation solved the problem of finding an n -cycle system (or hamilton cycle system) of K_n with a 2-regular leave. Later Bryant [12] in 2004 and McCauley [68] in 2008 gave different proofs for this problem.

In 2000, Fu and Rodger [31] found necessary and sufficient conditions for decomposing $K_n - F$ into 4-cycles, where F is an n vertex forest. In 2001 [32], the same authors solved the problem of obtaining 4-cycle systems of $K_n - F$ and $2K_n - F$, where F is a 2-regular leave, using the method of induction. Finally in 2001, Alspach and Gavlas [2] found necessary and sufficient conditions for half the cases of the problem of finding m -cycle systems of K_n and of $K_n - I$, where I is a 1-factor. In their remarkable paper they proved that the obvious necessary conditions, that each vertex in K_n have even degree and the total number of edges in K_n be divisible by the length of the cycle m , are also sufficient conditions to obtain the m -cycle system of K_n where m and n are both odd, and to obtain the m -cycle system of $K_n - I$ where m and n are both even. In 2002, Sajna [82] showed that if for all n satisfying the equation $m \leq n \leq 3m$ there exists an m -cycle system of $K_n - I$, when m is odd, then there exists an m -cycle system of $K_n - I$ for all values of $n \geq m$. In 2002, Sajana [83] settled the existence problem of necessary and sufficient conditions of m -cycle systems of G by solving the last two cases remaining in Alspach and Gavlas's paper. He proved that necessary conditions are also sufficient for the cases when m is even and $G = K_n$, and when m is odd and $G = K_n - I$.

The case where the leave F is any 2-regular graph and $m = 4$ was solved in [31]. The case where F is any graph with maximum degree 3 and $m = 4$ was solved in [34]. In 2003, Leach and Rodger [65] showed that for $n \geq 1$, $p \geq 2$ and any given set of integers s_1, s_2, \dots, s_q satisfying the conditions $s_j \leq p + 1$ for $1 \leq j \leq q$ and $\sum_{j=1}^q s_j = np$, there exists an n -cycle system of the complete p -partite graph $K(n, n, \dots, n)$ minus a 2-factor containing q cycles where the j^{th} cycle is of length s_j . In 2004, Ashe and Rodger [5] carried forward the research of Fu and Rodger on m -cycle systems of $K_n - F$, where F is a forest (solved in 2000). Ashe and Rodger found necessary and sufficient conditions for existence of a 6-cycle system of $K_n - F$, where F is any spanning forest of K_n . In 2004, Leach and Rodger

[66] showed that there exist n -cycle systems of $K_n - F$, where F is a 3-factor. They also proved that given any 2-factor of K_n there exists a 3-factor, F of K_n containing the given 2-factor which satisfies the previous statement. In 2005, Ashe and Rodger [6] continued their research on 6-cycles systems of K_n with a leave. They gave necessary and sufficient conditions to settle the existence of 6-cycle systems of $K_n - F$, where F is any 2-regular leave, not just a spanning subgraph of K_n like in their previous paper in 2004. This result was taken further in 2007 by Ashe, Leach and Rodger [7]. They gave necessary and sufficient conditions for obtaining 8-cycle systems of $K_n - F$, where F is any 2-regular leave. We have carried forward the reasearch done by Fu and Rodger in 2001, by settling the necessary and sufficient conditions for the 4-cycle systems of $K_n - F^*$, where F^* is a nearly regular leave in 2009. A *nearly 2-regular leave* is a 2-regular leave with the additional property that one special vertex, say ∞ having degree k , $k > 2$. The details of this result are given in Chapter 5 and can also be found in [85].

There are some results regarding m -cycle systems of $G - F$, where G is a complete multipartite graph and F is a leave. One of the first results about this problem was by Byrant, Leach and Rodger in 2005 [12]. They showed that there exists a n -cycle system of $K(n, n) - F$, where F is a 3-regular leave.

Another problem of graph theoretical importance was posed by Judson in 1899 [30]:
“Seven persons met at a summer resort, and agreed to remain as many days as there are ways of sitting at a round table, so that no one shall sit twice between the same two companions. They remained fifteen days. It is required to show in which way they may be seated.”

In 1900, Philbrick, [48] gave a solution to Judson’s problem for the cases when there are 6 people (can be seated on 10 days) and when there are 8 people (he showed that they can be seated as per Judson’s conditions for 21 days). In 1904, Safford, [50] gave the solution

to Judson’s original problem for 7 people.

In 1905, Dickson [25] gave a generalization of Judson’s problem using group theory. His problem statement was

“The general problem is to find all complete sets S_i^m of dihedrons”,

where S_k^m was defined to be a set of k mutually consistent dihedrons on m letters when every dihedron on m letters was inconsistent with at least one dihedron of S_k^m . He then went on to solve his problem (another version of Judson’s problem) for the cases when m (number of persons) was 6 (solution: 10 days), 8 (solution: 21 days), 10 (solution: 36 days) and 12 (solution: 55 days).

Rather than explain this notation, instead we consider the equivalent problem posed by Dudeney in 1905 [26] which can be described in graph theoretical terminology as follows:

“Seat the same n persons at a round table on $(n - 1)(n - 2)/2$ occasions so that no person shall ever have the same two neighbors twice. This is, of course, equivalent to saying that every person must sit once, and only once, between every possible pair”.

This problem is now known as *Dudeney’s Round Table problem*. In 1917, Dudeney [27] gave solutions for the cases $3 \leq n \leq 12$. He claimed to have solved the problem for the cases where $2 \leq n \leq 25$ and $n = 33$. In graph theory this problem is equivalent to asking for a set of hamilton cycles of K_n , with the added property that every 2-path lies in exactly one hamilton cycle. This set of hamilton cycles is also known as the *Dudeney set*. Some other generalizations of this problem could be formed by looking at x tables of size m each instead of considering one large table.

An (H, J) λ -covering of G is defined to be a set U of copies of the subgraph H in G such that each copy of J in G is contained in exactly λ of the copies of H in the set U . More informatively, this is also known as a set of copies of H that cover each copy of J in λG . There are various results about 2-path coverings; that is, the case where J is the 2-path, P_3 . One of the earliest results was by Hanani in 1960 [40]. He proved that the condition $n \equiv 2$ or $4 \pmod{6}$ is necessary and sufficient in order to obtain a 4-cycle system of K_n which covers each 2-path in K_n exactly once. In 1973, Huang and Rosa [47] solved Dudeney's Round table problem for the case when $n = p + 1$, p prime, and they mentioned that they found cyclic solutions for $n = 13, 15$. In 1983, Heinrich [41] showed that K_n can be decomposed into oriented 2-paths. She proved that whenever the two arcs in the 2-path have consistent orientation the result is true if and only if n is odd. Otherwise, the result holds for all $n \geq 4$. In 1987, Heinrich and Nonay [43] proved that there exists a set of 4-cycles, $(C_4, P_3)\lambda$ -covering of K_n if and only if one of the following conditions is satisfied: (1) n is even, (2) $n \equiv 1 \pmod{4}$ and $\lambda \equiv 0 \pmod{2}$, or (3) $n \equiv 3 \pmod{4}$ and $\lambda \equiv 0 \pmod{4}$. As a continuation of this result the same authors in 1988 [44] solved the problem of obtaining the minimum (maximum) number of 4-cycles in K_n such that each 2-path in K_n is contained in exactly λ 4-cycles. In 1992, Kobayashi and Nakamura [54] gave necessary and sufficient conditions to obtain an n -cycle system of K_{2n} such that each 2-path in K_{2n} lies in exactly two n -cycles. Later in 1993, these authors along with Kiyasu [53] gave the solution to Dudeney's Round Table problem for the case where n is even. A set of n -cycles of K_n is said to be a *double Dudeney set* if each 2-path in K_n is contained in exactly two n -cycles. Thus, these authors first constructed a double Dudeney set and then also gave the construction for a Dudeney set for the case when n is even. Heinrich, Langdeau and Verall [42] in 2000 solved the problem of finding n -cycle systems of K_n such that each 2-path in K_n is covered in exactly λ n -cycles. In this paper they also showed there exists an H decomposition of K_n such that each 2-path in K_n is covered in exactly λ copies of H , for each subgraph H of K_n .

having at most 4 vertices. Finally, they proved that there exists a 3-path decomposition of $K(n, n)$ covering each 2-path of $K(n, n)$ exactly once. In the following year, McGhee and Rodger [69] took the previous result further by giving two new constructions for the problem of finding a set of 3-paths in K_n such that each 2-path is contained in exactly one 3-path. They also solved a more general problem of finding an m -path decomposition of K_n covering $(m - 1)$ -paths. In 2001, Kobayashi and Nakamura [55] solved the problem of finding a set of 4-paths in K_n which cover each 2-path exactly once. The authors along with Nobuaki and Kiyasu [56] gave constructions for a double Dudeney set for the case when $n \geq 3$ is odd. Kobayashi et. al. [57] solved the problem of obtaining a set of 5-paths in K_n such that each 2-path is contained in exactly one 5-path in K_n . In 2004, McGhee and Rodger [70] proved that for all $n \geq 4$ there exists a set of 4-paths in K_n such that each 2-path is contained in exactly one 4-path. In the same year these authors [71] also gave a result about embedding coverings of 2-paths with 3-paths.

Kobayashi et. al. have defined the *uniform covering* of the 2-paths in K_n with m -paths (m -cycles) as a set S of m -paths (m -cycles) having the property that each 2-path in K_n lies in exactly one m -path (m -cycle) in S . In 2005, Kobayashi et. al. [58] proved that there exists a uniform covering of 2-paths with 5-paths in K_n . Akiyama, Kobayashi and Nakamura also proved that there exists a uniform covering of 2-paths with 6-paths in 2005 [1], 6-cycles in 2006 [59]. In Chapter 2, we have given the constructions for obtaining a 6-cycle system of $K_m \times K_n$ which covers each 2-path in $K(m, n)$ exactly once.

An m -cycle system of a graph G is said to be k -perfect if replacing each m -cycle c by the graph formed by replacing the edges in c with the edges joining vertices distance k apart in c produces another cycle system of G . Necessary and sufficient conditions for obtaining 2-perfect 6-cycle systems of K_s for all possible values of s was solved in [67]. The problem of finding non-isomorphic 2-perfect 6-cycle systems of K_s was solved for the case

$s = 13$ in [38]. This problem was taken further in [9], where the authors solved the problem of obtaining 2-perfect 6-cycle systems of λK_s for all $\lambda > 1$. For all λ , those authors later also solved the spectrum problem of 6-cycle systems of $2\lambda K_s$ for which the collection of distance 3 graphs obtained from each 6-cycle covers λK_s (see [10]).

The problem of finding a metamorphosis of λ -fold $K_{3,3}$ designs into λ -fold 6-cycle systems was solved in [11]. The idea of metamorphosis was taken further in [91], where the problem of metamorphosis of 2-fold 4-cycle system into the maximum packing of 2-fold 6-cycle system was solved.

An m -cycle system of a graph G is said to be *resolvable* if m -cycles can be partitioned into classes such that each vertex in $V(G)$ occurs exactly once in each class. Kobayashi and Nakamura proved that there exists a resolvable covering of 2-paths by 4-cycles in K_n when $n \equiv 0 \pmod{4}$ and a nearly resolvable covering when $n \equiv 2 \pmod{4}$ in [60]. Later, in 2002, the authors [61] also constructed a resolvable covering of 2-paths by m -cycles in K_n where, $n = p^e + 1$, p is a prime number, $e \geq 1$ and m is a divisor of n , $k \neq 1, 2$. In 2009, Danziger, Mendelson and Quattrocchi [23] proved that there exists a resolvable decomposition of K_n into the union of 2-paths.

Until now, we have considered the history of covering problems related to the complete graph and the complete multipartite graph. Now, we will turn our focus on coverings of the cartesian product of complete graphs. The *Cartesian product* of two graphs G_1 and G_2 is denoted by $G_1 \times G_2$ where $(u, v) \in V(G_1 \times G_2)$ if and only if $u \in V(G_1)$ and $v \in V(G_2)$, and $\{(u, v), (x, y)\} \in E(G_1 \times G_2)$ if and only if $u = x$ and $\{v, y\} \in E(G_2)$ or $v = y$ and $\{u, x\} \in E(G_1)$. One of the earliest results on the decompositions of the cartesian product of two complete graphs was by Myers [73] in 1972. He proved that product $K_n \times K_n$, is the sum of $n - 1$ spanning cycles. In 1978, Foregger [36] proved another useful result regarding the

hamilton cycle systems of the cartesian product of two complete graphs. Kötzig [64] proved that if G is the cartesian product of any regular graphs and one of the following conditions is true; at least one of the regular graphs has a 1-factorization or there exists at least two regular graphs containing a 1-factor then the graph G has a 1-factorization, in 1979. Wallis [90] proved that there exists a 1-factorization of the cartesian product of the Petersen graph with K_3 . In 1991, Stong [87] proved that the if G_1 and G_2 be graphs that are decomposable into n and m Hamilton cycles, respectively, with $n \leq m$. Then $G_1 \times G_2$ is Hamilton decomposable if one of the following holds: (1) $m \leq 3n$, (2) $n \geq 3$, (3) $|G_1|$ is even, or (4) $|G_2| \geq 6\lceil m/n \rceil - 3$, where $\lceil x \rceil$ is the least integer greater than or equal to x . Hoffman and Pike [46] gave necessary and sufficient conditions on n and m such that there exists a 4-cycle system of $K_n \times K_n$ in 1998. Chen [18] gave the formula for obtaining the number of spanning trees in the cartesian product graph of paths (or cycles) in 2003. Farrell and Pike [29] carried forward the research of Hoffman and Pike on cycle systems of the cartesian product of two complete graphs. In 2003, they gave necessary and sufficient conditions for constructing a 6-cycle system of $K_n \times K_m$. In Chapter 4, we give different and innovative constructions for the same result. In the following year Pike and Swain gave necessary and sufficient conditions for constructing an 8-cycle system of $K_n \times K_m$. Finally, Graham and Pike gave results about the maximum packings (minimum coverings) of $K_n \times K_m$ with 4-cycles in 2005 [39].

Fu and Huang [35] gave the conditions to find a maximum packing of $K_m \times K_n$ with hexagons in 2005. In Chapter 3, we give necessary and sufficient conditions for constructing the maximal *fair* 6-cycle system of $K_m \times K_n$.

We now, summarize the results mentioned in each of the following chapters to give the reader a flavor of the work done by the author in this dissertation.

Chapter 2: In this chapter we give necessary and sufficient conditions for obtaining a 6-cycle system of $K_m \times K_n$ which yields a 2-path covering of $K(m, n)$. In other words, we construct a (C_6, P_2) 1-covering of $K_m \times K_n$ which leads us to a (C_6, P_3) 1-covering of $K(m, n)$. We define the concept of *fairness* in order to do so. Constructing the *fair* 6-cycle system of $K_m \times K_n$ aids in the development of a 2-path covering of $K(m, n)$.

Chapter 3: After constructing a *fair* 6-cycle system of $K_m \times K_n$ in Chapter 2, we then focused our attention on the next best possible result regarding *fair* 6-cycle systems. So in this chapter, we give necessary and sufficient conditions to construct a maximum *fair* 6-cycle system of $K_m \times K_n$.

Chapter 4: In Chapter 4, we revisited the problem of finding the 6-cycle system of $K_m \times K_n$. We give necessary and sufficient conditions to solve this problem using unique and innovative constructions.

Chapter 5: In Chapter 5, we continued the work done by Fu and Rodger on 4-cycle systems of $K_n - F$, where F is any 2-regular leave. Taking this result further in literature, we found necessary and sufficient conditions for obtaining the 4-cycle system of $K_n - F^*$, where F^* is any nearly 2-regular leave. This result required several constructions for small values of n , which have been explained in detail in this chapter.

Chapter 6: Here we continued the work done for the author's Masters Thesis. We continued to solve the problem of obtaining the 4-cycle system of $L(K(a_1, a_2, \dots, a_p))$. Some new results about this problem are given here. For the background work on this problem the readers are referred to [76].

□

Chapter 2

6-cycle system of the Cartesian Product of $K_x \times K_y$ covering 2-paths in $K_{x,y}$

2.1 Introduction

In this chapter, the bipartite graph B with the bipartition $\{S, T\}$ of $V(B)$ is denoted by $K(S, T)$. Let $L(B)$ denote the line graph of B . The Cartesian product of two graphs G_1 and G_2 is denoted by $G_1 \times G_2$. The vertices of $K_i \times K_j$ will be considered as an array with i rows and j columns unless otherwise stated. We can also view the line graph of B as the Cartesian product of two complete graphs K_s and K_t . A k -path is a path having k edges and is traditionally denoted by P_{k+1} . An m -cycle is a cycle having m edges and is denoted by C_m .

Recall that (H, J) λ -covering of G is defined to be a set U of copies of the subgraph H in G such that each copy of J in G is contained in exactly λ copies of H in the set U . The problem of finding (H, J) λ -coverings of G for the case when $G = K_n$, $\lambda = 1$ and J is a 2-path have been solved for the following graphs H .

1. H is a 3-path [42, 69].
2. H is a 4-cycle [43].
3. H is a 4-path [55, 70]. The case where $G = K_n - E(P)$ for some P where $|E(P)|$ is as small as possible (this is known as the packing problem) has been solved for these values of H and J in [44].
4. H is a 5-path [58], and the case where $G = K_{2n}$ for these graphs H and J was solved in [57].
5. H is a 6-path [1].

6. H is a 6-cycle [59].

The problem of finding a (C_n, P_3) 1-covering of K_n is equivalent to solving the Dudeney problem. In 1905, Dudeney posed the problem [26] which asks for a seating arrangement of n people on $(n - 1)(n - 2)/2$ days such that no person can have the same two neighbors in more than one seating arrangement. This problem was solved for the case when n is even in [53]. In this case the set U of the copies of C_n in K_n is known as the Dudeney set. It is difficult to solve this problem for the case when n is odd. Hence, the problem of obtaining a double Dudeney set, D' was considered. A double Dudeney set is a set of hamilton cycles, C_n such that each P_3 lies in exactly two hamilton cycles in D' . In other words this problem asks for a (C_n, P_3) 2-covering of K_n . This problem was solved in [56]. The problem of finding a (C_n, P_3) 1-covering of K_{2n} was solved in [52].

Some embedding results have also been obtained. An (H, J) λ -covering, U_1 of a graph G_1 is said to be embedded in an (H, J) λ -covering, U_2 of a graph G_2 if the set U_1 of copies of the graph H in G_1 is a subset of the set U_2 of the copies of the graph H in G_2 . The problem of embedding a (P_4, P_3) 1-covering of K_n or $K_n - p$ into a (P_4, P_3) 1-covering of K_{n+m} or $K_{n+m} - p$, where p is a path of length 2 has been solved in [71].

There has also been some progress with respect to finding resolvable (C_m, P_3) λ -coverings of K_n . A resolvable (H, J) λ -covering of G is an (H, J) λ -covering of G with the added condition that the set U of the copies of H can be partitioned into classes such that each vertex of G occurs in exactly one graph in each class. In [43], the authors found the necessary and sufficient conditions for obtaining a resolvable (C_4, P_3) λ -covering of K_n for all λ . In [60], the problem of finding a resolvable (C_4, P_3) 1-covering of K_n was solved. This question was taken further in [61], where the authors solved the problem of obtaining a resolvable (C_k, P_3) 1-covering of K_n where $n = p^e + 1$, p is an odd prime and k is a divisor of n .

A cycle in $G_1 \times G_2$ is said to be *fair* if it has at most two vertices in each row and in each column. Notions of *fairness* in graph decompositions have arisen in various forms,

such as *equitable* [13] and *gregarious* [8] decompositions. A *fair* 6-cycle system of a graph G is defined to be a 6-cycle system of G in which each 6-cycle is *fair*. Here an added incentive is that a *fair* k -cycle in $K_s \times K_t = L(K(S, T))$ naturally corresponds to a cycle of length k in $K(S, T)$ as the following shows (see Lemma 7). In this chapter, we find necessary and sufficient conditions and give the required constructions to obtain a *fair* (C_6, P_2) 1-covering of $K_s \times K_t$ which yields a (C_6, P_3) 1-covering of $K(S, T)$. So, the motivation behind this result was our observation that a *fair* 6-cycle in $K_s \times K_t$ corresponds to a 6-cycle in $K(S, T)$ which covers six 2-paths in $K(S, T)$ corresponding to the 6 edges of the *fair* 6-cycle in $K_s \times K_t$, thus leading eventually to a construction of a (C_6, P_3) 1-covering of $K(S, T)$.

2.2 Notation

Let $\mathbb{N}_i = \{1, 2, \dots, i\}$. The number of 2-paths in the graph G is denoted by $P_3(G)$. In a 2-path, say (a, b, c) , the vertex, b , in the middle of the path is called the *middle* vertex. The number of 2-paths having their *middle* vertex in a set T is denoted by $m(T)$. It will help the reader to picture the graph $K_s \times K_t$ as a $\mathbb{Z}_s \times \mathbb{Z}_t$ array. An edge $\{(u_1, v_1), (u_2, v_2)\} \in E(K_s \times K_t)$ is called a *horizontal* edge if $u_1 = u_2$ and $\{v_1, v_2\} \in E(K_t)$. Similarly, an edge is called a *vertical* edge if $v_1 = v_2$ and $\{u_1, u_2\} \in E(K_s)$. Define $\text{diff}_p\{i, j\} = \min\{|i - j|, |p - i - j|\}$.

2.3 Preliminary Results

In this section we show that there exists a *fair* 6-cycle system of $K_s \times K_t$ for some small values of s and t .

Lemma 1. *There are 6-cycles in a 6-cycle system of $K_s \times K_t$*

Proof

Lemma 2. *There exists a fair 6-cycle system of $K_4 \times K_4$*

Proof Let $V(K_4 \times K_4) = \mathbb{Z}_4 \times \mathbb{Z}_4$. There are eight 6-cycles in every 6-cycle system of $K_4 \times K_4$. Define the following sets of 6-cycles,

$$\begin{aligned}
C_0 &= \{((0, j), (0, j + 1), (1, j + 1), (1, j + 2), (i, j + 2), (i, j)) \\
&\quad | i = 2 \text{ and } j \in \mathbb{Z}_2 \text{ or } i = 3 \text{ and } j \in \mathbb{Z}_4 \setminus \mathbb{Z}_2\} \\
C_1 &= \{((3, j), (3, j + 1), (2, j + 1), (2, j + 2), (i, j + 2), (i, j)) \\
&\quad | i = 0 \text{ and } j \in \mathbb{Z}_2 \text{ or } i = 1 \text{ and } j \in \mathbb{Z}_4 \setminus \mathbb{Z}_2\}
\end{aligned}$$

Note that each 6-cycle in C_0 and C_1 is *fair*. It is easy to check that each edge of $K_4 \times K_4$ occurs in a 6-cycle. Thus, $(V(K_4 \times K_4), \cup_{i \in \mathbb{Z}_2} C_i)$ is a *fair* 6-cycle system of $K_4 \times K_4$.

Lemma 3. *There exists a fair 6-cycle system of $K_6 \times K_6$*

Proof Let $V(K_6 \times K_6) = \mathbb{Z}_6 \times \mathbb{Z}_6$. There are thirty 6-cycles in every 6-cycle system of $K_6 \times K_6$. To obtain the required 6-cycle system, we make use of the difference edges in K_6 . We first construct three 6-cycles using the *vertical* and *horizontal* difference two edges only. We denote that set of 6-cycles by C_0 .

$$\begin{aligned}
C_0 &= \{((i, j), (i, j + 2), (i + 2, j + 2), (i + 2, j + 4), (i + 4, j + 4), \\
&\quad (i + 4, j)) | i = 0 \text{ and } j \in \{0, 2, 4\}\}
\end{aligned}$$

Next, we construct nine 6-cycles using *vertical* and *horizontal* difference one and two edges. We denote these by C_8, C_{10} and C_{12} . For each $i \in \{0, 2, 4\}$ let

$$\begin{aligned}
C_{i+8} &= \{((i, j), (i, j + 1), (i + 4, j + 1), (i + 4, j + 5), (i + 5, j + 5), \\
&\quad (i + 5, j)) | j \in \{0, 2, 4\}\}
\end{aligned}$$

Now, we construct eighteen 6-cycles using the *vertical* and *horizontal* difference one, two and three edges. Thus, we get the following sets of three cycles of length 6 each. For each $i \in \mathbb{Z}_6$ let

$$\begin{aligned}
C_{i+1} &= \{((i, j), (i, j + 1), (i + 1, j + 1), (i + 1, j + 3), (i + 3, j + 3), \\
&\quad (i + 3, j)) | j \in \{1, 3, 5\}\}
\end{aligned}$$

Thus, we have obtained thirty 6-cycles in $K_6 \times K_6$. All the 6-cycles given above are *fair*. One can easily check that each edge of $K_6 \times K_6$ occurs in a 6-cycle. Hence, $(V(K_6 \times K_6), \cup_{i \in \{\mathbb{Z}_7 \cup \{8, 10, 12\}\}} C_i)$ is a *fair* 6-cycle system of $K_6 \times K_6$.

Lemma 4. *There exists a fair 6-cycle system of $K_3 \times K_3$*

Proof Let $V(K_3 \times K_3) = \mathbb{Z}_3 \times \mathbb{Z}_3$. $(V(K_3 \times K_3), \{((0, 0), (0, 1), (2, 1), (2, 2), (1, 2), (1, 0)), ((0, 0), (0, 2), (1, 2), (1, 1), (2, 1), (2, 0)), ((0, 1), (0, 2), (2, 2), (2, 0), (1, 0), (1, 1))\})$ is a fair 6-cycle system of $K_3 \times K_3$.

Lemma 5. *There exists a fair 6-cycle system of $K_7 \times K_7$*

Proof Let $V(K_7 \times K_7) = \mathbb{Z}_7 \times \mathbb{Z}_7$. There are forty-nine 6-cycles in every 6-cycle system of $K_7 \times K_7$. The construction for this Lemma makes use of the method of differences.

Consider the following sets of ordered triples given in Table 2.1. As unordered sets these triples would form a *Steiner Triple System* on seven points, so we refer to this as an ordered *STS(7)*. These ordered triples are chosen so that they satisfy the properties

P_1 : For each $x, y \in \mathbb{Z}_7, x \neq y$ there is exactly one ordered triple containing x and y

P_2 : For each $i \in \mathbb{Z}_7$ and for each $j \in \mathbb{Z}_3$ there is exactly one triple containing i in position j

Table 2.1: An ordered STS(7)

(0,1,3)	(4,5,0)
(1,2,4)	(5,6,1)
(2,3,5)	(6,0,2)
(3,4,6)	

Making use of the ordered *STS(7)*, we construct a 6-cycle system of $K_7 \times K_7$, as follows. For each ordered triple (x, y, z) in Table 2.1 we construct the following set of 6-cycles:

$$C_{x,y,z} = \{((0 + i, x), (0 + i, y), (1 + i, y), (1 + i, z), (3 + i, z), (3 + i, x)) \mid i \in \mathbb{Z}_7\}$$

Any edge in $K_s \times K_t$ is of the form $\{(i_1, x), (i_2, x)\}$ or $\{(i, x), (i, y)\}$. First consider an edge of the form $\{(i_1, x), (i_2, x)\}$. If $\text{diff}_p(i_1, i_2) = 1, 2, 3$, then this edge occurs in a cycle in the set $C_{a,b,c}$ where $x = b, c$ and a respectively. Such an ordered triple exists by P_2 . Now, consider an edge of the form $\{(i, x), (i, y)\}$. By P_1 there exists exactly one ordered triple (a, b, c) such that $\{x, y\} \subset \{a, b, c\}$. Then $\{(i, x), (i, y)\}$ is in a cycle in $C_{a,b,c}$. We know that in the $STS(7)$ each difference edge appears exactly once and by ordering the $STS(7)$, we ensure that each difference edge is used once to construct a 6-cycle (this follows from the above discussion). Hence, the 6-cycles in $C_{x,y,z}$ are all edge disjoint.

Thus, we have obtained forty-nine 6-cycles and the reader can check that each of them is *fair*. Clearly, one can see that each edge of $K_7 \times K_7$ occurs in a 6-cycle. So, $(V(K_7 \times K_7), \cup_{(x,y,z) \in STS(7)} C_{x,y,z})$ is a *fair* 6-cycle system of $K_7 \times K_7$.

Lemma 6. *There exists a fair 6-cycle system of $K_9 \times K_9$*

Proof Let $V(K_9 \times K_9) = \mathbb{Z}_9 \times \mathbb{Z}_9$. In every 6-cycle system of $K_9 \times K_9$ there are 108 6-cycles.

Now, consider the following ordered *Steiner Triple System* on nine points, ordered $STS(9)$ denoted in Table 2.2. These ordered triples are chosen so that they satisfy the properties

P_1 : For each $x, y \in \mathbb{Z}_9, x \neq y$ there is exactly one ordered triple containing x and y

P_2 : For each $x \in \mathbb{Z}_9$ and for each $k \in \{1, 2, 3, 4\}$ there is exactly one ordered triple (a_0, a_1, a_2) containing x , say $x = a_i$, such that $k = \text{diff}_9\{a_i - a_{i-1}\}$ reducing the subscript mod 3

Making use of this ordered $STS(9)$, we obtain a 6-cycle system of $K_9 \times K_9$, as shown below. For each ordered triple (x, y, z) in Table 2.2, we construct the following set of 6-cycles where the calculations in the first co-ordinate are done (mod 9).

$$C_{x,y,z} = \{((0 + i, x), (0 + i, y), ((y - x) + i, y), ((y - x) + i, z), ((z - x) + i, z), ((z - x) + i, x)) \mid i \in \mathbb{Z}_9\}$$

Table 2.2: An ordered STS(9)

(0,1,2)	(0,3,6)	(0,4,8)	(2,6,4)
(3,4,5)	(1,4,7)	(3,7,2)	(3,1,8)
(6,7,8)	(2,5,8)	(6,1,5)	(0,7,5)

Any edge in $K_s \times K_t$ is of the form $\{(i_1, x), (i_2, x)\}$ or $\{(i, x), (i, y)\}$. First consider an edge of the form $\{(i_1, x), (i_2, x)\}$. If $\text{diff}_9(i_1, i_2) = 1, 2, 3, 4$, then by the property P_2 this edge occurs in exactly one cycle in the set C_{a_0, a_1, a_2} , where $x = a_i$ such that $k = \text{diff}_9\{a_i - a_{i-1}\}$, reducing the subscript (mod 3) respectively. Now, consider an edge of the form $\{(i, x), (i, y)\}$. By P_1 there exists exactly one ordered triple (a, b, c) such that $\{x, y\} \subset \{a, b, c\}$. Then $\{(i, x), (i, y)\}$ is in a cycle in $C_{a, b, c}$. As seen before, each difference edge in the $STS(9)$ appears exactly once and by ordering the $STS(9)$, we can ensure that each difference edge is used once to construct a 6-cycle (follows from the construction). Hence, the 6-cycles in $C_{x, y, z}$ are all edge disjoint.

Thus, we get 108 6-cycles and each of these 6-cycles is *fair*. It is easy to check that each edge of $K_9 \times K_9$ occurs in a 6-cycle. Hence, $(V(K_9 \times K_9), \cup_{(x, y, z) \in STS(9)} C_{x, y, z})$ is a *fair* 6-cycle system of $K_9 \times K_9$.

Lemma 7. *There exists a fair 6-cycle system of $K_s \times K_t$ if and only if there exists a (C_6, P_3) 1-covering of $K(s, t)$*

Proof. Define $V(K_s \times K_t) = \mathbb{Z}_s \times \mathbb{Z}_t$. Now, let us we assume that there exists a *fair* 6-cycle system of $K_s \times K_t$, $(\mathbb{Z}_s \times \mathbb{Z}_t, C)$. Any 6-cycle, $c \in C$ is of the form

$$c = ((x_0, y_0), (x_0, y_1), (x_1, y_1), (x_1, y_2), (x_2, y_2), (x_2, y_0))$$

where $x_i \in \mathbb{Z}_s, y_i \in \mathbb{Z}_t$ for $i \in \mathbb{Z}_3$.

Let $c' = (x_0, y_0, x_1, y_1, x_2, y_2)$ be the cycle in $K(S, T)$ that corresponds to c in $K_s \times K_t$. Define $C' = \{c' \mid c \in C\}$. Let (x_1, x_2, x_3) be a P_3 in $K(S, T)$; without loss of generality we can assume that $\{x_1, x_3\} \subset \mathbb{Z}_s$ and $x_2 \in \mathbb{Z}_t$. Let c_i be the cycle in C containing the edge $\{(x_1, x_2), (x_3, x_2)\}$. Then since c_i is *fair*,

$$c_i = ((x_1, x_2), (x_3, x_2), (x_3, x_4), (x_5, x_4), (x_5, x_6), (x_1, x_6))$$

for some $x_5 \in V(S)$, and $\{x_4, x_6\} \subset V(T)$

Corresponding to this c , there exists $c' = (x_1, x_2, x_3, x_4, x_5, x_6) \in C'$. Since, each 2-path occurs in at least one 6-cycle in C' , and since each c' covers 6 paths of length 2, it suffices to prove that

$$\begin{aligned} 6 | C' | &= 6 | C | = m(K(S, T)) \\ &= s \binom{t}{2} + t \binom{s}{2} \end{aligned}$$

So, let us count the number of *fair* 6-cycles in $K_s \times K_t$.

$$\begin{aligned} | C | &= st(s + t - 2)/12 \\ &= s \binom{t}{2} + t \binom{s}{2} / 6 \end{aligned}$$

Hence, proved.

Now, suppose that there exists a (C_6, P_3) 1-covering of $K(s, t)$. Then we will show that there exists a *fair* 6-cycle system of $K_s \times K_t$. By the reasoning given above each P_3 in $K(s, t)$ corresponds to an edge in $K_s \times K_t$ (no two paths of length 2 correspond to the same edge in $K_s \times K_t$). Clearly, a 6-cycle in $K(s, t)$ corresponds to a 6-cycle in $K_s \times K_t$ as shown earlier and by definition each vertex in the 6-cycle is distinct. Thus, each 6-cycle in $K(s, t)$ corresponds to a *fair* 6-cycle in $K_s \times K_t$ and a 1-covering of $K(x, y)$ implies that each edge in $K_s \times K_t$ is covered exactly once. Hence, if there exists a (C_6, P_3) 1-covering of $K(s, t)$ then there exists a *fair* 6-cycle system of $K_s \times K_t$.

Corollary 1. *There exists a (C_6, P_3) 1-covering of $K(4, 4)$.*

Proof The proof of this Corollary follows directly by applying Lemma 2 and Lemma 7.

Corollary 2. *There exists a (C_6, P_3) 1-covering of $K(6, 6)$.*

Proof The proof of this Corollary follows directly by applying Lemma 3 and Lemma 7.

Corollary 3. *There exists a (C_6, P_3) 1-covering of $K(7, 7)$.*

Proof The proof of this Corollary follows directly by applying Lemma 5 and Lemma 7.

Corollary 4. *There exists a (C_6, P_3) 1-covering of $K(9, 9)$.*

Proof The proof of this Corollary follows directly by applying Lemma 6 and Lemma 7.

2.4 Propositions

In this section we give the generalized constructions for the results obtained in the previous section. These constructions are then used to prove our main results.

Proposition 1. *There exists a fair 6-cycle system of $K_{6x} \times K_{6x}$*

Proof This result will be completed in the future work. □

Corollary 5. *There exists a (C_6, P_3) 1-covering of $K(6x, 6x)$*

Proof The proof of this Corollary follows by using Lemma 7 and Proposition 1.

Proposition 2. *There exists a fair 6-cycle system of $K_{6x+4} \times K_{6x+4}$*

Proof This result will be completed in the future work. □

Corollary 6. *There exists a (C_6, P_3) 1-covering of $K(6x + 4, 6x + 4)$*

Proof The proof of this Corollary follows by using Lemma 7 and Proposition 2.

Proposition 3. *There exists a fair 6-cycle system of $K_{6x+1} \times K_{6x+1}$*

Proof Let $V(K_{6x+1} \times K_{6x+1}) = \mathbb{Z}_{6x+1} \times \mathbb{Z}_{6x+1}$. There are $x(6x + 1)$, 6-cycles in every 6-cycle system of $K_{6x+1} \times K_{6x+1}$. We know that there exists a *Steiner Triple System* on $6x + 1 \equiv 1 \pmod{6}$ points. Now consider $STS(\mathbb{Z}_{6x+1})$. The total number of triples in this $STS(6x+1)$ is given by

$$\begin{aligned} &= \frac{((6x+1)(6x+1-1))/2}{3} \\ &= \frac{(6x+1)(3x)}{3} \\ &= (6x + 1)(x) \end{aligned}$$

Using each triple in $STS(\mathbb{Z}_{6x+1})$, we construct a 6-cycle system of $K_{6x+1} \times K_{6x+1}$ as follows. Let (a, b, c) be any triple in the $STS(\mathbb{Z}_{6x+1})$, then construct the following 6-cycles using it.

$$C_{a,b,c} = \{(i, a), (i, a + b), (i + b, a + b), (i + b, a + b + c), \\ (i + b + c, a + b + c), (i + b + c, a)\} \mid \text{for } i \in \mathbb{Z}_{6x+1}\}$$

Thus, we get $(6x + 1)^2(x)$ cycles of length 6 in $K_{6x+1} \times K_{6x+1}$ by this construction. Clearly, each 6-cycle in the above given cycle system is *fair*. Thus, there exists a *fair* 6-cycle system of $K_{6x+1} \times K_{6x+1}$. \square

Corollary 7. *There exists a (C_6, P_3) 1-covering of $K(6x + 1, 6x + 1)$*

Proof The proof of this Corollary follows by using Lemma 7 and Proposition 3.

Proposition 4. *There exists a fair 6-cycle system of $K_{6x+3} \times K_{6x+3}$*

Proof This result will be completed in the future work. \square

Corollary 8. *There exists a (C_6, P_3) 1-covering of $K(6x + 3, 6x + 3)$*

Proof The proof of this Corollary follows by using Lemma 7 and Proposition 4.

2.5 Main Results

Theorem 2.1. *There exists a fair 6-cycle system of $K_x \times K_y$ if and only if*

1. $x = y$

2. *If x is even then,*

(a) $x \equiv 0 \pmod{6}$ or

(b) $x \equiv 4 \pmod{6}$

3. *If x is odd then,*

(a) $x \equiv 1 \pmod{6}$ or

(b) $x \equiv 3 \pmod{6}$

Proof To prove the necessity of these conditions, we first count the total number of 2-paths in $K(X, Y)$. Let $V(X) = \{1, 2 \dots x\}$ and $V(Y) = \{1, 2 \dots y\}$. Clearly, each 2-path has the *middle vertex* in set X or in set Y . In order to count the number of 2-paths with the *middle vertex* in set X , first choose the *middle vertex*. Then, we have to choose two vertices from set Y to complete our path of length 2. Thus, $|P_3(X)|$ is given by

$$= x \times \binom{y}{2}$$

Similarly, $|P_3(Y)|$ is

$$= y \times \binom{x}{2}$$

So,

$$|P_3(K(X, Y))| = |P_3(X)| + |P_3(Y)|$$

Thus, $|P_3(K(X, Y))|$

$$\begin{aligned} &= (y \times \binom{x}{2}) + (x \times \binom{y}{2}) \\ &= \frac{xy}{2}(x-1+y-1) \\ &= \frac{xy}{2}(x+y-2) \end{aligned}$$

The number of 6-cycles in a 6-cycle system of $K_X \times K_Y$ is given by

$$\begin{aligned} &= (x \binom{y(y-1)}{2} + y \binom{x(x-1)}{2})/6 \\ &= (xy(x+y-2))/12 \end{aligned}$$

Suppose, that each 6-cycle lies on the same row or column of the graph $K_X \times K_Y$.

Such a 6-cycle covers fifteen 2-paths in $K(X, Y)$. So, the total number of 2-paths covered in $K(X, Y)$ by these types of 6-cycles is

$$(15 \lfloor \frac{(xy)(x+y-2)}{2} \rfloor) / 6 \neq \frac{xy(x+y-2)}{2}$$

Hence, it is not possible, to construct the required 6-cycle system using cycles of this type. Suppose, each 6-cycle in our 6-cycle system covers $*$ 2-paths exactly once, then clearly,

$$\left\{ \binom{xy(x+y-2)}{2} / 6 \right\} \times * = \frac{xy(x+y-2)}{2}$$

So, clearly,

$$* = 6$$

Thus, each 6-cycle should cover exactly six distinct 2-paths. The only way to do that is if each 6-cycle is a *fair* 6-cycle as shown below.

insert pic

Such a 6-cycle covers three 2-paths with the *middle vertex* in set $X = \{1, 2, \dots, x\}$ and three 2-paths with the *middle vertex* in set $Y = \{1, 2, \dots, y\}$ in $K_{X,Y}$. So,

$$\begin{aligned} \text{Number of 2-paths covered with the } \textit{middle vertex} \text{ in set } X &= \\ \text{Number of 2-paths covered with the } \textit{middle vertex} \text{ in set } Y & \end{aligned}$$

Thus,

$$|P_3(X)| = |P_3(Y)|$$

So,

$$\begin{aligned} (y \times \binom{x}{2}) &= [x \times \binom{y}{2}] \\ \Rightarrow y \binom{(x)(x-1)}{2} &= x \binom{(y)(y-1)}{2} \\ \Rightarrow x - 1 &= y - 1 \\ \Rightarrow x &= y \end{aligned}$$

Thus, $x = y$, which proves the necessity of condition 1. Now, suppose that x is even, then $x \equiv 0, 2 \text{ or } 4 \pmod{6}$. First consider the case, where $x \equiv 2 \pmod{6}$, say $x = 6a + 2$. Then the number of cycles of length 6 in $K_{6a+2} \times K_{6a+2}$ is

$$\begin{aligned}
&= (2(6a + 2) \times \frac{(6a+2)(6a+2-1)}{2})/6 \\
&= \frac{(6a+2)(6a+2)(6a+1)}{6} \\
&= \frac{2(3a+1)2(3a+1)(6a+1)}{6}
\end{aligned}$$

However, this is NOT an integer. So, $x \neq 6a + 2$. Now suppose, $x \equiv 0 \pmod{6}$. Say, $x = 6a$. Then the number of 6-cycles in $K_{6a} \times K_{6a}$ will be given by

$$\begin{aligned}
&= (2(6a) \times \frac{(6a)(6a-1)}{2})/6 \\
&= \frac{(6a)(6a)(6a-1)}{6} \\
&= (a)(6a)(6a - 1)
\end{aligned}$$

Thus, it is possible for $x \equiv 0 \pmod{6}$. $x \equiv 4 \pmod{6}$ is also possible by a similar reasoning. This proves the necessity of condition 2.

Now, suppose that x is odd, then $x \equiv 1, 3$ or $5 \pmod{6}$. First consider the case, where $x \equiv 5 \pmod{6}$, say $x = 6a + 5$. Then the number of cycles of length 6 in $K_{6a+5} \times K_{6a+5}$ is

$$\begin{aligned}
&= (2(6a + 5) \times \frac{(6a+5)(6a+5-1)}{2})/6 \\
&= \frac{(6a+5)(6a+5)(6a+4)}{6} \\
&= \frac{(6a+5)(6a+5)2(3a+2)}{6}
\end{aligned}$$

However, this is NOT an integer. So, $x \neq 6a + 5$. But, $x \equiv 1$ or $3 \pmod{6}$ by similar calculations. This proves the necessity of condition 3.

Now, to prove the sufficiency of these conditions, we make use of the earlier results in this chapter. Suppose, that x is even, clearly $x = y$ by condition 1. Then by condition 1,

1. $x \equiv 0 \pmod{6}$ or
2. $x \equiv 4 \pmod{6}$

Case 1 Let $x \equiv 0 \pmod{6}$, say $x = 6a$. Then using Proposition 1, there exists a *fair* 6-cycle system of $K_{6a} \times K_{6a}$.

Case 2 Let $x \equiv 4 \pmod{6}$, say $x = 6a + 4$. Then using Proposition 2, there exists a *fair* 6-cycle system of $K_{6a+4} \times K_{6a+4}$.

Now suppose, that x is *odd* and $x = y$ (by condition 1) and by condition 3,

1. $x \equiv 1 \pmod{6}$ or

2. $x \equiv 3 \pmod{6}$

Case 1 Let $x \equiv 1 \pmod{6}$, say $x = 6a + 1$. Then using Proposition 3, there exists a *fair* 6-cycle system of $K_{6a+1} \times K_{6a+1}$.

Case 2 Let $x \equiv 3 \pmod{6}$, say $x = 6a + 3$. Then using Proposition 4, there exists a *fair* 6-cycle system of $K_{6a+3} \times K_{6a+3}$.

This proves our Theorem. □

Theorem 2.2. *There exists a (C_6, P_3) 1-covering of $K(x, y)$ if and only if*

1. $x = y$

2. *If x is even then,*

(a) $x \equiv 0 \pmod{6}$ or

(b) $x \equiv 4 \pmod{6}$

3. *If x is odd then,*

(a) $x \equiv 1 \pmod{6}$ or

(b) $x \equiv 3 \pmod{6}$

Proof The proof of this Theorem follows from Lemma 7, Theorem 2.1 and Corollaries 5, 6, 7, and 8. □

Chapter 3

Maximum *fair* 6-cycle system of the Cartesian Product of two Complete Graphs $K_x \times K_y$

3.1 Introduction

In this chapter, we find necessary and sufficient conditions for obtaining a maximal *fair* 6-cycle system of $K_s \times K_t$. An m -cycle is a cycle having m edges. An m -cycle system of a graph G is denoted by the ordered set $(V(G), C)$, where $V(G)$ is the vertex set of G and C is a set of cycles of length m that partition the edge set of G , $E(G)$. A complete graph on s vertices is denoted by K_s . The cartesian product of two complete graphs K_s and K_t is denoted by $K_s \times K_t$. The problem of finding m -cycle systems of K_s has been solved for many values of m . For a survey of these results, we refer the reader to [79] (we will focus on the results involving 6-cycle systems in particular). D. Sotteau [84] proved a prominent result regarding m -cycle systems of $K(s, t)$, where $K(s, t)$ is the complete bipartite graph. A lot of work has been done on the problem for finding m -cycle systems of K_s minus some edges, called the leave. In [5], the authors found necessary and sufficient conditions for obtaining a 6-cycle system of $K_s - E(F)$, where F , the leave, is any spanning forest of K_s . The problem of finding a 6-cycle system of $K_s - E(L)$, where the leave L is any 2-regular (not necessarily spanning) subgraph of K_s was solved in [6]. The spectrum problem for finding a 6-cycle system of $L(K_s)$ was solved in [22].

An m -cycle system of a graph G , say C , is said to be maximal if $E(G) - E(C)$ does not contain an m -cycle. There have some results regarding a maximal set of m -cycle systems too. In 2000, the authors [14] found necessary and sufficient conditions for obtaining a maximal set of hamilton cycles in $K(s, s)$. In 2003, this problem was extended by

finding the maximal set of hamilton cycles in the complete multipartite graph, K_s^p (p parts of size s), [24]. The authors published the next part of this paper in 2007, [49]. In 2008 [33] the problem of obtaining the maximal set of hamilton cycles in $K_{2p} - F$ was solved. The latest result with respect to maximal m -cycle systems is [74], where the problem of finding the maximal set of hamilton cycles in the directed complete graph, D_s is solved. In 2006 [37], the authors solved the problem of finding the maximal cyclic 4-cycle packings and the minimal cyclic 4-cycle coverings of K_n .

The definitions of a *fair* m -cycle and a *fair* m -cycle system of a graph G can be found in the previous chapter. A maximum *fair* 6-cycle system of $K_s \times K_t$, C is defined to be a 6-cycle system of $K_s \times K_t$ containing the most number of *fair* 6-cycles among all 6-cycle systems of $K_s \times K_t$. A 6-cycle system (V, C_1) is said to be a maximum *fair* 6-cycle system if for each of the 6-cycle systems (V, C_2) of $K_s \times K_t$, (V, C_1) has at least as many *fair* 6-cycles as (V, C_2) . In the previous chapter we found the necessary and sufficient conditions for obtaining a *fair* 6-cycle system of $K_s \times K_t$ (which then gave us a (C_6, P_3) 1-covering of $K(S, T)$). So, the next natural problem to consider was to maximize the number of *fair* 6-cycles among the 6-cycle systems of $K_s \times K_t$. Thus, the results obtained in the previous chapter were a motivation for this chapter. In the next section of this chapter, we give the reader some useful notation. Then we give some preliminary results. Later, we give constructions for some propositions which will be used to prove our main results. Finally, we use all the tools constructed in the previous sections to prove our main Theorem.

3.2 Notation

The reader should refer to the notation section of the previous chapter for the notations used in this chapter. $MF(s, t)$ denotes the maximum number of disjoint *fair* 6-cycles in a 6-cycle system of $K_s \times K_t$. It will help the reader to picture the vertices of $K_s \times K_t$ in an $s \times t$ array in which if the vertex set $M(i | \mathbb{N}_j)$ is used then the vertex in position (i, j) of

the array is $x(i-1) + j$. All the edges in a graph, G except *vertical* and *horizontal* are said to be the *diagonal* edges. We have stated an important theorem by Sotteau that is used in our constructions below.

Theorem 3.1. [84] *There exists a 4-cycle system of $K_{a,b}$ and of $2K_{a,b}$ if and only if each vertex has even degree, the number of edges is divisible by 4, and $a, b \geq 2$.*

3.3 Preliminary Results

In this section we construct maximum *fair* 6-cycle systems of $K_s \times K_t$ for some small values of s and t . Before we proceed to those constructions, we will show the method for calculating the maximum number of *fair* 6-cycles in a 6-cycle system of $K_s \times K_t$. Since, the case where $s \geq t$ is solved in the previous chapter, here we can assume that $s < t$.

Lemma 8. *Suppose that $s \leq t$. Then the number of edge disjoint fair 6-cycles in $K_s \times K_t$ is at most $MF(s, t) = \lfloor ts(s-1)/6 \rfloor - \delta$ where*

$$\begin{aligned} \delta &= 2 \quad \text{if } ts(s-1)/2 = 4(\text{mod}6) \quad \text{and} \\ \delta &= 0 \quad \text{otherwise.} \end{aligned}$$

Proof Each *fair* 6-cycle uses an equal number of *vertical* and *horizontal* edges. Thus, in a *fair* 6-cycle, there are $6/2 = 3$ *vertical* edges. Similarly, there are $6/2 = 3$ *horizontal* edges. By our assumption $s \leq t$. So, the number of *vertical* edges is at most the number of *horizontal* edges, since, $|E(K_s)| \leq |E(K_t)|$. So, the maximum number of *fair* 6-cycles depends on the number of *vertical* edges. Thus, the maximum number of *fair* 6-cycles is

$$t(|E(K_s)|)/3 = ts(s-1)/6.$$

Lemma 9. *There exists a maximal fair 6-cycle system of $K_4 \times K_6$ that contains $MF(4, 6)$ fair 6-cycles*

Proof Let $V(K_4 \times K_6) = \mathbb{Z}_4 \times \mathbb{Z}_6$. There are 16 6-cycles in every 6-cycle system of $K_4 \times K_6$. By applying Lemma 8, the maximum number of *fair* 6-cycles in $K_4 \times K_6$ is

$MF(4, 6) = 36/3 = 12$. These can be obtained from the proof of Lemma 2 in the previous chapter. So, the remaining 6-cycles ($16 - 12 = 4$) can be any cycles of length 6.

Lemma 10. *There exists a maximal fair 6-cycle system of $K_4 \times K_{10}$ that contains $MF(4, 10)$ fair 6-cycles*

Proof Let $V(K_4 \times K_{10}) = \mathbb{Z}_4 \times \mathbb{Z}_{10}$. There are 40 6-cycles in every 6-cycle system of $K_4 \times K_{10}$. The maximum number of *fair* 6-cycles in $K_4 \times K_{10}$ is $60/3 = 20$, using Lemma 8. In order, to find the required maximal *fair* 6-cycle system, we embed one copy of $K_4 \times K_4$ and one copy of $K_4 \times K_6$ into $K_4 \times K_{10}$. Let $V(K_4 \times K_4) = \mathbb{Z}_4 \times \mathbb{Z}_4$ and $V(K_4 \times K_6) = \mathbb{Z}_4 \times (\mathbb{Z}_{10} \setminus \mathbb{Z}_4)$. We know that there exists a maximal *fair* 6-cycle system $(\mathbb{Z}_4 \times \mathbb{Z}_4, C_0)$ of $K_4 \times K_4$, from the proof of Lemma 2 given in Chapter 2. This system contains 8 *fair* 6-cycles. By applying Lemma 9, we know that there exists a maximum *fair* 6-cycle system $(\mathbb{Z}_4 \times (\mathbb{Z}_{10} \setminus \mathbb{Z}_4), C_1)$ of $K_4 \times K_6$ containing 12 *fair* 6-cycles. Thus, in all, we have $8 + 12 = 20 = ts(s - 1)/6$ *fair* 6-cycles. Now to complete the rest of the maximal *fair* 6-cycle system, we note that there is one copy of $K(4, 6)$ on each row of the array of vertices $\mathbb{Z}_4 \times \mathbb{Z}_{10}$. And to decompose $K(4, 6)$ into the 6-cycle system $(K(\{\{i\} \times \mathbb{Z}_4\}, \{\{i\} \times \mathbb{Z}_{10} \setminus \mathbb{Z}_4\}), C_{i+2})$ for each $i \in \mathbb{Z}_4$, we make use Theorem 3.1. This embedding of $K_4 \times K_4$ and $K_4 \times K_6$ in $\mathbb{Z}_4 \times \mathbb{Z}_{10}$, facilitated the construction of the required maximum *fair* 6-cycle system $(\mathbb{Z}_4 \times \mathbb{Z}_{10}, \cup_{i \in \mathbb{Z}_7} C_i)$ of $K_4 \times K_{10}$.

Lemma 11. *There exists a maximal fair 6-cycle system of $K_5 \times K_9$ that contains $MF(5, 9)$ fair 6-cycles*

Proof There are 45 6-cycles in every 6-cycle system of $K_5 \times K_9$. By Lemma 8 an upper bound for the maximum number of *fair* 6-cycles in any maximum *fair* 6-cycle system of $K_5 \times K_9$ is $ts(s - 1)/6 = 30$. To construct this maximum *fair* 6-cycle system with 30 *fair* 6-cycles begin by, considering two parallel classes of triples on $V(K_{\mathbb{Z}_9}) = M(3 | \mathbb{Z}_9)$.

$$\pi_1 = (i, i + 3, i + 6) \quad \forall i \in \{0, 1, 2\}$$

$$\pi_2 = (j, j + 1, j + 2) \quad \forall j \in \{0, 3, 6\}$$

We use each of these triples to induce 6 base cycles (π_1 induces B_0, B_1 and B_2 . And π_2 induces the other three) in $K_5 \times K_9$. And these base 6-cycles are rotated cyclically, which give rise to thirty *fair* 6-cycles in $K_5 \times K_9$. There are 180 *horizontal* edges and 90 *vertical* edges in $K_5 \times K_9$. To construct this cycle system, we assign subscripts to these triples to form our base cycles. These subscripts guide us in using the vertical edges in $K_5 \times K_9$. Now, let $V(K_5 \times K_9) = G(\mathbb{Z}_5 \times \mathbb{Z}_9)$.

$$\begin{aligned}\pi'_1 &= (\bigcup_{\forall i \in \{0,1\}} (i_1, i + 3_1, i + 6_2) \cup (2_2, 5_2, 8_1)) \\ \pi'_2 &= (\bigcup_{\forall j \in \{0,3\}} (j_2, j + 1_2, j + 2_1) \cup (6_1, 7_1, 8_2))\end{aligned}$$

For example, the base cycle B_0 is formed from the triple $(0_1, 3_1, 6_2)$ as follows:

$$B_0 = ((0, 0), (0 + 1, 0), (1, 0 + 3), (1 + 1, 3), (2, 0 + 6), (2 - 2, 6))$$

The other base blocks formed similarly are listed below:

$$B_i = \begin{cases} ((0, 1), (1, 1), (1, 4), (2, 4), (2, 7), (0, 7)) & \text{for } i = 1 \\ ((0, 2), (1, 2), (1, 5), (2, 5), (2, 8), (0, 8)) & \text{for } i = 2 \\ ((0, 0), (2, 0), (2, 1), (4, 1), (4, 2), (0, 2)) & \text{for } i = 3 \\ ((0, 2), (2, 2), (2, 3), (4, 3), (4, 4), (0, 4)) & \text{for } i = 4 \\ ((0, 6), (2, 6), (2, 7), (4, 7), (4, 8), (0, 8)) & \text{for } i = 5 \end{cases}$$

On vertical rotation each B_i gives rise to a set of corresponding *fair* 6-cycles C_i (say), given below:

$$C_i = \begin{cases} \bigcup_{i \in \mathbb{Z}_5} ((0+i, 0), (1+i, 0), (1+i, 3), (2+i, 3), (2+i, 6), (0+i, 6)) \\ \text{for } i = 0 \\ \bigcup_{i \in \mathbb{Z}_5} ((0+i, 1), (1+i, 1), (1+i, 4), (2+i, 4), (2+i, 7), (0+i, 7)) \\ \text{for } i = 1 \\ \bigcup_{i \in \mathbb{Z}_5} ((0+i, 2), (1+i, 2), (1+i, 5), (2+i, 5), (2+i, 8), (0+i, 8)) \\ \text{for } i = 2 \\ \bigcup_{i \in \mathbb{Z}_5} ((0+i, 0), (2+i, 0), (2+i, 1), (4+i, 1), (4+i, 2), (0+i, 2)) \\ \text{for } i = 3 \\ \bigcup_{i \in \mathbb{Z}_5} ((0+i, 2), (2+i, 2), (2+i, 3), (4+i, 3), (4+i, 4), (0+i, 4)) \\ \text{for } i = 4 \\ \bigcup_{i \in \mathbb{Z}_5} ((0+i, 6), (2+i, 6), (2+i, 7), (4+i, 7), (4+i, 8), (0+i, 8)) \\ \text{for } i = 5 \end{cases}$$

To form the triples, we had used all the *horizontal* edges in $K_5 \times K_9$. Now, to obtain the remaining 6-cycles, we go back to the structure of $M(3 \mid \mathbb{Z}_9)$ and make use of the *diagonal* edges in that graph. And, we get 15 6-cycles, three on each of the five rows in $K_5 \times K_9$ from the following cycle system.

$$C_6 = \{(i, j+6, i+3, j, i+6, j+3) \mid \text{for all } (i, j) \in \{(0+k, 1), (0+k, 2), (1+k, 2)\}, k \in \mathbb{Z}_5\}$$

Thus, $(V(K_5 \times K_9), \cup_{i \in \mathbb{Z}_7} C_i)$ is a maximal *fair* 6-cycle system of $K_5 \times K_9$.

Lemma 12. *There exists a maximal fair 6-cycle system of $K_5 \times K_{21}$ that contains $MF(5, 21)$ fair 6-cycles*

Proof There are 210 6-cycles in every 6-cycle system of $K_5 \times K_{21}$. By applying Lemma 8, there are at most 70 *fair* 6-cycles in any maximum *fair* 6-cycle system of $K_5 \times K_{21}$. To construct a system with 70 *fair* 6-cycles, consider two parallel classes of triples on $V(K_{\mathbb{Z}_{21}}) = M(7 \mid \mathbb{Z}_{21})$.

$$\pi_1 = (\bigcup_{i \in \mathbb{Z}_5} (i, i + 8, i + 16) \quad \cup \quad (5, 13, 14) \quad \cup \quad (6, 7, 15))$$

$$\pi_2 = (\bigcup_{i \in \mathbb{Z}_8 \setminus \mathbb{Z}_3} (j, j + 6, j + 12) \quad \cup \quad (10, 13, 19) \quad \cup \quad (1, 7, 20))$$

We use each of these triples to induce 14 base cycles (π_1 induces B_0, \dots, B_7 . And π_2 induces the other seven) in $K_5 \times K_{21}$. There are 1050 *horizontal* edges and 210 *vertical* edges in $K_5 \times K_{21}$. Then we will rotate these base cycles cyclically, to get 70 *fair* 6-cycles in $K_5 \times K_{21}$. First, we assign subscripts to these triples to form our base cycles. These subscripts help us use the *vertical* edges in $K_5 \times K_{21}$. For this, let $V(K_5 \times K_{21}) = G(\mathbb{Z}_5 \times \mathbb{Z}_{21})$.

$$\pi'_1 = (\bigcup_{i \in \mathbb{Z}_5} (i_1, i + 8_1, i + 16_2) \quad \cup \quad (5_1, 13_1, 14_2) \quad \cup \quad (6_1, 7_1, 15_2))$$

$$\pi'_2 = (\bigcup_{j \in \mathbb{Z}_8 \setminus \mathbb{Z}_3} (j_2, j + 6_2, j + 12_1) \quad \cup \quad (10_2, 13_2, 19_1) \quad \cup \quad (1_2, 7_2, 20_1))$$

Now, for example, the base cycle B_0 is formed from the triple $(0_1, 8_1, 16_2)$ follows:

$$B_0 = ((0, 0), (0 + 1, 0), (1, 0 + 8), (1 + 1, 8), (2, 8 + 8), (2 - 2, 16))$$

The other base blocks formed similarly are listed below:

$$B_i = \left\{ \begin{array}{ll} ((0, 1), (1, 1), (1, 9), (2, 9), (2, 17), (0, 17)) & \text{for } i = 1 \\ ((0, 2), (1, 2), (1, 10), (2, 10), (2, 18), (0, 18)) & \text{for } i = 2 \\ ((0, 3), (1, 3), (1, 11), (2, 11), (2, 19), (0, 19)) & \text{for } i = 3 \\ ((0, 4), (1, 4), (1, 12), (2, 12), (2, 20), (0, 20)) & \text{for } i = 4 \\ ((0, 5), (1, 5), (1, 13), (2, 13), (2, 14), (0, 14)) & \text{for } i = 5 \\ ((0, 6), (1, 6), (1, 7), (2, 7), (2, 15), (0, 15)) & \text{for } i = 6 \\ ((0, 2), (2, 2), (2, 8), (4, 8), (4, 14), (0, 14)) & \text{for } i = 7 \\ ((0, 3), (2, 3), (2, 9), (4, 9), (4, 15), (0, 15)) & \text{for } i = 8 \\ ((0, 4), (2, 4), (2, 10), (4, 10), (4, 16), (0, 16)) & \text{for } i = 9 \\ ((0, 5), (2, 5), (2, 11), (4, 11), (4, 17), (0, 17)) & \text{for } i = 10 \\ ((0, 6), (2, 6), (2, 12), (4, 12), (4, 18), (0, 18)) & \text{for } i = 11 \\ ((0, 0), (2, 0), (2, 13), (4, 13), (4, 19), (0, 19)) & \text{for } i = 12 \\ ((0, 1), (2, 1), (2, 7), (4, 7), (4, 20), (0, 20)) & \text{for } i = 13 \end{array} \right.$$

After vertical rotation each B_i gives rise to a set of corresponding *fair* 6-cycles C_j (say), given as below:

$$C_j = \left\{ \begin{array}{l} \bigcup_{i \in \mathbb{Z}_5} ((0 + i, j), (1 + i, j), (1 + i, j + 8), (2 + i, j + 8), \\ (2 + i, j + 16), (0 + i, j + 16)) \text{ for } j \in \mathbb{Z}_5 \\ \bigcup_{i \in \mathbb{Z}_5} ((0 + i, 5), (1 + i, 5), (1 + i, 13), (2 + i, 13), (2 + i, 14), \\ (0 + i, 14)), j = 5 \\ \bigcup_{i \in \mathbb{Z}_5} ((0 + i, 6), (1 + i, 6), (1 + i, 7), (2 + i, 7), (2 + i, 15), \\ (0 + i, 15)), j = 6 \\ \bigcup_{i \in \mathbb{Z}_5} ((0 + i, j - 5), (2 + i, j - 5), (2 + i, j + 1), (4 + i, j + 1), \\ (4 + i, j + 7), (0 + i, j + 7)) \text{ for } j \in \mathbb{Z}_{12} \setminus \mathbb{Z}_7 \\ \bigcup_{i \in \mathbb{Z}_5} ((0 + i, 0), (2 + i, 0), (2 + i, 13), (4 + i, 13), (4 + i, 19), \\ (0 + i, 19)), j = 12 \\ \bigcup_{i \in \mathbb{Z}_5} ((0 + i, 1), (2 + i, 1), (2 + i, 7), (4 + i, 7), (4 + i, 20), \\ (0 + i, 20)), j = 13 \end{array} \right.$$

At this point, we have used up all the vertical edges in $K_5 \times K_{21}$. Now, to obtain the remaining 6-cycles, we go back to the structure of $M(7 \mid \mathbb{Z}_{21})$. And embed one copy of the 6-cycle system of $K_3 \times K_7$ corresponding to $M(7 \mid \mathbb{Z}_{21})$ on each of the five rows of of $K_5 \times K_{21}$. There are 14, 6-cycles in $K_3 \times K_7$. Thus, we get $14 \times 5 = 70$, 6-cycles by this embedding for $i \in \mathbb{Z}_5$.

- $C_{i+14} =$ 6-cycle system of $K_3 \times K_7$ corresponding to $M(7 \mid \mathbb{Z}_{21})$ on $(\mathbb{Z}_{i+1} \setminus \mathbb{Z}_i) \times \mathbb{Z}_{21}$

Then we consider the unused edges in $M(7 \mid \mathbb{Z}_{21})$ as shown below to finish this proof.

- $C_\alpha = ((i, j + 14, i + 7, j, i + 14, j + 7) \mid ((i, j) = (m, m + 3),$
 $m \in \mathbb{Z}_4) \cup ((i, j) = (m + 4, m), m \in \mathbb{Z}_3)$

C_α gives us 7, 6-cycles. Now, place a copy of this set of 6-cycles on each of the *five* rows in $K_5 \times K_{21}$. So, let

- C_{19} = 6-cycle obtained by embedding a copy of C_α on each of the 5 rows in $K_5 \times K_{21}$

There are, $7 \times 5 = 35$, 6-cycles in C_{19} .

- $C_\beta = ((i, j + 14, i + 6, j, i + 14, j + 8) \mid (i, j) = (m, m + 1),$
 $m \in \mathbb{Z}_6 \setminus \mathbb{Z}_1) \cup (0, 15, 13, 1, 14, 9) \cup (6, 14, 12, 0, 20, 8)$

Now, embed a copy of C_β in each of the five rows in $K_5 \times K_{21}$, and call the new set of 6-cycles obtained, C_{20} . So,

- C_{20} = 6-cycle obtained by embedding a copy of C_β on each of the 5 rows in $K_5 \times K_{21}$

C_{20} also gives us $7 \times 5 = 35$ 6-cycles. So, now we have 210 6-cycles. And thus, $(V(K_5 \times K_{21}), \cup_{i \in \mathbb{Z}_{21}} C_i)$ is a maximum *fair* 6-cycle system of $K_5 \times K_{21}$.

Lemma 13. *There exists a maximum fair 6-cycle system of $K_5 \times K_{33}$ that contains $MF(5, 33)$ fair 6-cycles*

Proof There are 495 6-cycles in every 6-cycle system of $K_5 \times K_{33}$. The maximum number of *fair* 6-cycles in a maximum *fair* 6-cycle system of $K_5 \times K_{33}$ is $330/3 = 110$, using Lemma 8. To construct this maximum *fair* 6-cycle system, consider two parallel classes of triples on $V(K_{\mathbb{Z}_{33}}) = M(11 \mid \mathbb{Z}_{33})$.

$$\pi_1 = (\cup_{i \in \mathbb{Z}_8} (i, i + 12, i + 24) \quad \cup \quad (9, 21, 22) \quad \cup \quad (10, 11, 23))$$

$$\pi_2 = (\cup_{i \in \mathbb{Z}_{10} \setminus \mathbb{Z}_2} (j, j + 10, j + 20) \quad \cup \quad (0, 21, 31) \quad \cup \quad (1, 11, 32))$$

We use each of these triples to induce 22 base cycles (π_1 induces B_0, \dots, B_{11} . And π_2 induces the other eleven) in $K_5 \times K_{33}$. There are 2640 *horizontal* edges and 330 *vertical* edges in $K_5 \times K_{33}$. And these base *fair* 6-cycles are rotated cyclically, which give rise to 110 *fair* 6-cycles in $K_5 \times K_{33}$. First, we assign subscripts to these triples to form our base cycles, which help us use the *vertical* edges in $K_5 \times K_{33}$. For this, let $V(K_5 \times K_{33}) = G(\mathbb{Z}_5 \times \mathbb{Z}_{33})$.

$$\pi'_1 = (\cup_{i \in \mathbb{Z}_8} (i_1, i + 12_1, i + 24_2) \quad \cup \quad (9_1, 21_1, 22_2) \quad \cup \quad (10_1, 11_1, 23_2))$$

$$\pi'_2 = (\cup_{j \in \mathbb{Z}_{10} \setminus \mathbb{Z}_2} (j_2, j + 10_2, j + 20_1) \quad \cup \quad (0_2, 21_2, 31_1) \quad \cup \quad (1_2, 11_2, 32_1))$$

Now, for example, the base cycle B_0 is formed from the triple $(0_1, 12_1, 24_2)$ follows:

$$B_0 = ((0, 0), (0 + 1, 0), (1, 0 + 12), (1 + 1, 12), (2, 12 + 12), (2 - 2, 24))$$

The other base blocks formed similarly are listed below:

$$B_i = \left\{ \begin{array}{ll} ((0, 1), (1, 1), (1, 13), (2, 13), (2, 25), (0, 25)) & \text{for } i = 1 \\ ((0, 2), (1, 2), (1, 14), (2, 14), (2, 26), (0, 26)) & \text{for } i = 2 \\ ((0, 3), (1, 3), (1, 15), (2, 15), (2, 27), (0, 27)) & \text{for } i = 3 \\ ((0, 4), (1, 4), (1, 16), (2, 16), (2, 28), (0, 28)) & \text{for } i = 4 \\ ((0, 5), (1, 5), (1, 17), (2, 17), (2, 29), (0, 29)) & \text{for } i = 5 \\ ((0, 6), (1, 6), (1, 18), (2, 18), (2, 30), (0, 30)) & \text{for } i = 6 \\ ((0, 7), (1, 7), (1, 19), (2, 19), (2, 31), (0, 31)) & \text{for } i = 7 \\ ((0, 8), (1, 8), (1, 20), (2, 20), (2, 32), (0, 32)) & \text{for } i = 8 \\ ((0, 9), (1, 9), (1, 21), (2, 21), (2, 22), (0, 22)) & \text{for } i = 9 \\ ((0, 10), (1, 10), (1, 11), (2, 11), (2, 23), (0, 23)) & \text{for } i = 10 \\ ((0, 2), (2, 2), (2, 12), (4, 12), (4, 22), (0, 22)) & \text{for } i = 11 \\ ((0, 3), (2, 3), (2, 13), (4, 13), (4, 23), (0, 23)) & \text{for } i = 12 \\ ((0, 4), (2, 4), (2, 14), (4, 14), (4, 24), (0, 24)) & \text{for } i = 13 \\ ((0, 5), (2, 5), (2, 15), (4, 15), (4, 25), (0, 25)) & \text{for } i = 14 \\ ((0, 6), (2, 6), (2, 16), (4, 16), (4, 26), (0, 26)) & \text{for } i = 15 \\ ((0, 7), (2, 7), (2, 17), (4, 17), (4, 27), (0, 27)) & \text{for } i = 16 \\ ((0, 8), (2, 8), (2, 18), (4, 18), (4, 28), (0, 28)) & \text{for } i = 17 \\ ((0, 9), (2, 9), (2, 19), (4, 19), (4, 29), (0, 29)) & \text{for } i = 18 \\ ((0, 10), (2, 10), (2, 20), (4, 20), (4, 30), (0, 30)) & \text{for } i = 19 \\ ((0, 0), (2, 0), (2, 21), (4, 21), (4, 31), (0, 31)) & \text{for } i = 20 \\ ((0, 1), (2, 1), (2, 11), (4, 11), (4, 32), (0, 32)) & \text{for } i = 21 \end{array} \right.$$

After rotation each B_i gives rise to a set of *fair* 6-cycles C_j (say), given as below:

$$C_j = \left\{ \begin{array}{ll} \bigcup_{i \in \mathbb{Z}_5} ((0+i, 0), (1+i, 0), (1+i, 12), (2+i, 12), (2+i, 24), (0+i, 24)), & j = 0 \\ \bigcup_{i \in \mathbb{Z}_5} ((0+i, 1), (1+i, 1), (1+i, 13), (2+i, 13), (2+i, 25), (0+i, 25)), & j = 1 \\ \bigcup_{i \in \mathbb{Z}_5} ((0+i, 2), (1+i, 2), (1+i, 14), (2+i, 14), (2+i, 26), (0+i, 26)), & j = 2 \\ \bigcup_{i \in \mathbb{Z}_5} ((0+i, 3), (1+i, 3), (1+i, 15), (2+i, 15), (2+i, 27), (0+i, 27)), & j = 3 \\ \bigcup_{i \in \mathbb{Z}_5} ((0+i, 4), (1+i, 4), (1+i, 16), (2+i, 16), (2+i, 28), (0+i, 28)), & j = 4 \\ \bigcup_{i \in \mathbb{Z}_5} ((0+i, 5), (1+i, 5), (1+i, 17), (2+i, 17), (2+i, 29), (0+i, 29)), & j = 5 \\ \bigcup_{i \in \mathbb{Z}_5} ((0+i, 6), (1+i, 6), (1+i, 18), (2+i, 18), (2+i, 30), (0+i, 30)), & j = 6 \\ \bigcup_{i \in \mathbb{Z}_5} ((0+i, 7), (1+i, 7), (1+i, 19), (2+i, 19), (2+i, 31), (0+i, 31)), & j = 7 \\ \bigcup_{i \in \mathbb{Z}_5} ((0+i, 8), (1+i, 8), (1+i, 20), (2+i, 20), (2+i, 32), (0+i, 32)), & j = 8 \\ \bigcup_{i \in \mathbb{Z}_5} ((0+i, 9), (1+i, 9), (1+i, 21), (2+i, 21), (2+i, 22), (0+i, 22)), & j = 9 \\ \bigcup_{i \in \mathbb{Z}_5} ((0+i, 10), (1+i, 10), (1+i, 11), (2+i, 11), (2+i, 23), (0+i, 23)), & j = 10 \\ \bigcup_{i \in \mathbb{Z}_5} ((0+i, 2), (2+i, 2), (2+i, 12), (4+i, 12), (4+i, 22), (0+i, 22)), & j = 11 \\ \bigcup_{i \in \mathbb{Z}_5} ((0+i, 3), (2+i, 3), (2+i, 13), (4+i, 13), (4+i, 23), (0+i, 23)), & j = 12 \\ \bigcup_{i \in \mathbb{Z}_5} ((0+i, 4), (2+i, 4), (2+i, 14), (4+i, 14), (4+i, 24), (0+i, 24)), & j = 13 \\ \bigcup_{i \in \mathbb{Z}_5} ((0+i, 5), (2+i, 5), (2+i, 15), (4+i, 15), (4+i, 25), (0+i, 25)), & j = 14 \\ \bigcup_{i \in \mathbb{Z}_5} ((0+i, 6), (2+i, 6), (2+i, 16), (4+i, 16), (4+i, 26), (0+i, 26)), & j = 15 \\ \bigcup_{i \in \mathbb{Z}_5} ((0+i, 7), (2+i, 7), (2+i, 17), (4+i, 17), (4+i, 27), (0+i, 27)), & j = 16 \\ \bigcup_{i \in \mathbb{Z}_5} ((0+i, 8), (2+i, 8), (2+i, 18), (4+i, 18), (4+i, 28), (0+i, 28)), & j = 17 \\ \bigcup_{i \in \mathbb{Z}_5} ((0+i, 9), (2+i, 9), (2+i, 19), (4+i, 19), (4+i, 29), (0+i, 29)), & j = 18 \\ \bigcup_{i \in \mathbb{Z}_5} ((0+i, 10), (2+i, 10), (2+i, 20), (4+i, 20), (4+i, 30), (0+i, 30)), & j = 19 \\ \bigcup_{i \in \mathbb{Z}_5} ((0+i, 0), (2+i, 0), (2+i, 21), (4+i, 21), (4+i, 31), (0+i, 31)), & j = 20 \\ \bigcup_{i \in \mathbb{Z}_5} ((0+i, 1), (2+i, 1), (2+i, 11), (4+i, 11), (4+i, 32), (0+i, 32)), & j = 21 \end{array} \right.$$

At this point, we have used up all the vertical edges in $K_5 \times K_{33}$. Now, to obtain the remaining 6-cycles, we go back to the structure of $M(11 \mid \mathbb{Z}_{33})$. And embed the 6-cycle system of $K_3 \times K_{11}$ corresponding to $(M(11 \mid \mathbb{Z}_{33}), \cup_{i \in \mathbb{Z}_{27} \setminus \mathbb{Z}_{22}} C_i)$ on each of the five rows

of $K_5 \times K_{33}$. There are 33 6-cycles in every 6-cycle system of $K_3 \times K_{11}$. Thus, we get $33 \times 5 = 165$ 6-cycles by this embedding. For $i \in \mathbb{Z}_5$

- $C_{i+22} =$ 6-cycle system of $K_3 \times K_{11}$ corresponding to $M(11 \mid \mathbb{Z}_{33})$ on $\mathbb{Z}_{i+1} \setminus \mathbb{Z}_i \times \mathbb{Z}_{33}$

Now, we consider the unused edges in $M(11 \mid \mathbb{Z}_{33})$ as shown below to finish this proof.

- $C_\alpha = ((i, j + 11, i + 22, j, i + 11, j + 22) \mid \{(i, j) = (m, m + 5), m \in \mathbb{Z}_6\} \cup \{(i, j) = (m, m + 6), m \in \mathbb{Z}_5\})$

There are 11 6-cycles in C_α . Now we, embed one copy of C_α in each of the five rows in $K_5 \times K_{33}$.

- $C_{27} =$ 6-cycle obtained by embedding one copy of C_α on each of the 5 rows in $K_5 \times K_{33}$

Thus, C_{27} gives us $11 \times 5 = 55$ 6-cycles.

- $C_\beta = ((i, j + 22, i + 10, j, i + 22, j + 12) \mid \{(i, j) = (m, m + 1), m \in \mathbb{Z}_{10} \setminus \mathbb{Z}_1\} \cup (0, 23, 21, 1, 22, 13) \cup (10, 22, 20, 0, 32, 12))$

Now, embed one copy of C_β in each of the five rows of $K_5 \times K_{33}$, to get C_{28} .

- $C_{28} =$ 6-cycle obtained by embedding one copy of C_β on each of the 5 rows in $K_5 \times K_{33}$

And, we get $11 \times 5 = 55$ 6-cycles from C_{28} .

- $C_\gamma = ((i, j + 22, i + 10, j, i + 22, j + 11) \mid \{(i, j) = (m, m + 3), m \in \mathbb{Z}_8 \setminus \mathbb{Z}_1\} \cup \{(i, j) = (m + 8, m), m \in \mathbb{Z}_2\})$

On embedding C_γ into the five rows in $K_5 \times K_{33}$, we get another $11 \times 5 = 55$ 6-cycles, say C_{29} .

- $C_{29} =$ 6-cycle obtained by embedding one copy of C_γ on each of the 5 rows in $K_5 \times K_{33}$

- $C_\delta = ((i, j + 22, i + 10, j, i + 22, j + 11) \mid \{(i, j) = (m, m + 3), m \in \mathbb{Z}_8 \setminus \mathbb{Z}_1\} \cup \{(i, j) = (m + 8, m), m \in \mathbb{Z}_2\})$

And, finally we get, $11 \times 5 = 55$ 6-cycles by embedding C_δ in each of the five rows in $K_5 \times K_{33}$, C_{30} .

- $C_{30} = 6$ -cycle obtained by embedding a copy of C_δ on each of the 5 rows in $K_5 \times K_{33}$

Thus, we have 495 6-cycles now. Thus, $(V(K_5 \times K_{33}), \cup_{i \in \mathbb{Z}_{30}} C_i)$ a maximum *fair* 6-cycle system of $K_5 \times K_{33}$.

Lemma 14. *There exists a maximum fair 6-cycle system of $K_3 \times K_7$*

Proof There are 14, 6-cycles in every 6-cycle system of $K_3 \times K_7$. Let $V(K_3 \times K_7) = \mathbb{Z}_3 \times \mathbb{Z}_7$. In this case, the number of *fair* 6-cycles is $21/3 = 7$, using Lemma 8. But, it is not possible to pull out seven *fair* 6-cycles in the *-cycle system of $K_3 \times K_7$.

Thus, the maximum number of *fair* 6-cycles possible in the maximum *fair* 6-cycle system of $K_3 \times K_7$ is five. Let $V(K_3 \times K_7) = G(\mathbb{Z}_3 \times \mathbb{Z}_7)$.

- $C_0 = \cup_{i \in \mathbb{Z}_5} ((0, i), (0, i + 1), (1, i + 1), (1, i + 2), (2, i + 2), (2, i))$

The remaining 6-cycles in the maximum *fair* 6-cycle system of $K_3 \times K_7$ are obtained as follows:

- $C_1 = \{((0, 0), (0, 5), (0, 6), (1, 6), (1, 1), (1, 0)), ((1, 0), (1, 3), (1, 1), (2, 1), (2, 6), (2, 0))\}$
- $C_2 = \{((0, 0), (0, 4), (0, 1), (0, 5), (0, 2), (0, 6)), ((0, 0), (0, 3), (0, 1), (0, 6), (0, 4), (0, 2))\}$
- $C_3 = \{((1, 0), (1, 4), (1, 1), (1, 5), (1, 2), (1, 6)), ((1, 0), (1, 5), (1, 3), (1, 6), (1, 4), (1, 2))\}$
- $C_4 = \{((2, 0), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5)), ((2, 0), (2, 3), (2, 6), (2, 5), (2, 1), (2, 4))\}$

And so, $(V(K_3 \times K_7), \cup_{i \in \mathbb{Z}_5} C_i)$ is a maximum *fair* 6-cycle system of $K_3 \times K_7$.

3.4 Proposition

In this section, we give constructions for maximum *fair* 6-cycle systems of $K_s \times K_t$ for some generalized values of s and t .

Proposition 5. *There exists a maximum fair 6-cycle system of $K_{6x+1} \times K_{6x+1}$*

Proof To find the total number of 6-cycles in every 6-cycle system of $K_{6x+1} \times K_{6x+1}$, note that there is one copy of K_{6x+1} on each row in $K_{6x+1} \times K_{6x+1}$. And, there is one copy of K_{6x+1} on each column in $K_{6x+1} \times K_{6x+1}$. Thus, we have $2(6x+1)$ copies of K_{6x+1} in our graph. So, the total number of 6-cycles in every $K_{6x+1} \times K_{6x+1}$ is

$$\begin{aligned} &= \frac{2(6x+1)(6x+1)(6x+1-1)/2}{6} \\ &= \frac{(6x+1)(6x+1)(6x)}{6} \\ &= (x)(6x+3)(6x+1) \end{aligned}$$

Thus, there are $(x)(6x+3)(6x+1)$ 6-cycles in $K_{6x+1} \times K_{6x+1}$. And using Lemma 8, there are $(x)(6x+3)(6x+1)$ *fair* 6-cycles in every maximum *fair* 6-cycle system of $K_{6x+1} \times K_{6x+1}$. Thus, a *fair* 6-cycle system of $K_{6x+1} \times K_{6x+1}$ is the same as the maximal *fair* 6-cycle system of $K_{6x+1} \times K_{6x+1}$. For the construction of a maximum *fair* 6-cycle system of $K_{6x+1} \times K_{6x+1}$, we refer the reader to the previous chapter.

Proposition 6. *There exists a maximum fair 6-cycle system of $K_{6x+3} \times K_{6x+3}$*

Proof We first calculate the total number of 6-cycles in every 6-cycle system of $K_{6x+1} \times K_{6x+1}$. There is one copy of K_{6x+1} on each row in $K_{6x+3} \times K_{6x+3}$. And, there is one copy of K_{6x+1} on each column in $K_{6x+3} \times K_{6x+3}$. Thus, we have $2(6x+3)$ copies of K_{6x+3} in our graph. So, the total number of 6-cycles in every $K_{6x+3} \times K_{6x+3}$ is

$$\begin{aligned} &= \frac{2(6x+3)(6x+3)(6x+3-1)/2}{6} \\ &= \frac{(3)(2x+1)(6x+3)(6x+2)}{6} \\ &= (2x+1)(6x+3)(3x+1) \end{aligned}$$

Thus, there are $(2x + 1)(6x + 3)(3x + 1)$ 6-cycles in $K_{6x+3} \times K_{6x+3}$. And using Lemma 8, there are $(2x + 1)(6x + 3)(3x + 1)$ *fair* 6-cycles in every maximum *fair* 6-cycle system of $K_{6x+3} \times K_{6x+3}$. Thus, a *fair* 6-cycle system of $K_{6x+3} \times K_{6x+3}$ is the same as the maximal *fair* 6-cycle system of $K_{6x+3} \times K_{6x+3}$. For this construction, we refer the reader to the previous chapter. □

Chapter 4

6-cycle system of the Cartesian Product of two Complete Graphs

4.1 Introduction

All graphs considered in this chapter are simple (no loops or multiple edges) and finite. An m -cycle is defined as a cycle on m edges. An m -cycle system of G is a set of cycles of length m , such that each edge in G is contained in exactly one cycle. Considerable amount of research has been done in finding m -cycle systems of a graph G . Initially graph theorists were interested in the case when G was the complete graph on n vertices, denoted by K_n . Alspach and Gavalas found a pivotal result on m -cycle systems of K_n and $K_n - I$, [82].

Work has also been done in the case when the graph under consideration is the cartesian product of two graphs. In the cartesian product $G_1 \times G_2$, of two graphs G_1 and G_2 , (v_1, v_2) is adjacent to (u_1, u_2) iff $v_1 = u_1$ and v_2 is adjacent to u_2 in G_2 or $v_2 = u_2$ and v_1 is adjacent to u_1 in G_1 . There are several results for the problem of finding n -cycle system of $K_n \times K_n$ by [73, 36, 87]. The problem of finding a 1-factorization of $K_m \times K_n$ was solved by Ktzig [63] and Wallis [90]. Chen [18] solved the problem of finding the number of spanning trees in $K_m \times K_n$. The problems for obtaining k -cycle systems of $K_m \times K_n$ has been solved for the cases when $k = 4$ [46], 5 [72], 6 [29] and 8 [77]. Hoffman et. al. solved the problem of finding the 4-cycle system of the Cartesian product of two complete graphs, [46]. The problem of packing $K_m \times K_n$ with k -cycles has been been solved for the cases when $k = 4$ [39] and 6 [35].

In this chapter we find necessary and sufficient conditions for obtaining a 6-cycle system of $K_m \times K_n$. Let \mathbb{N}_x denote the first x natural numbers. For the rest of the notations used in this chapter the reader should refer to the notation section of Chapter 2.

And by definition, a *cycle system* decomposes the edge set of a graph into cycles, such that each edge is contained in exactly one cycle. Here we mention some important results used in this chapter.

Theorem 4.1. [84] *There exists a 4-cycle system of $K_{a,b}$ and of $2K_{a,b}$ if and only if each vertex has even degree, the number of edges is divisible by 4, and $a, b \geq 2$.*

Theorem 4.2. [17] *Billington's theorem*

Theorem 4.3. *Sajna's theorem*

4.2 Preliminary Results

In this section we solve the problem for obtaining a 6-cycle system of $K_s \times K_t$ for some small values of s and t .

Remark 1. The number of 6-cycles in a 6-cycle system of $K_s \times K_t$ is calculated as follows:

$$\begin{aligned} |E(K_s \times K_t)| / 6 &= (t[|E(K_s)|] + s[|E(K_t)|]) / 6 \\ &= (t[\frac{s(s-1)}{2}] + s[\frac{t(t-1)}{2}]) / 6 \end{aligned}$$

Lemma 15. *There exists a 6-cycle system of $K_2 \times K_6$*

Proof Let $V(K_2 \times K_6) = \mathbb{N}_{12}$. Using Remark 1, there are 6 6-cycles in $K_2 \times K_6$. $(V(K_2 \times K_6), \{(1, 3, 2, 4, 10, 7), (3, 5, 4, 6, 12, 9), (1, 2, 8, 11, 5, 6), (1, 4, 3, 6, 2, 5), (7, 11, 10, 9, 8, 12), (7, 9, 11, 12, 10, 8)\})$ is a 6-cycle system of $K_2 \times K_6$.

Lemma 16. *There exists a 6-cycle system of $K_4 \times K_6$*

Proof Let $V(K_4 \times K_6) = \mathbb{Z}_4 \times \mathbb{Z}_6$. There are 16 6-cycles in $K_4 \times K_6$ using Remark 1. The reader should refer to Chapter 3 for the proof of this Lemma.

Lemma 17. *There exists a 6-cycle system of $K_6 \times K_6$*

Proof There are 30 6-cycles in every 6-cycle system of $K_6 \times K_6$. The reader should refer to the proof given in Chapter 2 for this Lemma.

Lemma 18. *There exists a 6-cycle system of $K_4 \times K_4$*

Proof There are 8 6-cycles in every 6-cycle system of $K_4 \times K_4$. The reader should refer to the proof of this Lemma given in Chapter 2.

Lemma 19. *There exists a 6-cycle system of $K_4 \times K_{10}$*

Proof There are 40 6-cycles in the 6-cycle system of $K_4 \times K_{10}$. The reader should refer to Chapter 3 for the proof of this Lemma.

Lemma 20. *There exists a 6-cycle system of $K_{13} \times K_{13}$*

Proof There are 338 6-cycles in the 6-cycle system of $K_{13} \times K_{13}$. The proof of this Lemma is given in Chapter 2.

Lemma 21. *There exists a 6-cycle system of $K_7 \times K_7$*

Proof There are 49 6-cycles in this 6-cycle system. The detailed construction for this proof is given in Chapter 2.

Lemma 22. *There exists a 6-cycle system of $K_3 \times K_3$*

Proof There are 3 6-cycles in this 6-cycle system. The reader should refer to Chapter 2 for the detailed construction of this proof.

Lemma 23. *There exists a 6-cycle system of $K_3 \times K_7$*

Proof Let $V(K_7 \times K_7) = \mathbb{Z}_3 \times \mathbb{Z}_7$. There are 14 6-cycles in the 6-cycle system of $K_3 \times K_7$, using Remark 1. In order, to obtain this cycle system, we embed one copy of $K_2 \times K_6$ in $K_3 \times K_7$. Let $V(K_2 \times K_6) = \mathbb{Z}_2 \times \mathbb{Z}_6$. The 6-cycle system of $K_2 \times K_6$ exists using Lemma

15. So, define $C_0 = 6$ -cycle system of $\mathbb{Z}_2 \times \mathbb{Z}_6$ as a set of 6-cycles. C_0 gives rise to six cycles. Now, construct the following set of 6-cycles in $K_3 \times K_7$.

- $C_1 = ((0, i), (0, 6), (0, i + 1), (2, i + 1), (2, 6), (2, i))$ for $i \in \{1, 3\}$
- $C_2 = ((0, 6), (1, 6), (1, 1), (2, 1), (2, 0), (0, 0))$
- $C_3 = ((0, 5), (0, 6), (2, 6), (1, 6), (1, 5), (2, 5))$
- $C_4 = ((1, 0), (1, 6), (1, 2), (2, 2), (2, 3), (2, 0))$
- $C_5 = ((1, 3), (1, 6), (1, 4), (2, 4), (2, 5), (2, 3))$
- $C_6 = ((2, 0), (2, 4), (2, 1), (2, 2), (2, 5), (2, 6))$
- $C_7 = ((2, 0), (2, 2), (2, 4), (2, 3), (2, 1), (2, 5))$

$(\cup_{i \in \mathbb{N}_7} C_i)$ gives rise to eight 6-cycles. Thus $(V(K_3 \times K_7), \cup_{i \in \mathbb{Z}_8} C_i)$ is a 6-cycle system of $K_3 \times K_7$.

Lemma 24. *There exists a 6-cycle system of $K_3 \times K_{11}$.*

Proof Let $V(K_3 \times K_{11}) = \mathbb{Z}_3 \times \mathbb{Z}_{11}$. Using Remark 1 there are 33 6-cycles in the 6-cycle system of $K_3 \times K_{11}$. In this case, we first embed one copy of $K_1 \times K_9$ in $K_3 \times K_{11}$. Let $V(K_1 \times K_9) = \mathbb{Z}_1 \times \mathbb{Z}_9$. The 6-cycle system of $K_1 \times K_9$ exists by Lemma 25. And using that we construct our first set of 6-cycles, $C_0 = 6$ -cycle system of $G(\mathbb{Z}_1 \times \mathbb{Z}_9)$. C_0 contains 6 6-cycles. Similarly, we embed two more copies of $K_1 \times K_9$. First, let $V(K_1 \times K_9) = (\mathbb{Z}_1 \setminus \mathbb{Z}_0) \times \mathbb{Z}_9$. And based on this we construct a set of 6-cycles, $C_1 = 6$ -cycle system of $G((\mathbb{Z}_1 \setminus \mathbb{Z}_0) \times \mathbb{Z}_9)$. Finally, we embed another copy of $K_1 \times K_9$ in $K_3 \times K_{11}$. And, let, $V(K_1 \times K_9) = (\mathbb{Z}_2 \setminus \mathbb{Z}_1) \times \mathbb{Z}_9$. Let the set of 6-cycles obtained from this be $C_2 = 6$ -cycle system of $G((\mathbb{Z}_2 \setminus \mathbb{Z}_1) \times \mathbb{Z}_9)$. Thus, so far, we have constructed 18 6-cycles, six from each copy of $K_1 \times K_9$ embedded in to $K_3 \times K_{11}$. Now, we embed one copy of $K_3 \times K_3$ in $K_3 \times K_{11}$. The 6-cycle system of $K_3 \times K_3$ exists, using Lemma 22. Let $V(K_3 \times K_3) = \mathbb{Z}_3 \times (\mathbb{Z}_{11} \setminus \mathbb{Z}_8)$. We obtain 3 more 6-cycles from this

embedding, denoted by $C_3 = 6$ -cycle system of $G(\mathbb{Z}_3 \times (\mathbb{Z}_{11} \setminus \mathbb{Z}_8))$. At this stage, we have 21 6-cycles. We, construct the remaining 6-cycles as shown below.

- $C_4 = \{((0, i), (0, 9), (0, i + 1), (1, i + 1), (1, 9), (1, i)) \mid \text{for } i \in \{0, 2, 4, 6\}\}$
- $C_5 = \{((1, i), (1, 10), (1, i + 1), (2, i + 1), (2, 10), (2, i)) \mid \text{for } i \in \{0, 2, 4, 6\}\}$
- $C_6 = \{((0, i), (0, 10), (0, i + 1), (2, j + 1), (2, 9), (2, j)) \mid \text{for } i \in \{0, 2, 4, 6\}\}$

$(\cup_{i \in \mathbb{Z}_7 \setminus \mathbb{Z}_4} C_i)$ contains 12 6-cycles. Thus, $(V(K_3 \times K_{11}), \cup_{i \in \mathbb{Z}_7} C_i)$ is a 6-cycle system of $K_3 \times K_{11}$.

Lemma 25. *There exists a 6-cycle system of $K_1 \times K_9$*

Proof Let $(K_1 \times K_9) = \mathbb{Z}_9$. There are 6 6-cycles in the 6-cycle system of $K_1 \times K_9$ by Remark 1. Thus, $(\mathbb{Z}_9, \{(1, 8, 6, 0, 5, 2), (0, 2, 1, 5, 6, 7), (0, 1, 3, 5, 7, 4), (0, 3, 2, 6, 4, 8), (1, 6, 3, 8, 2, 7), (1, 4, 3, 7, 8, 5)\})$ is 6-cycle system of $K_1 \times K_9$.

Lemma 26. *There exists a 6-cycle system of $K_5 \times K_9$*

Proof There are 45 6-cycles in the 6-cycle system of $K_5 \times K_9$ using Remark 1. The reader should refer to Chapter 2 for the detailed construction for this Lemma.

Lemma 27. *There exists a 6-cycle system of $K_5 \times K_{21}$*

Proof There are 210 6-cycles in the 6-cycle system of $K_5 \times K_{21}$ by Remark 1. The reader should refer to Chapter 2 for this proof.

Lemma 28. *There exists a 6-cycle system of $K_5 \times K_{33}$*

Proof There are 495 6-cycles in the 6-cycle system of $K_5 \times K_{33}$ by Remark 1. The reader should refer to Chapter 2 for this construction.

4.3 Propositions

In this section we give generalized constructions for obtaining 6-cycle systems of $K_s \times K_t$ for some general values of s and t .

Proposition 7. *There exists a 6-cycle system of $K_{6x+2} \times K_{6y}$*

Proof The number of 6-cycles in the 6-cycle system of $K_{6x+2} \times K_{6y}$ is given by,

$$\begin{aligned}
 | E(K_{6x+2} \times K_{6y}) | &= (6x+2)[| E(K_{6y}) |] + (6y)[| E(K_{6x+2}) |] \\
 &= (6x+2)\left[\frac{6y(6y-1)}{2}\right] + (6y)\left[\frac{6x+2(6x+2-1)}{2}\right] \\
 &= (6x+2)[(3y)(6y-1)] + (6y)[(3x+1)(6x+1)] \\
 &= 6(3x+1)(y)[(6y-1) + (6x+1)] \\
 &= 6(3x+1)(y)[6y+6x] \\
 &= 36(3x+1)(y)[x+y]
 \end{aligned}$$

So, there are $6|36(3x+1)(y)[x+y] = 6(3x+1)(y)[x+y]$, 6-cycles in the 6-cycle system of $K_{6x+2} \times K_{6y}$. Now, let $V(K_{6x+2} \times K_{6y}) = G(3x+1, y)$, where each $v_{i,j} \in G(3x+1, y)$ is defined as follows:

$$v_{i,j} = \{K_2 \times K_6 \mid \text{for all } 1 \leq i \leq 3x+1, 1 \leq j \leq y\}$$

Using Lemma 15, there exists a 6-cycle system of $K_2 \times K_6$. The 6-cycle system of $K_2 \times K_6$ contains six 6-cycles. Thus, we get a set of 6-cycles given below.

- $C_0 = 6\text{-cycle system of } v_{i,j} \text{ for all } \{1 \leq i \leq 3x+1, 1 \leq j \leq y\}$

We get, $6(y)(3x+1)$, 6-cycles from C_0 . Now, to obtain the remaining 6-cycles we observe that the following edges do not appear in any 6-cycle yet.

1. There are $\binom{y}{2}$ copies of $K(6, 6)$, on each of the $6x+2$ rows in $K_{6x+2} \times K_{6y}$
2. There is one copy of $K(6x+2) - F$ for each of the $6y$ columns in $K_{6x+2} \times K_{6y}$

Using Sotteau's result, Theorem 4.1 there exists a 6-cycle system of $K(6, 6)$. There are six cycles of length 6, in the 6-cycle system of $K(6, 6)$. This gives us another set of 6-cycles, say C_1 .

- $C_1 = 6$ -cycle system of the $\binom{y}{2}$ copies of $K(6, 6)$, on each of the $6x + 2$ rows in $K_{6x+2} \times K_{6y}$

There are $3y(y-1)(6x+2)$, 6-cycles in C_1 . And, there exists a 6-cycle system of $K(6x+2) - F$, where F is a one factor of K_{6x+2} using Sajna's result, Theorem 4.3. There are $(x)(3x + 1)$, 6-cycles in the 6-cycle system of $K(6x + 2) - F$. Let us denote this set of 6-cycles by C_2 .

- $C_2 = 6$ -cycle system of $K(6x + 2) - F$ one on each of the $6y$ columns in $K_{6x+2} \times K_{6y}$

We get, $6y(x)(3x + 1)$ cycles of length 6, from C_2 . Thus, we have $6y(3x + 1)(x + y)$, 6-cycles now. And, thus $(G(3x + 1, y), \cup_{i \in \mathbb{Z}_3} C_i)$ is a 6-cycle system of $K_{6x+2} \times K_{6y}$.

Proposition 8. *There exists a 6-cycle system of $K_{6x+4} \times K_{6y}$*

Proof Let $V(K_{6x+4} \times K_{6y}) = G(x + 2, y)$, where each $v_{i,j} \in G(x + 2, y)$ is defined as follows:

$$v_{i,j} = \begin{cases} K_4 \times K_6 & \text{for } (i, j) \in G(1, y), \forall y \\ K_6 \times K_6 & \text{for } (i, j) \in G(m, y), 1 \leq m \leq x + 2 \end{cases}$$

And, there exists a 6-cycle system of $K_4 \times K_6$ using Lemma 16. Also, there exists a 6-cycle system of $K_6 \times K_6$, using Lemma 17. Now, to obtain the remaining 6-cycles, we observe that the following edges do not appear in any 6-cycle yet.

1. There are $\binom{y}{2}$ copies of $K(6, 6)$ on each of the $6x + 4$ rows in $K_{6x+4} \times K_{6y}$
2. There are x copies of $K(4, 6)$ and $\binom{x}{2}$ copies of $K(6, 6)$ on each of the $6y$ columns in $K_{6x+4} \times K_{6y}$

And using Sotteau's result, Theorem 4.1 we know that, there exist 6-cycle systems of $K(4, 6)$ and $K(6, 6)$. Thus, there exists a 6-cycle system of $K_{6x+4} \times K_{6y}$.

Proposition 9. *There exists a 6-cycle system of $K_{6x} \times K_{6y}$*

Proof The number of 6-cycles in the 6-cycle system of $K_{6x} \times K_{6y}$ is given by,

$$\begin{aligned}
| E(K_{6x} \times K_{6y}) | &= (6x)[| E(K_{6y}) |] + (6y)[| E(K_{6x}) |] \\
&= (6x)\left[\frac{6y(6y-1)}{2}\right] + (6y)\left[\frac{6x(6x-1)}{2}\right] \\
&= (6x)[(3y)(6y-1)] + (6y)[(3x)(6x-1)] \\
&= 18xy[(6y-1) + (6x-1)] \\
&= 36xy[3y + 3x - 1]
\end{aligned}$$

So, there are $6|36xy[3y + 3x - 1] = 6xy[3y + 3x - 1]$, 6-cycles in the 6-cycle system of $K_{6x} \times K_{6y}$. Let $V(K_{6x+4} \times K_{6y}) = G(x, y)$, where each $v_{i,j} \in G(x, y)$ is defined as follows:

$$v_{i,j} = \{K_6 \times K_6 \mid \text{for all } 1 \leq i \leq x, 1 \leq j \leq y\}$$

Using Lemma 17 there exists a 6-cycle system of $K_6 \times K_6$. And, there are 30, 6-cycles in $K_6 \times K_6$. Let, C_0 be the set of 6-cycles obtained from this.

- $C_0 = 6\text{-cycle system of } v_{i,j} \text{ for all } \{1 \leq i \leq 3x + 1, 1 \leq j \leq y\}$

To obtain the remaining 6-cycles, we observe that the following edges do not appear in any 6-cycle yet.

1. There are $\binom{y}{2}$ copies of $K(6, 6)$ on each of the $6x$ rows in $K_{6x} \times K_{6y}$
2. There are $\binom{x}{2}$ copies of $K(6, 6)$ on each of the $6y$ columns in $K_{6x} \times K_{6y}$

Using Sotteau's result, Theorem 4.1 we know that there exists 6-cycle systems of $K(6, 6)$. There are 6 cycles of length 6 in the 6-cycle system of $K(6, 6)$.

- $C_1 = 6\text{-cycle system of the } \binom{y}{2} \text{ copies of } K(6, 6) \text{ on each of the } 6x \text{ rows in } K_{6x} \times K_{6y}$

Clearly, there are $18xy(y-1)$, 6-cycles in this set. Finally, we form our last set of 6-cycles, C_2 , as given below:

- $C_2 = 6\text{-cycle system of the } \binom{x}{2} \text{ copies of } K(6, 6) \text{ on each of the } 6y \text{ columns in } K_{6x} \times K_{6y}$

So, we have $6xy[3x+3y-1]$, 6-cycles now. And hence, $(G(x, y), \cup_{i \in \mathbb{Z}_3} C_i)$ is a 6-cycle system of $K_{6x} \times K_{6y}$.

Proposition 10. *There exists a 6-cycle system of $K_{6x+4} \times K_{6y+4}$*

Proof Let $V(K_{6x+4} \times K_{6y+4}) = G(x+1, y+1)$, where each $v_{i,j} \in G(x+1, y+1)$ is defined as follows:

$$v_{i,j} = \begin{cases} K_4 \times K_6 & \text{for } (i, j) \in G(m, y+2-m), 1 \leq m \leq x, \forall y \\ K_4 \times K_4 & \text{for } (i, j) = (x+1, 1) \text{ if } x \neq y \\ K_6 \times K_4 & \text{for } (i, j) = (x+1, 1) \text{ if } x = y \\ K_6 \times K_4 & \text{for } (i, j) = (x+1, m), 2 \leq m \leq y+1 \\ K_6 \times K_6 & \text{otherwise} \end{cases}$$

And, there exists 6-cycle systems of $K_4 \times K_4$ (using Lemma 18), $K_4 \times K_6$ (using Lemma 16) and $K_6 \times K_6$ (using Lemma 17). To obtain the remaining 6-cycles, we observe that the following edges do not appear in any 6-cycle yet.

1. There are $\binom{y}{2}$ copies of $K(6, 6)$ and y copies of $K(4, 6)$ on each of the $6x+4$ rows in $K_{6x+4} \times K_{6y+4}$
2. There are $\binom{x}{2}$ copies of $K(6, 6)$ and x copies of $K(4, 6)$ on each of the $6y+4$ columns in $K_{6x+4} \times K_{6y+4}$

And using Sotteau's result, Theorem 4.1 we know that, there exist 6-cycle systems of $K(4, 6)$ and $K(6, 6)$. Hence, there exists a 6-cycle system of $K_{6x+4} \times K_{6y+4}$.

Proposition 11. *There exists a 6-cycle system of $K_{12x+1} \times K_{12y+1}$*

Proof The number of 6-cycles in the 6-cycle system of $K_{12x+1} \times K_{12y+1}$ is given by,

$$\begin{aligned}
| E(K_{12x+1} \times K_{12y+1}) | &= (12x + 1)[| E(K_{12y+1}) |] + \\
&\quad (12y + 1)[| E(K_{12x+1}) |] \\
&= (12x + 1)\left[\frac{(12y+1)(12y)}{2}\right] + \\
&\quad (12y + 1)\left[\frac{(12x+1)(12x)}{2}\right] \\
&= (12x + 1)[(6y)(12y + 1)] + \\
&\quad (12y + 1)[(6x)(12x + 1)] \\
&= 6(12x + 1)(12y + 1)(x + y)
\end{aligned}$$

So, there are $6|6(12x + 1)(12y + 1)(x + y) = (12x + 1)(12y + 1)(x + y)$, 6-cycles in the 6-cycle system of $K_{12x+1} \times K_{12y+1}$. There are $12x + 1$ copies of $K(12y + 1)$ on each of the $12x + 1$ rows in $K_{12x+1} \times K_{12y+1}$. The 6-cycle system of $K(12y + 1)$ exists by Lemma 20. And there are $(12y + 1)(y)$, 6-cycles in the 6-cycle system of $K(12y + 1)$. We denote this set of 6-cycles by C_0 , as given below.

- $C_0 = 6$ -cycle system of the $12x + 1$ copies of $K(12y + 1)$ on each of the $12x + 1$ rows in $K_{12x+1} \times K_{12y+1}$

So, we get $y(12y + 1)(12x + 1)$, 6-cycles in this set. Similarly, there are $12y + 1$ copies of $K(12x + 1)$ on each of the $12y + 1$ columns in $K_{12x+1} \times K_{12y+1}$. And, there exists a 6-cycle system of $K(12x + 1)$ using Lemma 20. There are $(12x + 1)(x)$, 6-cycles in the 6-cycle system of $K(12x + 1)$. Let this set of 6-cycles be C_1 .

- $C_1 = 6$ -cycle system of the $12y + 1$ copies of $K(12x + 1)$ on each of the $12y + 1$ columns in $K_{12x+1} \times K_{12y+1}$

So, we have $(12x + 1)(12y + 1)[x + y]$, 6-cycles now. And so, there exists a 6-cycle system of $K_{12x+1} \times K_{12y+1}$.

Proposition 12. *There exists a 6-cycle system of $K_{12x+7} \times K_{12y+7}$*

Proof The number of 6-cycles in the 6-cycle system of $K_{12x+7} \times K_{12y+7}$ is given by,

$$\begin{aligned}
| E(K_{12x+7} \times K_{12y+7}) | &= (12x + 7)[| E(K_{12y+7}) |] + \\
&\quad (12y + 7)[| E(K_{12x+7}) |] \\
&= (12x + 7)\left[\frac{(12y+7)(12y+7-1)}{2}\right] + \\
&\quad (12y + 7)\left[\frac{(12x+7)(12x+7-1)}{2}\right] \\
&= (12x + 7)[(12y + 7)(6y + 3)] + \\
&\quad (12y + 7)[(12x + 7)(6x + 3)] \\
&= 3(12x + 7)(12y + 7)[2x + 2y + 2] \\
&= 6(12x + 7)(12y + 7)[x + y + 1]
\end{aligned}$$

So, there are $6|6(12x + 7)(12y + 7)[x + y + 1]| = (12x + 7)(12y + 7)[x + y + 1]$, 6-cycles in the 6-cycle system of $K_{12x+7} \times K_{12y+7}$. Let $V(K_{12x+7} \times K_{12y+7}) = G(4x+7, 4y+7)$, where each $v_{i,j} \in$

$$G(4x+7, 4y+7) \text{ is defined as follows: } v_{i,j} = \begin{cases} K_7 \times K_7 & \text{for } (i, j) = (1, 1) \\ K_7 \times K_3 & \text{for } (i, j) = (1, m), 1 \leq m \leq 4y \\ K_3 \times K_7 & \text{for } (i, j) = (m, 1), 1 \leq m \leq 4x \\ K_3 \times K_3 & \text{otherwise} \end{cases}$$

And, there exists 6-cycle systems of $K_3 \times K_3$ (using Lemma 22), $K_3 \times K_7$ (using Lemma 23) and $K_7 \times K_7$ (using Lemma 21). We denote the set of 6-cycles obtained from this by C_0 .

- $C_0 = 6\text{-cycle system of } v_{i,j} \in G(4x + 7, 4y + 7)$

There are $49 + 14(4y) + 14(4x) + 3(4x)(4y)$, 6-cycles in C_0 . Now, to obtain the remaining 6-cycles, we observe that the following edges do not appear in any 6-cycle yet.

1. There is a copy of $K(7, 3 \dots 3)$ on each of the $12x + 7$ rows in $K_{12x+7} \times K_{12y+7}$
2. There is a copy of $K(7, 3 \dots 3)$ on each of the $12y + 7$ columns in $K_{12x+7} \times K_{12y+7}$

And using the Billington et. al. result, Theorem 4.2 we know that, there exists 6-cycle systems of $K(7, 3 \dots, 3)$. Also, there are $(12x + 7)[14y + 3y(4y - 1)]$, 6-cycles in the 6-cycle system of and $K(7, 3 \dots, 3)$.

- $C_1 = 6$ -cycle system of $K(7, 3 \dots 3)$ on each of the $12x + 7$ rows in $K_{12x+7} \times K_{12y+7}$

Finally,

- $C_2 = 6$ -cycle system of $K(7, 3 \dots 3)$ on each of the $12y + 7$ columns in $K_{12x+7} \times K_{12y+7}$

There are $(12y+7)[14x+3x(4x-1)]$, 6-cycles in C_2 . And so, we have $(12x+7)(12y+7)[x+y+1]$, 6-cycles in all. Hence, $(G(4x+7, 4y+7), \cup_{i \in \mathbb{Z}_3} C_i)$ is a 6-cycle system of $K_{12x+7} \times K_{12y+7}$.

Proposition 13. *There exists a 6-cycle system of $K_{12x+3} \times K_{12y+3}$*

Proof The number of 6-cycles in the 6-cycle system of $K_{12x+3} \times K_{12y+3}$ is given by,

$$\begin{aligned}
|E(K_{12x+3} \times K_{12y+3})| &= (12x+3)[|E(K_{12y+3})|] + \\
&\quad (12y+3)[|E(K_{12x+3})|] \\
&= (12x+3)\left[\frac{(12y+3)(12y+3-1)}{2}\right] + \\
&\quad (12y+3)\left[\frac{(12x+3)(12x+3-1)}{2}\right] \\
&= (12x+3)[(12y+3)(6y+1)] + \\
&\quad (12y+3)[(12x+3)(6x+1)] \\
&= (12x+3)(12y+3)[6x+6y+2] \\
&= 18(4x+1)(4y+1)[3x+3y+1]
\end{aligned}$$

So, there are $6|18(4x+1)(4y+1)[3x+3y+1] = 3(4x+1)(4y+1)[3x+3y+1]$, 6-cycles in the 6-cycle system of $K_{12x+3} \times K_{12y+3}$. Let $V(K_{12x+3} \times K_{12y+3}) = G(4x+1, 4y+1)$, where each $v_{i,j} \in G(4x+1, 4y+1)$ is defined as follows:

$$v_{i,j} = \{K_3 \times K_3 \mid 1 \leq i \leq 4x+1, 1 \leq j \leq 4y+1\}$$

And, there exists a 6-cycle system of $K_3 \times K_3$, using Lemma 22. There are 3, 6-cycles in the 6-cycle system of $K_3 \times K_3$. Let this set of 6-cycles be C_0 .

- $C_0 = 6$ -cycle system of $v_{i,j}$ for $\{1 \leq i \leq 4x+1, 1 \leq j \leq 4y+1\}$

There are $3(4x+1)(4y+1)$, 6-cycles in C_0 . To obtain the remaining 6-cycles, we observe that, the following edges do not appear in any 6-cycle yet.

1. There are $12x + 3$ copies of $K(3, 3, \dots, 3)$, one for each of the $12x + 3$ rows in $K_{12x+3} \times K_{12y+3}$
2. There are $12y + 3$ copies of $K(3, 3, \dots, 3)$, one for each of the $12y + 3$ columns in $K_{12x+3} \times K_{12y+3}$

Also, there exists a 6-cycle system of $K(3, 3, \dots, 3)$ and $K(3, 3, \dots, 3)$, using the result by Billington et. al., Theorem 4.2. So, we get, C_1 and C_2 , two sets of 6-cycles from the two corresponding types of edges given above respectively.

- $C_1 = 6$ -cycle system of $12x + 3$ copies of $K(3, 3, \dots, 3)$, one for each of the $12x + 3$ rows in $K_{12x+3} \times K_{12y+3}$

There are $(3y)(4y + 1)(12x + 3)$, 6-cycles in C_1 .

- $C_2 = 6$ -cycle system of $12y + 3$ copies of $K(3, 3, \dots, 3)$, one for each of the $12y + 3$ columns in $K_{12x+3} \times K_{12y+3}$

Finally, we get $(3x)(4x+1)(12y+3)$, 6-cycles in C_2 . Thus we have $3(4x+1)(4y+1)[3x+3y+1]$, 6-cycles in all. And so, $(G(4x + 1, 4y + 1), \cup_{i \in \mathbb{Z}_3} C_i)$ a 6-cycle system of $K_{12x+3} \times K_{12y+3}$.

Proposition 14. *There exists a 6-cycle system of $K_{12x+3} \times K_{12y+7}$*

Proof The number of 6-cycles in the 6-cycle system of $K_{12x+3} \times K_{12y+7}$ is given by,

$$\begin{aligned}
| E(K_{12x+3} \times K_{12y+7}) | &= (12x + 3)[| E(K_{12y+7}) |] + \\
&\quad (12y + 7)[| E(K_{12x+3}) |] \\
&= (12x + 3)\left[\frac{(12y+7)(12y+7-1)}{2}\right] + \\
&\quad (12y + 7)\left[\frac{(12x+3)(12x+3-1)}{2}\right] \\
&= (12x + 3)[(12y + 7)(6y + 3)] + \\
&\quad (12y + 7)[(12x + 3)(6x + 1)] \\
&= (12x + 3)(12y + 7)[6x + 6y + 4] \\
&= 6(4x + 1)(12y + 7)[3x + 3y + 2]
\end{aligned}$$

So, there are $6|6(4x+1)(12y+7)[3x+3y+2] = (4x+1)(12y+7)[3x+3y+2]$, 6-cycles in the 6-cycle system of $K_{12x+3} \times K_{12y+7}$. Now, let $V(K_{12x+3} \times K_{12y+7}) = G(4x+1, 4y+1)$, where each $v_{i,j} \in G(4x+1, 4y+1)$ is defined as follows:

$$v_{i,j} = \begin{cases} K_3 \times K_7 & \text{for } (i,j) = (m,1), 1 \leq m \leq 4x+1 \\ K_3 \times K_3 & \text{otherwise} \end{cases}$$

And, there exists a 6-cycle system of $K_3 \times K_3$, using Lemma 22 and a 6-cycle system of $K_3 \times K_7$, using Lemma 23. There are 6, 6-cycles and 14, 6-cycles in the 6-cycle systems of $K_3 \times K_3$ and $K_3 \times K_7$ respectively. We denote this set of 6-cycles as C_0 .

- $C_0 = 6$ -cycle system of $v_{i,j} \in G(4x+1, 4y+1)$

We get, $14(4x+1) + 3(4x+1)(4y)$, 6-cycles from C_0 . Now, to obtain the remaining 6-cycles, observe that, the following edges do not appear in any 6-cycle yet.

1. There are $12x+3$ copies of $K(7, 3, \dots, 3)$, one for each of the $12x+3$ rows in $K_{12x+3} \times K_{12y+7}$
2. There are $12y+7$ copies of $K(3, 3, \dots, 3)$, one for each of the $12y+7$ columns in $K_{12x+3} \times K_{12y+7}$

We know that, there exists a 6-cycle system of $K(7, 3, \dots, 3)$ using the result by Billington et. al., Theorem 4.2. Let this set of 6-cycles be C_1 .

- $C_1 = 6$ -cycle system of $12x+3$ copies of $K(7, 3, \dots, 3)$, one for each of the $12x+3$ rows in $K_{12x+3} \times K_{12y+7}$

There are $(12x+3)[14y+3y(4y-1)]$, 6-cycles in C_1 . Similarly, there exists a 6-cycle system of $K(3, 3, \dots, 3)$, using the result by Billington et. al., Theorem 4.2. And so, we get another set of 6-cycles, C_2 , given below.

- $C_2 = 6$ -cycle system of $12y+3$ copies of $K(3, 3, \dots, 3)$, one for each of the $12y+3$ columns in $K_{12x+3} \times K_{12y+3}$

And, we get $(12y+7)[(3x)(4x+1)]$, 6-cycles in C_2 . At this point, we have $(4x+1)(12y+7)(3x+3y+2)$, 6-cycles. And so, $(G(4x+1, 4y+1), \cup_{i \in \mathbb{Z}_3} C_i)$ is a 6-cycle system of $K_{12x+3} \times K_{12y+7}$.

Proposition 15. *There exists 6-cycle system of $K_{12x+3} \times K_{12y+11}$*

Proof The number of 6-cycles in the 6-cycle system of $K_{12x+3} \times K_{12y+11}$ is given by,

$$\begin{aligned}
|E(K_{12x+3} \times K_{12y+11})| &= (12x+3)[|E(K_{12y+11})|] + \\
&\quad (12y+11)[|E(K_{12x+3})|] \\
&= (12x+3)\left[\frac{(12y+11)(12y+11-1)}{2}\right] + \\
&\quad (12y+11)\left[\frac{(12x+3)(12x+3-1)}{2}\right] \\
&= (12x+3)[(12y+11)(6y+5)] + \\
&\quad (12y+11)[(12x+3)(6x+1)] \\
&= (12x+3)(12y+11)[6x+6y+6] \\
&= 6(12x+3)(12y+11)[x+y+1]
\end{aligned}$$

So, there are $6|6(12x+3)(12y+11)[x+y+1]| = (12x+3)(12y+11)[x+y+1]$, 6-cycles in the 6-cycle system of $K_{12x+3} \times K_{12y+11}$. First let $V(K_{12x+3} \times K_{12y+11}) = G(4x+1, 4y+1)$, where each $v_{i,j} \in G(4x+1, 4y+1)$ is defined as follows:

$$v_{i,j} = \begin{cases} K_3 \times K_{11} & \text{for } (i,j) = (m,1), 1 \leq m \leq 4x+1 \\ K_3 \times K_3 & \text{otherwise} \end{cases}$$

And, there exists a 6-cycle system of $K_3 \times K_3$, using Lemma 22 containing 3, 6-cycles. And there exists a 6-cycle system of $K_3 \times K_{11}$, using Lemma 24, containing 33, 6-cycles. Let C_0 be the set of 6-cycles obtained from this embedding.

- $C_0 = 6\text{-cycle system of } v_{i,j} \in G(4x+1, 4y+1)$

There are $33(4x+1) + 3(4x+1)(4y)$, 6-cycles in C_0 . In order to obtain the remaining 6-cycles, observe that, the following edges do not appear in any 6-cycle yet.

1. There are $12x + 3$ copies of $K(11, 3, \dots, 3)$, one for each of the $12x + 3$ rows in $K_{12x+3} \times K_{12y+11}$
2. There are $12y + 11$ copies of $K(3, 3, \dots, 3)$, one for each of the $12y + 11$ columns in $K_{12x+3} \times K_{12y+11}$

And, there exists a 6-cycle system of $K(11, 3, \dots, 3)$ using the result by Billington et. al., Theorem 4.2. Denote the 6-cycles obtained from this by C_1 .

- $C_1 = 6$ -cycle system of $12x + 3$ copies of $K(11, 3, \dots, 3)$, one for each of the $12x + 3$ rows in $K_{12x+3} \times K_{12y+11}$

There are $(12x + 3)[22y + 3y(4y - 1)]$, 6-cycles in C_1 . Similarly, there exists a 6-cycle system of $K(3, 3, \dots, 3)$, using the result by Billington et. al., Theorem 4.2. Using this fact, we get another set of 6-cycles, C_2 , given below.

- $C_2 = 6$ -cycle system of $12y + 11$ copies of $K(3, 3, \dots, 3)$, one for each of the $12y + 11$ columns in $K_{12x+3} \times K_{12y+11}$

Hence, we get another $(12y + 11)[(3x)(4x + 1)]$, 6-cycles from C_2 . Now, we have the required total number of 6-cycles, $(12x + 3)(12y + 11)[x + y + 1]$. Thus, $(G(4x + 1, 4y + 1), \cup_{i \in \mathbb{Z}_3} C_i)$ is a 6-cycle system of $K_{12x+3} \times K_{12y+11}$.

Proposition 16. *There exists a 6-cycle system of $K_{12x+9} \times K_{12y+1}$*

Proof The number of 6-cycles in the 6-cycle system of $K_{12x+9} \times K_{12y+1}$ is given by,

$$\begin{aligned}
| E(K_{12x+9} \times K_{12y+1}) | &= (12x + 9)[| E(K_{12y+1}) |] + \\
&\quad (12y + 1)[| E(K_{12x+9}) |] \\
&= (12x + 9)\left[\frac{(12y+1)(12y+1-1)}{2}\right] + \\
&\quad (12y + 1)\left[\frac{(12x+9)(12x+9-1)}{2}\right] \\
&= (12x + 9)[(12y + 1)(6y)] + \\
&\quad (12y + 1)[(12x + 9)(6x + 4)] \\
&= (12x + 9)(12y + 1)[6x + 6y + 4] \\
&= 6(4x + 3)(12y + 1)[3x + 3y + 2]
\end{aligned}$$

So, there are $6|6(4x+3)(12y+1)[3x+3y+2] = (4x+3)(12y+1)[3x+3y+2]$, 6-cycles in the 6-cycle system of $K_{12x+9} \times K_{12y+1}$. There are $12x+9$ copies of K_{12y+1} , one on each of the $12x+9$ rows in $K_{12x+9} \times K_{12y+1}$. We know that there exists a 6-cycle system of K_{12y+1} , using the result by Sajna, Theorem 4.3. We denote the 6-cycles obtained from this embedding by C_0 .

- $C_0 = 6$ -cycle system of $12x+9$ copies of K_{12y+1} , one on each of the $12x+9$ rows in $K_{12x+9} \times K_{12y+1}$

And, there are $y(12y+1)(12x+9)$, 6-cycles in C_0 . Next, note that there are $12y+1$ copies of K_{12x+9} , one on each of the $12y+1$ columns in $K_{12x+9} \times K_{12y+1}$. And we know that, there exist 6-cycle system K_{12x+9} , using the result by Sajna, Theorem 4.3. Using, this fact, we get another set of 6-cycles, C_1 given below.

- $C_1 = 6$ -cycle system of $12y+1$ copies of K_{12x+9} , one on each of the $12y+1$ columns in $K_{12x+9} \times K_{12y+1}$

We get, $(12y+1)(4x+3)(3x+2)$, 6-cycles from C_1 . So, now we have $(4x+3)(12y+1)[3x+3y+2]$, 6-cycles. Thus, by this construction there exists a 6-cycle system of $K_{12x+9} \times K_{12y+1}$.

Proposition 17. *There exists a 6-cycle system of $K_{12x+5} \times K_{12y+9}$, and*

1. $y \equiv 0 \pmod{3}$
2. $y \equiv 1 \pmod{3}$
3. $y \equiv 2 \pmod{3}$

Proof We shall prove each of these three cases in turn.

Case 1. Suppose $y = 3\alpha$.

First let $V(K_{12x+5} \times K_{12y+9}) = G(12x+1, 4\alpha+1)$, where each $v_{i,j} \in G(12x+1, 4\alpha+1)$ is defined as follows:

$$v_{i,j} = \begin{cases} K_5 \times K_9 & \text{for } (i,j) = (1,m), 1 \leq m \leq 4\alpha + 1 \\ K_1 \times K_9 & \text{otherwise} \end{cases}$$

And, there exists a 6-cycle system of $K_1 \times K_9$, using Lemma 25. Also, there exists a 6-cycle system of $K_5 \times K_9$, using Lemma 26. Now, to obtain the remaining 6-cycles, we observe that the following edges do not appear in any 6-cycle yet.

1. There is one copy of $K(9, \dots, 9)$, for each of the $12x + 5$ rows in $K_{12x+5} \times K_{12y+9}$
2. There is one copy of $K(5, 1, \dots, 1)$ for each of the $12y + 9$ columns in $K_{12x+5} \times K_{12y+9}$

And, there exists 6-cycle systems of $K(9, \dots, 9)$ and $K(5, 1, \dots, 1)$, using the result by Billington et. al., Theorem 4.2. Hence, there exists a 6-cycle system of $K_{12x+5} \times K_{12y+9}$.

Case 2. For this case, let $y = 3\alpha + 1$ (when $y = 1$, refer to Lemma 27).

And let $V(K_{12x+5} \times K_{12y+9}) = (G(12x - 3, 4\alpha + 1) \cup G'(2, 4) \cup G''(3, 12y + 9))$, as depicted in the figure. And let each $v_{i,j} \in G(12x - 3, 4\alpha + 1)$ be defined as follows:

$$v_{i,j} = \begin{cases} K_5 \times K_9 & \text{for } (i,j) = (1,m), 1 \leq m \leq 4\alpha + 1 \\ K_1 \times K_9 & \text{for } (i,j) = (p,q), 1 \leq p \leq 12x - 3, 1 \leq q \leq 4\alpha + 1 \end{cases}$$

And each $v'_{i,j} \in G'(2, 4)$ be defined as follows:

$$v'_{i,j} = \begin{cases} K_{11} \times K_3 & \text{for } (i,j) = (1,m), 1 \leq m \leq 4 \\ K_3 \times K_3 & \text{for } (i,j) = (2,m), 1 \leq m \leq 4 \end{cases}$$

Finally, each $v''_{i,j} \in G''(3, 12y + 9)$ is defined as follows:

$$v''_{i,j} = \{K_{12y+9} \mid \forall 1 \leq i \leq 3, 1 \leq j \leq 12y + 9\}$$

And, there exists 6-cycle systems of $K_5 \times K_9$ (using Lemma 23), $K_1 \times K_9$ (using Lemma 25), $K_3 \times K_{11}$ (using Lemma 24), $K_3 \times K_3$ (using Lemma 22) and K_{12y+9} (using Lemma 25). Now to obtain the remaining 6-cycles, we observe that the following edges do not appear in any 6-cycle yet.

1. There are $12x - 2$ copies of $K(9, \dots, 9, 3, 3, 3, 3)$, one for each of the first $12x - 2$ rows in $K_{12x+5} \times K_{12y+9}$
2. There are $12y - 3$ copies of $K(5, 1, \dots, 1)$, one for each of the first $12y - 3$ columns in $K_{12x+5} \times K_{12y+9}$
3. There are 12 copies of $K(11, 3, 1, 1, 1)$, one for each of the last 12 columns in $K_{12x+5} \times K_{12y+9}$

We know that, there exists a 6-cycle systems of $K(9, \dots, 9, 3, 3, 3, 3)$, $K(5, 1, \dots, 1)$ and $K(11, 3, 1, 1, 1)$, using the result by Billington et. al., Theorem 4.2. Thus, there exists a 6-cycle system of $K_{12x+5} \times K_{12y+9}$.

Case 3. For this case, let $y = 3\alpha + 2$ (when $y = 2$, refer to Lemma 28).

And let $V(K_{12x+5} \times K_{12y+9}) = (G(12x - 3, 4\alpha + 1) \cup G'(2, 8) \cup G''(3, 12y + 9))$, as depicted in the figure. And let each $v_{i,j} \in G(12x - 2, 4\alpha + 1)$ be defined as follows:

$$v_{i,j} = \begin{cases} K_5 \times K_9 & \text{for } (i, j) = (1, m), 1 \leq m \leq 4\alpha + 1 \\ K_1 \times K_9 & \text{otherwise} \end{cases}$$

And each $v'_{i,j} \in G'(2, 8)$ be defined as follows:

$$v'_{i,j} = \begin{cases} K_{11} \times K_3 & \text{for } (i, j) = (1, m), 1 \leq m \leq 8 \\ K_3 \times K_3 & \text{for } (i, j) = (a, b), a = 2, 1 \leq b \leq 8 \end{cases}$$

Finally, each $v''_{i,j} \in G''(3, 12y + 9)$ is defined as follows:

$$v''_{i,j} = \{K_{12y+9} \mid \forall 1 \leq i \leq 3, 1 \leq j \leq 12y + 9\}$$

And, there exists 6-cycle systems of $K_5 \times K_9$ (using Lemma 23), $K_1 \times K_9$ (using Lemma 25), $K_3 \times K_{11}$ (using Lemma 24), $K_3 \times K_3$ (using Lemma 22) and K_{12y+9} (using Lemma 25). Now to obtain the remaining 6-cycles, we observe that the following edges do not appear in any 6-cycle yet.

1. There are $12x+2$ copies of $K(9, \dots, 9, 3, 3, 3, 3, 3, 3, 3, 3)$, one for each of the first $12x+2$ rows in $K_{12x+5} \times K_{12y+9}$
2. There are $4\alpha + 1$ copies of $K(5, 1, \dots, 1)$, one for each of the first $4\alpha + 1$ columns in $K_{12x+5} \times K_{12y+9}$
3. There are 24 copies of $K(11, 3, 1, 1, 1)$, one for each of the last 24 columns in $K_{12x+5} \times K_{12y+9}$

There exists 6-cycle systems of $K(9, \dots, 9, 3, 3, 3, 3, 3, 3, 3, 3)$, $K(5, 1, \dots, 1)$ and $K(11, 3, 11, 1)$, using the result by Billington et. al., Theorem 4.2. Hence, there exists a 6-cycle system of $K_{12x+5} \times K_{12y+9}$.

Proposition 18. *There exists a 6-cycle system of $K_{12x+9} \times K_{12y+9}$*

Proof The number of 6-cycles in the 6-cycle system of $K_{12x+9} \times K_{12y+9}$ is given by,

$$\begin{aligned} | E(K_{12x+9} \times K_{12y+9}) | &= (12x + 9)[| E(K_{12y+9}) |] + \\ &\quad (12y + 9)[| E(K_{12x+9}) |] \\ &= (12x + 9) \left[\frac{(12y+9)(12y+9-1)}{2} \right] + \\ &\quad (12y + 9) \left[\frac{(12x+9)(12x+9-1)}{2} \right] \\ &= (12x + 9)[(12y + 9)(6y + 4)] + \\ &\quad (12y + 9)[(12x + 9)(6x + 4)] \\ &= (12x + 9)(12y + 9)[6x + 6y + 8] \\ &= 18(4x + 3)(4y + 3)[3x + 3y + 4] \end{aligned}$$

So, there are $6|18(4x+3)(4y+3)[3x+3y+4] = 3(4x+3)(4y+3)[3x+3y+4]$, 6-cycles in the 6-cycle system of $K_{12x+9} \times K_{12y+9}$. There are $12x+9$ copies of K_{12y+9} , one for each row in $K_{12x+9} \times K_{12y+9}$. And there are $12y+9$ copies of K_{12x+9} , one for each column in $K_{12x+9} \times K_{12y+9}$. Also, there exist 6-cycle systems of K_{12x+9} and K_{12y+9} , using Lemma 25. Based on this, we construct the following sets of 6-cycles, C_0 and C_1 respectively.

- $C_0 = 6$ -cycle system of $12x+9$ copies of K_{12y+9} , one on each of the $12x+9$ rows in $K_{12x+9} \times K_{12y+9}$

We get $(12x+9)(4y+3)(3y+2)$, 6-cycles from C_0 .

- $C_1 = 6$ -cycle system of $12y+9$ copies of K_{12x+9} , one on each of the $12y+9$ columns in $K_{12x+9} \times K_{12y+9}$

As seen in the previous case, we get, $(12y+9)(4x+3)(3x+2)$, 6-cycles from C_1 . Thus, there exists a 6-cycle system of $K_{12x+9} \times K_{12y+9}$ using this embedding of K_{12x+9} and K_{12y+9} .

Now, we prove the main result of this chapter.

Theorem 4.4. *There exists a 6-cycle system of $K_m \times K_n$ iff*

1. m, n are even

(a) $6 \mid m$ or $6 \mid n$

OR

(b) $m+n \equiv 2 \pmod{3}$

2. m, n are odd

(a) $m, n \not\equiv 0 \pmod{3}$ then $(m+n) \equiv 2 \pmod{12}$

OR

(b) m &/or $n \equiv 0 \pmod{3}$ then $m+n \equiv 2 \pmod{4}$

Proof To prove the necessity of these conditions we will first prove that either m and n are both even, or, m and n are both odd. Suppose that m and n are even and odd respectively. Say $m = 2x$ and $n = 2y + 1$. Then for any $v \in V(K_m \times K_n)$,

$$\deg(v) = [(2y+1-1) + (2x-1)]$$

So, the $\deg(v \in (K_m \times K_n))$ is odd. And, clearly to find an m -cycle system of any graph G , all vertices in G should have even degree. So it is not possible for m and n to be even and odd respectively. Hence, either both m and n are both even or both odd.

Now suppose that m and n are both even. Then,

$$\begin{aligned} |E(K_m \times K_n)| &= m\left[\frac{(n)(n-1)}{2}\right] + n\left[\frac{(m)(m-1)}{2}\right] \\ &= \frac{mn(m+n-2)}{2} \end{aligned}$$

And, to obtain a 6-cycle system of $K_m \times K_n$,

$$\begin{aligned} 6 &| |E(K_m \times K_n)| \\ \Rightarrow 6 &| \frac{mn(m+n-2)}{2} \end{aligned}$$

So, either

1. $6 | m$ or $6 | n$

OR

2. $m + n \equiv 2 \pmod{3}$

This proves the necessity of condition 1. Now, suppose that m and n are both odd, then if

1. $m, n \not\equiv 0 \pmod{3}$ then $m + n \equiv 2 \pmod{12}$

OR

2. $m \&/or n \equiv 0 \pmod{3}$ then $m + n \equiv 2 \pmod{4}$

And, this proves the necessity of condition 2. To prove the sufficiency, we first consider the case when m and n are both even. Then either

1. $6 \mid m$ or $6 \mid n$

OR

2. $m + n \equiv 2 \pmod{3}$

Case 1. Suppose $6 \mid m$ or $6 \mid n$ then there exists a 6-cycle system of $K_m \times K_n$ using Propositions 7 or 8 or 9.

Case 2. Suppose $m + n \equiv 2 \pmod{3}$ then there exists a 6-cycle system of $K_m \times K_n$ using Proposition 10.

Now, consider the case when both m and n are *odd*. Then either

1. $m, n \not\equiv 0 \pmod{3}$ then $(m + n) \equiv 2 \pmod{12}$

OR

2. $m \&/or n \equiv 0 \pmod{3}$ then $m + n \equiv 2 \pmod{4}$

Case 1. Suppose $m, n \not\equiv 0 \pmod{3}$ then, there exists a 6-cycle system of $K_m \times K_n$ using Propositions 11 or 12.

Case 2. If $m \&/or n \equiv 0 \pmod{3}$ then, there exists a 6-cycle system of $K_m \times K_n$ using Propositions 13, 14, 15, 16, 17 and 18.

This proves our Theorem. □

Chapter 5

Nearly 4-regular Leave of the Complete Graph on n vertices, K_n

5.1 Introduction

An m -cycle system of a graph G with vertex set $V(G)$ is an ordered pair $(V(G), S)$, where S is a set of edge-disjoint cycles of length m , such that each edge in G is contained in exactly one cycle in S .

Clearly, necessary conditions for an m -cycle system of G to exist are: m must divide $|E(G)|$; each vertex in G must have even degree; and if $|V(G)| > 1$ then $|V(G)| \geq m$. The existence problem of whether these conditions are sufficient was initially considered for the case where $G = K_n$. After many papers it was finally settled in [2, 45, 83], showing that these obvious necessary conditions are sufficient. Along the way, Sotteau [84] provided necessary and sufficient conditions for the case when $G = K(m, n)$. Let xG denote the graph with vertex set $V(G)$ in which, for all $u, v \in V(G)$, u and v are joined by xy edges iff they are joined by y edges in G .

Theorem 5.1. [84] *There exists a 4-cycle system of $K_{a,b}$ and of $2K_{a,b}$ if and only if each vertex has even degree, the number of edges is divisible by 4, and $a, b \geq 2$.*

To denote a 4-cycle system of $K_{m,n}$ with bipartition $\{A, B\}$ we write $(K(A, B), S)$. Let $K(a_1, a_2, \dots, a_p)$ denote a complete multipartite graph with p parts in which the i^{th} part has size a_i for $1 \leq i \leq p$. The line graph of a graph G , $L(G)$ is defined as follows. Every edge $uv \in E(G)$ is a vertex in $L(G)$ and two vertices are adjacent in $L(G)$ if the corresponding edges in G are adjacent. The existence of m -cycle systems of $L(K_n)$ was settled when $m \in \{4, 6\}$ in [16, 19, 20]. Also, there have been some results for obtaining m -cycle systems of $K(a_1, a_2, \dots, a_p)$, for example being settled when all parts have the same

size in [17] where m is even and when p is small and then: There is a companion result obtaining 4-cycle systems [78] of $L(K(a_1, a_2, \dots, a_n))$, but much remains to be done in this area.

When n is even, vertices in K_n have odd degree, so a natural companion of finding m -cycle systems of $G = K_n$ was to solve the case where G is the complete graph on n vertices with a one factor removed: $K_n - I$, [2]. (More generally, in this context the graph induced by edges removed from K_n is called a *leave*).

These results led to further questions, asking for which small graphs H does $K_n - H$ have an m -cycle system. Buchanan [15] solved the case where H is a 2-regular graph and $m = n$ (i.e. hamilton cycle). A new proof of this case was provided in [12, 68]. The case when H is 2-regular and $m = 3, 4$ and 6 were solved in [19], [32] and [6] respectively. The case where H has maximum degree 3 and $m = 4$ was solved in [34].

In this chapter we extend these results in literature by completely solving the case when $G = K_n - E(F^*)$ where F^* is a nearly 2-regular leave. A graph is said to be nearly 2-regular if all vertices have degree 2 except for one which has degree $k > 2$ (note that F^* need not be a spanning subgraph). Not only is this result of interest in it's own right in the context of the history of this problem, but, it also arose as a useful tool in studying the cycle systems of the line graphs of complete multipartite graphs.

5.2 Applications

This result has direct applications to neighbor designs [75]. Consider an experiment in serology in which we arrange the antigens in a petri dish and place the antiserum in the center of that dish. Then the results of this paper can be applied to the case when we are interested in observing reactions between most pairs of antigen-antigen reactions, the omitted pairs being specified by the leave H .

If S is a set of cycles then let $E(S)$ denote the set of edges in the cycles in S .

Section 2 of this chapter deals with constructions for some small values of n and cases where F^* contains cycles of lengths 4 and 5. These constructions are used to obtain 4-cycle systems for the larger values of n in Section 3. Finally, in Section 4 we combine the earlier results, proving the main result of this paper, Theorem 5.2.

Lemma 29. *There exists a 4-cycle system of $K_7 - E(\{(0, 1, 2, 3, 4)\})$*

Proof. $(\mathbb{Z}_7, \{(0, 2, 4, 6), (0, 3, 1, 5), (1, 4, 5, 6), (6, 2, 5, 3)\})$ is a 4-cycle system of $K_7 - E(\{(0,1,2,3,4)\})$.

Lemma 30. *There exists a 4-cycle system of $K_9 - E(\{(0, 1, 2, 3), (0, 4, 5, 6)\})$*

Proof. $(\mathbb{Z}_9, \{(0, 2, 4, 7), (0, 8, 2, 5), (1, 3, 4, 6), (1, 7, 6, 8), (1, 4, 8, 5), (2, 6, 3, 7), (3, 5, 7, 8)\})$ is a 4-cycle system of $K_9 - E(\{(0,1,2,3),(0,4,5,6)\})$.

Lemma 31. *There exists a 4-cycle system of $K_9 - E(\{(0, 1, 2, 3), (4, 5, 6, 7)\})$*

Proof. $(\mathbb{Z}_9, \{(0, 2, 5, 7), (0, 4, 1, 6), (0, 5, 1, 8), (1, 3, 8, 7), (2, 4, 3, 1), (2, 6, 4, 8), (3, 5, 8, 6)\})$ is a 4-cycle system of $K_9 - E(\{(0,1,2,3),(4,5,6,7)\})$.

Lemma 32. *There exists a 4-cycle system of $K_9 - E(\{(0, 1, 2, 3, 4), (0, 5, 6, 7), (0, 3, 6)\})$*

Proof. $(\mathbb{Z}_9, \{(1, 5, 2, 8), (1, 6, 2, 7), (1, 3, 7, 4), (2, 0, 8, 4), (3, 5, 7, 8), (4, 5, 8, 6)\})$ is a 4-cycle system of $K_9 - E(\{(0,1,2,3,4),(0,5,6,7),(0,3,6)\})$.

Lemma 33. *There exists a 4-cycle system of $K_9 - E(\{(0, 1, 2, 3, 4), (0, 5, 6, 7, 8), (0, 3, 7), (0, 2, 6)\})$*

Proof. $(\mathbb{Z}_9, \{(1, 4, 2, 5), (1, 3, 6, 8), (1, 7, 4, 6), (2, 7, 5, 8), (3, 5, 4, 8)\})$ is a 4-cycle system of $K_9 - E(\{(0,1,2,3,4),(0,5,6,7,8),(0,3,7),(0,2,6)\})$.

Lemma 34. *There exists a 4-cycle system of $K_{11} - E(\{(0, 1, 2, 3), (0, 4, 5, 6), (0, 7, 8, 9), (0, 2, 5)\})$*

Proof. $(\mathbb{Z}_{11}, \{(1, 3, 9, 4), (1, 5, 7, 9), (1, 7, 2, 6), (1, 8, 0, 10), (2, 4, 3, 8), (2, 9, 5, 10), (3, 5, 8, 6), (3, 7, 4, 10), (4, 6, 10, 8), (6, 7, 10, 9)\})$ is a 4-cycle system of $K_{11} - E(\{(0, 1, 2, 3), (0, 4, 5, 6), (0, 7, 8, 9), (0, 2, 5)\})$.

Lemma 35. *There exists a 4-cycle system of $K_{13} - E(\{(0, 1, 2, 3), (0, 4, 5, 6), (0, 7, 8, 9), (0, 10, 11, 12), (0, 2, 5), (0, 8, 11)\})$*

Proof. $(\mathbb{Z}_{13}, \{(1, 3, 4, 6), (1, 4, 7, 5), (1, 7, 2, 8), (1, 10, 2, 9), (1, 11, 2, 12), (2, 4, 6, 8), (3, 5, 9, 7), (3, 6, 10, 8), (3, 9, 4, 10), (3, 11, 4, 12), (5, 8, 12, 10), (5, 11, 6, 12), (6, 9, 10, 7), (9, 11, 7, 12)\})$ is a 4-cycle system of $K_{13} - E(\{(0, 1, 2, 3), (0, 4, 5, 6), (0, 7, 8, 9), (0, 10, 11, 12), (0, 2, 5), (0, 8, 11)\})$.

We now focus on some special cases in which the nearly 2-regular leave only contains 4-cycles.

Lemma 36. *There exists a 4-cycle system of $K_n - E(F^*)$ where F^* is nearly 2-regular, n and F^* are chosen to be any of the following.*

1. $n = 9$ and F^* consists of two 4-cycles, all of which intersect precisely in the vertex ∞ .
2. $n = 17$ and F^* consists of five 4-cycles, all of which intersect precisely in the vertex ∞ .
3. $n = 25$ and F^* consists of seven 4-cycles, all of which intersect precisely in the vertex ∞ .
4. $n = 25$ and F^* consists of eight 4-cycles, all of which intersect precisely in the vertex ∞ .

Proof. We will consider these four cases in turn.

1. Let $V(K_n) = \mathbb{Z}_8 \cup \{\infty\}$. Define

$$F_i^* = (\infty, 3i, 3i + 1, 3i + 2) \text{ for each } i \in \mathbb{Z}_2$$

and

$$F^* = (\cup_{i \in \mathbb{Z}_2} F_i^*)$$

Let $(\mathbb{Z}_8 \cup \{\infty\}, C_0)$ be a 4-cycle system of $K_9 - E(F^*)$ (see Lemma 30).

2. Let $V(K_n) = \mathbb{Z}_{16} \cup \{\infty\}$. Define

$$F_i^* = (\infty, 3i, 3i + 1, 3i + 2) \text{ for each } i \in \mathbb{Z}_5$$

and

$$F^* = (\cup_{i \in \mathbb{Z}_5} F_i^*)$$

Let $(\mathbb{Z}_6 \cup \{\infty, 7, 10\}, C_0)$ be a 4-cycle system of $K_9 - E(F_0^* \cup F_1^*)$ (see Lemma 30).

Similarly, let $((\mathbb{Z}_{16} \setminus \mathbb{Z}_6) \cup \{\infty\}, C_1)$ be a 4-cycle system of $K_{11} - E(F_2^* \cup F_3^* \cup \{\infty, 7, 10\})$

(see Lemma 34). Now, let $(K(\mathbb{Z}_6, (\cup_{x \in \{\mathbb{Z}_5 \setminus \mathbb{Z}_2\} \cup \{15\}} \{3x, 3x + 2\})), C_2)$ be a 4-cycle system of $K(6, 8)$, using Theorem 5.1. Then $(\cup_{i \in \mathbb{Z}_3} C_i)$ is a 4-cycle system of $K_{17} - E(F^*)$.

3. Let $V(K_n) = \mathbb{Z}_{24} \cup \{\infty\}$. Define

$$F_i^* = (\infty, 3i, 3i + 1, 3i + 2) \text{ for each } i \in \mathbb{Z}_7$$

and

$$F^* = (\cup_{i \in \mathbb{Z}_7} F_i^*)$$

Let $(\mathbb{Z}_{15} \cup \{\infty, 21\}, C_0)$ be a 4-cycle system of $K_{17} - E(\cup_{i \in \mathbb{Z}_5} F_i^*)$ (see Lemma 36(2)).

Similarly, let $((\mathbb{Z}_{24} \cup \{\infty\}) \setminus (\mathbb{Z}_{15} \cup \{21\}), C_1)$ be a 4-cycle system of $K_9 - E(\cup_{i \in \mathbb{Z}_7 \setminus \mathbb{Z}_5} F_i^*)$

(see Lemma 30). Now, let $(K((\mathbb{Z}_{24} \setminus (\mathbb{Z}_{15} \cup \{21\})), (\mathbb{Z}_{15} \cup \{21\})), C_2)$ be a 4-cycle system of $K(8, 16)$, using Theorem 5.1. Then $(\cup_{i \in \mathbb{Z}_3} C_i)$ is a 4-cycle system of $K_{25} - E(F^*)$.

4. Let $V(K_n) = \mathbb{Z}_{24} \cup \{\infty\}$. Define

$$F_i^* = (\infty, 3i, 3i + 1, 3i + 2) \text{ for each } i \in \mathbb{Z}_8$$

and

$$F^* = (\cup_{i \in \mathbb{Z}_8} F_i^*)$$

Let $(\mathbb{Z}_6 \cup \{\infty, 13, 16\}, C_0)$ be a 4-cycle system of $K_9 - E(F_0^* \cup F_1^*)$ (see Lemma 30). Similarly, let $((\mathbb{Z}_{12} \setminus \mathbb{Z}_6) \cup \{\infty, 19, 22\}, C_1)$ be a 4-cycle system of $K_9 - E(F_2^* \cup F_3^*)$ (see Lemma 30). Let $((\mathbb{Z}_{24} \setminus \mathbb{Z}_{12} \cup \{\infty\}), C_2)$ be a 4-cycle system of $K_{13} - E(\cup_{i \in \mathbb{Z}_8 \setminus \mathbb{Z}_4} (F_i^* \cup_{j \in \{4,6\}} (\infty, 3j + 1, 3(j+1) + 1)))$ (see Lemma 35). Using Theorem 5.1, $(K((\cup_{x \in \mathbb{Z}_8 \setminus \mathbb{Z}_4} \{3x, 3x + 2\}), (\mathbb{Z}_{12})), C_3)$ be a 4-cycle system of $K(8, 12)$. Similarly, $(K(\mathbb{Z}_6 \cup \{13, 16\}, \cup_{x \in \mathbb{Z}_4 \setminus \mathbb{Z}_2} \{3x, 3x + 1, 3x + 2\}), C_4)$ be a 4-cycle system of $K(6, 8)$. Finally, let $(K(\{19, 22\}, \mathbb{Z}_6), C_5)$ is a 4-cycle system of $K(2, 6)$. Then $(\cup_{i \in \mathbb{Z}_6} C_i)$ is a 4-cycle system of $K_{25} - E(F^*)$.

Lemma 37. *There exists a 4-cycle system of $K_n - E(F^*)$ where F^* is nearly 2-regular, n and F^* are chosen to be any of the following.*

1. $n = 9$ and F^* consists of two vertex disjoint 4-cycles, one of which contains the vertex ∞ .
2. $n = 17$ and F^* consists of four 4-cycles, three of which intersect precisely in the vertex ∞ and the other cycle does not contain ∞ .
3. $n = 17$ and F^* consists of five 4-cycles, four of which intersect precisely in the vertex ∞ and the other cycle does not contain ∞ .
4. $n = 25$ and F^* consists of seven 4-cycles, six of which intersect precisely in the vertex ∞ and the other cycle does not contain ∞ .

Proof. We will consider these four cases in turn.

1. Let $V(K_n) = \mathbb{Z}_8 \cup \{\infty\}$. Define

$$F_i^* = \begin{cases} (\infty, 0, 1, 2) & \text{for } i = 0, \\ (3, 4, 5, 6) & \text{for } i = 1 \end{cases}$$

and

$$F^* = (\cup_{i \in \mathbb{Z}_2} F_i^*)$$

Let $(\mathbb{Z}_8 \cup \{\infty\}, C_0)$ be a 4-cycle system of $K_9 - E(F^*)$ (see Lemma 31).

2. Let $V(K_n) = \mathbb{Z}_{16} \cup \{\infty\}$. Define

$$F_i^* = (\infty, 3i, 3i + 1, 3i + 2) \text{ for each } i \in \mathbb{Z}_3$$

and

$$F^* = ((\cup_{i \in \mathbb{Z}_3} (F_i^*)) \cup (9, 10, 11, 12))$$

Let $(\mathbb{Z}_{14} \setminus \mathbb{Z}_6 \cup \{\infty\}, C_0)$ be a 4-cycle system of $K_9 - E(F_2^* \cup (9, 10, 11, 12))$ (see Lemma 31). Similarly, let $((\mathbb{Z}_6 \cup \{\infty, 14, 15\}), C_1)$ be a 4-cycle system of $K_9 - E(\cup_{i \in \mathbb{Z}_2} F_i^*)$ (see Lemma 30). Now, let $(K((\mathbb{Z}_{14} \setminus \mathbb{Z}_6), (\mathbb{Z}_6 \cup \{14, 15\})), C_2)$ be a 4-cycle system of $K(8, 8)$, using Theorem 5.1. Then $(\cup_{i \in \mathbb{Z}_3} C_i)$ is a 4-cycle system of $K_{17} - E(F^*)$.

3. Let $V(K_n) = \mathbb{Z}_{16} \cup \{\infty\}$. Define

$$F_i^* = (\infty, 3i, 3i + 1, 3i + 2) \text{ for each } i \in \mathbb{Z}_4$$

and

$$F^* = ((\cup_{i \in \mathbb{Z}_4} (F_i^*)) \cup (12, 13, 14, 15))$$

Let $(\mathbb{Z}_6 \cup \{\infty, 12, 14\}, C_0)$ be a 4-cycle system of $K_9 - E(F_0^* \cup F_1^*)$ (see Lemma 30). Similarly, let $((\mathbb{Z}_{12} \setminus \mathbb{Z}_6) \cup \{\infty, 13, 15\}), C_1)$ be a 4-cycle system of $K_9 - E(\cup_{i \in \mathbb{Z}_4 \setminus \mathbb{Z}_2} F_i^*)$ (see Lemma 30). Using Theorem 5.1, let $(K(\{12, 14\}, (\mathbb{Z}_{12} \setminus \mathbb{Z}_6)), C_2)$ be a 4-cycle system

of $K(2, 6)$. Similarly, $(K(\{13, 15\}, \mathbb{Z}_6), C_3)$ be a 4-cycle system of $K(2, 6)$. Also, let $(K(\mathbb{Z}_6, \mathbb{Z}_{12} \setminus \mathbb{Z}_6), C_4)$ be a 4-cycle system of $K(6, 6)$. Then $(\cup_{i \in \mathbb{Z}_5} C_i)$ is a 4-cycle system of $K_{17} - E(F^*)$.

4. Let $V(K_n) = \mathbb{Z}_{24} \cup \{\infty\}$. Define

$$F_i^* = (\infty, 3i, 3i + 1, 3i + 2) \text{ for each } i \in \mathbb{Z}_6$$

and

$$F^* = ((\cup_{i \in \mathbb{Z}_6} (F_i^*)) \cup (18, 19, 20, 21))$$

Let $(\mathbb{Z}_6 \cup \{\infty, 18, 20\}, C_0)$ be a 4-cycle system of $K_9 - E(F_0^* \cup F_1^*)$ (see Lemma 30). Similarly, let $((\mathbb{Z}_{12} \setminus \mathbb{Z}_6) \cup \{\infty, 19, 21\}, C_1)$ be a 4-cycle system of $K_9 - E(\cup_{i \in \mathbb{Z}_4 \setminus \mathbb{Z}_2} F_i^*)$ (see Lemma 30). Let $((\mathbb{Z}_{18} \setminus \mathbb{Z}_{12}) \cup \{\infty, 22, 23\}, C_2)$ be a 4-cycle system of $K_9 - E(\cup_{i \in \mathbb{Z}_6 \setminus \mathbb{Z}_4} F_i^*)$ (see Lemma 30). Now, let $(K(\{18, 20\}, (\mathbb{Z}_{12} \setminus \mathbb{Z}_6)), C_3)$ be a 4-cycle system of $K(2, 6)$, using Theorem 5.1. Similarly, $(K(\{19, 21\}, \mathbb{Z}_6), C_4)$ be a 4-cycle system of $K(2, 6)$. And, let $(K((\mathbb{Z}_{18} \setminus \mathbb{Z}_{12}) \cup \{22, 23\}), (\mathbb{Z}_{12} \cup \{18, 19, 20, 21\}), C_5)$ be a 4-cycle system of $K(8, 16)$. Finally, let $(K(\mathbb{Z}_6, \mathbb{Z}_{12} \setminus \mathbb{Z}_6), C_6)$ be a 4-cycle system of $K(6, 6)$. Then $(\cup_{i \in \mathbb{Z}_7}, C_i)$ is a 4-cycle system of $K_{25} - E(F^*)$.

Now we turn to the cases in which all cycles in the nearly 2-regular leave have size 5.

Lemma 38. *There exists a 4-cycle system of $K_n - E(F^*)$ where F^* is a nearly 2-regular leave, n and F^* are chosen to be any of the following.*

1. $n = 7$ and F^* consists of one 5-cycle which contains the vertex ∞ .
2. $n = 17$ and F^* consists of four 5-cycles, all of which intersect precisely in the vertex ∞ .

Proof. We will consider these two cases in turn.

1. Let $V(K_n) = \mathbb{Z}_6 \cup \{\infty\}$. Define

$$F_0^* = (\infty, 0, 1, 2)$$

and

$$F^* = (F_0^*)$$

Let $(\mathbb{Z}_6 \cup \{\infty\}, C_0)$ be a 4-cycle system of $K_7 - E(F^*)$ (see Lemma 29).

2. Let $V(K_n) = \mathbb{Z}_{16} \cup \{\infty\}$. Define

$$F_i^* = (\infty, 4i, 4i + 1, 4i + 2, 4i + 3) \text{ for each } i \in \mathbb{Z}_4$$

and

$$F^* = (\cup_{i \in \mathbb{Z}_4} F_i^*)$$

Let $(\mathbb{Z}_4 \cup \{\infty, 9, 13\}, C_0)$ be a 4-cycle system of $K_7 - E(F_0^*)$ (see Lemma 29). Similarly,

let $(((\mathbb{Z}_8 \setminus \mathbb{Z}_4) \cup \{\infty, 10, 14\}), C_1)$ be a 4-cycle system of $K_7 - E(F_1^*)$ (see Lemma 29).

Let $(\mathbb{Z}_{16} \setminus \mathbb{Z}_8) \cup \{\infty\}, C_2)$ be a 4-cycle system of $K_9 - E(\cup_{i \in \mathbb{Z}_4} F_i^* \cup (\infty, 9, 13) \cup (\infty, 10, 14))$ (see Lemma 33). Now, let $(K((\mathbb{Z}_8 \setminus \mathbb{Z}_4), (\mathbb{Z}_4 \cup \{9, 13\})), C_3)$ be a 4-cycle system

of $K(4, 6)$, using Theorem 5.1. Similarly, let $(K(\{14, 10\}, \mathbb{Z}_4), C_4)$ be a 4-cycle system

of $K(2, 4)$. Finally, let $(K(\{8, 11, 12, 15\}, \mathbb{Z}_8), C_5)$ be a 4-cycle system of $K(4, 8)$.

Then $(\cup_{i \in \mathbb{Z}_6} C_i)$ is a 4-cycle system of $K_{17} - E(F^*)$.

Lemma 39. *There exists a 4-cycle system of $K_n - E(F^*)$ where F^* is a nearly 2-regular leave, n and F^* are chosen to be any of the following.*

1. $n = 29$ and F^* consists of six 5-cycles, two of which intersect precisely in the vertex ∞ and the other cycles do not contain ∞ .
2. $n = 35$ and F^* consists of seven 5-cycles, two of which intersect precisely in the vertex ∞ and the other cycles do not contain ∞ .

3. $n = 41$ and F^* consists of eight 5-cycles, two of which intersect precisely in the vertex ∞ and the other cycles do not contain ∞ .
4. $n = 47$ and F^* consists of nine 5-cycles, two of which intersect precisely in the vertex ∞ and the other cycles do not contain ∞ .

Proof. We will consider these four cases in turn.

1. Let $V(K_n) = \mathbb{Z}_{28} \cup \{\infty\}$. Define

$$F_i^* = \begin{cases} (\infty, 4i, 4i + 1, 4i + 2, 4i + 3) & \text{for each } i \in \mathbb{Z}_2, \\ (i, i + 1, i + 2, i + 3, i + 4) & \text{for each } i \in \{8, 13, 18, 23\}, \\ G_i^* = (8, i, i + 1, i + 2, i + 3) & \text{for each } i \in \{9, 14, 19, 24\}, \end{cases}$$

and

$$F^* = ((\cup_{i \in \mathbb{Z}_2 \cup \{8, 13, 18, 23\}} (F_i^*)))$$

Let $(\mathbb{Z}_{28} \setminus (\mathbb{Z}_8 \cup \{13, 18, 23\}), C_0)$ be a 4-cycle system of $K_{17} - E(\cup_{i \in \{9, 14, 19, 24\}} G_i^*)$ (see Lemma 38(2)). Let $(\mathbb{Z}_9 \cup \{13, 18, 23, \infty\}, C_1)$ be a 4-cycle system of $K_{13} - E(\cup_{i \in \mathbb{Z}_2} F_i^* \cup \{\infty, 13, 18, 23\})$ (see Lemma 13(5)). And, let $(K(\mathbb{Z}_9 \cup \{13, 18, 23\}), (\mathbb{Z}_{28} \setminus (\mathbb{Z}_8 \cup \{8, 13, 18, 23\})), C_2)$ be a 4-cycle system of $K(12, 16)$, using Theorem 5.1. Then $((\cup_{i \in \mathbb{Z}_3} C_i) \cup (8, 13, 18, 23) \cup \{(14, 8, 17, 0), (19, 8, 22, 0), (24, 8, 27, 0)\} \setminus \{(14, 13, 17, 0), (19, 18, 22, 0), (24, 23, 27, 0)\})$ be a 4-cycle system of $K_{29} - E(F^*)$.

2. Let $V(K_n) = \mathbb{Z}_{34} \cup \{\infty\}$. Define

$$F_i^* = \begin{cases} (5i, 5i + 1, 5i + 2, 5i + 3, 5i + 4) & \text{for each } i \in \mathbb{Z}_5 \\ (\infty, 4i + 1, 4i + 2, 4i + 3, 4i + 4) & \text{for each } i \in \mathbb{Z}_8 \setminus \mathbb{Z}_6 \end{cases}$$

and

$$F^* = (\cup_{i \in \mathbb{Z}_7 \setminus \{5\}} (F_i^*))$$

Let $(\mathbb{Z}_{20} \cup (\mathbb{Z}_{33} \setminus \mathbb{Z}_{25}) \cup \{\infty\}, C_0)$ be a 4-cycle system of $K_{29} - E(\cup_{i \in \mathbb{Z}_8 \setminus \{4,5\}} F_i^*)$ (see Lemma 39(1)). Similarly, let $(((\mathbb{Z}_{25} \setminus \mathbb{Z}_{20}) \cup \{\infty, 33\}), C_1)$ be a 4-cycle system of $K_7 - E(F_4^*)$ (see Lemma 29). Now, let $(K(((\mathbb{Z}_{25} \setminus \mathbb{Z}_{20}) \cup \{33\}), (\mathbb{Z}_{20} \cup (\mathbb{Z}_{33} \setminus \mathbb{Z}_{25}))), C_2)$ be a 4-cycle system of $K(6, 28)$, using Theorem 5.1. Then $(\cup_{i \in \mathbb{Z}_3} C_i)$ is a 4-cycle system of $K_{35} - E(F^*)$.

3. Let $V(K_n) = \mathbb{Z}_{40} \cup \{\infty\}$. Define

$$F_i^* = \begin{cases} (5i, 5i + 1, 5i + 2, 5i + 3, 5i + 4) & \text{for each } i \in \mathbb{Z}_6 \\ (\infty, 4i - 2, 4i - 1, 4i, 4i + 1) & \text{for each } i \in \mathbb{Z}_{10} \setminus \mathbb{Z}_8 \end{cases}$$

and

$$F^* = (\cup_{i \in \mathbb{Z}_{10} \setminus \{6,7\}} (F_i^*))$$

Let $(\mathbb{Z}_{25} \cup (\mathbb{Z}_{39} \setminus \mathbb{Z}_{30}) \cup \{\infty\}, C_0)$ be a 4-cycle system of $K_{35} - E(\cup_{i \in \mathbb{Z}_{10} \setminus \{5,6,7\}} F_i^*)$ (see Lemma 39(2)). Similarly, let $(((\mathbb{Z}_{30} \setminus \mathbb{Z}_{25}) \cup \{\infty, 39\}), C_1)$ be a 4-cycle system of $K_7 - E(F_5^*)$ (see Lemma 29). Now, let $(K(((\mathbb{Z}_{30} \setminus \mathbb{Z}_{25}) \cup \{39\}), (\mathbb{Z}_{25} \cup (\mathbb{Z}_{39} \setminus \mathbb{Z}_{30}))), C_2)$ be a 4-cycle system of $K(6, 34)$, using Theorem 5.1. Then $(\cup_{i \in \mathbb{Z}_3} C_i)$ is a 4-cycle system of $K_{41} - E(F^*)$.

4. Let $V(K_n) = \mathbb{Z}_{46} \cup \{\infty\}$. Define

$$F_i^* = \begin{cases} (5i, 5i + 1, 5i + 2, 5i + 3, 5i + 4) & \text{for each } i \in \mathbb{Z}_7 \\ (\infty, 4i - 1, 4i, 4i + 1, 4i + 2) & \text{for each } i \in \mathbb{Z}_{11} \setminus \mathbb{Z}_9 \end{cases}$$

and

$$F^* = (\cup_{i \in \mathbb{Z}_{11} \setminus \{7,8\}} (F_i^*))$$

Let $(\mathbb{Z}_{30} \cup (\mathbb{Z}_{45} \setminus \mathbb{Z}_{35}) \cup \{\infty\}, C_0)$ be a 4-cycle system of $K_{41} - E(\cup_{i \in \mathbb{Z}_{11} \setminus \{6,7,8\}} F_i^*)$ (see Lemma 39(3)). Similarly, let $(((\mathbb{Z}_{35} \setminus \mathbb{Z}_{30}) \cup \{\infty, 45\}), C_1)$ be a 4-cycle system of $K_7 - E(F_6^*)$ (see Lemma 29). Now, let $(K(((\mathbb{Z}_{35} \setminus \mathbb{Z}_{30}) \cup \{45\}), (\mathbb{Z}_{30} \cup (\mathbb{Z}_{45} \setminus \mathbb{Z}_{35}))), C_2)$ be a 4-cycle system of $K(6, 40)$, using Theorem 5.1. Then $(\cup_{i \in \mathbb{Z}_3} C_i)$ is a 4-cycle system of $K_{47} - E(F^*)$.

Lemma 40. *There exists a 4-cycle system of $K_n - E(F^*)$ where F^* is a nearly 2-regular leave, n and F^* are chosen to be any of the following.*

1. $n = 23$ and F^* consists of five 5-cycles, three of which intersect precisely in the vertex ∞ and the other cycles do not contain ∞ .
2. $n = 29$ and F^* consists of six 5-cycles, three of which intersect precisely in the vertex ∞ and the other cycles do not contain ∞ .
3. $n = 35$ and F^* consists of seven 5-cycles, three of which intersect precisely in the vertex ∞ and the other cycles do not contain ∞ .
4. $n = 41$ and F^* consists of eight 5-cycles, three of which intersect precisely in the vertex ∞ and the other cycles do not contain ∞ .

Proof. We will consider these four cases in turn.

1. Let $V(K_n) = \mathbb{Z}_{22} \cup \{\infty\}$. Define

$$F_i^* = \begin{cases} (5i, 5i + 1, 5i + 2, 5i + 3, 5i + 4) & \text{for each } i \in \mathbb{Z}_2, \\ (\infty, 4i - 2, 4i - 1, 4i, 4i + 1) & \text{for each } i \in \mathbb{Z}_6 \setminus \mathbb{Z}_3, \end{cases}$$

$$G_i^* = (\infty, i, i + 1, i + 2, i + 3) \quad \text{for each } i \in \{1, 6, 10, 14\},$$

and

$$F^* = (\cup_{i \in \mathbb{Z}_6 \setminus \{2\}} F_i^*)$$

Let $(((\mathbb{Z}_{18} \cup \{\infty\}) \setminus \{0, 5\}), C_0)$ be a 4-cycle system of $K_{17} - E(\cup_{i \in \{1, 6, 10, 14\}} G_i^*)$ (see Lemma 38(2)). And, $((\mathbb{Z}_{22} \setminus \mathbb{Z}_{18}) \cup \{\infty, 0, 5\}), C_1)$ be a 4-cycle system of $K_7 - E(F_5^*)$ (see Lemma 29). $(K((\mathbb{Z}_{22} \setminus \mathbb{Z}_{18}) \cup \{0, 5\}), \mathbb{Z}_{18} \setminus \{0, 5\}), C_2)$ be a 4-cycle system of $K(6, 16)$, using Theorem 5.1. Then, $((\cup_{i \in \mathbb{Z}_3} C_i) \cup \{(1, 5, 4, \infty), (6, 0, 9, \infty)\} \setminus \{(1, 5, 4, 0), (6, 0, 9, 5)\})$ is a 4-cycle system of $K_{23} - E(F^*)$.

2. Let $V(K_n) = \mathbb{Z}_{28} \cup \{\infty\}$. Define

$$F_i^* = \begin{cases} (5i, 5i + 1, 5i + 2, 5i + 3, 5i + 4) & \text{for each } i \in \mathbb{Z}_3, \\ (\infty, 4i - 1, 4i, 4i + 1, 4i + 2) & \text{for each } i \in \mathbb{Z}_7 \setminus \mathbb{Z}_4, \end{cases}$$

and

$$F^* = (\cup_{i \in \mathbb{Z}_7 \setminus \{3\}} (F_i^*))$$

Let $(\mathbb{Z}_{10} \cup (\mathbb{Z}_{27} \setminus \mathbb{Z}_{15}) \{ \infty \}, C_0)$ be a 4-cycle system of $K_{23} - E(\cup_{i \in \mathbb{Z}_7 \setminus \{2,3\}} F_i^*)$ (see Lemma 40(1)). Similarly, let $((\mathbb{Z}_{15} \setminus \mathbb{Z}_{10}) \cup \infty, 27\}$, $C_1)$ be a 4-cycle system of $K_7 - E(F_2^*)$ (see Lemma 29). Now, let $(K((\mathbb{Z}_{15} \setminus \mathbb{Z}_{10}) \cup \{27\}), (\mathbb{Z}_{10} \cup (\mathbb{Z}_{27} \setminus \mathbb{Z}_{15}))), C_2)$ be a 4-cycle system of $K(6, 22)$, using Theorem 5.1. Then $(\cup_{i \in \mathbb{Z}_3} C_i)$ is a 4-cycle system of $K_{29} - E(F^*)$.

3. Let $V(K_n) = \mathbb{Z}_{34} \cup \{ \infty \}$. Define

$$F_i^* = \begin{cases} (5i, 5i + 1, 5i + 2, 5i + 3, 5i + 4) & \text{for each } i \in \mathbb{Z}_4, \\ (\infty, 4i, 4i + 1, 4i + 2, 4i + 3) & \text{for each } i \in \mathbb{Z}_8 \setminus \mathbb{Z}_5, \end{cases}$$

and

$$F^* = (\cup_{i \in \mathbb{Z}_8 \setminus \{4\}} (F_i^*))$$

Let $(\mathbb{Z}_{15} \cup (\mathbb{Z}_{33} \setminus \mathbb{Z}_{20}) \{ \infty \}, C_0)$ be a 4-cycle system of $K_{29} - E(\cup_{i \in \mathbb{Z}_8 \setminus \{3,4\}} F_i^*)$ (see Lemma 40(2)). Similarly, let $((\mathbb{Z}_{20} \setminus \mathbb{Z}_{15}) \cup \{ \infty, 33 \}), C_1)$ be a 4-cycle system of $K_7 - E(F_3^*)$ (see Lemma 29). Now, let $(K((\mathbb{Z}_{20} \setminus \mathbb{Z}_{15}) \cup \{33\}), (\mathbb{Z}_{15} \cup (\mathbb{Z}_{33} \setminus \mathbb{Z}_{20}))), C_2)$ be a 4-cycle system of $K(6, 28)$, using Theorem 5.1. Then $(\cup_{i \in \mathbb{Z}_3} C_i)$ is a 4-cycle system of $K_{35} - E(F^*)$.

4. Let $V(K_n) = \mathbb{Z}_{40} \cup \{ \infty \}$. Define

$$F_i^* = \begin{cases} (5i, 5i + 1, 5i + 2, 5i + 3, 5i + 4) & \text{for each } i \in \mathbb{Z}_5, \\ (\infty, 4i - 3, 4i - 2, 4i - 1, 4i) & \text{for each } i \in \mathbb{Z}_{10} \setminus \mathbb{Z}_7, \end{cases}$$

and

$$F^* = (\cup_{i \in \mathbb{Z}_{10} \setminus \{5,6\}} (F_i^*))$$

Let $(\mathbb{Z}_{20} \cup (\mathbb{Z}_{39} \setminus \mathbb{Z}_{25}) \cup \{\infty\}, C_0)$ be a 4-cycle system of $K_{35} - E(\cup_{i \in \mathbb{Z}_{10} \setminus \{4,5,6\}} F_i^*)$ (see Lemma 39(3)). Similarly, let $(((\mathbb{Z}_{25} \setminus \mathbb{Z}_{20}) \cup \{\infty, 39\}), C_1)$ be a 4-cycle system of $K_7 - E(F_4^*)$ (see Lemma 29). Now, let $(K(((\mathbb{Z}_{25} \setminus \mathbb{Z}_{20}) \cup \{39\}), (\mathbb{Z}_{20} \cup (\mathbb{Z}_{39} \setminus \mathbb{Z}_{25}))), C_2)$ be a 4-cycle system of $K(6, 34)$, using Theorem 5.1. Then $(\cup_{i \in \mathbb{Z}_3} C_i)$ is a 4-cycle system of $K_{41} - E(F^*)$.

Finally, we turn to some small cases where the nearly 2-regular leave contains both 4-cycles and 5-cycles.

Lemma 41. *There exists a 4-cycle system of $K_n - E(F^*)$ where F^* is a nearly 2-regular leave which consists of x 4-cycles and y 5-cycles each intersecting in precisely the vertex ∞ where n , $x \leq 8$ and $y \leq 4$ can be chosen to be any of the following.*

1. $n = 23$, $x = 6$ and $y = 1$.
2. $n = 21$, $x = 4$ and $y = 2$.
3. $n = 19$, $x = 2$ and $y = 3$.
4. $n = 15$, $x = 3$ and $y = 1$.
5. $n = 13$, $x = 1$ and $y = 2$.
6. $n = 35$, $x = 7$ and $y = 3$.
7. $n = 29$, $x = 6$ and $y = 2$.
8. $n = 27$, $x = 4$ and $y = 3$.
9. $n = 23$, $x = 5$ and $y = 1$.
10. $n = 21$, $x = 3$ and $y = 2$.
11. $n = 19$, $x = 1$ and $y = 3$.

Proof. We will consider these eleven cases in turn.

1. Let $V(K_n) = \mathbb{Z}_{22} \cup \{\infty\}$. Define

$$F_i^* = \begin{cases} (\infty, 3i, 3i+1, 3i+2) & \text{for each } i \in \mathbb{Z}_6 \\ (\infty, 18, 19, 20, 21) & \text{for } i = 6 \end{cases}$$

and

$$F^* = (\cup_{i \in \mathbb{Z}_7} (F_i^*))$$

Let $(\mathbb{Z}_6 \cup \{\infty, 7, 10\}, C_0)$ be a 4-cycle system of $K_9 - E(\cup_{i \in \mathbb{Z}_2} F_i^*)$ (see Lemma 30). Similarly, let $(((\mathbb{Z}_{22} \setminus \mathbb{Z}_{18}) \cup \{\infty, 13, 16\}), C_1)$ be a 4-cycle system of $K_7 - E(F_6^*)$ (see Lemma 29). And $(((\mathbb{Z}_{18} \setminus \mathbb{Z}_6) \cup \{\infty\}), C_2)$ be a 4-cycle system of $K_{13} - E(\cup_{i \in \mathbb{Z}_6 \setminus \mathbb{Z}_2} (F_i^*) \cup (\infty, 7, 10) \cup (\infty, 13, 16))$ (see Lemma 35). Now, let $(K(\{18, 19, 20, 21\}, \mathbb{Z}_{18} \setminus \{13, 16\}), C_3)$ be the 4-cycle system of $K(4, 16)$, using Theorem 5.1. Similarly, $(K(\mathbb{Z}_6, \mathbb{Z}_{18} \setminus (\mathbb{Z}_6 \cup \{7, 10\})), C_4)$ be a 4-cycle system of $K(6, 10)$. Then $(\cup_{i \in \mathbb{Z}_5} C_i)$ is a 4-cycle system of $K_{23} - E(F^*)$.

2. Let $V(K_n) = \mathbb{Z}_{20} \cup \{\infty\}$. Define

$$F_i^* = \begin{cases} (\infty, 3i, 3i+1, 3i+2) & \text{for each } i \in \mathbb{Z}_4 \\ (\infty, 12, 13, 14, 15) & \text{for } i = 4 \\ (\infty, 16, 17, 18, 19) & \text{for } i = 5 \end{cases}$$

and

$$F^* = (\cup_{i \in \mathbb{Z}_6} (F_i^*))$$

Let $(\mathbb{Z}_{16} \setminus \mathbb{Z}_{12} \cup \{\infty, 1, 4\}, C_0)$ be a 4-cycle system of $K_7 - E(F_4^*)$ (see Lemma 29). Similarly, let $(((\mathbb{Z}_{20} \setminus \mathbb{Z}_{16}) \cup \{\infty, 7, 10\}), C_1)$ be a 4-cycle system of $K_7 - E(F_5^*)$ (see Lemma 29). And $(((\mathbb{Z}_{12} \cup \{\infty\}), C_2)$ be a 4-cycle system of $K_{13} - E(\cup_{i \in \mathbb{Z}_4} (F_i^*) \cup (\infty, 1, 4) \cup (\infty, 7, 10))$ (see Lemma 35). Now, let $(K(\{16, 17, 18, 19\}, \mathbb{Z}_{16} \setminus \{7, 10\}), C_3)$ be a 4-cycle system of $K(4, 14)$, using Theorem 5.1. Similarly, $(K(\{12, 13, 14, 15\},$

$\mathbb{Z}_{12} \setminus \{1, 4\}, C_4$) be a 4-cycle system of $K(4, 10)$. Then $(\cup_{i \in \mathbb{Z}_5} C_i)$ is a 4-cycle system of $K_{21} - E(F^*)$.

3. Let $V(K_n) = \mathbb{Z}_{18} \cup \{\infty\}$. Define

$$F_i^* = \begin{cases} (\infty, 3i, 3i+1, 3i+2) & \text{for each } i \in \mathbb{Z}_2 \\ (\infty, 6, 7, 8, 9) & \text{for } i = 2 \\ (\infty, 10, 11, 12, 13) & \text{for } i = 3 \\ (\infty, 14, 15, 16, 17) & \text{for } i = 4 \end{cases}$$

and

$$F^* = (\cup_{i \in \mathbb{Z}_5}, (F_i^*))$$

Let $(\mathbb{Z}_6 \cup \{\infty, 11, 16\}, C_0)$ be a 4-cycle system of $K_9 - E(\cup_{i \in \mathbb{Z}_2} F_i^*)$ (see Lemma 30). Similarly, let $(((\mathbb{Z}_{10} \setminus \mathbb{Z}_6) \cup \{\infty, 12, 15\}), C_1)$ be a 4-cycle system of $K_7 - E(F_2^*)$ (see Lemma 29). And $(((\mathbb{Z}_{18} \setminus \mathbb{Z}_{10}) \cup \{\infty\}), C_2)$ be a 4-cycle system of $K_9 - E(F_3^* \cup F_4^* \cup (\infty, 10, 15) \cup (\infty, 11, 16))$ (see Lemma 33). Now, let $(K(\mathbb{Z}_6, \mathbb{Z}_{18} \setminus (\mathbb{Z}_6 \cup \{11, 16\})), C_3)$ be a 4-cycle system of $K(6, 10)$, using Theorem 5.1. Similarly, $(K(\{6, 7, 8, 9\}, \mathbb{Z}_{18} \setminus (\mathbb{Z}_{10} \cup \{12, 15\})), C_4)$ be a 4-cycle system of $K(4, 6)$. Then $(\cup_{i \in \mathbb{Z}_5}, C_i)$ is a 4-cycle system of $K_{19} - E(F^*)$.

4. Let $V(K_n) = \mathbb{Z}_{14} \cup \{\infty\}$. Define

$$F_i^* = \begin{cases} (\infty, 3i, 3i+1, 3i+2) & \text{for each } i \in \mathbb{Z}_3 \\ (\infty, 9, 10, 11, 12) & \text{for } i = 3 \end{cases}$$

and

$$F^* = (\cup_{i \in \mathbb{Z}_4}, (F_i^*))$$

Let $(\mathbb{Z}_6 \cup \{\infty, 7, 10\}, C_0)$ be a 4-cycle system of $K_9 - E(\cup_{i \in \mathbb{Z}_2} F_i^*)$ (see Lemma 30). Similarly, let $(((\mathbb{Z}_{14} \setminus \mathbb{Z}_6) \cup \{\infty\}), C_1)$ be a 4-cycle system of $K_9 - E(F_2^* \cup F_3^* \cup (\infty, 7, 10))$

(see Lemma 32). Now, let $(K(\mathbb{Z}_6, \mathbb{Z}_{14} \setminus (\mathbb{Z}_6 \cup \{7, 10\})), C_2)$ be a 4-cycle system of $K(6, 6)$, using Theorem 5.1. Then $(\cup_{i \in \mathbb{Z}_3}, C_i)$ is a 4-cycle system of $K_{15} - E(F^*)$.

5. Let $V(K_n) = \mathbb{Z}_{12} \cup \{\infty\}$. Define

$$F_i^* = \begin{cases} (\infty, 4i, 4i + 1, 4i + 2, 4i + 3) & \text{for each } i \in \mathbb{Z}_2 \\ (\infty, 8, 9, 10) & \text{for } i = 2 \end{cases}$$

and

$$F^* = (\cup_{i \in \mathbb{Z}_3} (F_i^*))$$

Let $(\mathbb{Z}_4 \cup \{\infty, 6, 9\}, C_0)$ be a 4-cycle system of $K_7 - E(F_0^*)$ (see Lemma 29). Similarly, let $(((\mathbb{Z}_{12} \setminus \mathbb{Z}_4) \cup \{\infty\}), C_1)$ be a 4-cycle system of $K_9 - E((F_1^* \cup F_2^*) \cup (\infty, 6, 10))$ (see Lemma 32). Now, let $(K(\mathbb{Z}_4, \mathbb{Z}_{12} \setminus (\mathbb{Z}_4 \cup \{6, 9\})), C_2)$ be a 4-cycle system of $K(4, 6)$, using Theorem 5.1. Then $(\cup_{i \in \mathbb{Z}_3}, C_i)$ is a 4-cycle system of $K_{13} - E(F^*)$.

6. Let $V(K_n) = \mathbb{Z}_{34} \cup \{\infty\}$. Define

$$\begin{aligned} F_i^* &= (\infty, 3i, 3i + 1, 3i + 2) && \text{for each } i \in \mathbb{Z}_7 \\ G_i^* &= (\infty, 4i + 1, 4i + 2, 4i + 3, 4i + 4) && \text{for each } i \in \mathbb{Z}_8 \setminus \mathbb{Z}_5 \end{aligned}$$

and

$$F^* = ((\cup_{i \in \mathbb{Z}_7} (F_i^*)) \cup (\cup_{i \in \mathbb{Z}_8 \setminus \mathbb{Z}_5} (G_i^*)))$$

Let $(\mathbb{Z}_6 \cup \{\infty, 19, 22\}, C_0)$ be a 4-cycle system of $K_9 - E(\cup_{i \in \mathbb{Z}_2} F_i^*)$ (see Lemma 30). Similarly, let $(((\mathbb{Z}_{25} \setminus \mathbb{Z}_{18}) \cup \{\infty, 33\}), C_1)$ be a 4-cycle system of $K_9 - E(F_6^* \cup G_5^* \cup \{\infty, 19, 22\})$ (see Lemma 32). And $(((\mathbb{Z}_{29} \setminus \mathbb{Z}_{25}) \cup \{\infty, 7, 13\}), C_2)$ be a 4-cycle system of $K_7 - E(G_6^*)$ (see Lemma 29). Also, let $(((\mathbb{Z}_{33} \setminus \mathbb{Z}_{29}) \cup \{\infty, 10, 16\}), C_3)$ be a 4-cycle system of $K_7 - E(G_7^*)$ (see Lemma 29). Let $(((\mathbb{Z}_{18} \setminus \mathbb{Z}_6) \cup \{\infty\}), C_4)$ be a 4-cycle system of $K_{13} - E((\cup_{i \in \mathbb{Z}_6 \setminus \mathbb{Z}_2} F_i^*) \cup \{\infty, 7, 13\} \cup \{\infty, 10, 16\})$ (see Lemma 35). Now, let $(K(\mathbb{Z}_6, \mathbb{Z}_{34} \setminus (\mathbb{Z}_6 \cup \{19, 22\})), C_5)$ be a 4-cycle system of $K(6, 26)$, using Theorem 5.1.

Similarly, $(K((\mathbb{Z}_{25} \setminus \mathbb{Z}_{18}) \cup \{33\}, \mathbb{Z}_{33} \setminus (\mathbb{Z}_6 \cup (\mathbb{Z}_{25} \setminus \mathbb{Z}_{18}))), C_6)$ be a 4-cycle system of $K(8, 20)$. And $(K((\mathbb{Z}_{29} \setminus \mathbb{Z}_{25}), ((\mathbb{Z}_{18} \setminus \mathbb{Z}_6) \cup (\mathbb{Z}_{33} \setminus \mathbb{Z}_{29}))), C_7)$ be a 4-cycle system of $K(4, 16)$. Finally, let $(K((\mathbb{Z}_{33} \setminus \mathbb{Z}_{29}), (\mathbb{Z}_{18} \setminus (\mathbb{Z}_6 \cup \{7, 10, 13, 16\}))), C_8)$ be a 4-cycle system of $K(4, 8)$. Then $(\cup_{i \in \mathbb{Z}_9}, C_i)$ is a 4-cycle system of $K_{35} - E(F^*)$.

7. Let $V(K_n) = \mathbb{Z}_{28} \cup \{\infty\}$. Define

$$\begin{aligned} F_i^* &= (\infty, 3i, 3i + 1, 3i + 2) && \text{for each } i \in \mathbb{Z}_6 \\ G_i^* &= (\infty, 4i - 2, 4i - 1, 4i, 4i + 1) && \text{for each } i \in \mathbb{Z}_7 \setminus \mathbb{Z}_5 \end{aligned}$$

and

$$F^* = ((\cup_{i \in \mathbb{Z}_6} (F_i^*)) \cup (\cup_{i \in \mathbb{Z}_7 \setminus \mathbb{Z}_5} (G_i^*)))$$

Let $(\mathbb{Z}_{22} \setminus \mathbb{Z}_{18} \cup \{\infty, 1, 7\}, C_0)$ be a 4-cycle system of $K_7 - E(G_5^*)$ (see Lemma 29). $(\mathbb{Z}_{26} \setminus \mathbb{Z}_{22} \cup \{\infty, 4, 10\}, C_1)$ be a 4-cycle system of $K_7 - E(G_6^*)$ (see Lemma 29). Similarly, let $(((\mathbb{Z}_{18} \setminus \mathbb{Z}_{12}) \cup \{\infty, 26, 27\}), C_2)$ be a 4-cycle system of $K_9 - E(\cup_{i \in \mathbb{Z}_6 \setminus \mathbb{Z}_4} F_i^*)$ (see Lemma 30). $(\mathbb{Z}_{12} \cup \{\infty\}, C_3)$ be a 4-cycle system of $K_{13} - E(\cup_{i \in \mathbb{Z}_4} F_i^* \cup (\infty, 4, 10) \cup (\infty, 1, 7))$ (see Lemma 35). Using Theorem 5.1, let $(K((\mathbb{Z}_{22} \setminus \mathbb{Z}_{18}), (\mathbb{Z}_{28} \setminus ((\mathbb{Z}_{22} \setminus \mathbb{Z}_{18}) \cup \{1, 7\}))), C_4)$ be a 4-cycle system of $K(4, 22)$. And, $(K((\mathbb{Z}_{26} \setminus \mathbb{Z}_{22}), ((\mathbb{Z}_{18} \setminus \{10, 14\}) \cup \{26, 27\}))), C_5)$ be a 4-cycle system of $K(4, 16)$. Similarly, $(K((\mathbb{Z}_{18} \setminus \mathbb{Z}_{12}) \cup \{26, 27\}, \mathbb{Z}_{12}), C_6)$ be a 4-cycle system of $K(8, 12)$. Then $(\cup_{i \in \mathbb{Z}_7} C_i)$ is a 4-cycle system of $K_{29} - E(F^*)$.

8. Let $V(K_n) = \mathbb{Z}_{26} \cup \{\infty\}$. Define

$$\begin{aligned} F_i^* &= (\infty, 3i, 3i + 1, 3i + 2) && \text{for each } i \in \mathbb{Z}_4 \\ G_i^* &= (\infty, 4i, 4i + 1, 4i + 2, 4i + 3) && \text{for each } i \in \mathbb{Z}_6 \setminus \mathbb{Z}_3 \end{aligned}$$

and

$$F^* = ((\cup_{i \in \mathbb{Z}_4} (F_i^*)) \cup (\cup_{i \in \mathbb{Z}_6 \setminus \mathbb{Z}_3} (G_i^*)))$$

Let $(\mathbb{Z}_{16} \setminus \mathbb{Z}_{12} \cup \{\infty, 1, 7\}, C_0)$ be a 4-cycle system of $K_7 - E(G_3^*)$ (see Lemma 29). $(\mathbb{Z}_{20} \setminus \mathbb{Z}_{16} \cup \{\infty, 4, 10\}, C_1)$ be a 4-cycle system of $K_7 - E(G_4^*)$ (see Lemma 29). Similarly, let $((\mathbb{Z}_{24} \setminus \mathbb{Z}_{20}) \cup \{\infty, 24, 25\}, C_2)$ is a 4-cycle system of $K_7 - E(G_5^*)$ (see Lemma 29). $(\mathbb{Z}_{12} \cup \{\infty\}, C_3)$ be a 4-cycle system of $K_{13} - E(\cup_{i \in \mathbb{Z}_4} F_i^* \cup (\infty, 1, 7) \cup (\infty, 4, 10))$ (see Lemma 35). Using Theorem 5.1, let $(K((\mathbb{Z}_{16} \setminus \mathbb{Z}_{12}), (\mathbb{Z}_{26} \setminus ((\mathbb{Z}_{16} \setminus \mathbb{Z}_{12}) \cup \{1, 7\}))), C_4)$ be a 4-cycle system of $K(4, 20)$. And, $(K((\mathbb{Z}_{20} \setminus \mathbb{Z}_{16}), \mathbb{Z}_{26} \setminus ((\mathbb{Z}_{20} \setminus \mathbb{Z}_{12}) \cup \{4, 10\}))), C_5)$ be a 4-cycle system of $K(4, 16)$. Similarly, $(K((\mathbb{Z}_{26} \setminus \mathbb{Z}_{20}), \mathbb{Z}_{12}), C_6)$ is a 4-cycle system of $K(6, 12)$. Then $(\cup_{i \in \mathbb{Z}_7}, C_i)$ is a 4-cycle system of $K_{27} - E(F^*)$.

9. Let $V(K_n) = \mathbb{Z}_{22} \cup \{\infty\}$. Define

$$F_i^* = \begin{cases} (\infty, 3i, 3i + 1, 3i + 2) & \text{for each } i \in \mathbb{Z}_5 \\ (\infty, 15, 16, 17, 18) & \text{for } i = 5 \end{cases}$$

and

$$F^* = (\cup_{i \in \mathbb{Z}_6} (F_i^*))$$

Let $(\mathbb{Z}_6 \cup \{\infty, 13, 16\}, C_0)$ be a 4-cycle system of $K_9 - E(\cup_{i \in \mathbb{Z}_2} F_i^*)$ (see Lemma 30). $(\mathbb{Z}_{20} \setminus \mathbb{Z}_{12} \cup \{\infty\}, C_1)$ be a 4-cycle system of $K_9 - E(\cup_{i \in \mathbb{Z}_6 \setminus \mathbb{Z}_4} F_i^* \cup (\infty, 13, 16))$ (see Lemma 32). Similarly, let $((\mathbb{Z}_{12} \setminus \mathbb{Z}_6) \cup \{\infty, 20, 21\}, C_2)$ be a 4-cycle system of $K_9 - E(\cup_{i \in \mathbb{Z}_4 \setminus \mathbb{Z}_2} F_i^*)$ (see Lemma 30). Using Theorem 5.1, let $(K(\mathbb{Z}_6, (\mathbb{Z}_{22} \setminus (\mathbb{Z}_6 \cup \{13, 16\}))), C_3)$ be a 4-cycle system of $K(6, 14)$. By Theorem 5.1 let, $(K((\mathbb{Z}_{20} \setminus \mathbb{Z}_{12}), (\mathbb{Z}_{22} \setminus (\mathbb{Z}_6 \cup (\mathbb{Z}_{20} \setminus \mathbb{Z}_{12}))))), C_4)$ be a 4-cycle system of $K(8, 8)$. Then $(\cup_{i \in \mathbb{Z}_5}, C_i)$ is a 4-cycle system of $K_{23} - E(F^*)$.

10. Let $V(K_n) = \mathbb{Z}_{20} \cup \{\infty\}$. Define

$$F_i^* = (\infty, 3i, 3i + 1, 3i + 2) \quad \text{for each } i \in \mathbb{Z}_3$$

$$G_i^* = (\infty, 4i - 3, 4i - 2, 4i - 1, 4i) \quad \text{for each } i \in \mathbb{Z}_5 \setminus \mathbb{Z}_3$$

and

$$F^* = ((\cup_{i \in \mathbb{Z}_3} (F_i^*)) \cup (\cup_{i \in \mathbb{Z}_5 \setminus \mathbb{Z}_3} (G_i^*)))$$

Let $(\mathbb{Z}_6 \cup \{\infty, 7, 10\}, C_0)$ be a 4-cycle system of $K_9 - E(\cup_{i \in \mathbb{Z}_2} F_i^*)$ (see Lemma 30). $((\mathbb{Z}_{13} \setminus \mathbb{Z}_6) \cup \{\infty, 19\}, C_1)$ be a 4-cycle system of $K_9 - E(F_2^* \cup G_3^* \cup \{\infty, 7, 10\})$ (see Lemma 32). Similarly, let $((\mathbb{Z}_{19} \setminus \mathbb{Z}_{13}) \cup \{\infty\}, C_2)$ be a 4-cycle system of $K_7 - E(G_4^*)$ (see Lemma 29). Now, using Theorem 5.1, let $(K(\mathbb{Z}_6, (\mathbb{Z}_{20} \setminus (\mathbb{Z}_6 \cup \{7, 10\}))), C_3)$ be a 4-cycle system of $K(6, 12)$. Similarly, $(K((\mathbb{Z}_{19} \setminus \mathbb{Z}_{13}), (\mathbb{Z}_{13} \setminus \mathbb{Z}_6) \cup \{19\}), C_4)$ be a 4-cycle system of $K(6, 8)$. Then $(\cup_{i \in \mathbb{Z}_5}, C_i)$ is a 4-cycle system of $K_{21} - E(F^*)$.

11. Let $V(K_n) = \mathbb{Z}_{18} \cup \{\infty\}$. Define

$$\begin{aligned} F_i^* &= (\infty, 3i, 3i + 1, 3i + 2) && \text{for } i = 0 \\ G_i^* &= (\infty, 4i - 1, 4i, 4i + 1, 4i + 2) && \text{for each } i \in \mathbb{Z}_4 \setminus \mathbb{Z}_1 \end{aligned}$$

and

$$F^* = ((F_0^*) \cup (\cup_{i \in \mathbb{Z}_4 \setminus \mathbb{Z}_1} (G_i^*)))$$

Let $((\mathbb{Z}_{11} \setminus \mathbb{Z}_7) \cup \{\infty, 1, 4\}, C_0)$ be a 4-cycle system of $K_7 - E(G_2^*)$ (see Lemma 29). $((\mathbb{Z}_{17} \setminus \mathbb{Z}_{11}) \cup \{\infty\}, C_1)$ be a 4-cycle system of $K_7 - E(G_3^*)$ (see Lemma 29). Similarly, let $((\mathbb{Z}_7 \cup \{\infty, 17\}), C_2)$ be a 4-cycle system of $K_9 - E(F_0^* \cup G_1^* \cup \{\infty, 1, 4\})$ (see Lemma 32). Now, using Theorem 5.1 let, $(K((\mathbb{Z}_{11} \setminus \mathbb{Z}_7), (\mathbb{Z}_{18} \setminus ((\mathbb{Z}_{11} \setminus \mathbb{Z}_7) \cup \{1, 4\}))), C_3)$ be a 4-cycle system of $K(4, 12)$. Similarly, $(K((\mathbb{Z}_{17} \setminus \mathbb{Z}_{11}), (\mathbb{Z}_7 \cup \{17\})), C_4)$ be a 4-cycle system of $K(6, 8)$. Then $(\cup_{i \in \mathbb{Z}_5}, C_i)$ is a 4-cycle system of $K_{19} - E(F^*)$.

5.3 Proposition

In this section we give constructions for obtaining 4-cycle systems of $K_n - E(F^*)$ for some general values of n .

Proposition 19. *Suppose $n = 24x + y$, $x, y \in \mathbb{Z}^+$ and F^* consists of 4 cycles, all of which are incident with the vertex ∞ where $y \in \{1, 9, 17\}$ for some $x \in \mathbb{N}$. Then there exists a 4-cycle system of $K_n - E(F^*)$*

Proof. We know that $y \in \{1, 9, 17\}$ for some $x \in \mathbb{N}$. When $y \in \{9, 17\}$ the required 4-cycle system exists by Lemma 36. And the remaining 4-cycle systems exist by the following construction.

Let $V(K_n) = \{\{\infty\} \cup \{\mathbb{Z}_{24} \times \mathbb{Z}_x\} \cup \{\mathbb{Z}_{y-1}\}\}$. For each $z \in \mathbb{Z}_x$ let, $(\{\infty\} \cup (\mathbb{Z}_{24} \times \{z\}), C_z)$ be a 4-cycle system of order 25 which exists, by Lemma 36. $(\{\infty\} \cup \mathbb{Z}_{y-1}, C_y)$ be a 4-cycle system of order y which exists, by Lemma 36. By Theorem ??, for $0 \leq i < j < x$ let, $(\{\mathbb{Z}_{24} \times \{i\}\} \cup \{\mathbb{Z}_{24} \times \{j\}\}, C(i, j))$ be a 4-cycle system of (K_{24}, K_{24}) and for all $i \in \mathbb{Z}_x$, $(\{\mathbb{Z}_{24} \times \{i\}\} \cup \{\mathbb{Z}_{y-1}\}, C(i, x))$ be a 4-cycle system of (K_{24}, K_{y-1}) . Then $((\cup_{z \in \mathbb{Z}_{x+1}} C_z) \cup (\cup_{0 \leq i < j < x} C(i, j)) \cup (\cup_{i \in \mathbb{Z}_x} C(i, x)) \cup C_y)$ is a 4-cycle system of $K_n - E(F^*)$. \square

Proposition 20. *Suppose $n = 24x + y$, $x, y \in \mathbb{Z}^+$ and F^* consists of 4 cycles of which all but one is incident with the vertex ∞ where $y \in \{1, 9, 17\}$ for some $x \in \mathbb{N}$. Then there exists a 4-cycle system of $K_n - E(F^*)$*

Proof. We know that $y \in \{1, 9, 17\}$ for some $x \in \mathbb{N}$. When $y \in \{9, 17\}$, the required 4-cycles system exists by Lemma 36. And the remaining 4-cycle systems exist by the following construction.

Let $V(K_n) = \{\{\infty\} \cup \{\mathbb{Z}_{24} \times \mathbb{Z}_x\} \cup \{\mathbb{Z}_{y-1}\}\}$. For each $z \in \mathbb{Z}_x$ let, $(\{\infty\} \cup (\mathbb{Z}_{24} \times \{z\}), C_z)$ be a 4-cycle system of order 25 which exists, by Lemma 37. $(\{\infty\} \cup \mathbb{Z}_{y-1}, C_y)$ be a 4-cycle system of order y which exists, by Lemma 37. By Theorem ??, for $0 \leq i < j < x$ let, $(\{\mathbb{Z}_{24} \times \{i\}\} \cup \{\mathbb{Z}_{24} \times \{j\}\}, C(i, j))$ be a 4-cycle system of (K_{24}, K_{24}) and for all $i \in \mathbb{Z}_x$, $(\{\mathbb{Z}_{24} \times \{i\}\} \cup \{\mathbb{Z}_{y-1}\}, C(i, x))$ be a 4-cycle system of (K_{24}, K_{y-1}) . Then $((\cup_{z \in \mathbb{Z}_{x+1}} C_z) \cup (\cup_{0 \leq i < j < x} C(i, j)) \cup (\cup_{i \in \mathbb{Z}_x} C(i, x)) \cup C_y)$ is a 4-cycle system of $K_n - E(F^*)$. \square

Proposition 21. *Suppose $n = 16x + y$, $x, y \in \mathbb{Z}^+$ and F^* consists of 5 cycles, all of which are incident with the vertex ∞ where $y \in \{1, 7\}$ for some $x \in \mathbb{N}$. Then there exists a 4-cycle system of $K_n - E(F^*)$*

Proof. We know that $y \in \{1, 7\}$ for some $x \in \mathbb{N}$. When $y = 7$, the required 4-cycle system exists by Lemma 29. And, the remaining 4-cycle systems can be obtained by the following construction.

Let $V(K_n) = \{\{\infty\} \cup \{\mathbb{Z}_{16} \times \mathbb{Z}_x\} \cup \{\mathbb{Z}_{y-1}\}\}$. Using Lemma 38, we see that for each $z \in \mathbb{Z}_x$, $(\{\infty\} \cup \{\mathbb{Z}_{16} \times \{z\}\}, C_z)$ be a 4-cycle system of order 17. Similarly, $(\{\infty\} \cup \{\mathbb{Z}_{y-1}\}, C_y)$ be a 4-cycle system of order y using Lemma 38. Using Theorem ??, for $0 \leq i < j < x$, $(\{\mathbb{Z}_{16} \times \{i\}\} \cup \{\mathbb{Z}_{16} \times \{j\}\}, C(i, j))$ be a 4-cycle system of (K_{16}, K_{16}) and for all $i \in \mathbb{Z}_x$, $(\{\mathbb{Z}_{16} \times \{i\}\} \cup \{\mathbb{Z}_{y-1}\}, C(i, x))$ be a 4-cycle system of (K_{16}, K_{y-1}) . Thus $((\cup_{z \in \mathbb{Z}_{x+1}} C_z) \cup (\cup_{0 \leq i < j < x} C(i, j)) \cup (\cup_{i \in \mathbb{Z}_x} C(i, x)) \cup C_y)$ is a 4-cycle system of $K_n - E(F^*)$. \square

Proposition 22. *Suppose $n = 40x + y$, $x, y \in \mathbb{Z}^+$ and F^* consists of 5 cycles of which only two cycles intersect precisely in the vertex ∞ where $y \in \{1, 7, 29, 35\}$ for some $x \in \mathbb{N}$. Then there exists a 4-cycle system of $K_n - E(F^*)$*

Proof. We know that $y \in \{1, 7, 29, 35\}$ for some $x \in \mathbb{N}$. When $y \in \{29, 35\}$ the 4-cycle system exists by Lemma 39. The remaining 4-cycle systems can be obtained by the following construction.

Let $V(K_n) = \{\{\infty\} \cup \{\mathbb{Z}_{40} \times \mathbb{Z}_x\} \cup \{\mathbb{Z}_{y-1}\}\}$. Using Lemma 39, we see that for each $z \in \mathbb{Z}_x$, $(\{\infty\} \cup \{\mathbb{Z}_{40} \times \{z\}\}, C_z)$ be a 4-cycle system of order 41. Similarly, $(\{\infty\} \cup \{\mathbb{Z}_{y-1}\}, C_y)$ be a 4-cycle system of order 41 too. For $0 \leq i < j < x$, $(\{\mathbb{Z}_{40} \times \{i\}\} \cup \{\mathbb{Z}_{40} \times \{j\}\}, C(i, j))$ be a 4-cycle system of (K_{40}, K_{40}) and for all $i \in \mathbb{Z}_x$, $(\{\mathbb{Z}_{40} \times \{i\}\} \cup \{\mathbb{Z}_{y-1}\}, C(i, x))$ be a 4-cycle system of (K_{40}, K_{y-1}) . Thus $((\cup_{z \in \mathbb{Z}_{x+1}} C_z) \cup (\cup_{0 \leq i < j < x} C(i, j)) \cup (\cup_{i \in \mathbb{Z}_x} C(i, x)) \cup C_y)$ is a 4-cycle system of $K_n - E(F^*)$. \square

Proposition 23. *Suppose $n = 40x + y$, $x, y \in \mathbb{Z}^+$ and F^* consists of 5 cycles of which only three cycles intersect precisely in the vertex ∞ where $y \in \{1, 23, 29, 35\}$ for some $x \in \mathbb{N}$. Then there exists a 4-cycle system of $K_n - E(F^*)$*

Proof. We know that $y \in \{1, 23, 29, 35\}$ for some $x \in \mathbb{N}$. We know that $y \in \{23, 29, 35\}$ then there exists the required 4-cycle system by Lemma 40. The remaining 4-cycle systems can be obtained by the following construction.

Let $V(K_n) = \{\{\infty\} \cup \{\mathbb{Z}_{40} \times \mathbb{Z}_x\} \cup \{\mathbb{Z}_{y-1}\}\}$. Using Lemma 40, we see that for each $z \in \mathbb{Z}_x$, $(\{\infty\} \cup \{\mathbb{Z}_{40} \times \{z\}\}, C_z)$ be a 4-cycle system of order 41. Similarly, $(\{\infty\} \cup \{\mathbb{Z}_{y-1}\}, C_y)$ be a 4-cycle system of order 41 too. For $0 \leq i < j < x$, $(\{\mathbb{Z}_{40} \times \{i\}\} \cup \{\mathbb{Z}_{40} \times \{j\}\}, C(i, j))$ be a 4-cycle system of (K_{40}, K_{40}) and for all $i \in \mathbb{Z}_x$, $(\{\mathbb{Z}_{40} \times \{i\}\} \cup \{\mathbb{Z}_{y-1}\}, C(i, x))$ be a 4-cycle system of (K_{40}, K_{y-1}) . Thus $((\cup_{z \in \mathbb{Z}_{x+1}} C_z) \cup (\cup_{0 \leq i < j < x} C(i, j)) \cup (\cup_{i \in \mathbb{Z}_x} C(i, x)) \cup C_y)$ is a 4-cycle system of $K_n - E(F^*)$. \square

Proposition 24. *Suppose $n = 16x + y$, $x, y \in \mathbb{Z}^+$ and F^* consists of 5 cycles of which four or more cycles intersect precisely in the vertex ∞ where $y \in \{1\}$ for some $x \in \mathbb{N}$. Then there exists a 4-cycle system of $K_n - E(F^*)$*

Proof. The proof is similar to the proof of Proposition 19. \square

Proposition 25. *Suppose $n = 24x + y$, $x, y \in \mathbb{Z}^+$ and F^* consists of s 4 cycles and t 5 cycles all of which are incident with the vertex ∞ where $y \leq 4$ and $x \leq 8$. Then there exists a 4-cycle system of $K_n - E(F^*)$*

Proof. The proof is similar to the proof of Proposition 19. \square

And, now we prove the main result of this chapter.

Theorem 5.2. *There exists a 4-cycle system of $K_n - E(F^*)$ if and only if the following conditions are satisfied.*

1. n is odd, and
2. 4 divides $(K_n) - E(F^*)$

Proof. The necessity of condition (1) follows from the fact that in a 4-cycle system each vertex clearly has even degree. Condition (2) follows since each 4-cycle has four edges.

To prove the sufficiency we use induction on n , the total number of vertices. The result is trivially true when $n = 3$. Let $(\mathbb{Z}_4 \cup \{\infty\}, (0, 3, 1, 2))$ be the 4-cycle system of $K_5 - E((\infty, 0, 1) \cup (\infty, 2, 3))$ which has the only possible leave (of two 3-cycles) when $n = 5$. And the only possible leave when $n = 7$ is a 5-cycle which was considered in Lemma 29. So we can assume that $n = 2x + 1 \geq 9$, and suppose that for any odd $z < n$ and any nearly 2-regular leave \bar{F}^* for which, 4 divides $(K_z) - E(\bar{F}^*)$ there exists a 4-cycle system of $K_z - E(\bar{F}^*)$. It will be useful to notice that if

$$4 \text{ divides } (K_n - E(F^*)) \text{ then } 4 \text{ divides } (K_z - E(\bar{F}^*)) \quad (*)$$

where $z = n - 2, |E(\bar{F}^*)| = |E(F^*)| - 3$.

We construct the required 4-cycle systems by considering the following cases in turn. In each case, $V(K_n) = \mathbb{Z}_{n-1} \cup \{\infty\}$.

Case 1. F^* contains a cycle c , of length 3.

1. Suppose that c is incident with the vertex ∞ , say $c = (\infty, n - 2, n - 3)$. Then by (*) $(V(z), C_0)$ is the 4-cycle system of $K_{n-2} - E(\bar{F}^*)$, where $V(z) = \{\mathbb{Z}_{n-3} \cup \{\infty\}\}$ and $|E(\bar{F}^*)| = |E(F^*)| - c$. Also, $(K(\{n - 2, n - 3\}, V(z) \setminus \{\infty\}), C_1)$ will be a 4-cycle system of $K(2, n - 3)$ using Theorem 5.1. Then $(\cup_{i \in \mathbb{Z}_2}, C_i)$ is a 4-cycle system of $K_n - E(F^*)$.
2. Suppose that c is not incident with the vertex ∞ , say $c = (n - 2, n - 3, n - 4)$. Then $(V(z), C_0)$ be the 4-cycle system of $K_{n-2} - E(\bar{F}^*)$, where $V(z) = \{\mathbb{Z}_{n-3} \cup \{\infty\}\}$ and $|E(\bar{F}^*)| = |E(F^*)| - c$. Also, $(K(\{n - 2, n - 3\}, V(z) \setminus \{n - 4\}), C_1)$ be a 4-cycle system of $K(2, n - 3)$, using Theorem ???. Then $(\cup_{i \in \mathbb{Z}_2}, C_i)$ is a 4-cycle system of $K_n - E(F^*)$.

Case 2. F^* contains a cycle of length ≥ 6 .

The proof of this case is completely solved by [31] but is included here for the sake of continuity.

Let F^* contain a cycle of length l , where $l \geq 6$ (say $c = (n-1, n-2, \dots, n-l+1, n-l)$). Let $c_1 = (n-3, n-4, \dots, n-l)$. Let $E(\bar{F}^*) = (F^* - c) \cup c_1$. Then using the technique of induction, there exists a 6-cycle system of (\mathbb{Z}_{n-2}, C) of $K_{n-2} - E(\bar{F}^*)$. Now, let the edge $\{n-3, n-4\}$ be contained in the 4-cycle $c_2 = (n-3, n-4, a, b)$ contained in the cycle system C . Clearly, $b \neq n-l$ since $\{n-l, n-3\} \in c_1$. Let $C_1 = (C \setminus \{c_2\}) \cup \{(n-l, n-3, b, n-2), (b, a, n-3, n-1)\} \cup K(\{n-2, n-1\}, \mathbb{Z}_{n-2} \setminus \{n-1, n-3, b\})$. Then clearly, (\mathbb{Z}_n, C_1) is the 4-cycle system of $K_n - E(F^*)$.

Case 3. F^* contains only cycles of length 4.

1. Suppose that a cycle c , of length 4 is incident with the vertex ∞ , say $c = (\infty, n-2, n-3, n-4)$. And, suppose there exists a 4-cycle system of $K_n - E(F^*)$ when $n = 24x + y$, $x, y \in \mathbb{Z}^+$ and F^* consists of 4 cycles, all of which are incident with the vertex ∞ . Then $y \in \{1, 17, 9\}$ for some $x \in \mathbb{N}$ because if F^* contains 0, 1, 2 or 3 isolated vertices, condition (2) requires that $y = \{25, 17, 9, 25\}$ respectively. In each of these four cases by Lemma 36, the required 4-cycle system exists. The constructions for obtaining the 4-cycle systems for this case follows directly from the proof of Proposition 19.
2. Suppose that a cycle c , of length 4 is not incident with the vertex ∞ , say $c = (n-2, n-3, n-4, n-5)$. Now, suppose that there exists a 4-cycle system of $K_n - E(F^*)$ when $n = 24x + y$, $x, y \in \mathbb{Z}^+$ and F^* consists of 4 cycles, all of which all but one is incident with the vertex ∞ . Then $y \in \{17, 9, 1\}$ for some $x \in \mathbb{N}$ because if F^* contains 0, 1, 2 or 3 isolated vertices then condition (2) requires that $y = \{17, 9, 25, 17\}$ respectively. And the 4-cycle systems in these cases follow from Lemma 37. The

remaining constructions needed to complete the proof of this case follow directly from the proof of the Proposition 20.

Case 4. F^* contains only cycles of length 5.

1. F^* contains only cycles of length 5 of which only two cycles intersect precisely in the vertex ∞ . Suppose that there exists a 4-cycle system of $K_n - E(F^*)$ when $n = 40x + y$, $x, y \in \mathbb{Z}^+$ and F^* consists of 5 cycles, of which only two cycles intersect precisely in the vertex ∞ then $y \in \{29, 35, 11, 7\}$ for some $x \in \mathbb{N}$. Because, if F^* contains 0, 1, 2 or 3 isolated vertices then condition (2) requires that $y = \{29, 35, 41, 47\}$ respectively. The 4-cycle systems of these cases can be obtained from Lemma 39. The remaining constructions needed to complete this proof follow directly from Proposition 22.
2. F^* contains only cycles of length 5 of which only three cycles intersect precisely in the vertex ∞ . Now, suppose that there exists a 4-cycle system of $K_n - E(F^*)$ when $n = 40x + y$, $x, y \in \mathbb{Z}^+$ and F^* consists of 5 cycles of which only three cycles intersect precisely in the vertex ∞ then $y \in \{23, 29, 35, 1\}$ for some $x \in \mathbb{N}$. Because, if F^* contains 0, 1, 2 or 3 isolated vertices then condition (2) requires that $y = \{23, 29, 35, 41\}$ respectively. The 4-cycle systems of these cases follow from Lemma 40. The remaining constructions needed to complete this proof follow directly from Proposition 23.
3. F^* contains four or more cycles of length 5 which intersect precisely in the vertex ∞ . Suppose that there exists a 4-cycle system of $K_n - E(F^*)$ when $n = 16x + y$, $x, y \in \mathbb{Z}^+$ and F^* consists of 5 cycles, all of which are incident with the vertex ∞ then $y \in \{1, 7\}$ for some $x \in \mathbb{N}$. Because, if F^* contains 0 or 2 isolated vertices then condition (2) requires that $y = \{17, 7\}$ respectively. The 4-cycle systems for these two cases follow clearly from Lemma 38. The remaining constructions needed to complete this proof follow directly from Propositions 21 and Proposition 24.

Case 5. F^* contains only cycles of lengths 4 and 5 incident with the vertex ∞ .

The proof of this case follows directly from Lemma 41 and Proposition 25.

This proves our main result of this chapter.

□

Chapter 6

4-cycle system of the line graph of a complete multipartite graph: adding eight vertices

6.1 Introduction

In this chapter, we assume that there exists a 4-cycle system of $L(G)$, where G is a complete multipartite graph. Let $G = (V_1, V_2, \dots, V_p)$, and $|V(V_i)| = a_i$ for $1 \leq i \leq p$, where for $1 \leq i \leq p$, $V_i = \{v_{i,j} | 1 \leq j \leq a_i\}$. Also, we assume that the part size, a_i , is *odd* for $1 \leq i \leq p$ and that the number of parts, p , is *even*. Then, we obtain the results needed for a 4-cycle system of a new $L(G')$ obtained from $L(G)$ such that all but one parts are the same, the remaining part being increased in order by *eight* (i.e. we add *eight* vertices).

Let $G' = L(K(V'_1, V'_2, \dots, V'_p))$, where $V'_i = V_i$ for $2 \leq i \leq p$, and, $|V'_1| = |V_1| + 8$. The size of each part, $|V'_i|$ is denoted by a'_i . Also, each vertex in V'_i is denoted as $v_{i,j}'$ for $1 \leq i \leq p$ and $1 \leq j \leq a'_i$. Finally, we denote the clique formed by a vertex in the complete multipartite graph, say, v , by $c(v)$. We will need a result given by Sotteau, to prove our Theorem. Hence, it is stated below for reference.

Theorem 6.1. *There exists a 4-cycle system of $K_{a,b}$ and of $2K_{a,b}$ if and only if each vertex has even degree, the number of edges is divisible by 4, and $a, b \geq 2$.*

We now, look at the 4-cycle system of a complete graph on 9 vertices. We will use this Lemma to prove our main Theorem.

Lemma 42. *There exists a 4-cycle system of K_9 .*

Proof Let $V(K_9) = \mathbb{Z}_9$. Construct a base block $B_0 = (0, 1, 5, 2)$. Note that this base block contains exactly one edge of each of the four differences in K_9 . So, we can form the 4-cycle system $(\mathbb{Z}_9, \cup_{i \in \mathbb{Z}_9} (i, i+1, i+5, i+2))$ of K_9 .

Theorem 6.2. *If there exists a 4-cycle system of $G = L(K(V_1, \dots, V_p))$ then there exists a 4-cycle system of $G' = L(K(V'_1, \dots, V'_p))$*

Proof Before we look at the construction for this result, we make the following observation. The line graph of a graph G_1 , $L(G_1)$, can also be constructed as a union of the cliques of each vertex $v \in G_1$. We make use of this fact to decompose G' into cycles of length 4. We will denote the eight new vertices added to part V'_1 be denoted by v_i for $i \in \mathbb{N}_8$.

We first, decompose the cliques of all the vertices, $c(v_{1,j}')$ in part V'_1 . Now, note that the cliques of the vertices in part V_1 are the same as the cliques of the vertices in part V'_1 . We already have the 4-cycle system of $c(v_{1,j})$ for $v_{i,j} \in V_1$ from the 4-cycle system of G . Thus, we obtain the cycle systems of $c(v'_{1,j})$. Let C_0 denote the set of these 4-cycles.

- $C_0 = 4\text{-cycle system of } (\cup_{i \in \mathbb{N}_8} c(v_i)) \cup 4\text{-cycle system of } (c(v_{1,j}'))$

Now, the 4-cycle system of the $c(v_{i,j}')$ for $1 \leq j \leq a_i'$ and $2 \leq i \leq p$ can be constructed by a similar method. Consider the clique $c(v_{i,j}')$ formed by the vertex $v_j' \in V_i'$, for $1 \leq j \leq a_i'$ and $2 \leq i \leq p$. Suppose that, $|V(c(v_{i,j}))| = m$ for $v_{i,j} \in V_i$. Then clearly, m is odd. Since, by our assumption p is even and a_i' is odd for all $1 \leq i \leq p$. Thus, $|V(c(v_{i,j}))| = (p-1) * a_i$ is odd too. Thus,

$$|V(c(v_{i,j}'))| = m + 8 \quad \text{where } m \text{ is an odd number}$$

Also, there exists a 4-cycle of K_m (the edges in K_m have been used up in the 4-cycle system of G). If $|S| = 9$ then let $C(v_{i,j}; S)$ be a set of 4-cycles on the vertex set $\{\{v_{i,j}, s\} | s \in S\}$ in $c(v_{i,j})$ that form a 4-cycle system of K_9 using Lemma 42. Also, if $|S_1|$ and $|S_2|$ are even integers then let $C(v_{i,j}; S_1, S_2)$ be a set of 4-cycles of $K(\{\{v_{i,j}, s_1\} | s_1 \in$

$S_1\}$, $\{\{v_{i,j}, s_2\} | s_2 \in S_2\}$). Now, let C_1 be the set of 4-cycles obtained from the cliques $c(v_{i,j})$ for $1 \leq j \leq a_i'$ and $2 \leq i \leq p$, defined as follows:

$$\bullet C_1 = \left\{ \bigcup_{2 \leq i \leq p, 1 \leq j \leq a_i'} (C(v_{i,j}; \{v'_i | 1 \leq i \leq 8\} \cup \{v_{1,1}\})) \cup C(v_{i,j}; K(\{v'_i | 1 \leq i \leq 8\}, \{V(G') \setminus \{\{v'_i | 1 \leq i \leq 8\} \cup \{v_{1,1}\}\})) \right\}$$

Thus $(V(G'), \cup_{i \in \mathbb{Z}_2} C_i)$ is a 4-cycle system of G' . □

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