A New Solution to the Intersection Problem of Mendelsohn Triple Systems

by

Rachel Watson

A thesis submitted to the Graduate Faculty of
Auburn University
in partial fulfillment of the
requirements for the Degree of
Master of Science

Auburn, Alabama
May 7, 2012

Keywords: Mendelsohn triple system, intersection problem, cyclic triple

Approved by

Charles Lindner, Chair, Professor of Mathematics
Chris Rodger, Professor of Mathematics
Peter Johnson, Professor of Mathematics
Dean Hoffman, Professor of Mathematics

Abstract

This thesis gives a new and much simpler proof of the intersection problem for Mendelsohn triple systems.

THE INTERSECTION PROBLEM: For each $n \equiv 0$ or $1 (\text{mod } 3)$, $n \neq 6$, determine the set of all k such that there exists a pair of MTS(n) having exactly k cyclic triples in common. In what follows, we will set $I[n] = \{0, 1, 2, .., x = \frac{n(n-1)}{3}\} \setminus \{x-1, x-2, x-3, x-5\}$ and denote by $J[n] = \{k|$ there exists two MTS(n) having k cyclic triples in common$\}$.
In [2] it was shown that $J[3] = \{2\}, J[4] = \{0, 4\}$ and $J[n] = I[n]$ for all $n \geq 7$ ($n \equiv 0$ or $1 (\text{mod } 3)$, of course).

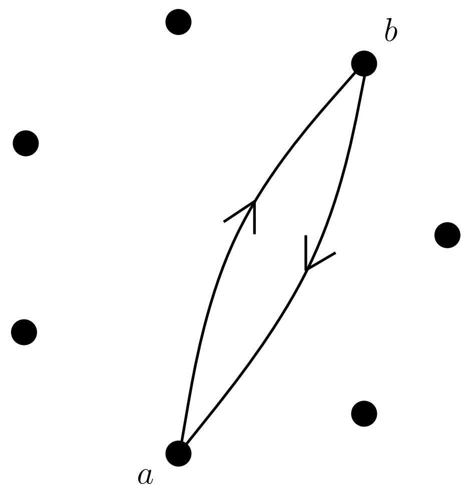
The objective of this thesis is a new and much simpler proof of the intersection problem using results completely different from those used in the original solution.

Table of Contents

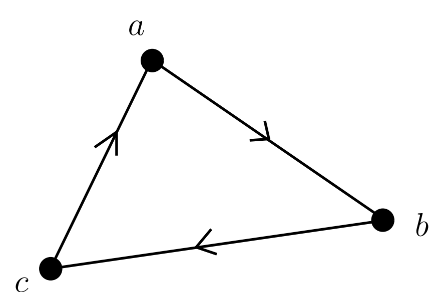
Chapter 1

Introduction and Outline

The complete directed graph D_n is the graph with n vertices in which each pair of distinct vertices are joined by two directed edges in opposite directions.

$D_n =$ 

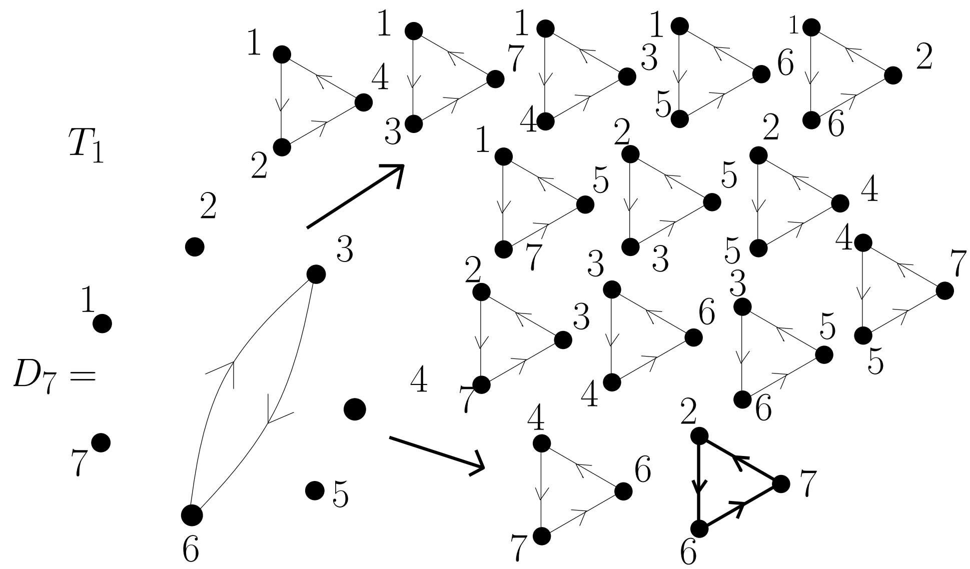
We will denote the directed edge from a to b by (a, b). A cyclic triple is a collection of three directed edges of the form $\{(a, b), (b, c), (c, a)\}$ where a, b, and c, are distinct.

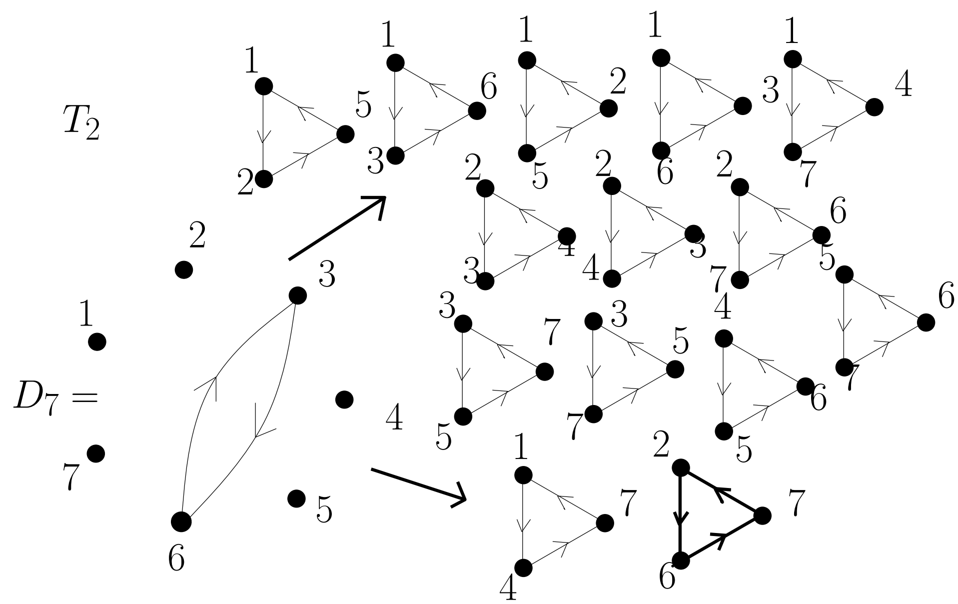


We will denote this cyclic triple by any cyclic shift of (a, b, c). Finally, a **Mendelsohn Triple System** (named after N.S. Mendelsohn[3]) of order n (MTS(n)) is a pair (S, T), where T is a collection of edge disjoint cyclic triples which partition D_n with vertex set S.

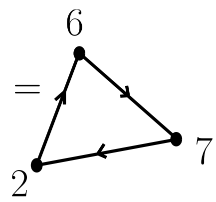
Example 1.1 *Two MTS* (7).

Let $S = \{1, 2, 3, 4, 5, 6, 7\}$ *and* T_1 *and* T_2 *be the following two MTS(7)s.*





It is immediate that the two MTS(7) in this example have exactly one cyclic triple in common, namely $(2, 6, 7)$.

$(2, 6, 7) =$ 

It is well-known that the spectrum for Mendelsohn Triple Systems is precisely the set of all $n \equiv 0$ or $1 \pmod 3$ EXCEPT for $n = 6$ (there does not exist a Mendelsohn Triple System of order 6), and if (S, T) is a MTS(n), $|T| = \frac{n(n-1)}{3}$.

THE INTERSECTION PROBLEM: For each $n \equiv 0$ or $1 \pmod 3$, $n \neq 6$, determine the set of all k such that there exists a pair of MTS(n) having exactly k cyclic triples in common. In what follows, we will set $I[n] = \{0, 1, 2, .., x = \frac{n(n-1)}{3}\} \setminus \{x-1, x-2, x-3, x-5\}$ and denote by $J[n] = \{k|$ there exists two MTS(n) having k cyclic triples in common$\}$.
In [2] it was shown that $J[3] = \{2\}, J[4] = \{0, 4\}$ and $J[n] = I[n]$ for all $n \geq 7$ ($n \equiv 0$ or $1 \pmod 3$, of course).

The objective of this thesis is a new and much more simple proof of the intersection problem using results completely different from those used in the original solution.

Chapter 2

The Intersection of Idempotent Quasigroups

A quasigroup $(Q, \circ)$ is said to be *idempotent* provided $x \circ x = x$ for all $x \in Q$. Two idempotent quasigroups are said to intersect in k products provided their tables agree in exactly k cells *off* of the main diagonal.

Example 2.1 (*Two idempotent quasigroups of order 6 intersecting in 4 products*).

$\circ_1$	1	2	3	4	5	6
1	1	6	2	5	3	4
2	4	2	5	6	1	3
3	2	4	3	1	6	5
4	5	3	6	4	2	1
5	6	1	4	3	5	2
6	3	5	1	2	4	6

$\circ_2$	1	2	3	4	5	6
1	1	5	6	2	4	3
2	6	2	5	1	3	4
3	5	4	3	6	2	1
4	3	1	2	4	6	5
5	4	6	1	3	5	2
6	2	3	4	5	1	6

In [1] H.L. Fu proved the following theorem.

Theorem 2.1 (*H.L. Fu* [1]). *If* $n \geq 6$, *there exists a pair of idempotent quasigroups of order* n *having* k *products in common if and only if* $k \in \{0, 1, 2, ..., x = n^2 - n\} \setminus \{x-1, x-2, x-3, x-5\}$. □

We will use this result to give a much simpler solution to the intersection problem beginning with order 18.

Chapter 3

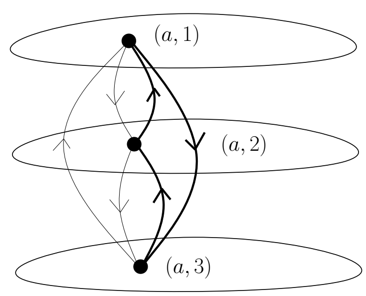
The Basic Constructions

We give three basic constructions in this chapter which will be used for all of the intersection results which follow.

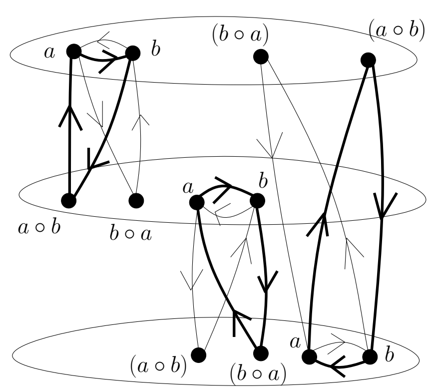
3.1 The $3n$ Construction

Let $(Q, \circ)$ be an idempotent quasigroup of order n, set $S = Q \times \{1, 2, 3\}$, and define a collection of cyclic triples T as follows:

1. $((a,1),(a,2),(a,3))$ and $((a,1),(a,3),(a,2)) \in T$ for all $a \in Q$; and



2. For each $a \neq b \in Q$ the six cyclic triples $((a,i),(b,i),(a \circ b, i+1))$, $((b,i),(a,i),(b \circ a, i+1)) \in T$.

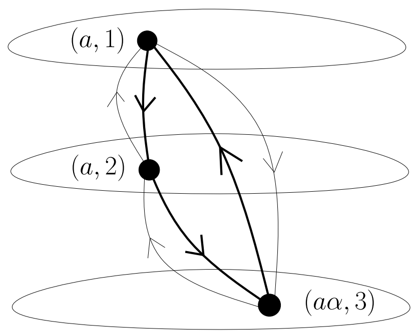


Then, (S, T) is a MTS($3n$).

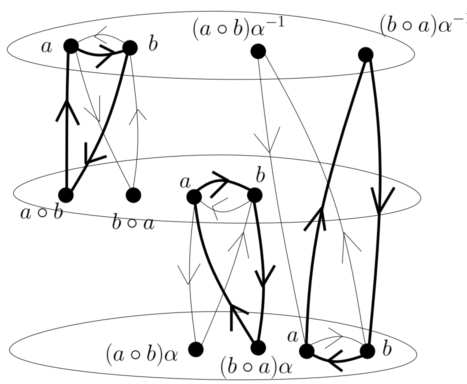
3.2 The $3n\alpha$ Construction:

Let $Q = \{1, 2, 3, ..., 3n\}$, $\alpha = (1, 2, 3, ..., 3n)$, and $(Q, \circ)$ an idempotent quasigroup of order n. Set $S = Q \times \{1, 2, 3\}$ and define a collection of cyclic triples $T\alpha$ as follows:

1. $((a, 1), (a, 2), (a\alpha, 3))$ and $((a, 1), (a\alpha, 3), (a, 2)) \in T\alpha$ for all $a \in Q$; and



2. For each $a \neq b \in Q$, the six cyclic triples $((a, 1), (b, 1), (a \circ b, 2))$, $((b, 1), (a, 1), (b \circ a, 2))$, $((a, 2), (b, 2), ((a \circ b)\alpha, 3))$, $((b, 2), (a, 2), ((b \circ a)\alpha, 3))$, $((a, 3), (b, 3), ((a \circ b)\alpha^{-1}, 1))$, $((b, 3), (a, 3), ((b \circ a)\alpha^{-1}, 1))$ belong to $T\alpha$.

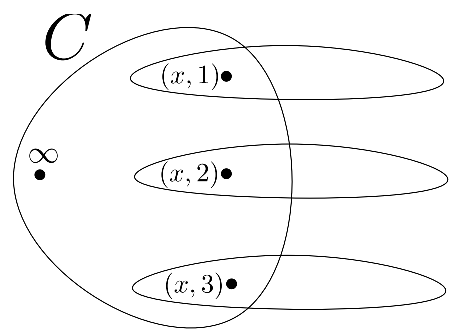


Then $(S, T\alpha)$ is a MTS($3n$).

3.3 The $3n + 1$ Construction:

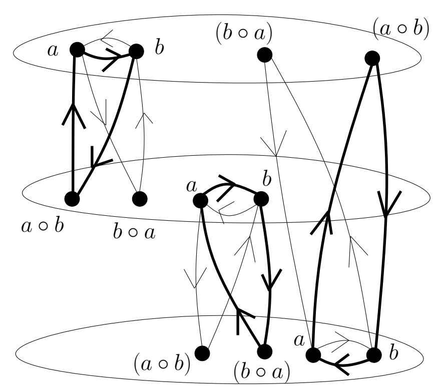
Let $(Q, \circ)$ be an idempotent quasigroup of order n, set $S = \{\infty\} \cup (Q \times \{1, 2, 3\})$, and define a collection of cyclic triples T as follows:

1. For each $x \in Q$, place a copy of $C = \{(\infty, 1, 2), (\infty, 1, 3), (\infty, 2, 3), (1, 2, 3)\}$ on $\{\infty\} \cup (\{x\} \times \{1, 2, 3\})$;



   and place these cyclic triples in T; and

2. for each $a \neq b \in Q$, place the six cyclic triples $((a, i), (b, i), (a \circ b, i+1))$, $((b, i), (a, i), (b \circ a, i+1)) \in T$.



Then (S, T) is a MTS$(3n + 1)$.

With these three constructions in hand, along with the results in Chapter 2, we can give a very simple and elegant solution to the intersection problem for Mendelsohn triple systems beginning with $n = 18$.

Chapter 4

The Solution for $3n + 1 \geq 19$

This is the easier of the two equivalence classes; so a good place to begin.

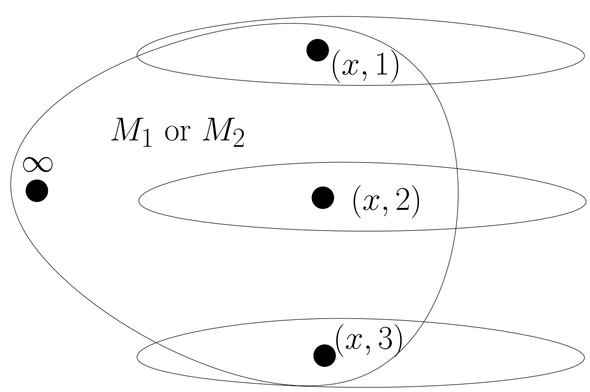
Let $(Q, \circ_{11})$, $(Q, \circ_{21})$, $(Q, \circ_{12})$, $(Q, \circ_{22})$, $(Q, \circ_{13})$, $(Q, \circ_{23})$ be any six idempotent quasigroups of order $n \geq 6$. Further, let M_1 and M_2 be the two Mendelsohn triple systems of order 4 defined below:

$$M_1 = \{(1, 2, 3), (2, 1, 4), (1, 3, 4), (3, 2, 4)\}, \textit{and}$$
$$M_2 = \{(1, 2, 4), (2, 1, 3), (1, 4, 3), (2, 3, 4)\}.$$

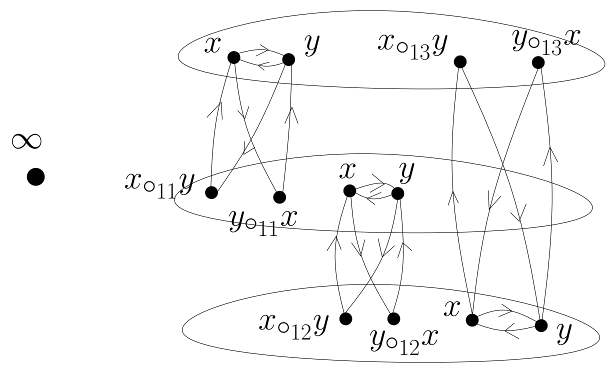
Then $M_1 \cap M_2 = \emptyset$.

Set $S = \{\infty\} \cup (Q \times \{1, 2, 3\})$ and define two MTS$(3n + 1)$s T_1 and T_2 as follows:

T_1**:** (i) For each $x \in Q$ place a copy of M_1 or M_2 on $\{\infty\} \cup (\{x\} \times \{1, 2, 3\})$ and place these cyclic triples in T_1.

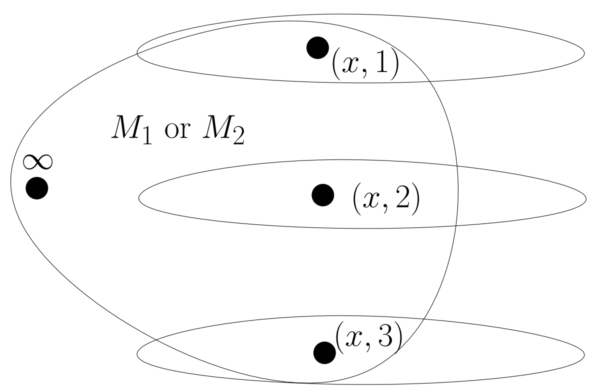


(*ii*) For each $x \neq y \in Q$ place the six cyclic triples $((x,i),(y,i),(x \circ_{1i} y, i+1))$ and $((y,i),(x,i),(y \circ_{1i} x, i+1))$ in T_1.

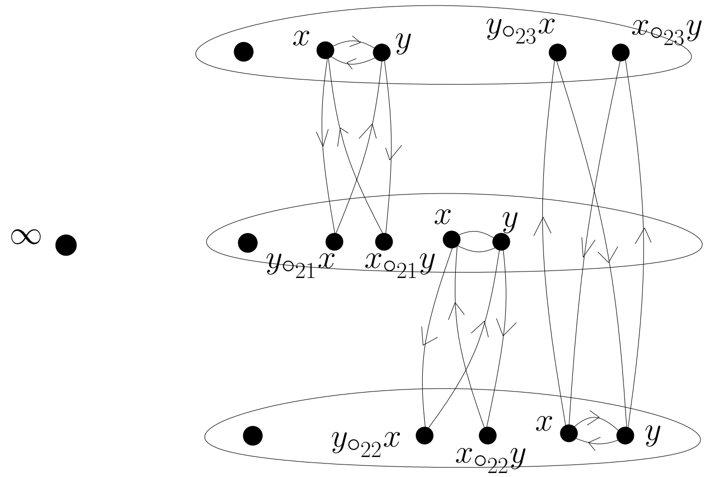


Then (S, T_1) is a MTS$(3n)$.

T_2**:** (*i*) For each $x \in Q$ place a copy of M_1 or M_2 on $\{\infty\} \cup (\{x\} \times \{1,2,3\})$ and place these cyclic triples in T_2.



(ii) For each $x \neq y \in Q$ place the six cyclic triples $((x,i),(y,i),(x \circ_{2i} y, i+1))$ and $((y,i),(x,i),(y \circ_{2i} x, i+1))$ in T_2.



Then (S, T_2) is a MTS$(3n)$.

It is immediate that the intersection number for (S, T_1) and (S, T_2) is $|T_1 \cap T_2| = \sum_{j=1}^{n} m + k_1 + k_2 + k_3$, where $m \in \{0, 4\}$ and $|(Q, \circ_{11}) \cap (Q, \circ_{21})| = k_1$, $|(Q, \circ_{12}) \cap (Q, \circ_{22})| = k_2$, $|(Q, \circ_{13}) \cap (Q, \circ_{23})| = k_3$. A straight-forward calculation shows that any $k \in I[3n+1]$ can be written in the form $\sum_{j=1}^{n} m + k_1 + k_2 + k_3$, where $k_1, k_2, k_3 \in \{0, 1, 2, ..., x = n^2 - n\} \setminus \{x-1, x-2, x-3, x-5\}$. Since $J[3n+1] \subseteq I[3n+1]$ (a necessary condition), it follows that $I[3n+1] \subseteq J[3n+1]$ so that $I[3n+1] = J[3n+1]$. We have the following result.

Lemma 4.1 $J[3n+1] = I[3n+1]$ *for all* $3n+1 \geq 19$. □

Chapter 5

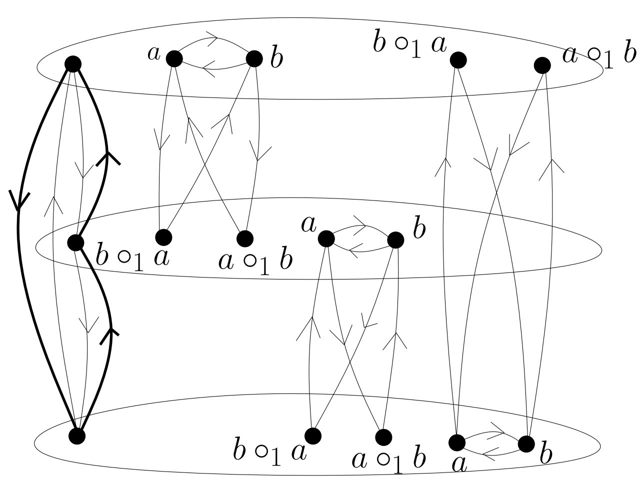
The Solution for $3n \geq 18$

There are two cases to consider here: (a) $k \leq 2n$, and (not too surprisingly) (b) $k \geq 2n$

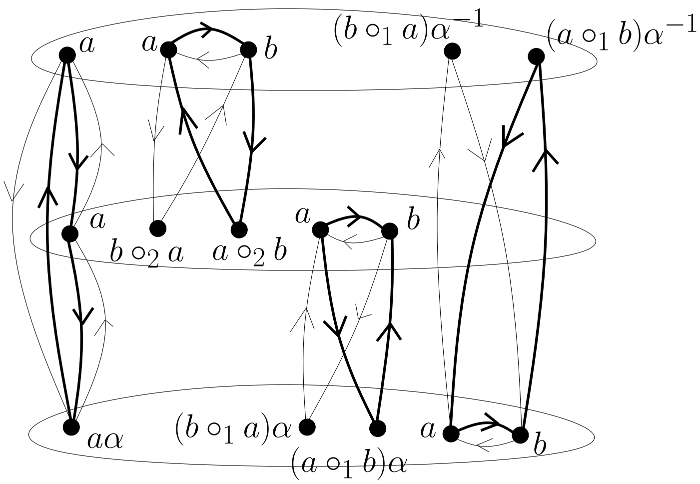
.

a) $k \leq 2n$. We will use the $3n$ and $3n\alpha$ Constructions here. Set $S = Q \times \{1, 2, 3\}$ and let $(Q, \circ_1)$ and $(Q, \circ_2)$ be a pair of idempotent quasigroups of order $n \geq 6$. Since $n \geq 6$, for any $k \in \{1, 2, 3, ..., x = n^2 - n\} \backslash \{x-1, x-2, x-3, x-5\}$, we can take $|(Q, \circ_1) \cap (Q, \circ_2)| = k$. It is important to note that any $k \leq 2n \in \{0, 1, 2, ..., x = n^2 - n\} \backslash \{x-1, x-2, x-3, x-5\}$. Now define two MTS($3n$)s T_1 and T_2 as follows:

T_1: Use the $3n$ Construction with $(Q, \circ_1)$.



T_2: Use the $3n\alpha$ Construction with $(Q, \circ_2)$ from the first to the second level; and $(Q, \circ_1)$ between the second and third, and third and first levels.



Clearly the intersection number for $T_1 \cap T_2$ is $|(Q, \circ_1) \cap (Q, \circ_2)| = k$.

b) $k \geq 2n$**.** In this case we use the $3n$ Construction with pairs of quasigroups as in the $3n+1$ solutions. It is immediate that any $k \geq 2n \in I[3n]$ can be written in the form $2n+k_1+k_2+k_3$ where $k_1, k_2, k_3 \in \{0, 1, 2, ..., x = n^2-n\} \setminus \{x-1, x-2, x-3, x-5\}$. Since $J[3n] \subseteq I[3n]$, it follows that $I[3n] \subseteq J[3n]$ and $I[3n] = J[3n]$. We have the following result.

Lemma 5.1 $J[3n] = I[3n]$ *for all* $3n \geq 18$. □

Chapter 6

Concluding Remarks

Combining Lemmas 4.1 and 5.1 gives the following result.

Theorem 6.1 *$J[n] = I[n]$ for all $n \equiv 0$ or $1 \pmod 3) \geq 18$, except $n = 6$ for which no MTS(6) exists.* □

The solution for the cases where $n \leq 16$ can be found in the original paper [2] and are handled by an eclectic collection of ad-hoc constructions. A quick glance at [2] will convince the reader that the solution for $n \geq 18$ given in this thesis is vastly superior to the original solution in its simplicity.

Bibliography

[1] Hang-Lin Fu, On the construction of certain types of latin squares with prescribed intersections, Ph. D. thesis, Auburn University, (1980).

[2] D.G. Hoffman and C.C. Lindner, Mendelsohn triple systems having a prescribed number of triples in common, Europ. J. Combinatorics, (1982), 51-61.

[3] N.S. Mendelsohn, A Natural Generalization of Steiner Triple Systems, in Computers in Number Theory, (A.O.L. Atkin and B.J. Birch, eds), Academic Press, London, 1971, 323-328 .