Competition Graphs

by

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Abstract

A competition graph is a simple graph G that correlates to a digraph D in the following way.

$$V(G) = V(D) \text{ and}$$

$$E(G) = \Big\{ \{u, v\} \mid \text{ there exists } x \in V(D) \text{ for which } \{(u, x), (v, x)\} \subseteq E(D) \Big\}.$$

Competition graphs were originally created in 1968 by Biologist Joel Cohen. In this paper we discuss four things. First, the use of linear algebra is considered with connection to competition graphs. Second, we generalize the idea of the competition graph into the m-step competition graph, and characterize P_n as an m-step competition graph. Third, we begin to characterize disjoint unions of graphs as m-step competition graphs. And last, we explore what happens if we treat m-step competition graphs as an infinite sequence, called a competition sequence.

Acknowledgments

I'd like to thank all of the professors on my committee. Especially my main advisor, Dr. Chris Rodger.

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Chapter 1

Introduction

This thesis concerns competition graphs. The Competition Graph $G = C(D)$ corresponding to the directed graph D is the simple graph defined in the following way.

$$V(G) = V(D) \text{ and}$$

$$E(G) = \Big\{\{u, v\} \mid \text{ there exists } x \in V(D) \text{ for which } \{(u, x), (v, x)\} \subseteq E(D)\Big\}.$$

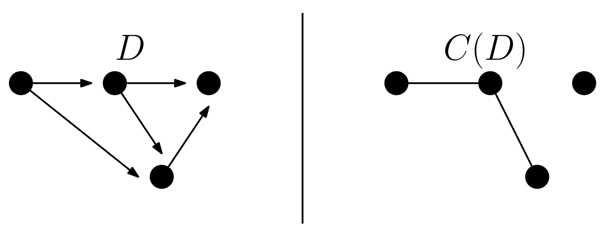


Figure 1.1: Competition Graph

The concept of competition graphs was first introduced by Joel Cohen[5]. Joel Cohen was a biologist, and his use for competition graphs was to model food webs. The digraph D would model a food web, and the arcs in this digraph modeled predator/prey relationships. The edges in the competition graph modeled two animals that shared prey. Due to this origin, when discussing competition graphs, we often refer to vertices as predators and prey. Vertex x preys on vertex y if $(x, y) \in E(D)$. Moreover, there are a few assumptions made about digraphs D because of this application. First, digraphs must be loopless because animals do not prey upon themselves. Secondly, all digraphs D must be acyclic because cycles do not occur naturally in the wilderness. It is worth noting, that both of these phenomena have actually occurred in natural food webs, but they are not very common. Moreover, another possible use for competition graphs is economic graphs. In this case, the digraph D models asset liability relationships in economic systems. Because of this, we must change the definition of the digraph slightly in this model.

$$E(G) = \Big\{ \{u, v\} \mid \text{ there exists } x \in V(D) \text{ for which } \{(x, v), (x, v)\} \in E(D) \Big\}$$

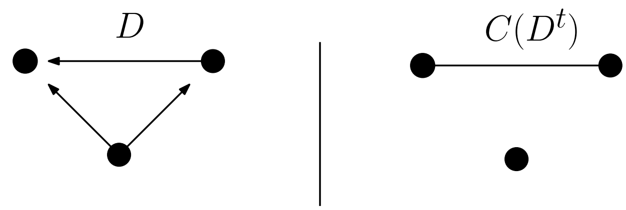


Figure 1.2: Economic Competition Graph

It is clear to see that this is equivalent to $C(D^t)$ where D^t is the digraph with all arcs reversed in D. However, early study of competition graphs kept these generalizations in place. For some of the early work in competition graphs see[12, 10, 5]. The major question in the early study of competition graphs was which simple graphs G could be realized as competition graphs? That is to say, given a simple graph G, does there exist a digraph D for which $C(D) = G$. It turns out that adding isolated vertices will eventually make it so. One naturally asks, if G could not be realized as a competition graph, how many isolated vertices must you add to make G realizable as a competition graph? This number of isolated vertices required is known as the *Competition Number*. The competition number of a simple graph G actually has been fully characterized, but we must introduce two definitions before we can state that result.

*C*lique: A *C*lique is a complete induced subgraph H of a simple graph G.

*C*lique Cover: A *C*lique Cover is a set of cliques in G with the property that every edge in G lies in at least one clique.

*C*lique Cover Number: The *C*lique Cover Number, $\psi(G)$, of a graph G is the minimum number of cliques required for which G can be covered with $\psi(G)$ cliques.

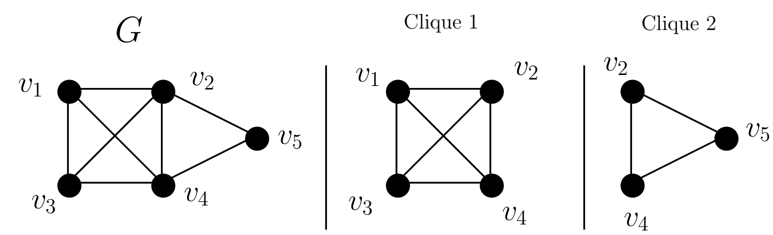


Figure 1.3: G with clique cover number 2

Theorem 1 *The competition number of a graph is* $\min\{0\ ,\ \psi(G) - |V(G)| + 2\}$.

This theorem will be more clear when the application of linear algebra to competition graphs is discussed.

Theorem 1 essentially ended the discussion on competition graphs themselves, but also led to the discussion of generalizations of competition graphs. There are many generalizations of competition graphs, but we are only concerned with two in this thesis.

The first generalization is to allow loops and cycles in the digraph D. However, this leads to a fairly obvious generalization of the previous theorem

Theorem 2 *The competition number of a graph is* $\min\{0\ ,\ \psi(G) - |V(G)|\}$ *if loops and cycles are allowed in* D.

The second generalization is much more exciting. This generalization was first introduced in [4]. If instead of looking at 1-step neighbors for competition, look at the more general m-step competition graph $G = C^m(D)$ of a digraph D. This changes the definition in the following way.

$$E(G) = \Big\{\{u,v\} \mid \text{ there exists } x \in V(D) \text{ for which there exist walks of length exactly } m \text{ from both } u \text{ and } v \text{ to } x\Big\}$$

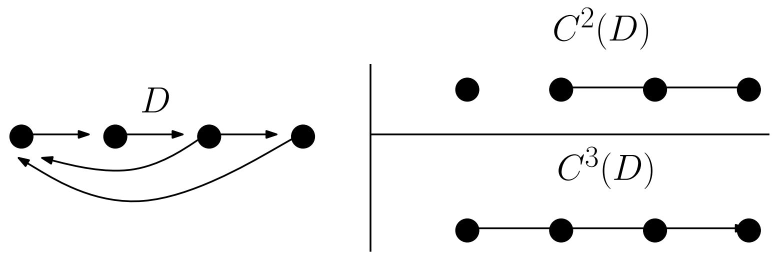


Figure 1.4: m-step Competition Graph Example

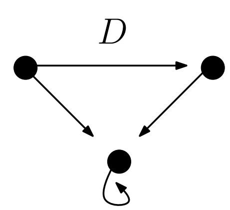
This will be the main topic of this thesis. Chapter 2 will introduce the linear algebra approach to this problem. Chapter 3 will discuss results characterizing paths and cycles as m-step competition graphs. Chapter 4 will introduce competition graphs as an infinite sequence. Chapter 5 will discuss further work to be done.

Chapter 2

Binary Linear Algebra

There are two approaches to the study of competition graphs: standard graph theory, and a linear algebra approach. In this chapter we introduce the linear algebra approach. When using linear algebra, it is appropriate to work over the boolean quasi-field $\mathcal{B}$. $\mathcal{B}$ is a quasi-field with the only elements being 0 and 1 in which addition and multiplication are defined as usual except that $1 + 1 = 1$. So for this entire chapter, assume all math is in $\mathcal{B}$.

Adjacency Matrix: For any directed graph D, the adjacency matrix A = A(D) is the $|V(D)| \times |V(D)|$ boolean matrix in which $(v_i, v_j) \in E(D)$ if and only if $A_{i,j} = 1$.



$$A = A(D)$$
$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Figure 2.1: The adjecency matrix $A(D)$ of a directed graph D

m-step digraph: An m-step digraph D^m of a directed graph D is a digraph for which $(v_i, v_j) \in E(D^m)$ if and only if there exists a directed walk of length m from v_i to v_j in D.

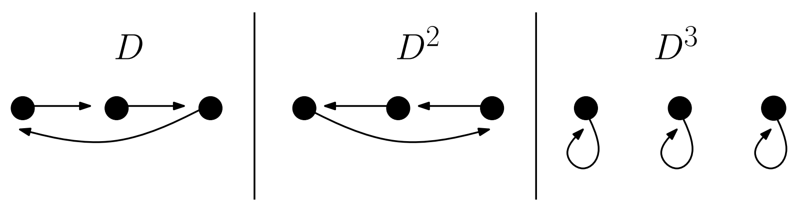


Figure 2.2: m-step Digraph Example

Obviously the adjacency matrix allows any digraph to be represented by a matrix, and therefore changes any graph theory problem into a linear algebra problem. However, this is only useful if some good properties hold. The following results are well known and can be found in most graph theory texts.

Observation 1 *A^m is the adjacency matrix of D^m*

Observation 2 *Two digraphs D_1 and D_2 with adjacency matrices A_1 and A_2 respectively are isomorphic if and only if A_1 and A_2 are isomorphic.*

There are also many more nice properties that hold. We will not delve deeply into the subject, but primitive and prime matrices have shown to be very useful in the study of competition graphs.

*P*rimitive Matrix: A matrix A is said to be *p*rimitive if there exists an integer m for which $A^m = J$ where J is the all 1s matrix.

*P*rime Matrix: A matrix A is said to be *p*rime if $A = B * C$ implies that either B or C is a permutation matrix.

For more work on prime and primitive matrices, see [7, 13, 6]. There is a very nice algorithm that will take A, the adjacency matrix of a digraph D, and transform it into $C(A)$ the adjacency matrix of $C(D)$. First we must notice that if $v_i, v_j, \ldots$ all prey upon a vertex x, then these vertices will all be in competition. Therefore $v_i, v_j, \ldots$ will form a clique in $C(D)$. We will need the definition of neighborhoods to state this formally.

$N^+(x) = \{v \mid (x, v) \in E(D)\}$

$N^-(x) = \{v \mid (v, x) \in E(D)\}$

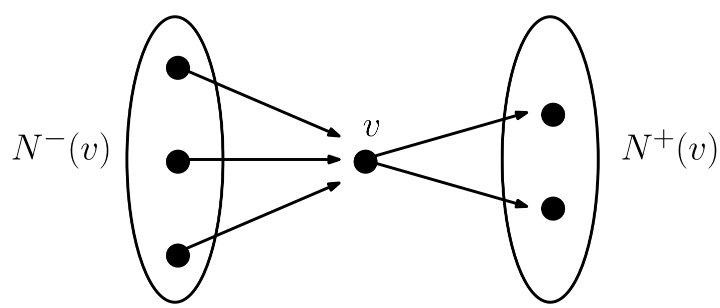


Figure 2.3: Neighborhood Example

It follows directly that all vertices in $N^{-}(x)$ will form a clique in $C(D)$ for all $x \in V(D)$. This totally exhausts all edges in $C(D)$. This leads to the following algorithmic construction for $C(A)$, the adjacency matrix for $C(D)$.

1. Start with the 0 matrix
2. Find all vertices in $N^{-}(v_1) = \{v_{i_1}, v_{i_2}, \ldots\}$
3. Replace the submatrix $M[\{i_1, i_2, \ldots\}, \{v_1, v_2, \ldots\}]$ with the J matrix.
4. Repeat step 2 for all $v_i \in V(D)$
5. Take the resulting matrix and subtract I.

Chapter 3

m-Step Competition Graph

This chapter will discuss results achieved regarding paths and cycles charterized as m-step competition graphs. We will also discuss unions of isomorphic graphs characterized as m-step competition graphs. We start with some general widely known results regarding m-step competition graphs.

Proposition 1 $C^m(D) = C(D^m)$

The following result appeared in [4]. This result is most easily seen by using the information introduced in the previous linear algebra chapter, as is the following result.

Corollary 1 *If G is an m-step competition graph, then G is also a k-step competition graph if k divides m.*

This follows directly from the observation that $C^m(D) = C^k(D^i)$ where $m = k * i$, which is portrayed nicely using adjacency matrices. Let A be the adjacency matrix of the digraph D for which $C^m(D) = G$. Then A^i is the adjacency matrix of a digraph D_i for which $C^k(D_i) = G$.

$$A \times A \times A \times A = A^4$$

$$A^2 \times A^2 = A^4$$

Figure 3.1: Use of adjacency matrices with $k = i = 2$

Theorem 3 *Let G be a triangle-free graph on n vertices. If G is an m-step competition graph for $m > n$, then G is S_n(the star graph with n vertices.)*

Theorem 3 was proved in [8]. Without going into details, it was proved that such graphs must have a "star forcing structure" through a beautiful induction technique. See Figure 3.2 for an example of a digraph D for which $C^m(D) = S_n$ for all values of m.

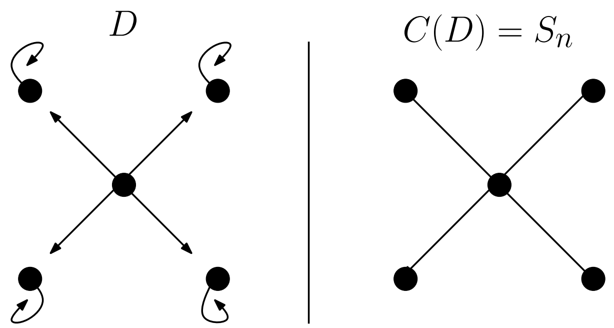


Figure 3.2: $C^i(D) = S_n$ for all $i \in \mathbb{Z}$

Lemma 1 *P_n is an $n-1$ and an $n-2$ competition graph.*

P_n is the $(n-1)$-step and $(n-2)$-step competition graph of the following directed graphs respectively defined by

$$n-1$$

$$\begin{cases} V(D) &= \{v_i \mid 1 \le i \le n\} \\ E(D) &= \{(v_i, v_{i+1}), (v_{n-1}, v_1) \mid 1 \le i \le n\} \end{cases}$$

$$n-2$$

$$\begin{cases} V(D) &= \{v_i \mid 1 \le i \le n\} \\ E(D) &= \{(v_i, v_{i+1}), (v_{n-1}, v_1), (v_{n-2}, v_n) \mid 1 \le i \le n\} \end{cases}$$

See Figure 3.3 for an example of P_7

Theorem 4 *P_n is a k-step competition graph if $n \equiv 1$(mod k) or $n \equiv 2$(mod k).*

The proof is actually very simple. It uses the previous observation along with Corollary 1. However, I wish to go a little more into detail, as well as give some direct construction of these graphs.

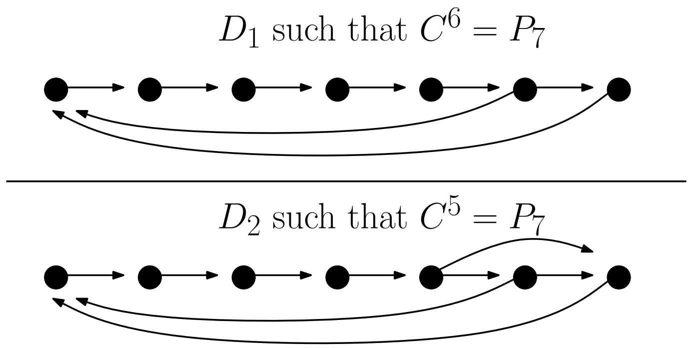
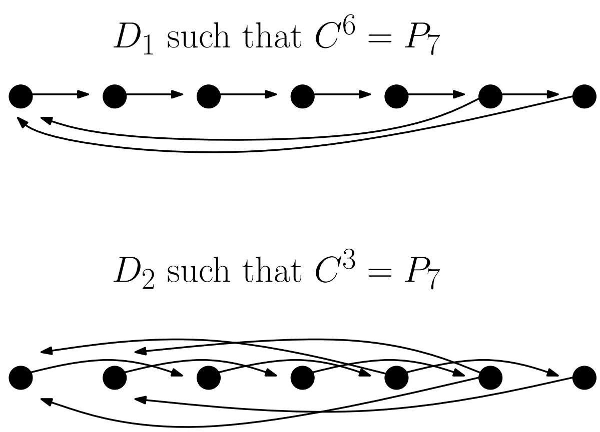


Figure 3.3: Digraphs D for which $C(D) = P_7$

The first construction utilizes linear algebra. If $n \equiv 1 (\text{mod } k)$, obtain the adjacency matrix A from the first digraph presented in Observation 1. Since $n \equiv 1 (\text{mod } k)$, there exists an integer i for which $k * i = n - 1$. $C^{n-1}(D) = C^k(D^i)$ by Corollary 1. A^i will be the adjacency matrix of D^i for which $C^k(D^i) = P_n$. If $n \equiv 2 (\text{mod } k)$, use the second digraph from Observation 1, and proceed likewise. See Figure 3.4 for an example for P_7.



A

$$\begin{bmatrix} 0&1&0&0&0&0&0 \\ 0&0&1&0&0&0&0 \\ 0&0&0&1&0&0&0 \\ 0&0&0&0&1&0&0 \\ 0&0&0&0&0&1&0 \\ 1&0&0&0&0&0&1 \\ 1&0&0&0&0&0&0 \end{bmatrix}$$

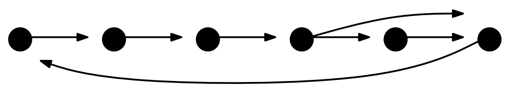
A^2

$$\begin{bmatrix} 0&0&1&0&0&0&0 \\ 0&0&0&1&0&0&0 \\ 0&0&0&0&1&0&0 \\ 0&0&0&0&0&1&0 \\ 1&0&0&0&0&0&1 \\ 1&1&0&0&0&0&0 \\ 0&1&0&0&0&0&0 \end{bmatrix}$$

Figure 3.4: Example where $n \equiv 1 (\text{mod } k)$.

The second construction is a direct Graph Theory construction. Let $k * i = n - 1$ or $k * i = n - 2$. The following is a construction for a digraph for which $C^i(D) = P_n$.

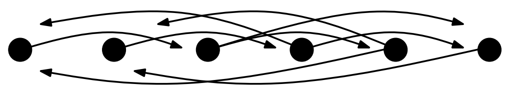
D_1 such than $C^4 = P_6$



A

$$\begin{bmatrix} 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&1\\ 1&0&0&0&0&1\\ 1&0&0&0&0&0 \end{bmatrix}$$

D_2 such that $C^2 = P_6$



A^2

$$\begin{bmatrix} 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&1\\ 1&0&0&0&0&1\\ 1&1&0&0&0&0\\ 0&1&0&0&0&0 \end{bmatrix}$$

Figure 3.5: Example where $n \equiv 2(\text{mod } k)$.

$$\begin{cases} V(D) &= \; v_i \mid 1 \leq i \leq n \\ E(D) &= \; \begin{cases} (v_i, v_{i+k}) & 1 \leq i \leq n \\ (v_{n-i}, v_{n-i+k+1}) & 1 \leq i \leq k \\ (v_{n-1-k}, v_n) & \text{if } n \equiv 2(\text{mod } m) \end{cases} \end{cases}$$

This begs the question: could Theorem 4 be strengthened to an if and only if statement? If not, what are some other values that work? That is a good question.

Theorem 5 *P_n is a k-step competition graph if and only if $n \equiv 1$(mod k) or $n \equiv 2$(mod k).*

The only if part of this statement was proved in [1].

The proof relies heavily on two definitions.

*A*nomaly: In Section 2, we learned that the columns of the adjacency matrix will create cliques in $C(D)$. Since P_n has clique cover number $n-1$, there is a bijective function from the cliques of P_n to $n-1$ of the columns of A. The other column is referred to as the *anomoly*.

*L*eaf: Since $C^m(D) = P_n$, There are only two vertices with degree 1. These vertices, l_1 and l_2, are known as the leaves of a graph. The proof is very long, but the basic construction is to force a certain structure on any digraph D for which $C^m(D) = P_n$ by using these basic lemmas.

1. Every vertex except maybe α has exactly 2 m-step predators.

2. Every vertex has a 1-step prey, and every vertex except maybe α has a 1-step predator.

3. If a pair of vertices share 2 m-step prey, one of these prey is α.

4. No vertex has 3 m-step predators.

5. If a vertex has 3 m-step prey, one of those prey is α.

6. If x has only one m-step prey, then x is a leaf.

7. If $N^i(u) \subseteq N^i(v)$ for any $i \leq m$ then u is a leaf.

8. If every m-step predator of x is also an m-step predator of y, then either x or y is α.

9. If two vertices share a k-step prey, they will share an m-step prey as well.

These statements are proved as follows.

1. These 2 m-step predators correspond to the cliques of P_n.

2. If x had no 1-step prey, x would not be in competition with any other vertex. If x has no 1-step predator, then it wouldn't have 2 m-step predators.

3. All vertices correspond to different cliques in the graph. Since these 2 m-step prey create the same clique, one of them is not needed in the bijective function from the cliques to the vertices. Pick either column and call it α.

4. This follows because $C^m(D)$ is triangle free.

5. If none were α, $C^m(D)$ would have a claw.

6. A more generalized concept will be proved in (7).

7. u is adjacent to v in $C^m(D)$. If u was adjacent to another vertex w, then v would be adjacent to w creating a triangle in $C^m(D)$.

8. Use a similar observation to that in (3).

9. If two vertices share a k-step prey, that prey has a 1-step prey. By induction, the two vertices will share an m-step prey as well.

3.1 Unions

The next section will cover taking the union of k different isomorphic graphs. We start with the C_n graph.

Theorem 6 *The spiked cycle is an m-step competition graph if and only if $m = 1$.*

This includes C_n itself. See Figure 3.6 an example of a spiked cycle with 3 pendant vertices.

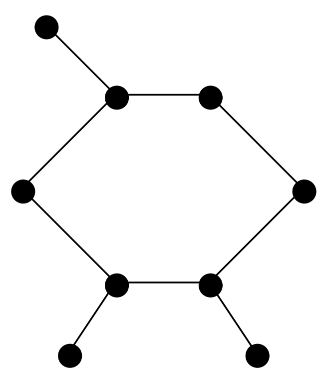


Figure 3.6: A C_6 graph with 3 pendant vertices.

This was proved in [4]. The proof is influenced heavily by linear algebra. Using some results from [6], it can be found that any such digraph D for which $C(D)$ is a spiked cycle must have a prime adjacency matrix.

This statement can be generalized to the union of m different cycles. For the following theorem, take $\bigcup_{i=1}^{m}(C_n)$ to mean the union of m vertex-disjoint C_n graphs.

Theorem 7 *$\bigcup_{i=1}^{m}(C_n)$ is a k-step competition graph if and only if k divides m.*

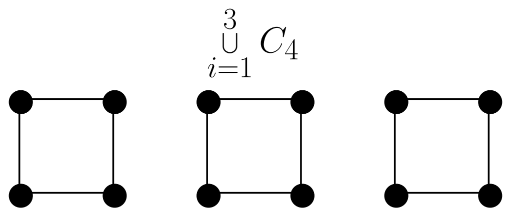


Figure 3.7: Union of 3 different disjoint C_4 graphs

Since $C^m(D)$ will need to be $\bigcup_{i=1}^{k} C_n$, we know a few things about D. Since the clique cover number of $\bigcup_{i=1}^{k} C_n$ is $k * n$, we know there is a bijective function from every vertex in D to the cliques in $C^m(D)$. This leads to the following observations.

1. For every vertex v, $d^+(v) \geq 1$.

2. For every vertex v, $d^+(v) \leq 2$.

3. For every vertex v, $d^-(v) \leq 1$.

4. For every vertex v, $d^-(v) \geq 2$.

5. If $N^+(u) \subseteq N^+(v)$ then $u = v$.

For (1), since every vertex is in competition in $C^m(D)$, every vertex must have an outgoing edge, else it would not be in competition. For (3), if a vertex had no in-degree, then it would have no m-step predators, contradicting the bijection from the edges to the vertices. For (2), if a vertex v had out degree 3 or higher, then it would have 3 or more out degree neighbors in D. This would create a claw in $C^m(D)$. For (4), if a vertex v had in-degree 3 or higher, there would be a triangle in $C^m(D)$. For (5), u will be in competition with v in $C^m(D)$. If u was in competition with another vertex w, v would be in competition with w too causing a triangle in $C^m(D)$. So u can only be in competition with v, which is a contradiction of the cycle structure.

It is possible to partition the vertices into two sets. $V_2 = \{v \mid d^+(v) = 2\}$ and $V_1 = \{v \mid d^+(v) = 1\}$. It is also possible to decompose V_1 even further into

$$V_{1,i} = \{v \mid d^+(v) = 1 \text{ And the shortest walk from } v \text{ to some vertex } u \in V_2 \text{ has length } i\}.$$

For emotionally satisfying reasons call $V_{1,0} = V_2$. Clearly $v \in V_{1,i}$ can only be adjacent to some vertex $u \in V_{1,i-1}$. But how many of these $V_{1,i}$ are non-empty? And how big are they? This leads to the following observations.

a. $V_{1,i} = \emptyset$ for all $i \geq m$.

b. $v \in V_2$ may only be adjacent to some vertex $u \in V_{1,m-1}$.

c. $|V_{1,i}| = |V_{1,j}|$ for all $0 \leq i, j \leq m - 1$.

For (a), if $i \geq m$, then v would only have 1 m-step neighbor, contradicting the cycle structure. For (b), suppose $v \in V_2$ was adjacent to some $v \in V_{1,i}$ for $i < m - 1$. Then, by (5), v would have 3 m-step neighbors, contradicting (2). For (c), without loss of generality, we can assume $|V_{1,i}| \neq |V_{1,i-1}|$. There are two cases.

Case 1: $|V_{1,i}| > |V_{1,i-1}|$. In this case, two vertices must be adjacent to the same vertex by the pigeon hole principle, but this contradicts (5).

Case 2: $|V_{1,i}| < |V_{1,i-1}|$. In this case, there must be one vertex in $V_{1,i-1}$ with in-degree 0, contradicting (3).

Due to the cyclic nature of D, $v \in V_{1,i}$ can only be in competition to some other $u \in V_{1,i}$ in $C^m(D)$. Moreover, these graphs will all be isomorphic. Therefore m must clearly divide the number of cycles there are, which is k.

This proof can be expanded a little bit in the following way. Let $\bigcup_{i=1}^{k}\bigcup_{i=1}^{m_k}(C_{n_i})$ be the disjoint union of m_1 C_{n_1} graphs, m_2 C_{n_2} graphs and so on.

Corollary 2 $\bigcup_{i=1}^{k}\bigcup_{i=1}^{m_k}(C_{n_i})$ *is a j-step competition graph if and only if j divides $g = GCD(m_1, \ldots, m_k)$.*

This theorem follows directly from the proof above. The digraph constraints are the same from above. See Figure 3.8 for an example of this.

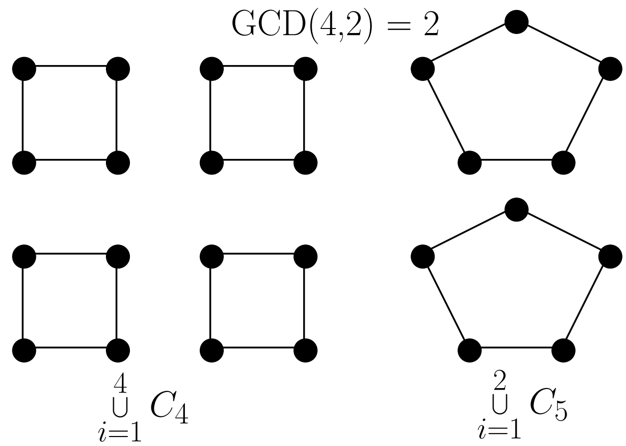


Figure 3.8: Union of 4 different disjoint C_4 graphs and 2 disjoint C_5 graphs

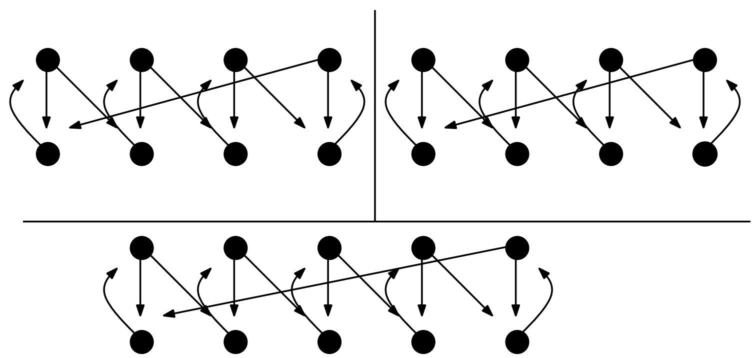


Figure 3.9: Digraphs for which $C^2(D)$ is the graph above

The next step is to generalize this theorem to spiked n cycles. Instead, let's generalize to any graph G as best we can.

Theorem 8 *Let G be any j-step competition graph. $\bigcup_{i=1}^{m}(G)$ is a k-step competition graph if k divides $m * j$.*

The proof involves the construction of a diagonal matrix $\mathcal{A}$ that is the adjacency matrix of a digraph D for which $C^{m*j}(D) = \bigcup_{i=1}^{m}(G)$. This construction along with Corollary 1 proves the theorem. In this digraph, let A be the adjacency matrix of D_G for which $C^j(D_G) = G$. Also let $n = |V(G)|$.

$$\mathcal{A} = \begin{array}{cc} \overbrace{\begin{bmatrix} 0_n & A & 0_n & \dots & \dots & 0_n \\ 0_n & 0_n & I_n & 0_n & \dots & 0_n \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\ I_n & 0_n & \dots & \dots & \dots & 0_n \end{bmatrix}}^{n*m} & \left.\vphantom{\begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix}}\right] n*m \end{array}$$

$$\mathcal{A}^{m*i} = \begin{array}{cc} \underbrace{\begin{bmatrix} A^m & 0_n & \dots & \dots & \dots & 0_n \\ 0_n & A^m & 0_n & \dots & \dots & 0_n \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\ 0_n & \dots & \dots & \dots & 0_n & A^m \end{bmatrix}}_{n*m} & \left.\vphantom{\begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix}}\right] n*m \end{array}$$

Figure 3.10: Block diagonal adjacency matrix for D for which $C^{m*i}(D) = \bigcup_{i=1}^{m}(G)$

Chapter 4

Completion

So far competition graphs have been restricted to one value m. Let us now observe an infinite sequence of competition graphs. This sequence $C(D), C^2(D), \ldots$ is called the the Competition Sequence of a graph D. Sometimes, for a certain value k, $G = C^k = C^{k+1} = \ldots$. This G is called the Final Graph of the Competition Sequence. A generalized idea of this, known as a *competition index*, was introduced and studied in [3, 2, 11].

Competition Index: When you look at the *competition sequence* of a digraph D, the length between the first repeated graphs in the sequence is known as the *index* of D.

The final graph is a special where *competition index* = 1.

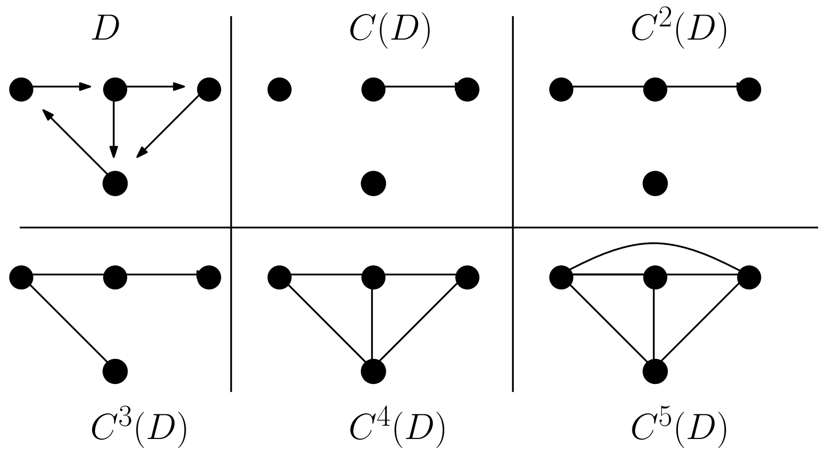


Figure 4.1: Competition Sequence of D. $C^i = K_n$ for all $i \geq 5$

Observation 3 *If K_n or $\overline{K_n}$ ever appears in a Competition Sequence, it will be the first instance of the Final Graph.*

If $C^k(D) = K_n$, every vertex is in competition with some vertex, therefore no vertex has out-degree 0. If vertices u, and v are in competition, then there exists a vertex x for which $\{(u,x),(v,x)\} \subseteq E(D^k)$. x has a prey, call it y. Therefore $\{(u,y),(v,y)\} \subseteq E(D^{k+1})$.

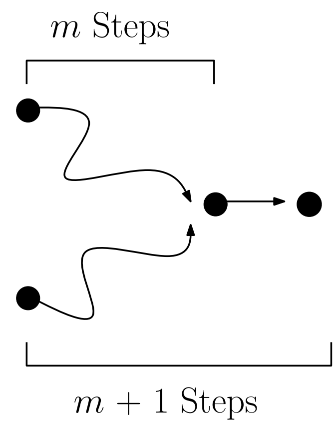


Figure 4.2: Pictoral example for Final Graph K_n

If $C^k(D) = \overline{K_n}$, we will prove by contradiction. If u and v are in competition in D^l for $l > k$. Without loss of generality we can say $l = k + 1$. There exists vertices u,v, and x for which there is an l-step path from both u to x and v to x. The second vertices on each path will be in competition in D^k. If these vertices are not unique, then u and v will be in competition in D^k. Either way, this contradicts $C^k(D) = \overline{K_n}$.

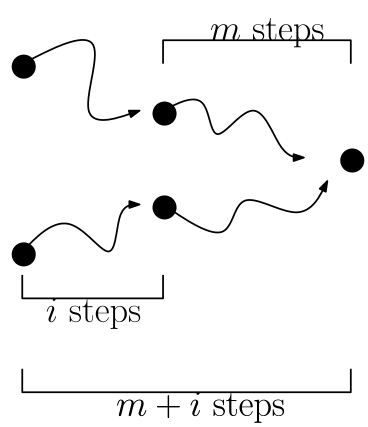


Figure 4.3: Pictoral example for Final Graph $\overline{K_n}$

It's pretty clear that a cycle-free digraph D will have Final Graph $\overline{K_n}$. However, there are non acyclic digraphs D with Final Graph $\overline{K_n}$. Also, there are digraphs that have final graphs other than K_n or $\overline{K_n}$. One naturally asks a few questions.

1. Can we classify digraphs D with Final Graph K_n?

2. Can we classify digraphs D with final graph $\overline{K_n}$?

3. Can we classify simple graphs G, for which G is the Final Graph of some digraph D?

4. Can we find some upper and lower bounds for when the Final Graph will be reached?

Question four suggests another definition. The *C*ompletion Number of a digraph D is the smallest integer k for which $C^k(D) = G$, where G is the Final Graph of D.

To answer Question 1, we must define a Primitive Graph, and a Primitive Set.

*P*rimitive Digraph: A digraph D is *p*rimitive if there exists an integer k for which at $D^k = K_n$. A digraph D is primitive if an only if its adjacency matrix is primitive.

*P*rimitive Set: A primitive set is an induced subgraph $C \subseteq D$ for which C is a primitive digraph.

Lemma 2 *If $C(D) = K_n$, there exists a vertex, v_e, for which there exists an $n-1$ path from every vertex $v \in v(D)$ to v_e.*

We know that since $C(D) = K_n$, given any two vertices u and v, there must be a third vertex x(maybe not distinct) for which $\{(u, x), (v, x)\} \subseteq E(D)$. We will create an algorithm that will find the vertex v_e.

1. Arbitrarily label all vertices in the graph v_i distinctly, call $V(G) = V_0$.

2. Form a set V_1 as follows: for every pair of vertices v_i and v_{i+1} arbitrarily chose one vertex v_i^1 for which v_i and v_{i+1} both prey upon v_i^1. Place v_i^1 in V_1.

3. Repeat to create V_{i+1} from V_i until you reach V_{n-1}.

A few observations must be made. While, v_i^1 may be equal to some v_j^1 for some $i \neq j$, this is fine because we're finding walks, not paths or trails. Also, at every step, $|V_i| = |V_{i-1}| - 1$. (When determining the size of these multisets, repetitions of elements are counted multiple times). Therefore, for $|V(D)| = |V_0| = n$, we know $|V_{n-1}| = 1$. This vertex, $v_1^{n-1} = v_e$, satisfies the lemma.

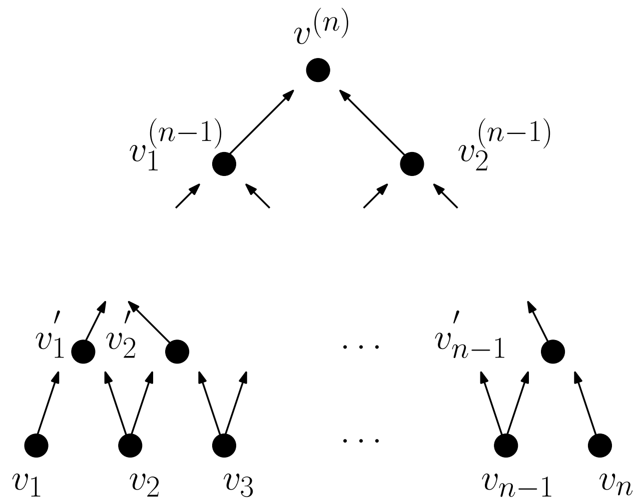


Figure 4.4: Example of the Algorithm

Theorem 9 *$C(D) = K_n$ if and only if D contains a primitive set P. Moreover, for every vertex $v \in V(D)$ there exists a vertex $p \in V(P)$ for which $(v, p) \in E(D)$.*

Take the vertex from the previous lemma and call it v. We must build a primitive set, we will start with v and $N^+(v)$. For all $u \in N^+(v)$, there exists a $n-1$ walk from u to v. v, $N^+(v)$ and all vertices in these walks will make the the primitive set.

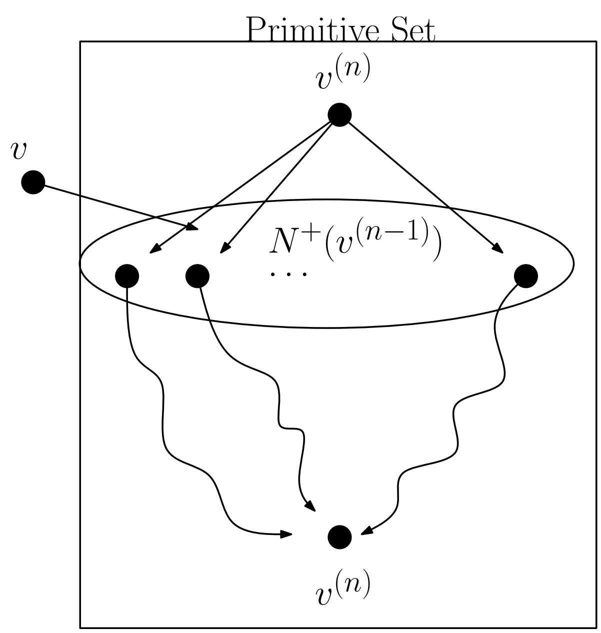


Figure 4.5: Construction of the primitive set

We must find an integer i for which there exists a walk from any vertex u to any other vertex w in our primitive set. Moreover, this walk may only contain vertices in the primitive set. Since $C(D) = K_n$, every vertex must be adjacent to v in $C(D)$. So therefore we know, every vertex $u \in V(G)$ must be adjacent to some vertex $w \in N^+(v)$. Therefore, any vertex in our primitive set may go to $N^+(v)$ and stay there as long as it wants. We know that you can get from any vertex $u \in N^+(v)$ back to v with a walk of length $n-1$. Moreover, this walk only uses vertices in our primitive set. However, we need to be able to reach any vertex, not just v. Chose a length of $2n-1$ for these walks. All vertices in our primitive set will be on some trail from $u \in N^+(v)$ to v. Moreover, this vertex can't be the $(n-1)^{th}$ or larger vertex in this trail, because that vertex is v. Let u be the i^{th} vetex on some trail for $i < n-1$. To get to u in $2n-1$ steps, walk around $N^+(v)$ for $n-i$ steps, then follow the trail to v in $n-1$ steps, then get back into $N^+(v)$ by the vertex that starts the trail containing u. Follow this trail for i steps to reach u. All together this trail was $(n-i)+(n-1)+i = 2n-1$ steps.

So $C(D) = K_n$ implies that D has a primitive set and that every vertex will be adjacent to some vertex in the primitive set. So therefore if $C^m(D) = K_n$, D will have a primitive set, because a set will either always be primitive or never be primitive. Also, every vertex will reach the primitive set in m steps.

Moreover, from this proof, it is easy to see that $C^m(D) = K_n$ at most $2n-1$ steps before the primitive set is J. So any bound for a primitive set becoming J is also a bound for a digraph D completing to its final graph of K_n. Furthermore, we can subtract at most $2n-1$ from all natural bounds of primitive bounds.

The next observation is in regards to question three.

Observation 4 *A simple graph G is the final graph of some digraph D if every maximal clique of G has at least one vertex exclusive to that maximal clique.*

The following is a construction for D that satisfies the observation. Let $\mathcal{P}_i$ be the vertices that are unique to maximal clique i. Form those vertices into a primitive set. Let vertices in more than one maximal clique be adjacent to all $\mathcal{P}_i$ for which $v \in \mathcal{K}_i$.

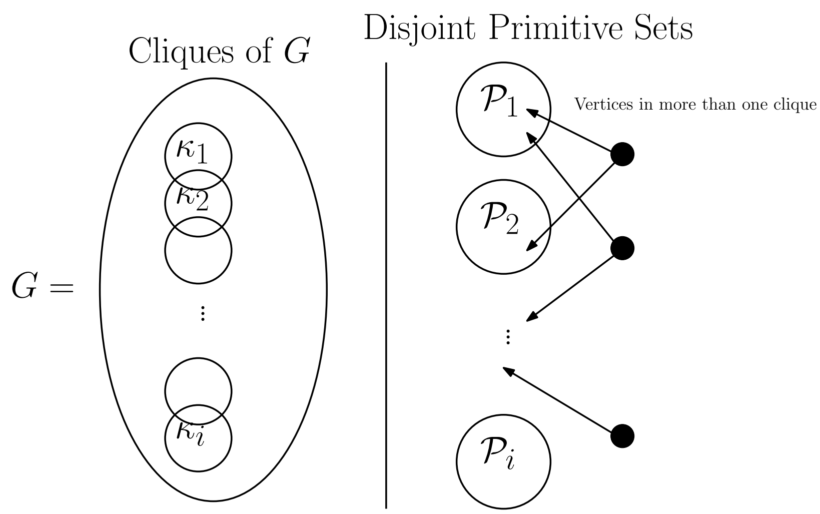


Figure 4.6: Digraph D who's final graph is G satisfying all maximal cliques having at least one vertex not in any other maximal clique.

However, the only if part of this statement is false.

False Conjecture 1 *A simple graph G is the final graph of some digraph D only if every maximal clique of G has at least one vertex exclusive to that maximal clique.*

See Figure 4.7 a counterexample.

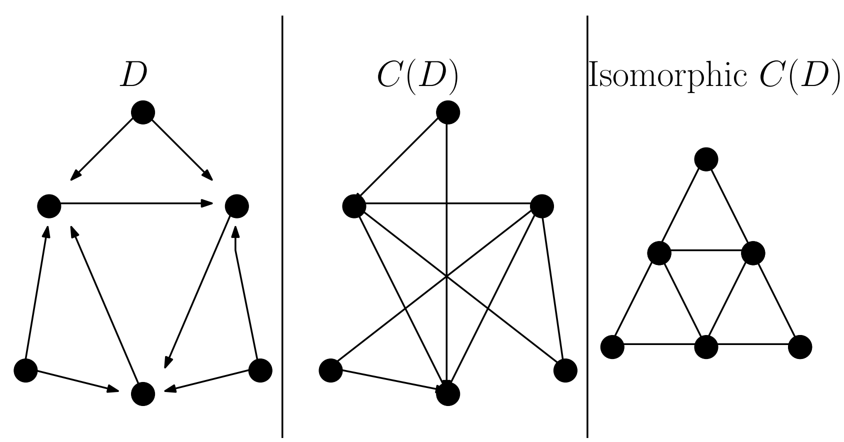


Figure 4.7: Digraph D who's Final Graph does not satisfy every maximal clique having at least one vertex not in any other maximal clique

This graph is also a counterexample to another conjecture about Question 2.

False Conjecture 2 *If a digraph D has no primitive set, then the final graph of D is $\overline{K_n}$.*

The previous graph also disproves this statement, because the digraph from the previous graph contains no primitive set.

There are many results on bounds for primitive graphs, and every one of those bounds is also a natural bound for the completion number. For more information see [7, 13]. However one would expect that the completion number would be strictly less than the primitive number. One interesting question is: given the set of graphs D that have the same competition graph, what are the range of completion numbers of these graphs? This brings on a few definitions

*B*iome: Given a simple graph G. The *b*iome of G, B_G, is the set of all digraphs D for which $C(D) = G$.

*C*ompetition Range: Given a biome, find the completion numbers for all digraphs D in said biome. The lower and upper bounds of these completion numbers are known as the *c*ompletion range.

Theorem 10 *For C_n, the upper bound in the completion range is $\lfloor n/2 \rfloor$. And for odd C_n the lower bound is $\lfloor \log_2(n) \rfloor$.*

From earlier theorems, we know a few things about any digraph D for which $C(D) = C_n$.

1. $d^+(v) = d^-(v) = 2$ for all $v \in V(G)$

2. $N^+(v) \subseteq N^+(u)$ implies that $u = v$.

Observation 5 *If $d^+(v) > \lfloor \frac{|V(G)|}{2} \rfloor$, then $C(D) = K_n$*

Since all v are adjacent to more than half of the vertices in G, they must share at least one vertex by the pigeon hole principle.

Observation 6 $m + 1 \leq |N^+(v)| \in D^m \leq 2^m$

By (1), every vertex starts off with 2 neighbors. Each of these neighbors has 2 neighbors, so 2^m is an obvious upper bound. However, by (2), these 2 neighbors must yield at least 3 2-step neighbors. Generalize this idea to get $m + 1$ as a natural lower bound.

So these bounds naturally give the bounds in the theorem, but can we actually reach these bounds? The following is a construction for a graph that will reach the upper bound for all C_n.

$$\begin{cases} V(D) &= \{v_i \mid 1 \leq i \leq n\} \\ E(D) &= \Big\{(v_i, v_i) \cup (v_i, v_{i+1})\Big\} \end{cases}$$

The following is a construction of a digraph that will reach the lower bound for odd C_n.

$$\begin{cases} V(D) &= \{v_i \mid 1 \leq i \leq n\} \\ E(D) &= \Big\{(v_i, v_{2i}) \cup (v_i, v_{2i+1})\Big\} \end{cases}$$

Obviously this will reach the bound, however, is $C(D) = C_n$? If n is even, then $C(D)$ will be a perfect matching. However, when n is odd, v_i will be adjacent to $v_{i+\lfloor \mathrm{n}/2 \rfloor}$ and $v_{i+\lceil \mathrm{n}/2 \rceil}$ This is equivalent to taking $\lfloor \frac{n}{2} \rfloor$ as a generator in the group $\mathcal{Z}_n$, which will generate every i because $\lfloor \frac{n}{2} \rfloor$ and n are relatively prime..

Chapter 5

Questions

This concludes the extent of our information on the subject of competition graphs. However there is much more work to be done. The following is a list of some interesting questions.

1. If the clique cover number of a graph G is $|V(G)|$, does this imply that G can only be realized as a 1-step competition graph?

   This is true for C_n and the spiked cycle. Any proof must use a bijective function from the vertices to the cliques of G. Forcing a prime matrix construction has been a useful technique in previous results, and may prove to be fruitful in the future. For an example of how this is done see [4]. For more information on prime matrices see [6].

2. Can we find a complete characterization of digraphs D for which the final graph of D is $\overline{K}_n$?

   The original thought was that G would have Final Graph $\overline{K}_n$ if and only if D had no primitive set. The original thought was disproved by Figure 4.7.

3. Can we make a complete characterization of graphs G for which G is the final graph of some digraph D?

   The original thought was disproved by 4.7.

4. Can we find completion range's for more graphs?

   This seems to be a very hard problem to generalize. The only result so far is C_n

5. Is Proposition 8 an only if statement as well?

6. Can we find a complete characterization of trees as m-step competition graphs? This seems to be a very hard problem to generalize. The proof for paths alone took more than 10 pages to prove, and it was very specific to the exact structure of P_n.

7. Given a Biome $\mathcal{B}(G)$, What graphs lie within $\{C(D^t) \mid D \in \mathcal{B}\}$.

   Sometimes $C(D^t) = C(D)$. Other times $C(D^t = \overline{C(D)}$. And rarely, $C(D^t)$ seems to have nothing to do with $C(D)$ altogether. The following are examples of each of these cases.

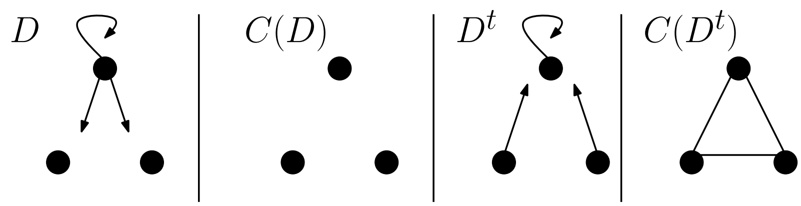


Figure 5.1: $\overline{C(D)} = C(D^t)$

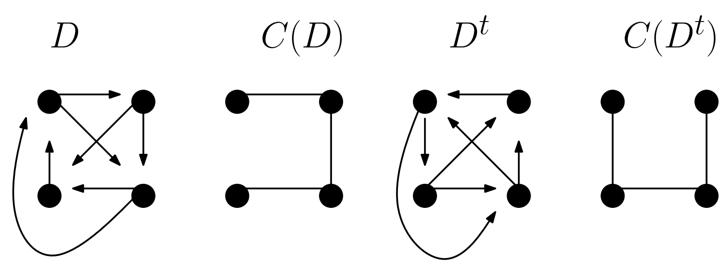


Figure 5.2: $C(D) = C(D^t)$

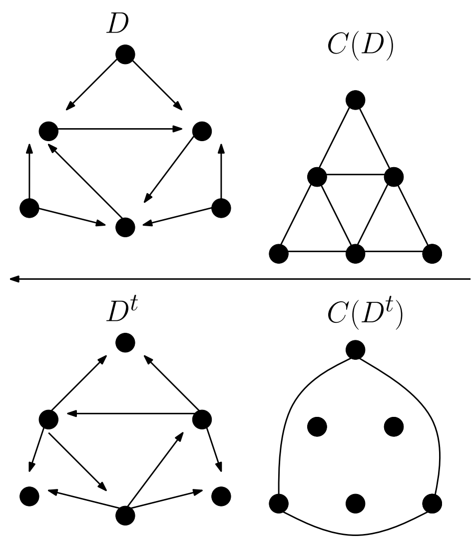


Figure 5.3: $C(D) \neq C(D^t)$ and $\overline{C(D)} \neq C(D^t)$

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