# Bell Numbers of Graphs

by

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Abstract

Let $G$ be a simple graph with vertex set $V(G)$. Let $\mathcal{F}$ be a family of graphs such that $K_1 \in \mathcal{F}$. Denote by $B(G;\mathcal{F})$ the number of unordered partitions of $V(G)$ such that each part induces a member of $\mathcal{F}$. We call $B(G;\mathcal{F})$ the Bell number of the graph $G$ with respect to the family $\mathcal{F}$. We investigate properties of this function for different families $\mathcal{F}$, and conditions on $\mathcal{F}$ for the function $B(G;\mathcal{F})$ to have certain properties.

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# Chapter 1

## Introduction

Suppose that $G$ is a simple graph and $\mathcal{F}$ is a family of simple graphs where $K_1 \in \mathcal{F}$. We define the Bell number of $G$ with respect to $\mathcal{F}$, denoted $B(G;\mathcal{F})$, to be the number of unordered partitions of $V(G)$ such that each member of the partition induces a member of $\mathcal{F}$. Such a partition is referred to as a $\mathcal{F}$-partition of $G$. Members of partitions are referred to as blocks (even if the partition is not a $\mathcal{F}$-partition).

This problem arose when I was an undergraduate working with Rhodes Peele at Auburn University at Montgomery. I was assigned the following problem. Due to Herbert Wilf [1], let $[n]$ denote the set of integers 1 through $n$:

*Let $f(n)$ be the number of* `subsets` *of $[n]$ that contain no two consecutive elements. Find the recurrence that is satisfied by these numbers, and then 'find' the numbers themselves. [1]*

But I misread `subsets` as `partitions`; that is, I thought that $f(n)$ was the number of unordered partitions of $[n]$ in which no partition set contained two consecutive integers. This task proved to be more involved than Wilf's problem. Dr. Peele suggested we view the problem as being about a simple graph $G$ where $V(G) = [n]$ and two distinct vertices $i, j$ are adjacent if and only if $|i - j| = 1$. In other words, $G$ is a path on $n$ vertices, and we seek to partition $V(G)$ such that no two vertices together in the same block have an edge between them - that is, each block is an independent set of vertices of $G$. It turns out that the number of such partitions is $(n-1)^{th}$ Bell number.

The $n^{th}$ Bell number, $b_n$, denotes the number of non-empty, unordered partitions of an $n$-element set. It is surprising that the number of partitions of the vertices of a path into independent sets followed the Bell sequence (with a shift in index), since these partitions are not unrestricted. Expanding this question to other graphs $G$, the Bell-like behavior motivated referring to this problem as finding the "Bell number" of $G$. With Dr. Johnson, this question was expanded from partitioning vertex sets into independent sets to partitioning into blocks which ecach induce a member of a specified family of graphs $\mathcal{F}$. Denoting the number of such partitions by $B(G;\mathcal{F})$, we call this the Bell number of $G$ with respect to the family $\mathcal{F}$.

### 1.1 Information on Set Partitions

The Bell numbers satisfy a well-known recurrence

$$b_n = \sum_{i=0}^{n-1} \binom{n-1}{i} b_i.$$

The proof of the recurrence is enlightening when it comes to certains proofs for the Bell numbers of graphs, so it is worthwhile to briefly examine it to build familiarity.

*Proof.* To partition the set $[n]$, consider which block will contain the element $n$. If $n$ is in a block by itself, then there is $\binom{n-1}{0}$ ways to choose that block, $n-1$ elements remain outside of the block, and there are $b_{n-1}$ ways to partition them. If $n$ is in a block with one other element, then there are $\binom{n-1}{1}$ ways to form that block, $n-2$ elements remain outside of the block, and there are $b_{n-2}$ ways to partition the rest. More generally, if $n$ is in a block with $k-1$ other elements, then there are $\binom{n-1}{k-1}$ ways to form the block, $n-k$ elements remain outside of the block, and there are $b_{n-k}$ ways to partition the rest. Double counting does not occur from this process, so the recurrence is evident. $\square$

Another way of representing the Bell numbers is in terms of Stirling Numbers of the Second Kind, denoted $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$, which indicates the number of unordered partitions of an $n$-element set into exactly $k$ blocks. Clearly,

$$b_n = \sum_{k=1}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\}$$

The corresponding notations that we will use for graphs $G$ are $B(G;\mathcal{F})$ and $S(G;\mathcal{F},k)$. For graphs, the latter is defined to be the number of unordered partitions of $V(G)$ into exactly $k$ blocks such that each block induces a member of $\mathcal{F}$. It is, of course, the case that the sum of $S(G;\mathcal{F},k)$ over $1 \leq k \leq |V(G)|$ yields $B(G;\mathcal{F})$.

# Chapter 2

# The Bell Number of a Graph

# With Respect to the Family $\mathcal{F}_0$

$$\mathcal{F}_0 := \{\overline{K}_n : n \in [n]\}$$

With $K_n$ denoting the complete graph on $n$ vertices, $\mathcal{F}_0$ is the collection of graphs with no edges and a positive number of vertices. The work I did with Rhodes Peele focused on $B(G;\mathcal{F}_0)$ for various graphs $G$. In particular, we considered trees, cycles, stars, and complements of paths. [2] More, we discovered certain properties of the $B(G;\mathcal{F}_0)$ function.

## 2.1 Deletion and Contraction Properties of $B(G;\mathcal{F}_0)$

For a simple graph $G$, the minimal $k$ for which $S(G;\mathcal{F}_0,k)$ is non-zero is the chromatic number $\chi(G)$. The chromatic polynomial $\chi(G;\lambda)$ of a graph is a polynomial whose evaluation at integer $m$ yields the number of functions $f: V(G) \to \{1,2,\ldots,m\}$ that properly color $G$ with at most $m$ colors. It is commonly written as follows.

$$\chi(G;\lambda) = \sum_{k=1}^{n(G)} p_k(G)\lambda_{(k)}$$

where $\lambda_{(k)} = \lambda(\lambda-1)\cdots(\lambda-k+1)$.[7]

Note that $S(G;\mathcal{F}_0,k) = p_k(G)$. It's possible to obtain the Bell number of a graph $G$ with respect to $\mathcal{F}_0$ if the quantities $p_k(G)$ are known for all $1 \le k \le |V(G)|$, but that's equivalent to finding the chromatic polynomial, which is known to be a difficult problem.

(cite something here)

The chromatic polynomial is known to have a very convenient deletion/contraction property which is shared by the function $B(G; \mathcal{F}_0)$. For a graph $G$, if $e \in E(G)$ then denote by $G \setminus e$ the graph $G$ with edge $e$ removed. Denote by $G \cdot e$ the graph $G$ with the ends of $e$ identified as a single vertex, which is adjacent to all vertices to which either of the ends of $e$ were adjacent; if duplicate edges are created, merge them into a single edge. Then the chromatic polynomial has the following property.

$$\chi(G; \lambda) = \chi(G \setminus e; \lambda) - \chi(G \cdot e; \lambda)$$

The reason this works is as follows. Partitions of $V(G \setminus e)$ into independent sets consist of two types: partitions where the ends of $e$ appear together in a block, and partitions where the ends of $e$ are in different blocks. Partitions of $V(G)$ into independent sets forbid the ends of $e$ from appearing in the same block, and partitions of $V(G \cdot e)$ are those where the ends of $e$ are always together in the same block (as they are identified as the same vertex). This means that

$$p_k(G) = p_k(G \setminus e) - p_k(G \cdot e) \text{ for } 1 \leq k \leq |V(G)|$$

Since these coefficients determine the Chromatic polynomial, it must be the case that

$$\chi(G; \lambda) = \chi(G \setminus e; \lambda) - \chi(G \cdot e; \lambda).$$

Moreover, those coefficients determine the Bell number, so it must also be the case that

$$B(G; \mathcal{F}_0) = B(G \setminus e; \mathcal{F}_0) - B(G \cdot e; \mathcal{F}_0).$$

**Question** Suppose that $\mathcal{F}$ is a family of graphs with $K_1 \in \mathcal{F}$. If for every graph $G$ and every $e \in E(G)$ it happens that $B(G;\mathcal{F}) = B(G \setminus e;\mathcal{F}) - B(G \cdot e;\mathcal{F})$, does it follow that $\mathcal{F} \subseteq \mathcal{F}_0$ ?

**Theorem 2.1.** *Let $G = (V, E)$ be a simple graph and $w$ a vertex not in $V$. Let $\mathcal{G}_w$ represent the collection of graphs obtained from $G$ by adding $w$ as a pendant vertex.*

$$\mathcal{G}_w = \{H_v : V(H) = V \cup \{w\} \text{ and } E(H) = E(G) \cup \{w, v\} \text{ for some } v \in V\}$$

*Then for any $u$, $v \in V$, $S(H_u;\mathcal{F}_0, k) = S(H_v;\mathcal{F}_0, k)$ for each $1 \leq k \leq |V(G)| + 1$. Consequently, $B(H_u;\mathcal{F}_0) = B(H_v;\mathcal{F}_0)$.*

*Proof.* Let $e = \{w, u\} \in E(H_u)$ and $f = \{w, v\} \in E(H_v)$. Suppose we wish to examine the partitions of $V(H_u)$ and $V(H_v)$ with respect to $\mathcal{F}_0$ into exactly $k$ blocks with $1 \leq k \leq |V(G)| + 1$. By deletion and contraction:

$$S(H_u;\mathcal{F}_0, k) = S(H_u \setminus e;\mathcal{F}_0, k) - S(H_u \cdot e;\mathcal{F}_0, k)$$

$$S(H_v;\mathcal{F}_0, k) = S(H_v \setminus f;\mathcal{F}_0, k) - S(H_v \cdot f;\mathcal{F}_0, k)$$

Alternatively, we may represent $H_u \setminus e$ by $G + w$ to represent the disjoint union of $G$ and $w$. And then we see that $H_v \setminus e = G + w$ as well. Since $H_u \cdot e \simeq G \simeq H_v \cdot f$,

$$S(H_u;\mathcal{F}_0, k) = S(G + w;\mathcal{F}_0, k) - S(G;\mathcal{F}_0, k)$$

$$S(H_v;\mathcal{F}_0, k) = S(G + w;\mathcal{F}_0, k) - S(G;\mathcal{F}_0, k)$$

Therefore,

$$S(H_u;\mathcal{F}_0, k) = S(H_v;\mathcal{F}_0, k).$$

Since $B(J;\mathcal{F}_0) = \sum_{k=1}^{n(J)} S(J;\mathcal{F}_0,k)$ for any graph $J$, we see that $B(H_u;\mathcal{F}_0) = B(H_v;\mathcal{F}_0)$.

$\square$

## 2.2 Multiplicative Property of $B(G;\mathcal{F}_0)$

Suppose that $G$ and $H$ are graphs with disjoint vertex sets and denote by $G \vee H$ the join of $G$ and $H$ where $V(G \vee H) = V(G) \cup V(H)$ and $E(G \vee H) = E(G) \cup E(H) \cup \{\{x,y\} : x \in V(G), y \in V(H)\}$. Then

$$B(G \vee H;\mathcal{F}_0) = B(G;\mathcal{F}_0) \cdot B(H;\mathcal{F}_0)$$

The property follows from the observation that if $p$ is a partition of $V(G)$ into independent sets and $q$ is a partition of $V(H)$ into independent sets, then $p \cup q$ is a partition of $V(G \vee H)$ into independent sets. Conversely, if $s$ is a partition of $V(G \vee H)$ into independent sets then no block of $s$ contains both a vertex of $V(G)$ and a vertex of $V(H)$, so $s$ can be sorted into blocks consisting only of vertices of $G$ and blocks consisting only of vertices of $H$. The former is a partition $p$ of $V(G)$ into independent sets and the latter is a parititon $q$ of $V(H)$ into independent sets.

Note also that if $\mathcal{F}' \subseteq \mathcal{F}_0$, then $B(G \vee H;\mathcal{F}') = B(G;\mathcal{F}') \cdot B(H;\mathcal{F}')$ by the same argument, because the number of elements in each block is irrelevant to the proof.

**Question** Let $\mathcal{F}$ be a family of graphs containing $K_1$. If $B(G \vee H;\mathcal{F}) = B(G;\mathcal{F}) \cdot B(H;\mathcal{F})$ for all simple graphs $G$ and $H$, is it necessarily the case that $\mathcal{F} \subseteq \mathcal{F}_0$?

### 2.3 $B(G; \mathcal{F}_0)$ for Particular Graphs

Let $n \in \mathbb{N}$. Denote by $K_n$ the complete graph on $n$ vertices, and $\overline{K}_n$ its complement, the graph with $n$ vertices and no edges. Then it is immediate that

$$B(K_n; \mathcal{F}_0) = 1 \text{ and } B(\overline{K}_n; \mathcal{F}_0) = b_n.$$

**Theorem 2.2.** *Let $T_n$ denote a tree on $n$ vertices. Then*

$$B(T_n; \mathcal{F}_0) = b_{n-1}$$

*Proof.* By the deletion and contraction property, for any graph $G$ and vertex $w$ not in $V(G)$: if $H_v, H_u \in \mathcal{G}_w$, then these two graphs will have the same Bell number wih respect to $\mathcal{F}_0$ by 2.1. Applying this to trees obtains the result, as follows.

The result is immediate for $n \leq 2$, so assume $n > 2$. Let $T_n$ be a tree on $n$ vertices. Choose any non-leaf $r \in V(T_n)$. Since any tree on at least two vertices has at least two leafs, let $l$ be a leaf of $T_n$ and $e$ its edge. Denote by $T'_n$ the graph obtained from $T_n$ by deleting $e$ and attaching $l$ to $r$. Let $G = T_n \setminus l$, then $T_n, T'_n \in \mathcal{G}_l$. So

$$B(T_n; \mathcal{F}_0) = B(T'; \mathcal{F}_0).$$

If we iterate this process, we can move all edges of $T_n$ to $r$. At each step, we choose a leaf $l$ not adjacent to $r$, delete its edge, and connect it to $r$. At each step, the Bell number is preserved. After at most $n-2$ steps, we'll arrive at $K_{1,n-1}$, the star with $n-1$ leafs. And so we see that for any tree $T_n$ on $n$ vertices,

$$B(T_n;\mathcal{F}_0) = B(K_{1,n-1};\mathcal{F}_0)$$

Since $K_{1,n-1} = K_1 \vee \overline{K}_{n-1}$,

$$B(K_{1,n-1};\mathcal{F}_0) = B(K_1 \vee \overline{K}_{n-1};\mathcal{F}_0) = B(K_1;\mathcal{F}_0)B(\overline{K}_{n-1};\mathcal{F}_0) = b_{n-1}$$

$\square$

**Theorem 2.3.** *Let $P_n$ denote a path on $n$ vertices, and $\overline{P}_n$ the complement of $P_n$ with respect to $K_n$. Then*

$$B(\overline{P}_n;\mathcal{F}_0) = f_{n+1}$$

*where $f_{n+1}$ denotes the $(n+1)^{th}$ Fibonacci number.*

*Proof.* Let the vertices of $P_n$, from one end to the other, be $\{1, 2, \ldots, n\}$. Consider partitioning $\overline{P}_n$ with respect to $\mathcal{F}_0$. Blocks containing element $n$ are either the singleton $\{n\}$ or the doubleton $\{n-1, n\}$. If $\{n\}$ is a block, then there are $B(\overline{P}_{n-1};\mathcal{F}_0)$ partitions of the remaining vertices. If $\{n-1, n\}$ is a block, then there are $B(\overline{P}_{n-2};\mathcal{F}_0)$ partitions of the remaining vertices. Therefore,

$$B(\overline{P}_n;\mathcal{F}_0) = B(\overline{P}_{n-1};\mathcal{F}_0) + B(\overline{P}_{n-2};\mathcal{F}_0).$$

Since $B(\overline{P}_1;\mathcal{F}_0) = 1$ and $B(\overline{P}_2;\mathcal{F}_0) = 2$, the recurrence shows us that the sequence $(B(\overline{P}_n;\mathcal{F}_0))$ is "Fibonacci-like". In particular, $B(\overline{P}_n;\mathcal{F}_0) = f_{n+1}$. $\square$

**Theorem 2.4.** *Let $K_{1,n}$ denote the star graph with $n$ leafs, $n \geq 0$. Denote by $\overline{K}_{1,n}$ its complement. Then*

$$B(\overline{K}_{1,n};\mathcal{F}_0) = n + 1$$

*Proof.* Let $r$ be the central vertex of $K_{1,n}$. Notice that $\overline{K}_{1,n} = K_1 + K_n$. Considering partitioning its vertices with respect to $\mathcal{F}_0$. None of the vertices in the $K_n$ component may appear in a block together, and $r$ in the $K_1$ component can appear in a block either by itself or with any of the vertices in the $K_n$ component. Therefore, there are $n+1$ ways to form a partition of $K_1 + K_n$, so $B(\overline{K}_{1,n}; \mathcal{F}_0) = n + 1$. □

*Corollary* Every positive integer is the Bell number, with respect to $\mathcal{F}_0$, of some graph.

*Corollary* For $n > 3$, there exist trees $T, R$ on $n$ vertices such that $B(\overline{T}; \mathcal{F}_0) \neq B(\overline{R}; \mathcal{F}_0)$.

*Proof.* Let $T = P_n$ and $R = K_{1,n-1}$. Done. □

**Theorem 2.5.** *Let $C_n$ denote a cycle on vertex set $\{1, 2, \ldots, n\}$, for $n \geq 3$. Then*

$$B(C_n; \mathcal{F}_0) = \sum_{k=0}^{n-2} (-1)^k b_{n-(k+1)}$$

*Proof.* Apply deletion and contraction repeatedly.

$$\begin{aligned}
B(C_n; \mathcal{F}_0) &= B(P_n; \mathcal{F}_0) - B(C_{n-1}; \mathcal{F}_0) \\
&= B(P_n; \mathcal{F}_0) - B(P_{n-1}; \mathcal{F}_0) + B(C_{n-2}; \mathcal{F}_0) \\
&= \vdots \\
&= B(P_n; \mathcal{F}_0) - B(P_{n-1}; \mathcal{F}_0) + \ldots + (-1)^{n-3} B(C_3; \mathcal{F}_0) \\
&= \textstyle\sum_{k=0}^{n-4} (-1)^k b_{n-(k+1)} + (-1)^{n-3} B(C_3; \mathcal{F}_0)
\end{aligned}$$

Since $B(C_3; \mathcal{F}_0) = 1 = b_2 - b_1$, the result follows.

□

**Theorem 2.6.** *Let $\overline{C}_n$ denote the complement of a cycle on $n \geq 4$ vertices. Then*

$$B(\overline{C}_n; \mathcal{F}_0) = f_n + 2f_{n-1}$$

*where $f_k$ denotes the $k^{th}$ Fibonacci number.*

*Proof.* Let $C_n$ have vertex set $\{1, 2, \ldots, n\}$, in order around the cycle, and consider partitioning wih respect to $\mathcal{F}_0$. In particular, consider vertex $n$. Vertex $n$ may be in a block by itself, with 1, or with $n-1$. However, 1 and $n-1$ cannot be in a block together. Therefore, we have three cases. If $n$ is in a block by itself, there are $B(\overline{P}_{n-1}; \mathcal{F}_0)$ ways to partition the remaining vertices. If either $\{1, n\}$ or $\{n-1, n\}$ are blocks, then there are $B(\overline{P}_{n-2}; \mathcal{F}_0)$ ways to partition the rest. Therefore,

$$B(\overline{C}_n; \mathcal{F}_0) = B(\overline{P}_{n-1}; \mathcal{F}_0) + 2B(\overline{P}_{n-2}; \mathcal{F}_0).$$

Since we know that $B(\overline{P}_n; \mathcal{F}_0) = f_{n+1}$, the result follows.

□

**Theorem 2.7.** *Let $M_n$ be a collection of $n$ vertex-disjoint edges. (That is, $M_n$ is a matching on $2n$ vertices.) Then, for $n \geq 2$,*

$$B(M_n; \mathcal{F}_0) = \sum_{k=0}^{n-1} \binom{n-1}{k} b_{n+k}.$$

*Proof.* Deletion and contraction can instead be thought of as insertion and contraction. Suppose $n \geq 2$ and $M_n$ is a matching on $2n$ vertices. Then the graph obtained from $M_n$ by inserting an edge between any two of its non-adjacent vertices results in $P_4 + M_{n-2}$. Deletion and contraction would tell us the Bell number of this graph is

$$B(P_4 + M_{n-2}; \mathcal{F}_0) = B(M_n; \mathcal{F}_0) - B(P_3 + M_{n-2}; \mathcal{F}_0).$$

Therefore,

$$B(P_4 + M_{n-2}; \mathcal{F}_0) + B(P_3 + M_{n-2}; \mathcal{F}_0) = B(M_n; \mathcal{F}_0).$$

Successive applications of insertion/contraction to terms of the form $B(P_i + M_j; \mathcal{F}_0)$ (for $j > 0$) will transform $P_i + M_j$ into a path, providing we choose to insert edges only between vertices of degree 1. The resulting coefficients on the terms are of importance. To see how they are generated, suppose that $n$ is sufficiently large to apply insertion/contraction several times. On each line, we will order the terms $B(P_i + M_j; \mathcal{F}_0)$ by decreasing $j$.

$$\begin{aligned}
B(M_n; \mathcal{F}_0) &= B(P_4 + M_{n-2}; \mathcal{F}_0) + B(P_3 + M_{n-2}; \mathcal{F}_0) \\
&= B(P_6 + M_{n-3}; \mathcal{F}_0) + B(P_5 + M_{n-3}; \mathcal{F}_0) + B(P_5 + M_{n-3}; \mathcal{F}_0) + B(P_4 + M_{n-3}; \mathcal{F}_0) \\
&= B(P_6 + M_{n-3}; \mathcal{F}_0) + 2B(P_5 + M_{n-3}; \mathcal{F}_0) + B(P_4 + M_{n-3}; \mathcal{F}_0)
\end{aligned}$$

Notice that under insertion/contraction, adjacent terms $B(P_i + M_j; \mathcal{F}_0)$ and $B(P_{i-1} + M_{j-2}; \mathcal{F}_0)$ each generate a term involving $B(P_{i+1} + M_{j-3}; \mathcal{F}_0)$. The consequence of this is after $i$ applications of insertion/contraction, the coefficients on the terms are precisely $\binom{n}{i}$.

$$\begin{aligned}
B(M_n; \mathcal{F}_0) &= B(P_6 + M_{n-3}; \mathcal{F}_0) + 2B(P_5 + M_{n-3}; \mathcal{F}_0) + B(P_4 + M_{n-3}; \mathcal{F}_0) \\
&= B(P_8 + M_{n-4}; \mathcal{F}_0) + 3B(P_7 + M_{n-4}; \mathcal{F}_0) + 3B(P_6 + M_{n-4}; \mathcal{F}_0) + B(P_5 + M_{n-4}; \mathcal{F}_0) \\
&= B(P_{10} + 4M_{n-5}; \mathcal{F}_0) + 4B(P_9 + M_{n-5}; \mathcal{F}_0) + 6B(P_8 + M_{n-5}; \mathcal{F}_0) \\
&\quad +4B(P_7 + M_{n-5}; \mathcal{F}_0) + B(P_6 + M_{n-5}; \mathcal{F}_0) \\
&= \vdots \\
&= \sum_{i=0}^{k} \binom{k}{i} B(P_{2k+2-i} + M_{n-k-1}; \mathcal{F}_0)
\end{aligned}$$

When $k = n-1$,

$$B(M_n; \mathcal{F}_0) = \sum_{i=0}^{n-1} \binom{n-1}{i} B(P_{2(n-1)+2-i} + M_0; \mathcal{F}_0)$$

Since $M_0$ is the empty graph, we're left with $B(P_{2n-i}; \mathcal{F}_0) = b_{2n-i-1}$ in the $i^{th}$ term, above. So,

$$B(M_n; \mathcal{F}_0) = \sum_{i=0}^{n-1} \binom{n-1}{i} b_{2n-i-1}$$

$$= \sum_{i=0}^{n-1} \binom{n-1}{n-1-i} b_{n+(n-1-i)} = \sum_{i=0}^{n-1} \binom{n-1}{k} b_{n+k}$$

$\square$

## 2.4 $B(G;\mathcal{F}_0)$ for Graphs with Two Components

The material in this section originally appeared in [3]. Here we consider graphs with two components.

**Theorem 2.8.** *Suppose that $D$ is a graph with components $G$ and $H$, so we can write $D = G + H$. Let $V(G)$ have size $n$ and $V(H)$ have size $m$. Then*

$$B(G+H;\mathcal{F}_0) = \sum_{i=\chi(G)}^{n} \sum_{j=\chi(H)}^{m} S(G;\mathcal{F}_0,i)S(H;\mathcal{F}_0,j)B(K_i+K_j;\mathcal{F}_0)$$

*where $\chi(G)$ and $\chi(H)$ are the chromatic numbers of $G$ and $H$ and $K_i, K_j$ are complete graphs on $i$ and $j$ vertices, respectively.*

*Proof.* Let $q = \{q_1, q_2, \ldots, q_i\}$ and $r = \{r_1, r_2, \ldots, r_j\}$ be $\mathcal{F}_0$-partitions of $G$ and $H$ respectively. We can form a $\mathcal{F}_0$-partition $p = \{p_1, p_2, \ldots, p_k\}$ of $G + H = D$ by combininig the blocks of $q$ and $r$ in such a way that each block of $p$ is either a block of $q$, a block of $r$, or a union of one block of $q$ and one block of $r$. So, we see that $\mathcal{F}_0$-partitions of $D$ may be constructed from any pair of partitions of its components.

Conversely, suppose that $p = \{p_1, p_2, \ldots, p_k\}$ is a $\mathcal{F}_0$-partition of $D$. For each $1 \le i \le k$, define $q_i = p_i \cap V(G)$ and $r_i = p_i \cap V(H)$. Let $q = \{q_i : q_i \neq \emptyset, 1 \le i \le k\}$ and $r = \{r_i : r_i \neq \emptyset, 1 \le i \le k\}$, and observe that $q$ and $r$ are $\mathcal{F}_0$-partitions of $G$ and $H$ respectively. Consequently, each $\mathcal{F}_0$-partition of $D$ arises from a pair $(q, r)$ of $\mathcal{F}_0$-partitions of $G$ and $H$. Further, if $a \subseteq V(G)$ is a non-empty independent set and $a \notin q$, then $a \cup r_i \neq p_j$ for any $i, j \in \{1, 2, \ldots, k\}$. Therefore, if $q'$ and $r'$ are $\mathcal{F}_0$-partitions of $G$ and $H$ respectively, then the $\mathcal{F}_0$-partitions of $D$ derived from $(q', r')$ and $(q, r)$ are the same if and only if $q = q'$ and $r = r'$.

Regarding the blocks of $q$ and $r$ as vertices of complete graphs $K_i$ and $K_j$ respectively, then we see that the partitions yielded by the construction above are in one-to-one correspondence with the partitions of $K_i+K_j$. Since there are $S(G;\mathcal{F}_0,i)$ $\mathcal{F}_0$-partitions of $G$ into $i$ blocks and $S(H;\mathcal{F}_0,j)$ $\mathcal{F}_0$-partitions of $H$ into $j$ blocks, there are $S(G;\mathcal{F}_0,i)S(H;\mathcal{F}_0,j)$ such pairs $(q,r)$ and $B(K_i+K_j;\mathcal{F}_0)S(G;\mathcal{F}_0,i)S(H;\mathcal{F}_0,j)$ partitions of $D$ that can be constructed from those pairs.

□

**Theorem 2.9.** *For complete graphs $K_i$ and $K_j$ (with $i \leq j$), $B(K_i+K_j;\mathcal{F}_0) = \sum_{l=0}^{i} \binom{i}{l} \frac{j!}{(j-l)!}$*

*Proof.* Note that $B(K_n;\mathcal{F}_0) = 1$, and that partition consists only of singletons. $\mathcal{F}_0$-partitions of $K_i + K_j$ will consist of singletons and only those doubletons consisting of one vertex from $V(K_i)$ and one vertex from $V(K_j)$.

Let $P$ and $Q$ be the $\mathcal{F}_0$-partitions of $K_i$ and $K_j$, respectively. We can construct a $\mathcal{F}_0$-partition of $K_i + K_j$ from blocks of $P$ and $Q$ such that each block of the resulting partition is either a block of $P$ (i.e., a vertex of $K_i$), a block of $Q$, or the union of exactly one block of $P$ and one block of $Q$.

There are $\binom{i}{l}$ ways to select $l$ blocks from $P$. If these $l$ blocks are used to form unions with blocks of $Q$ in a partition of $K_i + K_j$, then there are $\frac{j!}{(j-l)!}$ ways to choose blocks of $Q$ for the $l$ blocks of $P$. This $\mathcal{F}_0$-partition of $K_i + K_j$ will have $j+i-l$ blocks: $l$ blocks which are unions of blocks of $P$ and blocks of $Q$, $j-l$ blocks which are singletons of $Q$, and $i-l$ blocks which are singletons of $P$. Since there can be no other way to form $\mathcal{F}_0$-partitions of $K_i+K_j$ with $j+i-l$ blocks, we find that $S(K_i+K_j;\mathcal{F}_0,j+i-l) = \binom{i}{l}\frac{j!}{(j-l)!}$. Consequently, $B(K_i+K_j) = \sum_{l=0}^{i} S(K_i+K_j;\mathcal{F}_0,j+i-l)$
$= \sum_{l=0}^{i} \binom{i}{l}\frac{j!}{(j-l)!}$. □

**Theorem 2.10.** *Consider a graph $D$ with connected components $G$ and $H$.*

*Then* $S(G+H;\mathcal{F}_0,k) = \sum_{(i,j)\in\mathbb{N}^2} S(G;\mathcal{F}_0,i)S(H;\mathcal{F}_0,j)S(K_i+K_j;\mathcal{F}_0,k)$

*the sum taken over pairs* $(i,j)$ *subject to* $max(i,j) \le k \le i+j$, $\chi(G) \le i \le n(G)$ *and* $\chi(H) \le j \le n(H)$.

*Proof.* The method of construction here will be the same as in 2.8. Let $P = \{p_1, p_2, \ldots, p_i\}$ be a partition of $G$ with $i$ blocks and $Q = \{q_1, q_2, \ldots, q_j\}$ be a partition of $H$ with $j$ blocks. Then $P$ and $Q$ will be used to construct partitions of $G+H$ where blocks will be of the form $p_r$, $q_s$, or $p_r \cup q_s$. With this in mind, the pair $(P,Q)$ can construct partitions of $G+H$ with as few as $max(i,j)$ blocks and as many as $i+j$ blocks. In order to construct a partition with exactly $k$ blocks from $(P,Q)$, it is necessary that $max(i,j) \le k \le i+j$.

Suppose a pair of partitions $(P,Q)$ of $G, H$ respectively satisfies $max(i,j) \le k \le i+j$ for some chosen $k$. If we regard the blocks of $P$ and $Q$ as vertices of complete graphs $K_i$ and $K_j$ respectively, then the number of partitions with $k$ blocks yielded by $(P,Q)$ are in one-to-one correspondence with the partitions of $K_i + K_j$ with $k$ blocks. Since there are $B(G;\mathcal{F}_0,i)$, $B(H;\mathcal{F}_0,j)$ partitions of $G$ with $i$ blocks and $H$ with $j$ blocks, respectively, there are $B(G;\mathcal{F}_0,i)B(H;\mathcal{F}_0,j)$ such pairs $(P,Q)$ and thus there are $B(K_i + K_j;\mathcal{F}_0,k)B(G;\mathcal{F}_0,i)B(H;\mathcal{F}_0,j)$ partitions of $G+H$ with $k$ blocks that can be constructed from partitions of $G$ with $i$ blocks and partitions of $H$ with $j$ blocks.

As noted in the proof of 2.8, every partition of $G+H$ is derived from some pair $(P,Q)$ of partitions of $G, H$ separately, and collections of partitions derived from different pairs are disjoint. Since $max(i,j) \le k \le i+j$ is a necessary condition on $i$ and $j$ for a partition of $G+H$ to be constructed from a partition $P$ of $G$ with $i$ blocks and $Q$ of $H$ with $j$ blocks, and clearly there can be no such partitions unless $\chi(G) \le i \le n(G)$ and $\chi(H) \le j \le n(H)$, the sum in 2.9 counts each $k$-block partition of $G+H$ exactly once. □

## Chapter 3

## The Bell Number of a Graph

## With Respect to the Family $\mathcal{F}_t$

$$\mathcal{F}_t := \{T : T \text{ is a tree }\}$$

$\mathcal{F}_t$ is the collection of acyclic connected graphs, or trees. For this family, the Bell number $B(G;\mathcal{F}_t)$ is examined in the cases where $G$ is a complete graph, a tree, and a cycle. Moreover, exploitable properties of $B(G;\mathcal{F}_t)$ are found when $G$ has certain features.

It is first worthwhile to notice that the functions $B(G;\mathcal{F}_0)$ and $B(G;\mathcal{F}_t)$ behave very differently.

This material originally appeared in [4].

### 3.1 Complete Graphs

**Theorem 3.1.** $B(K_n;\mathcal{F}_t) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{2^k k!(n-2k)!}$

*Proof.* The subgraph induced by any collection of $m$ vertices (for $1 \le m \le n$) is $K_m$. Of these, only $K_1$ and $K_2$ are also trees, so no block of a $\mathcal{F}_t$-partition may contain more than 2 elements. We can count the number of $\mathcal{F}_t$-partitions of $K_n$ by counting the number of partitions of $\{1, 2, \ldots, n\}$ containing no blocks of size 2, exactly one block of size 2, exactly two blocks of size 2, and so on, up to exactly $\lfloor \frac{n}{2} \rfloor$ blocks of size 2.

In $K_n$, there is one $\mathcal{F}_t$-partition with no blocks of size 2. There are $\binom{n}{2}$ ways to choose one block of size 2, so there are precisely that many $\mathcal{F}_t$-partitions containing exactly 1 block of size 2. There are $\frac{1}{2!}\binom{n}{2}\binom{n-2}{2}$ ways to choose blocks of size 2, irrespective of order, so there are precisely that many $\mathcal{F}_t$-partitions containing exactly 2 blocks of size 2. The number of ways to select $k$ blocks of size 2, irrespective of order, is $\frac{1}{k!}\prod_{i=0}^{k-1}\binom{n-2i}{2}$. Notice, however, that $\prod_{i=0}^{k-1}\binom{n-2i}{2}$ collapses:

$$\begin{aligned}\prod_{i=0}^{k-1}\binom{n-2i}{2} &= \frac{n!}{2(n-2)!}\cdot\frac{(n-2)!}{2(n-4)!}\cdot\frac{(n-4)!}{2(n-6)!}\cdots\frac{(n-2k+4)!}{2(n-2k+2)!)}\cdot\frac{(n-2k+2)!}{2(n-2k)!}\\ &= \frac{n!}{2^k(n-2k)!}\end{aligned}$$

Thus, the number of $\mathcal{F}_t$-partitions with exactly $k$ blocks of size 2 is $\frac{n!}{2^k k!(n-2k)!}$. Summing over $k$ gives the result. □

*Remark.* Counting the $\mathcal{F}_t$-partitions of $K_n$ is equivalent to partitioning an $n$-element set into (unordered) sets of size 1 and 2, so it is unsurprising that this is not the first time this problem has arisen. Other interpretations can be found at [6].

### 3.2 Trees

**Theorem 3.2.** *If $H$ is a simple graph and $v \in V(H)$ a pendant vertex, then $B(H;\mathcal{F}_t) = 2B(H \setminus v;\mathcal{F}_t)$.*

*Proof.* If $p$ is a $\mathcal{F}_t$-partition of $H \setminus v$ then we can form a new partition by inserting $v$ either in a singleton set or in a block with its neighbor in $H$. This new partition will be a $\mathcal{F}_t$-partition of $H$, so from each $\mathcal{F}_t$-partition of $H \setminus v$ we can create two $\mathcal{F}_t$-partitions of $H$. These partitions will clearly be distinct. The reader can easily verify that all $\mathcal{F}_t$-partitions of $H$ are obtained in this way.

□

**Theorem 3.3.** *If $G$ is a tree on $n$ vertices, then $B(G;\mathcal{F}_t) = 2^{n-1}$.*

*Proof.* Since $G$ is a tree on $n$ vertices, $G$ can be built from a single vertex by sequentially appending pendant vertices $n-1$ times. Since $B(K_1; \mathcal{F}_t) = 1$, $B(G; \mathcal{F}_t) = 2^{n-1}$. $\square$

### 3.3 Cycles

**Theorem 3.4.** *If $C_n$ is a cycle on $n$ vertices ($n \geq 3$), then $B(C_n; \mathcal{F}_t) = 2^n - (n+1)$.*

*Remark.* We have opted to prove this using mathematical induction and 3.2. The interested reader may notice a shorter, direct proof.

*Proof.* Observe that $B(C_3; \mathcal{F}_t) = 4$, so the claim is true for $n = 3$. To proceed inductively, the following argument requires $n \geq 4$. It can be easily verified that $B(C_4; \mathcal{F}_t) = 11$, so the statement is true for a base case. Assume now that for some $n \in \mathbb{N}$, $n > 4$, $B(C_{n-1}; \mathcal{F}_t) = 2^{n-1} - n$.

We will count $B(C_n; \mathcal{F}_t)$ by constructing $\mathcal{F}_t$-partitions of $C_n$ from the $\mathcal{F}_t$-partitions of $C_{n-1}$. Let $V(C_n) = \{1, 2, \ldots, n\}$ and $V(C_{n-1}) = \{1, 2, \ldots, n-1\}$. A $\mathcal{F}_t$-partition $p$ of $C_{n-1}$ will be called "good" if 1 and $n-1$ do not appear in the same block. Otherwise, the partition is "bad".

In a good $p$, $n-1$ is either in a block with $n-2$ or it is in a block by itself. Likewise, 1 is either in a block with 2 or in a block by itself. Notice that, except for the partition $\overline{p} = \{\{1, 2, \ldots, n-1\}\}$, every $\mathcal{F}_t$-partition of $P_{n-1}$ (a path on $n-1$ vertices) is a good $\mathcal{F}_t$-partition of $C_{n-1}$, and every good partition is a $\mathcal{F}_t$-partition of $P_{n-1}$. Therefore, there are $B(P_{n-1}; \mathcal{F}_t) - 1$ good $\mathcal{F}_t$-partitions of $C_{n-1}$.

From each good $\mathcal{F}_t$-partition of $C_{n-1}$, we can construct three $\mathcal{F}_t$-partitions of $C_n$: $n$ can be inserted as a singleton block, $n$ can be inserted into a block with $n-1$, and $n$ can be inserted into a block with 1. In good partitions, $n-1$ and 1 are in separate blocks, so each choice produces a distinct partition. The partitions are $\mathcal{F}_t$-partitions of $C_n$ since a block inducing a tree in $C_{n-1}$ also induces a tree in $C_n$, and keeping 1 and $n-1$ apart has avoided inducing cycles. And so, good $\mathcal{F}_t$-partitions produce a total of $3(B(P_{n-1};\mathcal{F}_t)-1)$ $\mathcal{F}_t$-partitions of $C_n$.

In a bad $\mathcal{F}_t$-partition, 1 and $n-1$ appear in a block together, which means that block induces two disjoint paths in $C_n$. However, by inserting $n$ into such a block, since $n$ is a neighbor of both 1 and $n-1$ in $C_n$, the block now induces a tree. In $\mathcal{F}_t$-partitions of $C_{n-1}$, since 1 and $n-1$ either appear together in a block or in two separate blocks, we know that there must be $B(C_{n-1};\mathcal{F}_t)-(B(P_{n-1};\mathcal{F}_t)-1)$ bad $\mathcal{F}_t$-partitions of $C_{n-1}$. Each of these yields exactly one $\mathcal{F}_t$-partition of $C_n$, and each $\mathcal{F}_t$-partition produced is distinct.

Recall the partition $\overline{p}$. Being that it is not a $\mathcal{F}_t$ partition of $C_{n-1}$, it is neither good nor bad. However, the block $\{1,2,\ldots,n-1\}$ induces a tree in $C_n$ and this block does not appear in any good or bad partitions. The only way to construct from $\overline{p}$ a $\mathcal{F}_t$-partition of $C_n$ is to form $\overline{p}\cup\{\{n\}\}$.

Every $\mathcal{F}_t$-partition of $C_n$ is counted in this way. This produces

$$\begin{aligned}B(C_n;\mathcal{F}_t) &= 3(B(P_{n-1};\mathcal{F}_t)-1)+B(C_{n-1};\mathcal{F}_t)-(B(P_{n-1};\mathcal{F}_t)-1)+1\\ &= 2B(P_{n-1};\mathcal{F}_t)+B(C_{n-1};\mathcal{F}_t)-1\end{aligned}$$

Since $P_{n-1}$ is a tree, $B(P_{n-1};\mathcal{F}_t)=2^{n-2}$. By the inductive step, $B(C_{n-1};\mathcal{F}_t)=2^{n-1}-n$, so

$$B(C_n;\mathcal{F}_t)=2\cdot 2^{n-2}+2^{n-1}-n-1=2^n-(n+1).$$

□

*Remark.* Like 3.1, this theorem again provides new interpretation for an integer sequence. Other interpretations can be found at [6].

### 3.4 Combinations of Graphs

**Theorem 3.5.** *If $G$ is a graph with components $H_1$ and $H_2$, then $B(G;\mathcal{F}_t) = B(H_1;\mathcal{F}_t)B(H_2;\mathcal{F}_t)$*

*Proof.* If $p$ is a $\mathcal{F}_t$-partition of $H_1$ and $q$ is a $\mathcal{F}_t$-partition of $H_2$, then $p \cup q$ is a $\mathcal{F}_t$-partition of $G$. Conversely, if $r$ is a $\mathcal{F}_t$-partition of $G$ then each block of $r$ contains vertices either strictly from $V(H_1)$ or strictly from $V(H_2)$, so $r$ may be viewed as the union of a $\mathcal{F}_t$-partition of $H_1$ and a $\mathcal{F}_t$-partition of $H_2$. □

**Theorem 3.6.** *Suppose $G$ is a graph with cut vertex $v$, and let $H_1, H_2, \ldots, H_k$ be the components of $G \setminus v$. Define $I_j$ to be the subgraph of $G$ induced by $V(H_j) \cup \{v\}$. Then $B(G;\mathcal{F}_t) = \prod_{j=1}^{k} B(I_j;\mathcal{F}_t)$*

*Proof.* For $1 \leq j \leq k$, let $p_j$ denote a $\mathcal{F}_t$-partition of $I_j$ and $b_j$ denote the block of $p_j$ containing vertex $v$. Let $p = \cup_{j=1}^{k} p_j$ and $b = \cup_{j=1}^{k} b_j$. Form a new set collection $q$:

$$q = (p \setminus \{b_1, b_2, \ldots, b_k\}) \cup \{b\}$$

Observe that $q$ is a $\mathcal{F}_t$ partition of $G$, and $q$ is distinct for distinct choices of $p_j$.

Every $\mathcal{F}_t$-partition of $G$ is obtained in this way. Let $q$ be a $\mathcal{F}_t$-partition of $G$. Denote by $b$ the block containing $v$. For each block $c \neq b$, there is a $j \in \{1, 2, \ldots k\}$ such that $c \subseteq V(H_j)$. For each $j$, define $b_j = \{u \in b : u \in V(I_j)\}$ and $p_j = \{c \in q : c \subseteq V(H_j)\} \cup \{b_j\}$. This yields a collection $p_1, p_2, \ldots, p_k$ of partitions of the vertex sets of $I_1, I_2, \ldots, I_k$. It is straightforward that each $p_j$ is a $\mathcal{F}_t$-partition of $I_j$ and that $q$ arises from $p_1, p_2, \ldots, p_k$ in the way described above.

$\square$

**Theorem 3.7.** *Suppose $G$ is a connected graph with cut edge $e$, and let $H_1, H_2$ be the components of $G \setminus e$. Then $B(G; \mathcal{F}_t) = 2B(H_1; \mathcal{F}_t)B(H_2; \mathcal{F}_t)$.*

*Proof.* If either end of $e$ has degree 1, the result follows from 3.2. Assume that the ends of $e$ have degree larger than 1.

In a $\mathcal{F}_t$-partition of $G$, the end points of $e$ are either in a block together or they are in separate blocks. The $\mathcal{F}_t$-partitions of $G$ where the end points of $e$ are in separate blocks are exactly the $\mathcal{F}_t$-partitions of $G \setminus e$, so there are precisely $B(H_1; \mathcal{F}_t)B(H_2; \mathcal{F}_t)$ $\mathcal{F}_t$-partitions of $G$ of this type. The $\mathcal{F}_t$-partitions of $G$ where the end points of $e$ appear in the same block correspond to the $\mathcal{F}_t$-partitions of the graph $G \cdot e$ obtained by contracting the end points $v_1, v_2$ of $e$ to a single vertex $w$. Because $e$ was a cut edge, $w \in V(G \cdot e)$ is a cut vertex. Let $J_1$ be the subgraph of $G \cdot e$ induced by $\{w\} \cup V(H_1) \setminus \{v_1\}$, and $J_2$ be the subgraph of $G \cdot e$ induced by $\{w\} \cup V(H_2) \setminus \{v_2\}$. Then $B(G \cdot e; \mathcal{F}_t) = B(J_1; \mathcal{F}_t)B(J_2; \mathcal{F}_t)$. But clearly $J_1$ is isomorphic to $H_1$ and $J_2$ to $H_2$, so $B(H_1; \mathcal{F}_t)B(H_2; \mathcal{F}_t) = B(J_1; \mathcal{F}_t)B(J_2; \mathcal{F}_t)$. This gives us

$$B(G; \mathcal{F}_t) = B(G \setminus e; \mathcal{F}_t) + B(G \cdot e; \mathcal{F}_t)$$

$$B(G; \mathcal{F}_t) = 2B(H_1; \mathcal{F}_t)B(H_2; \mathcal{F}_t)$$

$\square$

**Questions** Which families $\mathcal{F}$ of graphs with $K_1 \in \mathcal{F}$ share some or all of the properties of $\mathcal{F}_t$ expressed in 3.2, 3.5, 3.6 or 3.7?

Chapter 4

The Bell Number of a Graph

With Respect to the Family $\mathcal{F}_f$

$$\mathcal{F}_f := \{F : F \text{ is a forest }\}$$

Many properties of $\mathcal{F}_t$ are lost by extending the family to forests, largely because the family contains graphs which are not connected. For example, Theorem 3.2 does not hold for the family $\mathcal{F}_f$ as it does for $\mathcal{F}_t$. To illustrate, let $G$ be a $K_3$ with a pendant vertex attached. If 3.2 held, then

$$B(G; \mathcal{F}_f) = 2B(K_3; \mathcal{F}_f)$$

$B(K_3; \mathcal{F}_f) = 4$, so we would find $B(G; \mathcal{F}_f) = 8$. However, we may find $B(G; \mathcal{F}_f)$ quickly by considering all (ordinary) partitions of the set $V(G)$, of which there are $b_4 = 15$, and eliminating those where a block induces $K_3$. There are exactly 2 of those: the partition consisting of $V(G)$ and the partition where the pendant vertex is isolated and the vertices of $K_3$ are in a block. Clearly, $B(G; \mathcal{F}_f) = 13 \neq 2B(G; \mathcal{F}_f)$.

$\mathcal{F}_f$ is an incredibly permissive family. It excludes only blocks which induce non-acyclic graphs. This makes it surprisingly easy to estabish Bell numbers, but less easy to identify convenient properties.

For example, supppose $G$ is any forest and $|V(G)| = n$. Then $B(G; \mathcal{F}_f) = b_n$ because absolutely every partition of $V(G)$ is a $\mathcal{F}_f$-partition. If $G = C_n$, then $B(G; \mathcal{F}_f) = b_n - 1$ because only $V(C_n)$ itself fails to be a $\mathcal{F}_f$-partition. If $G = K_n$, then $B(G; \mathcal{F}_f) = B(G; \mathcal{F}_t)$ since it is still the case that the only valid partitions are those whose blocks consist only of singletons and doubletons.

**Theorem 4.1.** *Suppose $G$ is a unicyclic graph with $|V(G)| = n$ and the cycle has size $k$. Then $B(G; \mathcal{F}_f) = b_n - b_{n-k+1}$*

*Proof.* The only partitions of $V(G)$ which are not $\mathcal{F}_f$-partitions are those in which some block induces a non-acyclic subgraph. Define $G'$ to be the graph obtained by identifying the vertices of the $k$-cycle in $G$ as the single vertex $v$. Then $B(G'; \mathcal{F}_f)$ is the number of partitions of this graph, and each of these corresponds to a non-$\mathcal{F}_f$-partition of $G$. Since $G'$ is a tree on $n - k + 1$ vertices, $B(G'; \mathcal{F}_f) = b_{n-k+1}$. The result follows. $\square$

The idea of shrinking cycles to single vertices allows this theorem to generalize.

**Theorem 4.2.** *Let $G$ be a graph where $|V(G)| = n$ and $V(G)$ has pairwise disjoint subsets $W_1, W_2, \ldots, W_l$ such that the subgraph of $G$ induced by $W_i$ ($1 \leq i \leq l$) is a cycle, and suppose that there is no other subset of $V(G)$ whose induced subgraph is a cycle. Refer to these cycles as $C_{k_1}, C_{k_2}, \ldots, C_{k_l}$ (where each $k_i$ is the cycle length), and let $K = \{k_1, k_2, \ldots, k_l\}$. Then*

$$B(G; \mathcal{F}_f) = b_n + \sum_{j=1}^{l} (-1)^j \Big( \sum_{\substack{S \subseteq K \\ |S| = j}} b_{n - (\sum_{x \in S} x) + j} \Big)$$

*Proof.* Let $A_i$ denote the collection of (ordinary) partitions $p$ of $V(G)$ such that some block of $p$ induces a subgraph containing $C_{k_i}$. Let $[l] = \{1, 2, \ldots, l\}$. If we know the size of $\bigcup_{i=1}^{l} A_i$, then we know how many partitions of $V(G)$ are not $\mathcal{F}_f$-partitions of $G$. The size of a union is well known to be

$$|\bigcup_{i=1}^{l} A_i| = \sum_{j=1}^{l} (-1)^{j-1} \Big( \sum_{\substack{R \subseteq [l] \\ |R|=j}} |\bigcap_{r \in R} A_r| \Big)$$

From here, consider the sizes of these different sets. The size of $A_i$, for any $1 \leq i \leq n$, is $b_{n-k_i+1}$. Identify all vertices of $C_{k_i}$ as a single vertex $c_i$. Consider the (ordinary) partitions of this new vertex set of size $n - k_i + 1$. Each of these partitions corresponds to a partition of $V(G)$ in which $C_{k_i}$ is in a block, and the subgraph of $G$ induced by that block will contain $C_{k_i}$.

For $i, j \in [l], i \neq j$, the size of $A_i \cap A_j$ can be found by shrinking the cycles $C_{k_i}$ and $C_{k_j}$ to single vertices $c_i$ and $c_j$, respectively. The resulting graph has $n - k_i - k_j + 2$ vertices, and there are $b_{n-k_i-k_j+2}$ (ordinary) partitions of that vertex set.

For $R \subseteq [l]$, the size of $\bigcap_{r \in R} A_r$ can be found by, for each $r \in R$, shrinking the cycle $C_r$ to a single point $c_r$. Then the number of partitions of that vertex set is $b_{n-(\sum_{r \in R} k_r)+|R|}$.

$\square$

## Chapter 5

## Families $\mathcal{F}$ and Their Properties

Throughout this chapter, $\mathcal{F}$ will denote a family of graphs containing $K_1$.

We have seen the families $\mathcal{F}_0$ and $\mathcal{F}_t$ exhibit a variety of properties. We now turn our attention to identifying other families that might share these properties.

### 5.0.1 Situations where $B(G;\mathcal{F})$ is Maximum

Recall that for any family $\mathcal{F}$ and any graph $G$ with $|V(G)| = n$, $B(G;\mathcal{F}) \leq b_n$. Equality occurs when every induced subgraph of $G$ is a member of $\mathcal{F}$. We also found that $B(G;\mathcal{F}_f) = b_{|V(G)|}$ for every $G \in \mathcal{F}_f$, but it was not the case that $B(G;\mathcal{F}_t) = b_{|V(G)|}$ for every $G \in \mathcal{F}_t$. Observe the following.

**Theorem 5.1.** *If $\mathcal{F}$ is a family of graphs containing $K_1$, then $B(G;\mathcal{F}) = b_{|V(G)|}$ for all $G \in \mathcal{F}$ if and only if $\mathcal{F}$ is closed under the operation of taking induced subgraphs.*

*Proof.* Suppose that $B(G;\mathcal{F}) = b_{|V(G)|}$ for all $G \in \mathcal{F}$. Then for all $G \in \mathcal{F}$, every subgraph of $G$ is a member of $\mathcal{F}$. So, $\mathcal{F}$ is closed under the operation of taking induced subgraphs. Conversely, if $\mathcal{F}$ is closed under the operation of taking induced subgraphs, then for every $G \in \mathcal{F}$, all of its induced subgraphs are members of $\mathcal{F}$. Therefore, $B(G;\mathcal{F}) = b_{|V(G)|}$. $\square$

### 5.0.2 Families with a Multiplicative Property

**Theorem 5.2.** *Let $G$ be a graph with connected components $H_1, H_2, \ldots, H_n$. If $\mathcal{F}$ is a family of connected graphs, then $B(G;\mathcal{F}) = \prod_{i=1}^{n} B(H_i;\mathcal{F})$.*

*Proof.* Recall the proof of 3.5 used only that each member of the family is a connected graph. The proof here is similar. □

### 5.0.3 Families with a Cut Vertex Property

In this section, $G$ will be a graph with a cut vertex $v$ such that $G \setminus v$ has connected components $H_1, H_2, \ldots, H_k$; respectively subgraphs $I_1, I_2, \ldots, I_k$ of $G$ are induced by $V(H_1) \cup \{v\}, V(H_2) \cup \{v\}, \ldots, V(H_k) \cup \{v\}$, respectively. For the sake of brevity, we will abridge this by referring to these special induced subgraphs as the **$v$-induced subgraphs of $G$**.

Recall the proof of 3.6. We began with a graph $G$ with cut vertex $v$ and $v$-induced subgraphs $I_1, I_2, \ldots, I_k$. From $\mathcal{F}_t$-partitions $p_1, p_2, \ldots, p_k$ of $I_1, I_2, \ldots, I_k$ (respectively), we constructed a $\mathcal{F}_t$-partition of $G$ by defining the block of $q$ containing $v$ to be the union of the blocks of $p_1, p_2, \ldots, p_k$ containing $v$. There are two reasons this construction is valid in proving 3.6. First, the construction of the block containing $v$ in $q$ requires that any time $X, Y \in \mathcal{F}_t$, if $x \in V(X)$ and $y \in V(Y)$ then the graph $X \cdot Y$ obtained by identifying $x = y$ is also a member of $\mathcal{F}_t$. Additionally, it is necessary that the graphs of $\mathcal{F}_t$ be connected as this means the only block of $q$ containing vertices of each of $I_1, I_2, \ldots, I_k$ is the block containing $v$. This observation of the essential properties of $\mathcal{F}_t$ used in the proof of 3.6 leads to the following generalization.

A family $\mathcal{F}$ of graphs containing $K_1$ will be said to have the Cut Vertex Property if and only if the statement obtained from 3.6 by replacing $\mathcal{F}_t$ by $\mathcal{F}$ is true.

**Theorem 5.3.** *Suppose that $\mathcal{F}$ is a family of connected graphs with the following properties. (1) For every $X, Y \in \mathcal{F}$, if $x \in V(X)$ and $y \in V(Y)$ then the graph $X \cdot Y \in \mathcal{F}$ where $X \cdot Y$*

*is the graph obtained by identifying the vertices $x = y$. (2) For any $X \in \mathcal{F}$ with cut vertex $x$, the $v$-induced subgraphs of $X$ are each members of $\mathcal{F}$. Then $\mathcal{F}$ has the Cut Vertex Property.*

*Proof.* The proof proceeds essentially the same as the one for $\mathcal{F}_t$. □

Our interest will be in identifying families $F$ with the Cut Vertex Property, but some of them can be observed right now, thanks to 5.3.

- $\{K_1\}$, a trivial family
- $\mathcal{F}_t$, the family of all trees
- $\mathcal{F}_c$, the family of all connected graphs

The relationship $\{K_1\} \subset \mathcal{F}_t \subset \mathcal{F}_c$ gives rise to several questions.

- Is there a family $\{K_1\} \subset \mathcal{F} \subset \mathcal{F}_t$ with this cut vertex property?
- Are there families $\{K_1\} \subset \mathcal{F} \subset \mathcal{F}_c$ with the cut vertex property other than the family $\mathcal{F}_t$ ?
- What are the necessary and sufficient conditions for the cut vertex property?

**Theorem 5.4.** *If $\mathcal{F}$ is a family of graphs such that $\{K_1\} \subset \mathcal{F} \subset \mathcal{F}_t$, then $\mathcal{F}$ does not have the cut vertex property.*

*Proof.* Let $T \in \mathcal{F}_t \setminus \mathcal{F}$ be such that $|V(T)|$ is minimal among all trees in $\mathcal{F}_t \setminus \mathcal{F}$.

*Case 1. $T \neq K_2$.*

Observe that $T$ has a cut vertex $v$. Consider $B(T; \mathcal{F})$. Since $T \in \mathcal{F}_t \setminus \mathcal{F}$ is of minimal order, every tree of order smaller than $T$ is in $\mathcal{F}$. Therefore, the only subtree of $T$ not in $\mathcal{F}$ is the tree induced by $V(T)$. Consequently, $B(T; \mathcal{F} \cup \{T\}) = B(T; \mathcal{F}_t)$ since every subtree of $T$ is a member of $\mathcal{F} \cup \{T\}$. If $T$ has $v$-induced subgraphs $I_1, I_2, \ldots, I_k$ then

$$B(T;\mathcal{F}) = B(T;\mathcal{F}\cup\{T\}) - 1 = \prod_{j=1}^{k} B(I_j;\mathcal{F}) - 1 \neq \prod_{j=1}^{k} B(I_j;\mathcal{F})$$

Thus we see that $\mathcal{F}$ does not have the cut vertex property.

*Case 2.* $T = K_2$

By way of contradiction, assume that $\mathcal{F}$ does have the cut vertex property.

Let $R$ be any tree with $|V(R)| \geq 3$. Since $R$ is a tree, it has at least 2 leafs. Let $v$ be a vertex adjacent to a leaf, and suppose it has $m$ neighbor leafs. Let $R$ have $v$-induced subgraphs $I_1, I_2, \ldots, I_k$ ordered so that $|V(I_i)| \leq |V(I_j)|$ whenever $i \leq j$. Since there are $m$ neighbor leafs to $v$, the subgraphs $I_1, I_2, \ldots, I_m$ will all be isomorphic to $K_2$. Since $K_2 \notin \mathcal{F}$, $B(K_2;\mathcal{F}) = 1$. As $\mathcal{F}$ has the cut vertex property,

$$B(R;\mathcal{F}) = \prod_{j=1}^{k} B(I_j;\mathcal{F}) = \prod_{j=m+1}^{k} B(I_j;\mathcal{F}).$$

We see that each leaf contributes nothing to the Bell number with respect to $\mathcal{F}$. Since each of $I_{m+1}, I_{m+2}, \ldots, I_k$ are trees larger than $K_2$, the process applied to $R$ can be iterated on each of them. The iterations halt when all $v$-induced subgraphs are isomorphic to $K_2$, and this leads us to conclude $B(R;\mathcal{F}) = 1$. But this tree $R$ was arbitrary, so we see that the Bell number of any tree with respect to $\mathcal{F}$ must be 1. That means $\mathcal{F} = \{K_1\}$, but we began by assuming $\{K_1\} \subset \mathcal{F}$. $\square$

**Corollary 5.1.** *If* $\{K_1\} \subset \mathcal{F} \subset \mathcal{F}_t$ *then* $\mathcal{F}$ *fails to have at least one of the properties (1) and (2) in 5.3.*

We have enough information to conclude that there are families $\{K_1\} \subset \mathcal{F} \subset \mathcal{F}_c$ with the Cut Vertex Property other than the family $\mathcal{F}_t$. For example, let $\mathcal{G}$ be a collection of graphs (possibly finite), not all trees. Generate a larger collection $\mathcal{G}$ (containing $K_1$) by applying the operations suggested by 5.3 properties (1) and (2). That is, for any graphs

$H, K \in \mathcal{G}$ and any $h \in V(H)$ and $k \in V(K)$, the graph obtained by identifying $h$ and $k$ as a single vertex ("gluing" $h$ and $k$) is also a member of the new collection. Moreover, if $L$ is a member of the new collection and $v$ is a cut vertex of $L$, then the $v$-induced subgraphs of $L$ must be members of the new collection. And if $L_1$ and $L_2$ are both members of the new collection, any graph obtained through a "gluing" manipulation (using only one vertex of each graph) must also be in the new collection.

In this way, we can see that there will indeed be many more distinct families with the Cut Vertex Property. For example, if $\{C_4\}$ is our initial collection, then applying the operations suggested by (1) and (2) will generate a family distinct from both $\mathcal{F}_t$ and the family of all connected graphs.

*Remark.* The Cut Edge Property of $\mathcal{F}_t$ can be considered for generalization as well. At the time of this writing, we will content ourselves with the discovery of the existence of families with the Cut Vertex Property apart from $\{K_1\}, \mathcal{F}_t$, and $\mathcal{F}_c$.

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