The Metamorphosis of Maximum Packings of $2K_n$ with Triples into Maximum Packings of $2K_n$ with 4-cycles

by

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Abstract

In this work, the problem of constructing a maximum packing of $2K_n$ with triples having a metamorphosis into a maximum packing of $2K_n$ with 4-cycles is concluded by solving the problem for every $n \geq 11$ such that $n \equiv 2, 5, 8,$ or 11 $(mod\ 12)$.

Acknowledgments

This work is dedicated to Curtis White, November 9, 1959 - April 4, 2012.

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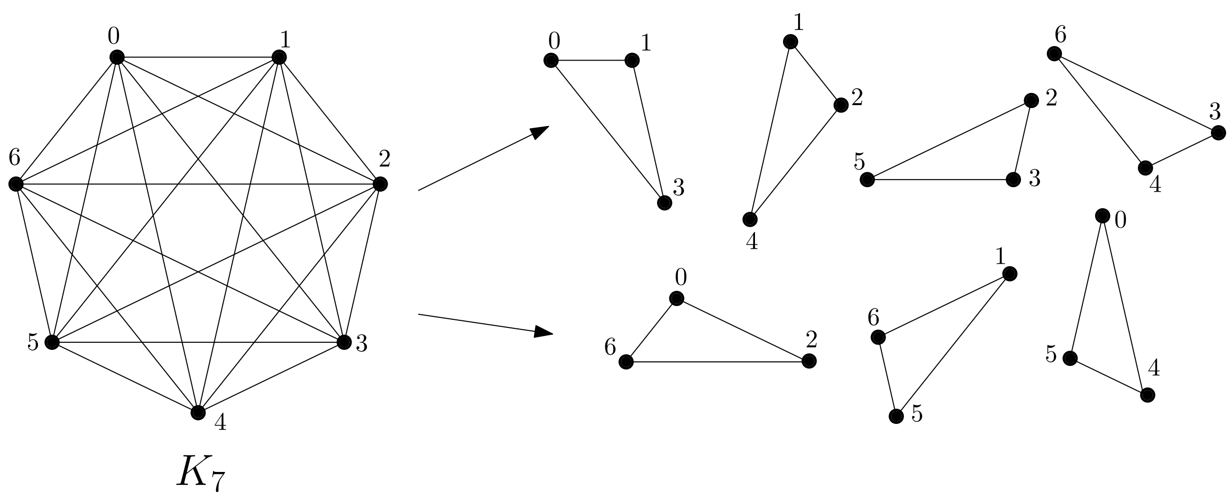
Chapter 1

Introduction

A *Steiner triple system* (more simply, *triple system*) *of order* n is a pair (S, T), where T is a collection of edge-disjoint triangles (or triples) which partitions the edge set of K_n with vertex set S.

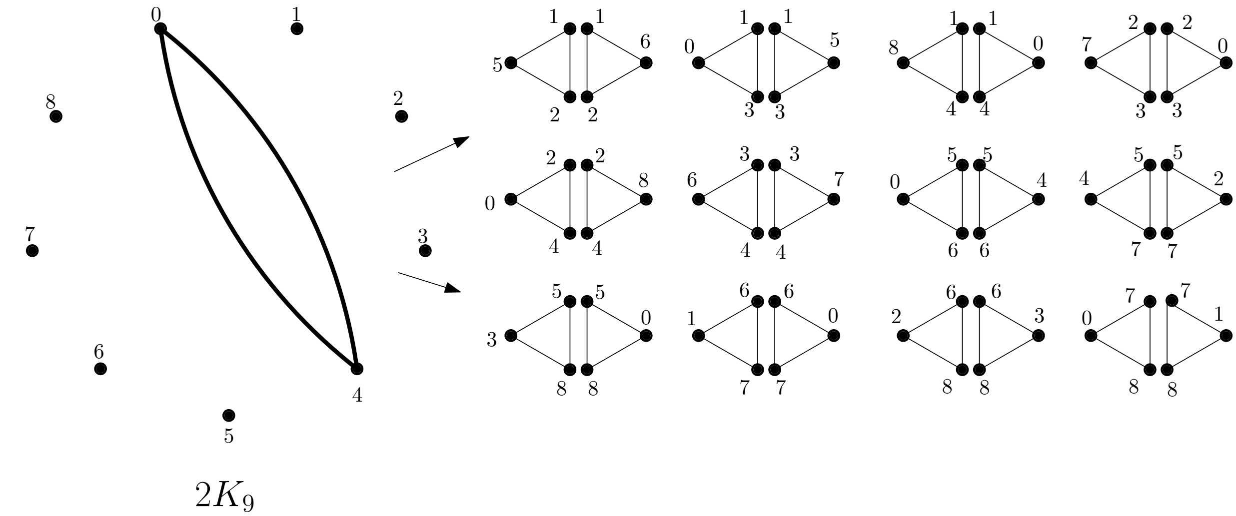
Example 1.1 (STS(7))**.**



It is well-known that the spectrum for Steiner triple systems is precisely the set of all $n \equiv 1$ or 3 $(mod\ 6)$ and that if (S, T) is a Steiner triple system of order n, $|T| = n(n-1)/6$ [2].

A *2-fold triple system of order* n is a pair (X, T), where T is a collection of edge-disjoint triples which partitions the edge set of $2K_n$ (every pair of vertices is joined by 2 edges) with vertex set X.

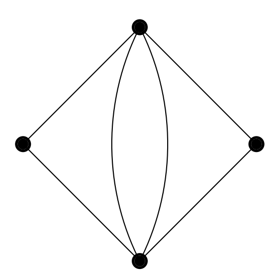
Example 1.2 (2-fold triple system of order 9)**.**



Just as for triple systems it is well-known that the spectrum for 2-fold triple systems is precisely the set of all $n \equiv 0$ or 1 $(mod\ 3)$ and that if (X, T) is a 2-fold triple system of order n, $|T| = n(n-1)/3$ [3].

Finally, a *2-fold 4-cycle system of order* n is a pair (X, C), where C is a collection of edge-disjoint 4-cycles which partitions the edge set of $2K_n$ with vertex set X. The spectrum for 2-fold 4-cycle systems is precisely the set of all $n \equiv 0$ or 1 $(mod\ 4)$ [4]. Since the spectra for 2-fold triple systems and 2-fold 4-cycle systems agree when $n \equiv 0, 1, 4,$ or 9 $(mod\ 12)$, an obvious question arises: Are there any connections between the two systems when they have the same order? The answer is yes!

In what follows, we will call the graph



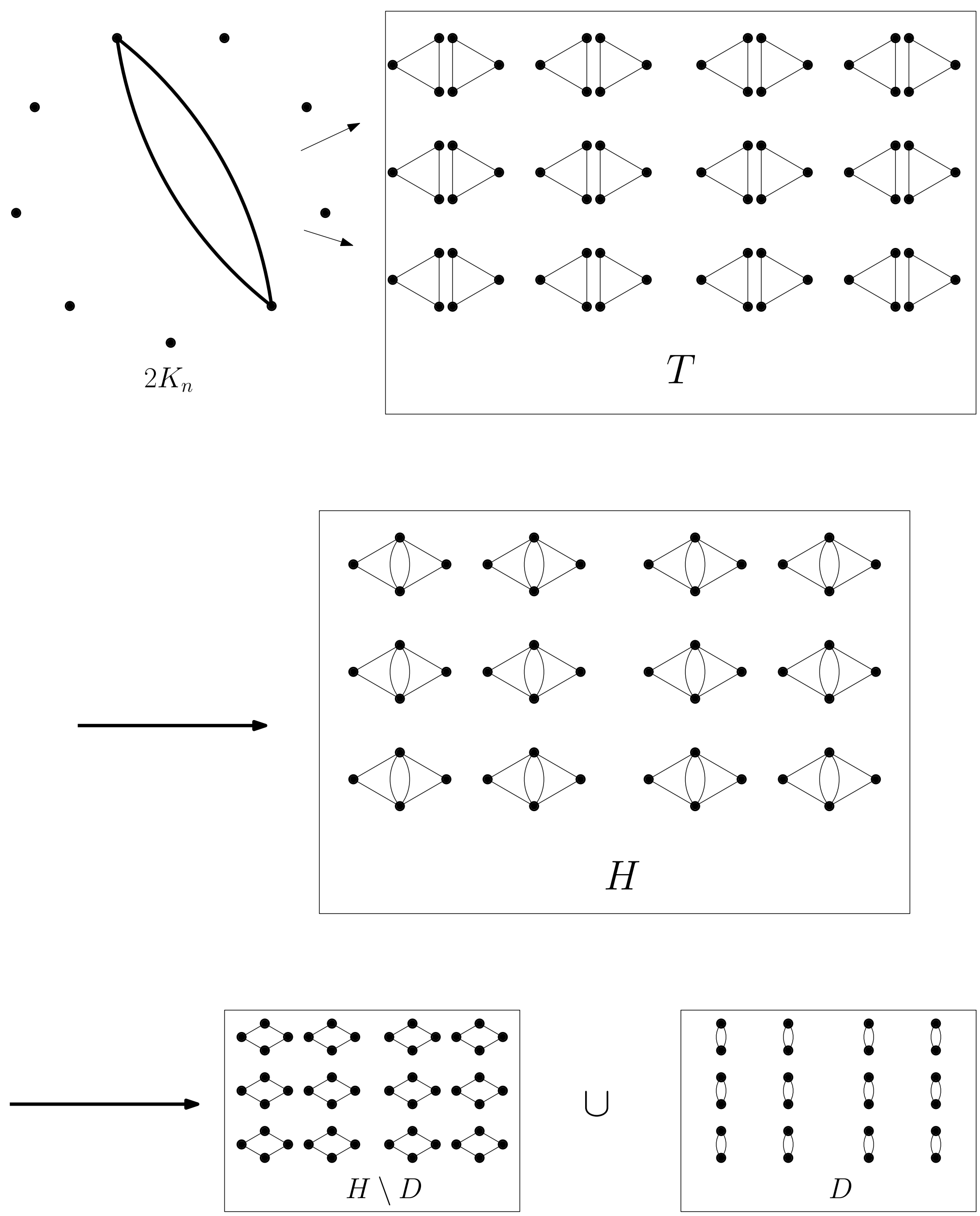
a *hinge*. Notice that if (X, T) is a 2-fold triple system, then $|T|$ is even. This yields the following problem for 2-fold triple systems.

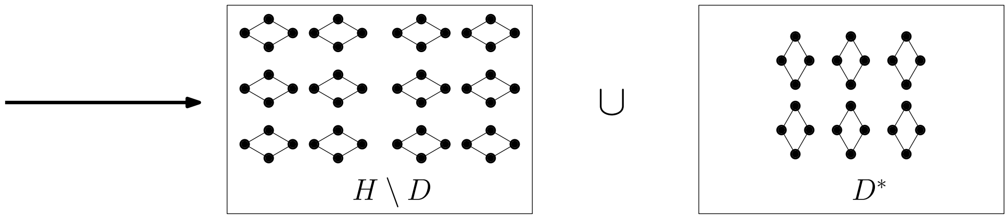
Suppose $n \equiv 0, 1, 4$, or 9 $(mod\ 12)$. Does there exist a 2-fold triple system (X, C) of order n such that:

(i) the triples can be paired into a collection of hinges, H, and

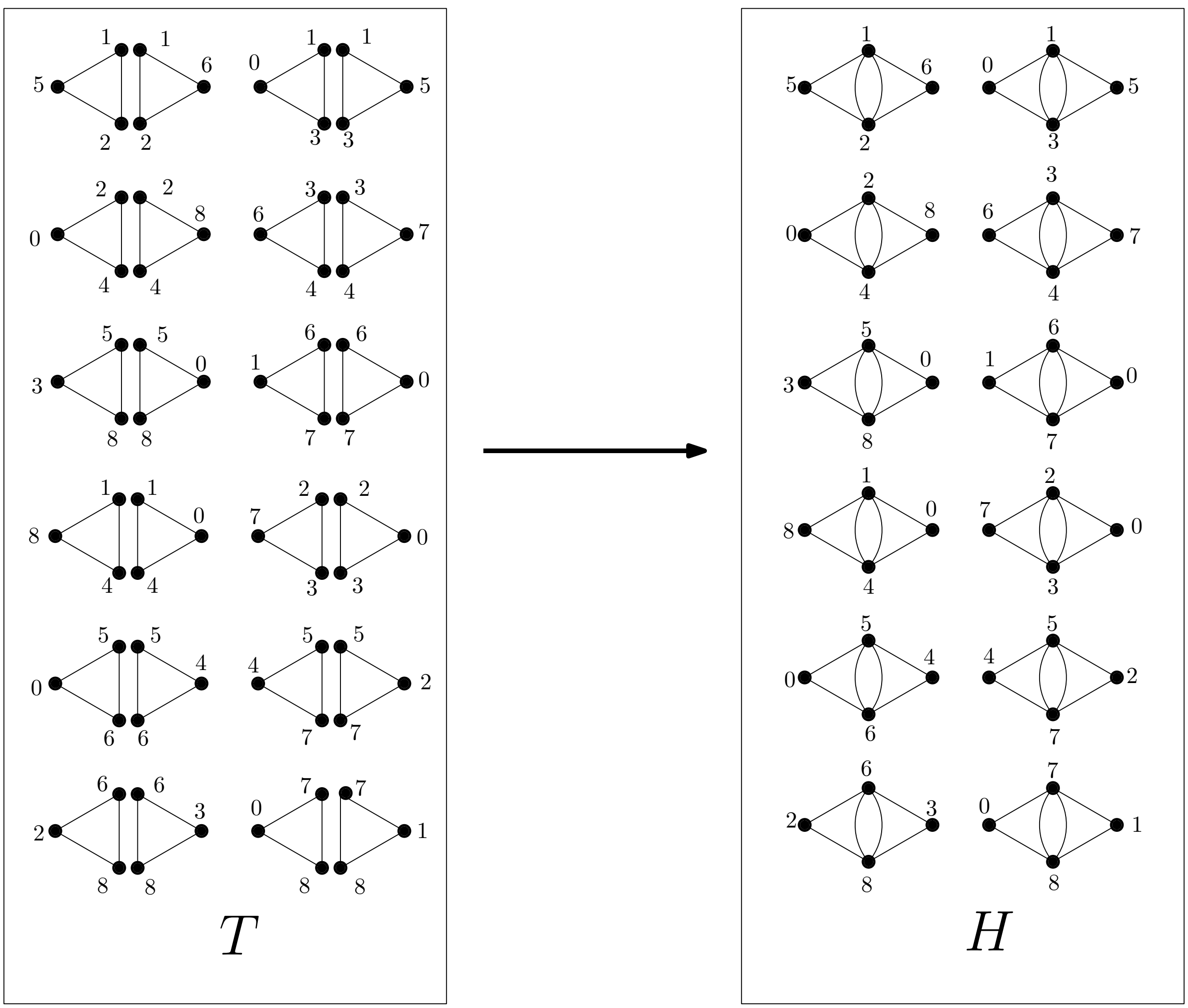
(ii) the collection of double edges, D, can be organized into a collection of 4-cycles D^*?

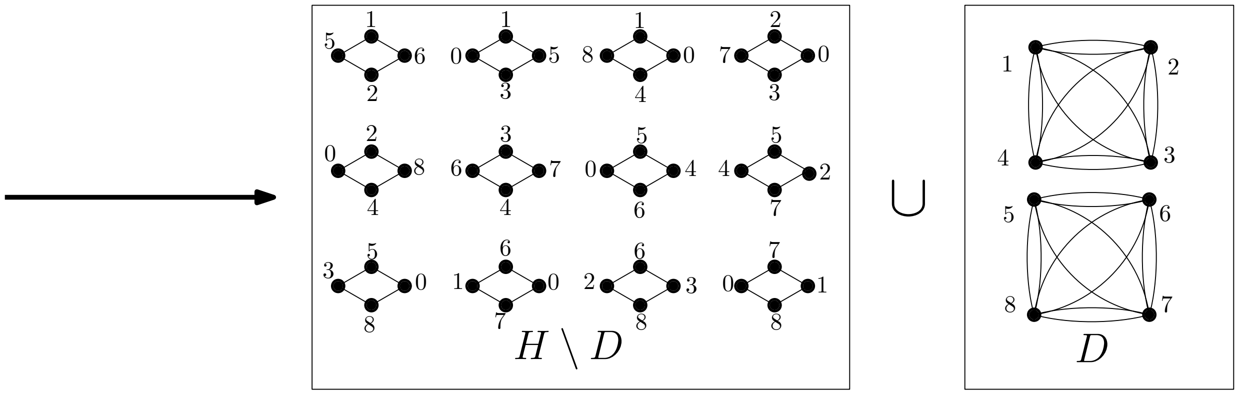
If so, $(X, (H \setminus D) \cup D^*)$ is a 2-fold 4-cycle system called a *metamorphosis* of the 2-fold triple system (X, C).

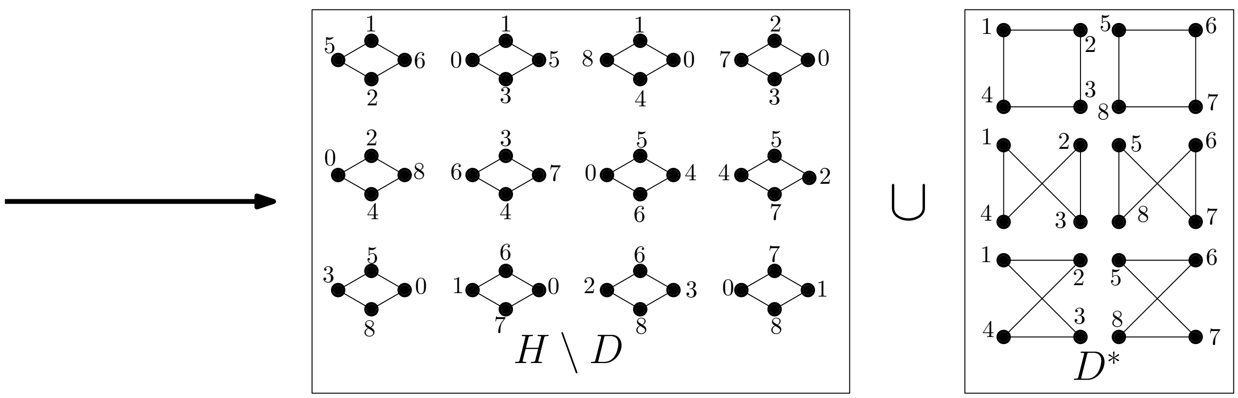




Example 1.3 (metamorphosis of Example 1.2 into a 2-fold 4-cycle system)**.**





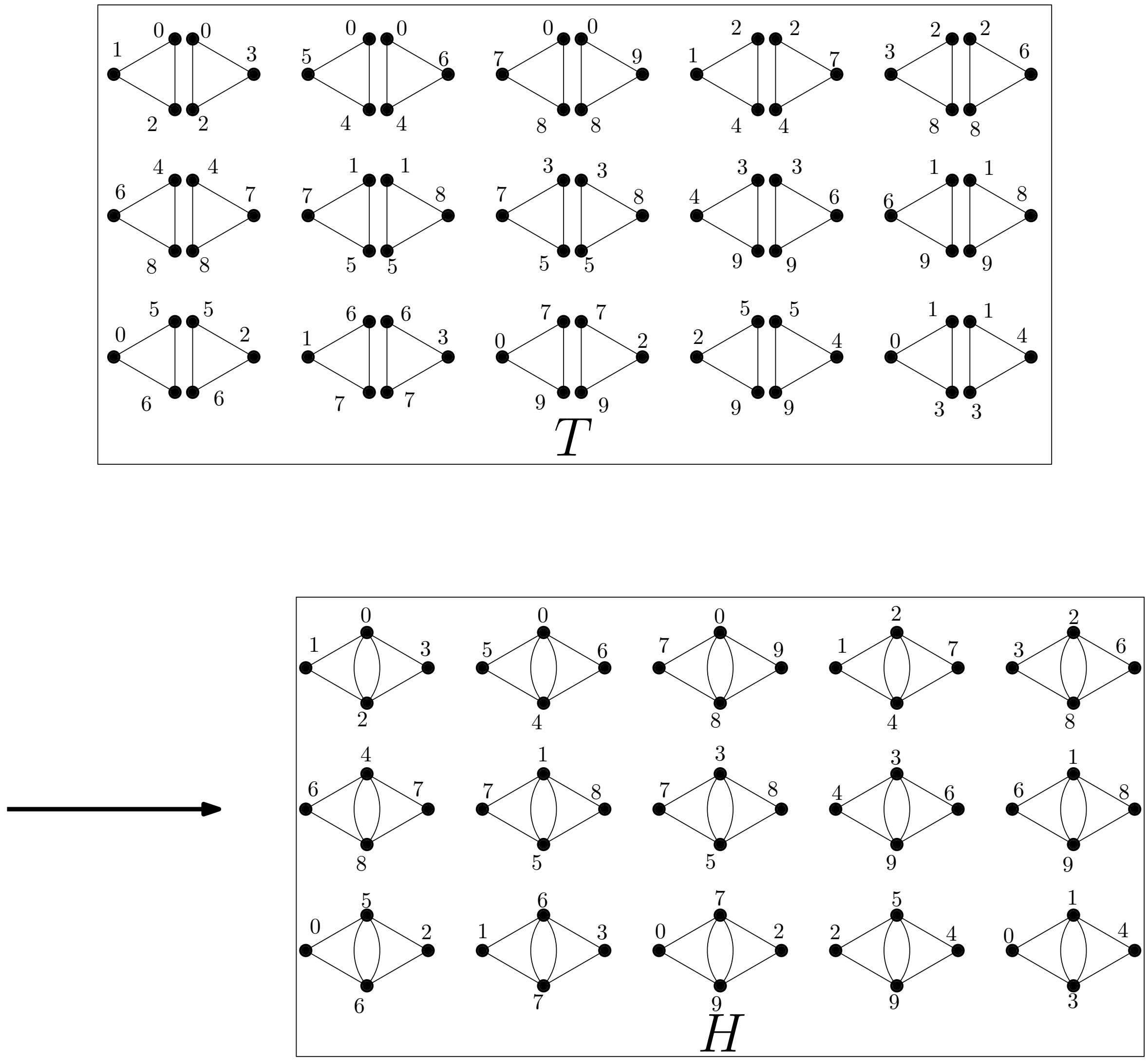


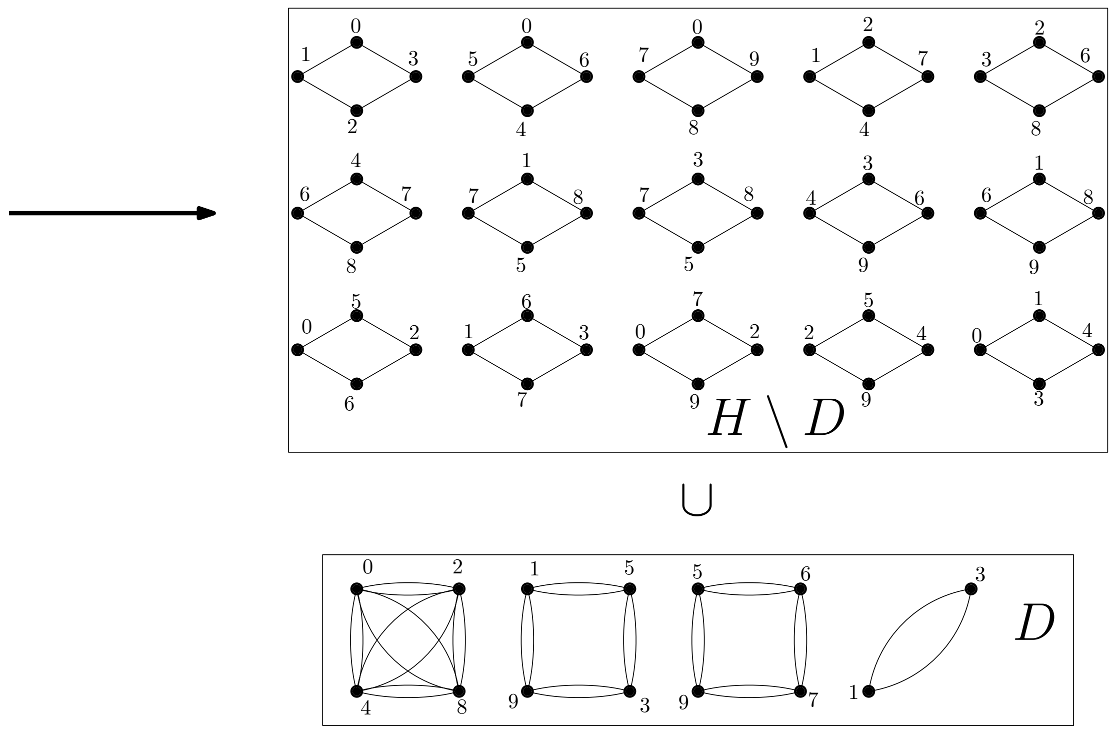
In [5], a complete solution is given of this problem.

Theorem 1.4 (M. Gionfriddo and C.C. Lindner [5])**.** *There exists a 2-fold triple system of every order* $n \equiv 0, 1, 4,$ *or* 9 $(mod\ 12)$ *having a metamorphosis into a 2-fold 4-cycle system.* ■

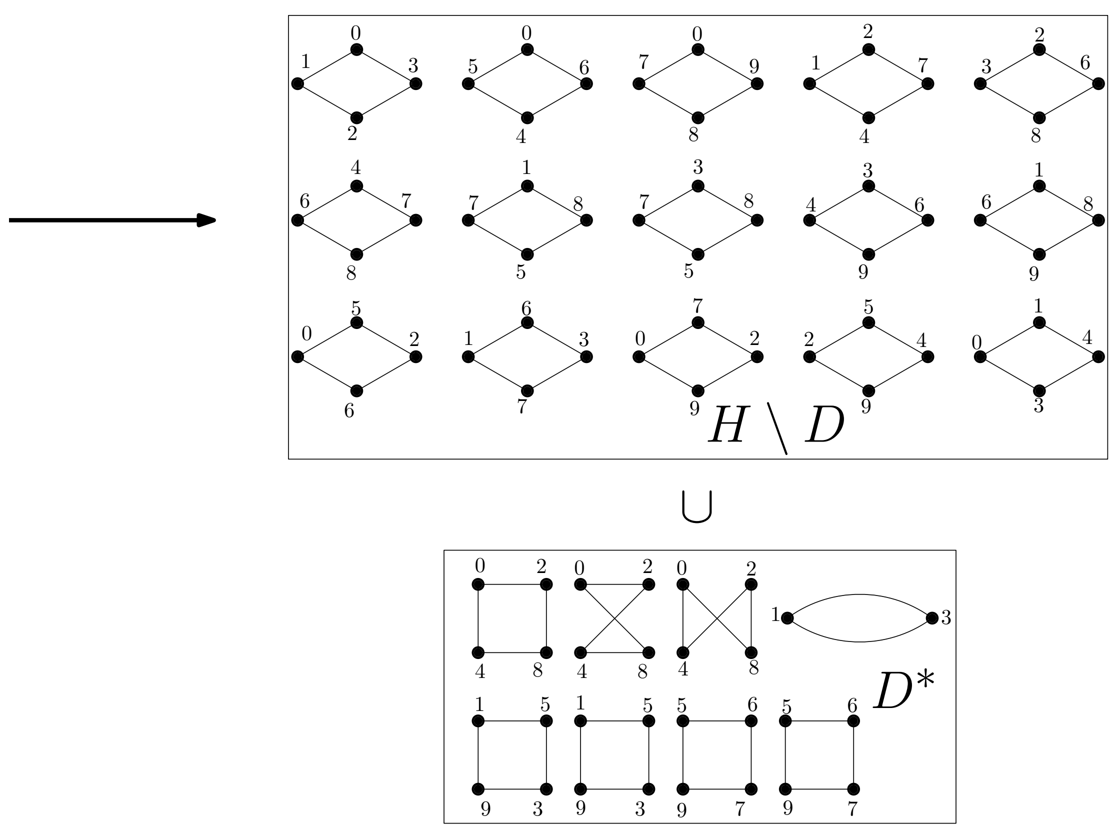
Now, when $n \equiv 3, 6, 7$, or 10 $(mod\ 12)$, a metamorphosis of a 2-fold triple system into a 4-cycle system is impossible, since 2-fold 4-cycle systems of these orders do not exist. However, a maximum packing of $2K_n$ with 4-cycles is possible [1]. In this case, the leave is always a double edge.

Example 1.5 (metamorphosis of a 2-fold triple system of order 10 into a maximum packing of $2K_{10}$ with 4-cycles).





$H \setminus D$
$\cup$
D



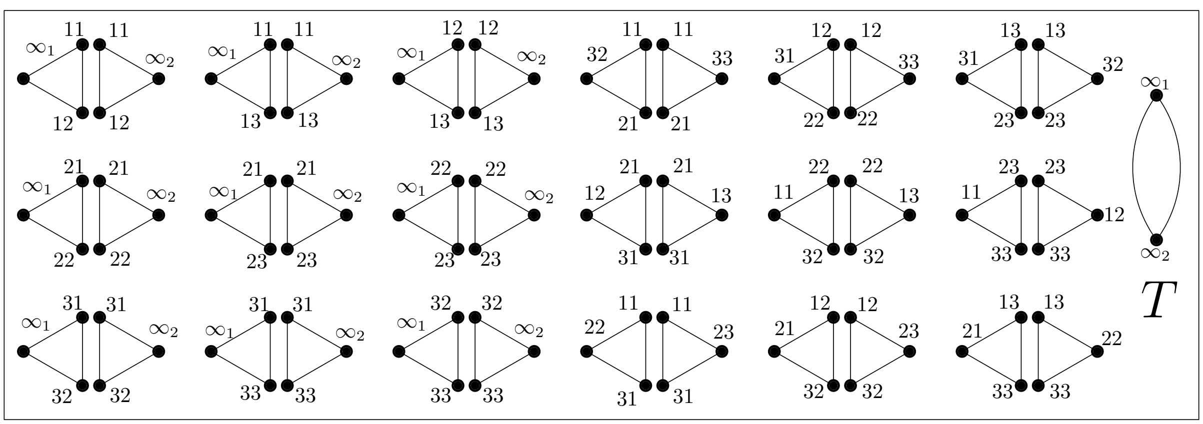
$H \setminus D$
$\cup$
D^*

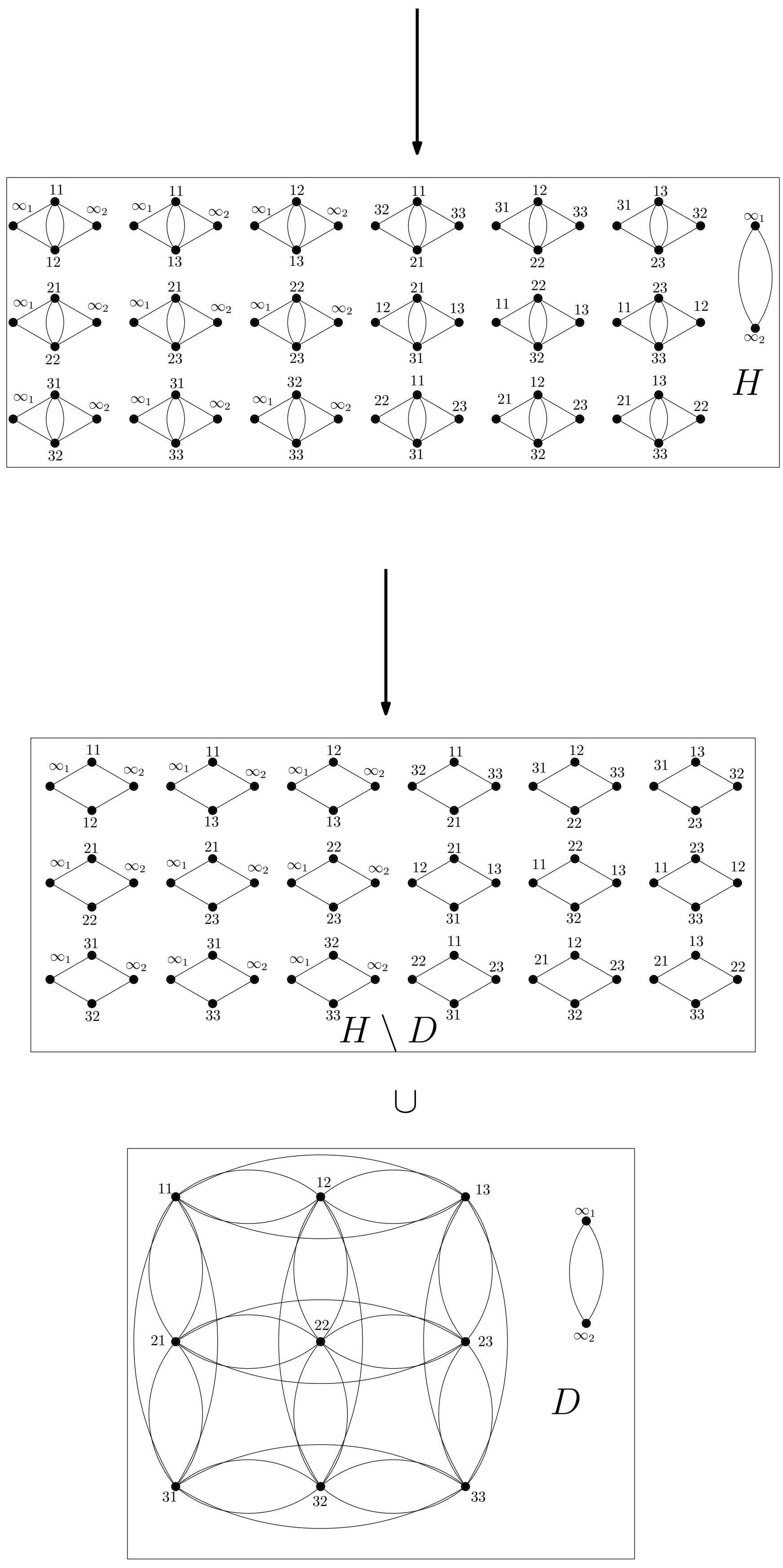
The following theorem is proved in [1].

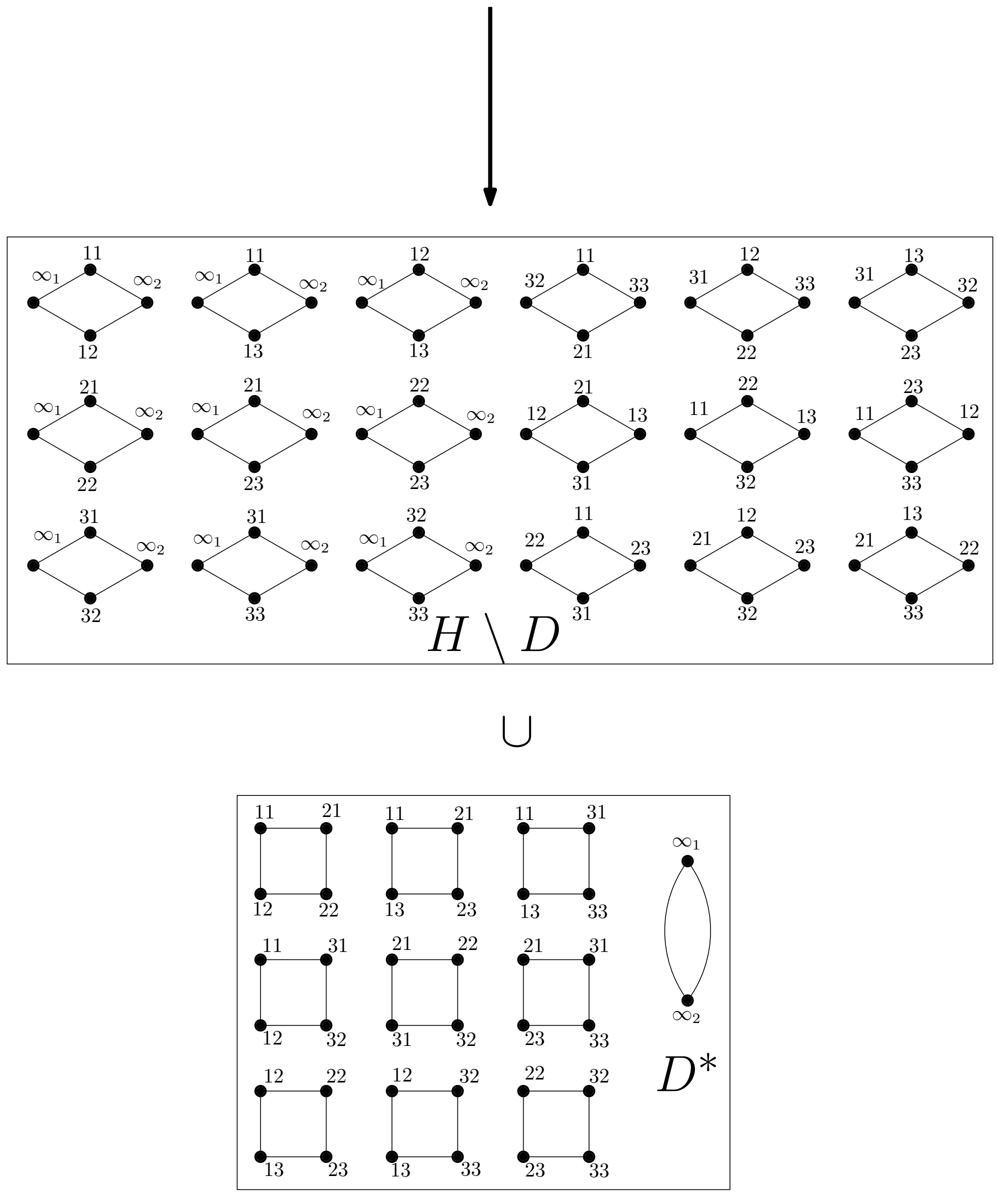
Theorem 1.6 (S.E. McClanahan)**.** *The spectrum for 2-fold triple systems having a metamorphosis into a maximum packing of* $2K_n$ *with 4-cycles is precisely the set of all* $n \equiv 3, 6, 7,$ *or* 10 (*mod* 12) ≥ 10. ■

The object of this thesis is the completion of the metamorphosis trilogy by constructing for every $n \equiv 2, 5, 8,$ or 11 (*mod* 12) a maximum packing of $2K_n$ with triples having a metamorphosis into a maximum packing of $2K_n$ with 4-cycles. A maximum packing of $2K_n$ with triples will always have an even number of triples with leave a double edge. (However, for $n \equiv 5$ or 8 (*mod* 12), a maximum packing of $2K_n$ with 4-cycles is a 2-fold 4-cycle system. In this case, the double edge leave in the maximum packing of $2K_n$ with triples is used in the metamorphosis into 4-cycles.)

Example 1.7 (metamorphosis of a maximum packing of $2K_{11}$ with triples into a maximum packing of $2K_{11}$ with 4-cycles)**.**



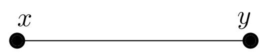
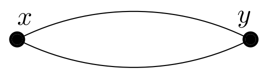
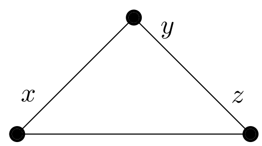
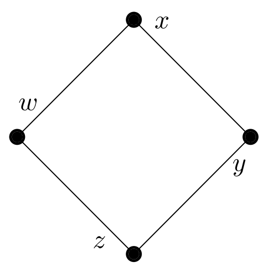
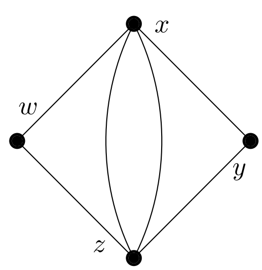

H
H \ D
∪
D



The following theorem gives a complete solution of the problem of constructing maximum packings of $2K_n$ with triples having metamorphoses into maximum packings of $2K_n$ with 4-cycles. As mentioned above, the object of this thesis is the proof of this theorem.

Theorem 1.8. *The spectrum for maximum packings of $2K_n$ with triples having a metamorphosis into a maximum packing of $2K_n$ with 4-cycles is the set of all $n \equiv 2, 5, 8,$ or 11 (mod 12) ≥ 8, except $n = 5$ and $n = 8$.* ■

In what follows, we will use the following notation:

	edge	(x, y) or (y, x)
	double edge	$< x, y >$ or $< y, x >$
	3-cycle or triple	Any cyclic shift of (x, y, z)
	4-cycle	Any cyclic shift of (x, y, z, w)
	hinge	$< x, z, y, w >$, $< x, z, w, y >$, $< z, x, y, w >$, or $< z, x, w, y >$

We will organize our results into 6 chapters: an introduction (this chapter), the $12n+11$ Construction, the $12n+5$ Construction, the $12n+2$ Construction, the $12n+8$ Construction, and a summary.

Chapter 2

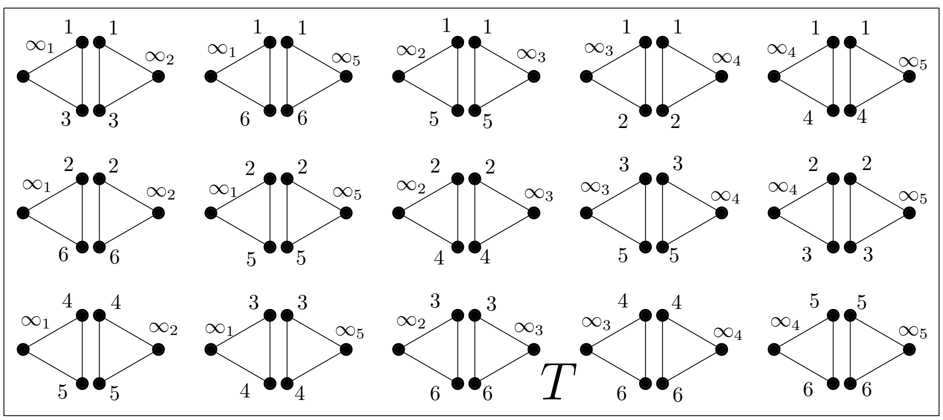
The $12n + 11$ Construction

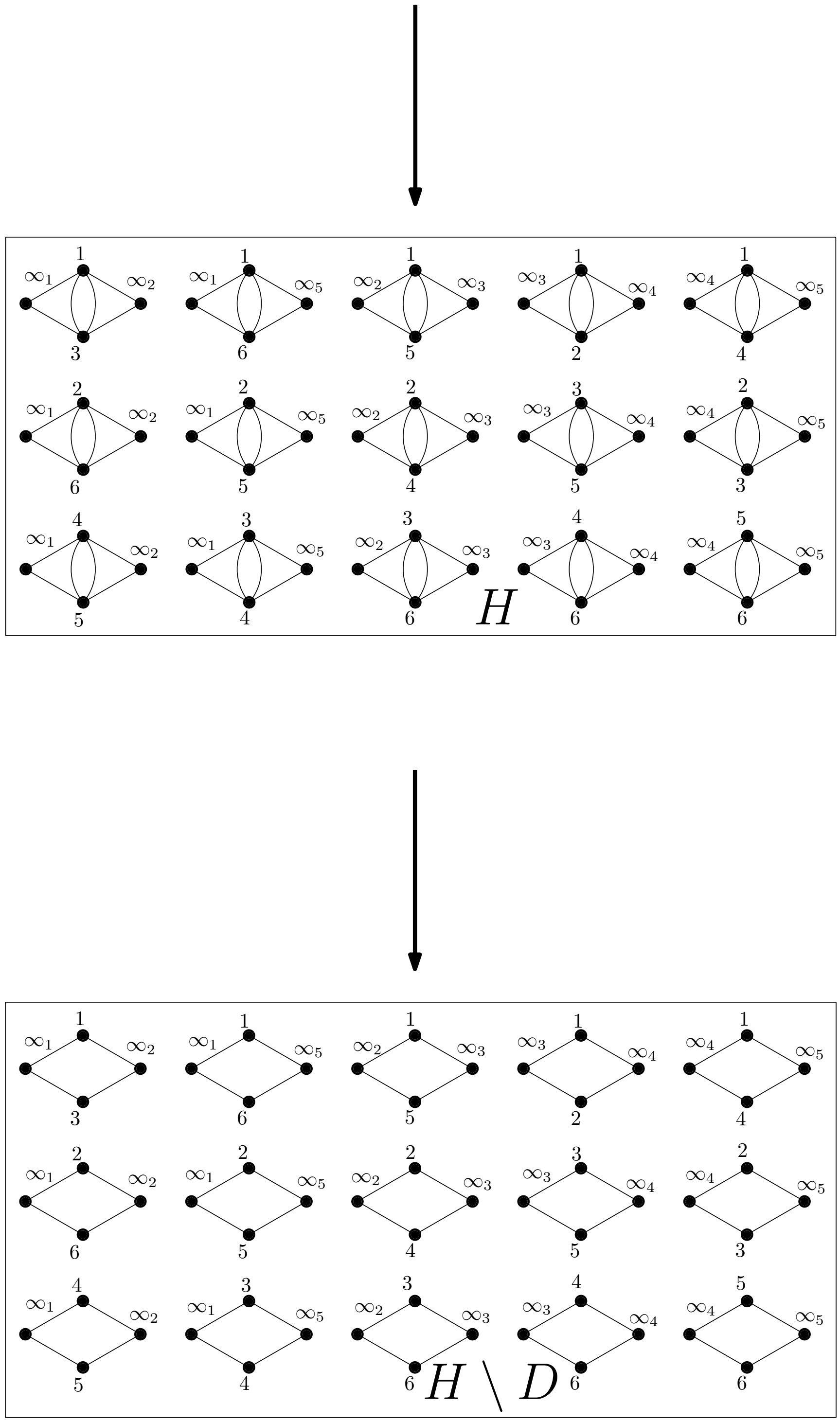
This is the easist construction out of the four basic constructions in this thesis; so, a perfect place to begin.

To begin with, Example 1.7 gives a metamorphosis of a maximum packing of $2K_{11}$ with triples into a maximum packing of $2K_{11}$ with 4-cycles. So, it is only necessary to give a construction for $12n + 11 \geq 23$.

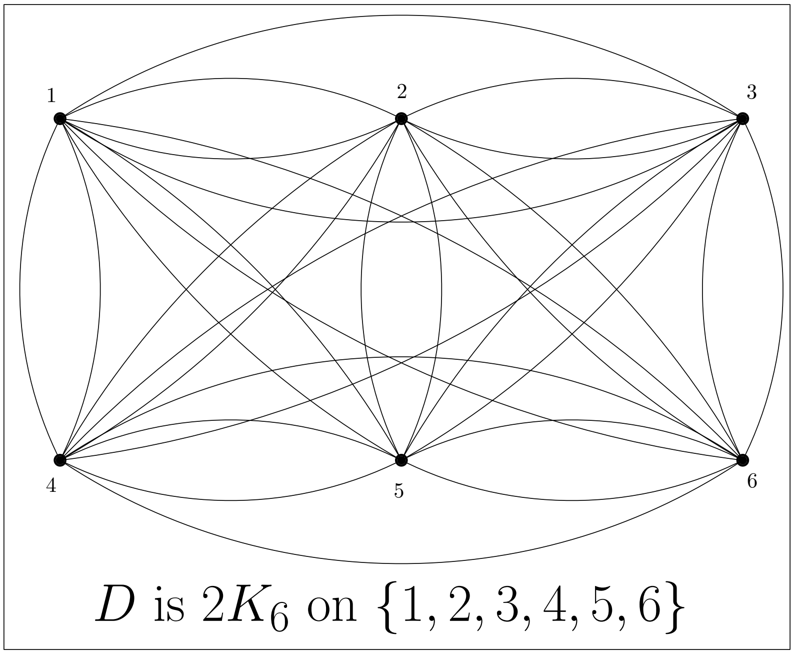
We begin with a necessary example.

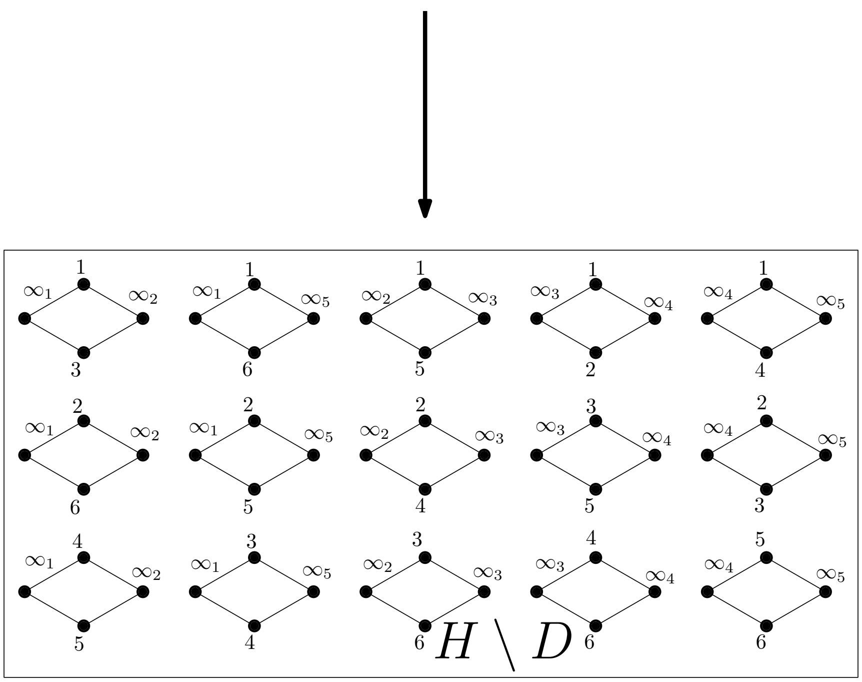
Example 2.1 (metamorphosis of a maximum packing of $2K_{11} \setminus 2K_5$ with triples into a maximum packing of $2K_{11} \setminus 2K_5$ with 4-cycles (the leave is a double edge))**.**



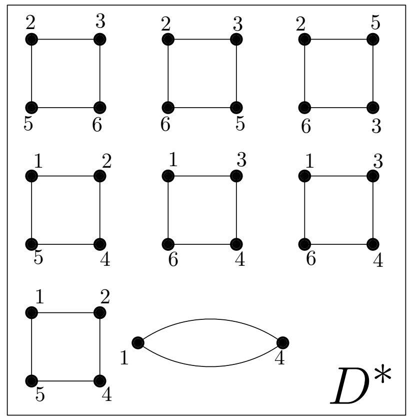

H
$H \setminus D$

$\cup$


1
2
3
4
5
6
D is $2K_6$ on $\{1,2,3,4,5,6\}$


∞_1 ∞_2 ∞_3 ∞_4 ∞_5
$H \setminus D$

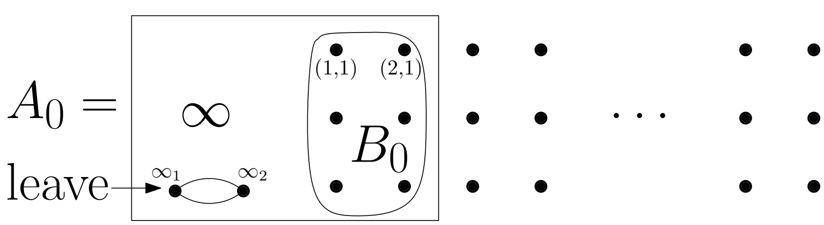
$\cup$



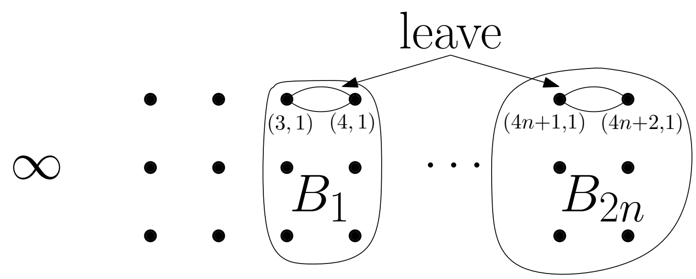
The 12n+11≥23 Construction

With the above example in hand, we can proceed to the $12n + 11 \geq 23$ Construction. Write $12n + 11 = 3(4n + 2) + 5$. Since $12n + 11 \geq 23$, $4n + 2 \geq 6$. This is important! Let $\infty = \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\}$ and $Q = \{1, 2, 3, \ldots, 4n + 2\}$. Let $H(Q) = \{h_0, h_1, \ldots, h_{2n}\}$ be a partition of Q into pairwise disjoint sets of size 2 (called *holes*), where $h_i = \{2i+1, 2i+2\}$. Let $(Q, \circ)$ be a commutative quasigroup of order $4n + 2$ with holes $H(Q)$ (see [6]) and set $X = \infty \cup (Q \times \{1, 2, 3\})$. For $0 \leq i \leq 2n$, let $B_i = h_i \times \{1, 2, 3\}$ and $A_i = \infty \cup B_i$. Define a collection of hinges as follows:

(1) Use the $n = 11$ example (Example 1.7) to find a hinge system (obvious definition) with our desired metamorphosis (A_0, H_0), where the leave is $< \infty_1, \infty_2 >$.



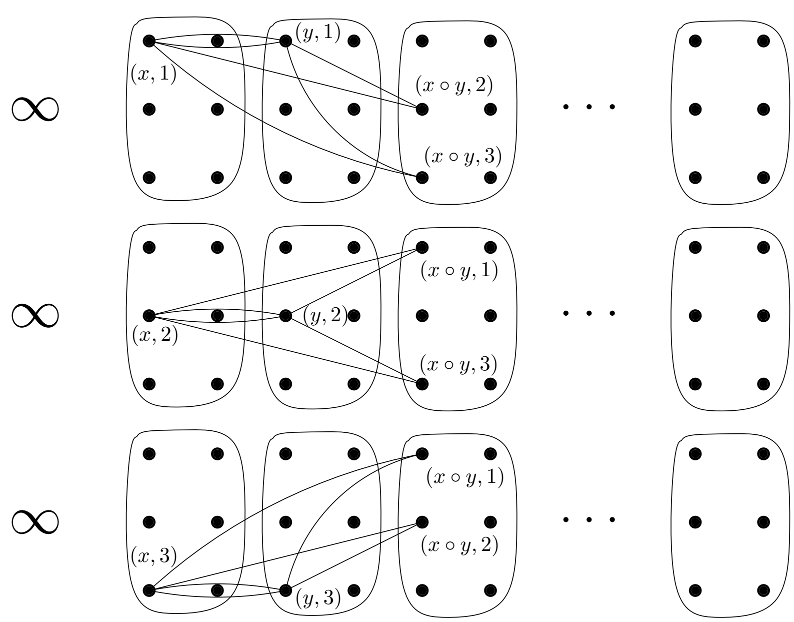
(2) For $1 \leq i \leq 2n$, use the $n = 11$ with a hole of size 5 (Example 2.1) Construction to find a hinge system with our desired metamorphosis (A_i, H_i), where the hole is ∞ and the leave is $< (1, 2i+1), (1, 2i+2) >$.



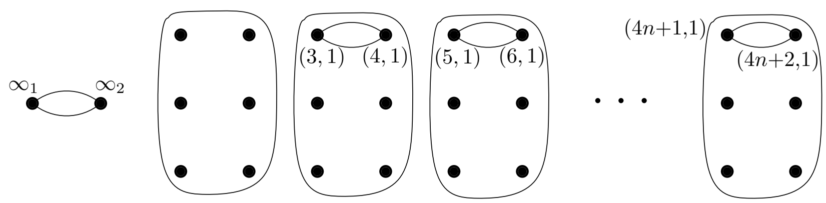
(3) We now need to use the edges between vertices in B_i and B_j for $i \neq j$. Define a collection of hinges $H_\star$ as follows. For $x \in h_i$, $y \in h_j$, $i \neq j$, place the hinges

$< (x, 1), (y, 1), (x \circ y, 2), (x \circ y, 3) >$, $< (x, 2), (y, 2), (x \circ y, 1), (x \circ y, 3) >$,

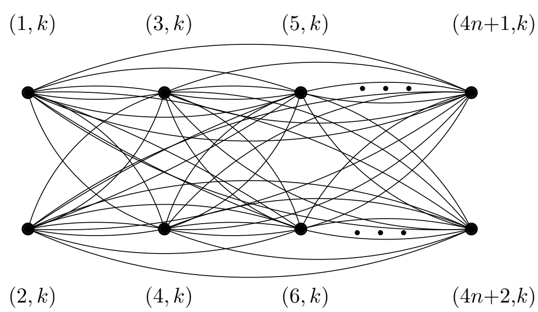
and $< (x, 3), (y, 3), (x \circ y, 1), (x \circ y, 2) >$ in $H_\star$.



It is straightforward to see that $(X = \bigcup_{i=0}^{2n} A_i, H_\star \cup \bigcup_{i=0}^{2n} H_i)$ is a hinge system of order $12n+11$ with leave $< \infty_1, \infty_2 >$. To proceed with our metamorphosis, first use the prescribed metamorphoses in (1) and (2) and place these 4-cycles in C. After removing the double edges from our hinges in $H_\star$ (from (3)), we still have the following edges remaining to use in our metamorphosis: $< \infty_1, \infty_2 >$, edges of the type $< (2i+1, 1), (2i+2, 1) >$, $1 \leq i \leq 2n$, and edges of the type $< (x, k), (y, k) >$, where $k \in \{1, 2, 3\}$ and $x \in h_i$, $y \in h_j$, $i \neq j$.



Edges left from (1) and (2)



Edges left from (3)

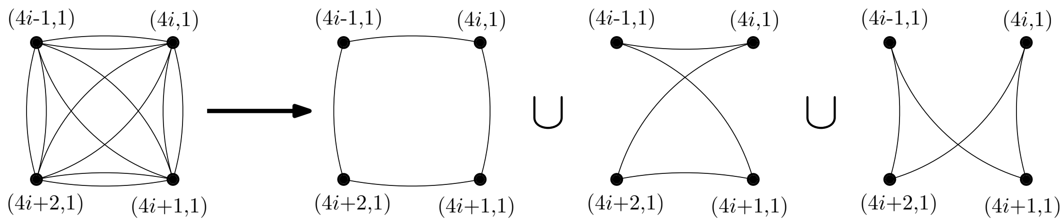
$2K_{2,2,\ldots,2}$ on $\{1,2,\ldots,4n+2\} \times \{k\}$

for $k \in \{1, 2, 3\}$

(4) For $k = 1$, we have $2K_{4n+2}$ on $\{(1,1), (2,1), \ldots, (4n+2, 1)\}$ minus $< (1,1), (2,1) >$.

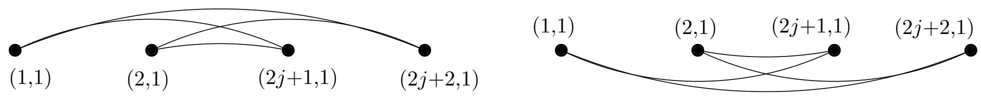
For $1 \leq i \leq n$, on $\{(4i-1, 1), (4i, 1), (4i+1, 1), (4i+2, 1)\}$, we have:

$$\left.\begin{array}{l} \{((4i-1,1),(4i,1),(4i+1,1),(4i+2,1)), \\ ((4i-1,1),(4i+1,1),(4i+2,1),(4i,1)), \\ ((4i-1,1),(4i+1,1),(4i,1),(4i+2,1))\} \end{array}\right. \subseteq C$$

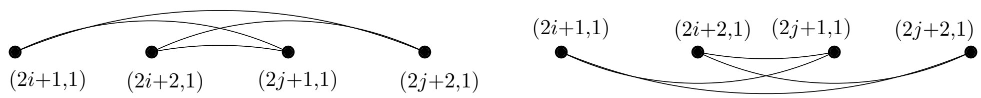


(5) For $0 \leq i < j \leq 2n$, $j-i > 1$ or $i = 0$, on $\{(2i+1,1),(2i+2,1),(2j+1,1),(2j+2,1)\}$, we have:

$$\begin{array}{l}\{((2i+1,1),(2j+2,1),(2i+2,1),(2j+1,1)),\\ ((2i+1,1),(2j+2,1),(2i+2,1),(2j+1,1))\}\end{array} \subseteq C.$$



$i = 0$ and $1 \leq j \leq 2n$



$1 \leq i < j \leq 2n$ and $j - i > 1$

(6) For $k = 2$ and $k = 3$, we have 2 copies of the complete $(2n+1)$-partite graph with all partite sets having size 2. Between each pair of partite sets $\{x,y\}$ and $\{z,w\}$ in these $2K_{2,2,\ldots,2}$, we have $\{(x,w,y,z),(x,w,y,z)\} \subseteq C$.

$\{x,y\} = h_i \times \{k\}$, $\{z,w\} = h_j \times \{k\}$, $i \neq j$, $k \in \{2,3\}$

The result is a maximum packing with 4-cycles of $2K_{12n+11}$ with vertex set X with leave $<\infty_1, \infty_2>$.

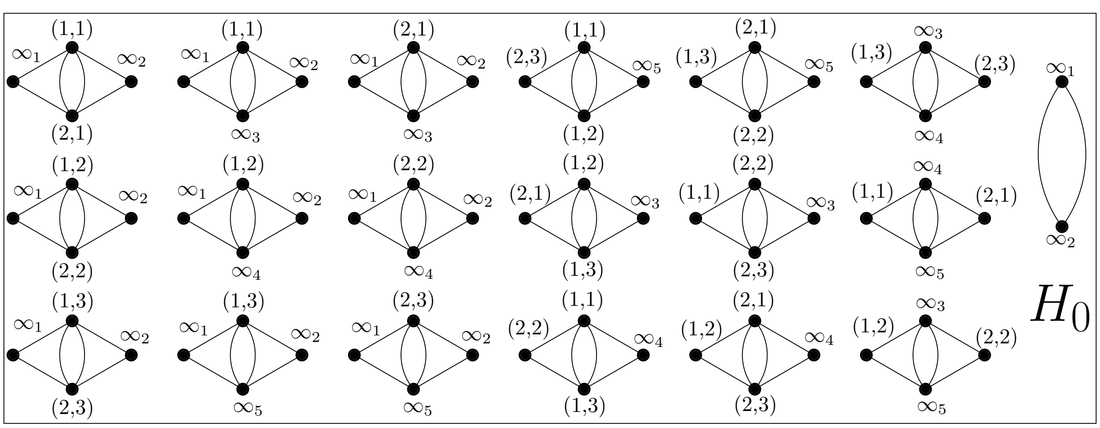
Theorem 2.2. *There exists a maximum packing of $2K_{12n+11}$ with triples having a metamorphosis into a maximum packing of $2K_{12n+11}$ with 4-cycles of every order $12n+11$.* ■

We illustrate the $12n+11$ Construction with the following example.

Example 2.3 (metamorphosis of a maximum packing of $2K_{23}$ with triples into a maximum packing of $2K_{23}$ with 4-cycles)**.**

(1) Rename the vertices of the hinge system in Example 1.7 to get (A_0, H_0) with leave $<\infty_1, \infty_2>$.

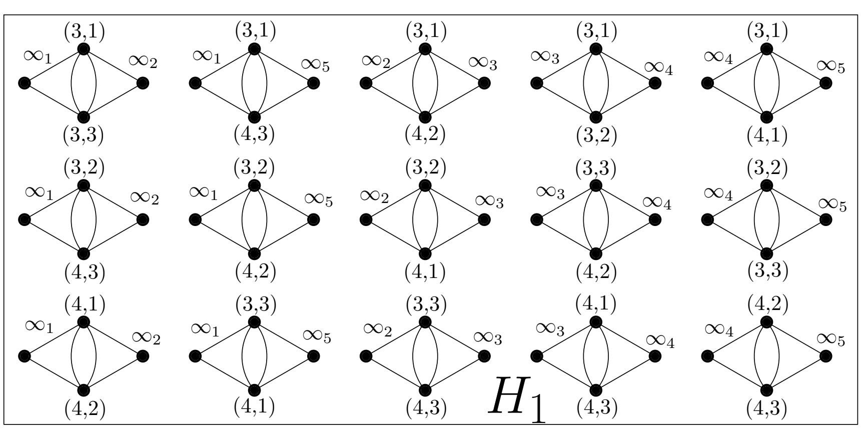
$$H_0 = \begin{array}{ll} \{<(1,1),(2,1),\infty_1,\infty_2>, & <(1,1),\infty_3,\infty_1,\infty_2>, \\ <(2,1),\infty_3,\infty_1,\infty_2>, & <(1,2),(2,2),\infty_1,\infty_2>, \\ <(1,2),\infty_4,\infty_1,\infty_2>, & <(2,2),\infty_4,\infty_1,\infty_2>, \\ <(1,3),(2,3),\infty_1,\infty_2>, & <(1,3),\infty_5,\infty_1,\infty_2>, \\ <(2,3),\infty_5,\infty_1,\infty_2>, & <(1,1),(1,2),(2,3),\infty_5>, \\ <(2,1),(2,2),(1,3),\infty_5>, & <\infty_3,\infty_4,(1,3),(2,3)>, \\ <(1,2),(1,3),(2,1),\infty_3>, & <(2,2),(2,3),(1,1),\infty_3>, \\ <\infty_4,\infty_5,(1,1),(2,1)>, & <(1,1),(1,3),(2,2),\infty_4>, \\ <(2,1),(2,3),(1,2),\infty_4>, & <\infty_3,\infty_5,(1,2),(2,2)>\} \end{array}$$



$$\left.\begin{array}{lll}
\{((1,1),\infty_1,(2,1),\infty_2), & ((1,1),\infty_1,\infty_3,\infty_2), & ((2,1),\infty_1,\infty_3,\infty_2), \\
((1,2),\infty_1,(2,2),\infty_2), & ((1,2),\infty_1,\infty_4,\infty_2), & ((2,2),\infty_1,\infty_4,\infty_2), \\
((1,3),\infty_1,(2,3),\infty_2), & ((1,3),\infty_1,\infty_5,\infty_2), & ((2,3),\infty_1,\infty_5,\infty_2), \\
((1,1),(2,3),(1,2),\infty_5), & ((2,1),(1,3),(2,2),\infty_5), & (\infty_3,(1,3),\infty_4,(2,3)), \\
((1,2),(2,1),(1,3),\infty_3), & ((2,2),(1,1),(2,3),\infty_3), & (\infty_4,(1,1),\infty_5,(2,1)), \\
((1,1),(2,2),(1,3),\infty_4), & ((2,1),(1,2),(2,3),\infty_4), & (\infty_3,(1,2),\infty_5,(2,2)), \\
((1,1),(1,2),(2,2),(2,1)), & ((1,1),(1,2),\infty_4,\infty_3), & ((1,1),(1,3),\infty_5,\infty_3), \\
((1,1),(1,3),(2,3),(2,1)), & ((1,2),(2,2),(2,3),(1,3)), & ((1,2),(1,3),\infty_5,\infty_4), \\
((2,1),(2,2),\infty_4,\infty_3), & ((2,1),(2,3),\infty_5,\infty_3), & ((2,2),(2,3),\infty_5,\infty_4)\}
\end{array}\right. \subseteq C$$

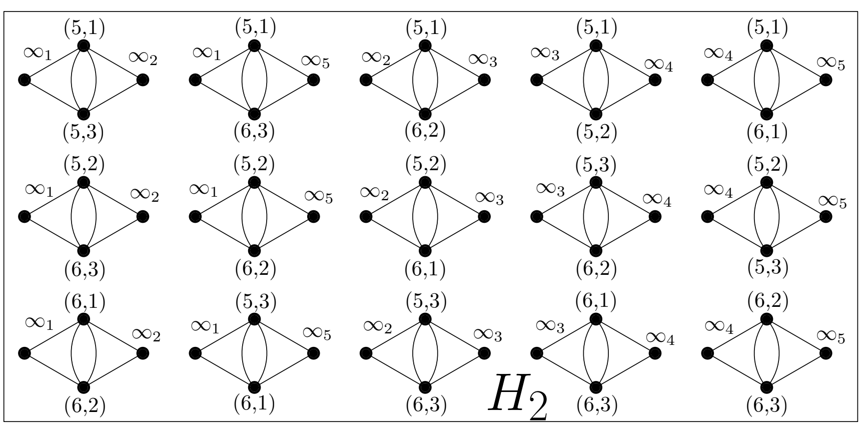
(2) For $1 \le i \le 2$, rename the vertices of the hinge system in Example 2.1 to get (A_i, H_i), where the hole is on ∞ and the leave is $< (2i+1,1),(2i+2,1) >$.

$$H_1 = \begin{array}{ll}
\{< (3,1),(3,3),\infty_1,\infty_2 >, & < (3,2),(4,3),\infty_1,\infty_2 >, \\
< (4,1),(4,2),\infty_1,\infty_2 >, & < (3,1),(4,3),\infty_1,\infty_5 >, \\
< (3,2),(4,2),\infty_1,\infty_5 >, & < (3,3),(4,1),\infty_1,\infty_5 >, \\
< (3,1),(4,2),\infty_2,\infty_3 >, & < (3,2),(4,1),\infty_2,\infty_3 >, \\
< (3,3),(4,3),\infty_2,\infty_3 >, & < (3,1),(3,2),\infty_3,\infty_4 >, \\
< (3,3),(4,2),\infty_3,\infty_4 >, & < (4,1),(4,3),\infty_3,\infty_4 >, \\
< (3,1),(4,1),\infty_4,\infty_5 >, & < (3,2),(3,3),\infty_4,\infty_5 >, \\
< (4,2),(4,3),\infty_4,\infty_5 >\} &
\end{array}$$



$$\begin{array}{lll}
\{(\infty_1,(3,1),\infty_2,(3,3)), & (\infty_1,(3,2),\infty_2,(4,3)), & (\infty_1,(4,1),\infty_2,(4,2)), \\
(\infty_1,(3,1),\infty_5,(4,3)), & (\infty_1,(3,2),\infty_5,(4,2)), & (\infty_1,(3,3),\infty_5,(4,1)), \\
(\infty_2,(3,1),\infty_3,(4,2)), & (\infty_2,(3,2),\infty_3,(4,1)), & (\infty_2,(3,3),\infty_3,(4,3)), \\
(\infty_3,(3,1),\infty_4,(3,2)), & (\infty_3,(3,3),\infty_4,(4,2)), & (\infty_3,(4,1),\infty_4,(4,3)), \\
(\infty_4,(3,1),\infty_5,(4,1)), & (\infty_4,(3,2),\infty_5,(3,3)), & (\infty_4,(4,2),\infty_5,(4,3)), \\
((3,2),(3,3),(4,2),(4,3)), & ((3,2),(4,3),(3,3),(4,2)), & \\
((3,2),(3,3),(4,3),(4,2)), & ((3,1),(3,2),(4,1),(4,2)), & \\
((3,1),(3,2),(4,1),(4,2)), & ((3,1),(3,3),(4,1),(4,3)), & \\
((3,1),(3,3),(4,1),(4,3))\} & \subseteq C &
\end{array}$$

$$H_2 = \begin{array}{ll}
\{< (5,1),(5,3),\infty_1,\infty_2 >, & < (5,2),(6,3),\infty_1,\infty_2 >, \\
< (6,1),(6,2),\infty_1,\infty_2 >, & < (5,1),(6,3),\infty_1,\infty_5 >, \\
< (5,2),(6,2),\infty_1,\infty_5 >, & < (5,3),(6,1),\infty_1,\infty_5 >, \\
< (5,1),(6,2),\infty_2,\infty_3 >, & < (5,2),(6,1),\infty_2,\infty_3 >, \\
< (5,3),(6,3),\infty_2,\infty_3 >, & < (5,1),(5,2),\infty_3,\infty_4 >, \\
< (5,3),(6,2),\infty_3,\infty_4 >, & < (6,1),(6,3),\infty_3,\infty_4 >, \\
< (5,1),(6,1),\infty_4,\infty_5 >, & < (5,2),(5,3),\infty_4,\infty_5 >, \\
< (6,2),(6,3),\infty_4,\infty_5 >\} &
\end{array}$$



$\{(\infty_1,(5,1),\infty_2,(5,3))$, $(\infty_1,(5,2),\infty_2,(6,3))$, $(\infty_1,(6,1),\infty_2,(6,2))$,

$(\infty_1,(5,1),\infty_5,(6,3))$, $(\infty_1,(5,2),\infty_5,(6,2))$, $(\infty_1,(5,3),\infty_5,(6,1))$,

$(\infty_2,(5,1),\infty_3,(6,2))$, $(\infty_2,(5,2),\infty_3,(6,1))$, $(\infty_2,(5,3),\infty_3,(6,3))$,

$(\infty_3,(5,1),\infty_4,(5,2))$, $(\infty_3,(5,3),\infty_4,(6,2))$, $(\infty_3,(6,1),\infty_4,(6,3))$,

$(\infty_4,(5,1),\infty_5,(6,1))$, $(\infty_4,(5,2),\infty_5,(5,3))$, $(\infty_4,(6,2),\infty_5,(6,3))$,

$((5,2),(5,3),(6,2),(6,3))$, $((5,2),(6,3),(5,3),(6,2))$,

$((5,2),(5,3),(6,3),(6,2))$, $((5,1),(5,2),(6,1),(6,2))$,

$((5,1),(5,2),(6,1),(6,2))$, $((5,1),(5,3),(6,1),(6,3))$,

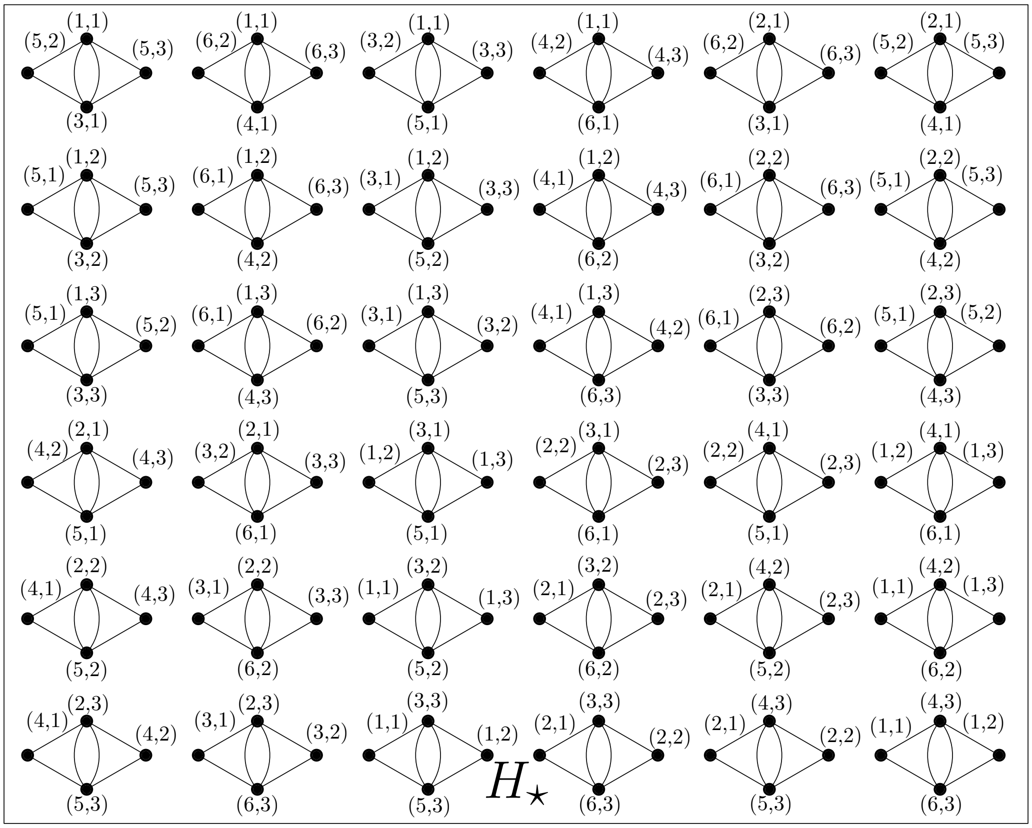
$((5,1),(5,3),(6,1),(6,3))\} \subseteq C$

(3) Let $Q = \{1,2,3,4,5,6\}$. Let $H(Q) = \{h_0, h_1, h_2\}$, where $h_i = \{2i+1, 2i+2\}$ for $0 \le i \le 2$. Define $(Q, \circ)$ to be the following symmetric quasigroup of order 6 with holes $H(Q)$:

$\circ$	1	2	3	4	5	6
1			5	6	3	4
2			6	5	4	3
3	5	6			1	2
4	6	5			2	1
5	3	4	1	2		
6	4	3	2	1		

We now need to use the edges between vertices in B_i and B_j for $i \neq j$. This is done as follows.

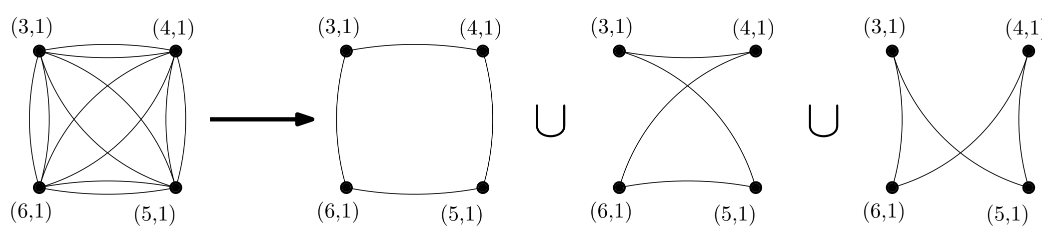
$$
H_\star = \begin{array}{ll}
\{<(1,1),(3,1),(5,2),(5,3)>, & <(1,2),(3,2),(5,1),(5,3)>, \\
<(1,3),(3,3),(5,1),(5,2)>, & <(1,1),(4,1),(6,2),(6,3)>, \\
<(1,2),(4,2),(6,1),(6,3)>, & <(1,3),(4,3),(6,1),(6,2)>, \\
<(1,1),(5,1),(3,2),(3,3)>, & <(1,2),(5,2),(3,1),(3,3)>, \\
<(1,3),(5,3),(3,1),(3,2)>, & <(1,1),(6,1),(4,2),(4,3)>, \\
<(1,2),(6,2),(4,1),(4,3)>, & <(1,3),(6,3),(4,1),(4,2)>, \\
<(2,1),(3,1),(6,2),(6,3)>, & <(2,2),(3,2),(6,1),(6,3)>, \\
<(2,3),(3,3),(6,1),(6,2)>, & <(2,1),(4,1),(5,2),(5,3)>, \\
<(2,2),(4,2),(5,1),(5,3)>, & <(2,3),(4,3),(5,1),(5,2)>, \\
<(2,1),(5,1),(4,2),(4,3)>, & <(2,2),(5,2),(4,1),(4,3)>, \\
<(2,3),(5,3),(4,1),(4,2)>, & <(2,1),(6,1),(3,2),(3,3)>, \\
<(2,2),(6,2),(3,1),(3,3)>, & <(2,3),(6,3),(3,1),(3,2)>, \\
<(3,1),(5,1),(1,2),(1,3)>, & <(3,2),(5,2),(1,1),(1,3)>, \\
<(3,3),(5,3),(1,1),(1,2)>, & <(3,1),(6,1),(2,2),(2,3)>, \\
<(3,2),(6,2),(2,1),(2,3)>, & <(3,3),(6,3),(2,1),(2,2)>, \\
<(4,1),(5,1),(2,2),(2,3)>, & <(4,2),(5,2),(2,1),(2,3)>, \\
<(4,3),(5,3),(2,1),(2,2)>, & <(4,1),(6,1),(1,2),(1,3)>, \\
<(4,2),(6,2),(1,1),(1,3)>, & <(4,3),(6,3),(1,1),(1,2)>\}
\end{array}
$$


$H_\star$

$$\left.\begin{array}{ll}
\{((1,1),(5,2),(3,1),(5,3)), & ((1,2),(5,1),(3,2),(5,3)), \\
((1,3),(5,1),(3,3),(5,2)), & ((1,1),(6,2),(4,1),(6,3)), \\
((1,2),(6,1),(4,2),(6,3)), & ((1,3),(6,1),(4,3),(6,2)), \\
((1,1),(3,2),(5,1),(3,3)), & ((1,2),(3,1),(5,2),(3,3)), \\
((1,3),(3,1),(5,3),(3,2)), & ((1,1),(4,2),(6,1),(4,3)), \\
((1,2),(4,1),(6,2),(4,3)), & ((1,3),(4,1),(6,3),(4,2)), \\
((2,1),(6,2),(3,1),(6,3)), & ((2,2),(6,1),(3,2),(6,3)), \\
((2,3),(6,1),(3,3),(6,2)), & ((2,1),(5,2),(4,1),(5,3)), \\
((2,2),(5,1),(4,2),(5,3)), & ((2,3),(5,1),(4,3),(5,2)), \\
((2,1),(4,2),(5,1),(4,3)), & ((2,2),(4,1),(5,2),(4,3)), \\
((2,3),(4,1),(5,3),(4,2)), & ((2,1),(3,2),(6,1),(3,3)), \\
((2,2),(3,1),(6,2),(3,3)), & ((2,3),(3,1),(6,3),(3,2)), \\
((3,1),(1,2),(5,1),(1,3)), & ((3,2),(1,1),(5,2),(1,3)), \\
((3,3),(1,1),(5,3),(1,2)), & ((3,1),(2,2),(6,1),(2,3)), \\
((3,2),(2,1),(6,2),(2,3)), & ((3,3),(2,1),(6,3),(2,2)), \\
((4,1),(2,2),(5,1),(2,3)), & ((4,2),(2,1),(5,2),(2,3)), \\
((4,3),(2,1),(5,3),(2,2)), & ((4,1),(1,2),(6,1),(1,3)), \\
((4,2),(1,1),(6,2),(1,3)), & ((4,3),(1,1),(6,3),(1,2))\}
\end{array}\right. \subseteq C$$

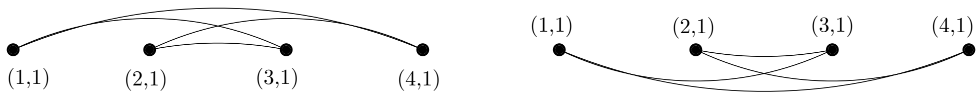
(4) For $k = 1$, we have $2K_6$ on $\{(1,1),(2,1),\ldots,(6,1)\}$ minus $< (1,1),(2,1) >$. Since $n = 1$ on $\{(3,1),(4,1),(5,1),(6,1)\}$ we have:

$$\{((3,1),(4,1),(5,1),(6,1)),((3,1),(5,1),(6,1),(4,1)),((3,1),(5,1),(4,1),(6,1))\} \subseteq C$$

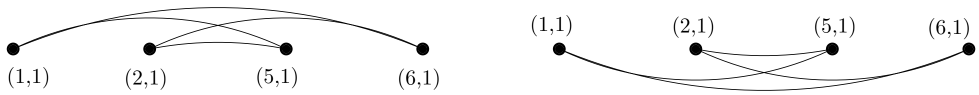


(5) For $1 \le j \le 2$, on $\{(1,1),(2,1),(2j+1,1),(2j+2,1)\}$, we have:

$\{((1,1),(2j+2,1),(2,1),(2j+1,1)),((1,1),(2j+2,1),(2,1),(2j+1,1))\} \subseteq C.$

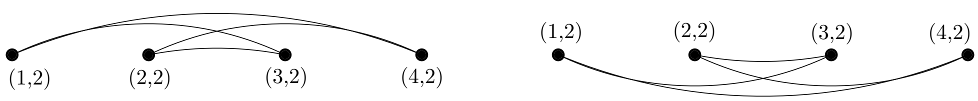


$i = 0$ and $j = 1$

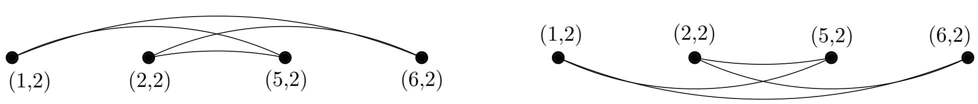


$i = 0$ and $j = 2$

(6) For $k = 2$ and $k = 3$, we have 2 copies of the complete tripartite graph with partite sets $h_0 \times \{k\}$, $h_1 \times \{k\}$, and $h_2 \times \{k\}$. Between each pair of partite sets $\{x, y\}$ and $\{z, w\}$ in these $2K_{2,2,2}$, we have $\{(x, w, y, z), (x, w, y, z)\} \subseteq C$.

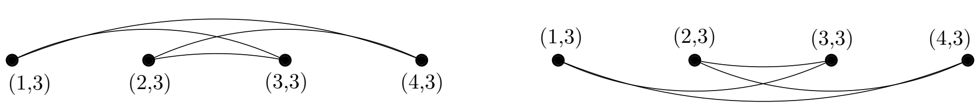


$\{x, y\} = h_0 \times \{2\}$, $\{z, w\} = h_1 \times \{2\}$

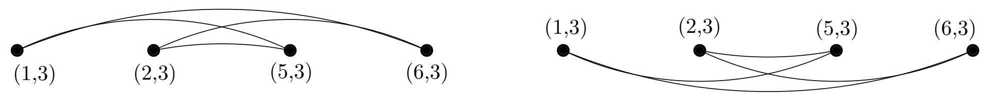


$\{x, y\} = h_0 \times \{2\}$, $\{z, w\} = h_2 \times \{2\}$

$\{x, y\} = h_1 \times \{2\}$, $\{z, w\} = h_2 \times \{2\}$



$\{x, y\} = h_0 \times \{3\}$, $\{z, w\} = h_1 \times \{3\}$



$$\{x, y\} = h_0 \times \{3\},\ \{z, w\} = h_2 \times \{3\}$$

$$\{x, y\} = h_1 \times \{3\},\ \{z, w\} = h_2 \times \{3\}$$

Chapter 3

The $12n + 5$ Construction

There does not exist a maximum packing of $2K_5$ with triples having a metamorphosis into a 2-fold 4-cycle system of order 5. So, we begin this chapter by showing the nonexistence for $n = 5$. If there were to be such a packing (and thus a packing with hinges), then we would have to be able to use the double edges from our hinges (along with the leave) to create two 4-cycles. Thus we would have a repeated 4-cycle. Let x be the vertex that does not appear in this 4-cycle. Our 3 hinges can cover at most 6 edges incident with x, but $d_{2K_5}(x) = 8$.

Before we can proceed to the $12n + 5$ Construction, we must first produce a maximum packing of $2K_{17}$ with triples having a metamorphosis into 2-fold 4-cycle system of order 17.

Example 3.1 (metamorphosis of a maximum packing of $2K_{17}$ with triples into a 2-fold 4-cycle system of order 17)**.** We will be decomposing $2K_{17}$ with vertex set $(\{1,2,3,4\} \times \{1,2,3,4\}) \cup \{\infty\}$. We will begin with the maximum packing with hinges, since it has an obvious correspondence to the maximum packing with triples. Our leave will be
$< (4,4), \infty >$.

$$
H = \begin{array}{ll}
\{< (1,1),(1,2),(1,4),\infty >, & < (2,1),(2,2),(2,4),\infty >, \\
< (3,1),(3,2),(3,4),\infty >, & < (4,1),(4,2),(4,4),\infty >, \\
< (1,3),\infty,(1,1),(1,2) >, & < (2,3),\infty,(2,1),(2,2) >, \\
< (3,3),\infty,(3,1),(3,2) >, & < (4,3),\infty,(4,1),(4,2) >, \\
< (1,4),(2,4),(4,4),\infty >, & < (3,4),\infty,(1,4),(2,4) >, \\
< (1,3),(1,4),(1,1),(1,2) >, & < (2,3),(2,4),(2,1),(2,2) >, \\
< (3,3),(3,4),(3,1),(3,2) >, & < (4,3),(4,4),(4,1),(4,2) >, \\
< (3,4),(4,4),(1,4),(2,4) >, & < (3,3),(4,3),(1,3),(2,3) >, \\
< (3,2),(4,2),(1,2),(2,2) >, & < (3,1),(4,1),(1,1),(2,1) >, \\
< (1,4),(2,3),(3,2),(4,1) >, & < (1,3),(2,4),(3,1),(4,2) >, \\
< (1,1),(2,2),(3,3),(4,4) >, & < (1,2),(2,1),(3,4),(4,3) >, \\
< (1,3),(2,3),(3,3),(4,3) >, & < (1,2),(2,2),(3,2),(4,2) >, \\
< (1,1),(2,1),(3,1),(4,1) >, & < (3,2),(4,1),(1,4),(2,3) >, \\
< (3,1),(4,2),(1,3),(2,4) >, & < (3,3),(4,4),(1,1),(2,2) >, \\
< (3,4),(4,3),(1,2),(2,1) >, & < (1,4),(2,1),(3,3),(4,2) >, \\
< (1,2),(2,3),(3,1),(4,4) >, & < (1,1),(2,4),(3,2),(4,3) >, \\
< (1,3),(2,2),(3,4),(4,1) >, & < (3,3),(4,2),(1,4),(2,1) >, \\
< (3,1),(4,4),(1,2),(2,3) >, & < (3,2),(4,3),(1,1),(2,4) >, \\
< (3,4),(4,1),(1,3),(2,2) >, & < (1,4),(2,2),(3,1),(4,3) >, \\
< (1,1),(2,3),(3,4),(4,2) >, & < (1,3),(2,1),(3,2),(4,4) >, \\
< (1,2),(2,4),(3,3),(4,1) >, & < (3,1),(4,3),(1,4),(2,2) >, \\
< (3,4),(4,2),(1,1),(2,3) >, & < (3,2),(4,4),(1,3),(2,1) >, \\
< (3,3),(4,1),(1,2),(2,4) >\} &
\end{array}
$$

Now, we will remove the double edges from these hinges to create 4-cycles.

$$
\left.
\begin{array}{ll}
\{((1,1),(1,4),(1,2),\infty)\,, & ((2,1),(2,4),(2,2),\infty)\,, \\
((3,1),(3,4),(3,2),\infty)\,, & ((4,1),(4,4),(4,2),\infty)\,, \\
((1,3),(1,1),\infty,(1,2))\,, & ((2,3),(2,1),\infty,(2,2))\,, \\
((3,3),(3,1),\infty,(3,2))\,, & ((4,3),(4,1),\infty,(4,2))\,, \\
((1,4),(4,4),(2,4),\infty)\,, & ((3,4),(1,4),\infty,(2,4))\,, \\
((1,3),(1,1),(1,4),(1,2))\,, & ((2,3),(2,1),(2,4),(2,2))\,, \\
((3,3),(3,1),(3,4),(3,2))\,, & ((4,3),(4,1),(4,4),(4,2))\,, \\
((3,4),(1,4),(4,4),(2,4))\,, & ((3,3),(1,3),(4,3),(2,3))\,, \\
((3,2),(1,2),(4,2),(2,2))\,, & ((3,1),(1,1),(4,1),(2,1))\,, \\
((1,4),(3,2),(2,3),(4,1))\,, & ((1,3),(3,1),(2,4),(4,2))\,, \\
((1,1),(3,3),(2,2),(4,4))\,, & ((1,2),(3,4),(2,1),(4,3))\,, \\
((1,3),(3,3),(2,3),(4,3))\,, & ((1,2),(3,2),(2,2),(4,2))\,, \\
((1,1),(3,1),(2,1),(4,1))\,, & ((3,2),(1,4),(4,1),(2,3))\,, \\
((3,1),(1,3),(4,2),(2,4))\,, & ((3,3),(1,1),(4,4),(2,2))\,, \\
((3,4),(1,2),(4,3),(2,1))\,, & ((1,4),(3,3),(2,1),(4,2))\,, \\
((1,2),(3,1),(2,3),(4,4))\,, & ((1,1),(3,2),(2,4),(4,3))\,, \\
((1,3),(3,4),(2,2),(4,1))\,, & ((3,3),(1,4),(4,2),(2,1))\,, \\
((3,1),(1,2),(4,4),(2,3))\,, & ((3,2),(1,1),(4,3),(2,4))\,, \\
((3,4),(1,3),(4,1),(2,2))\,, & ((1,4),(3,1),(2,2),(4,3))\,, \\
((1,1),(3,4),(2,3),(4,2))\,, & ((1,3),(3,2),(2,1),(4,4))\,, \\
((1,2),(3,3),(2,4),(4,1))\,, & ((3,1),(1,4),(4,3),(2,2))\,, \\
((3,4),(1,1),(4,2),(2,3))\,, & ((3,2),(1,3),(4,4),(2,1))\,, \\
((3,3),(1,2),(4,1),(2,4))\} &
\end{array}
\right. = H \setminus D \subseteq C
$$

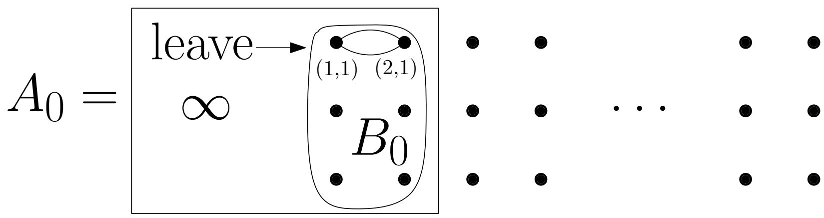
Finally, we will use the double edges (along with the leave) to create the last of our 4-cycles.

$$\left.\begin{array}{ll}
\{(\infty,(4,3),(3,4),(4,4))\,, & ((3,3),\infty,(4,4),(4,3))\,, \\
((3,4),(3,3),(4,3),\infty)\,, & ((4,4),(3,4),\infty,(3,3))\,, \\
((4,3),(4,4),(3,3),(3,4))\,, & (\infty,(1,3),(2,4),(2,3))\,, \\
(\infty,(1,3),(1,4),(2,3))\,, & ((1,3),(1,4),(2,4),(2,3))\,, \\
((1,3),(2,3),(1,4),(2,4))\,, & ((1,1),(2,3),(1,2),(2,4))\,, \\
((1,1),(2,3),(1,2),(2,4))\,, & ((2,1),(1,3),(2,2),(1,4))\,, \\
((2,1),(1,3),(2,2),(1,4))\,, & ((3,1),(4,3),(3,2),(4,4))\,, \\
((3,1),(4,3),(3,2),(4,4))\,, & ((4,1),(3,3),(4,2),(3,4))\,, \\
((4,1),(3,3),(4,2),(3,4))\,, & ((1,1),(2,1),(2,2),(1,2))\,, \\
((1,1),(2,1),(1,2),(2,2))\,, & ((1,1),(1,2),(2,1),(2,2))\,, \\
((3,1),(4,1),(4,2),(3,2))\,, & ((3,1),(4,1),(3,2),(4,2))\,, \\
((3,1),(3,2),(4,1),(4,2))\} &
\end{array}\right. \subseteq C$$

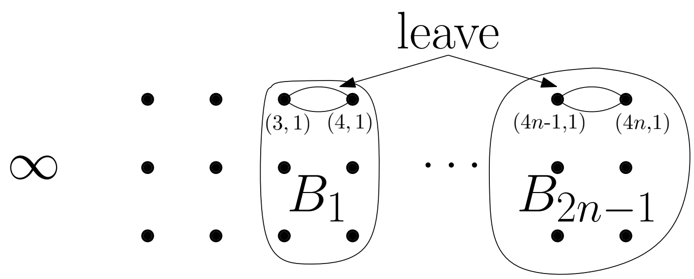
The 12n+5≥29 Construction

We can now proceed to the $12n+5 \geq 29$ Construction, which is simply a modification of the $12n+11 \geq 23$ Construction. Write $12n+5 = 3(4n)+5$. Since $12n+5 \geq 29$, $4n \geq 8$. This is important. Let $\boldsymbol{\infty} = \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\}$ and $Q = \{1, 2, 3, \ldots, 4n\}$. Let $H(Q) = \{h_0, h_1, \ldots, h_{2n-1}\}$ be a partition of Q into pairwise disjoint sets of size 2 (called *holes*), where $h_i = \{2i+1, 2i+2\}$. Let $(Q, \circ)$ be a commutative quasigroup of order $4n$ with holes $H(Q)$ (see [6]) and set $X = \boldsymbol{\infty} \cup (Q \times \{1,2,3\})$. For $0 \leq i \leq 2n-1$, let $B_i = h_i \times \{1,2,3\}$ and $A_i = \boldsymbol{\infty} \cup B_i$. Define a collection of hinges as follows:

(1) Use the Example 1.7 to find a hinge system with our desired metamorphosis (A_0, H_0), where the leave is $< (1,1), (2,1) >$.



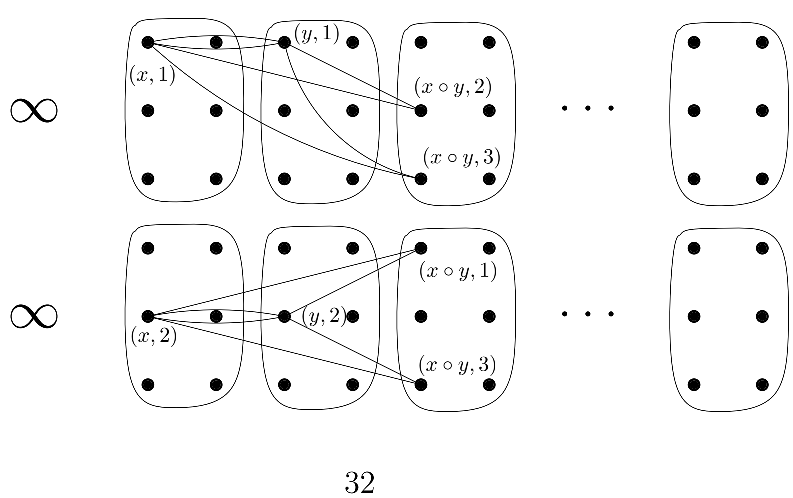
(2) For $1 \leq i \leq 2n-1$, use Example 2.1 to find a hinge system with our desired metamorphosis (A_i, H_i), where the hole is ∞ and the leave is $< (1, 2i+1), (1, 2i+2) >$.

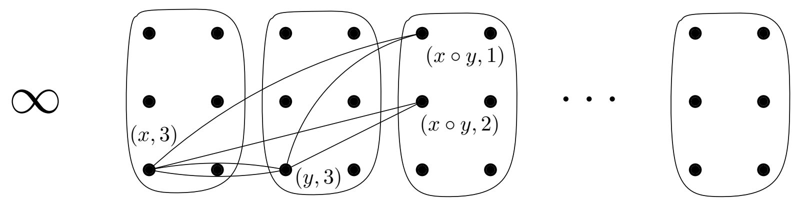


(3) We now need to use the edges between vertices in B_i and B_j for $i \neq j$. Define a collection of hinges $H_\star$ as follows. For $x \in h_i$, $y \in h_j$, $i \neq j$, place the hinges

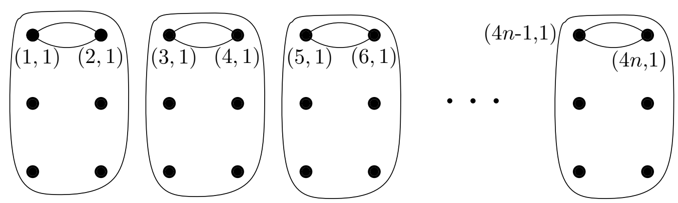
$< (x, 1), (y, 1), (x \circ y, 2), (x \circ y, 3) >$, $\quad < (x, 2), (y, 2), (x \circ y, 1), (x \circ y, 3) >$,

and $< (x, 3), (y, 3), (x \circ y, 1), (x \circ y, 2) >$ in $H_\star$.

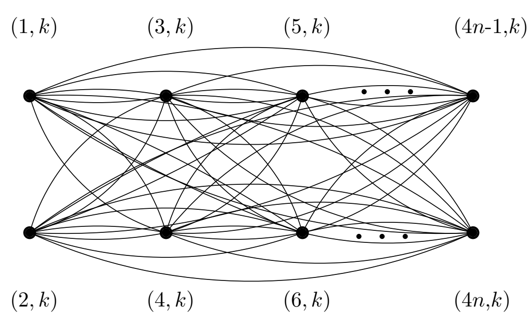




It is straightforward to see that $(X = \bigcup_{i=0}^{2n-1} A_i, H_\star \cup \bigcup_{i=0}^{2n-1} H_i)$ is a hinge system of order $12n+5$ with leave $< \infty_1, \infty_2 >$. To proceed with our metamorphosis, first use the prescribed metamorphoses in (1) and (2) and place these 4-cycles in C. After removing the double edges from our hinges in $H_\star$ (from (3)), we still have the following edges remaining to use in our metamorphosis: edges of type $< (2i+1,1), (2i+2,1) >$, $0 \le i \le 2n-1$, and edges of type $< (x,k), (y,k) >$, where $k \in \{1,2,3\}$ and $x \in h_i$, $y \in h_j$, $i \neq j$.



Edges left from (1) and (2)



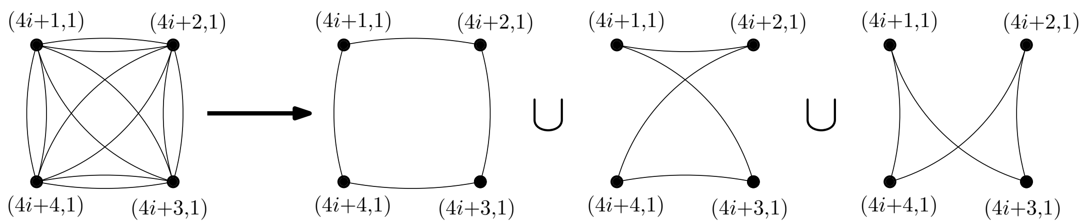
Edges left from (3)

$2K_{2,2,\ldots,2}$ on $\{1,2,\ldots,4n\}\times\{k\}$

for $k \in \{1,2,3\}$

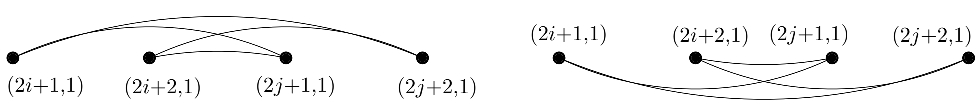
(4) For $k = 1$, we have $2K_{4n}$ on $\{(1,1),(2,1),\ldots,(4n,1)\}$. For $0 \le i \le n-1$, on $\{(4i+1,1),(4i+2,1),(4i+3,1),(4i+4,1)\}$, we have:

$$\left.\begin{array}{l}\{((4i+1,1),(4i+2,1),(4i+3,1),(4i+4,1)),\\ ((4i+1,1),(4i+3,1),(4i+4,1),(4i+2,1)),\\ ((4i+1,1),(4i+3,1),(4i+2,1),(4i+4,1))\}\end{array}\right. \subseteq C$$



(5) For $0 \le i < i+1 < j \le 2n-1$, on $\{(2i+1,1),(2i+2,1),(2j+1,1),(2j+2,1)\}$, we have:

$$\left.\begin{array}{l}\{((2i+1,1),(2j+2,1),(2i+2,1),(2j+1,1)),\\ ((2i+1,1),(2j+2,1),(2i+2,1),(2j+1,1))\}\end{array}\right. \subseteq C.$$



$$0 \le i < i+1 < j \le 2n-1$$

(6) For $k = 2$ and $k = 3$, we have 2 copies of the complete $(2n)$-partite graph with all partite sets having size 2. Between each pair of partite sets $\{x,y\}$ and $\{z,w\}$ in these $2K_{2,2,\ldots,2}$, we have $\{(x,w,y,z),(x,w,y,z)\} \subseteq C$.

$$\{x,y\} = h_i \times \{k\},\ \{z,w\} = h_j \times \{k\},\ i \ne j,\ k \in \{2,3\}$$

The result is a 2-fold 4-cycle system of order $12n+5$ with vertex set X.

Theorem 3.2. *There exists a maximum packing of* $2K_{12n+5}$ *with triples having a metamorphosis into a 2-fold 4-cycle system of order* $12n+5$*, for all* $12n+5 \geq 17$. ■

Chapter 4

The $12n+2$ Construction

A maximum packing of $2K_2$ with triples has a trivial metamorphosis into a maximum packing of $2K_2$ with 4-cycles; you simply have none of either! So, we begin this chapter by producing a maximum packing of $2K_{14}$ with triples having a metamorphosis into a maximum packing of $2K_{14}$ with 4-cycles.

Example 4.1 (metamorphosis of a maximum packing of $2K_{14}$ with triples into a maximum packing of $2K_{14}$ with 4-cycles)**.** We will decompose $2K_{14}$ with vertex set $\{0, 1, \ldots, 11\} \cup \{\infty_1, \infty_2\}$. We will begin with a maximum packing with hinges, since it has an obvious correspondence to the maximum packing with triples. Our leave will be $< \infty_1, \infty_2 >$.

$$H = \begin{array}{lll} \{< 0, \infty_1, 11, 1 >, & < 2, \infty_1, 1, 3 >, & < 4, \infty_1, 3, 5 >, \\ < 6, \infty_1, 5, 7 >, & < 8, \infty_1, 7, 9 >, & < 10, \infty_1, 9, 11 >, \\ < 0, \infty_2, 5, 7 >, & < 2, \infty_2, 7, 9 >, & < 4, \infty_2, 9, 11 >, \\ < 6, \infty_2, 11, 1 >, & < 8, \infty_2, 1, 3 >, & < 10, \infty_2, 3, 5 >, \\ < 0, 3, 11, 10 >, & < 1, 4, 0, 11 >, & < 2, 5, 1, 0 >, \\ < 3, 6, 2, 1 >, & < 4, 7, 3, 2 >, & < 5, 8, 4, 3 >, \\ < 6, 9, 5, 4 >, & < 7, 10, 6, 5 >, & < 8, 11, 7, 6 >, \\ < 9, 0, 8, 7 >, & < 10, 1, 9, 8 >, & < 11, 2, 10, 9 >, \\ < 0, 6, 2, 8 >, & < 1, 7, 3, 9 >, & < 2, 8, 4, 10 >, \\ < 3, 9, 5, 11 >, & < 4, 10, 6, 0 >, & < 5, 11, 7, 1 >\} \end{array}$$

Now, we will remove the double edges from these hinges to create 4-cycles.

$$\left.\begin{array}{lll}\{<0,11,\infty_1,1>, & <2,1,\infty_1,3>, & <4,3,\infty_1,5>, \\ <6,5,\infty_1,7>, & <8,7,\infty_1,9>, & <10,9,\infty_1,11>, \\ <0,5,\infty_2,7>, & <2,7,\infty_2,9>, & <4,9,\infty_2,11>, \\ <6,11,\infty_2,1>, & <8,1,\infty_2,3>, & <10,3,\infty_2,5>, \\ <0,11,3,10>, & <1,0,4,11>, & <2,1,5,0>, \\ <3,2,6,1>, & <4,3,7,2>, & <5,4,8,3>, \\ <6,5,9,4>, & <7,6,10,5>, & <8,7,11,6>, \\ <9,8,0,7>, & <10,9,1,8>, & <11,10,2,9>, \\ <0,2,6,8>, & <1,3,7,9>, & <2,4,8,10>, \\ <3,5,9,11>, & <4,6,10,0>, & <5,7,11,1>\} \end{array}\right. = H \setminus D \subseteq C$$

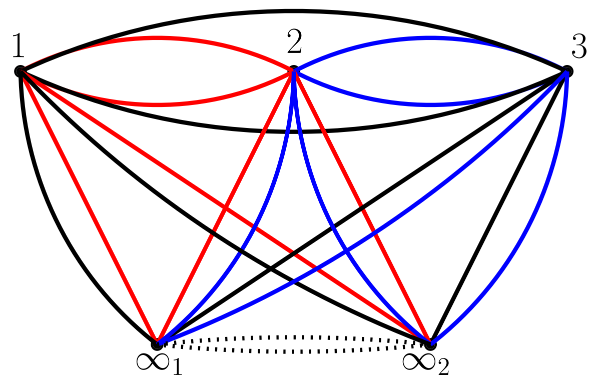
Finally, we will use the double edges (except the leave) to create the last of our 4-cycles.

$$\left.\begin{array}{llll}\{(0,\infty_1,2,\infty_2)\,, & (2,\infty_1,4,\infty_2)\,, & (4,\infty_1,6,\infty_2)\,, & (6,\infty_1,8,\infty_2)\,, \\ (8,\infty_1,10,\infty_2)\,, & (10,\infty_1,0,\infty_2)\,, & (0,3,6,9)\,, & (1,4,7,10)\,, \\ (2,5,8,11)\,, & (0,3,9,6)\,, & (1,4,10,7)\,, & (2,5,11,8)\,, \\ (0,6,3,9)\,, & (1,7,4,10)\,, & (2,8,5,11)\} & \end{array}\right. \subseteq C$$

Before we can begin the $12n+2 \geq 26$ Construction, we need to discuss some necessary ingredients.

Example 4.2 (maximum packing of $2K_5$ with triples)**.** We will decompose $2K_5$ with vertex set $\{1,2,3\} \cup \{\infty_1,\infty_2\}$. We begin with a maximum packing with hinges, since it has an obvious correspondence to the maximum packing with triples. Our leave will be $<\infty_1,\infty_2>$.

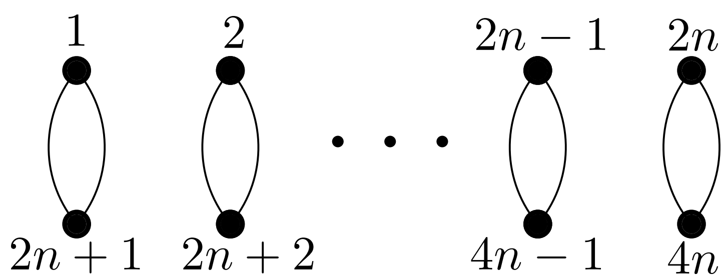
$H = \{<1,2,\infty_1,\infty_2>, <1,3,\infty_1,\infty_2>, <2,3,\infty_1,\infty_2>\}$



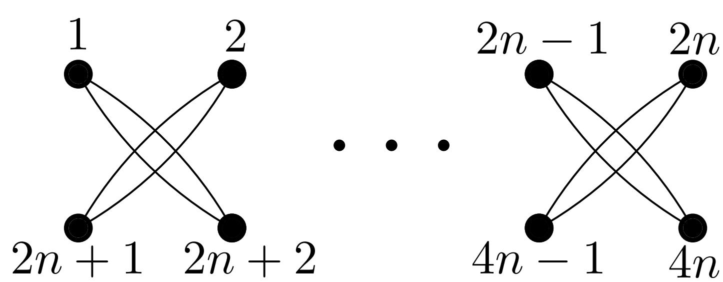
Although the above example does not have a metamorphosis into a 4-cycle system of order 5, it will be good enough. We will also require the following lemma:

Lemma 4.3. *We can always decompose $2K_{4n}$ into two double 1-factors and $(4n^2 - 3n)$ 4-cycles.*

Proof. We will prove this lemma by construction. Let $V(2K_{4n}) = \{1, 2, \ldots, 4n\}$. Our first double 1-factor will have edges of the form $< i, i + 2n >$ for $1 \leq i \leq 2n$.

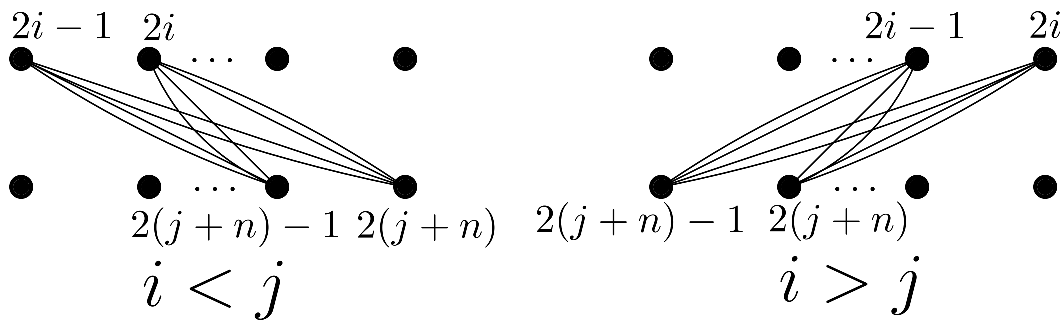


Our second double 1-factor will have edges of the form $< 2i - 1, 2(i + n) >$ and $< 2i, 2(i + n) - 1 >$ for $1 \leq i \leq n$.



For $n \equiv 0$ $(mod\ 2)$:

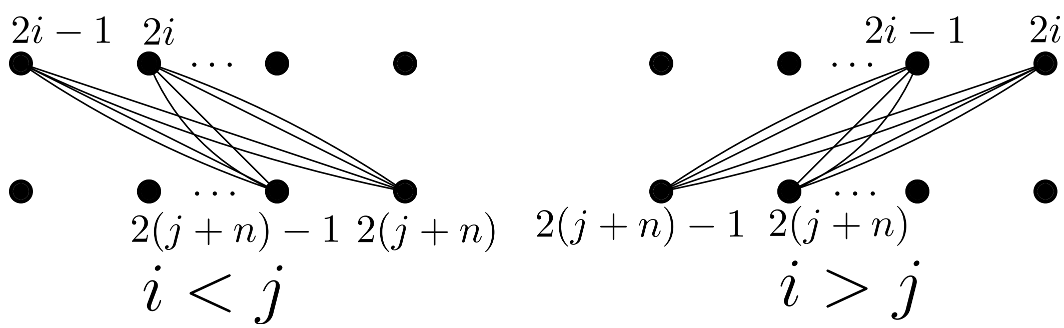
(1) Form two 4-cycles of the type $(2i-1, 2(j+n)-1, 2i, 2(j+n))$ for $1 \leq i \leq n$, $1 \leq j \leq n$, $i \neq j$.



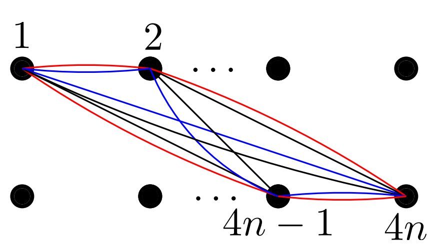
(2) The remaining edges form $2K_{2n} = 2K_{4k}$ on $\{1, 2, \ldots, 2n\}$ and $2K_{4k}$ on $\{2n+1, 2n+2, \ldots, 4n\}$. Decompose these as in the $12n+5$ Construction.

For $n \equiv 1 \ (mod\ 2)$:

(1) Form two 4-cycles of the type $(2i-1, 2(j+n)-1, 2i, 2(j+n))$ for $2 \leq i \leq n$, $1 \leq j \leq n-1$, $i \neq j$.



(2) Form the following 4-cycles: $(1, 2, 4n, 4n-1), (1, 4n, 2, 4n-1), (1, 2, 4n-1, 4n)$.



(3) The remaining edges form $2K_{2n} = 2K_{4k+2}$ on $\{1, 2, \ldots, 2n\}$ minus $< 1, 2 >$ and $2K_{4k+2}$ on $\{2n+1, 2n+2, \ldots, , 4n-1, 4n\}$ minus $< 4n-1, 4n >$. Decompose these as in the $12n + 11$ Construction. ■

The 12n+2≥26 Construction

We can now proceed to the $12n + 2 \geq 26$ Construction. Write $12n + 2 = 3(4n) + 2$. Let $\infty = \{\infty_1, \infty_2\}$ and $Q = \{1, 2, 3, \ldots, 4n\}$. Let $(Q, \circ)$ be an idempotent antisymmetric quasigroup of order $4n$ (that is, $i \circ i = i$ and $i \circ j \neq j \circ i$ for $i \neq j$) and set $X =$ $\infty \cup (Q \times \{1, 2, 3\})$. For $1 \leq i \leq 4n$, let $B_i = h_i \times \{1, 2, 3\}$ and $A_i = \infty \cup B_i$. Define a collection of hinges as follows:

(1) For $1 \leq i \leq 4n$, use the $n = 5$ example (Example 4.2) to find a hinge system (A_i, H_i), where

$H_i = \{< (i,1), (i,2), \infty_1, \infty_2 >, < (i,1), (i,3), \infty_1, \infty_2 >, < (i,2), (i,3), \infty_1, \infty_2 >\}$.

Now, let $\{((i,1), \infty_1, (i,2), \infty_2), ((i,1), \infty_1, (i,3), \infty_2), ((i,2), \infty_1, (i,3), \infty_2)\} \subseteq C$ and notice that we still have $2K_3$ remaining on $\{(i,1), (i,2), (i,3)\}$.

(2) We now need to use edges between distinct B_i. For $x < y$, define

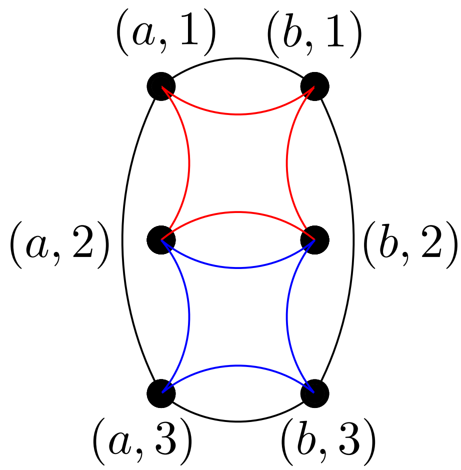
$H_\star = \{< (x,1), (y,1), (x \circ y, 2), (y \circ x, 2) >, < (x,2), (y,2), (x \circ y, 3), (y \circ x, 3) >,$
$< (x,3), (y,3), (x \circ y, 1), (y \circ x, 1) >\}$.

Now, let $\{((x,1), (x \circ y, 2), (y,1), (y \circ x, 2)), ((x,2), (x \circ y, 3), (y,2), (y \circ x, 3)),$
$((x,3), (x \circ y, 1), (y,3), (y \circ x, 1))\} \subseteq C$ and notice that we still have $2K_{4n}$ remaining on $\{1, 2, \ldots, 4n\} \times \{k\}$ for $k = 1, 2, 3$.

Also notice that $(X = \bigcup_{i=1}^{4n} A_i, H_\star \cup \bigcup_{i=1}^{4n} H_i)$ is a hinge system of order $12n + 2$ with leave $< \infty_1, \infty_2 >$.

(3) For $2K_{4n}$ on $\{1, 2, \ldots, 4n\} \times \{k\}$ for $k = 1, 2, 3$, we will form two double 1-factors, F_1 and F_2, and $(4n^2 - 3n)$ 4-cycles, in the manner described in Lemma 4.3.

For each $\{a, b\} \in F_1 \cup F_2$, let $\{((a,1),(b,1),(b,2),(a,2)), ((a,1),(b,1),(b,3),(a,3)), ((a,2),(b,2),(b,3),(a,3))\} \subseteq C$.



The result is a maximum packing with 4-cycles of $2K_{12n+2}$ with vertex set X with leave $< \infty_1, \infty_2 >$.

Theorem 4.4. *There exists a maximum packing of $2K_{12n+2}$ with triples having a metamorphosis into a maximum packing of $2K_{12n+2}$ with 4-cycles of every order $12n+2$.* ■

Chapter 5

The $12n + 8$ Construction

There does not exist a maximum packing of $2K_8$ with triples having a metamorphosis into a 2-fold 4-cycle system of order 8. So, we begin this chapter by showing the nonexistence for $n = 8$. If there were to be such a packing (and thus a packing with hinges), then we would have to be able to use the double edges from our hinges (along with the leave) to create five 4-cycles. This is equivalent to a simple graph G on 8 vertices with 10 edges having a double cover by 4-cycles.

If we have any hope of creating these 4-cycles, G can have no vertices of degree 1. Also, no vertex can have degree more than 4 in G, since $2K_8$ is 14-regular and each edge incident with a vertex v in G accounts for either four edges incident with v in $2K_8$ (if it corresponds to the double edge from a hinge in our packing) or two edges (if it corresponds to the leave). For this reason, G has at most two vertices of degree 4, and if it has two such vertices, then the edge between them must correspond to the leave.

Lemma 5.1. *There can be no repeated 4-cycles in our double cover.*

Proof. Suppose (a, b, c, d) is a repeated 4-cycle. Then none of $\{ab, bc, cd, ad\}$ can be used in another 4-cycle. Thus the remaining three 4-cycles must be a double cover of the remaining six edges, i.e. a double cover of K_4 with 4-cycles. Since K_4 is 3-regular, the vertex set of K_4 does not meet $\{a, b, c, d\}$, since that would force the degree of some vertex to be greater than 4 or for some edge to be used more than twice.

Now, nine of the ten edges in G are associated with hinges in our packing, so we need 36 edges from $2K_{4,4}$ to finish them; however, $2|E(K_{4,4})| = 32$. ■

Corollary 5.2. *Vertices of degree 2 in G cannot be adjacent.*

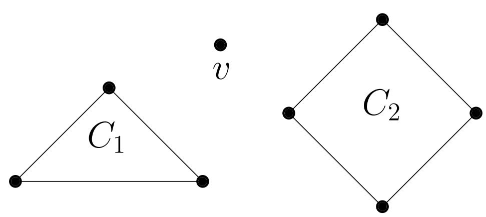
Proof. If there were a pair of adjacent vertices of degree 2 in G, they would have to be in a repeated 4-cycle in our double cover, a contradiction to Lemma 5.1. ■

Lemma 5.3. *G contains no 7-cycle*

Proof. Suppose G contains a 7-cycle, C. Let e, f, and g be the remaining 3 edges. Now, C contains no 4-cycles, so each 4-cycle must contain at least one of these edges. Without loss of generality, assume that e and f are both covered by some 4-cycle, since e, f, and g must each be covered by two 4-cycles and this means that there must be some 4-cycle that covers a pair of these edges. Now, there still must be two 4-cycles that cover g, but that requires a repeated 4-cycle, a contradiction to Lemma 5.1 ■

Corollary 5.4. *G has no vertex of degree 0.*

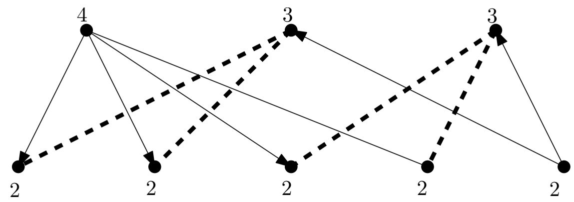
Proof. Suppose $d_G(v) = 0$. Note that this means that there is no other vertex of degree 0. If there were, neither our hinges from our packing nor the leave would cover the edges between v and this vertex. This also means that v appears in exactly 7 of the hinges in our packing to cover all edges incident with v in $2K_8$. Hence, by Lemma 5.3, we must have the following graph as a subgraph of G:



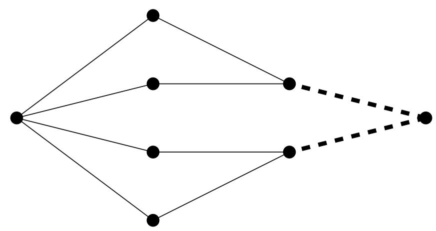
Now, all the edges in C_1 must be double covered by 4-cycles, and a 4-cycle can cover at most a pair of its edges. Thus the remaining 3 edges in G must go from C_1 to C_2 to cover the edges in C_1. However, this means we must repeat C_2 to double cover its edges, a contradiction to Lemma 5.1. ■

Lemma 5.5. *G cannot have $(4, 3, 3, 2, 2, 2, 2, 2)$ as its degree sequence.*

Proof. Suppose G has $(4, 3, 3, 2, 2, 2, 2, 2)$ as its degree sequence. Without loss of generality, since no vertices of degree 2 can be adjacent (Corollary 5.2), G must look like this:



Now, redraw G to look like this:



Note that no 4-cycle can contain the dotted edges. ■

Lemma 5.6. *G cannot have $(3,3,3,3,2,2,2,2)$ as its degree sequence.*

Proof. Suppose G has $(3,3,3,3,2,2,2,2)$ as its degree sequence. Call an edge *pure* if it is incident with two vertices of degree 3 and call it *mixed* otherwise. Note that there are exactly 2 pure edges (see below).

By Corollary 5.2, each 4-cycle in our double cover must use an even number of mixed edges, and thus it must also use an even number of pure edges. Hence, the pure edges must share a vertex, v, and both of these edges must be a part of two (distinct) 4-cycles. Now, note that the remaining edge at v must be mixed, but there is no way to cover it with a 4-cycle. ■

Theorem 5.7. *G does not have a double cover by 4-cycles.*

Proof. G must have one of the following degree sequences, none of which has our desired double cover by 4-cycles:

1. $(4,4,3,3,3,3,0,0)$ (corollary 5.4)

2. $(4,4,3,3,2,2,2,0)$ (corollary 5.4)

3. $(4,4,2,2,2,2,2,2)$ (corollary 5.2)

4. $(4,3,3,3,3,2,2,0)$ (corollary 5.4)

5. $(4,3,3,2,2,2,2,2)$ (lemma 5.5)

6. $(3,3,3,3,3,3,2,0)$ (corollary 5.4)

7. $(3,3,3,3,2,2,2,2)$ (lemma 5.6) ■

This leads us to the following conclusion:

Corollary 5.8. *There does not exist a maximum packing of $2K_8$ with triples having a metamorphosis into a 2-fold 4-cycle system of order 8.* ■

Before we can begin the $12n+8$ Construction, we must consider the case where $12n+8 = 32$.

Example 5.9 (metamorphosis of a maximum packing of $2K_{32}$ with triples into 2-fold 4-cycle system of order 32)**.**

We will be decomposing $2K_{32}$ with vertex set $\{1,2,\ldots,10\} \times \{1,2,3\} \cup \{\infty_1,\infty_2\}$. We will begin with the maximum packing with hinges, since it has an obvious correspondence to the maximum packing with triples. Our leave will be $<\infty_1,\infty_2>$.

$$
\left.
\begin{array}{ll}
\{< (10,1),(3,1),(1,1),(2,1) >, & < (10,1),(4,1),(5,1),(6,1) >, \\
< (10,1),(8,1),(7,1),(9,1) >, & < (3,1),(4,1),(1,1),(7,1) >, \\
< (3,1),(8,1),(2,1),(6,1) >, & < (4,1),(8,1),(6,1),(7,1) >, \\
< (1,1),(5,1),(7,1),(8,1) >, & < (2,1),(5,1),(7,1),(8,1) >, \\
< (2,1),(9,1),(4,1),(6,1) >, & < (1,1),(9,1),(6,1),(8,1) >, \\
< (5,1),(6,1),(10,1),(3,1) >, & < (6,1),(7,1),(1,1),(2,1) >, \\
< (7,1),(9,1),(10,1),(3,1) >, & < (5,1),(9,1),(3,1),(4,1) >, \\
< (1,1),(2,1),(10,1),(4,1) >, & < (10,2),(3,2),(1,2),(2,2) >, \\
< (10,2),(4,2),(5,2),(6,2) >, & < (10,2),(8,2),(7,2),(9,2) >, \\
< (3,2),(4,2),(1,2),(7,2) >, & < (3,2),(8,2),(2,2),(6,2) >, \\
< (4,2),(8,2),(6,2),(7,2) >, & < (1,2),(5,2),(7,2),(8,2) >, \\
< (2,2),(5,2),(7,2),(8,2) >, & < (2,2),(9,2),(4,2),(6,2) >, \\
< (1,2),(9,2),(6,2),(8,2) >, & < (5,2),(6,2),(10,2),(3,2) >, \\
< (6,2),(7,2),(1,2),(2,2) >, & < (7,2),(9,2),(10,2),(3,2) >, \\
< (5,2),(9,2),(3,2),(4,2) >, & < (1,2),(2,2),(10,2),(4,2) >, \\
< (10,3),(3,3),(1,3),(2,3) >, & < (10,3),(4,3),(5,3),(6,3) >, \\
< (10,3),(8,3),(7,3),(9,3) >, & < (3,3),(4,3),(1,3),(7,3) >, \\
< (3,3),(8,3),(2,3),(6,3) >, & < (4,3),(8,3),(6,3),(7,3) >, \\
< (1,3),(5,3),(7,3),(8,3) >, & < (2,3),(5,3),(7,3),(8,3) >, \\
< (2,3),(9,3),(4,3),(6,3) >, & < (1,3),(9,3),(6,3),(8,3) >, \\
< (5,3),(6,3),(10,3),(3,3) >, & < (6,3),(7,3),(1,3),(2,3) >, \\
< (7,3),(9,3),(10,3),(3,3) >, & < (5,3),(9,3),(3,3),(4,3) >, \\
< (1,3),(2,3),(10,3),(4,3) >\} &
\end{array}
\right. \quad \subseteq H
$$

$$
\left.
\begin{array}{ll}
\{< \infty_1, (1,1), (1,2), (2,2) >, & < \infty_1, (2,1), (1,3), (2,3) >, \\
< \infty_2, (1,1), (1,3), (2,3) >, & < \infty_2, (2,1), (1,2), (2,2) >, \\
< (1,2), (1,3), \infty_1, \infty_2 >, & < (1,2), (2,3), (1,1), (2,1) >, \\
< (2,2), (1,3), (1,1), (2,1) >, & < (2,2), (2,3), \infty_1, \infty_2 >, \\
< \infty_1, (3,1), (3,2), (4,2) >, & < \infty_1, (4,1), (3,3), (4,3) >, \\
< \infty_2, (3,1), (3,3), (4,3) >, & < \infty_2, (4,1), (3,2), (4,2) >, \\
< (3,2), (3,3), \infty_1, \infty_2 >, & < (3,2), (4,3), (3,1), (4,1) >, \\
< (4,2), (3,3), (3,1), (4,1) >, & < (4,2), (4,3), \infty_1, \infty_2 >, \\
< \infty_1, (5,1), (5,2), (6,2) >, & < \infty_1, (6,1), (5,3), (6,3) >, \\
< \infty_2, (5,1), (5,3), (6,3) >, & < \infty_2, (6,1), (5,2), (6,2) >, \\
< (5,2), (5,3), \infty_1, \infty_2 >, & < (5,2), (6,3), (5,1), (6,1) >, \\
< (6,2), (5,3), (5,1), (6,1) >, & < (6,2), (6,3), \infty_1, \infty_2 >, \\
< \infty_1, (7,1), (7,2), (8,2) >, & < \infty_1, (8,1), (7,3), (8,3) >, \\
< \infty_2, (7,1), (7,3), (8,3) >, & < \infty_2, (8,1), (7,2), (8,2) >, \\
< (7,2), (7,3), \infty_1, \infty_2 >, & < (7,2), (8,3), (7,1), (8,1) >, \\
< (8,2), (7,3), (7,1), (8,1) >, & < (8,2), (8,3), \infty_1, \infty_2 >, \\
< \infty_1, (9,1), (9,2), (10,2) >, & < \infty_1, (10,1), (9,3), (10,3) >, \\
< \infty_2, (9,1), (9,3), (10,3) >, & < \infty_2, (10,1), (9,2), (10,2) >, \\
< (9,2), (9,3), \infty_1, \infty_2 >, & < (9,2), (10,3), (9,1), (10,1) >, \\
< (10,2), (9,3), (9,1), (10,1) >, & < (10,2), (10,3), \infty_1, \infty_2 >\}
\end{array}
\right. \subseteq H
$$

$$
\left\{
\begin{array}{ll}
<(1,1),(3,2),(5,3),(6,3)>, & <(1,1),(4,2),(6,3),(5,3)>, \\
<(2,1),(3,2),(6,3),(5,3)>, & <(2,1),(4,2),(5,3),(6,3)>, \\
<(1,2),(3,1),(5,3),(6,3)>, & <(1,2),(4,1),(6,3),(5,3)>, \\
<(2,2),(3,1),(6,3),(5,3)>, & <(2,2),(4,1),(5,3),(6,3)>, \\
<(1,1),(5,2),(7,3),(8,3)>, & <(1,1),(6,2),(8,3),(7,3)>, \\
<(2,1),(5,2),(8,3),(7,3)>, & <(2,1),(6,2),(7,3),(8,3)>, \\
<(1,2),(5,1),(7,3),(8,3)>, & <(1,2),(6,1),(8,3),(7,3)>, \\
<(2,2),(5,1),(8,3),(7,3)>, & <(2,2),(6,1),(7,3),(8,3)>, \\
<(1,1),(7,2),(9,3),(10,3)>, & <(1,1),(8,2),(10,3),(9,3)>, \\
<(2,1),(7,2),(10,3),(9,3)>, & <(2,1),(8,2),(9,3),(10,3)>, \\
<(1,2),(7,1),(9,3),(10,3)>, & <(1,2),(8,1),(10,3),(9,3)>, \\
<(2,2),(7,1),(10,3),(9,3)>, & <(2,2),(8,1),(9,3),(10,3)>, \\
<(1,1),(9,2),(3,3),(4,3)>, & <(1,1),(10,2),(4,3),(3,3)>, \\
<(2,1),(9,2),(4,3),(3,3)>, & <(2,1),(10,2),(3,3),(4,3)>, \\
<(1,2),(9,1),(3,3),(4,3)>, & <(1,2),(10,1),(4,3),(3,3)>, \\
<(2,2),(9,1),(4,3),(3,3)>, & <(2,2),(10,1),(3,3),(4,3)>, \\
<(3,1),(5,2),(9,3),(10,3)>, & <(3,1),(6,2),(10,3),(9,3)>, \\
<(4,1),(5,2),(10,3),(9,3)>, & <(4,1),(6,2),(9,3),(10,3)>, \\
<(3,2),(5,1),(9,3),(10,3)>, & <(3,2),(6,1),(10,3),(9,3)>, \\
<(4,2),(5,1),(10,3),(9,3)>, & <(4,2),(6,1),(9,3),(10,3)>
\end{array}
\right\} \subseteq H
$$

$$\left.\begin{array}{ll}
\{<(3,1),(7,2),(1,3),(2,3)>, & <(3,1),(8,2),(2,3),(1,3)>, \\
<(4,1),(7,2),(2,3),(1,3)>, & <(4,1),(8,2),(1,3),(2,3)>, \\
<(3,2),(7,1),(1,3),(2,3)>, & <(3,2),(8,1),(2,3),(1,3)>, \\
<(4,2),(7,1),(2,3),(1,3)>, & <(4,2),(8,1),(1,3),(2,3)>, \\
<(3,1),(9,2),(7,3),(8,3)>, & <(3,1),(10,2),(8,3),(7,3)>, \\
<(4,1),(9,2),(8,3),(7,3)>, & <(4,1),(10,2),(7,3),(8,3)>, \\
<(3,2),(9,1),(7,3),(8,3)>, & <(3,2),(10,1),(8,3),(7,3)>, \\
<(4,2),(9,1),(8,3),(7,3)>, & <(4,2),(10,1),(7,3),(8,3)>, \\
<(5,1),(7,2),(3,3),(4,3)>, & <(5,1),(8,2),(4,3),(3,3)>, \\
<(6,1),(7,2),(4,3),(3,3)>, & <(6,1),(8,2),(3,3),(4,3)>, \\
<(5,2),(7,1),(3,3),(4,3)>, & <(5,2),(8,1),(4,3),(3,3)>, \\
<(6,2),(7,1),(4,3),(3,3)>, & <(6,2),(8,1),(3,3),(4,3)>, \\
<(5,1),(9,2),(1,3),(2,3)>, & <(5,1),(10,2),(2,3),(1,3)>, \\
<(6,1),(9,2),(2,3),(1,3)>, & <(6,1),(10,2),(1,3),(2,3)>, \\
<(5,2),(9,1),(1,3),(2,3)>, & <(5,2),(10,1),(2,3),(1,3)>, \\
<(6,2),(9,1),(2,3),(1,3)>, & <(6,2),(10,1),(1,3),(2,3)>, \\
<(7,1),(9,2),(5,3),(6,3)>, & <(7,1),(10,2),(6,3),(5,3)>, \\
<(8,1),(9,2),(6,3),(5,3)>, & <(8,1),(10,2),(5,3),(6,3)>, \\
<(7,2),(9,1),(5,3),(6,3)>, & <(7,2),(10,1),(6,3),(5,3)>, \\
<(8,2),(9,1),(6,3),(5,3)>, & <(8,2),(10,1),(5,3),(6,3)>\}
\end{array}\right. \subseteq H$$

Now, we will remove the double edges from these hinges to create 4-cycles.

$$
\left.\begin{array}{ll}
\{((10,1),(1,1),(3,1),(2,1))\,, & ((10,1),(5,1),(4,1),(6,1))\,, \\
((10,1),(7,1),(8,1),(9,1))\,, & ((3,1),(1,1),(4,1),(7,1))\,, \\
((3,1),(2,1),(8,1),(6,1))\,, & ((4,1),(6,1),(8,1),(7,1))\,, \\
((1,1),(7,1),(5,1),(8,1))\,, & ((2,1),(7,1),(5,1),(8,1))\,, \\
((2,1),(4,1),(9,1),(6,1))\,, & ((1,1),(6,1),(9,1),(8,1))\,, \\
((5,1),(10,1),(6,1),(3,1))\,, & ((6,1),(1,1),(7,1),(2,1))\,, \\
((7,1),(10,1),(9,1),(3,1))\,, & ((5,1),(3,1),(9,1),(4,1))\,, \\
((1,1),(10,1),(2,1),(4,1))\,, & ((10,2),(1,2),(3,2),(2,2))\,, \\
((10,2),(5,2),(4,2),(6,2))\,, & ((10,2),(7,2),(8,2),(9,2))\,, \\
((3,2),(1,2),(4,2),(7,2))\,, & ((3,2),(2,2),(8,2),(6,2))\,, \\
((4,2),(6,2),(8,2),(7,2))\,, & ((1,2),(7,2),(5,2),(8,2))\,, \\
((2,2),(7,2),(5,2),(8,2))\,, & ((2,2),(4,2),(9,2),(6,2))\,, \\
((1,2),(6,2),(9,2),(8,2))\,, & ((5,2),(10,2),(6,2),(3,2))\,, \\
((6,2),(1,2),(7,2),(2,2))\,, & ((7,2),(10,2),(9,2),(3,2))\,, \\
((5,2),(3,2),(9,2),(4,2))\,, & ((1,2),(10,2),(2,2),(4,2))\,, \\
((10,3),(1,3),(3,3),(2,3))\,, & ((10,3),(5,3),(4,3),(6,3))\,, \\
((10,3),(7,3),(8,3),(9,3))\,, & ((3,3),(1,3),(4,3),(7,3))\,, \\
((3,3),(2,3),(8,3),(6,3))\,, & ((4,3),(6,3),(8,3),(7,3))\,, \\
((1,3),(7,3),(5,3),(8,3))\,, & ((2,3),(7,3),(5,3),(8,3))\,, \\
((2,3),(4,3),(9,3),(6,3))\,, & ((1,3),(6,3),(9,3),(8,3))\,, \\
((5,3),(10,3),(6,3),(3,3))\,, & ((6,3),(1,3),(7,3),(2,3))\,, \\
((7,3),(10,3),(9,3),(3,3))\,, & ((5,3),(3,3),(9,3),(4,3))\,, \\
((1,3),(10,3),(2,3),(4,3))\} &
\end{array}\right. \subseteq H \setminus D \subseteq C
$$

$$
\left.
\begin{array}{ll}
\{(\infty_1,(1,2),(1,1),(2,2))\,, & (\infty_1,(1,3),(2,1),(2,3))\,, \\
(\infty_2,(1,3),(1,1),(2,3))\,, & (\infty_2,(1,2),(2,1),(2,2))\,, \\
((1,2),\infty_1,(1,3),\infty_2)\,, & ((1,2),(1,1),(2,3),(2,1))\,, \\
((2,2),(1,1),(1,3),(2,1))\,, & ((2,2),\infty_1,(2,3),\infty_2)\,, \\
(\infty_1,(3,2),(3,1),(4,2))\,, & (\infty_1,(3,3),(4,1),(4,3))\,, \\
(\infty_2,(3,3),(3,1),(4,3))\,, & (\infty_2,(3,2),(4,1),(4,2))\,, \\
((3,2),\infty_1,(3,3),\infty_2)\,, & ((3,2),(3,1),(4,3),(4,1))\,, \\
((4,2),(3,1),(3,3),(4,1))\,, & ((4,2),\infty_1,(4,3),\infty_2)\,, \\
(\infty_1,(5,2),(5,1),(6,2))\,, & (\infty_1,(5,3),(6,1),(6,3))\,, \\
(\infty_2,(5,3),(5,1),(6,3))\,, & (\infty_2,(5,2),(6,1),(6,2))\,, \\
((5,2),\infty_1,(5,3),\infty_2)\,, & ((5,2),(5,1),(6,3),(6,1))\,, \\
((6,2),(5,1),(5,3),(6,1))\,, & ((6,2),\infty_1,(6,3),\infty_2)\,, \\
(\infty_1,(7,2),(7,1),(8,2))\,, & (\infty_1,(7,3),(8,1),(8,3))\,, \\
(\infty_2,(7,3),(7,1),(8,3))\,, & (\infty_2,(7,2),(8,1),(8,2))\,, \\
((7,2),\infty_1,(7,3),\infty_2)\,, & ((7,2),(7,1),(8,3),(8,1))\,, \\
((8,2),(7,1),(7,3),(8,1))\,, & ((8,2),\infty_1,(8,3),\infty_2)\,, \\
(\infty_1,(9,2),(9,1),(10,2))\,, & (\infty_1,(9,3),(10,1),(10,3))\,, \\
(\infty_2,(9,3),(9,1),(10,3))\,, & (\infty_2,(9,2),(10,1),(10,2))\,, \\
((9,2),\infty_1,(9,3),\infty_2)\,, & ((9,2),(9,1),(10,3),(10,1))\,, \\
((10,2),(9,1),(9,3),(10,1))\,, & ((10,2),\infty_1,(10,3),\infty_2)\}
\end{array}
\right.
\subseteq H \setminus D \subseteq C
$$

$$\left.\begin{array}{ll}
\{((1,1),(5,3),(3,2),(6,3))\,, & ((1,1),(6,3),(4,2),(5,3))\,,\\
((2,1),(6,3),(3,2),(5,3))\,, & ((2,1),(5,3),(4,2),(6,3))\,,\\
((1,2),(5,3),(3,1),(6,3))\,, & ((1,2),(6,3),(4,1),(5,3))\,,\\
((2,2),(6,3),(3,1),(5,3))\,, & ((2,2),(5,3),(4,1),(6,3))\,,\\
((1,1),(7,3),(5,2),(8,3))\,, & ((1,1),(8,3),(6,2),(7,3))\,,\\
((2,1),(8,3),(5,2),(7,3))\,, & ((2,1),(7,3),(6,2),(8,3))\,,\\
((1,2),(7,3),(5,1),(8,3))\,, & ((1,2),(8,3),(6,1),(7,3))\,,\\
((2,2),(8,3),(5,1),(7,3))\,, & ((2,2),(7,3),(6,1),(8,3))\,,\\
((1,1),(9,3),(7,2),(10,3))\,, & ((1,1),(10,3),(8,2),(9,3))\,,\\
((2,1),(10,3),(7,2),(9,3))\,, & ((2,1),(9,3),(8,2),(10,3))\,,\\
((1,2),(9,3),(7,1),(10,3))\,, & ((1,2),(10,3),(8,1),(9,3))\,,\\
((2,2),(10,3),(7,1),(9,3))\,, & ((2,2),(9,3),(8,1),(10,3))\,,\\
((1,1),(3,3),(9,2),(4,3))\,, & ((1,1),(4,3),(10,2),(3,3))\,,\\
((2,1),(4,3),(9,2),(3,3))\,, & ((2,1),(3,3),(10,2),(4,3))\,,\\
((1,2),(3,3),(9,1),(4,3))\,, & ((1,2),(4,3),(10,1),(3,3))\,,\\
((2,2),(4,3),(9,1),(3,3))\,, & ((2,2),(3,3),(10,1),(4,3))\,,\\
((3,1),(9,3),(5,2),(10,3))\,, & ((3,1),(10,3),(6,2),(9,3))\,,\\
((4,1),(10,3),(5,2),(9,3))\,, & ((4,1),(9,3),(6,2),(10,3))\,,\\
((3,2),(9,3),(5,1),(10,3))\,, & ((3,2),(10,3),(6,1),(9,3))\,,\\
((4,2),(10,3),(5,1),(9,3))\,, & ((4,2),(9,3),(6,1),(10,3))\}
\end{array}\right. \subseteq H \setminus D \subseteq C$$

$$\left.\begin{array}{ll}
\{((3,1),(1,3),(7,2),(2,3))\,, & ((3,1),(2,3),(8,2),(1,3))\,, \\
((4,1),(2,3),(7,2),(1,3))\,, & ((4,1),(1,3),(8,2),(2,3))\,, \\
((3,2),(1,3),(7,1),(2,3))\,, & ((3,2),(2,3),(8,1),(1,3))\,, \\
((4,2),(2,3),(7,1),(1,3))\,, & ((4,2),(1,3),(8,1),(2,3))\,, \\
((3,1),(7,3),(9,2),(8,3))\,, & ((3,1),(8,3),(10,2),(7,3))\,, \\
((4,1),(8,3),(9,2),(7,3))\,, & ((4,1),(7,3),(10,2),(8,3))\,, \\
((3,2),(7,3),(9,1),(8,3))\,, & ((3,2),(8,3),(10,1),(7,3))\,, \\
((4,2),(8,3),(9,1),(7,3))\,, & ((4,2),(7,3),(10,1),(8,3))\,, \\
((5,1),(3,3),(7,2),(4,3))\,, & ((5,1),(4,3),(8,2),(3,3))\,, \\
((6,1),(4,3),(7,2),(3,3))\,, & ((6,1),(3,3),(8,2),(4,3))\,, \\
((5,2),(3,3),(7,1),(4,3))\,, & ((5,2),(4,3),(8,1),(3,3))\,, \\
((6,2),(4,3),(7,1),(3,3))\,, & ((6,2),(3,3),(8,1),(4,3))\,, \\
((5,1),(1,3),(9,2),(2,3))\,, & ((5,1),(2,3),(10,2),(1,3))\,, \\
((6,1),(2,3),(9,2),(1,3))\,, & ((6,1),(1,3),(10,2),(2,3))\,, \\
((5,2),(1,3),(9,1),(2,3))\,, & ((5,2),(2,3),(10,1),(1,3))\,, \\
((6,2),(2,3),(9,1),(1,3))\,, & ((6,2),(1,3),(10,1),(2,3))\,, \\
((7,1),(5,3),(9,2),(6,3))\,, & ((7,1),(6,3),(10,2),(5,3))\,, \\
((8,1),(6,3),(9,2),(5,3))\,, & ((8,1),(5,3),(10,2),(6,3))\,, \\
((7,2),(5,3),(9,1),(6,3))\,, & ((7,2),(6,3),(10,1),(5,3))\,, \\
((8,2),(6,3),(9,1),(5,3))\,, & ((8,2),(5,3),(10,1),(6,3))\}.
\end{array}\right. \subseteq H \setminus D \subseteq C$$

Finally, we will use the double edges (along with the leave) to create the last of our 4-cycles.

$$
\left.\begin{array}{ll}
\{((10,1),(3,1),(8,1),(4,1)), & ((10,1),(3,1),(4,1),(8,1)), \\
((10,1),(8,1),(3,1),(4,1)), & ((1,1),(5,1),(2,1),(9,1)), \\
((1,1),(5,1),(2,1),(9,1)), & ((5,1),(6,1),(7,1),(9,1)), \\
((5,1),(6,1),(7,1),(9,1)), & ((10,2),(3,2),(8,2),(4,2)), \\
((10,2),(3,2),(4,2),(8,2)), & ((10,2),(8,2),(3,2),(4,2)), \\
((1,2),(5,2),(2,2),(9,2)), & ((1,2),(5,2),(2,2),(9,2)), \\
((5,2),(6,2),(7,2),(9,2)), & ((5,2),(6,2),(7,2),(9,2)), \\
((10,3),(3,3),(8,3),(4,3)), & ((10,3),(3,3),(4,3),(8,3)), \\
((10,3),(8,3),(3,3),(4,3)), & ((1,3),(5,3),(2,3),(9,3)), \\
((1,3),(5,3),(2,3),(9,3)), & ((5,3),(6,3),(7,3),(9,3)), \\
((5,3),(6,3),(7,3),(9,3)), & (\infty_1,\infty_2,(2,1),(1,1)), \\
(\infty_1,\infty_2,(1,1),(2,1)), & (\infty_1,(1,1),\infty_2,(2,1)), \\
((1,2),(2,2),(2,3),(1,3)), & ((1,2),(2,2),(1,3),(2,3)), \\
((1,2),(1,3),(2,2),(2,3)), & (\infty_1,(3,1),\infty_2,(4,1)), \\
(\infty_1,(3,1),\infty_2,(4,1)), & ((3,2),(3,3),(4,2),(4,3)), \\
((3,2),(3,3),(4,2),(4,3)), & (\infty_1,(5,1),\infty_2,(6,1)), \\
(\infty_1,(5,1),\infty_2,(6,1)), & ((5,2),(5,3),(6,2),(6,3)), \\
((5,2),(5,3),(6,2),(6,3)), & (\infty_1,(7,1),\infty_2,(8,1)), \\
(\infty_1,(7,1),\infty_2,(8,1)), & ((7,2),(7,3),(8,2),(8,3)), \\
((7,2),(7,3),(8,2),(8,3)), & (\infty_1,(9,1),\infty_2,(10,1)), \\
(\infty_1,(9,1),\infty_2,(10,1)), & ((9,2),(9,3),(10,2),(10,3)), \\
((9,2),(9,3),(10,2),(10,3)), & ((1,1),(3,2),(2,1),(4,2))\}
\end{array}\right. \subseteq C
$$

$$
\left.\begin{array}{ll}
\{((1,1),(3,2),(2,1),(4,2)) & ((1,2),(3,1),(2,2),(4,1)) \\
((1,2),(3,1),(2,2),(4,1)) & ((1,1),(5,2),(2,1),(6,2)) \\
((1,1),(5,2),(2,1),(6,2)) & ((1,2),(5,1),(2,2),(6,1)) \\
((1,2),(5,1),(2,2),(6,1)) & ((1,1),(7,2),(2,1),(8,2)) \\
((1,1),(7,2),(2,1),(8,2)) & ((1,2),(7,1),(2,2),(8,1)) \\
((1,2),(7,1),(2,2),(8,1)) & ((1,1),(9,2),(2,1),(10,2)) \\
((1,1),(9,2),(2,1),(10,2)) & ((1,2),(9,1),(2,2),(10,1)) \\
((1,2),(9,1),(2,2),(10,1)) & ((3,1),(5,2),(4,1),(6,2)) \\
((3,1),(5,2),(4,1),(6,2)) & ((3,2),(5,1),(4,2),(6,1)) \\
((3,2),(5,1),(4,2),(6,1)) & ((3,1),(7,2),(4,1),(8,2)) \\
((3,1),(7,2),(4,1),(8,2)) & ((3,2),(7,1),(4,2),(8,1)) \\
((3,2),(7,1),(4,2),(8,1)) & ((3,1),(9,2),(4,1),(10,2)) \\
((3,1),(9,2),(4,1),(10,2)) & ((3,2),(9,1),(4,2),(10,1)) \\
((3,2),(9,1),(4,2),(10,1)) & ((5,1),(7,2),(6,1),(8,2)) \\
((5,1),(7,2),(6,1),(8,2)) & ((5,2),(7,1),(6,2),(8,1)) \\
((5,2),(7,1),(6,2),(8,1)) & ((5,1),(9,2),(6,1),(10,2)) \\
((5,1),(9,2),(6,1),(10,2)) & ((5,2),(9,1),(6,2),(10,1)) \\
((5,2),(9,1),(6,2),(10,1)) & ((7,1),(9,2),(8,1),(10,2)) \\
((7,1),(9,2),(8,1),(10,2)) & ((7,2),(9,1),(8,2),(10,1)) \\
((7,2),(9,1),(8,2),(10,1))\}. &
\end{array}\right. \subseteq C
$$

Now, we must consider the following examples that we will use in the $12n+8$ Construction.

Example 5.10 (metamorphosis of a maximum packing of $2K_{20}$ with triples into 2-fold 4-cycle system of order 20)**.**

We will decompose $2K_{20}$ with vertex set $\{1,2,3,4\} \times \{1,2,3,4,5\}$. We will begin with the maximum packing with hinges, since it has an obvious correspondence to the maximum packing with triples. Our leave will be $< (4,1),(4,2) >$.

$$\begin{array}{lll}
\{< (1,1),(1,2),(2,1),(2,2) >, & < (2,1),(2,2),(3,1),(3,2) >, & \\
< (3,1),(3,2),(1,1),(1,2) >, & < (1,1),(4,1),(2,1),(3,1) >, & \\
< (1,1),(4,2),(2,2),(3,2) >, & < (1,2),(4,1),(2,2),(3,1) >, & \\
< (1,2),(4,2),(2,1),(3,2) >, & < (3,1),(4,2),(2,1),(2,2) >, & \\
< (3,2),(4,1),(2,1),(2,2) >, & < (1,3),(1,4),(1,1),(1,2) > & \\
< (1,3),(1,5),(1,1),(1,2) >, & < (1,4),(1,5),(1,1),(1,2) >, & \\
< (2,3),(2,4),(2,1),(2,2) >, & < (2,3),(2,5),(2,1),(2,2) >, & \\
< (2,4),(2,5),(2,1),(2,2) >, & < (3,3),(3,4),(3,1),(3,2) >, & \\
< (3,3),(3,5),(3,1),(3,2) >, & < (3,4),(3,5),(3,1),(3,2) >, & \\
< (4,3),(4,4),(4,1),(4,2) >, & < (4,3),(4,5),(4,1),(4,2) >, & \\
< (4,4),(4,5),(4,1),(4,2) >, & < (1,1),(3,3),(2,4),(4,4) >, & \\
< (2,1),(4,3),(1,4),(3,4) >, & < (1,1),(3,4),(2,5),(4,5) >, & \subseteq H \\
< (2,1),(4,4),(1,5),(3,5) >, & < (1,1),(3,5),(2,3),(4,3) >, & \\
< (2,1),(4,5),(1,3),(3,3) >, & < (1,2),(3,3),(2,5),(4,5) >, & \\
< (2,2),(4,3),(1,5),(3,5) >, & < (1,2),(3,4),(2,3),(4,3) >, & \\
< (2,2),(4,4),(1,3),(3,3) >, & < (1,2),(3,5),(2,4),(4,4) >, & \\
< (2,2),(4,5),(1,4),(3,4) >, & < (1,3),(3,1),(2,4),(4,4) >, & \\
< (2,3),(4,1),(1,4),(3,4) >, & < (1,3),(3,2),(2,5),(4,5) >, & \\
< (2,3),(4,2),(1,5),(3,5) >, & < (1,3),(3,3),(2,3),(4,3) >, & \\
< (2,3),(4,3),(1,3),(3,3) >, & < (1,3),(3,4),(2,1),(4,1) >, & \\
< (2,3),(4,4),(1,1),(3,1) >, & < (1,3),(3,5),(2,2),(4,2) >, & \\
< (2,3),(4,5),(1,2),(3,2) >, & < (1,4),(3,1),(2,5),(4,5) >, & \\
< (2,3),(4,5),(1,2),(3,2) >, & < (1,4),(3,1),(2,5),(4,5) >\} &
\end{array}$$

$$
\left.\begin{array}{ll}
\{< (2,4),(4,1),(1,5),(3,5) >, & < (1,4),(3,2),(2,3),(4,3) >, \\
< (2,4),(4,2),(1,3),(3,3) >, & < (1,4),(3,3),(2,2),(4,2) >, \\
< (2,4),(4,3),(1,2),(3,2) >, & < (1,4),(3,4),(2,4),(4,4) >, \\
< (2,4),(4,4),(1,4),(3,4) >, & < (1,4),(3,5),(2,1),(4,1) > \\
< (2,4),(4,5),(1,1),(3,1) >, & < (1,5),(3,1),(2,3),(4,3) >, \\
< (2,5),(4,1),(1,3),(3,3) >, & < (1,5),(3,2),(2,4),(4,4) >, \\
< (2,5),(4,2),(1,4),(3,4) >, & < (1,5),(3,3),(2,1),(4,1) >, \\
< (2,5),(4,3),(1,1),(3,1) >, & < (1,5),(3,4),(2,2),(4,2) >, \\
< (2,5),(4,4),(1,2),(3,2) >, & < (1,5),(3,5),(2,5),(4,5) >, \\
< (2,5),(4,5),(1,5),(3,5) >\}. &
\end{array}\right. \subseteq H
$$

Now, we will remove the double edges from these hinges to create 4-cycles.

$$
\left.\begin{array}{ll}
\{((1,1),(2,1),(1,2),(2,2)), & ((2,1),(3,1),(2,2),(3,2)), \\
((3,1),(1,1),(3,2),(1,2)), & ((1,1),(2,1),(4,1),(3,1)), \\
((1,1),(2,2),(4,2),(3,2)), & ((1,2),(2,2),(4,1),(3,1)), \\
((1,2),(2,1),(4,2),(3,2)), & ((3,1),(2,1),(4,2),(2,2)), \\
((3,2),(2,1),(4,1),(2,2)), & ((1,3),(1,1),(1,4),(1,2)), \\
((1,3),(1,1),(1,5),(1,2)), & ((1,4),(1,1),(1,5),(1,2)), \\
((2,3),(2,1),(2,4),(2,2)), & ((2,3),(2,1),(2,5),(2,2)), \\
((2,4),(2,1),(2,5),(2,2)), & ((3,3),(3,1),(3,4),(3,2)), \\
((3,3),(3,1),(3,5),(3,2)), & ((3,4),(3,1),(3,5),(3,2)), \\
((4,3),(4,1),(4,4),(4,2)), & ((4,3),(4,1),(4,5),(4,2)), \\
((4,4),(4,1),(4,5),(4,2)), & ((1,1),(2,4),(3,3),(4,4)), \\
((2,1),(1,4),(4,3),(3,4)), & ((1,1),(2,5),(3,4),(4,5)), \\
((2,1),(1,5),(4,4),(3,5)), & ((1,1),(2,3),(3,5),(4,3)), \\
((2,1),(1,3),(4,5),(3,3)), & ((1,2),(2,5),(3,3),(4,5)), \\
((2,2),(1,5),(4,3),(3,5)), & ((1,2),(2,3),(3,4),(4,3))\}
\end{array}\right. \subseteq H \setminus D \subseteq C
$$

$$
\left.\begin{array}{ll}
\{((2,2),(1,3),(4,4),(3,3))\,, & ((1,2),(2,4),(3,5),(4,4))\,, \\
((2,2),(1,4),(4,5),(3,4))\,, & ((1,3),(2,4),(3,1),(4,4)) \\
((2,3),(1,4),(4,1),(3,4))\,, & ((1,3),(2,5),(3,2),(4,5))\,, \\
((2,3),(1,5),(4,2),(3,5))\,, & ((1,3),(2,3),(3,3),(4,3))\,, \\
((2,3),(1,3),(4,3),(3,3))\,, & ((1,3),(2,1),(3,4),(4,1))\,, \\
((2,3),(1,1),(4,4),(3,1))\,, & ((1,3),(2,2),(3,5),(4,2))\,, \\
((2,3),(1,2),(4,5),(3,2))\,, & ((1,4),(2,5),(3,1),(4,5))\,, \\
((2,4),(1,5),(4,1),(3,5))\,, & ((1,4),(2,3),(3,2),(4,3))\,, \\
((2,4),(1,3),(4,2),(3,3))\,, & ((1,4),(2,2),(3,3),(4,2))\,, \\
((2,4),(1,2),(4,3),(3,2))\,, & ((1,4),(2,4),(3,4),(4,4)) \\
((2,4),(1,4),(4,4),(3,4))\,, & ((1,4),(2,1),(3,5),(4,1))\,, \\
((2,4),(1,1),(4,5),(3,1))\,, & ((1,5),(2,3),(3,1),(4,3))\,, \\
((2,5),(1,3),(4,1),(3,3))\,, & ((1,5),(2,4),(3,2),(4,4))\,, \\
((2,5),(1,4),(4,2),(3,4))\,, & ((1,5),(2,1),(3,3),(4,1))\,, \\
((2,5),(1,1),(4,3),(3,1))\,, & ((1,5),(2,2),(3,4),(4,2))\,, \\
((2,5),(1,2),(4,4),(3,2))\,, & ((1,5),(2,5),(3,5),(4,5))\,, \\
((2,5),(1,5),(4,5),(3,5))\}. &
\end{array}\right. \subseteq H \setminus D \subseteq C
$$

Finally, we will use the double edges (along with the leave) to create the last of our 4-cycles.

$$\left.\begin{array}{ll}
\{((3,1),(3,2),(4,1),(4,2))\,, & ((3,1),(3,2),(4,1),(4,2))\,, \\
((1,1),(4,1),(1,2),(4,2))\,, & ((1,1),(4,1),(1,2),(4,2))\,, \\
((1,1),(1,2),(3,4),(3,3))\,, & ((2,1),(2,2),(4,4),(4,3))\,, \\
((1,1),(1,2),(3,5),(3,4))\,, & ((2,1),(2,2),(4,5),(4,4))\,, \\
((1,2),(3,3),(1,1),(3,5))\,, & ((2,2),(4,3),(2,1),(4,5))\,, \\
((1,2),(3,3),(3,5),(3,4))\,, & ((2,2),(4,3),(4,5),(4,4))\,, \\
((3,3),(3,4),(1,1),(3,5))\,, & ((4,3),(4,4),(2,1),(4,5))\,, \\
((1,3),(3,2),(1,4),(3,1))\,, & ((2,3),(4,2),(2,4),(4,1))\,, \\
((1,3),(3,2),(1,5),(3,1))\,, & ((2,3),(4,2),(2,5),(4,1))\,, \\
((1,4),(3,2),(1,5),(3,1))\,, & ((2,4),(4,2),(2,5),(4,1))\,, \\
((1,3),(3,4),(1,4),(3,5))\,, & ((2,3),(4,4),(2,4),(4,5))\,, \\
((1,4),(3,3),(1,5),(3,4))\,, & ((2,4),(4,3),(2,5),(4,4))\,, \\
((1,3),(3,3),(1,5),(3,5))\,, & ((2,3),(4,3),(2,5),(4,5))\,, \\
((1,3),(1,4),(1,5),(3,4))\,, & ((2,3),(2,4),(2,5),(4,4))\,, \\
((1,3),(1,5),(1,4),(3,3))\,, & ((2,3),(2,5),(2,4),(4,3))\,, \\
((1,3),(1,5),(3,5),(1,4))\,, & ((2,3),(2,5),(4,5),(2,4))\}.
\end{array}\right. \subseteq C$$

Example 5.11 (metamorphosis of a maximum packing of $2K_{20} \setminus 2K_8$ with triples into a maximum packing of $2K_{20} \setminus 2K_8$ with 4-cycles)**.** We will be decomposing $2K_{20}$ with vertex set $\{1,2,3,4,5\} \times \{1,2,3,4\}$ minus $2K_8$ on the vertex set $\{1,2\} \times \{1,2,3,4\}$. We will begin with the maximum packing with hinges, since it has an obvious correspondence to the maximum packing with triples.

$$\left.\begin{array}{ll}
\{<(3,1),(3,2),(3,3),(3,4)>, & <(3,3),(3,4),(3,1),(3,2)>, \\
<(4,1),(4,2),(4,3),(4,4)>, & <(4,3),(4,4),(4,1),(4,2)>, \\
<(5,1),(5,2),(5,3),(5,4)>, & <(5,3),(5,4),(5,1),(5,2)>\}
\end{array}\right. \subseteq H$$

$$\left.\begin{array}{ll}
\{<(3,1),(4,1),(1,1),(2,1)>, & <(3,1),(5,1),(1,1),(2,1)>, \\
<(4,1),(5,1),(1,1),(2,1)>, & <(3,1),(4,2),(1,3),(2,3)>, \\
<(3,1),(5,2),(1,3),(2,3)>, & <(4,1),(5,2),(1,3),(2,3)>, \\
<(3,1),(4,3),(1,4),(2,4)>, & <(3,1),(5,3),(1,4),(2,4)>, \\
<(4,1),(5,3),(1,4),(2,4)>, & <(3,1),(4,4),(1,2),(2,2)>, \\
<(3,1),(5,4),(1,2),(2,2)>, & <(4,1),(5,4),(1,2),(2,2)>, \\
<(3,2),(4,1),(1,4),(2,4)>, & <(3,2),(5,1),(1,4),(2,4)>, \\
<(4,2),(5,1),(1,4),(2,4)>, & <(3,2),(4,2),(1,2),(2,2)>, \\
<(3,2),(5,2),(1,2),(2,2)>, & <(4,2),(5,2),(1,2),(2,2)>, \\
<(3,2),(4,3),(1,1),(2,1)>, & <(3,2),(5,3),(1,1),(2,1)>, \\
<(4,2),(5,3),(1,1),(2,1)>, & <(3,2),(4,4),(1,3),(2,3)>, \\
<(3,2),(5,4),(1,3),(2,3)>, & <(4,2),(5,4),(1,3),(2,3)>, \\
<(3,3),(4,1),(1,2),(2,2)>, & <(3,3),(5,1),(1,2),(2,2)>, \\
<(4,3),(5,1),(1,2),(2,2)>, & <(3,3),(4,2),(1,4),(2,4)>, \\
<(3,3),(5,2),(1,4),(2,4)>, & <(4,3),(5,2),(1,4),(2,4)>, \\
<(3,3),(4,3),(1,3),(2,3)>, & <(3,3),(5,3),(1,3),(2,3)>, \\
<(4,3),(5,3),(1,3),(2,3)>, & <(3,3),(4,4),(1,1),(2,1)>, \\
<(3,3),(5,4),(1,1),(2,1)>, & <(4,3),(5,4),(1,1),(2,1)>, \\
<(3,4),(4,1),(1,3),(2,3)>, & <(3,4),(5,1),(1,3),(2,3)>, \\
<(4,4),(5,1),(1,3),(2,3)>, & <(3,4),(4,2),(1,1),(2,1)>, \\
<(3,4),(5,2),(1,1),(2,1)>, & <(4,4),(5,2),(1,1),(2,1)>, \\
<(3,4),(4,3),(1,2),(2,2)>, & <(3,4),(5,3),(1,2),(2,2)>, \\
<(4,4),(5,3),(1,2),(2,2)>, & <(3,4),(4,4),(1,4),(2,4)>, \\
<(3,4),(5,4),(1,4),(2,4)>, & <(4,4),(5,4),(1,4),(2,4)>\}.
\end{array}\right. \subseteq H$$

Now, we will remove the double edges from these hinges to create 4-cycles.

$$
\left.\begin{array}{ll}
\{((3,1),(3,3),(3,2),(3,4))\,, & ((3,3),(3,1),(3,4),(3,2))\,,\\
((4,1),(4,3),(4,2),(4,4))\,, & ((4,3),(4,1),(4,4),(4,2))\,,\\
((5,1),(5,3),(5,2),(5,4))\,, & ((5,3),(5,1),(5,4),(5,2))\,,\\
((3,1),(1,1),(4,1),(2,1))\,, & ((3,1),(1,1),(5,1),(2,1))\,,\\
((4,1),(1,1),(5,1),(2,1))\,, & ((3,1),(1,3),(4,2),(2,3))\,,\\
((3,1),(1,3),(5,2),(2,3))\,, & ((4,1),(1,3),(5,2),(2,3))\,,\\
((3,1),(1,4),(4,3),(2,4))\,, & ((3,1),(1,4),(5,3),(2,4))\,,\\
((4,1),(1,4),(5,3),(2,4))\,, & ((3,1),(1,2),(4,4),(2,2))\,,\\
((3,1),(1,2),(5,4),(2,2))\,, & ((4,1),(1,2),(5,4),(2,2))\,,\\
((3,2),(1,4),(4,1),(2,4))\,, & ((3,2),(1,4),(5,1),(2,4))\,,\\
((4,2),(1,4),(5,1),(2,4))\,, & ((3,2),(1,2),(4,2),(2,2))\,,\\
((3,2),(1,2),(5,2),(2,2))\,, & ((4,2),(1,2),(5,2),(2,2))\,,\\
((3,2),(1,1),(4,3),(2,1))\,, & ((3,2),(1,1),(5,3),(2,1))\,,\\
((4,2),(1,1),(5,3),(2,1))\,, & ((3,2),(1,3),(4,4),(2,3))\,,\\
((3,2),(1,3),(5,4),(2,3))\,, & ((4,2),(1,3),(5,4),(2,3))\,,\\
((3,3),(1,2),(4,1),(2,2))\,, & ((3,3),(1,2),(5,1),(2,2))\,,\\
((4,3),(1,2),(5,1),(2,2))\,, & ((3,3),(1,4),(4,2),(2,4))\,,\\
((3,3),(1,4),(5,2),(2,4))\,, & ((4,3),(1,4),(5,2),(2,4))\,,\\
((3,3),(1,3),(4,3),(2,3))\,, & ((3,3),(1,3),(5,3),(2,3))\,,\\
((4,3),(1,3),(5,3),(2,3))\,, & ((3,3),(1,1),(4,4),(2,1))\,,\\
((3,3),(1,1),(5,4),(2,1))\,, & ((4,3),(1,1),(5,4),(2,1))\,,\\
((3,4),(1,3),(4,1),(2,3))\,, & ((3,4),(1,3),(5,1),(2,3))\,,\\
((4,4),(1,3),(5,1),(2,3))\,, & ((3,4),(1,1),(4,2),(2,1))\,,\\
((3,4),(1,1),(5,2),(2,1))\,, & ((4,4),(1,1),(5,2),(2,1))\,,\\
((3,4),(1,2),(4,3),(2,2))\,, & ((3,4),(1,2),(5,3),(2,2))\,,\\
((4,4),(1,2),(5,3),(2,2))\,, & ((3,4),(1,4),(4,4),(2,4))\,,\\
((3,4),(1,4),(5,4),(2,4))\,, & ((4,4),(1,4),(5,4),(2,4))\}
\end{array}\right. = H \setminus D \subseteq C
$$

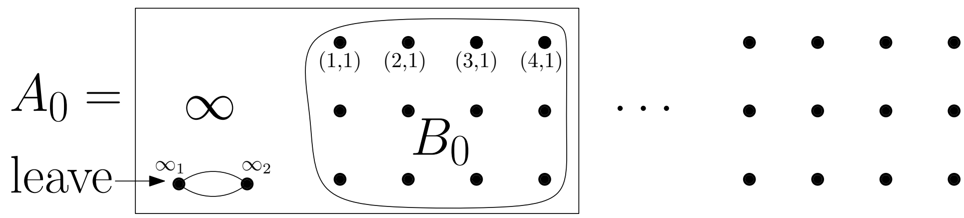
Finally, we will use the double edges to create the last of our 4-cycles.

$$\left.\begin{array}{ll}
\{((3,1),(3,2),(4,1),(4,2))\,, & ((3,1),(3,2),(4,2),(4,1))\,, \\
((3,1),(4,1),(3,2),(4,2))\,, & ((4,3),(4,4),(5,3),(5,4))\,, \\
((4,3),(4,4),(5,4),(5,3))\,, & ((4,3),(5,3),(4,4),(5,4))\,, \\
((3,3),(3,4),(5,1),(5,2))\,, & ((3,3),(3,4),(5,2),(5,1))\,, \\
((3,3),(5,1),(3,4),(5,2))\,, & ((4,1),(5,1),(4,2),(5,2))\,, \\
((4,1),(5,1),(4,2),(5,2))\,, & ((3,3),(4,3),(3,4),(4,4))\,, \\
((3,3),(4,3),(3,4),(4,4))\,, & ((3,1),(5,1),(3,2),(5,2))\,, \\
((3,1),(5,1),(3,2),(5,2))\,, & ((3,3),(5,3),(3,4),(5,4))\,, \\
((3,3),(5,3),(3,4),(5,4))\,, & ((3,1),(4,3),(3,2),(4,4))\,, \\
((3,1),(4,3),(3,2),(4,4))\,, & ((4,1),(5,3),(4,2),(5,4))\,, \\
((4,1),(5,3),(4,2),(5,4))\,, & ((3,3),(4,1),(3,4),(4,2))\,, \\
((3,3),(4,1),(3,4),(4,2))\,, & ((4,3),(5,1),(4,4),(5,2))\,, \\
((4,3),(5,1),(4,4),(5,2))\,, & ((3,1),(5,3),(3,2),(5,4))\,, \\
((3,1),(5,3),(3,2),(5,4))\}. &
\end{array}\right. \subseteq C$$

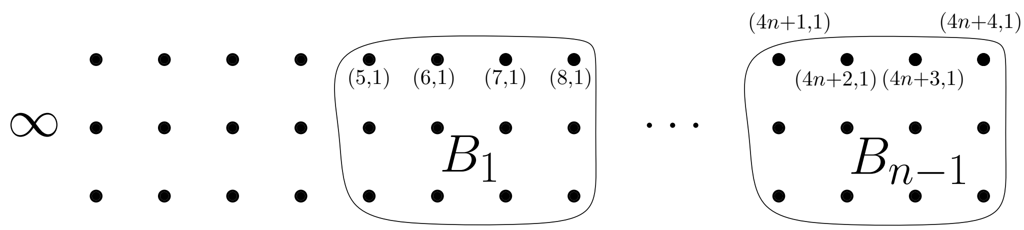
The 12n+8$\geq$44 Construction

With the above examples in hand, we can proceed to the $12n + 8 \geq 44$ Construction. Write $12n + 8 = 3(4n) + 8$. Since $12n + 8 \geq 44$, $4n \geq 12$. This is important. Let $\boldsymbol{\infty} = \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \infty_6, \infty_7, \infty_8\}$ and $Q = \{1, 2, 3, \ldots, 4n\}$. Let $H(Q) = \{h_0, h_1, \ldots, h_{n-1}\}$ be a partition of Q into pairwise disjoint sets of size 4 (called *holes*), where $h_i = \{4i + 1, 4i + 2, 4i + 3, 4i + 4\}$. Let $(Q, \circ)$ be a commutative quasigroup of order $4n$ with holes $H(Q)$ (see [6]) and set $X = \boldsymbol{\infty} \cup (Q \times \{1, 2, 3\})$. For $0 \leq i \leq n - 1$, let $B_i = h_i \times \{1, 2, 3\}$ and $A_i = \boldsymbol{\infty} \cup B_i$. Define a collection of hinges as follows:

(1) Use the $n = 20$ example (Example 5.10) to find a hinge system with our desired metamorphosis (A_0, H_0), where the leave is $< \infty_1, \infty_2 >$.

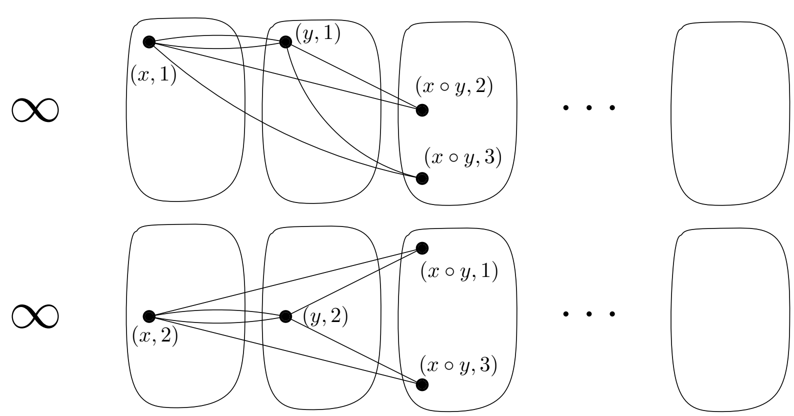


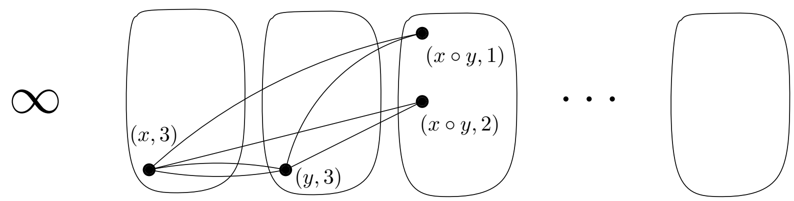
(2) For $1 \leq i \leq n-1$, use the $n = 20$ with a hole of size 8 Construction (Example 5.11) to find a hinge system with our desired metamorphosis (A_i, H_i), where the hole is ∞.



(3) We now need to use the edges between vertices in B_i and B_j for $i \neq j$. Define a collection of hinges $H_\star$ as follows. For $x \in h_i$, $y \in h_j$, $i \neq j$, place the hinges

$< (x,1),(y,1),(x \circ y, 2),(x \circ y, 3) >$, $\qquad < (x,2),(y,2),(x \circ y,1),(x \circ y,3) >$, and $< (x,3),(y,3),(x \circ y,1),(x \circ y,2) >\}$ in $H_\star$





It is straightforward to see that $(X = \bigcup_{i=1}^{n} A_i, H_\star \cup \bigcup_{i=1}^{n} H_i)$ is a hinge system of order $12n+8$ with leave $< \infty_1, \infty_2 >$. To proceed with our metamorphosis, first use the prescribed metamorphoses in (1) and (2) and places the 4-cycles in C. After removing the double edges from our hinges in $H_\star$ (from (3)), we still have edges of the type $< (x,k),(y,k) >$, where $k \in \{1,2,3\}$ and $x \in h_i$, $y \in h_j$, $i \neq j$ remaining to use in our metamorphosis.

(4) For $k \in \{1,2,3\}$, we have 2 copies of the complete n-partite graph with all partite sets having size 4. Between each pair of partite sets $\{x_1,x_2,x_3,x_4\}$ and $\{y_1,y_2,y_3,y_4\}$ in these $2K_{4,4,\ldots,4}$, we have:

$$\left.\begin{array}{llll}\{(x_1,y_2,x_2,y_1), & (x_1,y_2,x_2,y_1), & (x_1,y_4,x_2,y_3), & (x_1,y_4,x_2,y_3),\\ (x_3,y_4,x_4,y_3), & (x_3,y_4,x_4,y_3), & (x_3,y_2,x_4,y_1), & (x_3,y_2,x_4,y_1)\}\end{array}\right. \subseteq C.$$

The result is a 2-fold 4-cycle system of order $12n+8$ with vertex set X.

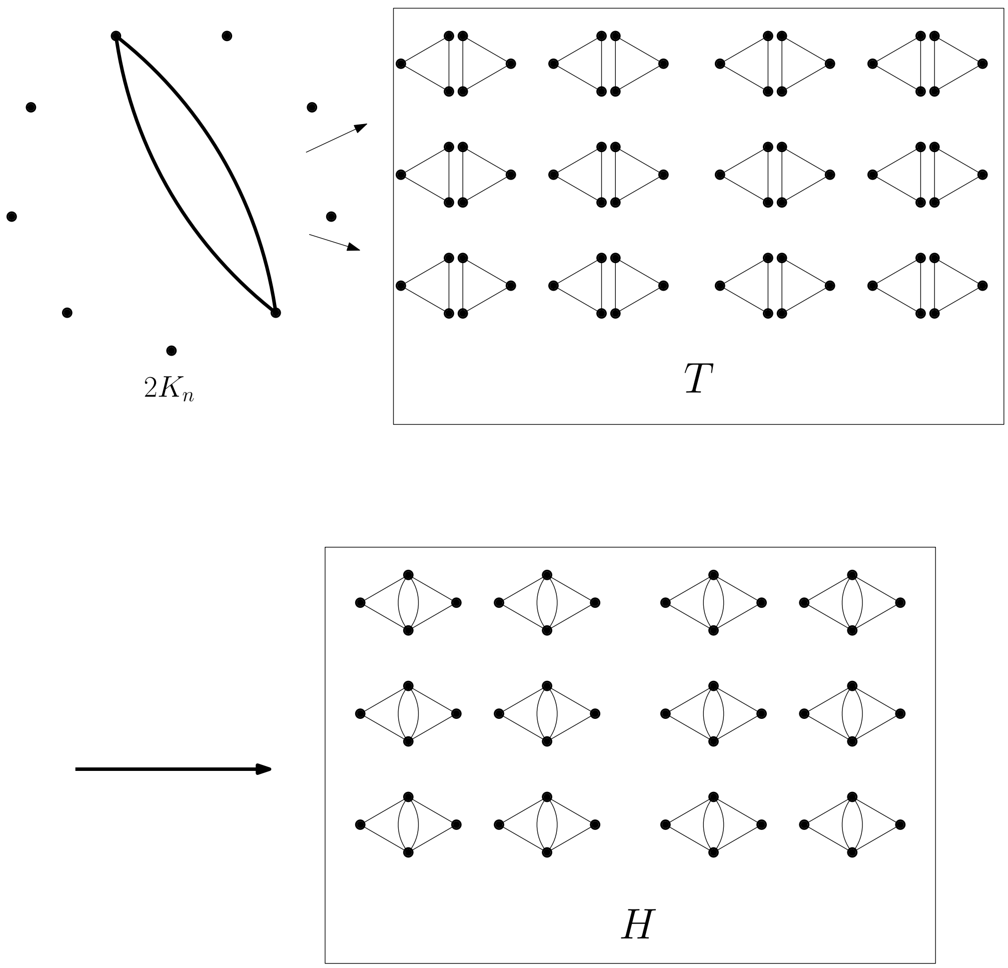
Theorem 5.12. *There exists a maximum packing of $2K_{12n+8}$ with triples having a metamorphosis into a 2-fold 4-cycle system of order $12n+8 \geq 20$.* ■

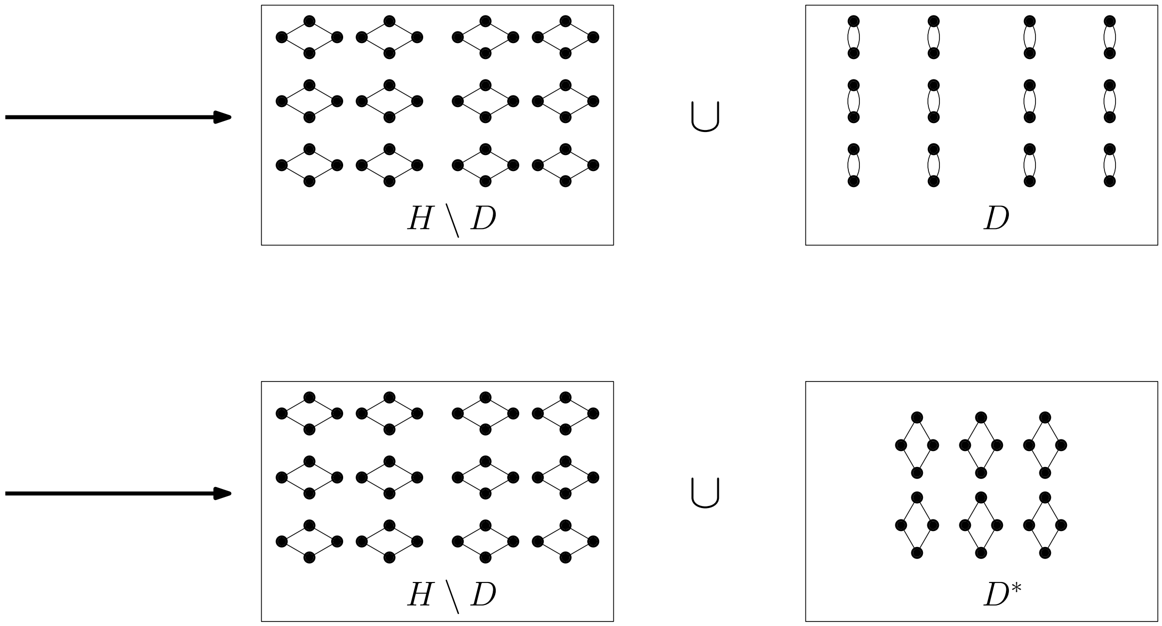
Chapter 6

Summary

Combining all of the results in Chapters 1, 2, 3, 4, and 5, we have a proof of Theorem 1.8.

Theorem (1.8)**.** *The spectrum for maximum packings of* $2K_n$ *with triples having a metamorphosis into a maximum packing of* $2K_n$ *with 4-cycles is the set of all* $n \equiv 2, 5, 8,$ *or* $11 \ (mod\ 12) \geq 8$*, except* $n = 5$ *and* $n = 8$. ■





maximum packing of $2K_n$ with 4-cycles

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