Stars and Hyperstars

by

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Abstract

A k-*star* is a complete bipartite graph $K_{1,k}$, and a hypergraph $G = (X, \mathcal{E})$ is a *hyperstar* with center C if $C \subseteq \bigcap_{E \in \mathcal{E}} E$. Given (hyper)graphs G and H, an H-decomposition of G is a partition of the edges of G into copies of H. In this work, we investigate different decomposition-like properties of stars and hyperstars. We discuss maximum packings of λK_n with k-stars, embeddings of partial k-star decompositions of K_n, and decompositions of complete hypergraphs and complete uniform hypergraphs into hyperstars with center-size 1.

We characterize the number of edges in a maximum packing of λK_n with k-stars for the cases $\lambda = 1$ for any n, and $\lambda > 1$ for $n \geq 2k$. For the case where $\lambda > 1$ with $n < 2k$ we give partial results. Our main result for an embedding of a partial k-star decomposition of K_n is a small embedding in which the number of new vertices depends only on k. The question of decomposing both complete hypergraphs and complete uniform hypergraphs into hyperstars has already been solved by Lonc [9], but we give an alternative proof.

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Chapter 1

Introduction

Let $\mathbb{N}$ denote the set of non-negative integers $\{0, 1, 2, \dots\}$. A *graph*, G, is a pair $(V(G), E(G))$ where $V(G)$ is a set whose elements are called *vertices*, and $E(G)$ (whose elements are called *edges*) is a collection of 1- or 2-element subsets of $V(G)$. The 1-element subsets included in $E(G)$ are called *loops*. We call G a *multigraph* if $E(G)$ is a multiset, i.e. $E(G)$ contains repeated elements. The standard way to represent a vertex is by a dot, and edges are represented by curves connecting dots. In figure 1.1 the black edges $\{1, 2\}$ and $\{2, 3\}$ are 2-elements subsets of the vertex set $\{1, 2, 3, 4\}$. The red edge $\{2\}$ is a loop, and the blue edge $\{3, 4\}$ is a repeated edge. It is also possible for a vertex not to be included in any edges, in which case the vertex is called *isolated*.

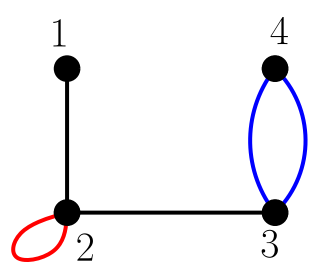

Figure 1.1: A multigraph.

An edge is said to be *incident* with all vertices that are endpoints of it. The *degree of a vertex* $v \in G$, denoted by $d_G(v)$, is the number of edges that are incident with that vertex (loops count as 2). If it is clear from the context what the graph G is, then we shorten the notation to $d(v)$.

There are many important classes of graphs. Here are a few that are studied in this work. The *complete graph on* n *vertices*, denoted K_n, is a graph with n vertices such that any pair of distinct vertices are joined by exactly one edge. We can generalize the complete graph by defining the *λ-fold complete graph on n vertices*, denoted λK_n, to be a graph on n vertices such that any pair of distinct vertices are joined by exactly λ edges. A *bipartite* graph is defined to be a graph without any cycles of odd length. Notice that a bipartite graph has the property that its vertices can be partitioned into two sets A and B such that there are no edges with both endpoints in one set in the partition. The *complete bipartite graph* with parts size n and m, denoted $K_{n,m}$, is a bipartite graph with bipartition (M, N) with $|M| = m$ and $|N| = n$ whose edge set is all edges with one end in M and one end in N. The graph that plays a central role in this work is the *k-star*, denoted S_k. A k-star is a complete bipartite graph with one part size 1 and one part size k. In figure 1.2 the blue vertex is called the *center* of the star. The other vertices are called *pendant*.

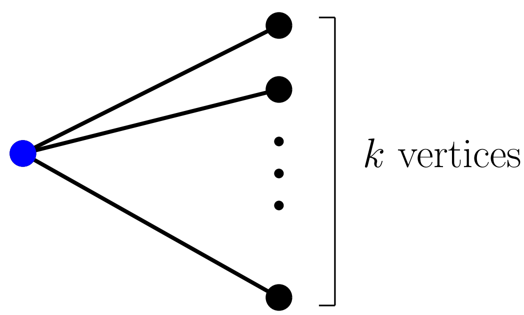

Figure 1.2: A k-star.

An *orientation* on a graph is an assignment of exactly one direction to each edge of the graph. This can be represented as an arrow on each edge (see figure 1.3). An edge that has been oriented is referred to as a *directed edge*, or sometimes called an *arc*. For a given directed edge, the vertex which the arrow is pointing away from is called the *tail* and the vertex which the arrow is pointing towards is called the *head*. A graph in which all of the edges have been oriented is called a *digraph*. Similarly to how we defined the degree of a vertex, we define the *outdegree of a vertex v in a digraph G*, denoted $d^+_G(v)$, to be the number of directed edges in G that have tail v. If the graph being referred to is clear from the context then we shorten the notation to $d^+(v)$.

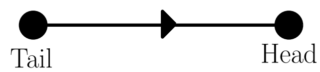

Figure 1.3: A directed edge, or arc.

A structure of central importance to the field of design theory is called a graph decomposition. Let G and H be two graphs. We define an *H-decomposition of G* to be a partition of the edges of G into copies of H. This is sometimes read as "a decomposition of G into H's". Graph decompositions have been widely studied for many different classes of graphs. In this work we will be studying problems related to decomposing complete graphs into k-stars. The problem of decomposing a complete graph into k-stars has been settled since 1979, and is stated as theorem 1.1. It should be noted that in 1975, Yamomoto et al. [13] independently published the result for $\lambda = 1$.

Theorem 1.1. *(Tarsi [11]) Let $n, k, \lambda \in \mathbb{N}$. There is a k-star decomposition of λK_n if and only if*

1. $\lambda n(n-1) \equiv 0 \pmod{2k}$
2. $n \geq 2k$ for $\lambda = 1$
3. $n \geq k+1$ for λ even
4. $n \geq k+1+\frac{k}{\lambda}$ for $\lambda \geq 3$ odd.

After characterizing the complete graphs that can be decomposed into k-stars, we next ask which complete bipartite graphs have this property. This problem was solved in much more generality by Hoffman and Liatti [7] in 1995. They actually characterized which complete bipartite graphs can be decomposed into a given (smaller) complete bipartite graph. As a k-star is a complete bipartite graph, their result solves our problem.

Theorem 1.2. *(D.G. Hoffman and Liatti [7]) Let a, b, n, m be positive integers. Let $g = \gcd(a, b)$, e, f be non-negative integers satisfying $ae - bf = g$ and let $h = ae + bf$. For each integer x, let:*

$$\alpha(x) = \left\lceil \frac{xf}{a} \right\rceil, \quad \beta(x) = \left\lfloor \frac{xe}{b} \right\rfloor, \quad \text{and } \gamma(x) = \frac{x}{ab}.$$

Then there is a $K_{a,b}$-decomposition of $K_{n,m}$ if and only if the following conditions are true:

$$ab|nm$$

$$g|n \text{ and } \alpha(n) \leq \beta(n)$$

$$g|m \text{ and } \alpha(m) \leq \beta(m)$$

$$n\alpha(m) + m\alpha(n) \leq h\gamma(nm) \leq n\beta(m) + m\beta(n)$$

The fact that theorem 1.2 can be used to characterize those complete bipartite graphs which can be decomposed into k-stars is not trivial, and requires proof. This result is presented as corollary 1.3.

Corollary 1.3. *$K_{n,m}$ has a k-star decomposition if and only if*

1. *$k|mn$, and*

2. *if $k > m$ then $k|n$ and*

   if $k > n$ then $k|m$.

Proof. We apply theorem 1.2 with $a = 1$ and $b = k$. We get $g = 1$, which yields that $e = 1$, $f = 0$ and $h = 1$. So for each integer x we have $\alpha(x) = 0$, $\beta(x) = \lfloor \frac{x}{k} \rfloor$ and $\gamma(x) = \frac{x}{k}$. The conditions are now:

$$k|nm$$

$$1|n \text{ and } 0 \leq \left\lfloor \frac{n}{k} \right\rfloor$$

$$1|m \text{ and } 0 \leq \left\lfloor \frac{m}{k} \right\rfloor$$

$$0 \leq \frac{nm}{k} \leq n \left\lfloor \frac{m}{k} \right\rfloor + m \left\lfloor \frac{n}{k} \right\rfloor$$

The first condition is the same as condition (1) as stated in the theorem. The middle two conditions are trivially satisfied, so we need to show the following:

$$\text{If } k > m \text{ then } k|n, \text{ and}$$
$$\text{if } k > n \text{ then } k|m$$
$$\Longleftrightarrow$$
$$0 \leq \frac{nm}{k} \leq n\left\lfloor \frac{m}{k} \right\rfloor + m\left\lfloor \frac{n}{k} \right\rfloor$$

To show this, we examine the latter inequality. Let $n = kp + s$ where $0 \leq s < k$, and $m = kq + r$ where $0 \leq r < k$. So we have

$$\begin{aligned} \frac{nm}{k} &\leq n\left\lfloor \frac{m}{k} \right\rfloor + m\left\lfloor \frac{n}{k} \right\rfloor \\ \frac{(kp+s)(kq+r)}{k} &\leq (kp+s)\left\lfloor \frac{kq+r}{k} \right\rfloor + (kq+r)\left\lfloor \frac{kp+s}{k} \right\rfloor \\ kpq + sq + pr + \frac{sr}{k} &\leq kpq + sq + kpq + pr \\ sr &\leq k^2pq \end{aligned}$$

But we know that $rs < k^2$, so we only require

$$\text{If either } p = 0 \text{ or } q = 0, \text{ then either } r = 0 \text{ or } s = 0$$

But $m, n > 0$ so:

If $p = 0$ then $r = 0$, and if $q = 0$ then $s = 0$, which is another way of saying: If $k > m$ then $k|n$, and if $k > n$ then $k|m$. □

Thus far, all of the results deal with decomposing graphs into k-stars. That is, all of the stars have the same size. A natural generalization is to ask when we can decompose a graph into stars of different sizes. Theorem 1.4 characterizes those complete graphs that can be decomposed into stars with prescribed sizes $m_1, m_2, \ldots, m_\ell$. This result turns out to be very useful when we study structures called maximum packings (defined in chapter

2). It is interesting to note that this wasn't published until 1996, roughly 20 years after the publication of the solution to the case where all stars have the same size.

Theorem 1.4. *(Lin and Shyu [8]) Let $m_1 \geq m_2 \geq \cdots \geq m_\ell$ be nonnegative integers with $n \geq 2m_1$. Necessary and sufficient conditions for K_n to be decomposed into stars $S_{m_1}, S_{m_2}, \ldots, S_{m_\ell}$ are*

(i) $\displaystyle\sum_{i=1}^{\ell} m_i = \binom{n}{2}$ *and*

(ii) $\displaystyle\sum_{i=1}^{p} m_i \leq \sum_{i=1}^{p}(n-i)$ *for* $p = 1, 2, \ldots, n-1$.

We now turn to the topic of computational complexity. We are interested in the decision question: Given a graph G and an integer $k > 1$, can G be decomposed into k-stars? To this end, we state the well-known result of Dor and Tarsi [2], published in 1997.

Theorem 1.5. *(Dor and Tarsi [2]) Given graphs G and H, deciding whether G has an H-decomposition is NP-complete whenever H contains a connected component with 3 or more edges.*

Therefore, our problem of deciding whether a given graph G has a k-star decomposition is NP-complete whenever $k \geq 3$. However, the complexity can be reduced given a little more information. To this end, we define the *central function* of a k-star decomposition of a graph G to be a function $c : V(G) \to \mathbb{N}$ such that for each $v \in V(G)$, $c(v)$ gives the number of k-stars centered at the vertex v. Hoffman [6] showed that the following decision question can be decided in polynomial time: Given a graph G and a central function $c : V(G) \to \mathbb{N}$, does G have a k-star decomposition with central function c ?

Given a graph G and a subset S of $V(G)$, let $\varepsilon(S)$ denote the number of edges with both ends in S.

Theorem 1.6. *(D.G. Hoffman [6]) Let $k \in \mathbb{N}$, $G = (V, E)$ be a loopless graph, and $c : V \to \mathbb{N}$. Then there is a k-star decomposition of G with central function c if and only if*

1. $k \sum_{v \in G} c(v) = \varepsilon(G)$

2. *For all* $\{x, y\} \in \binom{V}{2}$, $\mu(xy) \leq c(x) + c(y)$

3. *For all* $S \subseteq V$, $k \sum_{v \in S} c(v) \leq \varepsilon(S) + \sum_{\substack{x \in S \\ y \in V \setminus S}} \min(c(x), \mu(xy))$

where $\binom{V}{2}$ denotes the set of 2-element subsets of V, and $\mu(xy)$ denotes the number of edges with one endpoint x and the other endpoint y.

Chapter 2

Packing λK_n with k-stars

The problem of decomposing λK_n into k-stars has been solved. So if a k-star decomposition isn't possible, then how close can we get? This question gives rise to the following definition. Given graphs G and H, we define an *H-packing of G* to be a partition of the edges of G into some copies of H along with a set of edges L, called the *leave*. An H-packing is sometimes referred to as a *partial H-decomposition*. An H-packing is called *maximum* when $|L|$ is minimum, or equivalently, if the H-packing contains as many copies of H as possible. Our main question for this chapter is: Given $\lambda, n, k \in \mathbb{N}$ how many edges are in the leave of a maximum packing of λK_n with k-stars? In some cases, we will not only answer this question, but characterize what the leave graph looks like. We break it into two major cases: $\lambda = 1$ and $\lambda > 1$.

2.1 The case when $\lambda = 1$

Lemma 2.1. *Let $n, k \in \mathbb{N}$ where $n \geq 2k$. Then there are $\left\lfloor \frac{\binom{n}{2}}{k} \right\rfloor$ stars in a maximum packing of K_n with k-stars. Moreover, it is possible to have the leave graph be a star of size less than k.*

Proof. We must show that conditions (i) and (ii) hold from Theorem 1.4 with $m_1 = m_2 = \cdots = m_{\ell-1} = k$ and $m_\ell = \binom{n}{2} - k(\ell - 1)$ where $\ell = \left\lfloor \frac{\binom{n}{2}}{k} \right\rfloor + 1$. To verify (i) we see that

$$\sum_{i=1}^{\ell} m_i = \sum_{i=1}^{\ell-1} m_i + m_\ell = k(\ell - 1) + \binom{n}{2} - k(\ell - 1) = \binom{n}{2}$$

To show (ii) let $p \in \{1, \ldots, n-1\}$. First, note that $\ell \geq n-1$. To see this, suppose, to the contrary, that $\ell < n-1$. Then we have that $\sum_{i=1}^{\ell} m_i < k(n-2) + \binom{n}{2} - k(n-1) = \binom{n}{2} - k$, which contradicts (i). For $p = 1, 2, \ldots, n-1$ we have that $\ell \geq p$, and upon examining condition (ii), we need to show that

$$\sum_{i=1}^{p} m_i \leq \sum_{i=1}^{p} (n-i)$$
$$pk \leq pn - \frac{p(p+1)}{2}$$
$$\frac{p(p+1)}{2} \leq pn - pk$$

We have

$$\frac{p(p+1)}{2} \leq \frac{pn}{2} \leq p(n-k) = pn - pk$$

where the first inequality holds because $p+1 \leq n$ and the second inequality holds because $n/2 \leq n-k$. Thus, (ii) holds and we conclude that K_n can be decomposed into $\ell - 1$ k-stars and one star of size smaller than k. □

Lemma 2.1 shows that for $n \geq 2k$ a maximum packing of K_n with k-stars has less than k edges in the leave. However, it does not prove that the only configuration of these leave edges is a star, but merely finds a maximum packing with this (very desirable) property. It is not known to the author what other types of graphs the leave might be. On the contrary, lemma 2.3 exactly characterizes the leave graph of a maximum packing for the case when $n < 2k$. We must first state the results for the trivial cases.

Lemma 2.2. *Let $n, k \in \mathbb{N}$ with $n \leq k$. Then there are 0 stars in a maximum packing of K_n with k-stars. Moreover, the leave must be K_n.*

Proof. If $n \leq k$ then there are no k-stars in K_n, because a k-star has $k+1$ vertices. So a maximum packing with k-stars has no stars in it, and the leave is K_n. □

Lemma 2.3. *Let $n, k \in \mathbb{N}$ with $k < n < 2k$. There are $2n - 2k - 1$ stars in a maximum packing of K_n with k-stars. Moreover, the leave graph must be K_{2k-n+1}.*

Proof. The crucial observation we make is that when $n < 2k$, we can have at most one star centered at any given vertex. To see this, note that a star which is centered at vertex α must contain edges of the form αx for k other vertices x. Having 2 stars centered at α would require $|V(K_n) \setminus \{\alpha\}| \geq 2k$, however we know that $|V(K_n) \setminus \{\alpha\}| < 2k - 1$. Thus, there can be at most one star centered at vertex α. In figure 2.1 there are $n - 1$ total vertices on the right side of the graph.

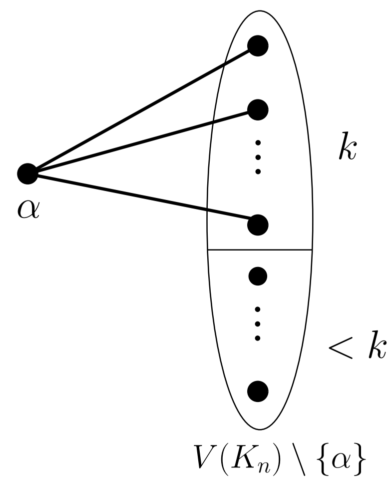

Figure 2.1: At most one star centered at each vertex.

Now, let b be the number of k-stars in a maximum packing of K_n. For notational purposes, we partition the vertices of K_n into two sets B and N, where B is the set of vertices that are the center of a star, and N is the set of vertices that are not the center of any star. It may be useful to note that $|B| = b$ and $|N| = n - b$. Let $v \in B$, then there are k edges used in the star centered at v. These edges can be of two types: either one endpoint is v and the other endpoint is in B, or one endpoint is v and the other endpoint is in N. (See figure 2.2).

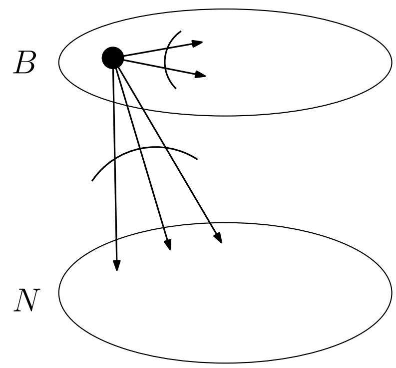

Figure 2.2: The different types of edges used in stars.

This argument holds for every $v \in B$, so we get the inequality

$$
\begin{aligned}
bk &\leq \binom{b}{2} + b(n-b) \\
b(k-n+b) &\leq \frac{b(b-1)}{2} \\
2(k-n+b) &\leq b-1 \\
b &\leq 2n-2k-1
\end{aligned}
$$

In a maximum packing we have b maximum. Therefore, if we can find a construction which achieves equality, i.e. $b = 2n-2k-1$, then we know it must be the best possible. We will find such a construction by finding an orientation of the edges within B as well as the edges with one endpoint in B and the other endpoint in N. The edges with both endpoints in N can't be used in any star, as there are no stars centered in N, so we need not orient them. In the construction, a star with center $v \in B$ will be guaranteed by $d^+(v) = k$ in the orientation. The pendant vertices of the star are the heads of the arcs with tail v.

The construction: First, take a regular tournament on B, where for each $v \in B$ we have $d^+(v) = \dfrac{b-1}{2} = n-k-1$. Note that we can always find such a regular tournament because B is a complete graph on an odd number of vertices. Also, notice that $k-n+b = k-n+(2n-2k-1) = n-k-1$, so for any given $v \in B$ we have used the least possible number of edges in B for the star centered at v. Therefore, for this star we must use all $n-b$ edges with one endpoint at v and the other endpoint in N. So the next step in the construction is to orient all of the edges with one endpoint in B and the other endpoint in N so that their tails are in B and their heads are in N.

We have that $n-b=2k-n+1$, so for each $v \in B$, $d^+(v)=(n-k-1)+(n-b)=n-k-1+2k-n+1=k$. Thus, we have exactly one star centered at each vertex in B. We have oriented all edges in K_n except those with both endpoints in N, and hence the leave is K_{2k-n+1}. □

Now that we have studied maximum packings of K_n, we can study these structures for higher values of λ.

2.2 The case when $\lambda > 1$

When $\lambda > 1$ we have a higher density of edges to deal with, but we also have considerably more freedom to center multiple stars at a vertex. While this allows for more creative constructions, it also complicates the problem of finding a maximum packing. Perhaps that is why the author finds the first main result of the section (theorem 2.5) both elegant and surprising.

2.2.1 What to do when $\lambda > 1$ and $n \geq 2k$

First we need a definition. A *multistar*, denoted $S(\lambda_1, \ldots, \lambda_n)$, is a star with n pendant vertices in which there are λ_i edges incident with the i^{th} pendant vertex. A multistar is called *edge-balanced* if $\max\limits_{i \neq j} |\lambda_i - \lambda_j| \leq 1$. The following lemma will help us to prove theorem 2.5.

Lemma 2.4. *If $n \geq k$ then the edge-balanced multistar $S(\lambda_1, \ldots, \lambda_n)$ has a maximum packing with k-stars in which the leave is a star of size less than k.*

Proof. Let $S = S(\lambda_1, \ldots, \lambda_n)$ be an edge-balanced multistar $\{v_1, \ldots, v_n\}$. The edge-balanced property implies that there can only be two distinct edge multiplicities. To justify this, assume that there are three distinct edges multiplicities. So there are three distinct integers i, j and k such that $\lambda_j < \lambda_k < \lambda_\ell$. But then $\lambda_\ell - \lambda_j > 1$, contradicting our assumption. Thus, there are only two values for the edges multiplicities, and they must be exactly one apart. So for each $1 \leq i \leq n$, $\lambda_i \in \{\lambda, \lambda+1\}$ for some $\lambda \in \mathbb{N}$. We can assume without loss of generality that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. To obtain the desired decomposition, for $0 \leq t \leq \left\lfloor \frac{|E(S)|}{k} \right\rfloor - 1$ we remove the stars with pendant vertices $v_{tk+1}, v_{tk+2}, \ldots, v_{tk+k}$ where the subscripts are considered mod n. This method of removing stars ensures that at each step, what remains is an edge-balanced multistar. Therefore, after the last star is removed we are left with an edge-balanced multistar with multiplicities 0 and 1, which is a star of size smaller than k. $\square$

Theorem 2.5. *Let $\lambda, n, k \in \mathbb{N}$ with $n \geq 2k$. Then there are $\left\lfloor \frac{\lambda\binom{n}{2}}{k} \right\rfloor$ stars in a maximum packing of λK_n with k-stars. Moreover, it is possible to have the leave graph be a star of size smaller than k.*

Proof. We first view λK_n as λ different copies of K_n, but all on the same vertex set $\{v_0, v_1, \ldots, v_{n-1}\}$. Let $b = \left\lfloor \frac{\binom{n}{2}}{k} \right\rfloor$ be the number of stars that are in a maximum packing of K_n. On each copy of K_n we apply lemma 2.1 with $m_1 = m_2 = \cdots = m_b = k$ and $m_{b+1} = \binom{n}{2} - bk$ to obtain a maximum packing with leave a star of size m_{b+1}. We denote the leave star in the i^{th} copy of K_n by S^i. It is important to note that we have the freedom to choose the center and pendant vertices of S^i in each copy of K_n. To this end, for each $0 \leq i \leq \lambda - 1$ let S^i have central vertex v_{n-1} and pendant vertices $v_{im_{b+1}}, v_{im_{b+1}+1}, \ldots, v_{(i+1)m_{b+1}-1}$ (note that the subscripts are considered mod $n-1$). (See figure 2.3). This ensures that S^{i+1} begins where S^i ends, i.e. that the leave edges are distributed over the vertices $v_0, \ldots, v_{n-2}$ as evenly as possible.

So we end up with an edge-balanced multistar centered at v_{n-1}, to which we can apply lemma 2.4 to obtain a maximum packing of with k-stars. Since the rest of the edges of λK_n are already partitioned into k-stars, the leave of the maximum packing of the edge-balanced

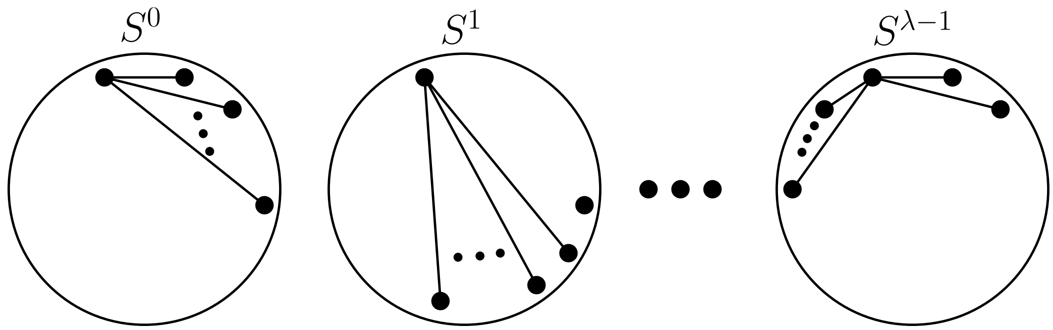

Figure 2.3: Strategic placement of the leave stars in order to build an edge-balanced multistar.

multistar is the leave of the maximum packing of λK_n. And we know that this leave is a star of size less than k; thus, we obtain the desired maximum packing. $\square$

2.2.2 Partial Results for when $\lambda > 1$ and $n < 2k$

We will first take care of the trivial cases with an analog of lemma 2.2.

Lemma 2.6. *Let $\lambda, n, k \in \mathbb{N}$ with $n \leq k$. Then there are 0 stars in a maximum packing of λK_n with k-stars. Moreover, the leave must be λK_n.*

Proof. A k-star requires $k+1$ vertices. Therefore, we have no k-stars in λK_n. So a maximum packing must have leave λK_n. $\square$

Our first partial result deals with the smallest non-trivial case, which is when there are $k+1$ vertices.

Theorem 2.7. *Let $\lambda, n, k \in \mathbb{N}$ with $\lambda > 1$ and $n = k+1$. If λ is even then λK_n can be decomposed into k-stars. If λ is odd then there are $k\left(\frac{\lambda-1}{2}\right) + \frac{\lambda+1}{2}$ stars in a maximum packing of λK_n with k-stars. Moreover, the leave graph must be K_k.*

Proof. **Case 1: λ is even.**

In this case we need only check conditions (1) and (3) from Theorem 1.1, which will guarantee a decomposition. Condition (3) is trivially satisfied since in this case $n = k+1 \geq k+1$. To show condition (1) we will substitute $n = k+1$ to obtain $\lambda n(n-1) = \lambda(k+1)k$. We have that λ is even, so $2k|\lambda(k+1)k$, and thus (1) holds.

Case 2: λ is odd.

First, notice that every k-star must contain every vertex, since $n = k+1$. For any vertex $v \in V(K_n)$, let $c(v)$ denote the number of k-stars centered at v. Now, for any two vertices, $u, v \in V(K_n)$, there are λ total edges between u and v, so $c(u) + c(v) \leq \lambda$. This leads us to note that there can be at most one vertex, $w \in V(K_n)$, such that $c(w) > \frac{\lambda}{2}$. We claim that the number of k-stars is maximum when $c(w) = \frac{\lambda+1}{2}$, and for every vertex $u \neq w$, $c(u) = \frac{\lambda-1}{2}$ (see figure 2.4).

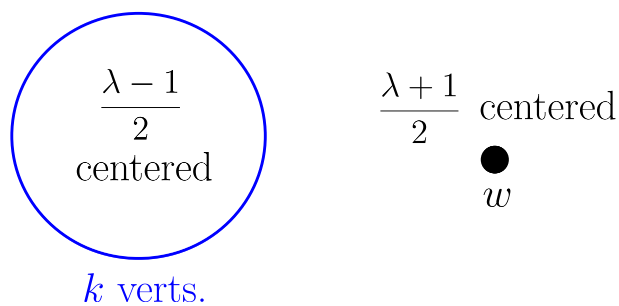

Figure 2.4: The number of k-stars centered at each vertex.

To show that this claim is true, first notice that all of the vertices in $V(K_n) \setminus \{w\}$ have the maximum number of k-stars centered at them. The only parameter that can change is $c(w)$. If $c(w)$ increases by 1, then we would have to decrease $c(u)$ by one for every $u \neq w$. But this would result in a net decrease of k-stars. So a maximum packing has the central function shown in figure 2.4. Now we must show that such a packing is possible.

The center of a star completely determines that star, since every vertex must be included in it. So we remove $\frac{\lambda+1}{2}$ stars centered at w. For each $u \in V(K_n) \setminus \{w\}$, we remove $\frac{\lambda-1}{2}$ stars centered at u. Between u and w we have used $\frac{\lambda+1}{2} + \frac{\lambda-1}{2} = \lambda$ edges. Between u and $v \in V(K_n) \setminus \{w\}$, for $u \neq v$, we have used $\frac{\lambda-1}{2} + \frac{\lambda-1}{2} = \lambda - 1$ edges. So we have exactly one edge left between any two vertices in $V(K_n) \setminus \{w\}$, and since there are exactly k

of these vertices our leave is a complete graph on k vertices. Our leave of K_k is represented as the blue vertices in figure 2.4. Thus we have obtained the desired maximum packing. $\square$

Our next partial result concerns the largest number of vertices that we can have before the more general theorems take over. This is when there are $2k-1$ vertices, but first we need a lemma.

Lemma 2.8. *Let $n, k \in \mathbb{N}$ with $n = 2k-1$. There are $\left\lfloor \frac{\binom{n}{2}}{k} \right\rfloor$ stars in a maximum packing of K_n with k-stars. Moreover, the leave graph is a single edge.*

Proof. We begin by investigating the upper bound on the number of stars in a maximum packing of K_n.

$$\binom{n}{2} = \frac{(2k-1)(2k-2)}{2} = \frac{4k^2-6k+2}{2} = 2k^2-3k+1 = k(2k-3)+1$$

The number of edges is $1 \pmod{k}$, so if we can find a construction that has $2k-3$ stars then we know it is a maximum packing. To this end, we group the vertices of K_n into two sets, A and B. We let $|A| = 2k-3$ and $|B| = 2$, recognizing that $|A \bigcup B| = 2k-1 = n$, so in fact A and B form a partition of the vertices of K_n. We wish to find a construction where each vertex of A is the center of exactly one k-star. Take a regular tournament on A, the existence of which was justified in the proof of lemma 2.3. Now for each $v \in A$ we have, from the regular tournament, that $d^+(v) = \dfrac{2k-3-1}{2} = k-2$. Next, we orient all of the edges between A and B with tails in A and heads in B. This adds 2 to the outdegree of every vertex in A, making $d^+(v) = k$. (See figure 2.5).

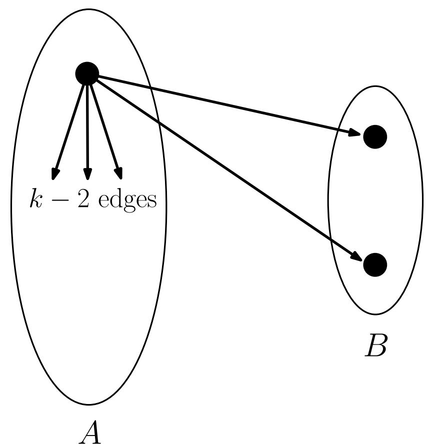

Figure 2.5: How the stars are constructed.

Every vertex in A has outdegree k so there is a star centered at each of them. The regular tournament ensures that we use every edge with both ends in A in a k-star, while the orientation on the edges between A and B ensures that we use all edges of that type. The only edges we haven't used are those with both ends in B. There is only one edge of that type, since $|B| = 2$. Thus, our leave is a single edge, and we have the desired maximum packing. □

It is important to note that in the proof of lemma 2.8, the set B is arbitrary. Thus we have control over which edge is the leave edge.

Theorem 2.9. *Let $\lambda \in \mathbb{N}$ and $k > 1$ be an integer. For $n = 2k-1$, a maximum packing of λK_n with k-stars has as its leave a loopless graph with $\lambda \pmod{k}$ edges.*

Proof. We first view λK_n as λ copies of K_n, each on the same vertex set. Now, on each copy of K_n we apply lemma 2.8 to obtain a k-star decomposition in which the leave has a single edge. We have the freedom to choose which edge becomes the leave edge. Let $\lambda = kr + s$ where $r, s \in \mathbb{N}$ and $0 \leq s \leq k-1$. In kr of the copies of K_n we will choose the leave edge to have one fixed endpoint $u \in V(K_n)$. The other endpoints of the edges will be the next k vertices in K_n. We take care to make sure that each one of these edges has multiplicity r (see figure 2.6). Clearly $r \leq \lambda$ since $k > 1$ and $s \geq 0$, so the multiplicity won't exceed λ.

This edge-balanced multistar can be trivially decomposed into r copies of S_k.

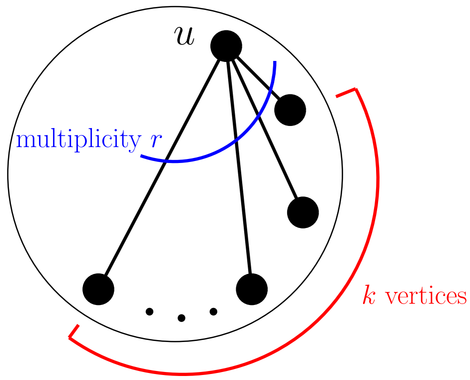

Figure 2.6: Organizing the leave edges into an edge-balanced multistar.

There are s more edges in the leave that we are free to arrange however we want, so long as they form a subgraph of λK_n. Note that $s \equiv \lambda \pmod{k}$, thus we obtain the desired leave. $\square$

Chapter 3

Embedding Partial k-star Decompositions

In this chapter we will investigate how to extend a partial decomposition to a decomposition. Given graphs G and H, let $\mathscr{A}$ be a partial H-decomposition of G – also called an H-packing of G. Let $\mathscr{B}$ be a partial H-decomposition of a graph G'. Then $\mathscr{B}$ is called an *embedding of* $\mathscr{A}$ if $\mathscr{A} \subseteq \mathscr{B}$ and G is a subgraph of G'. We are interested in the specific case when H is a k-star, and G and G' are complete graphs. We would also like to find the smallest possible graph G'. So our main question becomes: Given a partial k-star decomposition of K_n, find the smallest integer, t, in which we can embed the partial decomposition into a decomposition of K_{n+t}.

3.1 A Balanced Embedding

It was previously noted that k-stars correspond to orientations in which the center of the star is the tail of every edge (see figure 3.1). Therefore, in order to construct embeddings we will first give some results on the existence of certain types of orientations.

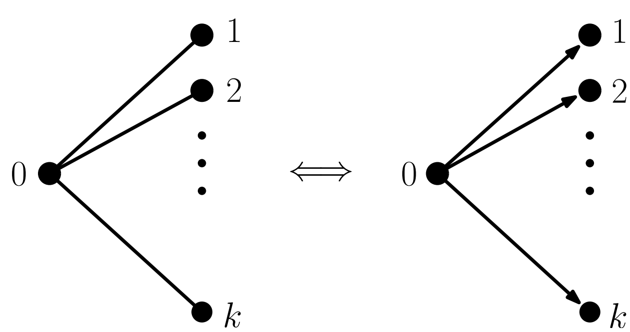

Figure 3.1: A k-star corresponds to an orientation.

Let $\mathbb{Z}$ denote the set of integers. If $f : V(G) \to \mathbb{Z}$ is a function, then let $f(S)$ denote $\sum_{v \in S} f(v)$. Recall that for any subset $S \subseteq V(G)$, $\varepsilon(S)$ denotes the number of edges with both ends in S.

We now define a well-known operation that can be performed on the edge set of a graph. Given a graph G, and some edge e of G, the graph $G \backslash e$ is the graph with vertex set $V(G)$ and edge set $E(G) \setminus \{e\}$. This operation is called "deletion by e," and the new graph obtained is referred to as "G delete e".

Lemma 3.1. *(D.G. Hoffman [5]) Let G be any graph and $f : V(G) \to \mathbb{Z}$. Then G has an orientation in which each vertex $v \in V(G)$ has outdegree $f(v)$ if and only if for every $S \subseteq V(G)$,*

$$\varepsilon(S) \leq f(S)$$

with equality if $S = V(G)$.

Proof. First we will discuss the necessity. Let $S \subseteq V(G)$ and f be defined as stated above. Every edge with both ends in S must be oriented, which means that each such edge contributes exactly 1 to the outdegree of some vertex in S. So the largest possible number of edges with both ends in S which can be oriented is $f(S)$. Therefore, the inequality holds. As for the case when $S = V(G)$, we must have all of the edges in G oriented, which means $f(S) \leq \varepsilon(S)$, and equality follows.

Next, we turn to sufficiency. Suppose the conditions on f and G hold. Let $v \in V(G)$, then we note that $\varepsilon(\{v\})$ is the number of loops at v. So $0 \leq \varepsilon(\{v\}) \leq f(v)$. We will continue by induction on $\varepsilon(G)$. If $\varepsilon(G) = 0$, then $f(G) = 0$. So $f(v) = 0$ for every $v \in V(G)$, thus there are no edges to orient and we are done. Let $e \in E(G)$ have endpoints u and v.

Case 1: $u = v$, i.e. e is a loop. Let $G' = G \setminus e$ and define $f' : V(G') \to \mathbb{N}$ by

$$f'(x) = \begin{cases} f(v) - 1 & \text{if } x = v \\ f(x) & \text{if } x \neq v. \end{cases}$$

So we get that for every $S \subseteq V(G)$

$$\sum_{x \in S} f'(x) = \begin{cases} f(S) - 1 & \text{if } v \in S \\ f(S) & \text{if } v \notin S \end{cases} \quad \text{and} \quad \varepsilon'(S) = \begin{cases} \varepsilon(S) - 1 & \text{if } v \in S \\ \varepsilon(S) & \text{if } v \notin S \end{cases}$$

where $\varepsilon'(S)$ denotes the number of edges in G' with both ends in S. So f' satisfies the hypotheses for G' which means that G' has an orientation with $d^+_{G'}(v) = f'(v)$ for every $v \in G'$. Now, orient e the only way possible, which produces the desired orientation on G.

Case 2: $u \neq v$, i.e. e is not a loop. For $A \subseteq V(G)$, we define A to be *critical* if $\varepsilon(A) = f(A)$.

Claim 3.2. *Let A and B be critical, then*

(I) $A \bigcap B$ is critical,

(II) $A \bigcup B$ is critical, and

(III) No edge of G has one end in $A \setminus B$ and the other end in $B \setminus A$.

Proof of claim: Assume A and B are critical. We make the following definitions, $a := f(A \setminus B)$, $b := f(B \setminus A)$, $c := f(A \bigcap B)$, α_1 is the number of edges with both ends in $A \setminus B$, α_2 is the number of edges with one end in $A \setminus B$ and one end in $A \bigcap B$, β_1 is the number of edges with both ends in B, β_2 is the number of edges with one end in $B \setminus A$ and one end in $A \bigcap B$, γ is the number of edges with both ends in $A \bigcap B$, and δ is the number of edges with one end in $A \setminus B$ and one end in $B \setminus A$. (See figure 3.2). We get the following equations

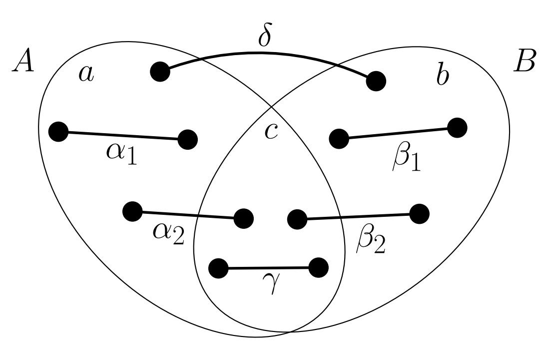

Figure 3.2: Different types of edges.

and inequalities since f satisfies the hypotheses of the lemma and A and B are critical

$$\alpha_1 \leq a \tag{EQ1}$$

$$\beta_1 \leq b \tag{EQ2}$$

$$\gamma \leq c \tag{EQ3}$$

$$\alpha_1 + \alpha_2 + \gamma = a + c \tag{EQ4}$$

$$\beta_1 + \beta_2 + \gamma = b + c \tag{EQ5}$$

$$\alpha_1 + \alpha_2 + \beta_1 + \beta_2 + \gamma + \delta \leq a + b + c \tag{EQ6}$$

From (EQ4) and (EQ5) we get $\alpha_1 + \alpha_2 = a + c - \gamma$ and $\beta_1 + \beta_2 = b + c + \gamma$, which we can substitute into (EQ6) to get

$$\begin{aligned} \alpha_1 + \alpha_2 + \beta_1 + \beta_2 + \gamma + \delta &\leq a + b + c \\ a + c - \gamma + b + c - \gamma + \gamma + \delta &\leq a + b + c \\ c - \gamma + \delta &\leq 0 \\ c + \delta &\leq \gamma \qquad \text{(EQ7)} \end{aligned}$$

But δ counts some number of edges, which forces $\delta \geq 0$. Combining this fact with (EQ3) and (EQ7) we obtain

$$c \leq c + \delta \leq \gamma \leq c.$$

So it must be the case that both $\gamma = c$ and $\delta = 0$. These two equations prove (I) and (III) from Claim 3.2, respectively. We also have that $\alpha_1+\alpha_2+\beta_1+\beta_2+\gamma = a+c+b+c-\gamma = a+b+c$, and thus (II) is true. Therefore, we have proven Claim 3.2.

Back to the proof of Case 2. Again, let $G' = G \setminus e$. Try orienting e with tail u and head v.

Define f' on $V(G')$ as follows:

$$f'(x) = \begin{cases} f(u) - 1 & \text{if } x = u \\ f(x) & \text{if } x \neq u. \end{cases}$$

Let $S \subseteq V(G)$, and we are interested in when the inequality $\varepsilon'(S) \leq f'(S)$ is true. Notice that $\varepsilon(S) \leq f(S)$, and both

$$\varepsilon'(S) = \begin{cases} \varepsilon(S) - 1 & \text{if } u, v \in S \\ \varepsilon(S) & \text{otherwise} \end{cases} \quad \text{and} \quad f'(S) = \begin{cases} f(S) - 1 & \text{if } u \in S \\ f(S) & \text{if } u \notin S. \end{cases}$$

The only situation in which $\varepsilon'(S) > f'(S)$ is when $u \in S$, $v \notin S$, and S is critical in G. So if our choice of orienting e with tail u and head v results in failure of our desired inequality, then we find a critical set S in G which contains u and not v. The argument works symmetrically if we choose to orient e with tail v and head u. In that case we find a critical set T in G which contains v and not u. It can't be the case that both choices fail, because that would result in two critical sets S and T in which there is an edge with one end, u, in $S \setminus T$ and one end, v, in $T \setminus S$. Thus, we can find an orientation of e which results in $\varepsilon'(S) \leq f'(S)$. Using our inductive hypothesis we get an orientation on G', and then by adding e back in – oriented appropriately – we find our desired orientation on G. $\square$

We now turn our attention towards complete graphs. Lemma 3.3 gives conditions for a complete graph on $n + t$ vertices with a hole of size n to obtain an orientation in which the vertices have prescribed outdegrees (see figure 3.3). This will allow us to embed partial k-star decompositions by considering them as being removed from a complete graph, thereby creating a hole. The function d will serve as the "patch" for the hole.

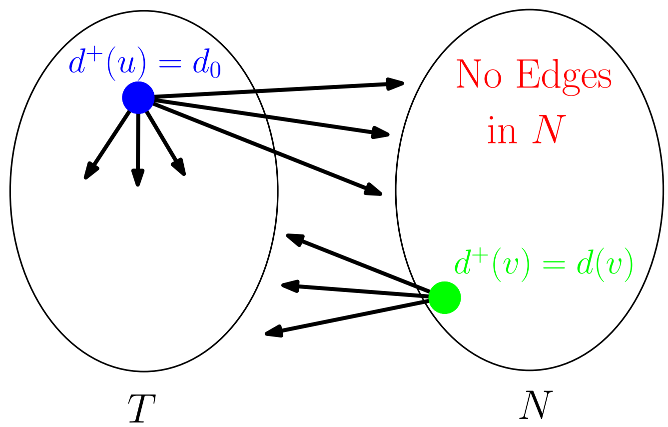

Figure 3.3: The orientation guaranteed by Lemma 3.3.

Lemma 3.3. *Let G be a graph whose vertex set is partitioned into sets N and T, and a pair of distinct vertices in G are adjacent if and only if they are not both in N. Let $|T| = t$ and $|N| = n$, $d : N \to \mathbb{N}$ and $d_0 \in \mathbb{N}$. Then G has an orientation in which each vertex $v \in N$ has outdegree $d(v)$ and each vertex in T has outdegree d_0 if and only if*

(i) $\forall v \in N\ d(v) \le t$, *and*

(ii) $td_0 + \sum_{v \in N} d(v) = \binom{t}{2} + tn.$

Proof. Assume G has such an orientation. To show (i) let $v \in N$. The only edges which can be oriented outwards from v are of the form vx where $x \in T$. There are only t such vertices in T, so $d(v) \le t$. To show (ii) we note that the LHS counts the total number of edges in G by counting all of the outdegrees of the vertices, while the RHS counts the total number of edges in G because we have a complete graph on t vertices along with a complete bipartite graph with parts of size t and n.

Now assume that (i) and (ii) hold. We will apply lemma 3.1 with

$$f(v) = \begin{cases} d(v) & \text{for } v \in N \\ d_0 & \text{for } v \in T. \end{cases}$$

Let $S \subseteq V(G)$. Then $S = I \bigcup J$ where $I = S \bigcap T$ and $J = S \bigcap N$, and we will say that $|I| = i$ and $|J| = j$. We have that $\varepsilon(S) = \binom{i}{2} + ij$ and $\sum_{v \in S} f(v) = d_0 i + d(J)$, so we get the following:

$$\begin{aligned} \varepsilon(S) \leq \sum_{v \in S} f(v) &\iff \binom{i}{2} + ij \leq d_0 i + d(J) \\ &\iff 0 \leq -\frac{1}{2} i^2 + (d_0 + j - \frac{1}{2}) i + d(J) \end{aligned} \tag{A}$$

The last inequality involves a concave-down quadratic in i, so we need only check the endpoints. When $i = 0$ we get $\binom{0}{2} + 0 \leq d(J)$, which is true. When $i = t$ we first note that $td_0 + d(J) = tn + \binom{t}{2} - \sum_{v \in N \setminus J} d(v)$. And, to show that (A) holds, we must show that

$$\begin{aligned} &\binom{t}{2} + tj \leq d_0 t + d(J) \\ &\binom{t}{2} + \sum_{v \in J} t \leq tn + \binom{t}{2} - \sum_{v \in N \setminus J} d(v) \\ &\sum_{v \in J} t + \sum_{v \in N \setminus J} d(v) \leq tn \end{aligned}$$

and we have that

$$\sum_{v \in J} t + \sum_{v \in N \setminus J} d(v) \leq \sum_{v \in J} t + \sum_{v \in N \setminus J} t = \sum_{v \in N} t = tn$$

The above line shows that (A) is true. Now we must show that equality holds when $S = V(G)$, in which case we get

$$\varepsilon(S) = \binom{t}{2} + tn \underbrace{=}_{by\text{(ii)}} td_0 + \sum_{v \in N} d(v) = \sum_{v \in S} f(v)$$

$\square$

The next theorem is our first result on embeddings. The type of decomposition constructed is sometimes referred to as "balanced" because the number of k-stars centered at each vertex is the same. We will utilize the tools we have developed for orientations to prove it.

Theorem 3.4. *Given a partial k-star decomposition of K_n and some $d_0 \in \mathbb{N}$ such that $d_0 \geq \frac{n-1}{k}$, the partial decomposition can be embedded into a k-star decomposition of K_{n+t} where $t = 2kd_0 - n + 1$ and every vertex has d_0 k-stars centered at it.*

Proof. Let $c : V(K_n) \to \mathbb{N}$ be the central function of the partial k-star decomposition of K_n. We must choose t such that K_{n+t} has a k-star decomposition. The number of edges in K_{n+t} must equal the sum of the edges used in all of the stars, and we obtain the following:

$$\begin{aligned} \binom{n+t}{2} &= (n+t)kd_0 \\ \frac{(n+t)(n+t-1)}{2} &= (n+t)kd_0 \\ n+t-1 &= 2kd_0 \\ t &= 2kd_0 + 1 - n \end{aligned}$$

Let $V(K_n) = \{v_1, \ldots, v_n\}$. Notice that $\forall v \in K_n$ the maximum number of k-stars that can be centered at v is $\lfloor \frac{n-1}{k} \rfloor$, so we can assume that the partial decomposition of K_n is maximal, i.e. the number of edges incident with v which are not used in the partial k-star decomposition is less than k. Take an arbitrary orientation of the edges in the compliment of the partial k-star decomposition of K_n. Let $f : V(K_n) \to \mathbb{N}$ be defined such that $f(v)$ equals the outdegree of v in this orientation. Notice that $\forall v \in V(K_n)$ $f(v) < k$, because the partial decomposition is maximal.

Let G be the graph with vertex partition $N = \{v_1, \ldots, v_n\}$ and $T = \{v_{n+1}, \ldots, v_{n+t}\}$, and two vertices are adjacent in G if and only if they are not both in N. An orientation of G

such that every vertex in N has outdegree $k(d_0 - c(v)) - f(v)$ and every vertex in T has outdegree kd_0 corresponds to an orientation of K_{n+t} such that every vertex has outdegree kd_0. Thus, finding such an orientation implies embedding our partial decomposition of K_n into a decomposition of K_{n+t}. We must apply lemma 3.3 with $d(v) = k(d_0 - c(v)) - f(v)$ and the outdegree in T being kd_0. To show (i), note that $d(v)$ is maximum when $kc(v) + f(v)$ is minimum, and this value might be 0. Thus, $d(v) \leq kd_0$. Now, by assumption we have:

$$\begin{aligned}
\frac{n-1}{k} &\leq d_0 \\
0 &\leq kd_0 + 1 - n \\
kd_0 &\leq 2kd_0 + 1 - n
\end{aligned}$$

Hence, $d(v) \leq kd_0 \leq 2kd_0 + 1 - n = t$ and (i) is true. To show (ii):

$$\begin{aligned}
tkd_0 + \sum_{v \in N} d(v) &= tkd_0 + \sum_{v \in N} k(d_0 - c(v)) - f(v) \\
&= tkd_0 + k\sum_{v \in N} d_0 - \sum_{v \in N} (kc(v) + f(v)) \\
&= tkd_0 + nkd_0 - \binom{n}{2} \\
&= \binom{n+t}{2} - \binom{n}{2} \\
&= \binom{t}{2} + tn
\end{aligned}$$

Hence, (ii) is true. Our orientation is guaranteed by lemma 3.3. $\square$

3.2 A Small Embedding

The main goal of this section is to produce an embedding of a partial k-star decomposition in which the number of new vertices depends only on k. We would also like the number of new vertices to be as small as possible. To this end, we make the following definition. Let G and H be graphs. If G and H have disjoint vertex sets, then let $G \vee H$ denote the graph with vertex set $V(G) \bigcup V(H)$ and edge set $E(G) \bigcup E(H) \bigcup \{uv : u \in V(G) \text{ and } v \in V(H)\}$. The first lemma is an embedding of a partial system into *another partial system*. While it is not the main result that we desire, it is a step in the right direction because the number of new vertices depends only on k. The construction also shrinks the leave down into a manageable number of vertices.

Lemma 3.5. *Any partial k-star decomposition of K_n can be embedded into a partial k-star decomposition of K_{n+2k-1} in such a way that the new leave is restricted to the $2k-1$ new vertices.*

Proof. Let $\mathscr{B}$ be a partial k-star decomposition of K_n with leave L. Let T be a complete graph on $2k-1$ vertices, and let $G = K_n \vee T$. Our goal is to build another collection, $\mathscr{B}'$, of k-stars in G that can be put together with $\mathscr{B}$ to form our embedding. We can assume that $\mathscr{B}$ is maximal, i.e. that every degree in L is at most $k-1$. (If there were some vertex $v \in V(L)$ with degree k or more, then we could center another k-star at v and include that star in the collection $\mathscr{B}'$.) Now, place an arbitrary orientation on the edges of L. For each $v \in L$, use $k - d_L(v)$ edges going from K_n to T with tails at v to form a k-star, and include this star in $\mathscr{B}'$. (See figure 3.4.) Note that this is possible because $d_L(v) \leq k-1$ and $|V(T)| = 2k-1$.

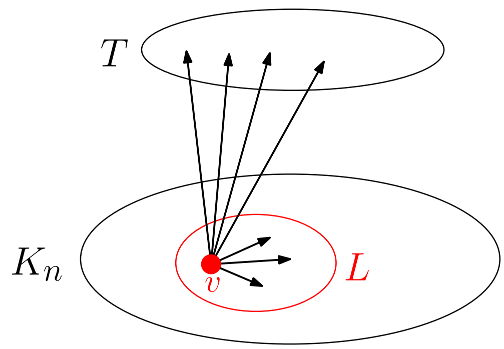

Figure 3.4: Extending the orientation of the leave to k-stars.

Now, let $w \in T$. For the rest of the edges with one end at w and the other end in K_n that haven't been used, orient them with tail at w. Form as many k-stars as possible with these edges, and include these edges in $\mathscr{B}'$. We will have some number $i < k$ of them left over. (See figure 3.5.)

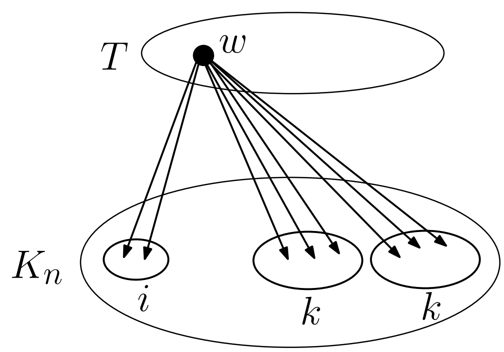

Figure 3.5: Using almost all edges between K_n and T.

At this point we have $d^+(w) = i < k$. We also have the edges with both ends in T yet to use in k-stars. To this end, take a regular tournament on T. Notice that now $d^+(w) = i + \dfrac{2k-2}{2} = i + k - 1$. We will use as many of these edges as possible in k-stars and include them in $\mathscr{B}'$. If $i \geq 1$ then we can center at least one more k-star at w. In this case we will have used all the edges with one end in K_n and the other end in T. Otherwise, $i = 0$ and all of these edges were used in the previous steps. We now have the partial k-star decomposition $\mathscr{B} \bigcup \mathscr{B}'$ of $K_n \vee T$, whose leave is a subset of T. $\square$

The following is a generalization of theorem 3.4, but we tailor it to make it useful for theorem 3.9. The major difference is that lemma 3.6 does not require that we begin with a partial k-star decomposition, but only that our number of edges is a multiple of k. Let $\mathbb{Z}^+$ denote the set of positive integers.

Lemma 3.6. *Fix $n, k \in \mathbb{Z}^+$. Let N be a complete graph on n vertices with a subgraph L which has the property that k divides $|E(N) \setminus E(L)|$. Let $f : V(N) \to \mathbb{N}$ and $g : V(L) \to \mathbb{N}$ be functions such that $\sum_{v \in N} kf(v) + g(v) = \binom{n}{2}$. For each $d_0 \in \mathbb{N}$ such that*

(i) *For every $v \in V(N)$ $kf(v) + g(v) \leq kd_0$, and*

(ii) *For every $v \in V(N)$ $n - 1 - kf(v) - g(v) \leq kd_0$*

then $K_t \vee N$ has a k-star decomposition where $t = 2kd_0 - n + 1$ and every vertex has d_0 stars centered at it.

Proof. Let G be the graph with vertex set $V(K_t) \bigcup V(N)$ and edge set $E(K_t \vee N) \setminus E(N)$. For every $v \in N$, define the function $d : V(N) \to \mathbb{N}$ such that $d(v) = d_0 k - kf(v) - g(v)$. Note that from condition (i) we obtain that for every $v \in N$

$$0 \leq kd_0 - kf(v) - g(v) = d(v)$$

Thus, $d(v)$ is non-negative. Notice that a k-star decomposition of $K_t \vee N$ in which every vertex has d_0 stars centered at it corresponds to an orientation of G in which every vertex in $V(K_t)$ has outdegree kd_0 and every vertex in $V(N)$ has outdegree $d(v)$.

We must choose t with the property that $K_t \vee N$ has a k-star decomposition, so the number of edges must be divisible by k. Thus, we obtain that

$$\begin{aligned} \binom{n+t}{2} &= (n+t)kd_0 \\ \frac{(n+t)(n+t-1)}{2} &= (n+t)kd_0 \\ n+t-1 &= 2kd_0 \\ t &= 2kd_0 + 1 - n \end{aligned}$$

By adding together the inequalities of conditions (i) and (ii) we get

$$\begin{gathered} kf(v) + g(v) + n - 1 - kf(v) - g(v) \leq kd_0 + kd_0 \\ 0 \leq 2kd_0 - n + 1 = t \end{gathered}$$

Thus, t is non-negative. Now we must show that d satisfies the conditions of lemma 3.3. By applying condition (ii) we see that for every $v \in G$

$$\begin{gathered} n - 1 - kf(v) - g(v) \leq kd_0 \\ -kf(v) - g(v) \leq kd_0 + n - 1 \\ kd_0 - kf(v) - g(v) \leq 2kd_0 + n - 1 \\ d(v) \leq t \end{gathered}$$

To see the condition on the total number of edges:

$$\begin{aligned}
tkd_0 + \sum_{v\in N} d(v) &= tkd_0 + \sum_{v\in N}(kd_0 - kf(v) - g(v)) \\
&= tkd_0 + nkd_0 - \sum_{v\in N}(kf(v) + g(v)) \\
&= (n+t)kd_0 - \binom{n}{2} \\
&= \binom{n+t}{2} - \binom{n}{2} \\
&= \binom{t}{2} + tn
\end{aligned}$$

Therefore, both conditions of lemma 3.3 are satisfied and G has the desired orientation. $\square$

For the rest of this chapter we will use the following notation:

$$k^* := \begin{cases} k & \text{if } k \text{ is odd} \\ 2k & \text{if } k \text{ is even} \end{cases}$$

We now state some results on the periodic nature of $\binom{n}{2} \pmod{k}$. This will assist us with obtaining a small embedding.

Lemma 3.7. *For every $n \in \mathbb{Z}^+$, the equivalence $\binom{n}{2} \equiv \binom{n+p}{2} \pmod{k}$ has smallest non-negative integer solution $p = k^*$.*

Proof. We first expand each side of the equivalence to obtain

$$\frac{n(n-1)}{2} \equiv \frac{(n+p)(n+p-1)}{2} \pmod{k}$$

$$n(n-1) \equiv (n+p)(n+p-1) \pmod{2k}$$

$$n^2 - n \equiv n^2 + 2np + p^2 - n - p \pmod{2k}$$

$$0 \equiv 2np + p^2 - p \pmod{2k}$$

So for all $n \in \mathbb{Z}^+$ we require that $2k | p(p + 2n - 1)$, which implies that

$$2k | p \gcd\{p + 2n - 1 : n \in \mathbb{Z}^+\}$$

To find the gcd we notice that $\{p + 2n - 1 : n \in \mathbb{Z}^+\} = \{p + 1, p + 3, p + 5, \ldots\}$. So $\gcd\{p + 2n - 1 : n \in \mathbb{Z}^+\} = \gcd\{p + 1, p + 3, p + 5, \ldots\} = \begin{cases} 2 & \text{if } p \text{ is odd} \\ 1 & \text{if } p \text{ is even} \end{cases}$

Thus, we require that

$$2k \text{ divides } \begin{cases} 2p & \text{if } p \text{ is odd} \\ p & \text{if } p \text{ is even} \end{cases}$$

If p is even, then the smallest value is $p = 2k$. If p is odd, then there is no solution for k even, but if k is odd then the smallest value is $p = k$.

□

Corollary 3.8. *If* $x \equiv n - 1 \pmod{k^*}$ *then* $\binom{x}{2} \equiv \binom{n + 2k - 1}{2} \pmod{k}$

Proof. Assume that $x \equiv n - 1 \pmod{k^*}$.
We have from Lemma 3.7 that $\binom{n-1}{2} \pmod{k}$ is periodic with period k^*. So, no matter the parity of k, we have

$$\binom{x}{2} \pmod{k} \equiv \binom{n-1}{2} \pmod{k} \equiv \binom{n - 1 + 2k}{2} \pmod{k}.$$

□

Recall that we defined $k^* = \begin{cases} k & \text{if } k \text{ is odd} \\ 2k & \text{if } k \text{ is even} \end{cases}$.

Theorem 3.9. *A partial k-star decomposition of K_n can be embedded into a k-star decomposition of K_{n+s} where $s \leq 6k + k^* - 4$*

Proof. Let N be a complete graph on n vertices with a partial k-star decomposition $\mathcal{A}$. So $\mathcal{A}$ is a set of edge-disjoint copies of k-stars in N.

Step 1: Apply lemma 3.5 to embed $\mathcal{A}$ into a partial k-star decomposition, $\mathcal{B}$, of $N \vee M$ where M is a complete graph on $2k-1$ vertices. Note that $\mathcal{B}$ is guaranteed to have the property that its leave is contained in M.

Step 2: Choose a set, X, of x vertices from $V(N \vee M)$ with the following properties:

(a) $V(M) \subseteq X$

(b) $x \equiv n-1 \pmod{k^*}$

(c) x is minimum with respect to the above properties.

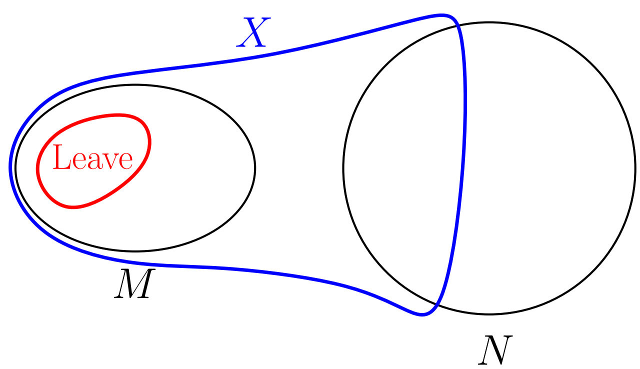

Figure 3.6: Choosing the set, X, of vertices.

Notice that by applying Corollary 3.8 to condition (b) we have that $\binom{x}{2} \equiv \binom{n+2k-1}{2} \pmod{k}$. Let $\overline{\mathcal{B}}$ denote the edges which are not used in copies of k-stars in $\mathcal{B}$. We also have that $\overline{\mathcal{B}} \subseteq E(M) \subseteq E(X)$, which means that

$$\binom{x}{2} - |\overline{\mathcal{B}}| \equiv \binom{n+2k-1}{2} - |\overline{\mathcal{B}}| \pmod{k} \iff$$
$$\binom{x}{2} \equiv \binom{n+2k-1}{2} \pmod{k}.$$

Since $\mathcal{B}$ is a partial k-star decomposition we have that $\binom{n+2k-1}{2} - |\overline{\mathcal{B}}| \equiv 0 \pmod{k}$. So condition (b) guarantees that $\binom{x}{2} - |\overline{\mathcal{B}}| \equiv 0 \pmod{k}$.

Step 3: We will embed the partial k-star decomposition on X into a k-star decomposition of a complete graph. Let L denote the leave. Let $g : V(L) \to \mathbb{N}$ represent the outdegrees of an arbitrary orientation on $E(L)$. Since we have condition (b), we can define a function $f : V(X) \to \mathbb{N}$ such that $\sum_{v \in X} kf(v) = |E(X) \setminus E(L)|$ and for each $v \in X$ we have $kf(v) + g(v) \leq x - 1$. In order to apply lemma 3.6 with $d_0 \geq \frac{x-1}{k}$ we note that for every $v \in V(X)$

$$kf(v) + g(v) \leq x - 1 \leq \frac{x-1}{k}k \leq d_0 k,$$

which implies that

$$x - 1 - kf(v) - g(v) \leq x - 1 \leq d_0 k.$$

Thus, the partial k-star decomposition on X can be embedded into a k-star decomposition on $T \vee X$ where T is a complete graph on t vertices.

Step 4: Finish the decomposition. The only thing we have left to show is that we can decompose the complete bipartite graph whose vertices come from T and $V(N) \setminus V(X)$, which we will call Y, into k-stars.

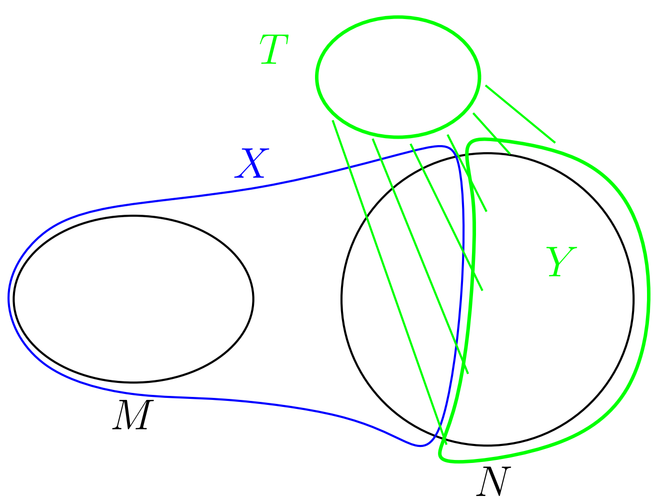

Figure 3.7: A k-star decomposition of the remaining bipartite graph with partition (T,Y).

Let $y = |V(N) \bigcap V(X)|$, and we have that $x = y + 2k - 1$. From corollary 1.3 we see that this complete bipartite graph has a decomposition into k-stars if and only if the following two conditions hold:

$$k|t(n-y)$$

and

$$\text{If } k > (n-y) \text{ then } k|t, \text{ and if } k > t \text{ then } k|(n-y)$$

To show the first condition, we note $n - y = n - x + 2k - 1 = (n-1) - x + 2k$, and recall that $x \equiv n - 1 \pmod{2k}$. Therefore, $n - 1 - x \equiv 0 \pmod{2k}$. Hence, $k|(n-y)$, which implies that $k|t(n-y)$. To show the second condition, we first note that if $k > t$ then we're done, because $k|(n-y)$. Also, due also to the fact that $k|(n-y)$, it can't be the case that $k > (n-y)$ so the second part of the second condition is trivially satisfied. Thus, our complete bipartite graph can be decomposed into k-stars, and this finishes the embedding.

Step 5: Count the total number of vertices. The smallest value for t is achieved when $d_0 = \lceil \frac{x-1}{k} \rceil$, so

$$t = 2kd_0 - x + 1 \leq 2k(\frac{x-1+k}{k}) - x + 1 = 2k + x - 1.$$

Also, from condition (b) we have

$$x \leq 2k - 1 + k^* - 1$$

which means that

$$t \leq 4k + k^* - 3$$

We have $n + 2k - 1 + t$ total vertices, and

$$n + 2k - 1 + t \leq n + 6k + k^* - 4$$

□

Chapter 4

Hyperstar Decompositions of Hypergraphs

4.1 What is a hyperstar?

The results in this section are all joint work with M. A. Bahmanian [1]. They have previously been obtained by Lonc [10], but we give an alternative proof.

In order to define what a hyperstar is we must first define a hypergraph. A *hypergraph* is a pair $(X, \mathcal{E})$ where X is a set and $\mathcal{E}$ is a collection of subsets of X. We refer to the elements of X as *vertices* and the elements of $\mathcal{E}$ as *hyperedges*. Note that a graph is, in fact, a hypergraph where the size of each edge is 2 – allowing that loops be considered as multisets. A multigraph is also a hypergraph where $\mathcal{E}$ is considered as a multiset. A hypergraph can also have multi-hyperedges, which means that a hyperedge can be repeated multiple times. Figure 4.1 is a hypergraph with vertex set $\{1, 2, 3, 4, 5, 6, 7, 8\}$, with a repeated edge $\{1, 2, 3\}$, and a loop $\{8\}$.

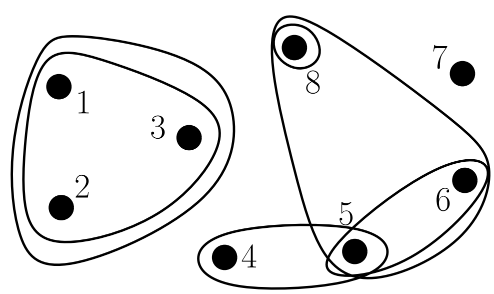

Figure 4.1: A hypergraph.

We can now define a hyperstar, which is a generalization of a star. A hypergraph $G = (X, \mathcal{E})$ is a *hyperstar* with center C if $C \subseteq \bigcap_{E \in \mathcal{E}} E$. The *size* of G is $|\mathcal{E}|$ and we say that G has *center* *size* $|C|$. A hyperstar of size k with center C is denoted by $\mathcal{S}_k(C)$. For brevity, $\mathcal{S}_k(v)$ denotes $\mathcal{S}_k(\{v\})$, and $\mathcal{S}_k$ denotes a hyperstar of size k and center size 1. A hyperstar can have more than one center, for example the hyperstar in figure 4.2 has centers $\{3,4\}$, $\{3\}$, $\{4\}$ and $\varnothing$. An interesting distinction between stars and hyperstars is that the edges of a hyperstar can intersect outside of the center. The edges $\{3,4,5,6\}$ and $\{3,4,6,7\}$ in the hyperstar in figure 4.2 have the vertex 6 in their intersection, but 6 is not in any center of the hyperstar.

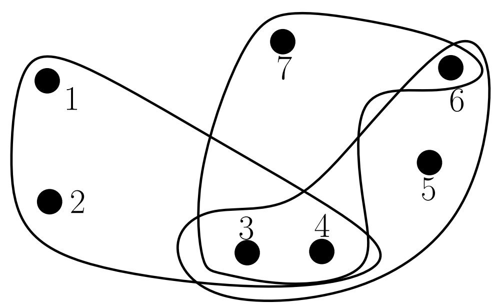

Figure 4.2: A hyperstar of size 3 with center size 2.

We can define a decomposition of a hypergraph much like how we defined it for a graph. Let $G = (X, \mathcal{E})$ be a hypergraph and let $\mathcal{H} = \{H_i\}_{i \in I}$ be a family of hypergraphs where $H_i = (X_i, \mathcal{E}_i)$. We say that G has an $\mathcal{H}$-*decomposition* if $\{\mathcal{E}_i : i \in I\}$ partitions $\mathcal{E}$ and each $\mathcal{E}_i$ forms an isomorphic copy of H_i.

For a positive integer r and a set Y, $\mathscr{P}(Y)$ denotes the set of all subsets of Y and $\mathscr{P}_r(Y)$ denotes the set of all r-subsets of Y. Let $G = (X, \mathcal{E})$ be a hypergraph, and let $\mathscr{C} \subseteq \bigcup_{E \in \mathcal{E}} \mathscr{P}(E)$. For every $T \subseteq \mathscr{C}$ we define the set

$$T(\mathcal{E}, \mathscr{C}) = \{E \in \mathcal{E} : \mathscr{P}(E) \bigcap \mathscr{C} \subseteq T\}.$$

The collection $\mathscr{C}$ will play the role of the set of centers of hyperstars in the desired hyperstar decomposition. For each $T \subseteq \mathscr{C}$ the relationship of the set $T(\mathcal{E}, \mathscr{C})$ with the hyperstar decomposition is a bit more subtle. Therefore, we will examine this set via an example.

(See example 4.1). For any given $T \subseteq \mathscr{C}$, the set $T(\mathcal{E}, \mathscr{C})$ is the set of all hyperedges whose intersection with $\mathscr{C}$ is a subset of T – this includes hyperedges whose intersection with $\mathscr{C}$ is empty, as the empty set is always a subset of T.

Example 4.1. *Examining* $T(\mathcal{E}, \mathscr{C})$ *where* $G = K_6^3$, $\mathscr{C} = \{\{1\}, \{2\}, \{3, 4\}\}$, *and* $T = \{\{2\}, \{3, 4\}\}$.

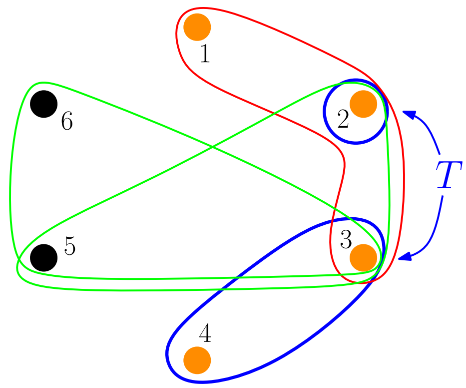

Figure 4.3: Determining whether a hyperedge is an element of $T(\mathcal{E}, \mathscr{C})$ or not.

The orange vertices are the vertices covered by $\mathscr{C}$. The way we can determine if a hyperedge is in $T(\mathcal{E}, \mathscr{C})$ or not is by looking at the intersection of $\mathscr{P}(E)$ with $\mathscr{C}$. This intersection will consist of subsets of vertices, and if these are all elements of T (blue subsets) then the edge is in $T(\mathcal{E}, \mathscr{C})$. In figure 4.3 the green edges are in $T(\mathcal{E}, \mathscr{C})$ and the red edges are not. This information is presented in table 4.1.

E	$\mathscr{P}(E) \bigcap \mathscr{C}$	Element of $T(\mathcal{E}, \mathscr{C})$?
$\{1, 2, 3\}$	$\{\{1\}, \{2\}\}$	no
$\{2, 3, 5\}$	$\{\{2\}\}$	yes
$\{3, 5, 6\}$	$\varnothing$	yes

Table 4.1: Categorizing the edges of figure 4.3.

This concludes the example.

Now we turn back to hyperstar decompositions. We will need the following theorem which gives necessary and sufficient conditions for a hypergraph to be decomposed into hyperstars with given sizes and centers. This is the result in which the set $T(\mathcal{E},\mathscr{C})$ realizes its full potential.

Theorem 4.2. *(Lonc [9]) Let $G=(V,\mathcal{E})$ be a hypergraph. Let $\mathscr{C} \subseteq \bigcup_{E\in\mathcal{E}} \mathscr{P}(E)$, and $\delta : \mathscr{C} \to \mathbb{Z}^+$ be a function. Then G has an $\{\mathcal{S}_{\delta(C)}(C) : C \in \mathscr{C}\}$-decomposition if and only if*

(i) $\mathscr{P}(E) \bigcap \mathscr{C} \neq \varnothing$ *for all* $E \in \mathcal{E}$,

(ii) $|\mathcal{E}| = \sum_{C\in\mathscr{C}} \delta(C)$, and

(iii) $|T(\mathcal{E},\mathscr{C})| \leq \sum_{C\in T} \delta(C)$ *for all* $T \subseteq \mathscr{C}$.

For the proof we refer the reader to Lonc [9], but we remark that the proof relies on a clever application of Hall's Marriage Theorem, which is a classical result in graph theory. It deals with finding matchings in bipartite graphs. A *matching* in a graph is a set of edges which have no common endpoints. Let G be a bipartite graph with bipartition (A,B). A *matching in G saturating A* is a matching in which each vertex of A is incident with exactly one edge. Hall's Theorem can be used to characterize those bipartite graphs which have a *perfect matching*, which is defined to be a matching which saturates all of the vertices. We state this theorem for the reader's mathematical enjoyment:

Theorem 4.3. *(P. Hall [3])*
Let G be a finite bipartite graph with vertex bipartition (A,B). Then G has a matching saturating A if and only if for every subset $S \subseteq A$, $|S| \leq |N(S)|$.

where $N(S)$ – called the *neighbor set of S* – denotes the set of vertices which are adjacent to at least one vertex in S. Note that when $|A|=|B|$ we get a matching saturating both A and B, which is a perfect matching. The original formulation of Hall's theorem was stated in the language of systems of distinct representatives. This result has been extended by M. Hall [4] to include an infinite number of sets.

4.2 Decomposing Complete Uniform Hypergraphs into Hyperstars

The *complete m-uniform hypergraph on n vertices*, denoted K_n^m, is a hypergraph with n vertices which has as its edge set all m-subsets of its vertices.

We start with a technical lemma which aids in the proof of the main result of this section:

Lemma 4.4. *Let n, ℓ be positive integers with $\ell \geq n+1$. Suppose $m_1 \geq \cdots \geq m_\ell$ is a sequence of positive integers such that for $k = 1, \ldots, n-1$*

$$\sum_{i=1}^{\ell} m_i = \binom{n}{m} \text{ and } \sum_{i=1}^{k} m_i \leq \binom{n}{m} - \binom{n-k}{m}.$$

If $m'_1 \geq \cdots \geq m'_{\ell-1}$ is a rearrangement of $m_1, \ldots, m_{n-1}, m_n + m_\ell, \ldots, m_{\ell-1}$, then for $k = 1, \ldots, n-1$

$$\sum_{i=1}^{\ell-1} m'_i = \binom{n}{m} \text{ and } \sum_{i=1}^{k} m'_i \leq \binom{n}{m} - \binom{n-k}{m}.$$

Proof. It is obvious that $\displaystyle\sum_{i=1}^{\ell-1} m'_i = \sum_{i=1}^{\ell} m_i = \binom{n}{m}$. Suppose $m'_t = m_n + m_\ell$ where $t \leq n$. It is easy to see that

$$m'_i = \begin{cases} m_i & \text{for } i = 1, \ldots, t-1 \\ m_{i-1} & \text{for } i = t+1, \ldots, n. \end{cases}$$

Therefore, for $j < t$, we have $\displaystyle\sum_{i=1}^{j} m'_i = \sum_{i=1}^{j} m_i \leq \binom{n}{m} - \binom{n-j}{m}$. Now, suppose to the contrary that there is some j with $t \leq j \leq n-1$, such that $\displaystyle\sum_{i=1}^{j} m'_i \geq \binom{n}{m} - \binom{n-j}{m} + 1$.

Then we have

$$\begin{aligned}
2m_n \geq m_n + m_\ell = m'_t &= \sum_{i=1}^{j} m'_i - \sum_{i=1}^{j-1} m_i \\
&\geq \binom{n}{m} - \binom{n-j}{m} + 1 - \binom{n}{m} + \binom{n-j+1}{m} \\
&= \binom{n-j}{m-1} + 1.
\end{aligned}$$

But

$$\begin{aligned}
\binom{n}{m} = \sum_{i=1}^{\ell-1} m'_i &= \sum_{i=1}^{j} m'_i + m'_{j+1} + \cdots + m'_{\ell-1} \\
&\geq \binom{n}{m} - \binom{n-j}{m} + 1 + (m_j + \cdots + m_{n-1}) + (m_{n+1} + \cdots + m_{\ell-1}) \\
&\geq \binom{n}{m} - \binom{n-j}{m} + 1 + (n-j)m_n + (m_{n+1} + \cdots + m_{\ell-1}) \\
&\geq \binom{n}{m} - \binom{n-j}{m} + 1 + \frac{1}{2}(n-j)\left(\binom{n-j}{m-1} + 1\right). \qquad \text{(M1)}
\end{aligned}$$

Now we show that

$$\frac{1}{2}(n-j)\left(\binom{n-j}{m-1} + 1\right) + 1 > \binom{n-j}{m}. \qquad \text{(M2)}$$

This together with (M1) implies that $\binom{n}{m} > \binom{n}{m}$, a contradiction. Let $p = n - j$. Note that since $j \leq n-1$, we have $p \geq 1$. To prove (M2) it is enough to show that

$$p\binom{p}{m-1} + p + 1 \geq 2\binom{p}{m}.$$

If $p \leq 2m-1$ then $\binom{p}{m} \leq \binom{p}{m-1}$ which implies $p\binom{p}{m-1} + p + 1 \geq 2\binom{p}{m}$. If $p > 2m-1$, then because $m \geq 2$ we have the following sequence of inequalities:

$$\begin{aligned} p(m-2) + 2(m-1) + \frac{(p+1)m!}{p(p-1)\dots(p-m+2)} &\geq 0 && \iff \\ pm + \frac{(p+1)m!}{p(p-1)\dots(p-m+2)} &\geq 2(p-m+1) && \iff \\ pmp(p-1)\dots(p-m+2) + (p+1)m! &\geq 2p(p-1)\dots(p-m+1) && \iff \\ p\left(\frac{p(p-1)\dots(p-m+2)}{(m-1)!}\right) + p + 1 &\geq 2\left(\frac{p(p-1)\dots(p-m+1)}{m!}\right) && \iff \\ p\binom{p}{m-1} + p + 1 &\geq 2\binom{p}{m}. \end{aligned}$$

This completes the proof. $\square$

Theorem 4.5. *Let $m_1 \geq \cdots \geq m_\ell$ be positive integers. Then K_n^m can be decomposed into hyperstars $\mathcal{S}_{m_1}, \dots, \mathcal{S}_{m_\ell}$ if and only if*

(i) $\ell \geq n - m + 1$,

(ii) $\displaystyle\sum_{i=1}^{\ell} m_i = \binom{n}{m}$, and

(iii) $\displaystyle\sum_{i=1}^{k} m_i \leq \binom{n}{m} - \binom{n-k}{m}$ for $k = 1, \dots, \ell - 1$.

Proof. Let $K_n^m = (V, \mathcal{E})$ where $V = \{v_1, \dots, v_n\}$. There are two cases:

Case 1: $\ell \leq n$. Let $\mathscr{C} = \{\{v_1\}, \dots, \{v_\ell\}\}$, and let $\delta : \mathscr{C} \to \mathbb{Z}^+$ be a function with $\delta(\{v_i\}) = m_i$ for $1 \leq i \leq \ell$. By Theorem 4.2, there is an $\{\mathcal{S}_{m_i}(v_i) : \{v_i\} \in \mathscr{C}, 1 \leq i \leq \ell\}$-decomposition of K_n^m if and only if

(1) $\mathscr{P}(E) \bigcap \mathscr{C} \neq \varnothing$ for all $E \in \mathcal{E}$,

(2) $|\mathcal{E}| = \displaystyle\sum_{\{v_i\} \in \mathscr{C}} \delta(\{v_i\})$, and

(3) $|T(\mathcal{E}, \mathscr{C})| \leq \displaystyle\sum_{\{v_i\} \in T} \delta(\{v_i\})$ for every $T \subseteq \mathscr{C}$.

It is easy to see that $\binom{n}{m} = |\mathcal{E}|$ and $\sum_{\{v_i\}\in\mathscr{C}} \delta(\{v_i\}) = \sum_{i=1}^{\ell} m_i$ and thus (2) and (ii) are equivalent. Now, we show that (1) and (i) are equivalent. If (i) holds, then we have $|\mathscr{P}_1(V)\setminus \mathscr{C}| \leq n-(n-m+1) = m-1$, so there can be no edge which does not meet $\mathscr{C}$. Suppose (1) holds. If by contrary $\ell < n-m+1$, then there are at least $n-(n-m) = m$ vertices which are not in $\{v_1,\ldots,v_\ell\}$. These m vertices form an edge E with the property $\mathscr{P}(E)\bigcap\mathscr{C} = \varnothing$, a contradiction. Therefore (1) and (i) are equivalent.

Now we show that (3) and (iii) are equivalent. Let $T \subseteq \mathscr{C}$ with $|T| = j$. Since $\mathscr{C}$ is a set of singletons we have that $T(\mathcal{E},\mathscr{C}) = \{E \in \mathcal{E} : \mathscr{P}_1(E)\bigcap\mathscr{C} \subseteq T\}$, which means that $T(\mathcal{E},\mathscr{C}) = \{E \in \mathcal{E} : \mathscr{P}_1(E)\bigcap(\mathscr{C}\setminus T) = \varnothing\} = \{E \in \mathcal{E} : E\bigcap(\bigcup(\mathscr{C}\setminus T)) = \varnothing\}$. Thus $T(\mathcal{E},\mathscr{C})$ induces a complete m-uniform hypergraph on $n-(\ell-j) = n-\ell+j$ vertices. Hence, $|T(\mathcal{E},\mathscr{C})| = \binom{n-\ell+j}{m}$. Assume that (3) holds and let $j \in \{1,\ldots,\ell-1\}$. By (3) we have that for all $T \subseteq \mathscr{C}$ with $|T| = j$

$$\binom{n-\ell+j}{m} = |T(\mathcal{E},\mathscr{C})| \leq \sum_{\{v_i\}\in T} \delta(\{v_i\}) = \sum_{\{v_i\}\in T} m_i.$$

Taking $T = \{\{v_{\ell-j+1}\},\ldots,\{v_\ell\}\}$ produces

$$\binom{n-\ell+j}{m} \leq \sum_{i=\ell-j+1}^{\ell} m_i = \binom{n}{m} - \sum_{i=1}^{\ell-j} m_i.$$

Thus, $\sum_{i=1}^{\ell-j} m_i \leq \binom{n}{m} - \binom{n-\ell+j}{m}$. Setting $k = \ell-j$ implies that $\sum_{i=1}^{k} m_i \leq \binom{n}{m} - \binom{n-k}{m}$ for $k = 1,\ldots,\ell-1$. Now, assume that (iii) holds and let $T \subseteq \mathscr{C}$ with $|T| = j$. Then (iii) yields

$$\sum_{i=1}^{\ell-j} m_i \leq \binom{n}{m} - \binom{n-\ell+j}{m} = \binom{n}{m} - |T(\mathcal{E},\mathscr{C})|.$$

So

$$|T(\mathcal{E},\mathscr{C})| \leq \binom{n}{m} - \sum_{i=1}^{\ell-j} m_i = \sum_{i=\ell-j+1}^{\ell} m_i \leq \sum_{\{v_i\}\in T} \delta(\{v_i\})$$

where the last inequality holds because the m_i's form a decreasing sequence. Therefore (iii) and (3) are equivalent.

Case 2: $\ell \geq n+1$. First suppose that K_n^m can be decomposed into hyperstars $\mathcal{S}_{m_1}, \dots, \mathcal{S}_{m_\ell}$. The necessity of (i) and (ii) is sufficiently obvious. By combining all the stars at the same center, one can obtain a decomposition of K_n^m into hyperstars $\mathcal{S}_{m_1''}, \dots, \mathcal{S}_{m_n''}$, where $m_1'' \geq \cdots \geq m_n''$, and each m_i'', $1 \leq i \leq n$, is obtained by adding some of the m_j's, $1 \leq j \leq \ell$. Therefore by Case 1, it is necessary that for $k = 1, \dots, n-1$

$$\sum_{i=1}^{k} m_i \leq \sum_{i=1}^{k} m_i'' \leq \binom{n}{m} - \binom{n-k}{m}.$$

Now assume (i)–(iii). Apply Lemma 4.4 $\ell - n$ times to the sequence $m_1, \dots, m_\ell$ to obtain a new sequence $m_1' \geq \cdots \geq m_n'$ of positive integers with the property that for $k = 1, \dots, n-1$

$$\sum_{i=1}^{n} m_i' = \binom{n}{m} \text{ and } \sum_{i=1}^{k} m_i' \leq \binom{n}{m} - \binom{n-k}{m}.$$

Applying Case 1 to $m_1', \dots, m_n'$ with $\mathscr{C} = \{\{v_1\}, \dots, \{v_n\}\}$, we obtain an $\{\mathcal{S}_{m_i'} : 1 \leq i \leq n\}$-decomposition of K_n^m. Note that each m_i' is a sum of at least one of the m_j's. So for each $i \in \{1, \dots, n\}$, $\mathcal{S}_{m_i'}$ can be split into hyperstars with sizes in the set $\{m_1, \dots, m_\ell\}$. Also, note that $m_1, \dots, m_\ell$ are each included as a summand for some m_i', which means that for each $i \in \{1, \dots, \ell\}$, $\mathcal{S}_{m_i}$ is a hyperstar in our decomposition. Thus, we have an $\{\mathcal{S}_{m_i} : 1 \leq i \leq \ell\}$ decomposition. □

It is worth noting that Theorem 4.5 implies the result of Lin and Shyu, which is presented as Theorem 1.4 in this work.

Corollary 4.6. *Let $m_1 \geq \cdots \geq m_\ell$ be positive integers. Then K_n can be decomposed into stars $S_{m_1}, \dots, S_{m_\ell}$ if and only if*

(1) $\ell \geq n-1$,

(2) $\sum_{i=1}^{\ell} m_i = \binom{n}{2}$, and

(3) $\sum_{i=1}^{k} m_i \leq \sum_{i=1}^{k}(n-i)$ *for* $k = 1, 2, \ldots, n-1$.

Proof. Apply Theorem 4.5 with $m = 2$. There are obvious equivalencies between (i) and (1), and between (ii) and (2). To show that (iii) and (3) are equivalent notice that for $k = 1, \ldots, n-1$ we have:

$$\sum_{i=1}^{k} m_i \leq \binom{n}{2} - \binom{n-k}{2} = \binom{k}{2} + k(n-k) = nk - \frac{k(k+1)}{2} = \sum_{i=1}^{k}(n-i).$$

Hence, (iii) and (3) are equivalent. □

4.3 Decomposing Complete Hypergraphs into Hyperstars

The *complete hypergraph on n vertices* has as its edge set all non-empty subsets of an n-set. The number of hyperedges in this hypergraph is $2^n - 1$. The main result of this section is much like the main result of the last section, namely, the result for complete m-uniform hypergraphs.

Theorem 4.7. *Let $m_1 \leq \cdots \leq m_\ell$ be positive integers. Then the hypergraph $G = (V, \mathscr{P}(V) \setminus \varnothing)$ with $|V| = n$ can be decomposed into hyperstars $\mathcal{S}_{m_1}, \ldots, \mathcal{S}_{m_\ell}$ if and only if*

(i) $\ell \geq n$,

(ii) $\sum_{i=1}^{\ell} m_i = 2^n - 1$, and

(iii) $\sum_{i=1}^{k} m_i \geq 2^k - 1$ *for* $k = 1, 2, \ldots, n-1$.

Proof. Let $V = \{v_1, \ldots, v_n\}$ and $\mathscr{C} = \{\{v_1\}, \ldots, \{v_n\}\}$. There are two cases:

Case 1: $\ell = n$. Let $\delta(\{v_i\}) = m_i$ for each $i = 1, \ldots, n$. By Theorem 4.2 there is an $\{\mathcal{S}_{m_i} : 1 \leq i \leq n\}$-decomposition of G if and only if

(1) $\mathscr{P}(E)\bigcap\mathscr{C}\neq\varnothing$ for all $E\in\mathcal{E}$,

(2) $|\mathcal{E}|=\sum_{C\in\mathscr{C}}\delta(C)$, and

(3) $|T(\mathcal{E},\mathscr{C})|\leq\sum_{C\in T}\delta(C)$ for all $T\subseteq\mathscr{C}$.

The equivalence of (1) and (i) is due to the fact that we have all of the hyperedges of size 1 in our hypergraph. Thus, every vertex must be the center of at least one hyperstar. The equivalence of (2) and (ii) is due to the fact that $|\mathcal{E}|=2^n-1$. Now, let $T\subseteq\mathscr{C}$ with $|T|=k$. It is easy to see that $|T(\mathcal{E},\mathscr{C})|=2^k-1$, since $T(\mathcal{E},\mathscr{C})=\{E\in\mathcal{E}:\mathscr{P}_1(E)\bigcap\mathscr{C}\subseteq T\}=\{E\in\mathcal{E}:\mathscr{P}_1(E)\subseteq T\}$. By the same reasoning used in the proof of Case 1 of Theorem 4.5 we have (3) is equivalent with (iii).

Case 2: $\ell\geq n+1$. First suppose that G can be decomposed into hyperstars $\mathcal{S}_{m_1},\ldots,\mathcal{S}_{m_\ell}$. The necessity of (i) and (ii) is obvious. By combining all the stars at the same center, one can obtain a decomposition of G into hyperstars $\mathcal{S}_{m''_1},\ldots,\mathcal{S}_{m''_n}$, where $m''_1\geq\cdots\geq m''_n$, and each m''_i, $1\leq i\leq n$, is obtained by adding some of the m_j's, $1\leq j\leq\ell$. Therefore by Case 1, it is necessary that for $k=1,2,\ldots,n-1$, $\sum_{i=1}^{k}m''_i\geq 2^k-1$ which is equivalent to $\sum_{i=k+1}^{n}m_i\leq\sum_{i=k+1}^{n}m''_i\leq 2^n-2^k$. Thus, $\sum_{i=1}^{k}m_i\geq 2^k-1$ for $k=1,2,\ldots,n-1$.

Now assume (i)–(iii). Let $m'_1\leq\cdots\leq m'_{\ell-1}$ be a rearrangement of $m_1+m_2,m_3,\ldots,m_\ell$. It is easy to see that

$$m'_i=\begin{cases} m_{i+2} & \text{for } i=1,\ldots,t-1\\ m_{i+1} & \text{for } i=t+1,\ldots,\ell-1.\end{cases}$$

It is clear that

$$\sum_{i=1}^{\ell-1}m'_i=\sum_{i=1}^{\ell}m_i=2^n-1.$$

Notice that for every $i\in\{1,\ldots,\ell-1\}$ we have that $m'_i\geq m_i$. Thus, for every $k\in\{1,\ldots,n-1\}$ we get

$$\sum_{i=1}^{k}m'_i\geq\sum_{i=1}^{k}m_i\geq 2^k-1.$$

Again, we can use lemma 4.4 $\ell - n$ times to obtain an appropriate sequence of m_i''s which produces an $\{\mathcal{S}_{m_i'} : 1 \leq i \leq n\}$-decomposition of our complete hypergraph. All we have left to do is to split the hyperstar centered at each vertex into an appropriate number of hyperstars of smaller sizes. The rest of the proof is analogous to the proof of Case 2 of Theorem 4.7. $\square$

As mentioned earlier, Lonc [10] obtained these results in 1992. His method of proof involved looking at packings and coverings of hypergraphs with hyperstars. He then used these structures to find the results of theorems 4.5 and 4.7.

Chapter 5

Concluding Remarks

We present a brief summary of the main results and open problems from each chapter. From chapter 2:

Result 1. *Let $n, k \in \mathbb{N}$ with $n \geq 2k$. Then there are $\left\lfloor \frac{\binom{n}{2}}{k} \right\rfloor$ stars in a maximum packing of K_n with k-stars. Moreover, it is possible to have the leave graph be a star of size smaller than k.*

Result 2. *Let $n, k \in \mathbb{N}$ with $k < n < 2k$. Then there are $2n - 2k - 1$ stars in a maximum packing of K_n with k-stars. Moreover, the leave graph must be K_{2k-n+1}.*

Result 3. *Let $\lambda, n, k \in \mathbb{N}$ with $n \geq 2k$. Then there are $\left\lfloor \frac{\lambda\binom{n}{2}}{k} \right\rfloor$ stars in a maximum packing of λK_n with k-stars. Moreover, it is possible to have the leave graph be a star of size smaller than k.*

Result 4. *Let $\lambda, n, k \in \mathbb{N}$ with $\lambda > 1$ and $n = k+1$. If λ is even then λK_n can be decomposed into k-stars. If λ is odd then there are $k\left(\frac{\lambda - 1}{2}\right) + \frac{\lambda + 1}{2}$ stars in a maximum packing of λK_n with k-stars. Moreover, the leave graph must be K_k.*

Result 5. *Let $\lambda, n, k \in \mathbb{N}$ with $\lambda > 1$ and $n = 2k - 1$. Then there are $\left\lfloor \frac{\lambda\binom{n}{2}}{k} \right\rfloor$ stars in a maximum packing of λK_n with k-stars. Moreover, the leave can be any subgraph of λK_n which has $\lambda \pmod{k}$ edges.*

For some of the above cases we have characterized the leave graph(s), but for some cases only the number of stars and at least one configuration of the leave is known. The astute reader will notice that there is a gap in the full solution of this problem. We must confront the following:

Open Problem 1. *Given $\lambda, n, k \in \mathbb{N}$ such that $\lambda > 1$ and $k+2 \leq n \leq 2k-2$, find a maximum packing of λK_n with k-stars.*

The author suggests attacking open problem 1 via orientations. There is a natural generalization of this question where we allow stars of different sizes. This is analogous to generalizing the theorem of Yamomoto [13] and Tarsi [12] to the theorem of Lin and Shyu [8]. We present it as an open problem.

Open Problem 2. *Given $m_1, m_2, \ldots, m_\ell \in \mathbb{Z}^+$ and $\lambda, n \in \mathbb{N}$, characterize the leave graph of a maximum packing of λK_n with stars of sizes $m_1, m_2, \ldots, m_\ell$.*

In chapter 3 our quest was to find a small embedding of a partial k-star decomposition of K_n which depended only on k. We achieved that feat, and here we state the main result.

Result 6. *A partial k-star decomposition of K_n can be embedded into a k-star decomposition of K_{n+s} where* $s \leq \begin{cases} 7k-4 & \text{if } k \text{ is odd} \\ 8k-4 & \text{if } k \text{ is even} \end{cases}$.

The satisfying part of this result is that the number of new vertices depends only on k. The embedding is on a relatively small number of vertices; however, this is not the smallest possible number. The author believes that it can be reduced. The methods used in this work may be of some use in reducing the number of new vertices, namely, relaxing the "balanced" condition of theorem 3.4. We note that any decomposition of K_{n+s} requires $\binom{n+s}{2} \equiv 0 \pmod{k}$. Combining this with the periodic nature of $\binom{n}{2} \pmod{k}$ (lemma 3.7) gives a good estimate for the smallest possible value of s to be about $2k$.

Open Problem 3. *Given a partial k-star decomposition of K_n, find an embedding into a k-star decomposition of K_{n+s} where s is no more than approximately $2k$.*

In chapter 4 we provided alternative proofs to some results obtained by Lonc [9]. Recall that we denote a hyperstar of size m_i with center size 1 by $\mathcal{S}_{m_i}$.

Result 7. *Let $m_1 \geq \cdots \geq m_\ell$ be positive integers. Then the complete m-uniform hypergraph on n vertices, K_n^m, can be decomposed into hyperstars $\mathcal{S}_{m_1}, \ldots, \mathcal{S}_{m_\ell}$ if and only if*

(i) $\ell \geq n - m + 1$,

(ii) $\displaystyle\sum_{i=1}^{\ell} m_i = \binom{n}{m}$, and

(iii) $\displaystyle\sum_{i=1}^{k} m_i \leq \binom{n}{m} - \binom{n-k}{m}$ for $k = 1, \ldots, \ell - 1$.

Recall that the *complete hypergraph on n vertices* is defined to be a hypergraph on n vertices whose edge set is all non-empty subsets of those vertices.

Result 8. *Let $m_1 \leq \cdots \leq m_\ell$ be positive integers. Then the complete hypergraph on n vertices can be decomposed into hyperstars $\mathcal{S}_{m_1}, \ldots, \mathcal{S}_{m_\ell}$ if and only if*

(i) $\ell \geq n$,

(ii) $\displaystyle\sum_{i=1}^{\ell} m_i = 2^n - 1$, *and*

(iii) $\displaystyle\sum_{i=1}^{k} m_i \geq 2^k - 1$ *for* $k = 1, 2, \ldots, n - 1$.

It is easy to generalize this problem by allowing hyperstars with different center sizes. However, it should be noted that even allowing centers of size 2 greatly complicates the problem. The author would suggest first looking into the following problem:

Open Problem 4. *Given $m_1, m_2, \ldots, m_\ell$, find necessary and sufficient conditions for the complete multipartite uniform hypergraph to obtain a decomposition into hyperstars of sizes $m_1, m_2, \ldots, m_\ell$ with centers of size 1.*

The above problem is more complicated than those problems solved in results 7 and 8, but it seems to be easier than allowing larger centers.

It is the author's decision to omit a summary of the results of chapter 5 to avoid the cycle of length $\aleph_0$ that would ensue.

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