Amalgamations and Detachments of Graphs and Hypergraphs

by

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Abstract

A *detachment* of a graph H is a graph obtained from H by splitting some or all of its vertices into more than one vertex. If g is a function from $V(H)$ into $\mathbb{N}$, then a *g-detachment* of H is a detachment of H in which each vertex u of H splits into $g(u)$ vertices. H is an *amalgamation* of G if there exists a function ϕ called an *amalgamation function* from $V(G)$ onto $V(H)$ and a bijection $\phi' : E(G) \to E(H)$ such that e joining u and v is in $E(G)$ iff $\phi'(e)$ joining $\phi(u)$ and $\phi(v)$ is in $E(H)$.

We prove that for a given edge-colored graph there exists a detachment so that the result is a graph in which the edges are shared among the vertices in ways that are fair with respect to several notions of balance (such as between pairs of vertices, degrees of vertices in both the graph and in each color class, etc.). The connectivity of color classes is also addressed. Most results in the literature on amalgamations focus on the detachments of amalgamated complete graphs and complete multipartite graphs. Many such results follow as immediate corollaries to the main result, which addresses amalgamations of graphs in general.

We exhibit some applications of this result in Hamiltonian decomposition of several families of graphs, and also we show that many known graph decomposition results can be obtained by a short proof using the main theorem. We study the companion embedding problems with many applications.

We then extend various results by Hilton, Nash-Williams and Rodger to hypergraphs. Such extensions provide a powerful tool to generalizes Baranyai's Theorems, and related results by Berge and Johnson.

We study several hypergraph embedding problems which will extend results of Brouwer, Schrijver, Baranyai, Häggkvist and Hellgren.

In connection with Baranyai-Katona conjecture, we provide necessary and sufficient conditions for a complete uniform hypergraph to be connected factorizable, answering a question by Katona.

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I would like to dedicate this work to Saeed Bahmanian and Mansour Hemayati.

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What are graph amalgamations?

1.1 Introduction

Edouard Lucas (1842–1891), the inventor of the Towers of Hanoi problem, discussed the *probléme de ronde* that asked the following [64]: Given $2n + 1$ people, is it possible to arrange them around a single table on n successive nights so that nobody is seated next to the same person on either side more than once? This problem is equivalent to a Hamiltonian decomposition of K_{2n+1}; that is partitioning the edge set of K_{2n+1} into spanning cycles. A solution to this problem for $n = 3$ is illustrated in Figure 1.1, which is due to Walecki. This can be easily generalized to any complete graph by "rotating" an initial cycle.
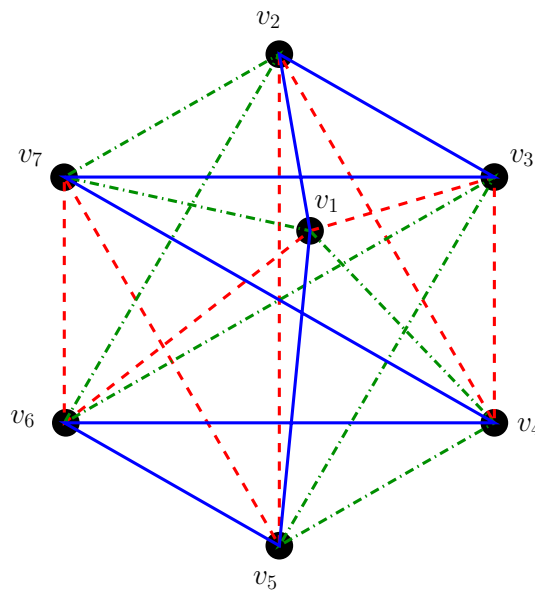
Figure 1.1: Walecki Construction

In 1984, Hilton [44] suggested a different approach to solving this problem, one of which is useful for solving another family of problems as well. He first fused all the vertices of K_n

(this is called amalgamation) which results in having $\binom{n}{2}$ loops incident with a vertex. Then he shared the loops evenly between different color classes. (In this dissertation, the ith color class of G is defined to be the spanning subgraph of G that contains precisely the edges colored i.) Finally he reversed the fusion by splitting the single vertex into n vertices (this is called detachment), so that each color class is a Hamiltonian cycle. This is illustrated in Figure 1.2 for K_7. It is not obvious how we can detach the loops so that each color class is a
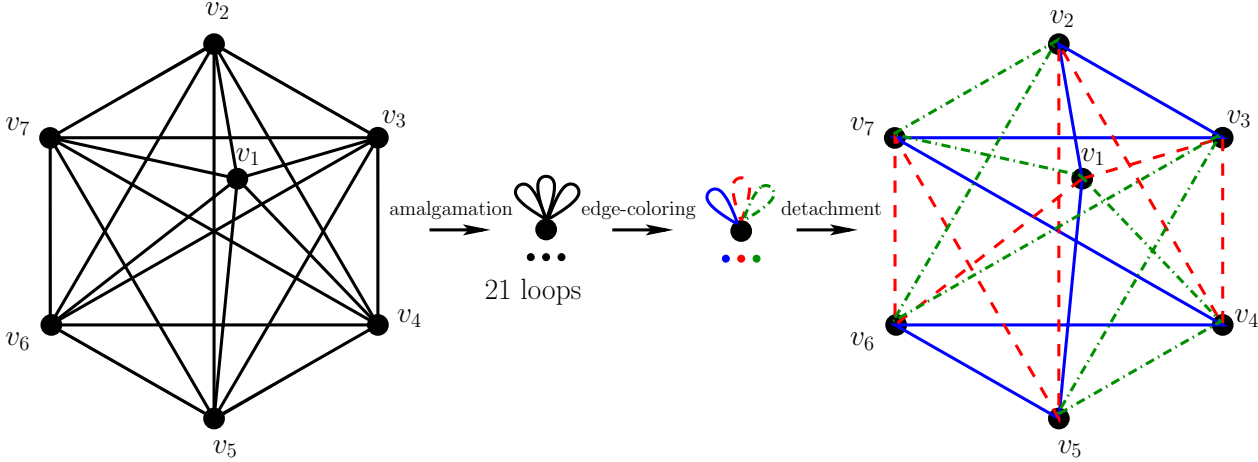
Figure 1.2: Hamiltonian decomposition of K_7

Hamiltonian cycle. The second problem that Hilton solved was an embedding problem [44]. Given an edge-coloring of K_m, in which each color class is a path, he used amalgamations to extend this coloring to an edge-coloring of K_{m+n}, so that each color class is a Hamiltonian cycle in K_{m+n} (so $m + n$ must be odd). The idea is to add a new vertex, say u to K_m incident with $\binom{n}{2}$ loops so that there are n edges between this vertex and every other vertex. Let us call this graph K_m^+. (In fact K_m^+ is an amalgamation of K_{m+n} in which all further n vertices are contracted in one point.) One can easily color all the edges incident with u so that the valency of u for each color class is exactly $2n$. Finally by detaching u into n vertices, say $u_1, \ldots, u_n$ and sharing the edges of each color class incident with u among $u_1, \ldots, u_n$ as evenly as possible and ensuring that each color class is connected, provides the desired

outcome: a Hamiltonian decomposition of K_{m+n}. This is illustrated for $m = 5, n = 2$ in Figure 1.3. To provide more explanation, first we give some definitions.
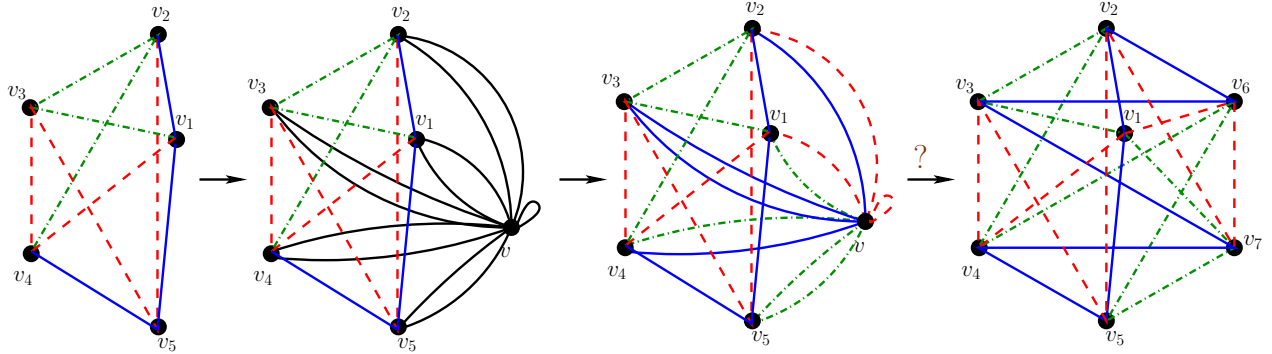
Figure 1.3: Embedding a path decomposition of K_5 into a Hamiltonian decomposition of K_7

Throughout this dissertation, all graphs are finite and undirected (possibly with loops and multiple edges). The letters G and H denote graphs. Sets may contain repeated elements (so are really multisets). Each edge is represented by a 2-element multisubset of the vertex set; in particular $\{u, u\}$ represents a loop on the vertex u. A *k-edge-coloring* of a graph G is a mapping $f : E(G) \to C$, where C is a set of k *colors* (often we use $C = \{1, \ldots, k\}$). It is often convenient to have empty color classes, so we do not require f to be surjective.

In this dissertation, $x \approx y$ means $\lfloor y \rfloor \leqslant x \leqslant \lceil y \rceil$, $\ell(u)$ denotes the number of loops incident with vertex u, $d(u)$ denotes the degree of vertex u (loops are considered to contribute two to the degree of the incident vertex), the subgraph of G induced by the edges colored j is denoted by $G(j)$, $\omega(G)$ is the number of components of G, the *multiplicity* of a pair of vertices u, v of G, denoted by $m(u, v)$, is the number of edges joining u and v in G, K_n denotes the complete graph with n vertices, and $K_{m,\ldots,m}$ denotes the complete multipartite graph each part having m vertices. If we replace every edge of G by λ multiple edges, then we denote the new graph by λG.

Informally speaking, *amalgamating* a finite graph G can be thought of as taking G, partitioning its vertices, then for each element of the partition squashing the vertices to form a single vertex in the amalgamated graph H. Any edge incident with an original vertex

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in G is then incident with the corresponding new vertex in H, and any edge joining two vertices that are squashed together in G becomes a loop on the new vertex in H.

More precisely, H is an *amalgamation* of G if there exists a function ϕ called an *amalgamation function* from $V(G)$ onto $V(H)$ and a bijection $\phi' : E(G) \rightarrow E(H)$ such that e joining u and v is in $E(G)$ if and only if $\phi'(e)$ joining $\phi(u)$ and $\phi(v)$ is in $E(H)$; We write $\phi(G) = H$. In particular, this requires that e be a loop in H if and only if, in G, it either is a loop or joins distinct vertices u, v, such that $\phi(u) = \phi(v)$. (Note that ϕ' is completely determined by ϕ.) Associated with ϕ is the *number function* $\eta : V(H) \rightarrow \mathbb{N}$ defined by $\eta(v) = |\phi^{-1}(v)|$, for each $v \in V(H)$. We also shall say that G is a *detachment* of H in which each vertex v of H splits (with respect to ϕ) into the vertices in $\phi^{-1}(\{v\})$ (see Figure 1.4).
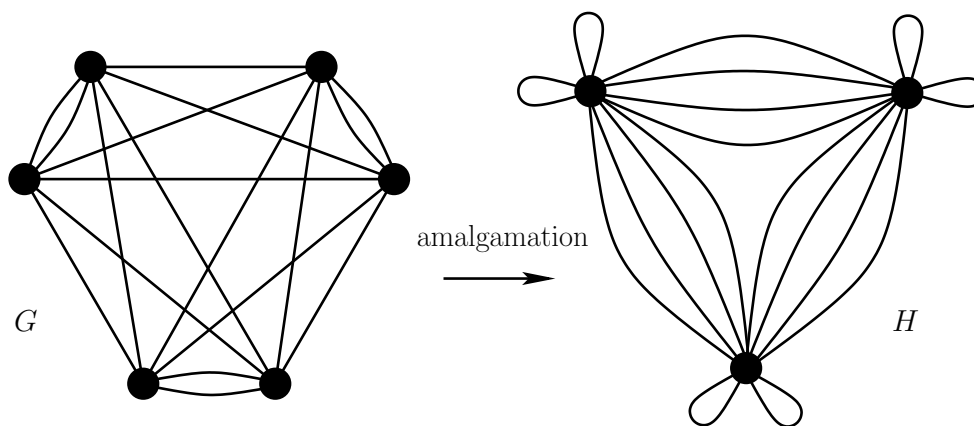
Figure 1.4: A graph G with one of its amalgamations H

A *detachment* of H is, intuitively speaking, a graph obtained from H by splitting some or all of its vertices into more than one vertex (see Figure 1.5). If η is a function from $V(H)$ into $\mathbb{N}$, then an *η-detachment* of H is a detachment of H in which each vertex u of H splits into $\eta(u)$ vertices. In other words, G is an η-detachment of H if there exists an amalgamation function ϕ of G onto H such that $|\phi^{-1}(\{u\})| = \eta(u)$ for every $u \in V(H)$. Some authors refer to detachments as *disentanglements* (see [58, 60, 61]).

Since two graphs G and H related in the above manner have an obvious bijection between the edges, an edge-coloring of G or H, naturally induces an edge-coloring on the
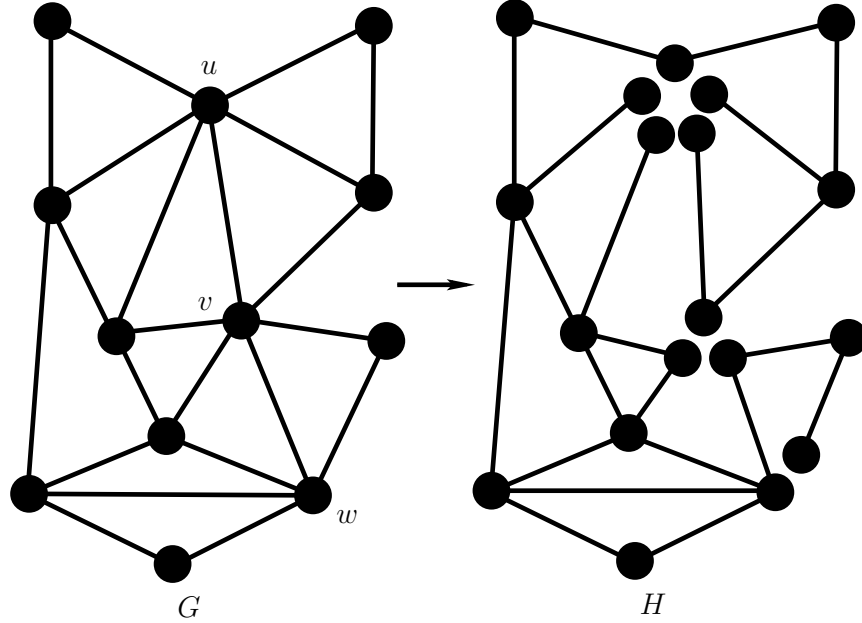
Figure 1.5: A graph G with one of its detachments H

other graph. Hence an amalgamation of a graph with colored edges is a graph with colored edges.

One of the most useful properties that one can obtain using the techniques described here, is that many graph parameters (such as colors, degrees, multiple edges) can be simultaneously shared evenly during the detachment process. This is often the most desirable property.

Theorem 1.1. (Bahmanian, Rodger [5, Theorem 3.1]) *Let H be a k-edge-colored graph and let η be a function from $V(H)$ into $\mathbb{N}$ such that for each $v \in V(H)$, $\eta(v) = 1$ implies $\ell_H(v) = 0$. Then there exists a loopless η-detachment G of H in which each $v \in V(H)$ is detached into $v_1, \ldots, v_{\eta(v)}$, such that G satisfies the following conditions:*

(A1) $d_G(u_i) \approx d_H(u)/\eta(u)$ *for each $u \in V(H)$ and $1 \leqslant i \leqslant \eta(u)$;*

(A2) $d_{G(j)}(u_i) \approx d_{H(j)}(u)/\eta(u)$ *for each $u \in V(H)$, $1 \leqslant i \leqslant \eta(u)$, and $1 \leqslant j \leqslant k$;*

(A3) $m_G(u_i, u_{i'}) \approx \ell_H(u)/\binom{\eta(u)}{2}$ *for each $u \in V(H)$ with $\eta(u) \geqslant 2$ and $1 \leqslant i < i' \leqslant \eta(u)$;*

(A4) $m_{G(j)}(u_i, u_{i'}) \approx \ell_{H(j)}(u)/\binom{\eta(u)}{2}$ *for each* $u \in V(H)$ *with* $\eta(u) \geqslant 2$, $1 \leqslant i < i' \leqslant \eta(u)$, *and* $1 \leqslant j \leqslant k$;

(A5) $m_G(u_i, v_{i'}) \approx m_H(u, v)/(\eta(u)\eta(v))$ *for every pair of distinct vertices* $u, v \in V(H)$, $1 \leqslant i \leqslant \eta(u)$, *and* $1 \leqslant i' \leqslant \eta(v)$;

(A6) $m_{G(j)}(u_i, v_{i'}) \approx m_{H(j)}(u, v)/(\eta(u)\eta(v))$ *for every pair of distinct vertices* $u, v \in V(H)$, $1 \leqslant i \leqslant \eta(u)$, $1 \leqslant i' \leqslant \eta(v)$, *and* $1 \leqslant j \leqslant k$;

(A7) *If for some* j, $1 \leqslant j \leqslant k$, $d_{H(j)}(u)/\eta(u)$ *is even for each* $u \in V(H)$, *then* $\omega(G(j)) = \omega(H(j))$.

The proof uses edge-coloring techniques and will be given in the next chapter. An edge-coloring of a multigraph is (i) *equalized* if the number of edges colored with any two colors differs by at most one, (ii) *balanced* if for each pair of vertices, among the edges joining the pair, the number of edges of each color differs by at most one from the number of edges of each other color, and (iii) *equitable* if, among the edges incident with each vertex, the number of edges of each color differs by at most one from the number of edges of each other color. In [80, 81, 82, 83] de Werra studied balanced equitable edge-coloring of bipartite graphs. The following result is used to prove Theorem 1.1.

Theorem 1.2. *Every bipartite graph has a balanced, equitable and equalized* k-*edge-coloring for each* $k \in \mathbb{N}$.

Here we show that this result is simply a consequence of Nash-Williams lemma. A family $\mathscr{A}$ of sets is *laminar* if, for every pair A, B of sets belonging to $\mathscr{A}$, either $A \subset B$, or $B \subset A$, or $A \cap B = \varnothing$.

Lemma 1.3. (Nash-Williams [70, Lemma 2]) *If* $\mathscr{A}, \mathscr{B}$ *are two laminar families of subsets of a finite set* S, *and* $n \in \mathbb{N}$, *then there exist a subset* A *of* S *such that for every* $P \in \mathscr{A} \cup \mathscr{B}$, $|A \cap P| \approx |P|/n$.

Proof of Theorem 1.2. Let B be a bipartite graph with vertex bipartition $\{V_1, V_2\}$. For $i = 1, 2$ define the laminar set L_i to consist of the following sets of subsets of edges of B: (i) The edges between each pair of vertices $v_1 \in V_1$ and $v_2 \in V_2$, (ii) For each $v \in V_i$, the edges incident with v, (iii) All the edges in B. Applying Lemma 1.3 with $n = k$ provides one color class. Remove these edges then reapply Lemma 1.3, with $n = k - 1$ to get the second class. Recursively proceeding in this way provides the k-edge-coloring of B. It is straightforward to see that this produces the result by observing that the edges in subsets defined in (i), (ii) and (iii) guarantee that the k-edge-coloring is balanced, equitable, and equalized respectively. □

1.2 Applications

In this section we demonstrate the power of Theorem 1.1. The results are not new, and many follow from earlier, more restrictive versions of Theorem 1.1. But the point of this section is to give the reader a feel for how amalgamations can be used.

Theorem 1.4. (Walecki [64]) *λK_n is Hamiltonian decomposable (with a 1-factor leave, respectively) if and only if $\lambda(n - 1)$ is even (odd, respectively).*

Proof. The necessity is obvious. To prove the sufficiency, let H be a graph with $V(H) = \{v\}$, $\ell(v) = \lambda\binom{n}{2}$ and $\eta(v) = n$, and let $k = \lfloor \lambda(n - 1)/2 \rfloor$. Color the loops so that $\ell_{H(j)}(v) = n$, for $1 \leqslant j \leqslant k$ (and $\ell_{H(k+1)}(v) = n/2$, if $\lambda(n - 1)$ is odd). Applying Theorem 1.1 completes the proof. □

The following result is essentially proved in [44], but the result is stated in less general terms.

Theorem 1.5. (Hilton [44]) *A k-edge-colored K_m can be embedded into a Hamiltonian decomposition of K_{m+n} (with a 1-factor leave, respectively) if and only if $(m + n - 1)$ is even (odd, respectively), $k = \lceil (m + n - 1)/2 \rceil$, and each color class of K_m (except one color class, say k, respectively) is a collection of at most n disjoint paths, (color class k consists of paths*

of length at most 1, at most n of which are of length 0, respectively), where isolated vertices in each color class are to be counted as paths of length 0.

Proof. The necessity is obvious. To prove the sufficiency, let $p_i \leqslant n$ be the number of paths colored i, $1 \leqslant i \leqslant k$. Form a graph H by adding a new vertex u to K_m so that $\ell(u) = \binom{n}{2}$, $m(u, v) = n$ for each $v \in V(K_m)$, and $\eta(u) = n$. Color the new edges incident with vertices in K_m so that $d_{H(j)}(v) = 2$ for $v \in V(K_m)$, $1 \leqslant j \leqslant k$ (if $m+n$ is even, do it so that $d_{H(k)}(v) = 1$ for $v \in V(K_m)$; so at most n such edges are incident with u by necessary conditions). Clearly, each color appears on an even number of such edges (except possibly color k when $m + n$ is odd). Color the loops so that $d_{H(j)}(u) = 2n$ for $1 \leqslant j \leqslant k$ (if $m+n$ is even, then the coloring must be so that $d_{H(k)}(u) = n$). This is possible since each color appears on $(2n - 2p_i)/2 \geqslant 0$ loops. Now applying Theorem 1.1 completes the proof. $\qquad\square$

A similar result can be obtained for embedding λK_m into a Hamiltonian decomposition of λK_{m+n}. A more general problem is the following enclosing problem

Problem 1. *Find necessary and sufficient conditions for enclosing an edge-colored λK_m into a Hamiltonian decomposition of μK_{m+n} for $\lambda < \mu$.*

An $(r_1, \ldots, r_k)$-*factorization* of a graph G is a partition (decomposition) $\{F_1, \ldots, F_k\}$ of $E(G)$ in which F_i is an r_i-factor for $i = 1, \ldots, k$. The following is a corollary of a strong result of Johnson [51] in which each color class can have a specified edge-connectivity. A special case of this is proved by Johnstone in [52].

Theorem 1.6. *λK_n is $(r_1, \ldots, r_k)$-factorizable if and only if $r_i n$ is even for $1 \leqslant i \leqslant k$, and $\sum_{i=1}^{k} r_i = \lambda(n-1)$. Moreover, for $1 \leqslant i \leqslant k$ each r_i-factor can be guaranteed to be connected if r_i is even.*

Proof. The necessity is obvious. To prove the sufficiency, start from the graph H as in the proof of Theorem 1.4, but color the loops so that $\ell_{H(j)}(v) = nr_j/2$ for $1 \leqslant j \leqslant k$. Then apply Theorem 1.1. $\qquad\square$

The following result was proved for the special case $r_1 = \ldots = r_k = r$ in [3, 74].

Theorem 1.7. *A k-edge-coloring of K_m can be embedded into an $(r_1, \ldots , r_k)$-factorization of K_{m+n} if and only if $r_i(m+n)$ is even for $1 \leqslant i \leqslant k$, $\sum_{i=1}^{k} r_i = m+n-1$, $d_{K_m(i)}(v) \leqslant r_{\sigma(i)}$ for each $v \in V(K_m)$, $1 \leqslant i \leqslant k$, and some permutation $\sigma \in S_k$, and $|E(K_m(i))| \geqslant r_{\sigma(i)}(m-n)/2$.*

Proof. The necessity is obvious. To prove the sufficiency, start from the graph H as in the proof of Theorem 1.5. Color the new edges incident with vertices in K_m so that $d_{H(j)}(v) = r_{\sigma(j)}$ for $v \in V(K_m)$, $1 \leqslant j \leqslant k$. Then color the loops incident with u so that $d_{H(j)}(u) = r_{\sigma(j)}m$ for $1 \leqslant j \leqslant k$ (the last necessary condition guarantees that the number of required loops is non-negative), and apply Theorem 1.1. $\square$

Problem 2. *Find necessary and sufficient conditions for enclosing an edge-colored λK_n into an $(r_1, \ldots , r_k)$-factorization of μK_{m+n} for $\lambda \leqslant \mu$.*

The case $\lambda = \mu$ can be obtained by altering the proof of Theorem 1.7 slightly.

Some of the above results can be easily generalized to complete multipartite graphs.

Theorem 1.8. (Laskar, Auerbach [57]) *$\lambda K_{n_1, \ldots , n_m}$ is Hamiltonian decomposable (with a 1-factor leave, respectively) if and only if $n_1 = \cdots = n_m := n$, and $\lambda n(m-1)$ is even (odd, respectively).*

Proof. The necessity is obvious. To prove the sufficiency, consider the graph $H := \lambda n^2 K_m$, and $\eta : V(H) \to \mathbb{N}$ with $\eta(v) = n$ for each $v \in V(H)$. Using Theorem 1.6, find a connected $2n$-factorization of H and apply Theorem 1.1. $\square$

Another very nice requirement that one can ask of a Hamiltonian decomposition of a complete multipartite graph is that it be fair; that is, in each Hamiltonian cycle, the number of edges between each pair of parts is within one of the number of edges between each other pair of parts. This result can be proved by being more careful in the construction of the edge-coloring of the graph H described in the proof of Theorem 2.5; ensure that for each

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color class the number of edges between each pair of vertices in H is within 1 of the number of edges between each other pair of vertices (one could think of this color class as being "equimultiple"). Leach and Rodger [59] used this approach to prove that

Theorem 1.9. $K_{n_1,\ldots,n_m}$ *is fair Hamiltonian decomposable if and only if $n_1 = \cdots = n_m := n$, and $n(m-1)$ is even.*

Problem 3. *Find necessary and sufficient conditions for enclosing a k-edge-colored $\lambda K_{n_1,\ldots,n_m}$ into a (fair) Hamiltonian decomposition of $\mu K_{n'_1,\ldots,n'_{m'}}$ for $n_i \leqslant n'_i$, $1 \leqslant i \leqslant m \leqslant m'$, and $\lambda \leqslant \mu$.*

Theorem 1.10. $\lambda K_{n_1,\ldots,n_m}$ *is $(r_1,\ldots,r_k)$-factorizable if and only if $n_1 = \cdots = n_m := n$, $r_i nm$ is even for $1 \leqslant i \leqslant k$, and $\sum_{i=1}^{k} r_i = \lambda n(m-1)$.*

Proof. The necessity is obvious. To prove the sufficiency, use Theorem 1.6 to find an $(nr_1,\ldots,nr_k)$-factorization of the graph H described in the proof of Theorem 1.8; then apply Theorem 1.1. □

Problem 4. *Find necessary and sufficient conditions for enclosing a k-edge-colored $\lambda K_{n_1,\ldots,n_m}$ into an $(r_1,\ldots,r_k)$-factorization of $\mu K_{n'_1,\ldots,n'_{m'}}$ for $n_i \leqslant n'_i$, $1 \leqslant i \leqslant m \leqslant m'$, and $\lambda \leqslant \mu$.*

The Oberwolfach problem $OP(r_1^{a_1},\ldots,r_k^{a_k})$ asks whether or not it is possible to partition the edge set of K_n, n odd, or K_n with a 1-factor removed when n is even, into isomorphic 2-factors such that each 2-factor consists of a_j cycles of length r_j, $1 \leqslant j \leqslant k$, and $n = \sum_{j=1}^{k} r_j a_j$. In [46] some new solutions to the Oberwolfach problem are given using the amalgamation technique.

Chapter 2

Multiply Balanced Edge Colorings of Multigraphs

2.1 Introduction

In this chapter, a theorem is proved that generalizes several existing amalgamation results in various ways. The main aim is to disentangle a given edge-colored amalgamated graph so that the result is a graph in which the edges are shared out among the vertices in ways that are fair with respect to several notions of balance (such as between pairs of vertices, degrees of vertices in the both graph and in each color class, etc). The connectivity of color classes is also addressed. Most results in the literature on amalgamations focus on the disentangling of amalgamated complete graphs and complete multipartite graphs. Many such results follow as immediate corollaries to the main result in this chapter, which addresses amalgamations of graphs in general, allowing for example the final graph to have multiple edges. A new corollary (see Chapter 3) of the main theorem is the settling of the existence of Hamilton decompositions of the family of graphs $K(a_1, \ldots, a_p; \lambda, \mu)$; such graphs arose naturally in statistical settings.

A graph is said to be: (i) *almost regular* if there is an integer d such that every vertex has degree d or $d + 1$, (ii) *equimultiple* if there is an integer d such that every pair of vertices has multiplicity d or $d + 1$, (iii) *P-almost-regular* (where $P = \{P_1, \ldots, P_r\}$ is a partition of $V(G)$) if for $1 \leqslant i \leqslant r$, there is an integer d_i such that each vertex in P_i has degree d_i or $d_i + 1$.

The main goal of this chapter is to prove Theorem 2.1. Informally, it states that for a given k-edge-colored graph H and a function $\eta : V(H) \to \mathbb{N}$, there exists a loopless η-detachment G of H with amalgamation function $\phi : V(G) \to V(H)$, η being the number function associated with ϕ, such that: (i) G and each of its color classes are P-almost-regular

where $P = \{\phi^{-1}(v) : v \in V(H)\}$, (ii) the subgraph of G induced by $\phi^{-1}(v)$ is equimultiple for each $v \in V(H)$, as are each of its color classes, (iii) the bipartite subgraph of G formed by the edges joining vertices in $\phi^{-1}(u)$ to the vertices in $\phi^{-1}(v)$ is equimultiple for every pair of distinct $u, v \in V(H)$, as are each of its color classes, and (iv) under certain conditions, the subgraph induced by each color class can be guaranteed to have the same number of components in G as in H. The conditions (ii) and (iii) can be used to force G to be multigraphs of interest, such as $\lambda K_n, \lambda K_{m,\ldots,m}$, or $K(a_1, \ldots, a_p; \lambda, \mu)$ (for the definition of $K(a_1, \ldots, a_p; \lambda, \mu)$, see Chapter 3). As in previous results, condition (iv) is especially useful in the context of Hamiltonian decompositions, since it can be used to force connected color classes in H to remain connected in G.

A Hamiltonian decomposition of a graph G is a partition of the edges of G into sets, each of which induces a spanning cycle. Hamiltonian decompositions have been studied since 1892, when Walecki [64] proved the classic result that K_n is Hamiltonian decomposable if and only if n is odd. In 1976 Laskar and Auerbach [57] settled the existence of Hamiltonian decomposition of the complete multipartite graph $K_{m,\ldots,m}$ and of $K_{m,\ldots,m} - F$ where F is a 1-factor. Nash-Williams [67] conjectured that every $2k$-regular graph with at most $4k + 1$ vertices has a Hamiltonian decomposition.

Several techniques have been used for finding Hamiltonian decompositions. The technique of vertex amalgamation, which was developed in the 1980s by Hilton and Rodger [44, 48], has proved to be very powerful in constructing Hamiltonian decompositions of various classes of graphs, especially in obtaining embedding results; see also [47, 51, 70, 74]. Buchanan [28] used amalgamations to prove that for any 2-factor U of K_n, n odd, $K_n - E(U)$ admits a Hamiltonian decomposition. Rodger and Leach [58] solved the corresponding existence problem for complete bipartite graphs, and obtained a solution for complete multipartite graphs when U has no small cycles [60]. See also [23, 66] for different approach to solve this problem. Detachments of graphs have also been studied in [18, 49], generalizing some results of Nash-Williams [68, 69].

The main theorem of this chapter, Theorem 2.1, not only generalizes several well-known graph amalgamation results, (for example, in [44, 48, 58, 61, 74], Theorem 1, Theorem 1, Theorem 3.1, Theorem 2.1 and Theorem 2.1 respectively all follow as immediate corollaries)), but also provides the right tool to find necessary and sufficient conditions for $K(a_1, \ldots, a_p; \lambda, \mu)$ to be Hamiltonian decomposable, as shown in Theorem 3.4. The latter graph, $K(a_1, \ldots, a_p; \lambda, \mu)$, is of particular interest to statisticians, who consider group divisible designs with two associate classes, beginning over 50 years ago with the work of Bose and Shimamoto [22]. Recently, partitions of the edges of $K(a_1, \ldots, a_p; \lambda, \mu)$ into sets, each of which induces a cycle of length m, have been extensively studied for small values of m [37, 38, 39]. Theorem 3.4 provides a companion to this work, settling the problem completely for longest (i.e. Hamiltonian) cycles with a really neat proof. When $a_1 = \ldots = a_p = a$, we denote $K(a_1, \ldots, a_p; \lambda, \mu)$ by $K(a^{(p)}; \lambda, \mu)$. Using Theorem 2.1, in Chapter 4 we will provide conditions under which one can embed an edge-colored $K(a^{(p)}; \lambda, \mu)$ into an edge-colored $K(a^{(p+r)}; \lambda, \mu)$ such that every color class of $K(a^{(p+r)}; \lambda, \mu)$ induces a Hamiltonian cycle. However obtaining such results will be much more complicated than for companion results for simple graphs, with a complete solution unlikely to be found in the near future.

We describe terminology and notation in Section 2.2. Then we prove the main result in Section 2.3.

2.2 Terminology and More Definitions

In this thesis, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{N}$ denotes the set of positive integers, and $\mathbb{Z}_k$ denotes the set of integers $\{1, \ldots, k\}$. If f is a function from a set X into a set Y and $y \in Y$, then $f^{-1}(y)$ denotes the set $\{x \in X : f(x) = y\}$, and $f^{-1}[y]$ denotes $\{x \in X : f(x) = y\} \backslash \{y\}$. If x, y are real numbers, then $\lfloor x \rfloor$ and $\lceil x \rceil$ denote the integers such that $x - 1 < \lfloor x \rfloor \leqslant x \leqslant \lceil x \rceil < x + 1$, and $x \approx y$ means $\lfloor y \rfloor \leqslant x \leqslant \lceil y \rceil$. We observe that for $x, y, z, x_1, \ldots x_n \in \mathbb{R}, a, b, c \in \mathbb{Z}$, and $n \in \mathbb{N}$: (i) $a \approx x$ implies $a \in \{\lfloor x \rfloor, \lceil x \rceil\}$, (ii) $x \approx y$ implies $x/n \approx y/n$ (iii) the relation $\approx$ is transitive (but not symmetric), (iv) $x_i \approx x$ for $1 \leqslant i \leqslant n$

implies $(\sum_{i=1}^{n} x_i)/n \approx x$, (v) $x \approx y$ and $y < a$ implies $x \leqslant a$, and (vi) $a = b - c$ and $c \approx x$, implies $a \approx b - x$. These properties of $\approx$ will be used in Section 2.3 when required without further explanation.

If G is a k-edge-colored graph, and if $u, v \in V(G)$ and $A, B \subset V(G)$ with $A \cap B = \varnothing$, then $m(A, B)$ denotes the total number of edges joining vertices in A to vertices in B. We refer to $m(A, B)$ as the *multiplicity* of pair A, B, naturally generalizing the multiplicity $m(u, v)$ of a pair of vertices u, v as used in [19]. In particular by $m(u, A)$ we mean $m(\{u\}, A)$. If G_1, G_2 are subgraphs of G with $V(G_1) = A$ and $V(G_2) = B$, then we let $m(G_1, G_2)$ denote $m(A, B)$, and $m(u, G_1)$ denote $m(\{u\}, A)$. The neighborhood of vertex v, written $N(v)$, denotes the set of all vertices adjacent to v (not including v).

2.3   Main Theorem

The main theorem below describes some strong properties that can be guaranteed to be satisfied by some detachment G of a given edge-colored graph H. Condition (A1) addresses the issue of P-almost-regularity (where P is a partition of $V(G)$), while conditions (A3) and (A5) address the equimultiplicity issue in G. Conditions (A1), (A3) and (A5) have companion conditions (A2), (A4) and (A6), respectively, that restricts the graphs considered to the color classes of G. Condition (A7) addresses the connectivity issue of each color class of G.

Theorem 2.1. (Bahmanian, Rodger [5, Theorem 3.1]) *Let H be a k-edge-colored graph and let η be a function from $V(H)$ into $\mathbb{N}$ such that for each $w \in V(H)$, $\eta(w) = 1$ implies $\ell_H(w) = 0$. Then there exists a loopless η-detachment G of H with amalgamation function $\psi : V(G) \to V(H)$, η being the number function associated with ψ, such that G satisfies the following conditions:*

(A1) $d_G(u) \approx d_H(w)/\eta(w)$ *for each $w \in V(H)$ and each $u \in \psi^{-1}(w)$;*

(A2) $d_{G(j)}(u) \approx d_{H(j)}(w)/\eta(w)$ *for each $w \in V(H)$, each $u \in \psi^{-1}(w)$ and each $j \in \mathbb{Z}_k$;*

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(A3) $m_G(u, u') \approx \ell_H(w)/\binom{\eta(w)}{2}$ for each $w \in V(H)$ with $\eta(w) \geqslant 2$ and every pair of distinct vertices $u, u' \in \psi^{-1}(w)$;

(A4) $m_{G(j)}(u, u') \approx \ell_{H(j)}(w)/\binom{\eta(w)}{2}$ for each $w \in V(H)$ with $\eta(w) \geqslant 2$, every pair of distinct vertices $u, u' \in \psi^{-1}(w)$ and each $j \in \mathbb{Z}_k$;

(A5) $m_G(u, v) \approx m_H(w, z)/(\eta(w)\eta(z))$ for every pair of distinct vertices $w, z \in V(H)$, each $u \in \psi^{-1}(w)$ and each $v \in \psi^{-1}(z)$;

(A6) $m_{G(j)}(u, v) \approx m_{H(j)}(w, z)/(\eta(w)\eta(z))$ for every pair of distinct vertices $w, z \in V(H)$, each $u \in \psi^{-1}(w)$, each $v \in \psi^{-1}(z)$ and each $j \in \mathbb{Z}_k$;

(A7) If for some $j \in \mathbb{Z}_k$, $d_{H(j)}(w)/\eta(w)$ is an even integer for each $w \in V(H)$, then $\omega(G(j)) = \omega(H(j))$.

Remark 2.2. All existing results in [44, 48, 58, 61, 74] study amalgamations for complete graphs or complete multipartite graphs. In these papers, Theorem 1, Theorem 1, Theorem 3.1, Theorem 2.1, and Theorem 2.1 respectively are all immediate corollaries of Theorem 2.3. Other results in the literature may have another focus, most notably in [51, 70, 74] where the edge-connectivity of each color class is specified; such results are not generalized by Theorem 2.3.

Proof. Let $H = (V, E)$ and let $n = \sum_{v \in V} (\eta(v) - 1)$. Our proof consists of the following major parts. First we shall describe the construction of a sequence of graphs $H_0 = H, H_1, \ldots, H_n$, where H_i is an amalgamation of H_{i+1} (so H_{i+1} is a detachment of H_i) for $0 \leqslant i \leqslant n - 1$ with amalgamation function ψ_i that combines a vertex with amalgamation number 1 with one other vertex. To construct each H_{i+1} from H_i we will use two bipartite graphs B_i, B_i'. Then we will observe some properties of B_i'. We will show that these properties will impose conditions on H_{i+1} in terms of H_i. The relations between H_{i+1} and H_i lead to conditions relating each H_i, $1 \leqslant i \leqslant n$ to the initial graph H. This will then show that H_n satisfies the conditions (A1)-(A7), so we can let $G = H_n$.

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Initially we let $H_0 = H, \eta_0 = \eta$, and we let ψ_0 be the identity function from V into V. Now assume that $H_0 = (V_0, E_0), \ldots, H_i = (V_i, E_i)$ and $\psi_0, \ldots, \psi_i$ have been defined for some $i \geqslant 0$. Also assume that $\eta_0 : V_0 \rightarrow \mathbb{N}, \ldots, \eta_i : V_i \rightarrow \mathbb{N}$ have been defined for some $i \geqslant 0$ such that for each $j = 0, \ldots, i$ and each $y \in V_j$, $\eta_j(y) = 1$ implies $\ell_{H_j}(y) = 0$. Let $\varphi_i = \psi_0 \ldots \psi_i$. If $i = n$, we terminate the construction, letting $G = H_n$ and $\psi = \varphi_n$. Otherwise, we can select a vertex y of H_i such that $\eta_i(y) \geqslant 2$. H_{i+1} is formed from H_i by detaching a vertex v_{i+1} with amalgamation number 1 from y.

To decide which edge (and loop) to detach from y and to move to v_{i+1}, we construct two sequences of bipartite graphs $B_0, \ldots, B_{n-1}$ and $B'_0, \ldots, B'_{n-1}$ together with a sequence $F_0, F_1, \ldots, F_{n-1}$ of sets of edges (possibly including loops) with $F_i \subset E(B'_i)$ for $i = 0, \ldots, n-1$; each edge in F_i corresponds to an edge in H_i which will have one end detached from y and joined to v_{i+1} when forming H_{i+1}.

Let $c_{i1}, \ldots, c_{ik}$ and $\mathcal{L}_i$ be distinct vertices which do not belong to V_i. Let B_i be a bipartite graph whose vertex bipartition is $\{Q_i, W_i\}$, where

$$Q_i = \{c_{i1}, \ldots, c_{ik}\} \text{ and } W_i = N_{H_i}(y) \cup \{\mathcal{L}_i\},$$

and whose edge set is

$$E(B_i) = \left(\bigcup_{\substack{\{y,u\} \in E(H_i(j)) \\ y \neq u}} \{\{c_{ij}, u\}\} \right) \bigcup \left(\bigcup_{\{y,y\} \in E(H_i(j))} \{\{c_{ij}, \mathcal{L}_i\}, \{c_{ij}, \mathcal{L}_i\}\} \right).$$

Intuitively speaking, for each color $j \in \mathbb{Z}_k$ and each vertex $u \in W_i \backslash \{\mathcal{L}_i\}$ an edge is placed between c_{ij} and u in B_i for each edge in $H_i(j)$ joining y to u. Moreover, two edges are placed between c_{ij} and $\mathcal{L}_i$ in B_i for each loop incident with y in $H_i(j)$. This is shown in Figure 2.1.
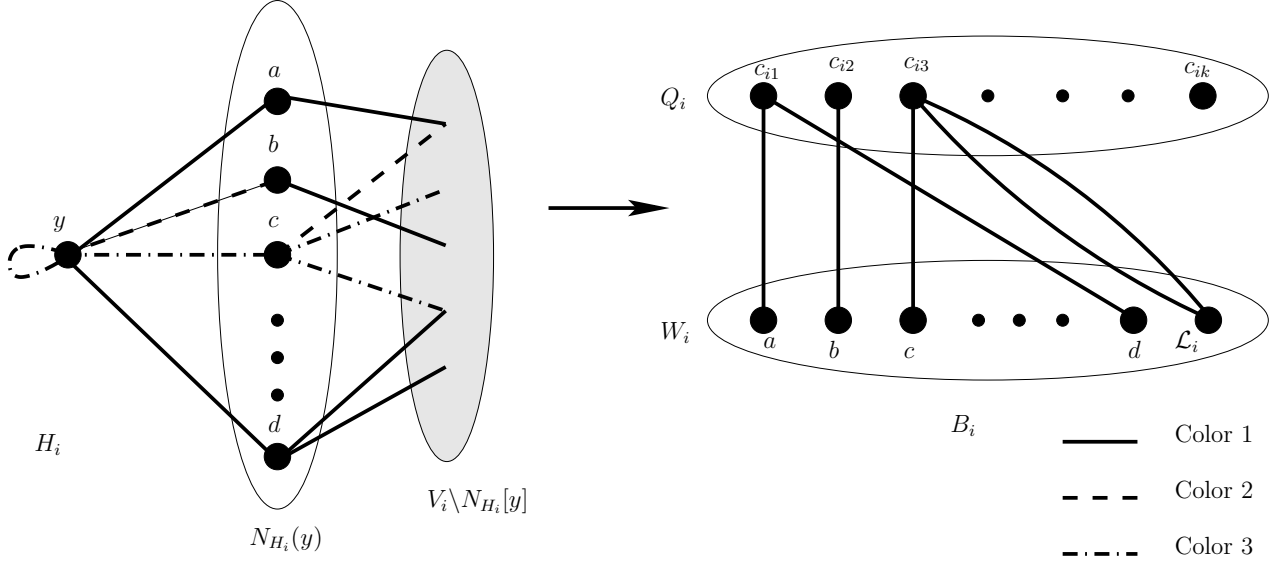
Figure 2.1: Construction of B_i from H_i

For B_i we have

$$d_{B_i}(v) = \begin{cases} d_{H_i(j)}(y) & \text{if } v = c_{ij} \text{ for some } j \in \mathbb{Z}_k \\ 2\ell_{H_i}(y) & \text{if } v = \mathcal{L}_i \\ m_{H_i}(y, v) & \text{otherwise.} \end{cases} \tag{2.1}$$

By Theorem 1.2 we can give B_i an equalized, equitable and balanced $\eta_i(y)$-edge-coloring $\mathcal{K}_i$. Since $\mathcal{K}_i$ is equitable, for each $1 \leqslant r \leqslant \eta_i(y)$, we have

$$d_{B_i(r)}(v) \approx \begin{cases} d_{H_i(j)}(y)/\eta_i(y) & \text{if } v = c_{ij} \text{ for some } j \in \mathbb{Z}_k \\ 2\ell_{H_i}(y)/\eta_i(y) & \text{if } v = \mathcal{L}_i \\ m_{H_i}(y, v)/\eta_i(y) & \text{otherwise.} \end{cases} \tag{2.2}$$

Now let T_i be formed by a subgraph of B_i induced by the edges colored 1 and 2. Since $\eta_i(y) \geqslant 2$, this is always possible. For each color $j \in \mathbb{Z}_k$ for which

$$\text{for all } v \in V_i, d_{H_i}(j)(v)/\eta_i(v) \text{ is an even integer,} \tag{2.3}$$

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define $\alpha_{ij} = d_{H_i(j)}(y)/\eta_i(y)$. By (2.2) for each color class r of $\mathcal{K}_i$, $d_{B_i(r)}(c_{ij}) \approx d_{H_i(j)}(y)/\eta_i(y)$. Therefore since two color classes of $\mathcal{K}_i$ are chosen to form T_i, if (2.3) is satisfied, then $d_{T_i}(c_{ij}) = 2d_{H_i(j)}(y)/\eta_i(y) = 2\alpha_{ij}$.

Let B_i' be the bipartite graph whose vertex bipartition is $\{Q_i', W_i\}$, obtained by splitting all the vertices c_{ij} in T_i for each $j \in \mathbb{Z}_k$ for which condition (2.3) holds, into α_{ij} vertices $c_{i,j,1}, \ldots, c_{i,j,\alpha_{ij}}$ all of degree 2 as described in (M1)-(M2) below. (We don't split vertices c_{ij} in T_i for $j \in \mathbb{Z}_k$ for which condition (2.3) does not hold; but they and their incident edges remain in B_i'.)

(M1) First, as many of $c_{i,j,t}$'s $1 \leqslant t \leqslant \alpha_{ij}$ as possible are joined by 2 edges to the same vertex in W_i;

(M2) Then, among all $c_{i,j,t}$'s $1 \leqslant t \leqslant \alpha_{ij}$ with valency less than 2, as many of them as possible are incident with two edges that correspond to edges in $H_i(j)$ that join y to vertices that are both in the same component of $H_i(j)\backslash\{y\}$.

For each $j \in \mathbb{Z}_k$ that satisfies condition (2.3), we let $\mathcal{C}_{ij} = \bigcup_{t=1}^{\alpha_{ij}} \{c_{i,j,t}\}$. Otherwise, we let $\mathcal{C}_{ij} = \{c_{ij}\}$. By Theorem 1.2, we can give B_i' an equalized, equitable and balanced 2-edge-coloring $\mathcal{K}_i'$. This gives us two color classes either of which can be chosen to be F_i, say the edges colored 1 are chosen. Since $\mathcal{K}_i'$ is equitable, we have

$$
d_{B_i'(1)}(v) \approx \begin{cases} d_{H_i(j)}(y)/\eta_i(y) & \text{if } v = c_{ij} \text{ for } j \in \mathbb{Z}_k \text{ for which (2.3) does not hold} \\ 1 & \text{if } v \in \mathcal{C}_{ij} \text{ for } j \in \mathbb{Z}_k \text{ for which (2.3) holds} \\ 2\ell_{H_i}(y)/\eta_i(y) & \text{if } v = \mathcal{L}_i \\ m_{H_i}(y,v)/\eta_i(y) & \text{otherwise.} \end{cases} \tag{2.4}
$$

Now we let

$$
A_{ij} = \Big(\bigcup_{\substack{\{c,v\}\in F_i \\ c\in\mathcal{C}_{ij}}} \{y,v\} \Big) \bigcup \Big(\bigcup_{\substack{\{c,\mathcal{L}_i\}\in F_i \\ c\in\mathcal{C}_{ij}}} \{y,y\} \Big)
$$

and

$$B_{ij} = \left(\bigcup_{\substack{\{c,v\}\in F_i \\ c\in \mathcal{C}_{ij}}} \{v_{i+1}, v\} \right) \bigcup \left(\bigcup_{\substack{\{c,\mathcal{L}_i\}\in F_i \\ c\in \mathcal{C}_{ij}}} \{v_{i+1}, y\} \right),$$

where v_{i+1} is a vertex which does not belong to V_i. Let $V_{i+1} = V_i \cup \{v_{i+1}\}$, and let ψ_{i+1} be a function from V_{i+1} onto V_i such that

$$\psi_{i+1}(v) = \begin{cases} y & \text{if } v = v_{i+1} \\ v & \text{otherwise.} \end{cases}$$

Let $H_{i+1} = (V_{i+1}, E_{i+1})$ be the ψ_{i+1}-detachment of H_i such that for each $j \in \mathbb{Z}_k$

$$E(H_{i+1}(j)) = (E(H_i(j)) \backslash A_{ij}) \cup B_{ij},$$

and $E_{i+1} = \bigcup_{j=1}^{k} E(H_{i+1}(j))$.

Intuitively speaking, H_{i+1} is formed as follows. Each edge $\{c, v\} \in F_i$ with $c \in \mathcal{C}_{ij}$ and $v \in W_i \backslash \{\mathcal{L}_i\}$ directly corresponds to an edge $\{y, v\}$ in $H_i(j)$; replace $\{y, v\}$ with the edge $\{v, v_{i+1}\}$ colored j in H_{i+1}. So in forming $H_{i+1}(j)$ from $H_i(j)$ the end of this edge is detached from v and joined to the new vertex v_{i+1} instead. Moreover, we remove $m_{B_i'(1)}(\mathcal{C}_{ij}, \mathcal{L}_i)$ loops colored j incident with y in H_i and we replace them with $m_{B_i'(1)}(\mathcal{C}_{ij}, \mathcal{L}_i)$ edges colored j joining y to v_{i+1} in H_{i+1}. Note that since $\mathcal{K}_i'$ is balanced, $\eta_i(y) \geqslant 2$ and $\lceil d_{B_i'}(\mathcal{L}_i)/2 \rceil \leqslant \lceil d_{B_i}(\mathcal{L}_i)/2 \rceil = \ell_{H_i}(y)$, at most half of the edges in B_i' incident with $\mathcal{L}_i$ are colored 1, so there are indeed $m_{B_i'(1)}(\mathcal{C}_{ij}, \mathcal{L}_i)$ loops incident with y in H_i (recall that each loop in H_i corresponds to two edges in B_i').

Obviously, ψ_{i+1} is an amalgamation function from H_{i+1} into H_i. Let η_{i+1} be the function from V_{i+1} into $\mathbb{N}$ such that

$$\eta_{i+1}(v) = \begin{cases} 1 & \text{if } v = v_{i+1} \\ \eta_i(v) - 1 & \text{if } v = y \\ \eta_i(v) & \text{otherwise.} \end{cases}$$

We now check that B_i', described above, satisfies the following conditions for each color $j \in \mathbb{Z}_k$:

(P1) $m_{B_i'(1)}(\mathcal{C}_{ij}, \mathcal{L}_i) \approx 2\ell_{H_i(j)}(y)/\eta_i(y)$;

(P2) $m_{B_i'(1)}(\mathcal{C}_{ij}, v) \approx m_{H_i(j)}(y, v)/\eta_i(y)$ for each $v \in W_i\backslash\{\mathcal{L}_i\}$;

(P3) $m_{B_i'(1)}(Q_i', W_i) \approx d_{H_i}(y)/\eta_i(y)$;

(P4) $m_{B_i'(1)}(\mathcal{C}_{ij}, W_i) \approx d_{H_i(j)}(y)/\eta_i(y)$.

In order to prove (P1) and (P2) first we show that

$$m_{B_i'(1)}(\mathcal{C}_{ij}, v) \approx \frac{m_{B_i'}(\mathcal{C}_{ij}, v)}{2} \text{ for each } v \in W_i'.$$

There are two cases:

- Case 1: $\mathcal{C}_{ij} = \{c_{ij}\}$. Since $\mathcal{K}_i'$ is balanced,

$$m_{B_i'(1)}(\mathcal{C}_{ij}, v) = m_{B_i'(1)}(c_{ij}, v) \approx \frac{m_{B_i'}(c_{ij}, v)}{2} = \frac{m_{B_i'}(\mathcal{C}_{ij}, v)}{2}.$$

- Case 2: $\mathcal{C}_{ij} = \bigcup\limits_{t=1}^{\alpha_{ij}} \{c_{i,j,t}\}$. By (M1), among all vertices in $\mathcal{C}_{ij}$, there are exactly $\lfloor m_{B_i'}(\mathcal{C}_{ij}, v)/2 \rfloor$ vertices of degree 2 which are joined to v (at most one vertex in $\mathcal{C}_{ij}$ is joined to v by one edge). Since $\mathcal{K}_i'$ is balanced(or equitable), among these vertices

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of degree 2, exactly one of them is joined to v by an edge colored 1. Therefore

$$m_{B_i'(1)}(\mathcal{C}_{ij}, v) = \sum_{t=1}^{\alpha_{ij}} m_{B_i'(1)}(c_{i,j,t}, v) \approx \frac{m_{B_i'}(\mathcal{C}_{ij}, v)}{2}.$$

Clearly $m_{B_i'}(\mathcal{C}_{ij}, v) = m_{T_i}(c_{ij}, v) = m_{B_i(1)}(c_{ij}, v) + m_{B_i(2)}(c_{ij}, v)$. If $v = \mathcal{L}_i$, from the definition of B_i it follows that $m_{B_i}(c_{ij}, \mathcal{L}_i) = 2\ell_{H_i(j)}(y)$. Since $\mathcal{K}_i$ is balanced, for each $1 \leqslant r \leqslant \eta_i(y)$ we have $m_{B_i(r)}(c_{ij}, \mathcal{L}_i) \approx 2\ell_{H_i(j)}(y)/\eta_i(y)$. Therefore

$$m_{B_i'(1)}(\mathcal{C}_{ij}, \mathcal{L}_i) \approx \frac{m_{B_i'}(\mathcal{C}_{ij}, \mathcal{L}_i)}{2} = \frac{m_{T_i}(c_{ij}, \mathcal{L}_i)}{2} \approx \frac{2\ell_{H_i(j)}(y)}{\eta_i(y)}.$$

This proves (P1).

Now let $v \in W_i \backslash \{\mathcal{L}_i\}$. From the definition of B_i it follows that $m_{B_i}(c_{ij}, v) = m_{H_i(j)}(y, v)$. Since $\mathcal{K}_i$ is balanced, for each $1 \leqslant r \leqslant \eta_i(y)$ we have $m_{B_i(r)}(c_{ij}, v) \approx m_{H_i(j)}(y, v)/\eta_i(y)$. Therefore

$$m_{B_i'(1)}(\mathcal{C}_{ij}, v) \approx \frac{m_{B_i'}(\mathcal{C}_{ij}, v)}{2} = \frac{m_{T_i}(c_{ij}, v)}{2} \approx \frac{m_{H_i(j)}(y, v)}{\eta_i(y)}.$$

This proves (P2).

Since $\mathcal{K}_i'$ is equalized, $m_{B_i'(1)}(Q_i', W_i) = |E(B_i'(1))| \approx m_{B_i'}(Q_i', W_i)/2$. Clearly $m_{B_i'}(Q_i', W_i) = |E(B_i')| = m_{T_i}(Q_i, W_i) = m_{B_i(1)}(Q_i, W_i) + m_{B_i(2)}(Q_i, W_i)$. From the definition of B_i it follows that $m_{B_i}(Q_i, W_i) = |E(B_i)| = d_{H_i}(y)$. Since $\mathcal{K}_i$ is equalized, for each $1 \leqslant r \leqslant \eta_i(y)$ we have $m_{B_i(r)}(Q_i, W_i) = |E(B_i(r))| \approx d_{H_i}(y)/\eta_i(y)$. Therefore

$$m_{B_i'(1)}(Q_i', W_i) \approx \frac{m_{B_i'}(Q_i', W_i)}{2} = \frac{m_{T_i}(Q_i, W_i)}{2} \approx \frac{d_{H_i}(y)}{\eta_i(y)}.$$

This proves (P3).

In order to prove (P4), there are two cases:

- Case 1: $\mathcal{C}_{ij} = \{c_{ij}\}$. From (2.4) it follows that

$$m_{B'_i(1)}(\mathcal{C}_{ij}, W_i) = m_{B'_i(1)}(c_{ij}, W_i) = d_{B'_i(1)}(c_{ij}) \approx \frac{d_{H_i(j)}(y)}{\eta_i(y)}.$$

- Case 2: $\mathcal{C}_{ij} = \bigcup\limits_{t=1}^{\alpha_{ij}} \{c_{i,j,t}\}$. In this case $m_{B'_i(1)}(\mathcal{C}_{ij}, W_i) = \sum\limits_{t=1}^{\alpha_{ij}} m_{B'_i(1)}(c_{i,j,t}, W_i)$. From (2.4) it follows that

$$m_{B'_i(1)}(\mathcal{C}_{ij}, W_i) = \sum_{t=1}^{\alpha_{ij}} 1 = \alpha_{ij} = \frac{d_{H_i(j)}(y)}{\eta_i(y)}.$$

This proves (P4).

Most of the conditions that H_{i+1} must satisfy, are numerical, and we consider them first. The reader who is more interested in the connectivity issue, namely property (A7), may wish to jump to the consideration of conditions (D1)-(D2) on the last three pages of this section.

Using (2.4) and (P1)-(P4), now we show that H_{i+1}, described above, satisfies the following conditions:

(B1) $\ell_{H_{i+1}}(y) \approx \ell_{H_i}(y)(\eta_{i+1}(y) - 1)/\eta_i(y)$;

(B2) $\ell_{H_{i+1}(j)}(y) \approx \ell_{H_i(j)}(y)(\eta_{i+1}(y) - 1)/\eta_i(y)$ for each $j \in \mathbb{Z}_k$;

(B3) (i) $d_{H_{i+1}}(y)/\eta_{i+1}(y) \approx d_{H_i}(y)/\eta_i(y)$,

  (ii) $d_{H_{i+1}}(v_{i+1}) \approx d_{H_i}(y)/\eta_i(y)$;

(B4) For each $j \in \mathbb{Z}_k$

  (i) $d_{H_{i+1}(j)}(y)/\eta_{i+1}(y) \approx d_{H_i(j)}(y)/\eta_i(y)$,

  (ii) $d_{H_{i+1}(j)}(v_{i+1}) \approx d_{H_i(j)}(y)/\eta_i(y)$;

(B5) For each $v \in N_{H_i}(y)$

  (i) $m_{H_{i+1}}(y, v)/\eta_{i+1}(y) \approx m_{H_i}(y, v)/\eta_i(y)$,

(ii) $m_{H_{i+1}}(v_{i+1}, v) \approx m_{H_i}(y, v)/\eta_i(y)$,

(iii) $m_{H_{i+1}}(y, v_{i+1})/\eta_{i+1}(y) \approx \ell_{H_i}(y)/\binom{\eta_i(y)}{2}$;

(B6) For each $v \in N_{H_i}(y)$, and each $j \in \mathbb{Z}_k$

(i) $m_{H_{i+1}(j)}(y, v)/\eta_{i+1}(y) \approx m_{H_i}(j)(y, v)/\eta_i(y)$,

(ii) $m_{H_{i+1}(j)}(v_{i+1}, v) \approx m_{H_{i(j)}}(y, v)/\eta_i(y)$,

(iii) $m_{H_{i+1}(j)}(y, v_{i+1})/\eta_{i+1}(y) \approx \ell_{H_i(j)}(y)/\binom{\eta_i(y)}{2}$.

Note that $\eta_{i+1}(y) = \eta_i(y) - 1$. Let us fix $v \in N_{H_i}(y)$, and $j \in \mathbb{Z}_k$.

From the construction of H_{i+1}, we have $\ell_{H_{i+1}}(y) = \ell_{H_i}(y) - d_{B_i'(1)}(\mathcal{L}_i)$. By (2.4), $d_{B_i'(1)}(\mathcal{L}_i) \approx 2\ell_{H_i}(y)/\eta_i(y)$. Hence

$$\ell_{H_{i+1}}(y) \approx \ell_{H_i}(y) - \frac{2\ell_{H_i}(y)}{\eta_i(y)} = \frac{\ell_{H_i}(y)(\eta_i(y) - 2)}{\eta_i(y)} = \frac{\ell_{H_i}(y)(\eta_{i+1}(y) - 1)}{\eta_i(y)}.$$

This completes the proof of (B1).

Clearly, $\ell_{H_{i+1}(j)}(y) = \ell_{H_i(j)}(y) - m_{B_i'(1)}(\mathcal{C}_{ij}, \mathcal{L}_i)$. By (P1), $m_{B_i'(1)}(\mathcal{C}_{ij}, \mathcal{L}_i) \approx 2\ell_{H_i(j)}(y)/\eta_i(y)$. Hence

$$\ell_{H_{i+1}(j)}(y) \approx \ell_{H_i(j)}(y) - \frac{2\ell_{H_i(j)}(y)}{\eta_i(y)} = \frac{\ell_{H_i(j)}(y)(\eta_i(y) - 2)}{\eta_i(y)} = \frac{\ell_{H_i(j)}(y)(\eta_{i+1}(y) - 1)}{\eta_i(y)}.$$

This completes the proof of (B2).

Construction of H_{i+1} follows that, $d_{H_{i+1}}(y) = d_{H_i}(y) - m_{B_i'(1)}(Q_i', W_i)$, and $d_{H_{i+1}}(v_{i+1}) = m_{B_i'(1)}(Q_i', W_i)$. By (P3), $m_{B_i'(1)}(Q_i', W_i) \approx d_{H_i}(y)/\eta_i(y)$. Hence

$$d_{H_{i+1}}(y) \approx d_{H_i}(y) - \frac{d_{H_i}(y)}{\eta_i(y)} = \frac{d_{H_i}(y)(\eta_i(y) - 1)}{\eta_i(y)} = \frac{d_{H_i}(y)\eta_{i+1}(y)}{\eta_i(y)},$$

and $d_{H_{i+1}}(v_{i+1}) \approx d_{H_i}(y)/\eta_i(y)$. This completes the proof of (B3).

From the construction of H_{i+1}, we have that $d_{H_{i+1}(j)}(y) = d_{H_i(j)}(y) - m_{B_i'(1)}(\mathcal{C}_{ij}, W_i)$, and $d_{H_{i+1}(j)}(v_{i+1}) = m_{B_i'(1)}(\mathcal{C}_{ij}, W_i)$. By (P4), $m_{B_i'(1)}(\mathcal{C}_{ij}, W_i) \approx d_{H_i(j)}(y)/\eta_i(y)$. Hence

$$d_{H_{i+1}(j)}(y) \approx d_{H_i(j)}(y) - \frac{d_{H_i(j)}(y)}{\eta_i(y)} = \frac{d_{H_i(j)}(y)(\eta_i(y) - 1)}{\eta_i(y)} = \frac{d_{H_i(j)}(y)\eta_{i+1}(y)}{\eta_i(y)},$$

and $d_{H_{i+1}(j)}(v_{i+1}) \approx d_{H_i(j)}(y)/\eta_i(y)$. This completes the proof of (B4).

It is easy to see that, $m_{H_{i+1}}(y, v) = m_{H_i}(y, v) - m_{B_i'(1)}(Q_i', v) = m_{H_i}(y, v) - d_{B_i'(1)}(v)$, and $m_{H_{i+1}}(v_{i+1}, v) = d_{B_i'(1)}(v)$. By (2.4), $d_{B_i'(1)}(v) \approx m_{H_i}(y, v)/\eta_i(y)$. Hence

$$m_{H_{i+1}}(y, v) \approx m_{H_i}(y, v) - \frac{m_{H_i}(y, v)}{\eta_i(y)} = \frac{m_{H_i}(y, v)(\eta_i(y) - 1)}{\eta_i(y)} = \frac{m_{H_i}(y, v)\eta_{i+1}(y)}{\eta_i(y)},$$

and $m_{H_{i+1}}(v_{i+1}, v) \approx m_{H_i}(y, v)/\eta_i(y)$. Moreover, $m_{H_{i+1}}(y, v_{i+1}) = m_{B_i'(1)}(Q_i', \mathcal{L}_i) = d_{B_i'(1)}(\mathcal{L}_i)$. By (2.4), $d_{B_i'(1)}(\mathcal{L}_i) \approx 2\ell_{H_i}(y)/\eta_i(y)$. Therefore $m_{H_{i+1}}(y, v_{i+1}) \approx 2\ell_{H_i}(y)/\eta_i(y)$. Hence

$$\frac{m_{H_{i+1}}(y, v_{i+1})}{\eta_{i+1}(y)} \approx \frac{2\ell_{H_i}(y)}{\eta_i(y)\eta_{i+1}(y)} = \frac{\ell_{H_i}(y)}{\binom{\eta_i(y)}{2}}.$$

This completes the proof of (B5).

Finally, from the construction of H_{i+1}, $m_{H_{i+1}(j)}(y, v) = m_{H_i(j)}(y, v) - m_{B_i'(1)}(\mathcal{C}_{ij}, v)$, and $m_{H_{i+1}(j)}(v_{i+1}, v) = m_{B_i'(1)}(\mathcal{C}_{ij}, v)$. By (P2), $m_{B_i'(1)}(\mathcal{C}_{ij}, v) \approx m_{H_i(j)}(y, v)/\eta_i(y)$. Hence

$$m_{H_{i+1}}(y, v) \approx m_{H_i(j)}(y, v) - \frac{m_{H_i(j)}(y, v)}{\eta_i(y)} = \frac{m_{H_i(j)}(y, v)(\eta_i(y) - 1)}{\eta_i(y)} = \frac{m_{H_i(j)}(y, v)\eta_{i+1}(y)}{\eta_i(y)},$$

and $m_{H_{i+1}(j)}(v_{i+1}, v) \approx m_{H_i(j)}(y, v)/\eta_i(y)$. Moreover, $m_{H_{i+1}(j)}(y, v_{i+1}) = m_{B_i'(1)}(\mathcal{C}_{ij}, \mathcal{L}_i)$. By (P1), $m_{B_i'(1)}(\mathcal{C}_{ij}, \mathcal{L}_i) \approx 2\ell_{H_i(j)}(y)/\eta_i(y)$. Therefore $m_{H_{i+1}(j)}(y, v_{i+1}) \approx 2\ell_{H_i(j)}(y)/\eta_i(y)$. Hence

$$\frac{m_{H_{i+1}(j)}(y, v_{i+1})}{\eta_{i+1}(y)} \approx \frac{2\ell_{H_i(j)}(y)}{\eta_i(y)\eta_{i+1}(y)} = \frac{\ell_{H_i(j)}(y)}{\binom{\eta_i(y)}{2}}.$$

This completes the proof of (B6).

Recall that $\varphi_i = \psi_0 \ldots \psi_i$, that $\psi_0 : V \to V$, and that $\psi_i : V_i \to V_{i-1}$ for $i > 0$. Therefore $\varphi_i : V_i \to V$ and thus $\varphi_i^{-1} : V \to V_i$. Now we use (B1)-(B6) to prove that for $0 \leqslant i \leqslant n$, H_i satisfies the following conditions:

(C1)   (i) $\ell_{H_i}(w)/\binom{\eta_i(w)}{2} \approx \ell_H(w)/\binom{\eta(w)}{2}$ for each $w \in V$ with $\eta(w) \geqslant 2$, $\eta_i(w) \geqslant 2$,

    (ii) $\ell_{H_i}(w) = \ell_{H_i}(v_r) = 0$ for each $w \in V$ with $\eta_i(w) = 1$ and each $1 \leqslant r \leqslant i$;

(C2) $\ell_{H_i(j)}(w)/\binom{\eta_i(w)}{2} \approx \ell_{H(j)}(w)/\binom{\eta(w)}{2}$ for each $w \in V$ with $\eta(w) \geqslant 2$, $\eta_i(w) \geqslant 2$ and each $j \in \mathbb{Z}_k$;

(C3) For each $w \in V$

    (i) $d_{H_i}(w)/\eta_i(w) \approx d_H(w)/\eta(w)$,

    (ii) $d_{H_i}(v_r) \approx d_H(w)/\eta(w)$ for each $v_r \in \varphi_i^{-1}[w]$;

(C4) For each $w \in V$ and each $j \in \mathbb{Z}_k$

    (i) $d_{H_i(j)}(w)/\eta_i(w) \approx d_{H(j)}(w)/\eta(w)$,

    (ii) $d_{H_i(j)}(v_r) \approx d_{H(j)}(w)/\eta(w)$ for each $v_r \in \varphi_i^{-1}[w]$;

(C5) For each $w \in V$

    (i) $m_{H_i}(w, v_r)/\eta_i(w) \approx \ell_H(w)/\binom{\eta(w)}{2}$ for each $v_r \in \varphi_i^{-1}[w]$,

    (ii) $m_{H_i}(v_r, v_s) \approx \ell_H(w)/\binom{\eta(w)}{2}$ for every pair of distinct vertices $v_r, v_s \in \varphi_i^{-1}[w]$;

(C6) For each $w \in V$, and each $j \in \mathbb{Z}_k$

    (i) $m_{H_i(j)}(w, v_r)/\eta_i(w) \approx \ell_{H(j)}(w)/\binom{\eta(w)}{2}$ for each $v_r \in \varphi_i^{-1}[w]$,

(ii) $m_{H_i(j)}(v_r, v_s) \approx \ell_{H(j)}(w)/\binom{\eta(w)}{2}$ for every pair of distinct vertices $v_r, v_s \in \varphi_i^{-1}[w]$;

(C7) For every pair of distinct vertices $w, z \in V$

    (i) $m_{H_i}(w, z)/(\eta_i(w)\eta_i(z)) \approx m_H(w, z)/(\eta(w)\eta(z))$,

    (ii) $m_{H_i}(v_r, v_s) \approx m_H(w, z)/(\eta(w)\eta(z))$ for each $v_r \in \varphi_i^{-1}[w]$ and each $v_s \in \varphi_i^{-1}[z]$,

    (iii) $m_{H_i}(w, v_s)/\eta_i(w) \approx m_H(w, z)/(\eta(w)\eta(z))$ for each $v_s \in \varphi_i^{-1}[z]$;

(C8) For every pair of distinct vertices $w, z \in V$, and each $j \in \mathbb{Z}_k$

    (i) $m_{H_i(j)}(w, z)/(\eta_i(w)\eta_i(z)) \approx m_{H(j)}(w, z)/(\eta(w)\eta(z))$,

    (ii) $m_{H_i(j)}(v_r, v_s) \approx m_{H(j)}(w, z)/(\eta(w)\eta(z))$ for each $v_r \in \varphi_i^{-1}[w]$ and each $v_s \in \varphi_i^{-1}[z]$,

    (iii) $m_{H_i(j)}(w, v_s)/\eta_i(w) \approx m_{H(j)}(w, z)/(\eta(w)\eta(z))$ for each $v_s \in \varphi_i^{-1}[z]$.

Let w, z be an arbitrary pair of distinct vertices of V, and let $j \in \mathbb{Z}_k$. We prove (C1)-(C8) by induction. Let us first verify (C1)-(C8) for $i = 0$. Recall that $H_0 = H$, and $\eta_0(w) = \eta(w)$.

If $\eta(w) \geqslant 2$, obviously $\ell_{H_0}(w)/\binom{\eta_0(w)}{2} = \ell_H(w)/\binom{\eta(w)}{2}$. If $\eta(w) = 1$, by hypothesis of Theorem 2.1, $\ell_H(w) = 0$. This proves (C1) for $i = 0$. (C2) can be proved in a similar way. Obviously $d_{H_0}(w)/\eta_0(w) = d_H(w)/\eta(w)$ and (C3)(ii) is obvious, so this proves (C3) for $i = 0$. The proof for (C4) is similar and (C5)-(C8) are sufficiently obvious.

Now we will show that if H_i satisfies the conditions (C1) - (C8) for some $i < n$, then H_{i+1} (formed from H_i by detaching v_{i+1} from the vertex y) satisfies these conditions by replacing i with $i + 1$; we denote the corresponding conditions for H_{i+1} by (C1)'-(C8)'. If $\eta_{i+1}(w) = \eta_i(w)$, then (C1)'-(C6)' are obviously true. So we just check (C1)'-(C6)' in the case where $w = y$. Also if $\eta_{i+1}(w) = \eta_i(w)$ and $\eta_{i+1}(z) = \eta_i(z)$, then (C7)'-(C8)' are clearly true. So in order to prove (C7)' - (C8)' we shall assume that either $\eta_{i+1}(w) = \eta_i(w) - 1$ or $\eta_{i+1}(z) = \eta_i(z) - 1$. (so $y \in \{w, z\}$; the asymmetry in condition (iii) of (C7)' and (C8)' prevents us from assuming that $w = y$.)

(C1)' If $\eta_{i+1}(y) \geqslant 2$, by (B1) $\ell_{H_{i+1}}(y) \approx \ell_{H_i}(y)(\eta_{i+1}(y)-1)/\eta_i(y)$, and by (C1)(i) of the induction hypothesis, $\ell_{H_i}(y)/\binom{\eta_i(y)}{2} \approx \ell_H(y)/\binom{\eta(y)}{2}$. Also note that $\binom{\eta_i(y)}{2} = \eta_i(y)(\eta_i(y)-1)/2$.

Therefore

$$\frac{\ell_{H_{i+1}}(y)}{\binom{\eta_{i+1}(y)}{2}} \approx \frac{\ell_{H_i}(y)(\eta_{i+1}(y) - 1)}{\binom{\eta_{i+1}(y)}{2}\eta_i(y)} = \frac{\ell_{H_i}(y)}{\binom{\eta_i(y)}{2}} \approx \frac{\ell_H(y)}{\binom{\eta(y)}{2}}.$$

This proves (C1)$'$(i).

Clearly $\ell_{H_{i+1}}(v_{i+1}) = 0$ and $\ell_{H_{i+1}}(v_r) = \ell_{H_i}(v_r) = 0$ for each $1 \leqslant r \leqslant i$. Therefore $\ell_{H_{i+1}}(v_r) = 0$ for each $1 \leqslant r \leqslant i + 1$. Also if $\eta_{i+1}(y) = 1$, by (B1) $\ell_{H_{i+1}}(y) = 0$. This proves (C1)$'$(ii).

(C2)$'$ The proof is similar to the proof of (C1)$'$(i), following from (B2) and (C2) of the induction hypothesis.

(C3)$'$ By (B3)(i), $d_{H_{i+1}}(y)/\eta_{i+1}(y) \approx d_{H_i}(y)/\eta_i(y)$, and by (C3)(i) of the induction hypothesis, $d_{H_i}(y)/\eta_i(y) \approx d_H(y)/\eta(y)$. Therefore

$$\frac{d_{H_{i+1}}(y)}{\eta_{i+1}(y)} \approx \frac{d_H(y)}{\eta(y)}.$$

This proves (C3)$'$(i).

By (B3)(ii), $d_{H_{i+1}}(v_{i+1}) \approx d_{H_i}(y)/\eta_i(y)$, and by (C3)(ii) of the induction hypothesis, $d_{H_i}(v_r) \approx d_H(y)/\eta(y)$ for each $v_r \in \varphi_i^{-1}[y]$. Since in forming H_{i+1} no edge is detached from v_r for each $v_r \in \varphi_i^{-1}[y]$, we have $d_{H_{i+1}}(v_r) = d_{H_i}(v_r)$. Therefore $d_{H_{i+1}}(v_r) \approx d_H(y)/\eta(y)$ for each $v_r \in \varphi_{i+1}^{-1}[y]$. This proves (C3)$'$(ii).

(C4)$'$ The proof is similar to the proof of (C3)$'$, following from (B4) and (C4) of the induction hypothesis.

(C5)′ By (B5)(i), $m_{H_{i+1}}(y, v_r)/\eta_{i+1}(y) \approx m_{H_i}(y, v_r)/\eta_i(y)$ for each $v_r \in \varphi_i^{-1}[y]$. By (C5)(i) of the induction hypothesis, $m_{H_i}(y, v_r)/\eta_i(y) \approx \ell_H(y)/\binom{\eta(y)}{2}$ for each $v_r \in \varphi_i^{-1}[y]$. Therefore

$$\frac{m_{H_{i+1}}(y, v_r)}{\eta_{i+1}(y)} \approx \frac{\ell_H(y)}{\binom{\eta(y)}{2}}.$$

for each $v_r \in \varphi_i^{-1}[y]$. Moreover, by (B5)(iii) $m_{H_{i+1}}(y, v_{i+1})/\eta_{i+1}(y) \approx \ell_{H_i}(y)/\binom{\eta_i(y)}{2}$, and by (C1)(i) of the induction hypothesis, $\ell_{H_i}(y) \approx \ell_H(y)\binom{\eta_i(y)}{2}/\binom{\eta(y)}{2}$. Therefore

$$\frac{m_{H_{i+1}}(y, v_{i+1})}{\eta_{i+1}(y)} \approx \frac{\ell_H(y)\binom{\eta_i(y)}{2}}{\binom{\eta(y)}{2}\binom{\eta_i(y)}{2}} = \frac{\ell_H(y)}{\binom{\eta(y)}{2}}.$$

This proves (C5)′(i).

By (B5)(ii), $m_{H_{i+1}}(v_{i+1}, v_r) \approx m_{H_i}(y, v_r)/\eta_i(y)$ for each $v_r \in \varphi_i^{-1}[y]$. By (C5)(i) of the induction hypothesis, $m_{H_i}(y, v_r)/\eta_i(y) \approx \ell_H(y)/\binom{\eta(y)}{2}$ for each $v_r \in \varphi_i^{-1}[y]$. Therefore

$$m_{H_{i+1}}(v_{i+1}, v_r) \approx \frac{\ell_H(y)}{\binom{\eta(y)}{2}}$$

for each $v_r \in \varphi_i^{-1}[y]$. By (C5)(ii) of the induction hypothesis, $m_{H_i}(v_r, v_s) \approx \ell_H(y)/\binom{\eta(y)}{2}$ for every pair of distinct vertices $v_r, v_s \in \varphi_i^{-1}[y]$. Since in forming H_{i+1} no edge is detached from v_r for each $v_r \in \varphi_i^{-1}[y]$, we have $m_{H_{i+1}}(v_r, v_s) = m_{H_i}(v_r, v_s)$. Therefore

$$m_{H_{i+1}}(v_r, v_s) \approx \frac{\ell_H(y)}{\binom{\eta(y)}{2}}$$

for every pair of distinct vertices $v_r, v_s \in \varphi_{i+1}^{-1}[y]$. This proves (C5)′(ii).

(C6)′ The proof is similar to the proof of (C5)′, following from (B6) and (C6) of the induction hypothesis.

(C7)′ If $z \notin N_H(w)$ then $m_H(w, z) = 0$ and (C7)′ is trivial. So we assume that $z \in N_H(w)$.

(i) If $\eta_{i+1}(w) = \eta_i(w) - 1$ (so $w = y$), by (B5)(i) $m_{H_{i+1}}(y,z)/\eta_{i+1}(y) \approx m_{H_i}(y,z)/\eta_i(y)$, and since $\eta_{i+1}(z) = \eta_i(z)$, we have $m_{H_{i+1}}(y,z)/(\eta_{i+1}(y)\eta_{i+1}(z)) \approx m_{H_i}(y,z)/(\eta_i(y)\eta_i(z))$. By (C7)(i) of the induction hypothesis, $m_{H_i}(y,z)/(\eta_i(y)\eta_i(z)) \approx m_H(y,z)/(\eta(y)\eta(z))$. Therefore

$$\frac{m_{H_{i+1}}(y,z)}{\eta_{i+1}(y)\eta_{i+1}(z)} \approx \frac{m_H(y,z)}{\eta(y)\eta(z)}.$$

The other case, $\eta_{i+1}(z) = \eta_i(z) - 1$), is similar. This proves (C7)'(i).

(ii) By (C7)(ii) of the induction hypothesis $m_{H_i}(v_r, v_s) \approx m_H(w,z)/(\eta(w)\eta(z))$ for each $v_r \in \varphi_i^{-1}[w]$ and each $v_s \in \varphi_i^{-1}[z] = \varphi_{i+1}^{-1}[z]$. Since in forming H_{i+1} no edge is detached from v_r and v_s for each $v_r \in \varphi_i^{-1}[w]$ and each $v_s \in \varphi_i^{-1}[z]$, we have $m_{H_{i+1}}(v_r, v_s) = m_{H_i}(v_r, v_s)$. Therefore $m_{H_{i+1}}(v_r, v_s) \approx m_H(w,z)/(\eta(w)\eta(z))$ for each $v_r \in \varphi_i^{-1}[w]$ and each $v_s \in \varphi_{i+1}^{-1}[z]$. If $\eta_{i+1}(y) = \eta_i(y) - 1$ (so $w = y$), by (B5)(ii) $m_{H_{i+1}}(v_{i+1}, v_s) \approx m_{H_i}(y, v_s)/\eta_i(y)$ for each $v_s \in \varphi_i^{-1}[z] = \varphi_{i+1}^{-1}[z]$. By (C7)(iii) of induction hypothesis, $m_{H_i}(y, v_s)/\eta_i(y) \approx m_H(y,z)/(\eta(y)\eta(z))$. So

$$m_{H_{i+1}}(v_{i+1}, v_s) \approx \frac{m_H(y,z)}{\eta(y)\eta(z)}.$$

The other case, $\eta_{i+1}(z) = \eta_i(z) - 1$, is similar. This proves (C7)'(ii).

(iii) If $\eta_{i+1}(y) = \eta_i(y) - 1$ (so $w = y$), then by (B5)(i) $m_{H_{i+1}}(y, v_s)/\eta_{i+1}(y) \approx m_{H_i}(y, v_s)/\eta_i(y)$ for each $v_s \in \varphi_i^{-1}[z] = \varphi_{i+1}^{-1}[z]$. But by (C7)(iii) of induction hypothesis, $m_{H_i}(y, v_s)/\eta_i(y) \approx m_H(y,z)/(\eta(y)\eta(z))$ for each $v_s \in \varphi_i^{-1}[z]$. Therefore

$$\frac{m_{H_{i+1}}(y, v_s)}{\eta_{i+1}(y)} \approx \frac{m_H(y,z)}{\eta(y)\eta(z)}$$

for each $v_s \in \varphi_{i+1}^{-1}[z]$. If $\eta_{i+1}(z) = \eta_i(z) - 1$ (so $z = y$), then since in forming H_{i+1} no edge is detached from v_s for each $v_s \in \varphi_i^{-1}[y]$, we have $m_{H_{i+1}}(w, v_s) = m_{H_i}(w, v_s)$ for each $v_s \in \varphi_i^{-1}[y]$. Therefore $m_{H_{i+1}}(w, v_s)/\eta_{i+1}(w) = m_{H_i}(w, v_s)/\eta_i(w)$ for each

$v_s \in \varphi_i^{-1}[y]$. Moreover, by (B5)(ii) $m_{H_{i+1}}(w, v_{i+1}) \approx m_{H_i}(w, y)/\eta_i(y)$. Therefore $m_{H_{i+1}}(w, v_{i+1})/\eta_{i+1}(w) \approx m_{H_i}(w, y)/(\eta_i(w)\eta_i(y))$. By (C7)(i) of induction hypothesis, $m_{H_i}(w, y)/(\eta_i(w)\eta_i(y)) = m_H(w, y)/(\eta(w)\eta(y))$. Hence

$$\frac{m_{H_{i+1}}(w, v_{i+1})}{\eta_{i+1}(w)} \approx \frac{m_H(w, y)}{\eta(w)\eta(y)}.$$

This proves (C7)$'$(iii).

(C8)$'$ The proof is similar to the proof of (C7)$'$, following from (B6) and (C8) of the induction hypothesis.

As a result of (C1)-(C8), we prove that G is loopless, and satisfies conditions (A1)-(A6) of Theorem 2.1. Recall that $H_n = G$, $\varphi_n = \psi$, and $\eta_n(w) = 1$ for each $w \in V$. Let w, z be an arbitrary pair of distinct vertices of V, and let $j \in \mathbb{Z}_k$. Now in (C1)-(C8) we let $i = n$. From C1(ii) it is immediate that G is loopless.

From (C3)(i) it follows that $d_{H_n}(w)/\eta_n(w) \approx d_H(w)/\eta(w)$, so $d_G(w) \approx d_H(w)/\eta(w)$. From (C3)(ii), $d_{H_n}(v_r) \approx d_H(w)/\eta(w)$ for each $v_r \in \varphi_n^{-1}[w]$, so $d_G(v_r) \approx d_H(w)/\eta(w)$ for each $v_r \in \psi^{-1}[w]$. Therefore G satisfies (A1).

From (C5)(i) it follows that $m_{H_n}(w, v_r)/\eta_n(w) \approx \ell_H(w)/\binom{\eta(w)}{2}$ for each $v_r \in \varphi_n^{-1}[w]$, so $m_G(w, v_r) \approx \ell_H(w)/\binom{\eta(w)}{2}$ for each $v_r \in \psi^{-1}[w]$. From (C5)(ii), $m_{H_n}(v_r, v_s) \approx \ell_H(w)/\binom{\eta(w)}{2}$ for every pair of distinct vertices $v_r, v_s \in \varphi_n^{-1}[w]$, so $m_G(v_r, v_s) \approx \ell_H(w)/\binom{\eta(w)}{2}$ for every pair of distinct vertices $v_r, v_s \in \psi^{-1}[w]$. Therefore G satisfies (A3).

From (C7)(i) it follows that $m_{H_n}(w, z)/(\eta_n(w)\eta_n(z)) \approx m_H(w, z)/(\eta(w)\eta(z))$, so $m_G(w, z) \approx m_H(w, z)/(\eta(w)\eta(z))$. From (C7)(ii), $m_{H_n}(v_r, v_s) \approx m_H(w, z)/(\eta(w)\eta(z))$ for each $v_r \in \varphi_n^{-1}[w]$ and each $v_s \in \varphi_n^{-1}[z]$, so $m_G(v_r, v_s) \approx m_H(w, z)/(\eta(w)\eta(z))$ for each $v_r \in \psi^{-1}[w]$ and each $v_s \in \psi^{-1}[z]$. From (C7)(iii) it follows that $m_{H_n}(v_r, z)/\eta_n(z) \approx m_H(w, z)/(\eta(w)\eta(z))$ for each $v_r \in \varphi_n^{-1}[w]$, so $m_G(v_r, z) \approx m_H(w, z)/(\eta(w)\eta(z))$ for each $v_r \in \psi^{-1}[w]$. From (C7)(iii), $m_{H_n}(w, v_s)/\eta_m(w) \approx m_H(w, z)/(\eta(w)\eta(z))$ for each $v_s \in \varphi_n^{-1}[z]$, so $m_G(w, v_s) \approx m_H(w, z)/(\eta(w)\eta(z))$ for each $v_s \in \psi^{-1}[z]$. Therefore G satisfies (A5).

A similar argument shows that G satisfies (A2), (A4), (A6). In order to prove that G satisfies the last condition (A7) of Theorem 2.1, it suffices to show that if for some $j \in \mathbb{Z}_k$, $d_{H_i(j)}(v)/\eta_i(v)$ is even for all $v \in V_i$, then

(D1) $d_{H_{i+1}(j)}(v)/\eta_{i+1}(v)$ is an even integer for all $v \in V_{i+1}$, and

(D2) $\omega(H_{i+1}(j)) = \omega(H_i(j))$.

For then, if for each $v \in V(H) = V_0$, $d_{H(j)}(v)/\eta(v) = d_{H_0(j)}(v)/\eta_0(v)$ is an even integer, then it follows inductively that for each $0 \leqslant r \leqslant n$ and each $v \in V_r$, $d_{H_r(j)}(v)/\eta_r(v)$ is an even integer and

$$\omega(H_r(j)) = \omega(H_0(j)).$$

Therefore $\omega(G(j)) = \omega(H_n(j)) = \omega(H_0(j)) = \omega(H(j))$. This will complete the proof of Theorem 2.1.

So we now establish (D1) and (D2). Let $j \in \mathbb{Z}_k$ be a color for which for all $v \in V_i$, $d_{H_i(j)}(v)/\eta_i(v)$ is an even integer. Recall that y is the vertex for which $\eta_{i+1}(y) = \eta_i(y) - 1$. To establish (D1), there are three cases to consider:

- Case 1: $v \notin \{y, v_{i+1}\}$. Clearly $d_{H_{i+1}(j)}(v) = d_{H_i(j)}(v)$ and $\eta_{i+1}(v) = \eta_i(v)$. So $d_{H_{i+1}(j)}(v)/\eta_{i+1}(v) = d_{H_i(j)}(v)/\eta_i(v)$ which is an even integer.

- Case 2: $v = y$. From (B4)(i), it follows that $d_{H_{i+1}(j)}(y)/\eta_{i+1}(y) = d_{H_i(j)}(y)/\eta_i(y)$ which is an even integer.

- Case 3: $v = v_{i+1}$. From (B4)(ii), it follows that $d_{H_{i+1}(j)}(v_{i+1}) = d_{H_i(j)}(y)/\eta_i(y)$ which is an even integer.

This proves (D1).

In order to prove (D2), let $H_i^y(j)$ be the component of $H_i(j)$ which contains y. It is enough to show that $\omega(H_{i+1}^y(j)) = \omega(H_i^y(j))$. Let $\omega_{ij} = \omega(H_i^y(j)\backslash\{y\})$ and let $\Gamma_{i,j,1}, \ldots, \Gamma_{i,j,\omega_{ij}}$ be the vertex sets of the components of $H_i^y(j)\backslash\{y\}$. Note that $\Gamma_{i,j,r}$ is a subset of $V(B_i)$,

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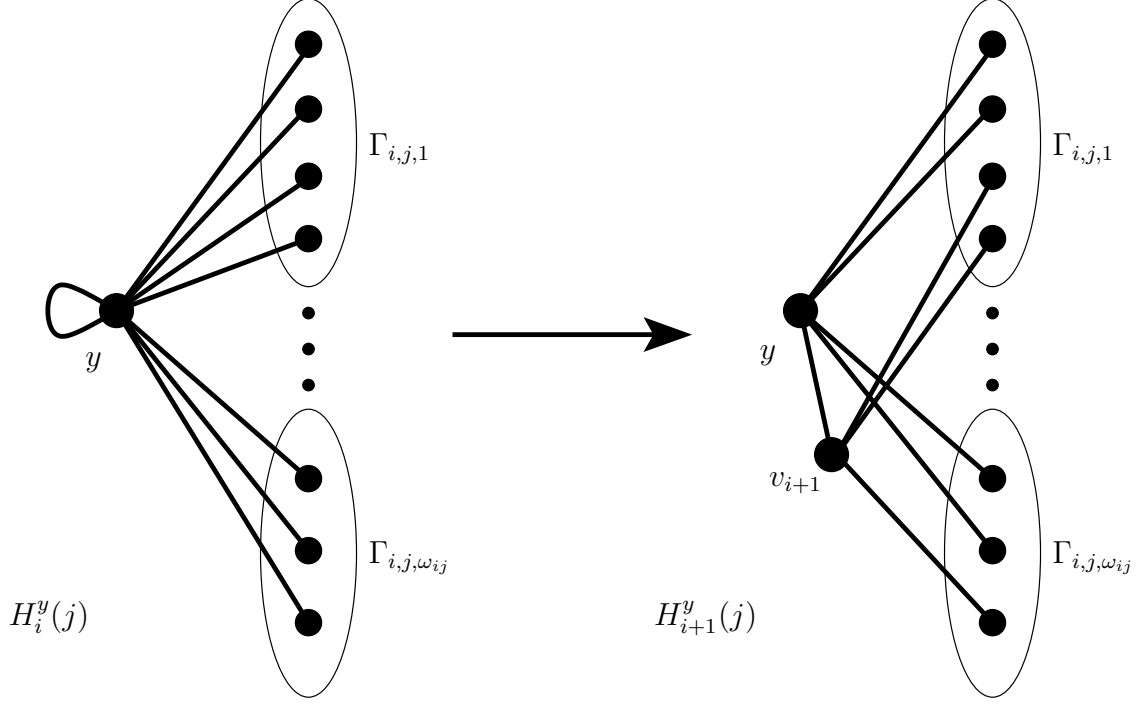
Figure 2.2: Detachment of $H_i^y(j)$ into $H_{i+1}^y(j)$

of $V(T_i)$, and of $V(B_i')$ for $1 \leqslant r \leqslant \omega_{ij}$. Since $d_{H_i(j)}(v)/\eta_i(v)$ is an even integer for each $v \in V_i$, it follows that $d_{H_i(j)}(v)$ is an even integer for each $v \in V_i$. Therefore $H_i(j)$ is an even graph (all vertices are of even degree). Since $d_{H_i(j)}(y)$ is even, so is $d_{H_i(j)}(y) - 2\ell_{H_i(j)}(y)$. Since $H_i(j)$ is an even graph, and the sum of the degree of the vertices in any graph must be even, it follows that $m_{H_i(j)}(y, \Gamma_{i,j,t}) = m_{B_i}(c_{ij}, \Gamma_{i,j,t})$ is even for $1 \leqslant t \leqslant \omega_{ij}$. (In fact every edge cut in $H_i(j)$ is even.) Now from (M2) it follows that for each t, $1 \leqslant t \leqslant \omega_{ij}$, $m_{B_i'(1)}(\mathcal{C}_{ij}, \Gamma_{i,j,t}) \approx m_{B_i'}(\mathcal{C}_{ij}, \Gamma_{i,j,t})/2$. There are two cases to consider:

- Case 1: $m_{T_i}(c_{ij}, \Gamma_{i,j,t}) = m_{B_i}(c_{ij}, \Gamma_{i,j,t})$. In this case we have

$$m_{B_i'(1)}(\mathcal{C}_{ij}, \Gamma_{i,j,t}) = \frac{m_{B_i'}(\mathcal{C}_{ij}, \Gamma_{i,j,t})}{2} = \frac{m_{T_i}(c_{ij}, \Gamma_{i,j,t})}{2} = \frac{m_{B_i}(c_{ij}, \Gamma_{i,j,t})}{2}.$$

- Case 2: $m_{T_i}(c_{ij}, \Gamma_{i,j,t}) < m_{B_i}(c_{ij}, \Gamma_{i,j,t})$. In this case we have

$$m_{B_i'(1)}(\mathcal{C}_{ij}, \Gamma_{i,j,t}) \approx \frac{m_{B_i'}(\mathcal{C}_{ij}, \Gamma_{i,j,t})}{2} = \frac{m_{T_i}(c_{ij}, \Gamma_{i,j,t})}{2} < \frac{m_{B_i}(c_{ij}, \Gamma_{i,j,t})}{2}.$$

Therefore in both cases $m_{B_i'(1)}(\mathcal{C}_{ij}, \Gamma_{i,j,t}) \leqslant m_{B_i}(c_{ij}, \Gamma_{i,j,t})/2$ for $1 \leqslant t \leqslant \omega_{ij}$. This is shown in Figure 2.2. This means, at most half of the edges joining y to $\Gamma_{i,j,t}$, $1 \leqslant t \leqslant \omega_{ij}$, are moved to v_{i+1} in forming H_{i+1}. So from each vertex $u \neq v_{i+1}$ in $H_{i+1}^y(j)$, there is a path of edges colored j from u to y. Moreover, v_{i+1} is either adjacent with y or is adjacent with another vertex in $H_{i+1}^y(j)$, so v_{i+1} is also joined to y by a path of edges colored j. Therefore $\omega(H_{i+1}^y(j)) = \omega(H_i^y(j))$. This proves (D2) and the proof of Theorem 2.1 is complete. $\square$

Chapter 3

Hamiltonian Decomposition of $K(n_1, \ldots, n_m; \lambda, \mu)$

Let $n_1, \ldots, n_m \in \mathbb{N}$, and $\lambda, \mu \in \mathbb{N} \cup \{0\}$. Let $G = K(n_1, \ldots, n_m; \lambda, \mu)$ denote a graph with m parts, the ith part having size n_i, in which multiplicity of each pair of vertices in the same part (in different parts) is λ (μ, respectively). In other words, G is a graph with m parts $V_1, \ldots, V_m$, with $|V_i| = n_i$ for $1 \leqslant i \leqslant m$, $m_G(u, v) = \lambda$ for every pair of distinct vertices $u, v \in V_i$ for $1 \leqslant i \leqslant m$, and $m_G(u, v) = \mu$ for each $u \in V_i, v \in V_j$ for $1 \leqslant i < j \leqslant m$.

When $n_1 = \ldots = n_m = n$, we denote $K(n_1, \ldots, n_m; \lambda, \mu)$ by $K(n^{(m)}; \lambda, \mu)$. In [5], we settled the existence of Hamiltonian decomposition for $K(n_1, \ldots, n_m; \lambda, \mu)$, a graph of particular interest to statisticians, who consider group divisible designs with two associate classes.

Example 3.1. *Figure 3.1 illustrates a Hamiltonian decomposition of $K(2^{(3)}; 2, 1)$.*

In this chapter, we present a constructive proof of this existence and we also solve the companion problem; that is the Hamiltonian decompositions problem for $K(n_1, \ldots, n_m; \lambda, \mu)$ when it is a regular graph of odd degree (see [9]). The details are provided in order that the reader may become more familiar with the nuances of using amalgamations.

A graph G is said to be even if all of its vertices have even degree. Let k be a positive integer. We say that G has an evenly-equitable k-edge-coloring if G has a k-edge-coloring for which, for each $v \in V(G)$

(i) $d_{G(i)}(v)$ is even for $1 \leqslant i \leqslant k$, and

(ii) $|d_{G(i)}(v) - d_{G(j)}(v)| \in \{0, 2\}$ for $1 \leqslant i, j \leqslant k$.
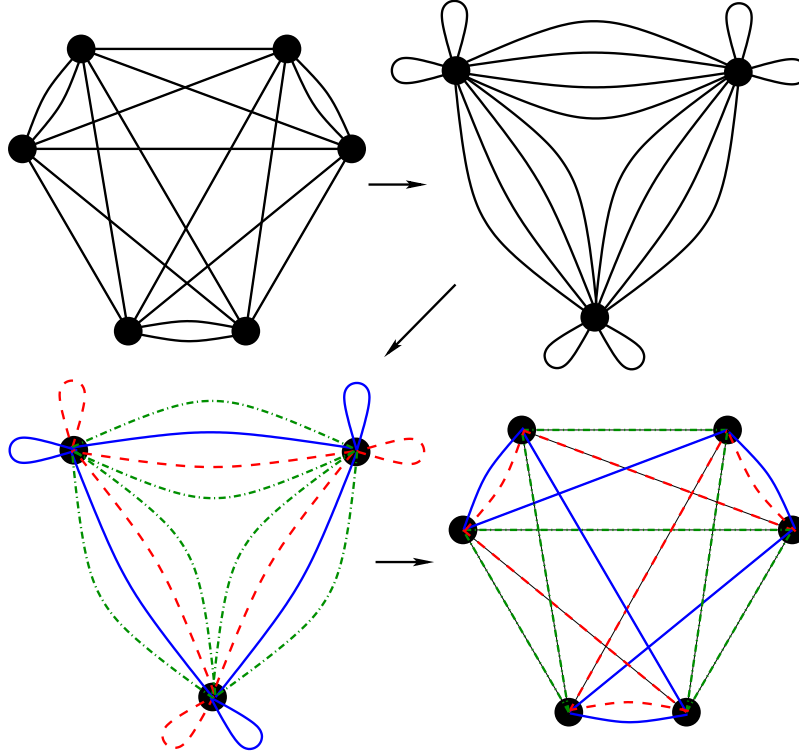
Figure 3.1: A Hamiltonian Decomposition of $K(2^{(3)}; 2, 1)$

We need the following theorem of Hilton [43]. (It may help to recall that the definition of k-edge-coloring allows some color classes to be empty. It is also worth noting that the following theorem is true even if the graph contains loops.)

Theorem 3.2. (Hilton [43, Theorem 8]) *Each finite even graph has an evenly-equitable k-edge-coloring for each positive integer k.*

3.1 Hamiltonian Decomposition of $K(n_1 \ldots, n_m; \lambda, \mu)$

Walecki's construction for Hamiltonian decomposition of K_n and $K_n - F$ where F is a 1-factor [64], easily provides the following result:

Theorem 3.3. *The graph λK_n is Hamiltonian decomposable if and only if $\lambda(n-1)$ is an even integer.*

Using this result, together with Theorem 3.2 and Theorem 2.1, now we are able to find necessary and sufficient conditions for $K(n_1 \ldots, n_m; \lambda, \mu)$ to be Hamiltonian decomposable. Let us first look at some trivial cases:

(i) If $m = 1$, then $G = \lambda K_{n_1}$ which by Theorem 3.3, is Hamiltonian decomposable if and only if $\lambda(n_1 - 1)$ is even.

(ii) If $m > 1, \mu = 0$, then $G = \bigcup_{i=1}^{m} \lambda K_{n_i}$. Clearly G is disconnected and so is not Hamiltonian decomposable.

(iii) If $n_i = 1$ for $1 \leqslant i \leqslant m$, then $G = \mu K_m$ which is Hamiltonian decomposable if and only if $\mu(m - 1)$ is even.

(iv) If $\lambda = \mu$, then $G = \lambda K_{n_1 + \cdots + n_m}$ which is Hamiltonian decomposable if and only if $\lambda(\sum_{i=1}^{m} n_i - 1)$ is even.

We exclude the above four cases from our theorem:

Theorem 3.4. (Bahmanian, Rodger [5, Theorem 4.3]) *Let $m > 1$, $\lambda \geqslant 0$, and $\mu \geqslant 1$, with $\lambda \neq \mu$ be integers. Let $n_1, \ldots, n_m$ be positive integers with $n_1 \leqslant \ldots \leqslant n_m$, and $n_m \geqslant 2$. Then $G = K(n_1, \ldots, n_m; \lambda, \mu)$ is Hamiltonian decomposable if and only if the following conditions are satisfied:*

(i) *$n_i = n_j := n$ for $1 \leqslant i < j \leqslant m$;*

(ii) *$\lambda(n - 1) + \mu n(m - 1)$ is an even integer;*

(iii) *$\lambda \leqslant \mu n(m - 1)$.*

Proof. Let $s = \sum_{i=1}^{m} n_i$. To prove the necessity, suppose G is Hamiltonian decomposable. For $v \in V_i$, $1 \leqslant i \leqslant m$, we have $d_G(v) = \lambda(n_i - 1) + \mu(s - n_i)$. Since G is Hamiltonian decomposable, it is regular. So we have $\lambda(n_i - 1) + \mu(s - n_i) = \lambda(n_j - 1) + \mu(s - n_j)$ for every pair $1 \leqslant i < j \leqslant m$. Equivalently $\lambda(n_i - n_j) = \mu(n_i - n_j)$. So $(\lambda - \mu)(n_i - n_j) = 0$

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and since $\lambda \neq \mu$, we have $n_i = n_j := n$ for every pair $1 \leqslant i < j \leqslant m$. So we can assume that $G = K(n^{(m)}; \lambda, \mu)$. Therefore $s = mn$ and $d_G(v) = \lambda(n-1) + \mu(mn-n) = \lambda(n-1) + \mu n(m-1)$. Now by the Hamiltonian decomposability of G, the degree of each vertex

$$\lambda(n-1) + \mu n(m-1) \text{ is an even integer.}$$

By the preceding paragraph, the number of Hamiltonian cycles of G is $\frac{1}{2}(\lambda(n-1) + \mu n(m-1))$. Let us say that an edge is pure if both of its endpoints belong to the same part. Each Hamiltonian cycle passes through every vertex of every part exactly once. Hence each Hamiltonian cycle contains at most $(n-1)$ pure edges from each part. Since the total number of pure edges in each part is $\lambda\binom{n}{2}$, we have

$$\lambda\binom{n}{2} \leqslant \frac{(n-1)}{2}(\lambda(n-1) + \mu n(m-1)).$$

So,

$$\frac{\lambda n(n-1)}{2} \leqslant \frac{(n-1)}{2}(\lambda(n-1) + \mu n(m-1)).$$

Since $n > 1$, it implies that $\lambda n \leqslant \lambda(n-1) + \mu n(m-1)$. Thus $\lambda \leqslant \mu n(m-1)$. Therefore conditions (i)-(iii) are necessary. Note that the necessity of condition (iii) can also be seen as an edge-connectivity issue. Of course G has edge-connectivity at most $\mu n^2(m-1)$, as deleting all the edges incident with vertices in a fixed part disconnects the graph. Since G has a Hamiltonian decomposition, it clearly has degree equal to its edge-connectivity. Therefore, the degree of G, namely $\lambda(n-1) + \mu n(m-1)$, is at most $\mu n^2(m-1)$.

To prove the sufficiency, suppose conditions (i)-(iii) are satisfied and let H be a graph with $|V(H)| = m$, $\ell_H(y) = \lambda\binom{n}{2}$ for every $y \in V(H)$, and $m_H(y, z) = \mu n^2$ for every pair $y, z \in V(H)$ and let η be a function from $V(H)$ into $\mathbb{N}$ with $\eta(y) = n$ for all $y \in V(H)$. We note that H is $(\lambda n(n-1) + \mu n^2(m-1))$-regular. It is easy to see that H is an amalgamation of G. In what follows we shall find an appropriate edge-coloring for H and then we shall

apply Theorem 2.1, to show that H has a η-detachment G in which every color class induces a Hamiltonian cycle.

Let H^* be the spanning subgraph of H whose edges are the non-loop edges of H. It is easy to see that $H^* \cong \mu n^2 K_m$. We claim that $\mu n(m-1)$ is even. To see this, suppose $\mu n(m-1)$ is odd; then a is odd and $\lambda(n-1)$ is even. But then $\lambda(n-1) + \mu n(m-1)$ is odd, contradicting condition (ii) of the theorem. Therefore $\mu n^2(m-1)$ is even and thus by Theorem 3.3, H^* is Hamiltonian decomposable.

Since $\mu n^2 K_m$ is $\mu n^2(m-1)$-regular, it is decomposable into $\mu n^2(m-1)/2$ Hamiltonian cycles by Theorem 3.3. Now define $k = \big(\lambda(n-1) + \mu n(m-1)\big)/2$. From (ii), k is an integer. Now since $n > 1$ and $\mu n(m-1) \geqslant \lambda$, we have the following sequence of equivalences:

$$(n-1)(\mu n(m-1) - \lambda) \geqslant 0 \Leftrightarrow \mu n(m-1)(n-1) - \lambda(n-1) \geqslant 0 \Leftrightarrow$$

$$\mu n^2(m-1) - \lambda(n-1) - \mu n(m-1) \geqslant 0 \Leftrightarrow \frac{\mu n^2(m-1)}{2} \geqslant \frac{\lambda(n-1) + \mu n(m-1)}{2}.$$

Hence, the number of Hamiltonian cycles in H^* is at least k. Now let $\mathcal{C}_1, \ldots, \mathcal{C}_k$ be k arbitrary Hamilton cycles of a Hamiltonian decomposition of H^*. Let $\mathcal{K}^*$ be a (partial) k-edge-coloring of H^* such that all edges of each cycle $\mathcal{C}_i$ are colored i, for each $i \in \mathbb{Z}_k$. Now let H^{**} be the spanning subgraph of H whose edges are all the edges of H that are uncolored in H^*. Recall that H is $2nk$-regular, so for each $v \in V(H^{**})$ we have $d_{H^{**}}(v) = 2nk - 2k = 2(n-1)k$. Therefore H^{**} is an even graph and so by Theorem 3.2 it has an evenly-equitable edge-coloring $\mathcal{K}^{**}$ with k colors $1, \ldots, k$ (Note that we are using the same colors we used to color edges of H^*). Therefore for each j, $1 \leqslant j \leqslant k$, and for each $y \in V(H^{**})$, we have $d_{H^{**}(j)}(y) = 2(n-1)k/k = 2(n-1)$. Now we can define the edges coloring $\mathcal{K} : E(H) \to \mathbb{Z}_k$ for H as below:

$$\mathcal{K}(e) = \begin{cases} \mathcal{K}^*(e) & \text{if } e \in E(H^*) \backslash E(H^{**}), \\ \mathcal{K}^{**}(e) & \text{if } e \in E(H^{**}). \end{cases}$$

So for each $j \in \mathbb{Z}_k$, for each $y \in V(H)$, we have $d_{H(j)}(y) = 2 + 2(n - 1) = 2n$. Note that since all edges of each Hamiltonian cycle $\mathcal{C}_j$ are colored j, $1 \leqslant j \leqslant k$, each color class $H(j)$ is connected.

So we have a k-edge-colored graph H for which, for each $y, z \in V(H), y \neq z$, and each $j \in \mathbb{Z}_k$, $\eta(y) = n \geqslant 2$, $\ell_H(y) = \lambda\binom{n}{2}$, $m_H(y, z) = \mu n^2$, $d_H(y) = 2nk$, $d_{H(j)}(y) = 2n$, $\omega(H(j)) = 1$.

Now by Theorem 2.1 there exists a loopless η-detachment G^* of H with amalgamation function $\psi : V(G^*) \to V(H)$, η being the number function associated with ψ, such that for each $y, z \in V(H), y \neq z$, and each $j \in \mathbb{Z}_k$ the following conditions are satisfied:

- $m_{G^*}(u, u') = \lambda\binom{n}{2}/\binom{n}{2} = \lambda$ for every pair of distinct vertices $u, u' \in \psi^{-1}(y)$;

- $m_{G^*}(u, v) = \mu n^2/(nn) = \mu$ for each $u \in \psi^{-1}(y)$ and each $v \in \psi^{-1}(z)$;

- $d_{G^*(j)}(u) = 2n/n = 2$ for each $u \in \psi^{-1}(y)$;

- $\omega(G^*(j)) = \omega(H(j)) = 1$, since $d_{H(j)}(y)/\eta(y) = 2n/n = 2$.

From the first two conditions it follows that $G^* \cong K(n^{(m)}; \lambda, \mu) = G$. The last two conditions tells us that each color class is 2-regular and connected, respectively; that is each color class is a Hamiltonian cycle. So we obtained a Hamiltonian decomposition of $K(n^{(m)}; \lambda, \mu)$ and the proof is complete. $\qquad \square$

Remark 3.5. We may prove the necessity of condition (iii) of Theorem 3.4 by a different counting argument. Let us say an edge is mixed if its endpoints are from different parts of G. Each Hamiltonian cycle starts from a vertex of a part V_i for some $1 \leqslant i \leqslant m$ and it will pass through every part at least once and it will eventually come back to the initial vertex in V_i. Hence each Hamiltonian cycle contains at least m mixed edges. On the other hand, the total number of mixed edges is $\mu n^2 \binom{m}{2}$. Therefore,

$$\mu n^2 \binom{m}{2} \geqslant m\frac{1}{2}(\lambda(n - 1) + \mu n(m - 1)).$$

So,

$$\frac{\mu n^2 m(m-1)}{2} \geqslant \frac{m(\lambda(n-1) + \mu n(m-1))}{2}.$$

It implies that, $\mu n(m-1)(n-1) - \lambda(n-1) \geqslant 0$, so $(n-1)(\mu n(m-1) - \lambda) \geqslant 0$ and since $n > 1$, we have $\lambda \leqslant \mu n(m-1)$.

Remark 3.6. Observe that the equality in condition (iii) of Theorem 3.4 holds if and only if for each Hamiltonian decomposition, each Hamiltonian cycle contains exactly $(n-1)$ pure edges from every part, and exactly m mixed edges.

3.2 Hamiltonian Decomposition of $K(n_1, \ldots, n_m; \lambda, \mu)$ with a 1-factor leave

Let us first look at some trivial cases:

(i) If $m = 1$, then $G = \lambda K_{n_1}$ which by Theorem 1.4, is decomposable into Hamiltonian cycles and a single 1-factor if and only if $\lambda(n_1 - 1)$ is odd.

(ii) If $m > 1$, $\mu = 0$, then $G = \bigcup\limits_{i=1}^{m} \lambda K_{n_i}$. Clearly G is disconnected it does not have any Hamiltonian cycle.

(iii) If $n_i = 1$ for $1 \leqslant i \leqslant m$, then $G = \mu K_m$ which is decomposable into Hamiltonian cycles and a single 1-factor if and only if $\mu(m-1)$ is odd.

(iv) If $\lambda = \mu$, then $G = \lambda K_{n_1 + \cdots + n_m}$ which is decomposable into Hamiltonian cycles and a single 1-factor if and only if $\lambda(\sum\limits_{i=1}^{m} n_i - 1)$ is even.

(v) If $\lambda = 0$, and $n_i = n$ for $1 \leqslant i \leqslant m$, then $G = \mu K_{\underbrace{n, \ldots, n}_{m}}$ which is decomposable into Hamiltonian cycles and a single 1-factor if and only if $\mu n(m-1)$ is odd (see [57]).

We exclude the above five cases from our theorem:

Theorem 3.7. *Let $m > 1$. Let $n_1, \ldots, n_m$ be positive integers with $n_1 \leqslant \ldots \leqslant n_m$, and $n_m \geqslant 2$, and $\lambda, \mu \geqslant 1$ with $\lambda \neq \mu$. Then $G = K(n_1, \ldots, n_m; \lambda, \mu)$ is decomposable into Hamiltonian cycles and a single 1-factor if and only if the following conditions are satisfied:*

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(i) $n_i = n_j := n$ *for* $1 \leqslant i < j \leqslant m$;

(ii) $\lambda(n-1) + \mu n(m-1)$ *is an odd integer*;

(iii) $\lambda \leqslant \mu n(m-1)$ *if* $n \geqslant 3$, *and* $\lambda - 1 \leqslant 2\mu(m-1)$ *otherwise*.

Proof. Let $s = \sum_{i=1}^{m} n_i$. To prove the necessity, suppose G is Hamiltonian decomposable. For $v \in V_i$, $1 \leqslant i \leqslant m$, we have $d_G(v) = \lambda(n_i - 1) + \mu(s - n_i)$. Since G is Hamiltonian decomposable, it is regular. So we have $\lambda(n_i - 1) + \mu(s - n_i) = \lambda(n_j - 1) + \mu(s - n_j)$ for $1 \leqslant i < j \leqslant m$. It follows that $n_i = n_j := n$ for $1 \leqslant i < j \leqslant m$. So we can assume that $G = K(n^{(m)}; \lambda, \mu)$. Therefore $d_G(v) = \lambda(n-1) + \mu n(m-1)$. Now since G is decomposable into Hamiltonian cycles and a single 1-factor

$$\lambda(n-1) + \mu n(m-1) \text{ is an odd integer.}$$

By the preceding paragraph, the number of Hamiltonian cycles of G is $\big(\lambda(n-1) + \mu n(m-1) - 1\big)/2$. Let us say that an edge is pure if both of its endpoints belong to the same part. Each Hamiltonian cycle passes through every vertex of every part exactly once. Hence each Hamiltonian cycle contains at most $n - 1$ pure edges from each part. Since the total number of pure edges in each part is $\lambda\binom{n}{2}$, and a 1-factor contains at most $\lfloor a/2 \rfloor$ pure edges from each part, we have

$$\lambda \binom{n}{2} \leqslant \frac{(n-1)}{2} \big(\lambda(n-1) + \mu n(m-1) - 1\big) + \lfloor \frac{n}{2} \rfloor.$$

So,
$$\frac{\lambda n(n-1)}{2} \leqslant \frac{(n-1)}{2} \big(\lambda(n-1) + \mu n(m-1) - 1\big) + \lfloor \frac{n}{2} \rfloor.$$

Since $n > 1$, it implies that

$$\lambda n \leqslant \lambda(n-1) + \mu n(m-1) - 1 + \frac{2\lfloor \frac{n}{2} \rfloor}{n-1}.$$

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It follows that if n is odd, then we have $\lambda \leqslant \mu n(m-1)$, and if $n > 2$ is even, then we have $\lambda \leqslant \mu n(m-1) + 1/(n-1)$, which is equivalent to $\lambda \leqslant \mu n(m-1)$. Moreover, if $n = 2$, then we have $\lambda - 1 \leqslant 2\mu(m-1)$. Therefore conditions (i)-(iii) are necessary.

To prove the sufficiency, suppose conditions (i)-(iii) are satisfied. We first solve the special case of $n = 2$. Since $\lambda + 2\mu(m-1)$ is odd, so is λ. Also $\lambda - 1 \leqslant 2\mu(m-1)$. Therefore, by Theorem 3.4, $K(2^{(m)}; \lambda - 1, \mu)$ is Hamiltonian decomposable. Adding an edge to each part of $K(2^{(m)}; \lambda - 1, \mu)$ (which is a 1-factor) will form $K(2^{(m)}; \lambda, \mu)$. Thus we obtain a decomposition of $K(2^{(m)}; \lambda, \mu)$ into Hamiltonian cycles and a single 1-factor. To prove the sufficiency for $n \geqslant 3$, let H be a graph with $|V(H)| = m$, $\ell_H(y) = \lambda \binom{n}{2}$ for every $y \in V(H)$, and $m_H(y,z) = \mu n^2$ for every pair $y, z \in V(H)$ and let η be a function from $V(H)$ into $\mathbb{N}$ with $\eta(y) = n$ for all $y \in V(H)$. Now define $k = \big(\lambda(n-1) + \mu n(m-1) - 1\big)/2$. From (ii), k is an integer. We note that H is $(2k+1)n$-regular. In what follows we shall find an appropriate edge-coloring for H and then we shall apply Theorem 2.1, to show that H has an η-detachment G in which every color class except one induces a Hamiltonian cycle, the exceptional color class being a -factor.

Let H^* be the spanning subgraph of H whose edges are the non-loop edges of H. It is easy to see that $H^* \cong \mu n^2 K_m$. We shall find a $(k+1)$-edge-coloring for H. There are two cases to consider, but first we observe that:

$$
\begin{aligned}
(n-1)\big(\mu n(m-1) - \lambda\big) &\geqslant 0 \iff \\
\mu n(m-1)(n-1) - \lambda(n-1) &\geqslant 0 \iff \\
\mu n^2(m-1) - \lambda(n-1) - \mu n(m-1) &\geqslant 0 \iff \\
\frac{\mu n^2(m-1)}{2} &\geqslant \frac{\lambda(n-1) + \mu n(m-1)}{2}.
\end{aligned}
\tag{3.1}
$$

- **Case 1: n is even.** It follows that $\mu n^2(m-1)$ is even and thus by Theorem 1.4, H^* is decomposable into $\frac{\mu n^2(m-1)}{2}$ Hamiltonian cycles. Now since $n > 1$, and since by (iii) $\mu n(m-1) \geqslant \lambda$, by (3.1) it follows that the number of Hamiltonian cycles in H^*

is greater than k. Now let $\mathcal{C}_1, \ldots, \mathcal{C}_k$ be k arbitrary Hamilton cycles of a Hamiltonian decomposition of H^*. Let $\mathcal{K}^*$ be a (partial) k-edge-coloring of H^* such that all edges of each cycle $\mathcal{C}_i$ are colored i, for each $1 \leqslant i \leqslant k$. Now let $\mathcal{L}$ be a spanning subgraph of H in which every vertex is incident with $n/2$ loops (observe that $\lambda\binom{n}{2} \geqslant n/2$; so the graph $\mathcal{L}$ consists only of loops. Now let H^{**} be the spanning subgraph of H whose edges are all edges in $E(H) \backslash E(\mathcal{L})$ that are uncolored in H^*. Recall that H is $(2k+1)n$-regular, so for each $v \in V(H^{**})$ we have $d_{H^{**}}(v) = (2k+1)n - 2k - 2(n/2) = 2k(n-1)$. Therefore H^{**} is an even graph and so by Theorem 3.2 it has an evenly-equitable edge-coloring $\mathcal{K}^{**}$ with k colors $1, \ldots, k$ (Note that we are using the same colors we used to color edges of H^*). Therefore for each j, $1 \leqslant j \leqslant k$, and for each $y \in V(H^{**})$, we have $d_{H^{**}(j)}(y) = 2(n-1)k/k = 2(n-1)$. Now we can define the $(k+1)$-edges coloring $\mathcal{K}$ for H as below:

$$
\mathcal{K}(e) := \begin{cases}
\mathcal{K}^*(e) & \text{if } e \in E(H^*) \backslash E(H^{**}), \\
\mathcal{K}^{**}(e) & \text{if } e \in E(H^{**}), \\
k+1 & \text{if } e \in E(\mathcal{L}).
\end{cases}
$$

So for each $y \in V(H)$,

$$
d_{H(j)}(y) = \begin{cases}
2(n-1) + 2 = 2n & \text{if } 1 \leqslant j \leqslant k, \\
2(n/2) = n & \text{if } j = k+1.
\end{cases}
$$

- **Case 2: n is odd.** Since $\lambda(n-1)$ is even, and by (ii), $\lambda(n-1) + \mu n(m-1)$ is odd, it follows that $\mu n(m-1)$ is odd. So $\mu n^2(m-1)$ is odd. Thus by Theorem 1.4, H^* is decomposable into $\big(\mu n^2(m-1) - 1\big)/2$ Hamiltonian cycles and a single 1-factor F.

By (3.1), it follows that

$$
\frac{\mu n^2(m-1) - 1}{2} \geqslant \frac{\lambda(n-1) + \mu n(m-1) - 1}{2} = k.
$$

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Hence, the number of Hamiltonian cycles in H^* is at least k. Now let $\mathcal{C}_1, \ldots, \mathcal{C}_k$ be k arbitrary Hamilton cycles of a Hamiltonian decomposition of H^*. Let $\mathcal{K}^*$ be a (partial) k-edge-coloring of H^* such that all edges of each cycle $\mathcal{C}_i$ are colored i, for each $1 \leqslant i \leqslant k$, and the single 1-factor F is colored $k+1$. Now let $\mathcal{L}$ be a spanning subgraph of H in which every vertex is incident with $(n-1)/2$ loops (observe that $\lambda\binom{n}{2} \geqslant (n-1)/2$). Now let H^{**} be the spanning subgraph of H whose edges are all edges in $E(H) \backslash E(\mathcal{L})$ that are uncolored in H^*. Recall that H is $(2k+1)n$-regular, so for each $v \in V(H^{**})$ we have $d_{H^{**}}(v) = (2k+1)n - 2k - 1 - 2(n-1)/2 = 2k(n-1)$. Therefore H^{**} is an even graph and so by Theorem 3.2 it has an evenly-equitable edge-coloring $\mathcal{K}^{**}$ with k colors $1, \ldots, k$ (Note that we are using the same colors we used to color edges of H^*). Therefore for each j, $1 \leqslant j \leqslant k$, and for each $y \in V(H^{**})$, we have $d_{H^{**}(j)}(y) = 2(n-1)k/k = 2(n-1)$. Now we can define the $(k+1)$-edges coloring $\mathcal{K}$ for H as below:

$$
\mathcal{K}(e) := \begin{cases} \mathcal{K}^*(e) & \text{if } e \in E(H^*) \backslash E(H^{**}), \\ \mathcal{K}^{**}(e) & \text{if } e \in E(H^{**}), \\ k+1 & \text{if } e \in E(\mathcal{L}). \end{cases}
$$

So for each $y \in V(H)$,

$$
d_{H(j)}(y) = \begin{cases} 2(n-1) + 2 = 2n & \text{if } 1 \leqslant j \leqslant k, \\ 1 + 2(n-1)/2 = n & \text{if } j = k+1. \end{cases}
$$

Note that since all edges of each Hamiltonian cycle $\mathcal{C}_j$ are colored j, $1 \leqslant j \leqslant k$, each color class $H(j)$ is connected for $1 \leqslant j \leqslant k$. Therefore in both cases, we have a $(k+1)$-edge-colored graph H for which, for each $y, z \in V(H), y \neq z, \eta(y) = n \geqslant 2, \ell_H(y) = \lambda\binom{n}{2}, m_H(y, z) = \mu n^2$,

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$d_H(y) = (2k+1)n$, $\omega(H(j)) = 1$ for each $1 \leqslant j \leqslant k$, and

$$
d_{H(j)}(y) = \begin{cases} 2n & \text{if } 1 \leqslant j \leqslant k, \\ n & \text{if } j = k+1. \end{cases}
$$

Now by Theorem 2.1, there exists a loopless η-detachment G^* of H in which each $v \in V(H)$ is detached into $v_1, \ldots, v_{\eta(v)}$ such that for each $u, v \in V(H), u \neq v$ the following conditions are satisfied:

- $m_{G^*}(u_i, u_{i'}) = \lambda \binom{n}{2} / \binom{n}{2} = \lambda$ for $1 \leqslant i < i' \leqslant \eta(u)$;

- $m_{G^*}(u_i, v_{i'}) = \mu n^2 / (nn) = \mu$ for $1 \leqslant i \leqslant \eta(u)$ and $1 \leqslant i' \leqslant \eta(v)$;

- $d_{G^*(j)}(u_i) = \begin{cases} 2n/n = 2 & \text{if } 1 \leqslant j \leqslant k, \\ n/n = 1 & \text{if } j = k+1, \end{cases}$ for $1 \leqslant i \leqslant \eta(u)$;

- $\omega(G^*(j)) = \omega(H(j)) = 1$ for each $1 \leqslant j \leqslant k$, since $d_{H(j)}(u)/\eta(u) = 2n/n = 2$ for $1 \leqslant j \leqslant k$.

From the first two conditions it follows that $G \cong K(n^{(m)}; \lambda, \mu)$. The last two conditions tells us that each color class $1 \leqslant j \leqslant k$ is 2-regular and connected respectively; that is each color class $1 \leqslant j \leqslant k$ is a Hamiltonian cycle. Furthermore, the color class $k+1$ is 1-regular. So we obtained a decomposition of $K(n^{(m)}; \lambda, \mu)$ into Hamiltonian cycles and a single 1-factor. $\square$

Chapter 4

Embedding an Edge-colored $K(a^{(p)}; \lambda, \mu)$ into a Hamiltonian Decomposition of

$$K(a^{(p+r)}; \lambda, \mu)$$

4.1 Introduction

Recall that $K(a^{(p)}; \lambda, \mu)$ is a graph with p parts, each part having size a, in which the multiplicity of each pair of vertices in the same part (in different parts) is λ (μ, respectively). In this chapter we consider the following embedding problem: When can a graph decomposition of $K(a^{(p)}; \lambda, \mu)$ be extended to a Hamiltonian decomposition of $K(a^{(p+r)}; \lambda, \mu)$ for $r > 0$? A general result is proved, which is then used to solve the embedding problem for all $r \geqslant \frac{\lambda}{\mu a} + \frac{p-1}{a-1}$. The problem is also solved when r is as small as possible in two different senses, namely when $r = 1$ and when $r = \frac{\lambda}{\mu a} - p + 1$.

Let $G = (V, E)$ be a graph and let $H = \{H_i\}_{i \in I}$ be a family of graphs where $H_i = (V_i, E_i)$. We say that G has an H-*decomposition* if $\{E_i : i \in I\}$ partitions E and each E_i induces an isomorphic copy of H_i. Graph decomposition in general has been studied for many classes of graphs. The decomposition of a graph into paths [79], cycles [76] or stars [78] has been of special interest over the years. Of particular interest is the decomposition of a graph into Hamiltonian cycles; that is a *Hamiltonian Decomposition*. In 1892 Walecki [64] proved the classic result that the complete graph K_n is decomposable into Hamiltonian cycles if and only if n is odd. Laskar and Auerbach [57] settled the existence of Hamiltonian decomposition of the complete multipartite graph $K_{m,...,m}$. Alspach, Gavlas, and Šajna [1, 76, 77] collectively solved the problem of decomposing the complete graph into isomorphic cycles, but the problem remains open for different cycle lengths.

Another challenge is the companion embedding problem:

Let $H = \{H_i\}_{i \in I}$ and $H^* = \{H_j^*\}_{j \in J}$ be two families of graphs. Given a graph G with an H-decomposition and a graph G^* which is a supergraph of G (or G is a subgraph of G^*), under what circumstances one can extend the H-decomposition of G into an H^*-decomposition of G^*? In other words, given an edge-coloring of G (that can be considered as a decomposition when each color class induces a graph in H), is it possible to extend this coloring to an edge-coloring of G^* so that each color class of G^* induces a graph in H^*?

In this direction, Hilton [44] found necessary and sufficient conditions for a decomposition of K_m to be embedded into a Hamiltonian decomposition of K_{m+n}, which later was generalized by Nash-Williams [70]. Hilton and Rodger [48] considered the embedding of Hamiltonian decompositions for complete multipartite graphs. For embedding factorizations see [47, 74], where the connectivity of the graphs in H^* is one defining property. We also note that embedding problems first were studied for latin squares by M. Hall [41]. For extensions of Hall's theorem see [2, 3, 63].

The graph $K(a_1, \ldots, a_p; \lambda, \mu)$ is of particular interest to statisticians, who consider group divisible designs with two associate classes, beginning over 50 years ago with the work of Bose and Shimamoto [22]. Decompositions of $K(a_1, \ldots, a_p; \lambda, \mu)$ into cycles of length m have been studied for small values of m [37, 38, 39]. Recently, Bahmanian and Rodger have settled the existence problem completely for longest (i.e. Hamiltonian) cycles in [5]. In this chapter, we study conditions under which one can embed a decomposition of $K(a^{(p)}; \lambda, \mu)$ into a Hamiltonian decomposition of $K(a^{(p+r)}; \lambda, \mu)$ for $r > 0$. Our proof is largely based on our results in [5] (see Theorem 2.1).

4.2 Amalgamation and Graph Embedding

Recall that a *detachment* of H is, intuitively speaking, a graph G obtained from H by splitting some or all of its vertices into more than one vertex. That is, to each vertex α of H there corresponds a subset V_α of $V(G)$ such that an edge joining two vertices α and β in H will join some element of V_α to some element of V_β. If η is a function from $V(H)$ into

$\mathbb{N}$ (the set of positive integers), then an η-*detachment* of H is a detachment of H in which each vertex u of H splits into $\eta(u)$ vertices. For a more precise definition of amalgamation and detachment, we refer the reader to Chapter 1.

Since two graphs G and H related in the above manner have an obvious bijection between the edges, an edge-coloring of G or H, naturally induces an edge-coloring on the other graph. Hence an amalgamation of a graph with colored edges is a graph with colored edges.

The technique of vertex amalgamation, which was developed in the 1980s by Rodger and Hilton, has proved to be very powerful in decomposing of various classes of graphs. For a survey about the method of amalgamation and embedding partial edge-colorings we refer the reader to [4]. In [5], the authors proved a general detachment theorem for multigraphs. For the purpose of this chapter we use a very special case of this theorem as follows:

Theorem 4.1. *Let H be a k-edge-colored graph all of whose color classes are connected, and let η be a function from $V(H)$ into $\mathbb{N}$ such that for each $v \in V(H)$: (i) $\eta(v) = 1$ implies $\ell_H(v) = 0$, (ii) $d_{H(j)}(v)/\eta(v)$ is an even integer for $1 \leqslant j \leqslant k$, (iii) $\binom{\eta(v)}{2}$ divides $\ell_H(v)$, and (iv) $\eta(v)\eta(w)$ divides $m_H(v,w)$ for each $w \in V(H) \backslash \{v\}$. Then there exists a loopless η-detachment G of H in which each $v \in V(H)$ is detached into $v_1, \ldots, v_{\eta(v)}$, all of whose color classes are connected, and for $v \in V(H)$:*

(i) *$m_G(v_i, v_{i'}) = \ell_H(v)/\binom{\eta(v)}{2}$ for $1 \leqslant i < i' \leqslant \eta(v)$ if $\eta(v) \geqslant 2$,*

(ii) *$m_G(v_i, w_{i'}) = m_H(v, w)/(\eta(v)\eta(w))$ for $w \in V(H) \backslash \{v\}$, $1 \leqslant i \leqslant \eta(v)$ and $1 \leqslant i' \leqslant \eta(w)$, and*

(iii) *$d_{G(j)}(v_i) = d_{H(j)}(v)/\eta(v)$ for $1 \leqslant i \leqslant \eta(v)$ and $1 \leqslant j \leqslant k$.*

Here is our main result:

Theorem 4.2. (Bahmanian, Rodger [10, Theorem 2]) *Let* $G = K(a^{(p)}; \lambda, \mu)$ *with* $a > 1$, $\lambda \geqslant 0$, $\mu \geqslant 1$, $\lambda \neq \mu$, $r \geqslant 1$, *and let* $\omega_j = \omega(G(j))$. *For* $1 \leqslant j \leqslant k$, *define*

$$s_j \equiv \omega_j \pmod{r} \text{ with } 1 \leqslant s_j \leqslant r, \tag{4.1}$$

and suppose

$$\sum_{j=1}^{k} s_j \geqslant kr - \mu a^2 \binom{r}{2}. \tag{4.2}$$

Then a k-*edge-coloring of* G *can be embedded into a Hamiltonian decomposition of* $G^* = K(a^{(p+r)}; \lambda, \mu)$ *if and only if:*

(i) $k = \big(\lambda(a-1) + \mu a(p+r-1)\big)/2$,

(ii) $\lambda \leqslant \mu a(p+r-1)$,

(iii) *Every component of* $G(j)$ *is a path (possibly of length 0) for* $1 \leqslant j \leqslant k$, *and*

(iv) $\omega_j \leqslant ar$ *for* $1 \leqslant j \leqslant k$.

Proof. By Theorem 3.4, for $K(a^{(p+r)}; \lambda, \mu)$ to be Hamiltonian decomposable, conditions (i) and (ii) are necessary and sufficient. (Condition (i) follows since k must be $d_{G^*}(v)/2$. Condition (ii) follows since each Hamiltonian cycle must use at least $p + r$ mixed edges, so there must be sufficiently many mixed edges for all pure edges to be used.) For $1 \leqslant j \leqslant k$, for $G(j)$ to be extendable to a Hamiltonian cycle in $K(a^{(p+r)}; \lambda, \mu)$, the degree of each vertex has to be at most 2, and thus every component must be a path. Moreover, since each new vertex can link together two disjoint paths, the number of components of every color class can not exceed the number of new vertices, ar. This proves the necessity of (i)–(iv).

Let $G = (V, E)$, and let u be a vertex distinct from vertices in V. Define the new graph $G_1 = (V_1, E_1)$ with $V_1 = V \cup \{u\}$ by adding to G the vertex u incident with $\mu a^2 \binom{r}{2}$ loops, and adding μar edges between u and each vertex v in V (see Figure 4.1). Note that for each $v \in V$, $d_{G_1}(v) = \lambda(a-1) + \mu a(p-1) + \mu ar = \lambda(a-1) + \mu a(p+r-1) = 2k$. Now we extend
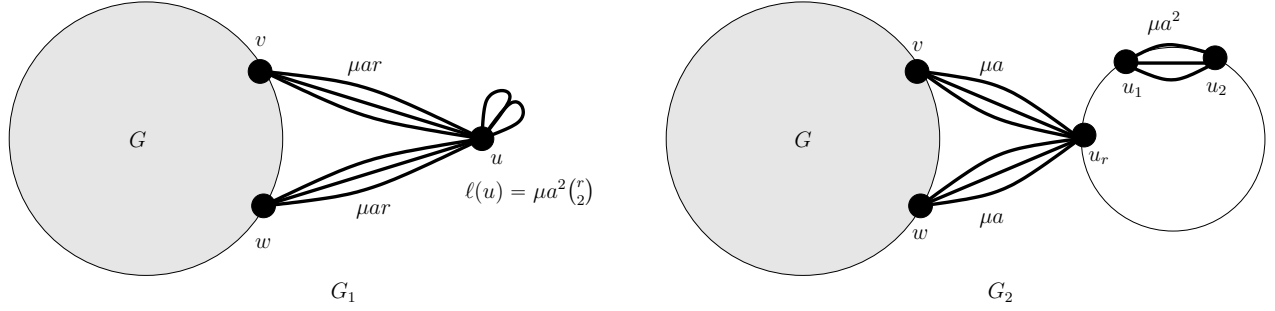
Figure 4.1: G_1 and its detachment G_2

the k-edge-coloring of G to a $(k+1)$-edge-coloring of G_1 as follows:

(A1) Each edge in G has the same color as it does in G_1,

(A2) For every $v \in V$, color the $\mu a r$ edges between v and u so that $d_{G_1(j)}(v) = 2$ for $1 \leqslant j \leqslant k$. Since $d_{G(j)}(v) \leqslant 2$ for $1 \leqslant j \leqslant k$, and since $d_{G_1}(v) = 2k$, this can be done. Notice that for every component of $G(j)$ (which is a path), exactly two edges (from end points of the path) are connected to u; so at this point $d_{G_1(j)}(u) = 2\omega_j$ for $1 \leqslant j \leqslant k$.

(A3) For $1 \leqslant j \leqslant k$ color $r - s_j$ ($\geqslant 0$) loops with j. This coloring of loops can be done, since by condition (2) of the theorem we have:

$$\sum_{j=1}^{k} s_j \geqslant kr - \mu a^2 \binom{r}{2} \iff \sum_{j=1}^{k} r - \sum_{j=1}^{k} s_j \leqslant \mu a^2 \binom{r}{2}$$

$$\iff \sum_{j=1}^{k} (r - s_j) \leqslant \mu a^2 \binom{r}{2} = \ell_{G_1}(u).$$

Moreover we color the remaining $\sum_{j=1}^{k} s_j - kr + \mu a^2 \binom{r}{2}$ ($\geqslant 0$) loops with the new color $k+1$. Thus for $1 \leqslant j \leqslant k$,

$$d_{G_1(j)}(u) = 2\omega_j + 2(r - s_j) = 2r + 2(\omega_j - s_j),$$

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and $d_{G_1(k+1)}(u) = 2\left(\sum_{j=1}^{k} s_j - kr + \mu a^2 \binom{r}{2}\right)$. By (1) $d_{G_1(j)}(u)$ is an even multiple of r for $1 \leqslant j \leqslant k$. Now to show that $d_{G_1(k+1)}(u)$ is an even multiple of r, first we show that $\sum_{j=1}^{k} \omega_j = \mu a^2 pr/2$.

$$
\begin{aligned}
\sum_{j=1}^{k} \omega_j &= \sum_{j=1}^{k} (pa - |E(G(j))|) \\
&= kpa - |E| \\
&= pa(\lambda(a-1) + \mu a(p + r - 1))/2 - pa(\lambda(a-1) + \mu a(p-1))/2 \\
&= \mu a^2 pr/2.
\end{aligned}
$$

Notice that $\mu a(p + r - 1)$ is even, since otherwise, in particular a would be odd, so k would not be an integer. Thus,

$$
\begin{aligned}
d_{G_1(k+1)}(u) &\equiv 2\sum_{j=1}^{k} \omega_j + \mu a^2 r(r - 1) \\
&\equiv \mu a^2 pr + \mu a^2 r(r - 1) \\
&\equiv \mu a^2 r(p + r - 1) \\
&\equiv 0 \pmod{2r}.
\end{aligned}
$$

Let $b_1, \ldots, b_{k+1}$ be even integers such that $d_{G_1(j)}(u) = b_j r$ for $1 \leqslant j \leqslant k+1$. Note that for $1 \leqslant j \leqslant k$, we have

$$
b_j/2 = 1 + \frac{\omega_j - s_j}{r} \leqslant 1 + \lfloor \frac{ar - 1}{r} \rfloor \leqslant 1 + (a - 1) = a.
$$

Since each component of $G(j)$ is joined to u in $G_1(j)$, each color class of G_1 is connected. Let η be a function from V_1 into $\mathbb{N}$ such that $\eta(v) = 1$ for each $v \in V$, and $\eta(u) = r$. Now by Theorem 4.1, there exists an η-detachment G_2 of G_1, all of whose color classes are connected, (see Figure 4.1) in which u is detached into r new vertices $u_1, \ldots, u_r$ such that:

(a) $m_{G_2}(u_i, u_{i'}) = \mu a^2 \binom{r}{2}/\binom{r}{2} = \mu a^2$, for $1 \leqslant i < i' \leqslant r$;

(b) $m_{G_2}(u_i, v) = \mu a r / r = \mu a$ for each $v \in V$ and each i, $1 \leqslant i \leqslant r$;

(c) $d_{G_2(j)}(u_i) = b_j r / r = b_j$ for $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant k + 1$.

We observe that $d_{G_2}(u_i) = ap(\mu a) + (r - 1)\mu a^2 = \mu a^2(p + r - 1)$ for $1 \leqslant i \leqslant r$, and is even. Note that by (c), $d_{G_2(j)}(u_i) = d_{G_2(j)}(u_{i'})$ and is even for $1 \leqslant i \leqslant i' \leqslant r$, and we know that $d_{G_2}(u_i) \leqslant 2ka$ for $1 \leqslant i \leqslant r$. Therefore, since $G(k + 1)$ is an even graph, (so it has a 2-factorization), we can recolor each 2-factor of color class $k + 1$ with a color j, $1 \leqslant j \leqslant k$ such that $d_{G_2(j)}(u_i) \leqslant 2a$. We let $b'_1, \ldots, b'_k$ be even integers such that in the resulting edge-coloring of G_2 obtained from recoloring the color class $k + 1$, $d_{G_2(j)}(u) = b'_j r$ for $1 \leqslant j \leqslant k$.

Now we define the new graph G_3 by adding $\lambda\binom{a}{2}$ loops on every vertex u_i in G_2, for $1 \leqslant i \leqslant r$ (see Figure 4.2). We extend the k-edge-coloring of G_2 to a k-edge-coloring of G_3
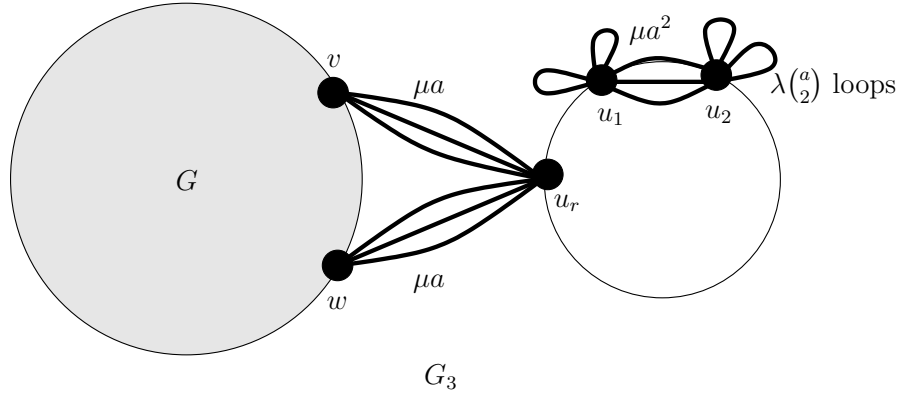
Figure 4.2: G_3

such that:

(B1) Each edge in G_2 has the same color at it does in G_3,

(B2) For $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant k$, there are $a - b'_j/2$ ($\geqslant 0$) loops incident with u_i colored j.

This is possible, for the following reason:

$$
\begin{aligned}
\sum_{j=1}^{k}(a - b'_j/2) &= ka - \frac{1}{2}\sum_{j=1}^{k} d_{G_2(j)}(u_1) \\
&= ka - \frac{1}{2}d_{G_2}(u_1) \\
&= ka - \frac{1}{2}\mu a^2(p + r - 1) \\
&= \frac{a}{2}\left(\lambda(a - 1) + \mu a(p + r - 1)\right) - \frac{1}{2}\mu a^2(p + r - 1) \\
&= \frac{a}{2}\lambda(a - 1) = \ell_{G_3}(u_1).
\end{aligned}
$$

Since for $1 \leqslant j \leqslant k$, $G_2(j)$ is a connected spanning subgraph of $G_3(j)$, $G_3(j)$ is also connected. Let η' be a function from V_3 into $\mathbb{N}$ such that $\eta'(v) = 1$ for each $v \in V$, and $\eta'(u_i) = a$ for $1 \leqslant i \leqslant r$. Now by Theorem 4.1, there exists an η'-detachment G_4 of G_3, all of whose color classes are connected, in which u_i is detached into a new vertices $u_{i1}, \ldots, u_{ia}$ for $1 \leqslant i \leqslant r$ such that:

- $m_{G_4}(u_{ij}, u_{ij'}) = \lambda\binom{a}{2}/\binom{a}{2} = \lambda$ for $1 \leqslant i \leqslant r$ and $1 \leqslant j < j' \leqslant a$;

- $m_{G_4}(u_{ij}, u_{i'j'}) = \mu a^2/a^2 = \mu$ for $1 \leqslant i < i' \leqslant r$ and $1 \leqslant j < j' \leqslant a$;

- $m_{G_4}(u_{ij}, v) = \mu a/a = \mu$ for each $v \in V$ and for $1 \leqslant i \leqslant r$; and

- $d_{G_4(j)}(u_{ii'}) = 2a/a = 2$ for $1 \leqslant i \leqslant r$, $1 \leqslant i' \leqslant a$.

Therefore $G_4 = K(a^{(p+r)}; \lambda, \mu)$, and each color class in G_4 is a Hamiltonian cycle, so the proof is complete. $\qquad\square$

A natural perspective of this embedding problem is to keep a, p, λ and μ fixed, and ask for which values of r the embedding is possible. The following result completely settles this question for all $r \geqslant \frac{\lambda(a-1)+\mu a(p-1)}{\mu a(a-1)}$.

Theorem 4.3. (Bahmanian, Rodger [10, Theorem 3]) *Let* $G = K(a^{(p)}; \lambda, \mu)$ *with* $a > 1$, $\lambda \geqslant 0$, $\mu \geqslant 1$, $\lambda \neq \mu$, *and*

$$r \geqslant \frac{\lambda(a-1) + \mu a(p-1)}{\mu a(a-1)}. \tag{4.3}$$

Then a k-edge-coloring of G can be embedded into a Hamiltonian decomposition of $K(a^{(p+r)}; \lambda, \mu)$ if and only if (i)–(iv) *of Theorem* 4.2 *are satisfied.*

Proof. It is enough to show that (4.3) implies (4.2). Since $s_j \geqslant 1$ for $1 \leqslant j \leqslant k$, $\sum_{j=1}^{k} s_j \geqslant k$. Thus, if we show that $k \geqslant kr - \mu a^2 \binom{r}{2}$, we are done. This is true by the following sequence of equivalences:

$$k(r-1) \leqslant \mu a^2 \binom{r}{2} \quad \Longleftrightarrow$$
$$(r-1)\big(\lambda(a-1) + \mu a(p+r-1)\big) \leqslant \mu a^2 r(r-1) \quad \Longleftrightarrow$$
$$\lambda(a-1) \leqslant \mu a(ar - p - r + 1) = \mu a\big(r(a-1) - (p-1)\big) \quad \Longleftrightarrow$$
$$\lambda(a-1)/(\mu a) \leqslant r(a-1) - (p-1) \quad \Longleftrightarrow$$
$$r \geqslant \frac{\lambda(a-1) + \mu a(p-1)}{\mu a(a-1)}.$$

$\square$

Another immediate corollary of Theorem 4.2 is the following complete solution to the embedding problem when $r = 1$:

Corollary 4.4. *Let* $G = K(a^{(p)}; \lambda, \mu)$ *with* $a > 1$, $\lambda \geqslant 0$, $\mu \geqslant 1$, $\lambda \neq \mu$. *Then a k-edge-coloring of G can be embedded into a Hamiltonian decomposition of $K(a^{(p+1)}; \lambda, \mu)$ if and only if:*

(i) $k = \big(\lambda(a-1) + \mu a p\big)/2$,

(ii) $\lambda \leqslant \mu a p$,

(iii) *Every component of $G(j)$ is a path (possibly of length 0) for $1 \leqslant j \leqslant k$, and*

(iv) $\omega_j \leqslant a$ for $1 \leqslant j \leqslant k$.

Proof. Since $r = 1$, we have $s_1 = \ldots = s_k = 1$, so $k = \sum_{j=1}^{k} s_j = k - \mu a^2 \binom{1}{2} = k$, and thus condition (4.2) of Theorem 4.2 is satisfied. □

Proposition 4.5. *Whenever $\lambda \leqslant \mu a$ and $p \leqslant a$, the embedding problem is completely solved for all values of $r \geqslant 1$.*

Proof. Condition 4.3 can be rewritten as $r \geqslant \frac{\lambda}{\mu a} + \frac{p-1}{a-1}$. Since we are assuming that $\lambda \leqslant \mu a$ and $p \leqslant a$, we have $\frac{\lambda}{\mu a} + \frac{p-1}{a-1} \leqslant 2$. Therefore the result follows from Theorem 4.3 and Corollary 4.4. □

Example 4.6. A k-edge-coloring of $K(10^{(7)}; 2, 5)$ can be embedded into a Hamiltonian decomposition of $K(10^{(7+r)}; 2, 5)$ for $r \geqslant 1$ if and only if (i)–(iv) of Theorem 4.2 are satisfied.

The following result completely settles the embedding problem for the smallest value of r in another sense, namely with respect to the inequality (ii) of Theorem 4.2; so it settles the case where $\lambda = \mu a(p + r - 1)$, or equivalently where $r = \frac{\lambda}{\mu a} - p + 1$. The proof is similar to that of Theorem 4.2, so only an outline of the proof is provided, the details being omitted. The proof of the necessity of condition (ii) of Theorem 4.2 shows that, in a Hamiltonian decomposition of $K(a^{(p)}; \lambda a(p + r - 1), \lambda)$, each Hamiltonian cycle contains exactly $a - 1$ pure edges from each part, and exactly $p + r$ mixed edges.

Theorem 4.7. (Bahmanian, Rodger [10, Theorem 4]) *Let $a > 1$ and $r, \mu \geqslant 1$. A k-edge-coloring of $G = K(a^{(p)}; \mu a(p + r - 1), \mu)$ can be embedded into a Hamiltonian decomposition of $G^* = K(a^{(p+r)}; \mu a(p + r - 1), \mu)$ if and only if:*

(i) $k = \mu a^2 (p + r - 1)/2$,

(ii) *Every component of $G(j)$ is a path (possibly of length 0) for $1 \leqslant j \leqslant k$,*

(iii) *$G(j)$ has exactly $a - 1$ pure edges from each part, and at most $p - 1$ mixed edges for $1 \leqslant j \leqslant k$, and*

(iv) $\omega_j \leqslant r$ for $1 \leqslant j \leqslant k$.

Proof. The necessity of (i)–(iii) follows as described in Theorem 4.2. Let m_j be the number of mixed edges in $G(j)$. To extend each component P of $G(j)$ to a Hamiltonian cycle in G^*, two mixed edges have to join P to the new vertices, and since each Hamiltonian cycle in G^* contains exactly $p + r$ mixed edges, we have that

$$m_j + 2\omega_j \leqslant p + r. \tag{4.4}$$

Since $G(j)$ is a collection of paths, for $1 \leqslant j \leqslant k$, we have $|V(G(j))| = |E(G(j))| + \omega_j$. Therefore $ap = m_j + p(a - 1) + \omega_j$ and thus

$$m_j + \omega_j = p. \tag{4.5}$$

Combining (4.4) and (4.5) implies (iv).

To prove sufficiency, we define the graph G_1 as it is defined in Theorem 4.2. We extend the k-edge-coloring of G to a k-edge-coloring of G_1 such that $d_{G_1(j)}(v) = 2$ for each $v \in V$ and $1 \leqslant j \leqslant k$. This is possible by the same argument as in Theorem 4.2. At this point $d_{G_1(j)}(u) = 2\omega_j \leqslant 2r$ for $1 \leqslant j \leqslant k$. So we can color the loops incident with u such that $d_{G_1(j)}(u) = 2r$ for $1 \leqslant j \leqslant k$, simply by assigning the color j to $r - \omega_j$ loops.

Now we detach the vertex u into r new vertices $u_1, \ldots, u_r$ to obtain the new graph G_2 (as we did in the proof of Theorem 4.2). Note that $d_{G_2(j)}(u_i) = 2r/r = 2$ for each i, $1 \leqslant i \leqslant r$ and each j, $1 \leqslant j \leqslant k$. Now we define the new graph G_3 by adding $a - 1$ loops of color j, $1 \leqslant j \leqslant k$, on every vertex u_i in G_2, for each i, $1 \leqslant i \leqslant r$. So we have $d_{G_3(j)}(u_i) = 2a$. Using Theorem 4.1, detach each vertex u_i into a new vertices $u_{i1}, \ldots, u_{ia}$ for $1 \leqslant i \leqslant r$, to obtain the new graph G_4 in which, $G_4(j)$ is connected and $d_{G_4(j)}(u_{ii'}) = 2a/a = 2$ for $1 \leqslant j \leqslant k$, $1 \leqslant i \leqslant r$, $1 \leqslant i' \leqslant a$. This completes the proof. $\square$

Chapter 5

Detachments of Amalgamated 3-uniform Hypergraphs : Factorization Consequences

5.1 Introduction

A *detachment* of a hypergraph $\mathscr{F}$ is, informally speaking, a hypergraph obtained from $\mathscr{F}$ by splitting some or all of its vertices into more than one vertex. If $\mathscr{G}$ is a detachment of $\mathscr{F}$, then $\mathscr{F}$ is an *amalgamation* of $\mathscr{G}$. Amalgamating $\mathscr{G}$, intuitively speaking, can be thought of as taking $\mathscr{G}$, partitioning its vertices, then for each element of the partition squashing the vertices to form a single vertex in the amalgamated hypergraph $\mathscr{F}$. We shall give more precise definition for amalgamation and detachment in Section 5.2.

Perhaps the most interesting use of amalgamations has been to prove embedding results; see, for example [2, 3, 47, 51, 70, 74]. Detachments of graphs have also been studied in [18, 49], generalizing some results of Nash-Williams [69, 68]. For a survey about the method of amalgamation and embedding partial edge-colorings we refer the reader to [4].

Most of the results in graph amalgamation have used edge-coloring techniques due to de Werra [80, 81, 82, 83], however Nash-Williams [70] proved a lemma (see Lemma 1.3) to generalize theorems of Hilton and Rodger. In this chapter we apply Nash-Williams technique to produce a general detachment theorem for 3-uniform hypergraphs (see Theorem 5.3). This result is not only a substantial generalization of previous amalgamation theorems, but also yields several consequences on factorizations of complete 3-uniform multipartite (multi)hypergraphs. To demonstrate the power of our detachment theorem, we show that the complete 3-uniform n-partite multi-hypergraph $\lambda K^3_{m_1,\dots,m_n}$ can be expressed as the union $\mathscr{G}_1 \cup \dots \cup \mathscr{G}_k$ of k edge-disjoint factors, where for $i = 1, \dots, k$, $\mathscr{G}_i$ is r_i-regular, if and only if:

(i) $m_i = m_j := m$ for all $1 \leqslant i, j \leqslant k$,

(ii) $3 \mid r_i mn$ for each i, $1 \leqslant i \leqslant k$, and

(iii) $\sum_{i=1}^{k} r_i = \lambda \binom{n-1}{2} m^2$.

It is expected that Theorem 5.3 can be used to provide conditions under which one can embed a k-edge-colored complete 3-uniform hypergraph K_n^3 into an edge-colored K_{n+m}^3 such that i^{th} color class of K_{n+m}^3 induces an r_i-factor for $i = 1, \ldots, k$. However obtaining such results will require more advanced edge-coloring techniques and it will be much more complicated than for companion results for simple graphs, with a complete solution unlikely to be found in the near future (see [11]).

In connection with Kirkman's famous Fifteen Schoolgirls Problem [56], Sylvester remarked in 1850 that the complete 3-uniform hypergraph with 15 vertices, is 1-factorizable. Several generalizations of this problem were solved during the last 70 years (see for example [71, 73, 15, 16]). It was Baranyai, who died tragically in his youth, who settled this 120-year-old problem (1-factorization of complete uniform hypergraphs) ingeniously [15, 16].

Baranyai's proof actually yields a method for constructing a 1-factorization recursively. However, this approach would not be very efficient and its complexity is exponential [53]. Baranyi's original theorem was spurred by Peltesohn's result [71] which was a direct construction, and it was polynomial time to implement. Brouwer and Schrijver gave an elegant proof for 1-factorizations of the complete uniform hypergraph for which the algorithm is more efficient [25]. Our construction leads to an algorithm similar to that of Brouwer and Schrijver. This is discussed briefly in Section 5.6, but for more details we refer the reader to Chapter 10.

Notation and more precise definitions will be given in Section 5.2. Any undefined term may be found in [20]. In Section 5.3, we state our main result and we postpone its proof to Section 5.5. In Section 5.4, we exhibit some applications of our result by providing several factorization theorems for 3-uniform (multi)hypergraphs. The key idea used in proving the main theorem is short and is given in 5.5.1. The rest of Section 5.5 is devoted to the verification of all conditions in Theorem 5.3.

5.2   Notation and More Precise Definitions

For the purpose of this chapter, a *hypergraph* $\mathcal{G}$ is an ordered quintuple $(V(\mathcal{G}), E(\mathcal{G}), H(\mathcal{G}),$ $\psi, \phi)$ where $V(\mathcal{G}), E(\mathcal{G}), H(\mathcal{G})$ are disjoint finite sets, $\psi : H(\mathcal{G}) \to V(\mathcal{G})$ is a function and $\phi : H(\mathcal{G}) \to E(\mathcal{G})$ is a surjection. Elements of $V(\mathcal{G}), E(\mathcal{G}), H(\mathcal{G})$ are called *vertices*, *hyper-edges* and *hinges* of $\mathcal{G}$, respectively. A vertex v and hinge h are said to be *incident* with each other if $\psi(h) = v$. A hyperedge e and hinge h are said to be *incident* with each other if $\phi(h) = e$. A hinge h is said to *attach* the hyperedge $\phi(h)$ to the vertex $\psi(h)$. In this manner, the vertex $\phi(h)$ and the hyperedge $\psi(h)$ are said to be *incident* with each other. If $e \in E(\mathcal{G})$, and e is incident with n hinges $h_1, \ldots, h_n$ for some $n \in \mathbb{N}$, then the hyperedge e is said to *join* (not necessarily distinct) vertices $\psi(h_1), \ldots, \psi(h_n)$. If $v \in V(\mathcal{G})$, then the number of hinges incident with v is called the *degree* of v and is denoted by $d_{\mathcal{G}}(v)$.

The number of vertices incident with a hyperedge e, denoted by $|e|$, is called the *size* of e. If $|e| = 1$ then e is called a *loop*. If for all hyperedges e of $\mathcal{G}$, $|e| \leqslant 2$ and $|\phi^{-1}(e)| = 2$, then $\mathcal{G}$ is a *graph*. If $n > 1$ and $e_1, \ldots, e_n$ are n distinct hyperedges of $\mathcal{G}$, incident with the same set of vertices, then $e_1, \ldots, e_n$ is said to be *multiple hyperedges*. A *multi-hypergraph* is a hypergraph with multiple hyperedges.

Thus a hypergraph, in the sense of our definition is a generalization of a finite hypergraph as usually defined, but for convenience, we imagine each hyperedge of a hypergraph to be attached to the vertices which it joins by in-between objects called hinges. In fact if for every edge e, $|e| = |\phi^{-1}(e)|$, then our definition is essentially the same as the usual definition. One can think of a hypergraph as a bipartite multigraph, where E forms one class, V forms other class, and the hinges H form the edges. A hypergraph may be drawn as a set of points representing the vertices. An edge is represented by a simple closed curve enclosing its incident vertices. A hinge is represented by a small line attached to the vertex incident with it (see Figure 5.1).

Example 5.1. Let $\mathscr{F} = (V, E, H, \psi, \phi)$, with $V = \{v_i : 1 \leqslant i \leqslant 7\}, E = \{e_1, e_2, e_3\}, H = \{h_i : 1 \leqslant i \leqslant 9\}$, such that $\psi(h_1) = v_1, \psi(h_2) = \psi(h_3) = v_2, \psi(h_4) = v_3, \psi(h_5) = \psi(h_6) = \psi(h_7) = v_4, \psi(h_8) = v_5, \psi(h_9) = v_6$ and $\phi(h_1) = e_1, \phi(h_2) = \phi(h_3) = \phi(h_4) = \phi(h_5) = \phi(h_6) = e_2, \phi(h_7) = \phi(h_8) = \phi(h_9) = e_3$. Moreover $|e_1| = 1, |e_2| = |e_3| = 3$, and $d(v_1) = d(v_3) = d(v_5) = d(v_6) = 1, d(v_2) = 4, d(v_4) = 3, d(v_7) = 0$.
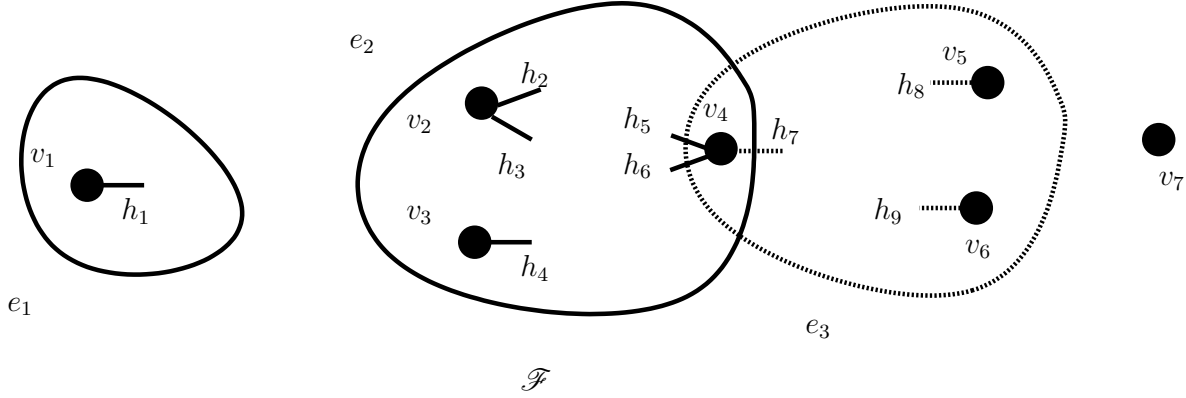
Figure 5.1: Representation of a hypergraph $\mathscr{F}$

Throughout this chapter, the letters $\mathscr{F}$ and $\mathscr{G}$ denote hypergraphs (possibly with loops and multiple hyperedges). The set of hinges of $\mathscr{G}$ which are incident with a vertex v (a hyperedge e), is denoted by $H(\mathscr{G}, v)$ ($H(\mathscr{G}, e)$, respectively). Thus if $e \in E(\mathscr{G})$, then $H(\mathscr{G}, e) = \phi^{-1}(e)$. If $v \in V(\mathscr{G})$, then $H(\mathscr{G}, v) = \psi^{-1}(v)$, and $|H(\mathscr{G}, v)|$ is the degree $d(v)$ of v. If S is a subset of $V(\mathscr{G})$ or $E(\mathscr{G})$, then $H(\mathscr{G}, S)$ denotes the set of those hinges of $\mathscr{G}$ which are incident with an element of S. If $S_1 \subset V(\mathscr{G})$ and $S_2 \subset E(\mathscr{G})$, then $H(\mathscr{G}, S_1, S_2)$ denotes $H(\mathscr{G}, S_1) \cap H(\mathscr{G}, S_2)$. If $v \in V(\mathscr{G})$ and $S \subset E(\mathscr{G})$, then $H(\mathscr{G}, v, S)$ denotes $H(\mathscr{G}, \{v\}, S)$. To avoid ambiguity, subscripts may be used to indicate the hypergraph in which hypergraph-theoretic notation should be interpreted — for example, $d_{\mathscr{G}}(v)$.

Let $\mathscr{G}$ be a hypergraph in which each hyperedge is incident with exactly three hinges. If u, v, w are three (not necessarily distinct) vertices of $\mathscr{G}$, then $\nabla(u, v, w)$ denotes the set of hyperedges which are incident with u, v, w. For each hyperedge e incident with three hinges h_1, h_2, h_3 there are three possibilities (see Figure 5.2):

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(i) e is incident with exactly one vertex u. In this case u is incident with h_1, h_2, h_3. We denote $\nabla(u, u, u)$ by $\nabla(u^3)$.

(ii) e is incident with exactly two distinct vertices u, v. In this case one of the vertices, say u is incident with two hinges, say h_1, h_2 and v is incident with h_3. We denote $\nabla(u, u, v)$ by $\nabla(u^2, v)$.

(iii) e is incident with three distinct vertices u, v and w.

For *multiplicity* we use $m(.)$ rather than $|\nabla(.)|$. A hypergraph $\mathscr{G}$ is said to be *k-uniform*
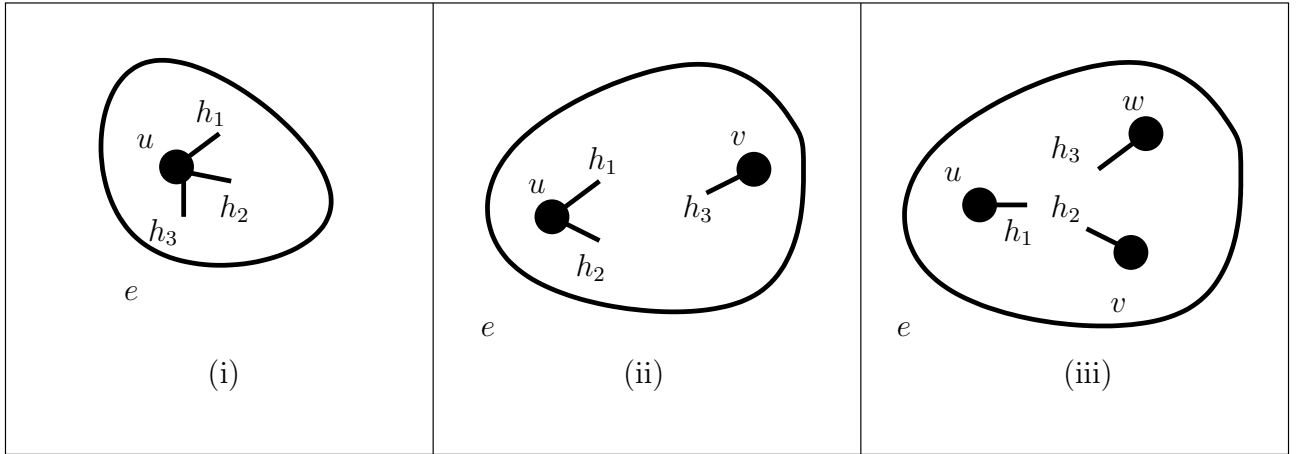
Figure 5.2: The three types of edges in a hypergraph $\mathscr{G}$ in which $|H(\mathscr{G}, e)| = 3$ for every edge e

if $|e| = |H(\mathscr{G}, e)| = k$ for each $e \in E(\mathscr{G})$. A k-uniform hypergraph with n vertices is said to be *complete*, denoted by K_n^k, if every k distinct vertices are incident within one edge. A 3-uniform hypergraph with vertex partition $\{V_1, \ldots, V_n\}$ with $|V_i| = m_i$ for $i = 1, \ldots, n$, is said to be (i) *n-partite*, if every edge is incident with at most one vertex of each part, and (ii) *complete n-partite*, denoted by $K_{m_1, \ldots, m_n}^3$, if it is n-partite and every three distinct vertices from three different parts are incident.

If we replace every hyperedge of $\mathscr{G}$ by λ ($\geqslant 2$) multiple hyperedges, then we denote the new (multi) hypergraph by $\lambda\mathscr{G}$. A *k-hyperedge-coloring* of $\mathscr{G}$ is a mapping $K : E(\mathscr{G}) \to C$,

where C is a set of k *colors* (often we use $C = \{1, \ldots, k\}$), and the hyperedges of one color form a *color class*. The sub-hypergraph of $\mathscr{G}$ induced by the color class j is denoted by $\mathscr{G}(j)$.

A hypergraph $\mathscr{G}$ is said to be (i) *regular* if there is an integer d such that every vertex has degree d, and (ii) *k-regular* if every vertex has degree k. A *factor* of $\mathscr{G}$ is a regular spanning sub-hypergraph of $\mathscr{G}$. A *k-factor* is a k-regular factor. A *factorization* is a decomposition (partition) of $E(\mathscr{G})$ into factor(s). Let $r_1, \ldots, r_k$ be (not necessarily distinct) positive integers. An *$(r_1, \ldots, r_k)$-factorization* is a factorization in which there is one r_i-factor for $i = 1, \ldots, k$. An (r)-factorization is called simply an *r-factorization*. A hypergraph $\mathscr{G}$ is said to be *factorizable* if it has a factorization. The definition for *k-factorizable* and *$(r_1, \ldots, r_k)$-factorizable* hypergraphs is similar.

If $\mathscr{F} = (V, E, H, \psi, \phi)$ is a hypergraph and Ψ is a function from V onto a set W, then we shall say that the hypergraph $\mathscr{G} = (W, E, H, \Psi \circ \psi, \phi)$ is an *amalgamation* of $\mathscr{F}$ and that $\mathscr{F}$ is a *detachment* of $\mathscr{G}$. In this manner, Ψ is called an *amalgamation function*, and $\mathscr{G}$ is the *Ψ-amalgamation* of $\mathscr{F}$. Associated with Ψ is the *number function* $g : W \to \mathbb{N}$ defined by $g(w) = |\Psi^{-1}(w)|$, for each $w \in W$, and we shall say that $\mathscr{F}$ is a *g-detachment* of $\mathscr{G}$. Intuitively speaking, a g-detachment of $\mathscr{G}$ is obtained by splitting each $u \in V(\mathscr{G})$ into $g(u)$ vertices. Thus $\mathscr{F}$ and $\mathscr{G}$ have the same hyperedges and hinges, and each vertex v of $\mathscr{G}$ is obtained by identifying those vertices of $\mathscr{F}$ which belong to the set $\Psi^{-1}(v)$. In this process, a hinge incident with a vertex u and a hyperedge e in $\mathscr{F}$ becomes incident with the vertex $\Psi(u)$ and the edge e in $\mathscr{G}$. Since two hypergraphs $\mathscr{F}$ and $\mathscr{G}$ related in the above manner have the same hyperedges, coloring the hyperedges of one of them is the same thing as coloring the hyperedges of the other. Hence an amalgamation of a hypergraph with colored hyperedges is a hypergraph with colored hyperedges.

Example 5.2. Let $\mathscr{F}$ be the hypergraph of Example 5.1. Let $\Psi : V \to \{w_1, w_2, w_3, w_4\}$ be the function with $\Psi(v_1) = \Psi(v_7) = w_1$, $\Psi(v_2) = w_2$, $\Psi(v_3) = \Psi(v_4) = w_3$, $\Psi(v_5) = \Psi(v_6) = w_4$. The hypergraph $\mathscr{G}$ in Figure 5.3 is the Ψ-amalgamation of $\mathscr{F}$.
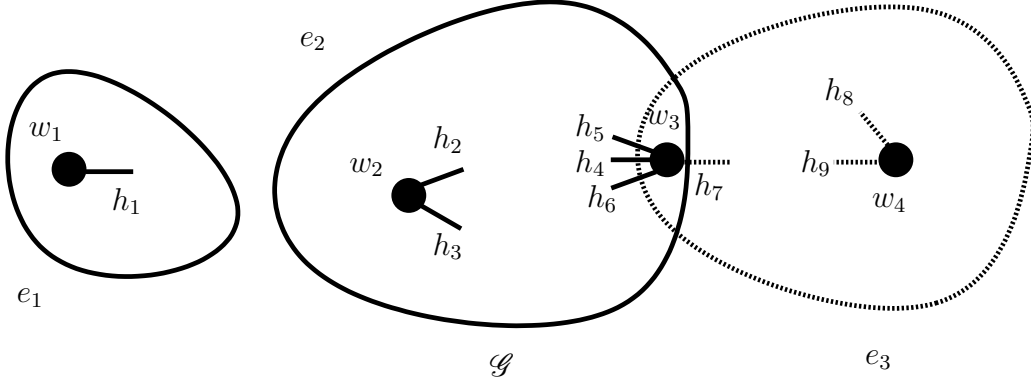
Figure 5.3: Amalgamation $\mathcal{G}$ of the hypergraph $\mathcal{F}$ in Example 5.1

5.3   Statement of the Main Theorem

In the remainder of this chapter, all hypergraphs are either 3-uniform or are amalgama-tions of 3-uniform hypergraphs. That is, for every hypergraph $\mathcal{F}$ we have

$$1 \leqslant |e| \leqslant |H(\mathcal{F}, e)| = 3 \text{ for every } e \text{ in } \mathcal{F}. \tag{5.1}$$

Therefore every edge is of one the types shown in Figure 5.2. For $g : V(\mathcal{F}) \to \mathbb{N}$, we define the symmetric function $\tilde{g} : V^3(\mathcal{F}) \to \mathbb{N}$ such that for distinct $x, y, z \in V(\mathcal{F})$, $\tilde{g}(x, x, x) = \binom{g(x)}{3}$, $\tilde{g}(x, x, y) = \binom{g(x)}{2}g(y)$, and $\tilde{g}(x, y, z) = g(x)g(y)g(z)$. Also we assume that for each $x \in V(\mathcal{F})$, $g(x) \leqslant 2$ implies $m_{\mathcal{F}}(x^3) = 0$, and $g(x) = 1$ implies $m_{\mathcal{F}}(x^2, y) = 0$ for every $y \in V(\mathcal{F})$.

Theorem 5.3. *Let $\mathcal{F}$ be a k-hyperedge-colored hypergraph and let g be a function from $V(\mathcal{F})$ into $\mathbb{N}$. Then there exists a 3-uniform g-detachment $\mathcal{G}$ (possibly with multiple hyperedges) of $\mathcal{F}$ with amalgamation function $\Psi : V(\mathcal{G}) \to V(\mathcal{F})$, g being the number function associated with Ψ, such that $\mathcal{G}$ satisfies the following conditions:*

(A1) *$d_{\mathcal{G}}(u) \approx d_{\mathcal{F}}(x)/g(x)$ for each $x \in V(\mathcal{F})$ and each $u \in \Psi^{-1}(x)$;*

(A2) *$d_{\mathcal{G}(j)}(u) \approx d_{\mathcal{F}(j)}(x)/g(x)$ for each $x \in V(\mathcal{F})$, each $u \in \Psi^{-1}(x)$ and each $j \in \{1, \ldots, k\}$;*

(A3) $m_{\mathscr{G}}(u,v,w) \approx m_{\mathscr{F}}(x,y,z)/\tilde{g}(x,y,z)$ *for every* $x,y,z \in V(\mathscr{F})$ *with* $g(x) \geqslant 3$ *if* $x = y = z$, *and* $g(x) \geqslant 2$ *if* $|\{x,y,z\}| = 2$, *and every triple of distinct vertices* u,v,w *with* $u \in \Psi^{-1}(x)$, $v \in \Psi^{-1}(y)$ *and* $w \in \Psi^{-1}(z)$;

(A4) $m_{\mathscr{G}(j)}(u,v,w) \approx m_{\mathscr{F}(j)}(x,y,z)/\tilde{g}(x,y,z)$ *for every* $x,y,z \in V(\mathscr{F})$ *with* $g(x) \geqslant 3$ *if* $x = y = z$, *and* $g(x) \geqslant 2$ *if* $|\{x,y,z\}| = 2$, *every triple of distinct vertices* u,v,w *with* $u \in \Psi^{-1}(x)$, $v \in \Psi^{-1}(y)$ *and* $w \in \Psi^{-1}(z)$ *and each* $j \in \{1,\dots,k\}$.

5.4 Factorization Consequences

Throughout this section $n \geqslant 3$. It is easy to see that every factorizable hypergraph must be regular. If $\mathscr{G}$ is a 3-uniform hypergraph with an r-factor, since each edge contributes 3 to the sum of the degree of all vertices in an r-factor, $r|V(\mathscr{G})|$ must be divisible by 3.

5.4.1 Factorizations of λK_n^3

We first note that λK_n^3 is $\lambda\binom{n-1}{2}$-regular, and $|E(\lambda K_n^3)| = \lambda\binom{n}{3}$. Throughout this section, $\mathscr{F}$ is a hypergraph consisting of a single vertex x and $\lambda\binom{n}{3}$ loops incident with x, and $g : V(\mathscr{F}) \to \mathbb{N}$ is a function with $g(x) = n$. Note that λK_n^3 is a g-detachment of $\mathscr{F}$.

Theorem 5.4. λK_n^3 *is* $(r_1,\dots,r_k)$-*factorizable if and only if*

(i) $3 \mid r_i n$ *for each* i, $1 \leqslant i \leqslant k$, *and*

(ii) $\sum_{i=1}^{k} r_i = \lambda\binom{n-1}{2}$.

Proof. Suppose first that λK_n^3 is $(r_1,\dots,r_k)$-factorizable. The existence of each r_i-factor implies that $3 \mid r_i n$ for each i, $1 \leqslant i \leqslant k$. Since each r_i-factor is an r_i-regular spanning sub-hypergraph and λK_n^3 is $\lambda\binom{n-1}{2}$-regular, we must have $\sum_{i=1}^{k} r_i = \lambda\binom{n-1}{2}$.

Now assume (i)–(ii). We find a k-hyperedge-coloring for $\mathscr{F}$ such that $m_{\mathscr{F}(j)}(x^3) = r_j n/3$ for each $j \in \{1, \ldots, k\}$. It is possible, because

$$
\begin{aligned}
\sum_{j=1}^{k} m_{\mathscr{F}(j)}(x^3) &= \sum_{j=1}^{k} \frac{r_j n}{3} = \frac{n}{3} \sum_{j=1}^{k} r_j \\
&= \frac{\lambda n}{3} \binom{n-1}{2} = \lambda \binom{n}{3} = m_{\mathscr{F}}(x^3).
\end{aligned}
$$

Now by Theorem 5.3, there exists a 3-uniform g-detachment $\mathscr{G}$ of $\mathscr{F}$ with n vertices, say $v_1, \ldots, v_n$ such that by (A2) $d_{\mathscr{G}(j)}(v_i) = r_j n/n = r_j$ for each $i = 1, \ldots, n$ and each $j \in \{1, \ldots, k\}$; and by (A3) $m_{\mathscr{G}}(v_r, v_s, v_t) = \lambda\binom{n}{3}/\binom{n}{3} = \lambda$ for distinct r, s, t, $1 \leqslant r, s, t \leqslant n$. Therefore $\mathscr{G} \cong \lambda K_n^3$ and each color class i is an r_i-factor for $i = 1, \ldots, k$. $\qquad\square$

5.4.2 Factorizations of $K_{m_1,\ldots,m_n}^3$

We denote $K_{\underbrace{m, \ldots, m}_{n}}^3$ by $K_{m,\ldots,m}^3$ (so we don't write the under-brace when it is not ambiguous). We first note that $\lambda K_{m,\ldots,m}^3$ is a $\lambda\binom{n-1}{2}m^2$-regular hypergraph with nm vertices and $\lambda\binom{n}{3}m^3$ edges. Throughout this section, $\mathscr{F} = \lambda m^3 K_n^3$ with vertex set $V(\mathscr{F}) = \{x_1, \ldots, x_n\}$, and $g : V(\mathscr{F}) \to \mathbb{N}$ is a function with $g(x_i) = m$ for $i = 1, \ldots, n$. We observe that $\lambda K_{m,\ldots,m}^3$ is a g-detachment of $\mathscr{F}$.

Theorem 5.5. $\lambda K_{m_1,\ldots,m_n}^3$ is $(r_1, \ldots, r_k)$-factorizable if and only if

(i) $m_i = m_j := m$ for $1 \leqslant i < j \leqslant n$,

(ii) $3 \mid r_i mn$ for each i, $1 \leqslant i \leqslant k$, and

(iii) $\sum_{i=1}^{k} r_i = \lambda\binom{n-1}{2}m^2$.

Proof. Suppose first that $\lambda K_{m_1,\ldots,m_n}^3$ is r-factorizable (so it is regular). Let u and v be two vertices from two different parts, say p^{th} and q^{th} parts respectively. Then we have the following sequence of equivalences:

$$d(u) = d(v) \qquad\qquad \Longleftrightarrow$$

$$\sum_{\substack{1 \leqslant i < j \leqslant n \\ i,j \neq p}} m_i m_j = \sum_{\substack{1 \leqslant i < j \leqslant n \\ i,j \neq q}} m_i m_j \qquad\qquad \Longleftrightarrow$$

$$m_q \sum_{\substack{1 \leqslant i \leqslant n \\ i \neq p,q}} m_i + \sum_{\substack{1 \leqslant i < j \leqslant n \\ i,j \notin \{p,q\}}} m_i m_j = m_p \sum_{\substack{1 \leqslant i \leqslant n \\ i \neq p,q}} m_i + \sum_{\substack{1 \leqslant i < j \leqslant n \\ i,j \notin \{p,q\}}} m_i m_j \qquad\qquad \Longleftrightarrow$$

$$m_q \sum_{\substack{1 \leqslant i \leqslant n \\ i \neq p,q}} m_i = m_p \sum_{\substack{1 \leqslant i \leqslant n \\ i \neq p,q}} m_i \qquad\qquad \Longleftrightarrow$$

$$(m_p - m_q) \sum_{\substack{1 \leqslant i \leqslant n \\ i \neq p,q}} m_i = 0 \qquad\qquad \Longleftrightarrow$$

$$m_p = m_q := m. \qquad\qquad (n \geqslant 3)$$

This proves (i). The existence of each r_i-factor implies that $3 \mid r_i mn$ for each i, $1 \leqslant i \leqslant k$. Since each r_i-factor is an r_i-regular spanning sub-hypergraph and $K^3_{m,\dots,m}$ is $\lambda \binom{n-1}{2} m^2$-regular, we must have $\sum_{i=1}^{k} r_i = \lambda \binom{n-1}{2} m^2$.

Now assume (i)–(iii). Since $3 \mid r_i mn$ for each i, $1 \leqslant i \leqslant k$ and $\sum_{i=1}^{k} mr_i = \lambda \binom{n-1}{2} m^3$, by Theorem 5.4, $\mathscr{F}$ is $(mr_1, \dots, mr_k)$-factorizable. Therefore we can find a k-hyperedge-coloring for $\mathscr{F}$ such that

$$d_{\mathscr{F}(j)}(x) = r_j m \quad \forall j \in \{1, \dots, k\}.$$

Now by Theorem 5.3, there exists a 3-uniform g-detachment $\mathscr{G}$ of $\mathscr{F}$ with mn vertices, say x_{ij}, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$ ($x_{i1}, \dots, x_{im}$ are obtained by splitting x_i into m vertices for $i = 1, \dots, n$) such that by (A2) $d_{\mathscr{G}(t)}(x_{ij}) = r_t m/m = r_t$ for each $i = 1, \dots, n$, $j = 1, \dots, m$, and each $t \in \{1, \dots, k\}$; by (A3) $m_{\mathscr{G}}(x_{ij}, x_{ij'}, x_{ij''}) = 0$ for $i = 1 \dots, n$ and distinct j, j', j'', $1 \leqslant j, j', j'' \leqslant m$, if $m \geqslant 3$; by (A3) $m_{\mathscr{G}}(x_{ij}, x_{ij'}, x_{i'j''}) = 0$ for distinct i, i', $1 \leqslant i, i' \leqslant n$ and distinct j, j', $1 \leqslant j, j', j'' \leqslant m$, if $m \geqslant 2$; and by (A3) $m_{\mathscr{G}}(x_{ij}, x_{i'j'}, x_{i''j''}) = \lambda m^3/(mmm) = \lambda$ for distinct i, i', i'', $1 \leqslant i, i', i'' \leqslant n$ and $1 \leqslant j, j', j'' \leqslant m$. Therefore $\mathscr{G} \cong \lambda K^3_{m,\dots,m}$ and each color class i is an r_i-factor for each $i \in \{1, \dots, k\}$. $\qquad\square$

5.5 Proof of the Main Theorem

Recall that $x \approx y$ means $\lfloor y \rfloor \leqslant x \leqslant \lceil y \rceil$. We observe that for $x, y \in \mathbb{R}, a, b, c \in \mathbb{Z}$, and $n \in \mathbb{N}$ (i) $a \approx x$ implies $a \in \{\lfloor x \rfloor, \lceil x \rceil\}$, (ii) $x \approx y$ implies $x/n \approx y/n$ (iii) the relation $\approx$ is transitive (but not symmetric), and (vi) $a = b - c$ and $c \approx x$, implies $a \approx b - x$. These properties of $\approx$ will be used in this section when required without further explanation.

Let $\mathscr{F} = (V, E, H, \psi, \phi)$. Let $n = \sum_{v \in V}(g(v) - 1)$. Our proof of Theorem 5.3 consists of the following major parts. First, in Section 5.5.1 we shall describe the construction of a sequence $\mathscr{F}_0 = \mathscr{F}, \mathscr{F}_1, \ldots, \mathscr{F}_n$ of hypergraphs where $\mathscr{F}_i$ is an amalgamation of $\mathscr{F}_{i+1}$ (so $\mathscr{F}_{i+1}$ is a detachment of $\mathscr{F}_i$) for $0 \leqslant i \leqslant n - 1$ with amalgamation function Φ_i that combines a vertex with amalgamation number 1 with one other vertex. To construct each $\mathscr{F}_{i+1}$ from $\mathscr{F}_i$ we will use two laminar families $\mathscr{A}_i$ and $\mathscr{B}_i$. In Section 5.5.2 we shall observe some properties of $\mathscr{F}_{i+1}$ in terms of $\mathscr{F}_i$. As we will see in Section 5.5.3, the relations between $\mathscr{F}_{i+1}$ and $\mathscr{F}_i$ lead to conditions relating each $\mathscr{F}_i$, $1 \leqslant i \leqslant n$ to the initial hypergraph $\mathscr{F}$. Finally, in Section 5.5.4 we will show that $\mathscr{F}_n$ satisfies the conditions (A1)–(A4), so we can let $\mathscr{G} = \mathscr{F}_n$.

5.5.1 Construction of $\mathscr{G}$

Initially we let $\mathscr{F}_0 = \mathscr{F}$ and $g_0 = g$, and we let Φ_0 be the identity function from V into V. Now assume that $\mathscr{F}_0 = (V_0, E_0, H_0, \psi_0, \phi_0), \ldots, \mathscr{F}_i = (V_i, E_i, H_i, \psi_i, \phi_i)$ and $\Phi_0, \ldots, \Phi_i$ have been defined for some $i \geqslant 0$. Also assume that $g_0 : V_0 \to \mathbb{N}, \ldots, g_i : V_i \to \mathbb{N}$ have been defined such that for each $j = 0, \ldots, i$ and each $x \in V_j$, $g_j(x) \leqslant 2$ implies $m_{\mathscr{F}_j}(x^3) = 0$, and $g_j(x) = 1$ implies $m_{\mathscr{F}_j}(x^2, y) = 0$ for every $y \in V_j$. Let $\Psi_i = \Phi_0 \ldots \Phi_i$. If $i = n$, we terminate the construction, letting $\mathscr{G} = \mathscr{F}_n$ and $\Psi = \Psi_n$.

If $i < n$, we can select a vertex α of $\mathscr{F}_i$ such that $g_i(\alpha) \geqslant 2$. As we will see, $\mathscr{F}_{i+1}$ is formed from $\mathscr{F}_i$ by detaching a vertex v_{i+1} with amalgamation number 1 from α.

Let $H_{ij} = H(\mathscr{F}_i(j), \alpha)$ for $j = 1, \ldots, k$. If $e \in E_i$ incident with α, we let $H_{ij}^e = H(\mathscr{F}_i(j), \alpha, e)$ for $j = 1, \ldots, k$. Recall that by (5.1), $|H_{ij}^e| \leqslant 3$. Intuitively speaking, H_{ij} is

the set of all hinges which are incident with α and a hyperedge colored j, and H_{ij}^e is a subset of H_{ij} consisting of only those hinges incident with a single hyperedge e colored j. Now let

$$
\begin{aligned}
\mathscr{A}_i \;=\; & \{H(\mathscr{F}_i, \alpha)\} \\
& \bigcup \;\{H_{i1}, \ldots, H_{ik}\} \\
& \bigcup \;\{H_{ij}^e : e \in \nabla(\alpha^2, y), y \in V_i, 1 \leqslant j \leqslant k\}.
\end{aligned}
\tag{5.2}
$$

Note that

$$
\begin{aligned}
\{H_{ij}^e : e \in \nabla(\alpha^2, y), y \in V_i, 1 \leqslant j \leqslant k\} \;=\; & \{H_{ij}^e : e \in \nabla(\alpha^3), 1 \leqslant j \leqslant k\} \\
& \bigcup \;\{H_{ij}^e : e \in \nabla(\alpha^2, y), y \in V_i \backslash \{\alpha\}, 1 \leqslant j \leqslant k\}.
\end{aligned}
$$

If $u, v \in V_i$, let $H_i^{uv} = H(\mathscr{F}_i, \nabla(\alpha, u, v))$ and $H_{ij}^{uv} = H(\mathscr{F}_i(j), \alpha, \nabla(\alpha, u, v))$ for $j = 1, \ldots, k$. Now let

$$
\begin{aligned}
\mathscr{B}_i \;=\; & \{H_i^{uv} : u, v \in V_i\} \\
& \bigcup \;\{H_{ij}^{uv} : u, v \in V_i, 1 \leqslant j \leqslant k\}.
\end{aligned}
\tag{5.3}
$$

It is easy to see that both $\mathscr{A}_i$ and $\mathscr{B}_i$ are laminar families of subsets of $H(\mathscr{F}_i, \alpha)$. Then, by Lemma 1.3, there exists a subset Z_i of $H(\mathscr{F}_i, \alpha)$ such that

$$
|Z_i \cap P| \approx |P|/g_i(\alpha), \text{ for every } P \in \mathscr{A}_i \cup \mathscr{B}_i.
\tag{5.4}
$$

Let v_{i+1} be a vertex which does not belong to V_i and let $V_{i+1} = V_i \cup \{v_{i+1}\}$. Let Φ_{i+1} be the function from V_{i+1} onto V_i such that $\Phi_{i+1}(v) = v$ for every $v \in V_i$ and $\Phi_{i+1}(v_{i+1}) = \alpha$. Let $\mathscr{F}_{i+1}$ be the detachment of $\mathscr{F}_i$ under Φ_{i+1} ($\mathscr{F}_i$ is the Φ_{i+1}-amalgamation of $\mathscr{F}_{i+1}$) such that $V(\mathscr{F}_{i+1}) = V_{i+1}$, and

$$H(\mathscr{F}_{i+1}, v_{i+1}) = Z_i, H(\mathscr{F}_{i+1}, \alpha) = H(\mathscr{F}_i, \alpha) \backslash Z_i. \tag{5.5}$$

In fact, $\mathscr{F}_{i+1}$ is obtained from $\mathscr{F}_i$ by splitting α into two vertices α and v_{i+1} in such a way that hinges which were incident with α in $\mathscr{F}_i$ become incident in $\mathscr{F}_{i+1}$ with α or v_{i+1} according as they do not or do belong to Z_i, respectively. Obviously, Ψ_i is an amalgamation function from $\mathscr{F}_{i+1}$ into $\mathscr{F}_i$. Let g_{i+1} be the function from V_{i+1} into $\mathbb{N}$, such that $g_{i+1}(v_{i+1}) = 1, g_{i+1}(\alpha) = g_i(\alpha) - 1, g_{i+1}(v) = g_i(v)$ for every $v \in V_i \backslash \{\alpha\}$. This finishes the construction of $\mathscr{F}_{i+1}$. Now, we explore some relations between $\mathscr{F}_{i+1}$ and $\mathscr{F}_i$. In the remainder of this chapter, $d_i(.), m_i(.), d(.),$ and $m(.)$ will denote $d_{\mathscr{F}_i}(.), m_{\mathscr{F}_i}(.), d_{\mathscr{F}}(.),$ and $m_{\mathscr{F}}(.)$, respectively.

5.5.2   Relations Between $\mathscr{F}_{i+1}$ and $\mathscr{F}_i$

The hypergraph $\mathscr{F}_{i+1}$, described in 5.5.1, satisfies the following conditions:

(B1) $d_{i+1}(\alpha) \approx d_i(\alpha)g_{i+1}(\alpha)/g_i(\alpha)$;

(B2) $d_{i+1}(v_{i+1}) \approx d_i(\alpha)/g_i(\alpha)$;

(B3) $m_{i+1}(\alpha, v^2) \approx m_i(\alpha, v^2)g_{i+1}(\alpha)/g_i(\alpha)$ for each $v \in V_i \backslash \{\alpha\}$;

(B4) $m_{i+1}(v_{i+1}, v^2) \approx m_i(\alpha, v^2)/g_i(\alpha)$ for each $v \in V_i \backslash \{\alpha\}$;

(B5) $m_{i+1}(\alpha, u, v) \approx m_i(\alpha, u, v)g_{i+1}(\alpha)/g_i(\alpha)$ for every pair of distinct vertices $u, v \in V_i \backslash \{\alpha\}$;

(B6) $m_{i+1}(v_{i+1}, u, v) \approx m_i(\alpha, u, v)/g_i(\alpha)$ for every pair of distinct vertices $u, v \in V_i \backslash \{\alpha\}$;

(B7) $m_{i+1}(v_{i+1}^2, v) = 0$ for each $v \in V_i \backslash \{\alpha\}$;

(B8) $m_{i+1}(\alpha, v_{i+1}, v) \approx 2m_i(\alpha^2, v)/g_i(\alpha)$ for each $v \in V_i \backslash \{\alpha\}$;

(B9) $m_{i+1}(\alpha^2, v) \approx m_i(\alpha^2, v)(g_{i+1}(\alpha) - 1)/g_i(\alpha)$ for each $v \in V_i \backslash \{\alpha\}$;

(B10) $m_{i+1}(v_{i+1}^3) = m_{i+1}(v_{i+1}^2, \alpha) = 0$;

(B11) $m_{i+1}(\alpha^3) \approx m_i(\alpha^3)(g_{i+1}(\alpha) - 2)/g_i(\alpha)$;

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(B12) $m_{i+1}(v_{i+1}, \alpha^2) \approx 3m_i(\alpha^3)/g_i(\alpha)$.

Proof. Since $H(\mathscr{F}_i, \alpha) \in \mathscr{A}_i$, from (5.5) it follows that

$$
\begin{aligned}
d_{i+1}(v_{i+1}) &= |H(\mathscr{F}_{i+1}, v_{i+1})| = |Z_i| = |Z_i \cap H(\mathscr{F}_i, \alpha)| \\
&\approx |H(\mathscr{F}_i, \alpha)|/g_i(\alpha) = d_i(\alpha)/g_i(\alpha), \\
d_{i+1}(\alpha) &= |H(\mathscr{F}_{i+1}, \alpha)| = |H(\mathscr{F}_i, \alpha)| - |Z_i| \\
&\approx d_i(\alpha) - d_i(\alpha)/g_i(\alpha) = (g_i(\alpha) - 1)d_i(\alpha)/g_i(\alpha) \\
&= d_i(\alpha)g_{i+1}(\alpha)/g_i(\alpha).
\end{aligned}
$$

This proves (B1) and (B2).

If $v \in V_i \backslash \{\alpha\}$, then $H_i^{vv} \in \mathscr{B}_i$ and so

$$
\begin{aligned}
m_{i+1}(v_{i+1}, v^2) &= |Z_i \cap H_i^{vv}| \approx |H_i^{vv}|/g_i(\alpha) = m_i(\alpha, v^2)/g_i(\alpha), \\
m_{i+1}(\alpha, v^2) &= |H_i^{vv}| - |Z_i \cap H_i^{vv}| \approx m_i(\alpha, v^2) - m_i(\alpha, v^2)/g_i(\alpha) \\
&= (g_i(\alpha) - 1)m_i(\alpha, v^2)/g_i(\alpha) \\
&= m_i(\alpha, v^2)g_{i+1}(\alpha)/g_i(\alpha).
\end{aligned}
$$

This proves (B3) and (B4) (see Figure 5.4(i)).

If u, v are a pair of distinct vertices in $V_i \backslash \{\alpha\}$, then $H_i^{uv} \in \mathscr{B}_i$ and so

$$
\begin{aligned}
m_{i+1}(v_{i+1}, u, v) &= |Z_i \cap H_i^{uv}| \approx |H_i^{uv}|/g_i(\alpha) = m_i(\alpha, u, v)/g_i(\alpha), \\
m_{i+1}(\alpha, u, v) &= |H_i^{uv}| - |Z_i \cap H_i^{uv}| \\
&\approx m_i(\alpha, u, v) - m_i(\alpha, u, v)/g_i(\alpha) \\
&= (g_i(\alpha) - 1)m_i(\alpha, u, v)/g_i(\alpha) \\
&= m_i(\alpha, u, v)g_{i+1}(\alpha)/g_i(\alpha).
\end{aligned}
$$

This proves (B5) and (B6) (see Figure 5.4(ii)).

If $v \in V_i \backslash \{\alpha\}$, and $e \in \nabla_{\mathscr{F}_i(j)}(\alpha^2, v)$, then $H_{ij}^e \in \mathscr{A}_i$, so

$$\left| Z_i \cap H_{ij}^e \right| \approx |H_{ij}^e| / g_i(\alpha) = 2/g_i(\alpha) \leqslant 1.$$

Therefore either $|Z_i \cap H_{ij}^e| = 1$ and consequently $e \in \nabla_{\mathscr{F}_{i+1}}(v_{i+1}, \alpha, v)$ or $Z_i \cap H_{ij}^e = \varnothing$ and consequently $e \in \nabla_{\mathscr{F}_{i+1}}(\alpha^2, v)$. Therefore

$$\nabla_{\mathscr{F}_{i+1}}(v_{i+1}^2, v) = \varnothing.$$

This proves (B7) (see Figure 5.4(iii)). Moreover, since $H_i^{\alpha v} \in \mathscr{B}_i$
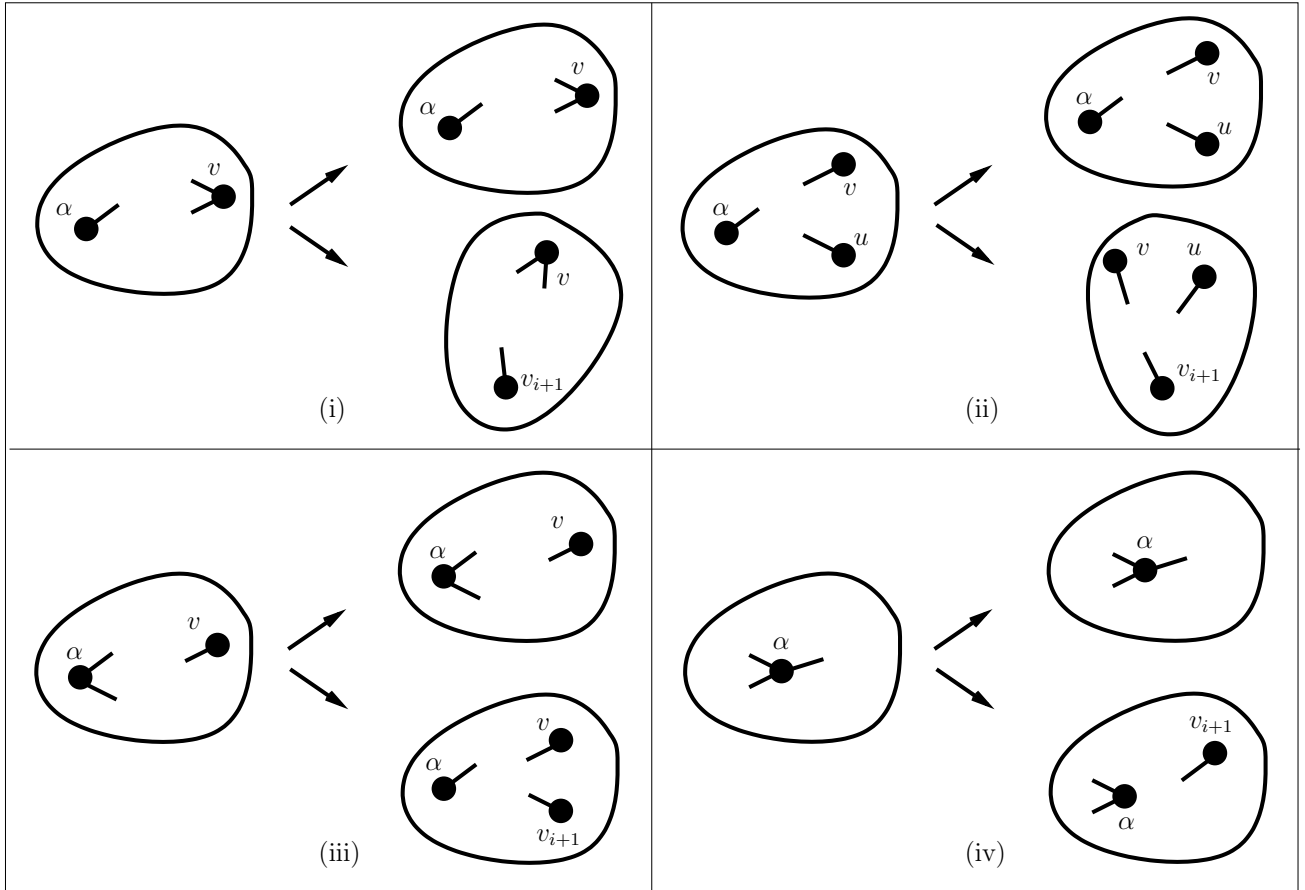
Figure 5.4: The four possibilities for detachment of a single edge incident with α

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$$m_{i+1}(\alpha, v_{i+1}, v) = |Z_i \cap H_i^{\alpha v}| \approx |H_i^{\alpha v}|/g_i(\alpha) = 2m_i(\alpha^2, v)/g_i(\alpha),$$

$$m_{i+1}(\alpha^2, v) = m_i(\alpha^2, v) - |Z_i \cap H_i^{\alpha v}|$$

$$\approx m_i(\alpha^2, v) - 2m_i(\alpha, u, v)/g_i(\alpha)$$

$$= (g_i(\alpha) - 2)m_i(\alpha^2, v)/g_i(\alpha)$$

$$= m_i(\alpha^2, v)(g_{i+1}(\alpha) - 1)/g_i(\alpha).$$

This proves (B8) and (B9). We note that from (B9) it follows that if $g_{i+1}(\alpha) = 1$, then $m_{i+1}(\alpha^2, v) = 0$.

If e is a loop in $\mathscr{F}_i(j)$ incident with α, (so $g_i(\alpha) \geqslant 3$,) then $H_{ij}^e \in \mathscr{A}_i$. So

$$|Z_i \cap H_{ij}^e| \approx |H_{ij}^e|/g_i(\alpha) = 3/g_i(\alpha) \leqslant 1.$$

Therefore either $|Z_i \cap H_{ij}^e| = 1$ and consequently $e \in \nabla_{\mathscr{F}_{i+1}}(\alpha^2, v_{i+1})$ or $Z_i \cap H_{ij}^e = \varnothing$ and consequently $e \in \nabla_{\mathscr{F}_{i+1}}(\alpha^3)$. Therefore

$$\nabla_{\mathscr{F}_{i+1}}(v_{i+1}^3) = \nabla_{\mathscr{F}_{i+1}}(v_{i+1}^2, \alpha) = \varnothing.$$

This proves (B10) (see Figure 5.4(iv)). Moreover,

$$m_{i+1}(\alpha^2, v_{i+1}) = |Z_i \cap H_i^{\alpha\alpha}| \approx |H_i^{\alpha\alpha}|/g_i(\alpha) = 3m_i(\alpha^3)/g_i(\alpha),$$

$$m_{i+1}(\alpha^3) = m_i(\alpha^3) - |Z_i \cap H_i^{\alpha\alpha}| \approx m_i(\alpha^3) - 3m_i(\alpha^3)/g_i(\alpha)$$

$$= (g_i(\alpha) - 3)m_i(\alpha^3)/g_i(\alpha) = m_i(\alpha^3)(g_{i+1}(\alpha) - 2)/g_i(\alpha).$$

This proves (B11) and (B12). We may note that from (B11) it follows that if $g_{i+1}(\alpha) = 2$, then $m_{i+1}(\alpha^3) = 0$. $\qquad\square$

A similar statement can be proved for every color class: Let us fix $j \in \{1, \ldots, k\}$, and let u, v be a pair of distinct vertices in $V_i \backslash \{\alpha\}$. The colored version of (B7) and (B10) is trivial.

Since $H_{ij} \in \mathscr{A}_i$, $H_{ij}^{vv} \in \mathscr{B}_i$, $H_{ij}^{uv} \in \mathscr{B}_i$, $H_{ij}^{\alpha v} \in \mathscr{B}_i$, $H_{ij}^{\alpha \alpha} \in \mathscr{B}_i$, respectively, we can obtain a colored version for (B1) and (B2), (B3) and (B4), (B5) and (B6), (B8) and (B9), and (B11) and (B12), respectively.

5.5.3 Relations Between $\mathscr{F}_i$ and $\mathscr{F}$

Recall that $\Psi_i = \Phi_0 \dots \Phi_i$, that $\Phi_0 : V \to V$, and that $\Phi_i : V_i \to V_{i-1}$ for $i > 0$. Therefore $\Psi_i : V_i \to V$ and thus $\Psi_i^{-1} : V \to V_i$.

Now we use (B1)–(B12) to prove that the hypergraph $\mathscr{F}_i$ satisfies the following conditions for $0 \leqslant i \leqslant n$:

(D1) $d_i(x)/g_i(x) \approx d(x)/g(x)$ for each $x \in V$;

(D2) $d_i(v_r) \approx d(x)/g(x)$ for each $x \in V$ and each $v_r \in \Psi_i^{-1}[x]$;

(D3) $m_i(x^3)/\binom{g_i(x)}{3} \approx m(x^3)/\binom{g(x)}{3}$ for each $x \in V$ with $g(x) \geqslant 3$ if $g_i(x) \geqslant 3$, and $m_i(x^3) = 0$ otherwise;

(D4) $m_i(v_r^3) = 0$ for each $x \in V$ and each $v_r \in \Psi_i^{-1}[x]$;

(D5) $m_i(x^2, v_r)/\binom{g_i(x)}{2} \approx m(x^3)/\binom{g(x)}{3}$ for each $x \in V$ with $g(x) \geqslant 3$ and each $v_r \in \Psi_i^{-1}[x]$ if $g_i(x) \geqslant 2$, and $m_i(x^2, v_r) = 0$ otherwise;

(D6) $m_i(x, v_r, v_s)/g_i(x) \approx m(x^3)/\binom{g(x)}{3}$ for each $x \in V$ with $g(x) \geqslant 3$ and every pair of distinct vertices $v_r, v_s \in \Psi_i^{-1}[x]$;

(D7) $m_i(v_r, v_s, v_t) \approx m(x^3)/\binom{g(x)}{3}$ for each $x \in V$ with $g(x) \geqslant 3$ and every triple of distinct vertices $v_r, v_s, v_t \in \Psi_i^{-1}[x]$;

(D8) $m_i(x^2, y)/(\binom{g_i(x)}{2}g_i(y)) \approx m(x^2, y)/(\binom{g(x)}{2}g(y))$ for every pair of distinct vertices $x, y \in V$ with $g(x) \geqslant 2$ if $g_i(x) \geqslant 2$, and $m_i(x^2, y) = 0$ otherwise;

(D9) $m_i(x^2, v_t)/\binom{g_i(x)}{2} \approx m(x^2, y)/(\binom{g(x)}{2}g(y))$ for every pair of distinct vertices $x, y \in V$ with $g(x) \geqslant 2$ and each $v_t \in \Psi_i^{-1}[y]$ if $g_i(x) \geqslant 2$, and $m_i(x^2, v_t) = 0$ otherwise;

(D10) $m_i(x, v_r, y)/(g_i(x)g_i(y)) \approx m(x^2, y)/((\binom{g(x)}{2})g(y))$ for every pair of distinct vertices $x, y \in V$ with $g(x) \geqslant 2$ and each $v_r \in \Psi_i^{-1}[x]$;

(D11) $m_i(x, v_r, v_t)/g_i(x) \approx m(x^2, y)/((\binom{g(x)}{2})g(y))$ for every pair of distinct vertices $x, y \in V$ with $g(x) \geqslant 2$, each $v_r \in \Psi_i^{-1}[x]$ and each $v_t \in \Psi_i^{-1}[y]$;

(D12) $m_i(v_r, v_s, y)/g_i(y) \approx m(x^2, y)/((\binom{g(x)}{2})g(y))$ for every pair of distinct vertices $x, y \in V$ with $g(x) \geqslant 2$ and every pair of distinct vertices $v_r, v_s \in \Psi_i^{-1}[x]$;

(D13) $m_i(v_r, v_s, v_t) \approx m(x^2, y)/((\binom{g(x)}{2})g(y))$ for every pair of distinct vertices $x, y \in V$ with $g(x) \geqslant 2$, every pair of distinct vertices $v_r, v_s \in \Psi_i^{-1}[x]$ and each $v_t \in \Psi_i^{-1}[y]$;

(D14) $m_i(x, y, z)/(g_i(x)g_i(y)g_i(z)) \approx m(x, y, z)/(g(x)g(y)g(z))$ for every triple of distinct vertices $x, y, z \in V$;

(D15) $m_i(x, y, v_t)/(g_i(x)g_i(y)) \approx m(x, y, z)/(g(x)g(y)g(z))$ for every triple of distinct vertices $x, y, z \in V$ and each $v_t \in \Psi_i^{-1}[z]$;

(D16) $m_i(x, v_s, v_t)/g_i(x) \approx m(x, y, z)/(g(x)g(y)g(z))$ for every triple of distinct vertices $x, y, z \in V$, each $v_s \in \Psi_i^{-1}[y]$ and each $v_t \in \Psi_i^{-1}[z]$;

(D17) $m_i(v_r, v_s, v_t) \approx m(x, y, z)/(g(x)g(y)g(z))$ for every triple of distinct vertices $x, y, z \in V$, each $v_r \in \Psi_i^{-1}[x]$, each $v_s \in \Psi_i^{-1}[y]$ and each $v_t \in \Psi_i^{-1}[z]$.

Proof. Let x, y, z be an arbitrary triple of distinct vertices of V. We prove (D1)–(D17) by induction. To verify (D1)–(D17) for $i = 0$, recall that $\mathscr{F}_0 = \mathscr{F}$, and $g_0(x) = g(x)$.

Obviously $d_0(x)/g_0(x) = d(x)/g(x)$, and this proves (D1) for $i = 0$. (D2) is trivial. If $g(x) \geqslant 3$, obviously $m_0(x^3)/(\binom{g_0(x)}{3}) = m(x^3)/(\binom{g(x)}{3})$, and if $g(x) \leqslant 2$, by hypothesis of Theorem 5.3, $m(x^3) = 0$. This proves (D3) for $i = 0$. The proof of (D4)–(D17) for $i = 0$ is similar and can be verified easily.

Now we will show that if $\mathscr{F}_i$ satisfies the conditions (D1)–(D17) for some $i < n$, then $\mathscr{F}_{i+1}$ (formed from $\mathscr{F}_i$ by detaching v_{i+1} from the vertex α) satisfies these conditions by

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replacing i with $i+1$; we denote the corresponding conditions for $\mathscr{F}_{i+1}$ by (D1)′–(D17)′. If $g_{i+1}(x) = g_i(x)$, then (D1)′–(D7)′ are obviously true. So we just check (D1)′–(D7)′ in the case where $x = \alpha$. Also if $g_{i+1}(x) = g_i(x)$ and $g_{i+1}(y) = g_i(y)$, then (D8)′–(D13)′ are clearly true. So in order to prove (D8)′–(D13)′, we shall assume that either $g_{i+1}(x) = g_i(x) - 1$ or $g_{i+1}(y) = g_i(y) - 1$ (so $\alpha \in \{x, y\}$). Similarly, if $g_{i+1}(x) = g_i(y), g_{i+1}(y) = g_i(y)$, and $g_{i+1}(z) = g_i(z)$, then (D14)′–(D17)′ are true. Therefore to prove (D14)′–(D17)′ we shall assume that either $g_{i+1}(x) = g_i(x) - 1$ or $g_{i+1}(y) = g_i(y) - 1$ or $g_{i+1}(z) = g_i(z) - 1$ (so $\alpha \in \{x, y, z\}$).

(D1)′ By (B1), $d_{i+1}(\alpha)/g_{i+1}(\alpha) \approx d_i(\alpha)/g_i(\alpha)$, and by (D1) of the induction hypothesis $d_i(\alpha)/g_i(\alpha) \approx d(\alpha)/g(\alpha)$. Therefore

$$\frac{d_{i+1}(\alpha)}{g_{i+1}(\alpha)} \overset{\text{(B1)}}{\approx} \frac{d_i(\alpha)}{g_i(\alpha)} \overset{\text{(D1)}}{\approx} \frac{d(\alpha)}{g(\alpha)}.$$

This proves (D1)′.

(D2)′ By (B2), $d_{i+1}(v_{i+1}) \approx d_i(\alpha)/g_i(\alpha)$, and by (D1) of the induction hypothesis $d_i(\alpha)/g_i(\alpha) \approx d(\alpha)/g(\alpha)$. Therefore

$$d_{i+1}(v_{i+1}) \overset{\text{(B2)}}{\approx} \frac{d_i(\alpha)}{g_i(\alpha)} \overset{\text{(D1)}}{\approx} \frac{d(\alpha)}{g(\alpha)}.$$

Since in forming $\mathscr{F}_{i+1}$ no hyperedge is detached from v_r for each $v_r \in \Psi_i^{-1}[\alpha]$, we have $d_{i+1}(v_r) = d_i(v_r)$. By (D2) of the induction hypothesis $d_i(v_r) \approx d(\alpha)/g(\alpha)$ for each $v_r \in \Psi_i^{-1}[\alpha]$. Therefore

$$d_{i+1}(v_r) = d_i(v_r) \overset{\text{(D2)}}{\approx} \frac{d(\alpha)}{g(\alpha)}$$

for each $v_r \in \Psi_i^{-1}[\alpha]$. This proves (D2)′.

(D3)′ Suppose $g(\alpha) \geqslant 3$. If $g_{i+1}(\alpha) \geqslant 3$, by (B11)

$$\frac{m_{i+1}(\alpha^3)}{\binom{g_{i+1}(\alpha)}{3}} \overset{\text{(B11)}}{\approx} \frac{m_i(\alpha^3)(g_{i+1}(\alpha) - 2)}{g_i(\alpha)\binom{g_{i+1}(\alpha)}{3}}$$
$$= \frac{m_i(\alpha^3)(g_{i+1}(\alpha) - 2)}{g_i(\alpha)g_{i+1}(\alpha)(g_{i+1}(\alpha) - 1)(g_{i+1}(\alpha) - 2)/6}$$
$$= \frac{m_i(\alpha^3)}{\binom{g_i(\alpha)}{3}}.$$

Since $g_i(\alpha) \geqslant 4 > 3$, by (D3) of the induction hypothesis $m_i(\alpha^3)/\binom{g_i(\alpha)}{3} \approx m(\alpha^3)/\binom{g(\alpha)}{3}$.

Therefore

$$\frac{m_{i+1}(\alpha^3)}{\binom{g_{i+1}(\alpha)}{3}} \overset{\text{(B11)}}{\approx} \frac{m_i(\alpha^3)}{\binom{g_i(\alpha)}{3}} \overset{\text{(D3)}}{\approx} \frac{m(\alpha^3)}{\binom{g(\alpha)}{3}}.$$

If $g_{i+1}(\alpha) < 3$, by (B11) $m_{i+1}(\alpha^3) = 0$. This proves (D3)′.

(D4)′ By (B10), $m_{i+1}(v_{i+1}^3) = 0$. Moreover, $m_{i+1}(v_r^3) = m_i(v_r^3) = 0$ for each $1 \leqslant r \leqslant i$. This proves (D4)′.

(D5)′ Suppose $g(\alpha) \geqslant 3$. If $g_{i+1}(\alpha) \geqslant 2$, by (B12)

$$\frac{m_{i+1}(\alpha^2, v_{i+1})}{\binom{g_{i+1}(\alpha)}{2}} \overset{\text{(B12)}}{\approx} \frac{3m_i(\alpha^3)}{g_i(\alpha)\binom{g_{i+1}(\alpha)}{2}}$$
$$= \frac{3m_i(\alpha^3)}{g_i(\alpha)g_{i+1}(\alpha)(g_{i+1}(\alpha) - 1)/2}$$
$$= \frac{m_i(\alpha^3)}{\binom{g_i(\alpha)}{3}}.$$

Since $g_i(\alpha) \geqslant 3$, by (D3) of the induction hypothesis $m_i(\alpha^3)/\binom{g_i(\alpha)}{3} \approx m(\alpha^3)/\binom{g(\alpha)}{3}$.

Therefore

$$\frac{m_{i+1}(\alpha^2, v_{i+1})}{\binom{g_{i+1}(\alpha)}{2}} \overset{\text{(B12)}}{\approx} \frac{m_i(\alpha^3)}{\binom{g_i(\alpha)}{3}} \overset{\text{(D3)}}{\approx} \frac{m(\alpha^3)}{\binom{g(\alpha)}{3}}.$$

By (B9) for each $v_r \in \Psi_i^{-1}[\alpha]$

$$
\begin{aligned}
\frac{m_{i+1}(\alpha^2, v_r)}{\binom{g_{i+1}(\alpha)}{2}} &\overset{\text{(B9)}}{\approx} \frac{m_i(\alpha^2, v_r)(g_{i+1}(\alpha) - 1)}{g_i(\alpha)\binom{g_{i+1}(\alpha)}{2}} \\
&= \frac{m_i(\alpha^2, v_r)(g_{i+1}(\alpha) - 1)}{g_i(\alpha)g_{i+1}(\alpha)(g_{i+1}(\alpha) - 1)/2} \\
&= \frac{m_i(\alpha^2, v_r)}{\binom{g_i(\alpha)}{2}}.
\end{aligned}
$$

Since $g_i(\alpha) \geqslant 3 > 2$, by (D5) of the induction hypothesis we have $m_i(\alpha^2, v_r)/\binom{g_i(\alpha)}{2} \approx m(\alpha^3)/\binom{g(\alpha)}{3}$ for each $v_r \in \Psi_i^{-1}[\alpha]$. Therefore

$$
\frac{m_{i+1}(\alpha^2, v_r)}{\binom{g_{i+1}(\alpha)}{2}} \overset{\text{(B9)}}{\approx} \frac{m_i(\alpha^2, v_r)}{\binom{g_i(\alpha)}{2}} \overset{\text{(D5)}}{\approx} \frac{m(\alpha^3)}{\binom{g(\alpha)}{3}}
$$

for each $v_r \in \Psi_i^{-1}[\alpha]$. If $g_{i+1}(\alpha) = 1$, by (B9) it follows that $m_{i+1}(\alpha^2, v_r) = 0$ for each $v_r \in \Psi_{i+1}^{-1}[\alpha]$. This proves (D5)$'$.

(D6)$'$ Suppose $g(\alpha) \geqslant 3$ and v_r, v_s are a pair of distinct vertices in $\Psi_i^{-1}[\alpha]$. From (B5) it follows that $m_{i+1}(\alpha, v_r, v_s)/g_{i+1}(\alpha) \approx m_i(\alpha, v_r, v_s)/g_i(\alpha)$. By (D6) of the induction hypothesis $m_i(\alpha, v_r, v_s)/g_i(\alpha) \approx m(\alpha^3)/\binom{g(\alpha)}{3}$. Therefore

$$
\frac{m_{i+1}(\alpha, v_r, v_s)}{g_{i+1}(\alpha)} \overset{\text{(B5)}}{\approx} \frac{m_i(\alpha, v_r, v_s)}{g_i(\alpha)} \overset{\text{(D6)}}{\approx} \frac{m(\alpha^3)}{\binom{g(\alpha)}{3}}.
$$

From (B8) it follows that

$$
\frac{m_{i+1}(\alpha, v_r, v_{i+1})}{g_{i+1}(\alpha)} \overset{\text{(B8)}}{\approx} \frac{2m_i(\alpha^2, v_r)}{g_i(\alpha)g_{i+1}(\alpha)} = \frac{m_i(\alpha^2, v_r)}{\binom{g_i(\alpha)}{2}}.
$$

By (D5) of the induction hypothesis $m_i(\alpha^2, v_r)/\binom{g_i(\alpha)}{2} \approx m(\alpha^3)/\binom{g(\alpha)}{3}$. Therefore

$$
\frac{m_{i+1}(\alpha, v_r, v_{i+1})}{g_{i+1}(\alpha)} \overset{\text{(B8)}}{\approx} \frac{m_i(\alpha^2, v_r)}{\binom{g_i(\alpha)}{2}} \overset{\text{(D5)}}{\approx} \frac{m(\alpha^3)}{\binom{g(\alpha)}{3}}.
$$

This proves (D6)′.

(D7)′ Suppose $g(\alpha) \geqslant 3$ and v_r, v_s, v_t are a triple of distinct vertices in $\Psi_i^{-1}[\alpha]$. Since in forming $\mathscr{F}_{i+1}$ no hyperedge is detached from v_r, v_s, v_t, we have $m_{i+1}(v_r, v_s, v_t) = m_i(v_r, v_s, v_t)$. But by (D7) of the induction hypothesis, $m_i(v_r, v_s, v_t) \approx m(\alpha^3)/\binom{g(\alpha)}{3}$. Therefore

$$m_{i+1}(v_r, v_s, v_t) = m_i(v_r, v_s, v_t) \overset{\text{(D7)}}{\approx} \frac{m(\alpha^3)}{\binom{g(\alpha)}{3}}.$$

By (B6) $m_{i+1}(v_r, v_s, v_{i+1}) \approx m_i(\alpha, v_r, v_s)/g_i(\alpha)$. By (D6) of the induction hypothesis $m_i(\alpha, v_r, v_s)/g_i(\alpha) \approx m(\alpha^3)/\binom{g(\alpha)}{3}$. Therefore

$$m_{i+1}(v_r, v_s, v_{i+1}) \overset{\text{(B6)}}{\approx} \frac{m_i(\alpha, v_r, v_s)}{g_i(\alpha)} \overset{\text{(D6)}}{\approx} \frac{m(\alpha^3)}{\binom{g(\alpha)}{3}}.$$

This proves (D7)′.

(D8)′ **Case 1:** If $g_{i+1}(x) = g_i(x) - 1$ (so $x = \alpha$), by (B9) $m_{i+1}(\alpha^2, y) \approx m_i(\alpha^2, y)(g_{i+1}(\alpha) - 1)/g_i(\alpha)$ which is 0 if $g_{i+1}(\alpha) = 1$. If $g_{i+1}(\alpha) \geqslant 2$, by (B9)

$$
\begin{aligned}
\frac{m_{i+1}(\alpha^2, y)}{\binom{g_{i+1}(\alpha)}{2} g_{i+1}(y)} \quad &\overset{\text{(B9)}}{\approx} \quad \frac{m_i(\alpha^2, y)(g_{i+1}(\alpha) - 1)}{g_i(\alpha)\binom{g_{i+1}(\alpha)}{2} g_{i+1}(y)} \\
&= \quad \frac{m_i(\alpha^2, y)(g_{i+1}(\alpha) - 1)}{g_i(\alpha)g_{i+1}(\alpha)(g_{i+1}(\alpha) - 1)g_i(y)/2} \\
&= \quad \frac{m_i(\alpha^2, y)}{\binom{g_i(\alpha)}{2} g_i(y)}.
\end{aligned}
$$

Since $g_i(\alpha) \geqslant 3 > 2$, by (D8) of the induction hypothesis $m_i(\alpha^2, y)/(\binom{g_i(\alpha)}{2} g_i(y)) \approx m(\alpha^2, y)/(\binom{g(\alpha)}{2} g(y))$. Therefore

$$\frac{m_{i+1}(\alpha^2, y)}{\binom{g_{i+1}(\alpha)}{2} g_{i+1}(y)} \overset{\text{(B9)}}{\approx} \frac{m_i(\alpha^2, y)}{\binom{g_i(\alpha)}{2} g_i(y)} \overset{\text{(D8)}}{\approx} \frac{m(\alpha^2, y)}{\binom{g(\alpha)}{2} g(y)}.$$

Case 2: If $g_{i+1}(y) = g_i(y) - 1$ (so $y = \alpha$), by (B3) $m_{i+1}(x^2, \alpha) \approx m_i(x^2, \alpha)g_{i+1}(\alpha)/g_i(\alpha)$ which is 0 by (D8) of the induction hypothesis, if $g_{i+1}(x) = g_i(x) = 1$. If $g_{i+1}(x) \geqslant 2$,

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by (B3) and (D8) of the induction hypothesis

$$\frac{m_{i+1}(x^2,\alpha)}{\binom{g_{i+1}(x)}{2}g_{i+1}(\alpha)} \overset{\text{(B3)}}{\approx} \frac{m_i(x^2,\alpha)}{\binom{g_{i+1}(x)}{2}g_i(\alpha)} = \frac{m_i(x^2,\alpha)}{\binom{g_i(x)}{2}g_i(\alpha)} \overset{\text{(D8)}}{\approx} \frac{m(x^2,\alpha)}{\binom{g(x)}{2}g(\alpha)}.$$

This proves (D8)′.

(D9)′ Suppose $v_t \in \Psi_i^{-1}[y]$. There are two cases:

Case 1: If $g_{i+1}(x) = g_i(x) - 1$ (so $x = \alpha$), by (B9) $m_{i+1}(\alpha^2, v_t) \approx m_i(\alpha^2, v_t)(g_{i+1}(\alpha) - 1)/g_i(\alpha)$ which is 0 if $g_{i+1}(\alpha) = 1$. If $g_{i+1}(\alpha) \geqslant 2$, by (B9)

$$\begin{aligned}
\frac{m_{i+1}(\alpha^2, v_t)}{\binom{g_{i+1}(\alpha)}{2}} &\overset{\text{(B9)}}{\approx} \frac{m_i(\alpha^2, v_t)(g_{i+1}(\alpha) - 1)}{g_i(\alpha)\binom{g_{i+1}(\alpha)}{2}} \\
&= \frac{m_i(\alpha^2, v_t)(g_{i+1}(\alpha) - 1)}{g_i(\alpha)g_{i+1}(\alpha)(g_{i+1}(\alpha) - 1)/2} \\
&= \frac{m_i(\alpha^2, v_t)}{\binom{g_i(\alpha)}{2}}.
\end{aligned}$$

Since $g_i(\alpha) \geqslant 3 > 2$, by (D9) of the induction hypothesis we have $m_i(\alpha^2, v_t)/\binom{g_i(\alpha)}{2} \approx m(\alpha^2, y)/(\binom{g(\alpha)}{2}g(y))$. Therefore

$$\frac{m_{i+1}(\alpha^2, v_t)}{\binom{g_{i+1}(\alpha)}{2}} \overset{\text{(B9)}}{\approx} \frac{m_i(\alpha^2, v_t)}{\binom{g_i(\alpha)}{2}} \overset{\text{(D9)}}{\approx} \frac{m(\alpha^2, y)}{\binom{g(\alpha)}{2}g(y)}.$$

Case 2: If $g_{i+1}(y) = g_i(y) - 1$ (so $y = \alpha$), since in forming $\mathscr{F}_{i+1}$ no hyperedge is detached from v_t and x, we have $m_{i+1}(x^2, v_t) = m_i(x^2, v_t)$ which is 0 by (D9) of the induction hypothesis, if $g_{i+1}(x) = g_i(x) = 1$. If $g_{i+1}(x) \geqslant 2$, by (D9) of the induction hypothesis

$$\frac{m_{i+1}(x^2, v_t)}{\binom{g_{i+1}(x)}{2}} = \frac{m_i(x^2, v_t)}{\binom{g_i(x)}{2}} \overset{\text{(D9)}}{\approx} \frac{m(x^2, \alpha)}{\binom{g(x)}{2}g(\alpha)}.$$

By (B4), $m_{i+1}(v_{i+1}, x^2) \approx m_i(\alpha, x^2)/g_i(\alpha)$ which is 0 by (D8) of the induction hypothesis, if $g_{i+1}(x) = g_i(x) = 1$. If $g_{i+1}(x) \geqslant 2$, by (B4) and (D8) of the induction

hypothesis

$$\frac{m_{i+1}(x^2, v_{i+1})}{\binom{g_{i+1}(x)}{2}} \overset{(B4)}{\approx} \frac{m_i(x^2, \alpha)}{\binom{g_{i+1}(x)}{2}g_i(\alpha)} = \frac{m_i(x^2, \alpha)}{\binom{g_i(x)}{2}g_i(\alpha)} \overset{(D8)}{\approx} \frac{m_i(x^2, \alpha)}{\binom{g(x)}{2}g(\alpha)}.$$

This proves (D9)′.

(D10)′ Suppose $v_r \in \Psi_i^{-1}[x]$. There are two cases:

Case 1: If $g_{i+1}(x) = g_i(x)-1$ (so $x = \alpha$), by (B5) $m_{i+1}(\alpha, v_r, y)/g_{i+1}(\alpha) \approx m_i(\alpha, v_r, y)/g_i(\alpha)$.

Therefore by (D10) of the induction hypothesis

$$\begin{aligned}
\frac{m_{i+1}(\alpha, v_r, y)}{g_{i+1}(\alpha)g_{i+1}(y)} &\overset{(B5)}{\approx} \frac{m_i(\alpha, v_r, y)}{g_i(\alpha)g_{i+1}(y)} \\
&= \frac{m_i(\alpha, v_r, y)}{g_i(\alpha)g_i(y)} \overset{(D10)}{\approx} \frac{m(\alpha^2, y)}{\binom{g(\alpha)}{2}g(y)}.
\end{aligned}$$

By (B8) $m_{i+1}(\alpha, v_{i+1}, y) \approx 2m_i(\alpha^2, y)/g_i(\alpha)$. Therefore since $g_i(\alpha) \geqslant 2$, by (D8) of the induction hypothesis

$$\begin{aligned}
\frac{m_{i+1}(\alpha, v_{i+1}, y)}{g_{i+1}(\alpha)g_{i+1}(y)} &\overset{(B8)}{\approx} \frac{2m_i(\alpha^2, y)}{g_i(\alpha)g_{i+1}(\alpha)g_{i+1}(y)} \\
&= \frac{m_i(\alpha^2, y)}{\binom{g_i(\alpha)}{2}g_i(y)} \overset{(D8)}{\approx} \frac{m(\alpha^2, y)}{\binom{g(\alpha)}{2}g(y)}.
\end{aligned}$$

Case 2: If $g_{i+1}(y) = g_i(y) - 1$ (so $y = \alpha$), by (B5) we have $m_{i+1}(x, v_r, \alpha)/g_{i+1}(\alpha) \approx m_i(x, v_r, \alpha)/g_i(\alpha)$. Therefore by (D10) of the induction hypothesis

$$\begin{aligned}
\frac{m_{i+1}(x, v_r, \alpha)}{g_{i+1}(x)g_{i+1}(\alpha)} &\overset{(B5)}{\approx} \frac{m_i(x, v_r, \alpha)}{g_{i+1}(x)g_i(\alpha)} \\
&= \frac{m_i(x, v_r, \alpha)}{g_i(x)g_i(\alpha)} \overset{(D10)}{\approx} \frac{m(x^2, \alpha)}{\binom{g(x)}{2}g(\alpha)}.
\end{aligned}$$

This proves (D10)′.

(D11)′ Suppose $v_r \in \Psi_i^{-1}[x], v_t \in \Psi_i^{-1}[y]$. There are two cases:

Case 1: If $g_{i+1}(x) = g_i(x) - 1$ (so $x = \alpha$), by (B5) and (D11) of the induction

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hypothesis

$$\frac{m_{i+1}(\alpha, v_r, v_t)}{g_{i+1}(\alpha)} \overset{\text{(B5)}}{\approx} \frac{m_i(\alpha, v_r, v_t)}{g_i(\alpha)} \overset{\text{(D11)}}{\approx} \frac{m(\alpha^2, y)}{\binom{g(\alpha)}{2} g(y)}.$$

By (B8) $m_{i+1}(\alpha, v_{i+1}, v_t) \approx 2m_i(\alpha^2, v_t)/g_i(\alpha)$. Therefore by (D10) of the induction hypothesis

$$\begin{aligned}
\frac{m_{i+1}(\alpha, v_{i+1}, v_t)}{g_{i+1}(\alpha)} \quad &\overset{\text{(B8)}}{\approx} \quad \frac{2m_i(\alpha^2, v_t)}{g_i(\alpha) g_{i+1}(\alpha)} \\
&= \quad \frac{m_i(\alpha, v_r, y)}{\binom{g_i(\alpha)}{2}} \overset{\text{(D10)}}{\approx} \frac{m(\alpha^2, y)}{\binom{g(\alpha)}{2} g(y)}.
\end{aligned}$$

Case 2: If $g_{i+1}(y) = g_i(y) - 1$ (so $y = \alpha$), since in forming $\mathscr{F}_{i+1}$ no hyperedge is detached from x, v_r and v_t, we have $m_{i+1}(x, v_r, v_t) = m_i(x, v_r, v_t)$. Therefore by (D11) of the induction hypothesis

$$\frac{m_{i+1}(x, v_r, v_t)}{g_{i+1}(x)} = \frac{m_i(x, v_r, v_t)}{g_{i+1}(x)} = \frac{m_i(x, v_r, v_t)}{g_i(x)} \overset{\text{(D11)}}{\approx} \frac{m(x^2, \alpha)}{\binom{g(x)}{2} g(\alpha)}.$$

By (B6) $m_{i+1}(v_{i+1}, x, v_r) \approx m_i(\alpha, x, v_r)/g_i(\alpha)$. Therefore by (D10) of the induction hypothesis

$$\frac{m_{i+1}(x, v_r, v_{i+1})}{g_{i+1}(x)} \overset{\text{(B6)}}{\approx} \frac{m_i(x, v_r, \alpha)}{g_{i+1}(x) g_i(\alpha)} = \frac{m_i(x, v_r, \alpha)}{g_i(x) g_i(\alpha)} \overset{\text{(D10)}}{\approx} \frac{m(x^2, \alpha)}{\binom{g(x)}{2} g(\alpha)}.$$

This proves (D11)$'$.

(D12)$'$ Suppose $v_r, v_s \in \Psi_i^{-1}[x]$. There are two cases:

Case 1: If $g_{i+1}(x) = g_i(x) - 1$ (so $x = \alpha$), since in forming $\mathscr{F}_{i+1}$ no hyperedge is detached from v_r, v_s and y, we have $m_{i+1}(v_r, v_s, y) = m_i(v_r, v_s, y)$. Therefore by (D12) of the induction hypothesis

$$\frac{m_{i+1}(v_r, v_s, y)}{g_{i+1}(y)} = \frac{m_i(v_r, v_s, y)}{g_{i+1}(y)} = \frac{m_i(v_r, v_s, y)}{g_i(y)} \overset{\text{(D12)}}{\approx} \frac{m(\alpha^2, y)}{\binom{g(\alpha)}{2} g(y)}.$$

By (B6) $m_{i+1}(v_{i+1}, v_r, y) \approx m_i(\alpha, v_r, y)/g_i(\alpha)$. Therefore by (D10) of the induction hypothesis

$$\frac{m_{i+1}(v_{i+1}, v_r, y)}{g_{i+1}(y)} \overset{\text{(B6)}}{\approx} \frac{m_i(\alpha, v_r, y)}{g_i(\alpha)g_{i+1}(y)} = \frac{m_i(\alpha, v_r, y)}{g_i(\alpha)g_i(y)} \overset{\text{(D10)}}{\approx} \frac{m(\alpha^2, y)}{\binom{g(\alpha)}{2}g(y)}.$$

Case 2: If $g_{i+1}(y) = g_i(y)-1$ (so $y = \alpha$), by (B5) and (D12) of the induction hypothesis

$$\frac{m_{i+1}(v_r, v_s, \alpha)}{g_{i+1}(\alpha)} \overset{\text{(B5)}}{\approx} \frac{m_i(v_r, v_s, \alpha)}{g_i(\alpha)} \overset{\text{(D12)}}{\approx} \frac{m(x^2, \alpha)}{\binom{g(x)}{2}g(\alpha)}.$$

This proves (D12)$'$.

(D13)$'$ Suppose $v_r, v_s \in \Psi_i^{-1}[x], v_t \in \Psi_i^{-1}[y]$. Since in forming $\mathscr{F}_{i+1}$ no hyperedge is detached from v_r, v_s and v_t, we have $m_{i+1}(v_r, v_s, v_t) = m_i(v_r, v_s, v_t)$. Therefore by (D13) of the induction hypothesis

$$m_{i+1}(v_r, v_s, v_t) \overset{\text{(D13)}}{\approx} \frac{m(x^2, y)}{\binom{g(x)}{2}g(y)}.$$

If $g_{i+1}(x) = g_i(x) - 1$ (so $x = \alpha$), by (B6) and (D11) of the induction hypothesis

$$m_{i+1}(v_r, v_{i+1}, v_t) \overset{\text{(B6)}}{\approx} \frac{m_i(\alpha, v_r, v_t)}{g_i(\alpha)} \overset{\text{(D11)}}{\approx} \frac{m(\alpha^2, y)}{\binom{g(\alpha)}{2}g(y)}.$$

If $g_{i+1}(y) = g_i(y) - 1$ (so $y = \alpha$), by (B6) and (D12) of the induction hypothesis

$$m_{i+1}(v_r, v_s, v_{i+1}) \overset{\text{(B6)}}{\approx} \frac{m_i(\alpha, v_r, v_s)}{g_i(y)} \overset{\text{(D12)}}{\approx} \frac{m(x^2, \alpha)}{\binom{g(x)}{2}g(\alpha)}.$$

This proves (D13)$'$.

(D14)′ If $g_{i+1}(x) = g_i(x) - 1$ (so $x = \alpha$) , by (B5) $m_{i+1}(\alpha, y, z)/g_{i+1}(\alpha) \approx m_i(\alpha, y, z)/g_i(\alpha)$.

Therefore by (D14) of the induction hypothesis

$$
\frac{m_{i+1}(\alpha, y, z)}{g_{i+1}(\alpha)g_{i+1}(y)g_{i+1}(z)} \overset{\text{(B5)}}{\approx} \frac{m_i(\alpha, y, z)}{g_i(\alpha)g_{i+1}(y)g_{i+1}(z)}
$$
$$
= \frac{m_i(\alpha, y, z)}{g_i(\alpha)g_i(y)g_i(z)} \overset{\text{(D14)}}{\approx} \frac{m(\alpha, y, z)}{g(\alpha)g(y)g(z)}.
$$

There are two other cases ($g_{i+1}(y) = g_i(y) - 1$ and $g_{i+1}(z) = g_i(z) - 1$) for which the proof is similar. This proves (D14)′.

(D15)′ Suppose $v_t \in \Psi_i^{-1}[z]$. There are three cases:

Case 1: If $g_{i+1}(x) = g_i(x) - 1$ (so $x = \alpha$) , by (B5) $m_{i+1}(\alpha, y, v_t)/g_{i+1}(\alpha) \approx m_i(\alpha, y, v_t)/g_i(\alpha)$.
Therefore by (D15) of the induction hypothesis

$$
\frac{m_{i+1}(\alpha, y, v_t)}{g_{i+1}(\alpha)g_{i+1}(y)} \overset{\text{(B5)}}{\approx} \frac{m_i(\alpha, y, v_t)}{g_i(\alpha)g_{i+1}(y)}
$$
$$
= \frac{m_i(\alpha, y, v_t)}{g_i(\alpha)g_i(y)} \overset{\text{(D15)}}{\approx} \frac{m(\alpha, y, z)}{g(\alpha)g(y)g(z)}.
$$

Case 2: If $g_{i+1}(y) = g_i(y) - 1$ (so $y = \alpha$), the proof is similar to that of case 1.

Case 3: If $g_{i+1}(z) = g_i(z) - 1$ (so $z = \alpha$), since in forming $\mathscr{F}_{i+1}$ no hyperedge is detached from x, y and v_t, we have $m_{i+1}(x, y, v_t) \approx m_i(x, y, v_t)$. Therefore by (D15) of the induction hypothesis

$$
\frac{m_{i+1}(x, y, v_t)}{g_{i+1}(x)g_{i+1}(y)} = \frac{m_i(x, y, v_t)}{g_i(x)g_i(y)} \overset{\text{(D15)}}{\approx} \frac{m(x, y, \alpha)}{g(x)g(y)g(\alpha)}.
$$

By (B6) $m_{i+1}(x, y, v_{i+1}) \approx m_i(x, y, \alpha)/g_i(\alpha)$. Therefore by (D14) of the induction hypothesis

$$
\frac{m_{i+1}(x, y, v_{i+1})}{g_{i+1}(x)g_{i+1}(y)} \overset{\text{(B6)}}{\approx} \frac{m_i(x, y, \alpha)}{g_{i+1}(x)g_{i+1}(y)g_i(\alpha)}
$$
$$
= \frac{m_i(x, y, \alpha)}{g_i(x)g_i(y)g_i(\alpha)} \overset{\text{(D14)}}{\approx} \frac{m(x, y, \alpha)}{g(x)g(y)g(\alpha)}.
$$

This proves (D15)$'$.

(D16)$'$ Suppose $v_s \in \Psi_i^{-1}[y], v_t \in \Psi_i^{-1}[z]$. There are three cases:

Case 1: If $g_{i+1}(x) = g_i(x) - 1$ (so $x = \alpha$) , by (B5) and (D16) of the induction hypothesis

$$\frac{m_{i+1}(\alpha, v_s, v_t)}{g_{i+1}(\alpha)} \overset{\text{(B5)}}{\approx} \frac{m_i(\alpha, v_s, v_t)}{g_i(\alpha)} \overset{\text{(D16)}}{\approx} \frac{m(\alpha, y, z)}{g(\alpha)g(y)g(z)}.$$

Case 2: If $g_{i+1}(y) = g_i(y) - 1$ (so $y = \alpha$), since in forming $\mathscr{F}_{i+1}$ no hyperedge is detached from x, v_s and v_t, we have $m_{i+1}(x, v_s, v_t) = m_i(x, v_s, v_t)$. Therefore by (D16) of the induction hypothesis

$$\frac{m_{i+1}(x, v_s, v_t)}{g_{i+1}(x)} = \frac{m_i(x, v_s, v_t)}{g_i(x)} \overset{\text{(D16)}}{\approx} \frac{m(x, \alpha, z)}{g(x)g(\alpha)g(z)}.$$

By (B6) $m_{i+1}(x, v_{i+1}, v_t) \approx m_i(x, \alpha, v_t)/g_i(\alpha)$. Therefore by (D15) of the induction hypothesis

$$\frac{m_{i+1}(x, v_{i+1}, v_t)}{g_{i+1}(x)} \overset{\text{(B6)}}{\approx} \frac{m_i(x, \alpha, v_t)}{g_{i+1}(x)g_i(\alpha)}$$
$$= \frac{m_i(x, \alpha, v_t)}{g_i(x)g_i(\alpha)} \overset{\text{(D15)}}{\approx} \frac{m(x, \alpha, z)}{g(x)g(\alpha)g(z)}.$$

Case 3: If $g_{i+1}(z) = g_i(z) - 1$ (so $z = \alpha$), the proof is similar to that of case 2. This proves (D16)$'$.

(D17)$'$ Suppose $v_r \in \Psi_i^{-1}[x], v_s \in \Psi_i^{-1}[y], v_t \in \Psi_i^{-1}[z]$. Since in forming $\mathscr{F}_{i+1}$ no hyperedge is detached from v_r, v_s and v_t, we have $m_{i+1}(v_r, v_s, v_t) = m_i(v_r, v_s, v_t)$. Therefore by (D17) of the induction hypothesis

$$m_{i+1}(v_r, v_s, v_t) \overset{\text{(D17)}}{\approx} \frac{m(x, y, z)}{g(x)g(y)g(z)}.$$

If $g_{i+1}(x) = g_i(x) - 1$ (so $x = \alpha$) , by (B6) and (D16) of the induction hypothesis

$$m_{i+1}(v_{i+1}, v_s, v_t) \overset{\text{(B6)}}{\approx} \frac{m_i(\alpha, v_s, v_t)}{g_i(\alpha)} \overset{\text{(D16)}}{\approx} \frac{m(\alpha, y, z)}{g(\alpha)g(y)g(z)}.$$

There are two other cases $(g_{i+1}(y) = g_i(y) - 1$ and $g_{i+1}(z) = g_i(z) - 1)$ for which the proof is similar. This proves (D17)$'$.

$\square$

A similar statement can be proved for every color class simply by restricting each relation above to a color class $j \in \{1, \ldots, k\}$.

5.5.4 Relations Between $\mathscr{G} = \mathscr{F}_n$ and $\mathscr{F}$

Recall that $\mathscr{G} = \mathscr{F}_n, \Psi = \Psi_n$ and $g_n(x) = 1$ for each $x \in V$. We claim that $\mathscr{G}$ satisfies all conditions stated in Theorem 5.3.

Obviously $\mathscr{G}$ is a g-detachment of $\mathscr{F}$. Let x, y, z be an arbitrary triple of distinct vertices of V, and let $j \in \{1, \ldots, k\}$. Now in (D1)–(D17) we let $i = n$. From (D3) and (D4) it is immediate that $\mathscr{G}$ is loopless. From (D5), (D8) and (D9) it follows that $\mathscr{G}$ has no hyperedge of size 2. Thus $\mathscr{G}$ is a 3-uniform hypergraph.

From (D1) it follows that $d_{\mathscr{F}_n}(x)/g_n(x) \approx d(x)/g(x)$, so $d_{\mathscr{G}}(x) \approx d(x)/g(x)$. From (D2), $d_{\mathscr{F}_n}(v_r) \approx d(x)/g(x)$ for each $v_r \in \Psi_n^{-1}[x]$, so $d_{\mathscr{G}}(v_r) \approx d(x)/g(x)$ for each $v_r \in \Psi^{-1}[x]$. Therefore $\mathscr{G}$ satisfies (A1).

A similar argument shows that (A2) follows from the colored version of (D1) and (D2), (A3) follows from (D6), (D7), and (D10)–(D17), and (A4) follows from the colored version of (D6), (D7), and (D10)–(D17). This completes the proof of Theorem 5.3.

5.6 Algorithmic Aspects

To construct an r-factorization for λK_n^3, we start with an amalgamation of λK_n^3 in which all hyperedges are loops. We color the hyperedges among $k := \lambda \binom{n-1}{2}/r$ color classes

as evenly as possible, and apply Theorem 5.3. In Theorem 5.3, we detach vertices in $n-1$ steps. At each step, to decide how to share edges (and hinges) among the new vertices, we define two sets $\mathscr{A}$ and $\mathscr{B}$ whose sizes are no more than $1+k+\binom{n}{3}$ and $(k+1)\binom{n}{2}$, respectively, and use Nash-Willimas lemma. Nash-Williams lemma builds a graph of size $O(n^3)$ (or more precisely of size $|\mathscr{A}|+|\mathscr{B}|$) and finds a set Z with a polynomial time algorithm. The set Z tells us exactly how to share edges (and hinges) among the new vertices. Therefore, our construction is polynomial in $\binom{n}{3}$, the output size for the problem.

Chapter 6

Mathematicians Collaboration Problem

6.1   Introduction

In a mathematics workshop with mn mathematicians in n different areas, each area consisting of m mathematicians, we want to create a collaboration network. For this purpose, we would like to schedule daily meetings between groups of size three, so that (i) two persons of the same area meet one person of another area, (ii) each person has exactly r meetings each day, and (iii) every two persons of the same area have exactly λ meetings with each person of another area by the end of the workshop. We show that this can be done if: $3 \mid rm$, $2 \mid rnm$ and $r \mid 3\lambda(n-1)\binom{m}{2}$.

Let $\mathscr{K}^3_{n \times m}$ denote a hypergraph with vertex partition $\{V_i : 1 \leqslant i \leqslant n\}$, so that $V_i = \{x_{ij} : 1 \leqslant j \leqslant m\}$ for $1 \leqslant i \leqslant n$, and with edge set $E = \{\{x_{ij}, x_{ij'}, x_{kl}\} : 1 \leqslant j < j' \leqslant m, 1 \leqslant i, k \leqslant n, i \neq k, 1 \leqslant l \leqslant m\}$. One may notice that finding an r-factorization for $\lambda\mathscr{K}^3_{n \times m}$ is equivalent to scheduling the meetings between mathematicians with the above restrictions.

In this chapter we use hypergraph amalgamation to solve our Mathematicians Collaboration Problem.

Example 6.1. Let $\mathscr{F} = (V, E, H, \psi, \phi)$, with $V = \{v_i : 1 \leqslant i \leqslant 6\}, E = \{e_1, e_2, e_3\}, H = \{h_i : 1 \leqslant i \leqslant 9\}$, such that $\psi(h_i) = v_i$ for $1 \leqslant i \leqslant 6$, $\psi(h_7) = v_1, \psi(h_8) = v_3, \psi(h_9) = v_5$ and $\phi(h_5) = \phi(h_6) = \phi(h_7) = e_1, \phi(h_1) = \phi(h_2) = \phi(h_8) = e_2, \phi(h_3) = \phi(h_4) = \phi(h_9) = e_3$. Let $\Psi : V \to \{w_1, w_2, w_3\}$ be the function with $\Psi(v_1) = \Psi(v_6) = w_1$, $\Psi(v_2) = \Psi(v_3) = w_2$, $\Psi(v_4) = \Psi(v_5) = w_3$. The hypergraph $\mathscr{G}$ in Figure 6.1 is the Ψ-amalgamation of $\mathscr{F}$.

In the remainder of this chapter, we assume that $n \geqslant 3$, $m \geqslant 2$, and all hypergraphs are either 3-uniform or are amalgamations of 3-uniform hypergraphs. Notice that for every
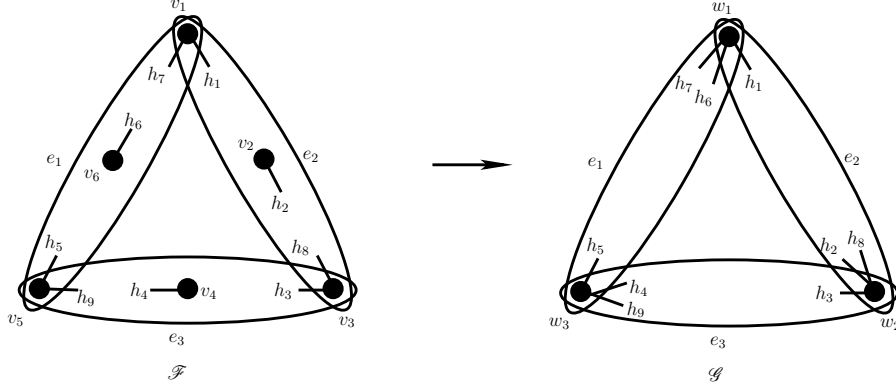
Figure 6.1: A visual representation of a hypergraph $\mathscr{F}$ with an amalgamation $\mathscr{G}$

hypergraph $\mathscr{G}$ we have

$$1 \leqslant |e| \leqslant |\phi^{-1}(e)| = 3 \text{ for every } e \text{ in } \mathscr{G}. \tag{6.1}$$

If u, v, w are three (not necessarily distinct) vertices of $\mathscr{G}$, then $\mathscr{E}(u, v, w)$ denotes the set of hyperedges that join u, v, w. For a graph G, we denote the set of edges joining a pair of vertices u, v by $E(u, v)$.

In [6], the author proved a general detachment theorem for hypergraphs. For the purpose of this chapter we use a very special case of this theorem as follows:

Theorem 6.2. *Let $\mathscr{F}$ be a k-hyperedge-colored hypergraph and let g be a function from $V(\mathscr{F})$ into $\mathbb{N}$ such that for $x, y, z \in V(\mathscr{F})$: (i) $g(x) \leqslant 2$ implies $\mathscr{E}(x, x, x) = \varnothing$, (ii) $g(x) = 1$ implies $\mathscr{E}(x, x, y) = \varnothing$, and (iii) $g(x)$ divides $d_{\mathscr{F}(j)}(x)$, $\binom{g(x)}{3}$ divides $|\mathscr{E}(x, x, x)|$, $\binom{g(x)}{2}g(y)$ divides $|\mathscr{E}(x, x, y)|$, and $g(x)g(y)g(z)$ divides $|\mathscr{E}(x, y, z)|$. Then there exists a 3-uniform g-detachment $\mathscr{G}$ of $\mathscr{F}$ in which each $v \in V(\mathscr{F})$ is detached into $v_1, \ldots, v_{g(v)}$ such that $\mathscr{G}$ satisfies the following conditions for distinct $x, y, z \in V(\mathscr{F})$:*

(A1) $d_{\mathscr{G}(j)}(x_i) = d_{\mathscr{F}(j)}(x)/g(x)$ *for* $1 \leqslant i \leqslant g(x)$ *and* $1 \leqslant j \leqslant k$;

(A2) $|\mathscr{E}_{\mathscr{G}}(x_i, x_{i'}, x_{i''})| = |\mathscr{E}_{\mathscr{F}}(x, x, x)|/\binom{g(x)}{3}$ *for* $1 \leqslant i < i' < i'' \leqslant g(x)$*, if* $g(x) \geqslant 3$;

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(A3) $|\mathscr{E}_{\mathscr{G}}(x_i, x_{i'}, y_{i''})| = |\mathscr{E}_{\mathscr{F}}(x, x, y)|/(\binom{g(x)}{2}g(y))$ *for* $1 \leqslant i < i' \leqslant g(x)$ *if* $g(x) \geqslant 2$, *and*

$\quad 1 \leqslant i'' \leqslant g(y)$;

(A4) $|\mathscr{E}_{\mathscr{G}}(x_i, y_{i'}, z_{i''})| = |\mathscr{E}_{\mathscr{F}}(x, y, z)|/(g(x)g(y)g(z))$ *for* $1 \leqslant i \leqslant g(x)$, $1 \leqslant i' \leqslant g(y)$ *and*

$\quad 1 \leqslant i'' \leqslant g(z)$.

6.2 Proof of the Main Theorem

Let K_n^* denote the hypergraph with n vertices in which $|\mathscr{E}(u, u, v)| = 1$, and $\mathscr{E}(u, u, u) = \mathscr{E}(u, v, w) = \varnothing$ for distinct vertices u, v, w. First we need the following simple lemma:

Lemma 6.3. *If* $2 \mid r_i n$ *for* $1 \leqslant i \leqslant k$, *and* $\sum_{i=1}^{k} r_i = \lambda(n-1)$, *then* λK_n^* *is* $(3r_1, \ldots, 3r_k)$-*factorizable.*

Proof. Let $G = \lambda K_n$ with vertex set V. Since $2 \mid r_i n$ for $1 \leqslant i \leqslant k$, and $\sum_{i=1}^{k} r_i = \lambda(n-1)$, G is $(r_1, \ldots, r_k)$-factorizable (see [52], or [51]). So we can find a k-edge-coloring for G such that $d_{G(i)}(v) = r_i$ for every $v \in V$ and every color $1 \leqslant i \leqslant k$. Now we form a hypergraph $\mathscr{H}$ with vertex set V, such that $|\mathscr{E}_{\mathscr{H}(i)}(u, u, v)| = |E_{G(i)}(u, v)|$ for every pair of distinct vertices $u, v \in V$. It is easy to see that $\mathscr{H} = \lambda K_n^*$ and $d_{\mathscr{H}(i)}(v) = 3r_i$ for every $v \in V$ and every color $1 \leqslant i \leqslant k$. Thus we obtain a $(3r_1, \ldots, 3r_k)$-factorization for λK_n^*. $\qquad \square$

Notice that $\lambda \mathscr{K}_{m \times n}^3$ is a $3\lambda(n-1)\binom{m}{2}$-regular hypergraph with nm vertices and $2\lambda m \binom{n}{2}\binom{m}{2}$ edges.

Theorem 6.4. $\lambda \mathscr{K}_{m \times n}^3$ *is* $(r_1, \ldots, r_k)$-*factorizable if*

(i) $3 \mid r_i m$ *for* $1 \leqslant i \leqslant k$,

(ii) $2 \mid r_i mn$ *for* $1 \leqslant i \leqslant k$, *and*

(iii) $\sum_{i=1}^{k} r_i = 3\lambda(n-1)\binom{m}{2}$.

Proof. Let $\mathscr{F} = \lambda m \binom{m}{2} K_n^*$. Note that $\mathscr{F}$ is an amalgamation of $\lambda \mathscr{K}_{m\times n}^3$. Since for $1 \leqslant i \leqslant k$,

$2 \mid \frac{r_i mn}{3}$ and $\sum_{i=1}^k \frac{r_i m}{3} = \lambda m(n-1)\binom{m}{2}$, by Lemma 6.3, $\mathscr{F}$ is $(mr_1, \ldots, mr_k)$-factorizable.

Thus, we can find a k-hyperedge-coloring for $\mathscr{F}$ such that

$$d_{\mathscr{F}(j)}(x) = mr_i \quad 1 \leqslant i \leqslant k.$$

Let $g : V(\mathscr{F}) \to \mathbb{N}$ be a function with $g(x_i) = m$ for $i = 1, \ldots, n$. Note that $\mathscr{K}_{m\times n}^3$ is a g-detachment of $\mathscr{F}$. Now by Theorem 6.2, there exists a 3-uniform g-detachment $\mathscr{G}$ of $\mathscr{F}$ with mn vertices, say x_{ij}, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$ ($x_{i1}, \ldots, x_{im}$ are obtained by splitting x_i into m vertices for $i = 1, \ldots, n$) such that

- $d_{\mathscr{G}(t)}(x_{ij}) = r_t$ for $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$, and $1 \leqslant t \leqslant k$;

- $\mathscr{E}_{\mathscr{G}}(x_{ij}, x_{ij'}, x_{ij''}) = \varnothing$ for $1 \leqslant i \leqslant n$ and $1 \leqslant j < j' < j'' \leqslant m$, if $m \geqslant 3$;

- $|\mathscr{E}_{\mathscr{G}}(x_{ij}, x_{ij'}, x_{i'j''})| = \lambda$ for $1 \leqslant i < i' \leqslant n$, $1 \leqslant j < j' \leqslant m$, and $1 \leqslant j'' \leqslant m$; and

- $\mathscr{E}_{\mathscr{G}}(x_{ij}, x_{i'j'}, x_{i''j''}) = \varnothing$ for $1 \leqslant i < i' < i'' \leqslant n$ and $1 \leqslant j, j', j'' \leqslant m$.

Therefore $\mathscr{G} \cong \lambda \mathscr{K}_{m\times n}^3$ and the ith color class is an r_i-factor for $1 \leqslant i \leqslant k$. $\qquad \square$

In particular we solve the Mathematicians Collaboration Problem in the following case.

Corollary 6.5. $\lambda \mathscr{K}_{m\times n}^3$ *is r-factorizable if*

(i) $3 \mid rm$,

(ii) $2 \mid rnm$, *and*

(iii) $r \mid 3\lambda(n-1)\binom{m}{2}$.

We define $\mathscr{K}_{m_1,\ldots,m_n}^3$ similar to $\mathscr{K}_{m\times n}^3$ with the difference that in $\mathscr{K}_{m_1,\ldots,m_n}^3$ we allow different parts to have different sizes.

Conjecture 6.6. $\lambda \mathscr{K}_{m_1,\ldots,m_n}^3$ *is $(r_1, \ldots, r_k)$-factorizable if and only if*

(i) $m_i = m_j := m$ for $1 \leqslant i < j \leqslant n$,

(ii) $3 \mid r_i mn$ for each i, $1 \leqslant i \leqslant k$, and

(iii) $\sum_{i=1}^{k} r_i = 3\lambda(n-1)\binom{m}{2}$.

We prove the necessity as follows. Since $\lambda\mathscr{K}_{m\times n}^3$ is factorizable, it must be regular. Let u and v be two vertices from two different parts, say p^{th} and q^{th} parts respectively. Then we have the following sequence of equivalences:

$$d(u) = d(v) \qquad \Longleftrightarrow$$

$$\sum_{\substack{1\leqslant i\leqslant n\\ i\neq p}} \binom{m_i}{2} + (m_p - 1)\sum_{\substack{1\leqslant i\leqslant n\\ i\neq p}} m_i =$$

$$\sum_{\substack{1\leqslant i\leqslant n\\ i\neq q}} \binom{m_i}{2} + (m_q - 1)\sum_{\substack{1\leqslant i\leqslant n\\ i\neq q}} m_i \qquad \Longleftrightarrow$$

$$\binom{m_q}{2} + \sum_{\substack{1\leqslant i\leqslant n\\ i\neq p,q}} \binom{m_i}{2} + (m_p - 1)(m_q + \sum_{\substack{1\leqslant i\leqslant n\\ i\neq p,q}} m_i) =$$

$$\binom{m_p}{2} + \sum_{\substack{1\leqslant i\leqslant n\\ i\neq p,q}} \binom{m_i}{2} + (m_q - 1)(m_p + \sum_{\substack{1\leqslant i\leqslant n\\ i\neq p,q}} m_i) \qquad \Longleftrightarrow$$

$$\binom{m_p}{2} - \binom{m_q}{2} + m_p m_q - m_p - m_p m_q + m_q + (m_p - m_q)\sum_{\substack{1\leqslant i\leqslant n\\ i\neq p,q}} m_i) = 0 \qquad \Longleftrightarrow$$

$$m_p^2 - m_q^2 - 3m_p + 3m_q + 2(m_p - m_q)\sum_{\substack{1\leqslant i\leqslant n\\ i\neq p,q}} m_i) = 0 \qquad \Longleftrightarrow$$

$$(m_p - m_q)(m_p + m_q - 3 + 2\sum_{\substack{1\leqslant i\leqslant n\\ i\neq p,q}} m_i) = 0 \qquad \Longleftrightarrow$$

$$m_p = m_q := m.$$

This proves (i). The existence of an r_i-factor implies that $3 \mid r_i mn$ for $1 \leqslant i \leqslant k$. Since each r_i-factor is an r_i-regular spanning sub-hypergraph and $\lambda\mathscr{K}_{m\times n}^3$ is $3\lambda(n-1)\binom{m}{2}$-regular, we must have $\sum_{i=1}^{k} r_i = 3\lambda(n-1)\binom{m}{2}$.

It is not difficult to show that $\mathscr{K}_{3,3,3}^3$ has a unique 1-factorization, but it does not satisfy condition (ii) of Theorem 6.4. There are many other examples of this kind, but none of them gives us a general construction.

Chapter 7

Embedding factorizations for 3-uniform hypergraphs

7.1 Introduction

In this chapter, two results are obtained on a hypergraph embedding problem. The proof technique is itself of interest, being the first time amalgamations have been used to address the embedding of hypergraphs.

The first result finds necessary and sufficient conditions for the embedding a hyperedge-colored copy of the complete 3-uniform hypergraph of order m, K_m^3, into an r-factorization of K_n^3, providing that $n > 2m + (-1 + \sqrt{8m^2 - 16m - 7})/2$.

The second result finds necessary and sufficient conditions for an embedding when not only are the colors of the hyperedges of K_m^3 given, but also the colors of all the "pieces" of hyperedges on these m vertices are prescribed (the "pieces" of hyperedges are eventually extended to hyperedges of size 3 in K_n^3 by adding new vertices to the hyperedges of size 1 and 2 during the embedding process).

Both these results make progress towards settling an old question of Cameron on completing partial 1-factorizations of hypergraphs.

Let $\mathscr{G}$ be a hypergraph, and let $\mathscr{H}$ be a family of hypergraphs. We say that $\mathscr{G}$ has an $\mathscr{H}$-decomposition if there exists a partition $\{E(\mathscr{H}_1), \ldots, E(\mathscr{H}_m)\}$ of $E(\mathscr{G})$ such that $\mathscr{H}_i$ is isomorphic to a hypergraph in $\mathscr{H}$ for $1 \leqslant i \leqslant m$.

The general setting for this chapter is as follows. Let $\mathscr{H}$ and $\mathscr{H}^*$ be two families of hypergraphs. Given a hypergraph $\mathscr{G}$ with an $\mathscr{H}$-decomposition and a hypergraph $\mathscr{G}^*$ which is a super-hypergraph of $\mathscr{G}$, under what circumstances can one extend the $\mathscr{H}$-decomposition of $\mathscr{G}$ into an $\mathscr{H}^*$-decomposition of $\mathscr{G}^*$? In other words, given a hyperedge-coloring of $\mathscr{G}$ in which each color class induces a hypergraph in $\mathscr{H}$, is it possible to extend this coloring to a

hyperedge-coloring of $\mathscr{G}^*$ so that each color class of $\mathscr{G}^*$ induces a hypergraph in $\mathscr{H}^*$? Most naturally, $\mathscr{G}$ is usually taken to be the complete h-uniform hypergraph on m vertices, K_m^h.

Solving this problem requires knowledge about hypergraph decompositions; compared to graph decompositions, very little is known about these, even for special cases. Perhaps the best evidence for this difficulty is the long standing open problem of Sylvester in 1850 (in connection with Kirkman's famous Fifteen Schoolgirls Problem [56]) which asks whether it is possible to find a 1-factorization of K_n^h (see the next section for definitions). It took 120 years before Baranyai finally settled this conjecture [15]. After Baranyai's proof appeared, in 1976 Cameron [29] asked the following question:

> Under what conditions can partial 1-factorizations of K_m^h be extended to 1-factorizations of K_n^h?

This problem is wide open and to the authors' best knowledge, the only partial results address the very special case of embedding a 1-factorization of K_m^h into a 1-factorization of K_n^h [17, 40].

Here we make some progress toward settling this problem, considering the following related general embedding problem that is natural in its own right. When can a hyperedge-coloring of a given hypergraph $\mathscr{G}$ on m vertices be embedded into a hyperedge-coloring of K_n^3 in such a way that each color class forms an r-factor? So the special case when $r = 1$ and $\mathscr{G} = K_m^h$ addresses the Cameron question in the situation where the given partial 1-factors are all defined on a set of m vertices.

In Section 7.3, we assume that precisely the hyperedges of size 3 on m vertices have been colored; that is, the given hypergraph is $\mathscr{G} = K_m^3$), giving a complete solution if $n > 2m + (-1 + \sqrt{8m^2 - 16m - 7})/2$ (see Theorem 7.3). Lemma 7.4 then shows that Theorem 7.3 is not true if this bound on n is replaced by $n \geqslant 2m - 1$. In Section 7.4 we assume that not only the hyperedges of size 3 are colored, but so are all the "pieces" of hyperedges of K_n^3 that contain one or two of the given m vertices (i.e. $n - m$ and $\binom{n-m}{2}$ copies of the hyperedges in K_m^2 and K_m^1, respectively); these pieces are built up to hyperedges

of size 3 when the new vertices are added. In this case the problem is completely solved in Section 7.4, providing necessary and sufficient conditions (see Theorem 7.5).

The results in this chapter supplement embedding results for graphs. Such results typically take a given edge-coloring of all the edges of a smaller complete graph and extend it to an edge-coloring of all the edges of a bigger complete graph in such a way that each color class is one of a given family of graphs. Hilton [44] used amalgamations to solve the problem of embedding an edge-coloring of K_m into a Hamiltonian decomposition of K_n. This was later generalized by Nash-Williams [70]. Hilton and Rodger [48] considered the embedding problem for Hamiltonian decompositions of complete multipartite graphs. For embeddings of factorizations in which connectivity is also addressed, see [47, 51, 74].

It is worth remarking that embeddings of combinatorial structures with the same flavor as results found in this chapter have a long history. For example, in his 1945 paper [41], Hall proved that every $p \times n$ latin rectangle on n symbols can be embedded in a latin square of size n. Following this classic embedding theorem, in 1951 Ryser generalized Hall's result to $p \times q$ latin rectangles on n symbols [75]. Ryser's result is equivalent to embedding a proper edge-coloring of the complete bipartite graph $K_{p,q}$ into a 1-factorization of $K_{n,n}$. Doyen and Wilson [35] solved the embedding problem for Steiner triple systems (K_3-decompostions of K_n), then Bryant and Horsley [27] addressed the embedding of partial designs, proving Lindner's conjecture [62] that any partial Steiner triple system of order u, $PSTS(u)$, can be embedded in an $STS(v)$ if $v \equiv 1, 3 \pmod 6$ and $v \geqslant 2u + 1$. ($2u + 1$ is best possible in the sense that for all $u \geqslant 9$ there exists a $PSTS(u)$ that can not be embedded in an $STS(v)$ for any $v < 2u + 1$.)

7.2 Detachments of Amalgamated Hypergraphs

Note that a hypergraph as defined here corresponds to a hypergraph as usually defined providing hyperedges are allowed to contain vertices multiple times. We imagine each hyperedge of a hypergraph to be attached to the vertices which it joins by in-between objects

called hinges. A hypergraph may be drawn as a set of points representing the vertices. A hyperedge is represented by a simple closed curve enclosing its incident vertices. A hinge is represented by a small line attached to the vertex incident with it (see Figure 7.1).

Example 7.1. Let $\mathscr{F} = (V, E, H, \psi, \phi)$, with $V = \{v_i : 1 \leqslant i \leqslant 8\}, E = \{e_1, e_2, e_3\}, H = \{h_i : 1 \leqslant i \leqslant 9\}$, such that for $1 \leqslant i \leqslant 8$, $\psi(h_i) = v_i$, $\psi(h_9) = v_6$ and $\phi(h_1) = \phi(h_2) = \phi(h_3) = e_1, \phi(h_4) = \phi(h_5) = \phi(h_6) = e_2, \phi(h_7) = \phi(h_8) = \phi(h_9) = e_3$. Let $\Psi : V \to \{w_1, w_2, w_3, w_4\}$ be the function with $\Psi(v_1) = \Psi(v_2) = \Psi(v_3) = w_1$, $\Psi(v_4) = w_2$, $\Psi(v_5) = \Psi(v_6) = w_3$, $\Psi(v_7) = \Psi(v_8) = w_4$. The hypergraph $\mathscr{G}$ is the Ψ-amalgamation of $\mathscr{F}$ (see Figure 7.1).
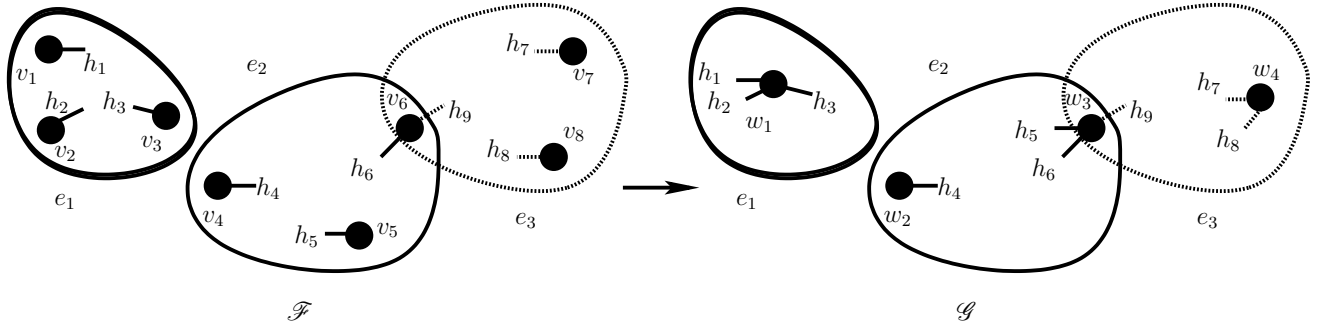
Figure 7.1: A visual representation of a hypergraph $\mathscr{F}$ with an amalgamation $\mathscr{G}$

In the remainder of this chapter, all hypergraphs are either 3-uniform or are amalgamations of 3-uniform hypergraphs. This implies that for every hypergraph $\mathscr{G}$ we have

$$1 \leqslant |e| \leqslant |\phi^{-1}(e)| = 3 \text{ for every } e \text{ in } \mathscr{G}. \tag{7.1}$$

If u, v, w are three (not necessarily distinct) vertices of $\mathscr{G}$, then $m(u, v, w)$ denotes the number of hyperedges that join u, v, and w. For convenience, we let $m(u^2, v) = m(u, u, v)$, and $m(u^3) = m(u, u, u)$. If we think of an edge as a multiset, then $m(u^2, v)$ (or $m(u^3)$) counts the multiplicity of an edge of the form $\{u, u, v\}$ (or $\{u, u, u\}$, respectively).

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For the purpose of this chapter, we need the following result which is a special case of both Theorem **3.1** in [6], and Theorem **4.1** in [8]. To state it, some notation must be introduced.

For $g : V(\mathscr{F}) \to \mathbb{N}$, we define the symmetric function $\tilde{g} : V^3(\mathscr{F}) \to \mathbb{N}$ such that for distinct $x, y, z \in V(\mathscr{F})$, $\tilde{g}(x, x, x) = \binom{g(x)}{3}$, $\tilde{g}(x, x, y) = \binom{g(x)}{2} g(y)$, and $\tilde{g}(x, y, z) = g(x)g(y)g(z)$. Also we assume that for each $x \in V(\mathscr{F})$, $g(x) \leqslant 2$ implies $m_{\mathscr{F}}(x^3) = 0$, and $g(x) = 1$ implies $m_{\mathscr{F}}(x^2, y) = 0$ for every $y \in V(\mathscr{F})$.

Theorem 7.2. (Bahmanian [6, Theorem 3.1]) *Let $\mathscr{F}$ be a k-hyperedge-colored hypergraph and let g be a function from $V(\mathscr{F})$ into $\mathbb{N}$. Then there exists a 3-uniform g-detachment $\mathscr{G}$ of $\mathscr{F}$ with amalgamation function $\Psi : V(\mathscr{G}) \to V(\mathscr{F})$, g being the number function associated with Ψ, such that:*

(A1) *for each $x \in V(\mathscr{F})$, each $u \in \Psi^{-1}(x)$ and each $j \in \{1, \ldots, k\}$*

$$d_{\mathscr{G}(j)}(u) \approx \frac{d_{\mathscr{F}(j)}(x)}{g(x)}; \ \ and$$

(A2) *for every $x, y, z \in V(\mathscr{F})$, with $g(x) \geqslant 3$ if $x = y = z$, and $g(x) \geqslant 2$ if $|\{x, y, z\}| = 2$, and every triple of distinct vertices u, v, w with $u \in \Psi^{-1}(x)$, $v \in \Psi^{-1}(y)$ and $w \in \Psi^{-1}(z)$,*

$$m_{\mathscr{G}}(u, v, w) \approx \frac{m_{\mathscr{F}}(x, y, z)}{\tilde{g}(x, y, z)}.$$

7.3 Embedding Partial Hyperedge-colorings into Factorizations

In this section we completely solve the embedding problem in the case where all the hyperedges of size 3 on a set of m vertices have been colored, providing n is big enough. We then show that some lower bound on n is needed, since the necessary conditions of Theorem 7.3 are not sufficient if $n = 2m - 1$.

Theorem 7.3. *Suppose that $n > 2m + (-1 + \sqrt{8m^2 - 16m - 7})/2$. A q-hyperedge-coloring of $\mathscr{F} = K_m^3$ can be embedded into an r-factorization of $\mathscr{G} = K_n^3$ if and only if*

(i) $3 \mid rn$,

(ii) $r \mid \binom{n-1}{2}$,

(iii) $q \leqslant \binom{n-1}{2}/r$, *and*

(iv) $d_{\mathscr{F}(j)}(v) \leqslant r$ *for each* $v \in V(\mathscr{F})$ *and* $1 \leqslant j \leqslant q$.

Proof. To prove the necessity, suppose that $\mathscr{F}$ with $V = V(\mathscr{F})$ can be embedded into an r-factorization of $\mathscr{G}$. Since each edge contributes 3 to the the sum of the degrees of the vertices in an r-factor, $r|V(\mathscr{G})|$ must be divisible by 3 which implies (i). Since each r-factor is an r-regular spanning sub-hypergraph and $\mathscr{G}$ is $\binom{n-1}{2}$-regular, we must have $r \mid \binom{n-1}{2}$, which is condition (ii). This r-factorization requires exactly $k = \binom{n-1}{2}/r$ colors which is condition (iii), and to be able to extend each color class to an r-factor we need condition (iv).

Now assume that conditions (i)–(iv) are true. By Baranyai's theorem [15], the case of $m \leqslant 3$ is trivial, and so we may assume that $m \geqslant 4$. Let $e_j = |E(\mathscr{F}(j))|$ for $1 \leqslant j \leqslant k$. In what follows, we extend the hyperedge-coloring of $\mathscr{F}$ into a k-hyperedge-coloring of an amalgamation of $\mathscr{G}$, and then we apply Theorem 7.2 to obtain the detachment $\mathscr{G}$ in which each color class is an r-factor. The hyperedges added in steps (I), (II), and (III) correspond to the hyperedges in $\mathscr{G}$ that contain one, two, and three new vertices, respectively.

(I) Let $\mathscr{F}_1$ be a hypergraph formed by adding a new vertex u and hyperedges to $\mathscr{F}$ such that $m(u, v, w) = n - m$ for every pair of distinct vertices $v, w \in V$. Of course the hyperedges in $E(\mathscr{F}) \cap E(\mathscr{F}_1)$ are already colored. We color greedily as many of the added $(n - m)\binom{m}{2}$ hyperedges as possible, ensuring that $d_{\mathscr{F}_1(j)}(v) \leqslant r$ for $1 \leqslant j \leqslant k$. Suppose there exists a hyperedge incident with u,v and w that is not colored. Then for $1 \leqslant j \leqslant k$ either $d_{\mathscr{F}_1(j)}(v) = r$ or $d_{\mathscr{F}_1(j)}(w) = r$, so $d_{\mathscr{F}_1(j)}(v) + d_{\mathscr{F}_1(j)}(w) \geqslant r$ for

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every $1 \leqslant j \leqslant k$. Therefore $2\binom{m-1}{2} + 2(n-m)(m-1) - 2 = d_{\mathscr{F}_1}(v) + d_{\mathscr{F}_1}(w) - 2 \geqslant \sum_{j=1}^{k} \left(d_{\mathscr{F}_1(j)}(v) + d_{\mathscr{F}_1(j)}(w) \right) \geqslant \sum_{j=1}^{k} r = kr = \binom{n-1}{2}$, in which the first inequality follows from that fact that at least one hyperedge incident with v and w is not colored. So, $2(m-1)(m-2) + 4(n-m)(m-1) - 4 \geqslant (n-1)(n-2)$. Thus $n^2 - 4nm + n + 2m^2 + 2m + 2 \leqslant 0$. So

$$n \leqslant 2m + (-1 + \sqrt{8m^2 - 16m - 7})/2,$$

a contradiction. So all hyperedges can be colored greedily. Let f_j be the number of hyperedges of color j in some such coloring for $1 \leqslant j \leqslant k$.

(II) Let $\mathscr{F}_2$ be a hypergraph formed by adding $m\binom{n-m}{2}$ further hyperedges to $\mathscr{F}_1$ so that $m(u^2, v) = \binom{n-m}{2}$ for each $v \in V$. Note that for each $v \in V$,

$$\begin{aligned} d_{\mathscr{F}_2}(v) &= \binom{m-1}{2} + (m-1)(n-m) + \binom{n-m}{2} \\ &= \binom{n-1}{2} = rk. \end{aligned}$$

Since $d_{\mathscr{F}_1(j)}(v) \leqslant r$ for $v \in V$ and $1 \leqslant j \leqslant k$, to ensure that $d_{\mathscr{F}_2(j)}(v) = r$, we color $r - d_{\mathscr{F}_1(j)}(v)(\geqslant 0)$ hyperedges incident with v that were added in forming $\mathscr{F}_2$ from $\mathscr{F}_1$ with color j for each $v \in V$ and $1 \leqslant j \leqslant k$. So the coloring we perform in this step results in all the newly added hyperedges being colored. Let g_j denote the number of such hyperedges of color j for $1 \leqslant j \leqslant k$.

(III) Let $\mathscr{F}_3$ be the hypergraph formed by adding $\binom{n-m}{3}$ further hyperedges to $\mathscr{F}_2$ so that $m(u^3) = \binom{n-m}{3}$. Let $\ell_j := r(n/3 - m) + f_j + 2e_j$ for $1 \leqslant j \leqslant k$. We claim that $\ell_j \geqslant 0$ for $1 \leqslant j \leqslant k$. To prove this, it is enough to show that $n \geqslant 3m$. Since $m \geqslant 4 > (3 + \sqrt{17})/2$, we have $m^2 - 3m - 2 \geqslant 0$. Therefore, $8m^2 - 16m - 7 \geqslant 4m^2 - 4m + 1$, and thus $\sqrt{8m^2 - 16m - 7} \geqslant 2m - 1$, which implies $(1 + \sqrt{8m^2 - 16m - 7})/2 \geqslant m$,

and consequently we have $\lfloor (1 + \sqrt{8m^2 - 16m - 7})/2 \rfloor \geqslant m$. Since $n > 2m + \lfloor (1 + \sqrt{8m^2 - 16m - 7})/2 \rfloor$, we have $n \geqslant 3m$.

Now we color the added hyperedges such that there are exactly ℓ_j further hyperedges colored j for $1 \leqslant j \leqslant k$. This is possible because

$$
\begin{aligned}
\sum_{j=1}^{k} \ell_j &= \sum_{j=1}^{k} \left(r(n/3 - m) + f_j + 2e_j \right) \\
&= rk(n/3 - m) + \sum_{j=1}^{k} f_j + 2\sum_{j=1}^{k} e_j \\
&= \binom{n-1}{2}(n/3 - m) + (n - m)\binom{m}{2} + 2\binom{m}{3} \\
&= n^3/6 - n^2 m/2 - n^2/2 + nm^2/2 + nm \\
&\quad + n/3 - m^3/6 - m^2/2 - m/3 \\
&= \binom{n-m}{3} = m_{\mathscr{F}_3}(u^3).
\end{aligned}
$$

Let us fix $j \in \{1, \dots, k\}$. Since $d_{\mathscr{F}_3(j)}(v) = r$ for $v \in V$, we have

$$
rm = \sum_{v \in V} d_{\mathscr{F}_3(j)}(v) = 3e_j + 2f_j + g_j. \tag{7.2}
$$

On the other hand,

$$
\begin{aligned}
d_{\mathscr{F}_3(j)}(u) &= 3\ell_j + 2g_j + f_j = r(n - 3m) + 3f_j + 6e_j + 2g_j + f_j \\
&= r(n - 3m) + 4f_j + 6e_j + 2g_j.
\end{aligned}
$$

This together with (7.2) implies that for $1 \leqslant j \leqslant k$,

$$
d_{\mathscr{F}_3(j)}(u) = r(n - 3m) + 2rm = r(n - m).
$$

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(IV) Let $g : V(\mathscr{F}_3) \to \mathbb{N}$ be a function with $g(u) = n - m$, and $g(v) = 1$ for each $v \in V$. By Theorem 7.2, there exists a 3-uniform g-detachment $\mathscr{G}^*$ of $\mathscr{F}_3$ with $n - m$ new vertices, say $u_1, \ldots, u_{n-m}$ detached from u such that

- $d_{\mathscr{G}^*(j)}(v) = d_{\mathscr{F}_3(j)}(v)/g(v) = r/1 = r$ and $d_{\mathscr{G}^*(j)}(u_i) = d_{\mathscr{F}_3(j)}(u)/g(u) = r(n-m)/(n-m) = r$ for $1 \leqslant i \leqslant n-m$ and $1 \leqslant j \leqslant k$;

- $m_{\mathscr{G}^*}(u_i, u_{i'}, u_{i''}) = m_{\mathscr{F}_3}(u^3)/\binom{g(u)}{3} = \binom{n-m}{3}/\binom{n-m}{3} = 1$ for $1 \leqslant i < i' < i'' \leqslant n-m$;

- $m_{\mathscr{G}^*}(u_i, u_{i'}, v) = m_{\mathscr{F}_3}(u^2, v)/\left(\binom{g(u)}{2}g(v)\right) = \binom{n-m}{2}/\binom{n-m}{2} = 1$ for $1 \leqslant i < i' \leqslant n - m$, and $v \in V$, and

- $m_{\mathscr{G}^*}(u_i, v, w) = m_{\mathscr{F}_3}(u, v, w)/\left(g(u)g(v)g(w)\right) = (n-m)/(n-m) = 1$ for $1 \leqslant i \leqslant n - m$ and distinct $v, w \in V$.

Therefore $\mathscr{G}^* \cong \mathscr{G} = K_n^3$ and each color class is an r-factor. This completes the proof. $\qquad \square$

Lemma 7.4. *Conditions* (i)–(iv) *of Theorem 7.3 are not sufficient if $n = 2m - 1$.*

Proof. Suppose that the hyperedge-coloring of K_m^3 induces an r-factorization. Then in the embedding, the sub-hypergraph of K_n^3 on the new $n - m$ vertices induced by the hyperedges having the original colors clearly has an r-factorization (each of the colors induces an r-factor). Therefore $n - m \geqslant m$, or equivalently $n \geqslant 2m$. So if r is chosen so that $3 \mid r$ and $r \mid m - 1$, then it is easy to check that conditions (i)–(iv) of Theorem 7.3 are satisfied when $n = 2m - 1$, yet no embedding is possible. $\qquad \square$

7.4 Extending Restrictions of Partial Edge-colorings

If every hyperedge e of the hypergraph $\mathscr{G}$ is replaced with λ ($\geqslant 2$) copies of e then denote the resulting (multi) hypergraph by $\lambda \mathscr{G}$. If $\mathscr{G}_1, \ldots, \mathscr{G}_t$ are hypergraphs on the vertex set V with edge sets $E(\mathscr{G}_1) \ldots, E(\mathscr{G}_t)$ respectively, then let $\bigcup_{i=1}^t \mathscr{G}_i$ be the hypergraph with vertex set V and edge set $\bigcup_{i=1}^t E(\mathscr{G}_i)$.

In this section we completely solve the embedding problem in the case where all the hyperedges in $\mathscr{F} = K_m^3 \cup (n-m)K_m^2 \cup \binom{n-m}{2}K_m^1$ on a set of m vertices have been colored, regardless of the size of n. One can think of the given colored hyperedges as being all the "pieces" of hyperedges on these m vertices that are eventually extended to hyperedges of size 3 by adding the new $n-m$ vertices during the embedding process.

Let $E^i(\mathscr{G}(j))$ denote the set of hyperedges of size i and color j in $\mathscr{G}$.

Theorem 7.5. *A k-hyperedge-coloring of $\mathscr{F} = K_m^3 \cup (n-m)K_m^2 \cup \binom{n-m}{2}K_m^1$ with $V = V(\mathscr{F})$ can be extended to an r-factorization of $\mathscr{G} = K_n^3$ if and only if*

(i) $3 \mid rn$,

(ii) $r \mid \binom{n-1}{2}$,

(iii) $k = \binom{n-1}{2}/r$,

(iv) $d_{\mathscr{F}(j)}(v) = r$ *for each $v \in V$ and $1 \leqslant j \leqslant k$, and*

(v) $|E^2(\mathscr{F}(j))| + 2|E^3(\mathscr{F}(j))| \geqslant r(m - n/3)$ *for $1 \leqslant j \leqslant k$.*

Proof. First, suppose that $\mathscr{F}$ can be embedded into an r-factorization of $\mathscr{G}$. The necessity of (i)–(iv) follow as described in the proof of Theorem 7.3; equalities in this result replace the inequalities there because the colors of all hyperedges restricted to $\mathscr{F}$ have been prescribed in this case. Let us fix $j \in \{1, \ldots, k\}$. Let $e_j, f_j, g_j,$ and ℓ_j be the number of hyperedges in $E(\mathscr{G}(j))$ that are incident with exactly 3, 2, 1 and 0 vertices in V, respectively. It is easy to see that $e_j = |E^3(\mathscr{F}(j))|$ and $f_j = |E^2(\mathscr{F}(j))|$. Since $r(n - m) = 3\ell_j + 2g_j + f_j$, and $rm = g_j + 2f_j + 3e_j$, we have $r(n - 3m) = 3\ell_j - 3f_j - 6e_j$, and thus $\ell_j = r(n/3 - m) + f_j + 2e_j$, but since $\ell_j \geqslant 0$, we must have $2e_j + f_j \geqslant r(m - n/3)$. This proves (v).

To prove the sufficiency, assume that conditions (i)–(v) are true. Let $\mathscr{F}'$ be a hypergraph formed by adding a new vertex u to $\mathscr{F}$ with $m(u^3) = \binom{n-m}{3}$, and extending each hyperedge of size one or two to a hyperedge incident with u of size two or three, respectively. We extend the hyperedges of size one (two, respectively) such that u is incident with two (one, respectively) hinges within that hyperedge. Ignoring colorings, $\mathscr{F}'$ is isomorphic to $\mathscr{F}_3$ in

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the proof of Theorem 7.3, and $\mathscr{F}'$ is an amalgamation of $\mathscr{G}$. We color $r(n/3 - m) + f_j + 2e_j$ of the new hyperedges with color j. This coloring results in all the newly added hyperedges being colored. The rest of the proof is identical to part (IV) of Theorem 7.3. $\qquad\square$

Chapter 8

Detachments of Hypergraphs: The Berge-Johnson Problem

8.1 Introduction

Intuitively speaking, a detachment of a hypergraph is formed by splitting each vertex into one or more subvertices, and sharing the incident edges arbitrarily among the subvertices. As the main result of this chapter (see Theorem 8.2), we prove that for a given edge-colored hypergraph $\mathscr{F}$, there exists a detachment $\mathscr{G}$ such that the degree of each vertex and the multiplicity of each edge in $\mathscr{F}$ (and each color class of $\mathscr{F}$) are shared fairly among the subvertices in $\mathscr{G}$ (and each color class of $\mathscr{G}$, respectively). This result is not only interesting by itself and generalizes various graph theoretic results (see for example [5, 44, 48, 51, 58, 61, 70, 74]), but also is used to obtain extensions of existing results on edge-decompositions of hypergraphs by Bermond, Baranyai [15, 16], Berge and Johnson [21, 50], and Brouwer and Tijdeman [24, 26].

Given a set N of n elements, Berge and Johnson [21, 50] addressed the question of when do there exist disjoint partitions of N, each partition containing only subsets of h or fewer elements, such that every subset of N having h or fewer elements is in exactly one partition. Here we state the problem in a more general setting with the hypergraph theoretic notation.

Let $(\lambda_1 \ldots, \lambda_m) K_{p_1,\ldots,p_n}^{h_1,\ldots,h_m}$ be a hypergraph with vertex partition $\{V_1,\ldots,V_n\}$, $|V_i| = p_i$ for $1 \leqslant i \leqslant n$ such that there are λ_i edges of size h_i incident with every h_i vertices, at most one vertex from each part for $1 \leqslant i \leqslant m$ (so no edge is incident with more than one vertex of a part). We use our detachment theorem to show that the obvious necessary conditions for $(\lambda_1 \ldots, \lambda_m) K_{p_1,\ldots,p_n}^{h_1,\ldots,h_m}$ to be expressed as the union $\mathscr{G}_1 \cup \ldots \cup \mathscr{G}_k$ of k edge-disjoint factors, where for $1 \leqslant i \leqslant k$, $\mathscr{G}_i$ is r_i-regular, are also sufficient. Baranyai [15, 16] solved the case of $h_1 = \cdots = h_m$, $\lambda_1 = \ldots, \lambda_m = 1$, $p_1 = \cdots = p_m$, $r_1 = \cdots = r_k$. Berge and Johnson [21, 50],

(and later Brouwer and Tijdeman [24, 26], respectively) considered (and solved, respectively) the case of $h_i = i$, $1 \leqslant i \leqslant m$, $p_1 = \cdots = p_m = \lambda_1 = \cdots = \lambda_m = r_1 = \cdots = r_k = 1$. We also extend our result to the case where each $\mathscr{G}_i$ is almost regular.

In the next two sections, we give more precise definitions along with terminology. In Section 8.4, we state our main result, followed by the proof in Section 8.5. In the last section, we show the usefulness of the main result on decompositions of various classes of hypergraphs. We defer the applications of the main result in solving embedding problems to a future paper.

8.2 Terminology and Precise Definitions

For a multiset A and $u \in A$, let $\mu_A(u)$ denote the multiplicity of u in A, and let $|A| = \sum_{u \in A} \mu_A(u)$. For multisets $A_1, \ldots, A_n$, we define $A = \bigcup_{i=1}^n A_i$ by $\mu_A(u) = \sum_{i=1}^n \mu_{A_i}(u)$. We may use abbreviations such as $\{u^r\}$ for $\{\underbrace{u, \ldots, u}_{r}\}$ — for example $\{u^2, v, w^2\} \cup \{u, w^2\} = \{u^3, v, w^4\}$.

For the purpose of this chapter, a *hypergraph* $\mathscr{G}$ is an ordered quintuple $(V(\mathscr{G}), E(\mathscr{G}), H(\mathscr{G}), \psi, \phi)$ where $V(\mathscr{G}), E(\mathscr{G}), H(\mathscr{G})$ are disjoint finite sets, $\psi : H(\mathscr{G}) \to V(\mathscr{G})$ is a function and $\phi : H(\mathscr{G}) \to E(\mathscr{G})$ is a surjection. Elements of $V(\mathscr{G}), E(\mathscr{G}), H(\mathscr{G})$ are called *vertices, edges* and *hinges* of $\mathscr{G}$, respectively. A vertex v (edge e, respectively) and hinge h are said to be *incident* with each other if $\psi(h) = v$ ($\phi(h) = e$, respectively). A hinge h is said to *attach* the edge $\phi(h)$ to the vertex $\psi(h)$. In this manner, the vertex $\psi(h)$ and the edge $\phi(h)$ are said to be *incident* with each other. If $e \in E(\mathscr{G})$, and e is incident with n hinges $h_1, \ldots, h_n$ for some $n \in \mathbb{N}$, then the edge e is said to *join* (not necessarily distinct) vertices $\psi(h_1), \ldots, \psi(h_n)$. If $v \in V(\mathscr{G})$, then the number of hinges incident with v (i.e. $|\psi^{-1}(v)|$) is called the *degree* of v and is denoted by $d(v)$. The number of (distinct) vertices incident with an edge e, denoted by $|e|$, is called the *size* of e. If for all edges e of $\mathscr{G}$, $|e| \leqslant 2$ and $|\phi^{-1}(e)| = 2$, then $\mathscr{G}$ is a *graph*.

Thus a hypergraph, in the sense of our definition, is a generalization of a hypergraph as it is usually defined. In fact, if for every edge e, $|e| = |\phi^{-1}(e)|$, then our definition is essentially the same as the usual definition. Here for convenience, we imagine each edge of a hypergraph to be attached to the vertices which it joins by in-between objects called hinges. Readers from a graph theory background may think of this as a bipartite multigraph with vertex bipartition $\{V, E\}$, in which the hinges form the edges. A hypergraph may be drawn as a set of points representing the vertices. A hyperedge is represented by a simple closed curve enclosing its incident vertices. A hinge is represented by a small line attached to the vertex incident with it (see Figure 8.1).

The set of hinges of $\mathscr{G}$ which are incident with a vertex v (and an edge e, respectively), is denoted by $H(v)$ ($H(v, e)$, respectively). Thus if $v \in V(\mathscr{G})$, then $H(v) = \psi^{-1}(v)$, and $|H(v)|$ is the degree $d(v)$ of v. If U is a multi-subset of $V(\mathscr{G})$, and $u \in V(\mathscr{G})$, let $E(U)$ denote the set of edges e with $|\phi^{-1}(e)| = |U|$ joining vertices in U. More precisely, $E(U) = \{e \in E(\mathscr{G})|$ for all $v \in V(\mathscr{G}), |H(v, e)| = \mu_U(v)\}$. For $U_1, \ldots, U_n \subset V$ where for $1 \leqslant i \leqslant n$ each U_i is a multiset, let $E(U_1, \ldots, U_n)$ denote $E(\bigcup_{i=1}^{n} U_i)$. We write $m(U)$ for $|E(U)|$ and call it the *multiplicity* of U. For simplicity, $E(u^r, U)$ denotes $E(\{u^r\}, U)$, and $m(u_1^{m_1}, \ldots, u_r^{m_r})$ denotes $m(\{u_1^{m_1}, \ldots, u_r^{m_r}\})$. The set of hinges that are incident with u and an edge in $E(u^r, U)$ is denoted by $H(u^r, U)$.

Example 8.1. Let $\mathscr{G} = (V, E, H, \psi, \phi)$, with $V = \{v_1, v_2, v_3, v_4, v_5\}, E = \{e_1, e_2, e_3\}, H = \{h_i, 1 \leqslant i \leqslant 7\}$, such that $\psi(h_1) = \psi(h_2) = v_1, \psi(h_3) = v_2, \psi(h_4) = \psi(h_5) = v_3, \psi(h_6) = v_4, \psi(h_7) = v_5$ and $\phi(h_1) = \phi(h_2) = \phi(h_3) = \phi(h_4) = e_1, \phi(h_5) = \phi(h_6) = e_2, \phi(h_7) = e_3$. We have:

- $|e_1| = 3, |e_2| = 2, |e_3| = 1$,

- $d(v_1) = d(v_3) = 2, d(v_2) = d(v_4) = d(v_5) = 1$,

- $H(v_1) = \{h_1, h_2\}, H(v_2) = \{h_3\}, H(v_3) = \{h_4, h_5\}$,

- $H(v_3, e_1) = \{h_4\}, H(v_3, e_2) = \{h_5\}, H(v_3, e_3) = \varnothing$,
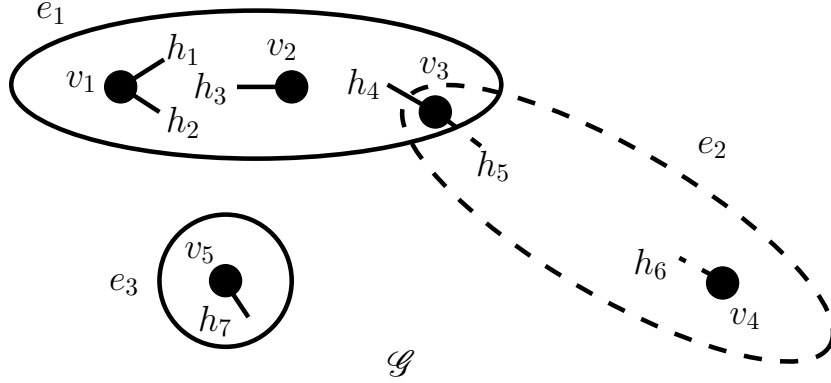
Figure 8.1: Representation of a hypergraph $\mathscr{G}$

- $E(\{v_1, v_2, v_3\}) = \varnothing, E(\{v_1^2, v_2, v_3\}) = E(v_1^2, \{v_2, v_3\}) = \{e_1\},$

- $m(v_1, v_2, v_3) = 0, m(v_1^2, v_2, v_3) = 1,$

- $H(v_1^2, \{v_2, v_3\}) = \{h_1, h_2\}, H(v_1, \{v_2, v_3\}) = \varnothing, H(v_3, \{v_1^2, v_2\}) = \{h_4\}.$

A *k-edge-coloring* of $\mathscr{G}$ is a mapping $f : E(\mathscr{G}) \to C$, where C is a set of k *colors* (often we use $C = \{1, \ldots, k\}$), and the edges of one color form a *color class*. The sub-hypergraph of $\mathscr{G}$ induced by the color class j is denoted by $\mathscr{G}(j)$. To avoid ambiguity, subscripts may be used to indicate the hypergraph in which hypergraph-theoretic notation should be interpreted — for example, $d_{\mathscr{G}}(v)$, $E_{\mathscr{G}}(v^2, w)$, $H_{\mathscr{G}}(v)$.

8.3 Amalgamations and Detachments

If $\mathscr{F} = (V, E, H, \psi, \phi)$ is a hypergraph and Ψ is a function from V onto a set W, then we shall say that the hypergraph $\mathscr{G} = (W, E, H, \Psi \circ \psi, \phi)$ is an *amalgamation* of $\mathscr{F}$ and that $\mathscr{F}$ is a *detachment* of $\mathscr{G}$. Associated with Ψ is the *number function* $g : W \to \mathbb{N}$ defined by $g(w) = |\Psi^{-1}(w)|$, for each $w \in W$; being more specific, we may also say that $\mathscr{F}$ is a *g-detachment* of $\mathscr{G}$. Intuitively speaking, a *g*-detachment of $\mathscr{G}$ is obtained by splitting each $u \in V(\mathscr{G})$ into $g(u)$ vertices. Thus $\mathscr{F}$ and $\mathscr{G}$ have the same edges and hinges, and each vertex v of $\mathscr{G}$ is obtained by identifying those vertices of $\mathscr{F}$ which belong to the set $\Psi^{-1}(v)$.

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In this process, a hinge incident with a vertex u and an edge e in $\mathscr{F}$ becomes incident with the vertex $\Psi(u)$ and the edge e in $\mathscr{G}$.

There are quite a lot of other papers on amalgamations and some highlights include [36, 42, 45, 44, 48, 51, 70, 74].

8.4 Main Result

A function $g : V(\mathscr{G}) \to \mathbb{N}$ is said to be *simple* if

$$|H(v, e)| \leqslant g(v) \text{ for } v \in V(\mathscr{G}), e \in E(\mathscr{G}).$$

A hypergraph $\mathscr{G}$ is said to be *simple* if $g : V(\mathscr{G}) \to \mathbb{N}$ with $g(v) = 1$ for $v \in V(\mathscr{G})$ is simple. It is clear that for a hypergraph $\mathscr{F}$ and a function $g : V(\mathscr{F}) \to \mathbb{N}$, there exists a simple g-detachment if and only if g is simple.

Theorem 8.2. *Let $\mathscr{F}$ be a k-edge-colored hypergraph and let $g : V(\mathscr{F}) \to \mathbb{N}$ be a simple function. Then there exists a simple g-detachment $\mathscr{G}$ (possibly with multiple edges) of $\mathscr{F}$ with amalgamation function $\Psi : V(\mathscr{G}) \to V(\mathscr{F})$, g being the number function associated with Ψ, such that:*

(A1) *for each $u \in V(\mathscr{F})$ and each $v \in \Psi^{-1}(u)$*

$$d_{\mathscr{G}}(v) \approx \frac{d_{\mathscr{F}}(u)}{g(u)};$$

(A2) *for each $u \in V(\mathscr{F})$, each $v \in \Psi^{-1}(u)$ and $1 \leqslant j \leqslant k$*

$$d_{\mathscr{G}(j)}(v) \approx \frac{d_{\mathscr{F}(j)}(u)}{g(u)};$$

(A3) *for distinct $u_1, \ldots, u_r \in V(\mathscr{F})$ and $U_i \subset \Psi^{-1}(u_i)$ with $|U_i| = m_i \leqslant g(u_i)$ for $1 \leqslant i \leqslant r$*

$$m_{\mathscr{G}}(U_1, \ldots, U_r) \approx \frac{m_{\mathscr{F}}(u_1^{m_1}, \ldots, u_r^{m_r})}{\Pi_{i=1}^r \binom{g(u_i)}{m_i}};$$

(A4) *for distinct $u_1, \ldots, u_r \in V(\mathscr{F})$ and $U_i \subset \Psi^{-1}(u_i)$ with $|U_i| = m_i \leqslant g(u_i)$ for $1 \leqslant i \leqslant r$*
and $1 \leqslant j \leqslant k$

$$m_{\mathscr{G}(j)}(U_1, \ldots, U_r) \approx \frac{m_{\mathscr{F}(j)}(u_1^{m_1}, \ldots, u_r^{m_r})}{\Pi_{i=1}^r \binom{g(u_i)}{m_i}}.$$

8.5 Proof of Theorem 8.2

8.5.1 Inductive Construction of $\mathscr{G}$

Let $\mathscr{F} = (V, E, H, \psi, \phi)$. Let $n = \sum_{v \in V}(g(v) - 1)$. Initially we let $\mathscr{F}_0 = \mathscr{F}$ and $g_0 = g$, and we let Φ_0 be the identity function from V into V. Now assume that $\mathscr{F}_0 = (V_0, E_0, H_0, \psi_0, \phi_0), \ldots, \mathscr{F}_i = (V_i, E_i, H_i, \psi_i, \phi_i)$ and $\Phi_0, \ldots, \Phi_i$ have been defined for some $i \geqslant 0$. Also assume that the simple functions $g_0 : V_0 \to \mathbb{N}, \ldots, g_i : V_i \to \mathbb{N}$ have been defined for some $i \geqslant 0$. Let $\Psi_i = \Phi_0 \ldots \Phi_i$. If $i = n$, we terminate the construction, letting $\mathscr{G} = \mathscr{F}_n$ and $\Psi = \Psi_n$.

If $i < n$, we can select a vertex α of $\mathscr{F}_i$ such that $g_i(\alpha) \geqslant 2$. As we will see, $\mathscr{F}_{i+1}$ is formed from $\mathscr{F}_i$ by splitting off a vertex v_{i+1} from α so that we end up with α and v_{i+1}. Let

$$
\begin{aligned}
\mathscr{A}_i \quad = \quad & \{H_{\mathscr{F}_i}(\alpha)\} \\
\bigcup \quad & \{H_{\mathscr{F}_i(1)}(\alpha), \ldots, H_{\mathscr{F}_i(k)}(\alpha)\} \\
\bigcup \quad & \{H_{\mathscr{F}_i(j)}(\alpha, e) : e \in E_{\mathscr{F}_i(j)}(\alpha), 1 \leqslant j \leqslant k\}, \quad (8.1)
\end{aligned}
$$

and let

$$
\begin{aligned}
\mathscr{B}_i \quad = \quad & \{H_{\mathscr{F}_i}(\alpha^t, U) : t \geqslant 1, U \subset V_i \backslash \{\alpha\}\} \\
\bigcup \quad & \{H_{\mathscr{F}_i(j)}(\alpha^t, U) : t \geqslant 1, U \subset V_i \backslash \{\alpha\}, 1 \leqslant j \leqslant k\}. \quad (8.2)
\end{aligned}
$$

It is easy to see that both $\mathscr{A}_i$ and $\mathscr{B}_i$ are laminar families of subsets of $H(\mathscr{F}_i, \alpha)$. Therefore, by Lemma 1.3, there exists a subset Z_i of $H(\mathscr{F}_i, \alpha)$ such that

$$|Z_i \cap P| \approx |P|/g_i(\alpha), \text{ for every } P \in \mathscr{A}_i \cup \mathscr{B}_i. \tag{8.3}$$

Let v_{i+1} be a vertex which does not belong to V_i and let $V_{i+1} = V_i \cup \{v_{i+1}\}$. Let Φ_{i+1} be the function from V_{i+1} onto V_i such that $\Phi_{i+1}(v) = v$ for every $v \in V_i$ and $\Phi_{i+1}(v_{i+1}) = \alpha$. Let $\mathscr{F}_{i+1}$ be the detachment of $\mathscr{F}_i$ under Φ_{i+1} such that $V(\mathscr{F}_{i+1}) = V_{i+1}$, and

$$H_{\mathscr{F}_{i+1}}(v_{i+1}) = Z_i, H_{\mathscr{F}_{i+1}}(\alpha) = H_{\mathscr{F}_i}(\alpha) \backslash Z_i. \tag{8.4}$$

In fact, $\mathscr{F}_{i+1}$ is obtained from $\mathscr{F}_i$ by splitting α into two vertices α and v_{i+1} in such a way that hinges which were incident with α in $\mathscr{F}_i$ become incident in $\mathscr{F}_{i+1}$ with α or v_{i+1} according as they do not or do belong to Z_i, respectively. Obviously, Ψ_i is an amalgamation function from $\mathscr{F}_{i+1}$ into $\mathscr{F}_i$. Let g_{i+1} be the function from V_{i+1} into $\mathbb{N}$, such that $g_{i+1}(v_{i+1}) = 1, g_{i+1}(\alpha) = g_i(\alpha) - 1$, and $g_{i+1}(v) = g_i(v)$ for every $v \in V_i \backslash \{\alpha\}$. This finishes the construction of $\mathscr{F}_{i+1}$.

8.5.2 Relations Between $\mathscr{F}_{i+1}$ and $\mathscr{F}_i$

The hypergraph $\mathscr{F}_{i+1}$, satisfies the following conditions:

(B1) $d_{\mathscr{F}_{i+1}}(\alpha) \approx d_{\mathscr{F}_i}(\alpha) g_{i+1}(\alpha)/g_i(\alpha)$;

(B2) $d_{\mathscr{F}_{i+1}}(v_{i+1}) \approx d_{\mathscr{F}_i}(\alpha)/g_i(\alpha)$;

(B3) $m_{\mathscr{F}_{i+1}}(v_{i+1}^s, \alpha^t, U) = 0$ for $s \geqslant 2$, and $t \geqslant 0$;

(B4) $m_{\mathscr{F}_{i+1}}(\alpha^t, U) \approx m_{\mathscr{F}_i}(\alpha^t, U)(g_i(\alpha) - t)/g_i(\alpha)$ for each $U \subset V_i \backslash \{\alpha\}$, and $g_i(\alpha) \geqslant t \geqslant 1$;

(B5) $m_{\mathscr{F}_{i+1}}(\alpha^t, v_{i+1}, U) \approx (t+1)m_{\mathscr{F}_i}(\alpha^{t+1}, U)/g_i(\alpha)$ for each $U \subset V_i \backslash \{\alpha\}$, and $t \geqslant 0$.

Proof. Since $H_{\mathscr{F}_i}(\alpha) \in \mathscr{A}_i$, from (8.4) it follows that

$$
\begin{aligned}
d_{\mathscr{F}_{i+1}}(v_{i+1}) &= |H_{\mathscr{F}_{i+1}}(v_{i+1})| = |Z_i| = |Z_i \cap H_{\mathscr{F}_i}(\alpha)| \\
&\approx |H_{\mathscr{F}_i}(\alpha)|/g_i(\alpha) = d_{\mathscr{F}_i}(\alpha)/g_i(\alpha), \\
d_{\mathscr{F}_{i+1}}(\alpha) &= |H_{\mathscr{F}_{i+1}}(\alpha)| = |H_{\mathscr{F}_i}(\alpha)| - |Z_i| \\
&\approx d_{\mathscr{F}_i}(\alpha) - d_{\mathscr{F}_i}(\alpha)/g_i(\alpha) = (g_i(\alpha) - 1)d_{\mathscr{F}_i}(\alpha)/g_i(\alpha) \\
&= d_{\mathscr{F}_i}(\alpha)g_{i+1}(\alpha)/g_i(\alpha).
\end{aligned}
$$

This proves (B1) and (B2).

If $t \geqslant 1, U \subset V_i\backslash\{\alpha\}$, and $e \in E_{\mathscr{F}_i}(\alpha^t, U)$, then for some j, $1 \leqslant j \leqslant k$, $H_{\mathscr{F}_i(j)}(\alpha, e) \in \mathscr{A}_i$, so

$$
\left|Z_i \cap H_{\mathscr{F}_i(j)}(\alpha, e)\right| \approx |H_{\mathscr{F}_i(j)}(\alpha, e)|/g_i(\alpha) = t/g_i(\alpha) \leqslant 1,
$$

where the inequality implies from the fact that g_i is simple. Therefore either $|Z_i \cap H_{\mathscr{F}_i(j)}(\alpha, e)| = 1$ and consequently $e \in E_{\mathscr{F}_{i+1}}(\alpha^{t-1}, v_{i+1}, U)$ or $Z_i \cap H_{\mathscr{F}_i(j)}(\alpha, e) = \varnothing$ and consequently $e \in E_{\mathscr{F}_{i+1}}(\alpha^t, U)$. Therefore

$$
m_{\mathscr{F}_{i+1}}(v_{i+1}^s, \alpha^r, U) = 0,
$$

for $r \geqslant 1$, and $s \geqslant 2$. This proves (B3). Moreover, since $H_{\mathscr{F}_i}(\alpha^t, U) \in \mathscr{B}_i$, we have

$$
\begin{aligned}
m_{\mathscr{F}_{i+1}}(\alpha^{t-1}, v_{i+1}, U) &= |Z_i \cap H_{\mathscr{F}_i}(\alpha^t, U)| \approx |H_{\mathscr{F}_i}(\alpha^t, U)|/g_i(\alpha) = t m_{\mathscr{F}_i}(\alpha^t, U)/g_i(\alpha), \\
m_{\mathscr{F}_{i+1}}(\alpha^t, U) &\approx m_{\mathscr{F}_i}(\alpha^t, U) - |H_{\mathscr{F}_i}(\alpha^t, U)|/g_i(\alpha) = m_{\mathscr{F}_i}(\alpha^t, U) - t m_{\mathscr{F}_i}(\alpha^t, U)/g_i(\alpha) \\
&= m_{\mathscr{F}_i}(\alpha^t, U)(g_i(\alpha) - t)/g_i(\alpha).
\end{aligned}
$$

This proves (B4) and (B5). $\qquad\square$

Let us fix $j \in \{1, \ldots, k\}$. It is enough to replace $\mathscr{F}_i$ with $\mathscr{F}_i(j)$ in the statement and the proof of (B1)–(B5) to obtain companion conditions, say (C1)–(C5) for each color class.

8.5.3 Relations Between $\mathscr{F}_i$ and $\mathscr{F}$

Recall that $\Psi_i = \Phi_0 \ldots \Phi_i$, that $\Phi_0 : V \to V$, and that $\Phi_i : V_i \to V_{i-1}$ for $i > 0$. Therefore $\Psi_i : V_i \to V$ and thus $\Psi_i^{-1} : V \to V_i$. Now we use (B1)–(B5) to prove that the hypergraph $\mathscr{F}_i$ satisfies the following conditions for $0 \leqslant i \leqslant n$:

(D1) $d_{\mathscr{F}_i}(v)/g_i(v) \approx d_{\mathscr{F}}(u)/g(u)$ for each $u \in V$ and each $v \in \Psi_i^{-1}(u)$;

(D2) $m_{\mathscr{F}_i}(u_1^{a_1}, U_1, \ldots, u_r^{a_r}, U_r)/\Pi_{j=1}^r \binom{g_i(u_j)}{a_j} \approx m_{\mathscr{F}}(u_1^{m_1}, \ldots, u_r^{m_r})/\Pi_{j=1}^r \binom{g(u_j)}{m_j}$ for distinct vertices $u_1, \ldots, u_r \in V$, $a_j \geqslant 0$, $U_j \subset \Psi_i^{-1}(u_j)\backslash\{u_j\}$ with $1 \leqslant m_j = a_j + |U_j| \leqslant g(u_j)$, $1 \leqslant j \leqslant r$ if $g_i(u_j) \geqslant a_j$, $1 \leqslant j \leqslant r$.

Proof. The proof is by induction. Recall that $\mathscr{F}_0 = \mathscr{F}$, and $g_0(u) = g(u)$ for $u \in V$. Thus, (D1) and (D2) are trivial for $i = 0$. Now we will show that if $\mathscr{F}_i$ satisfies the conditions (D1) and (D2) for some $i < n$, then $\mathscr{F}_{i+1}$ satisfies these conditions by replacing i with $i + 1$; we denote the corresponding conditions for $\mathscr{F}_{i+1}$ by (D1)$'$ and (D2)$'$.

Let $u \in V$. If $g_{i+1}(u) = g_i(u)$, then (D1)$'$ is obviously true. So we just check (D1)$'$ in the case where $u = \alpha$. By (B1) and (D1) we have $d_{\mathscr{F}_{i+1}}(\alpha)/g_{i+1}(\alpha) \approx d_{\mathscr{F}_i}(\alpha)/g_i(\alpha) \approx d_{\mathscr{F}}(\alpha)/g(\alpha)$. Moreover, from (B2) and (D1) it follows that $d_{\mathscr{F}_{i+1}}(v_{i+1}) \approx d_{\mathscr{F}_i}(\alpha)/g_i(\alpha) \approx d_{\mathscr{F}}(\alpha)/g(\alpha)$. Since in forming $\mathscr{F}_{i+1}$ no edge is detached from v_r for each $v_r \in \Psi_i^{-1}(\alpha)\backslash\{\alpha\}$, we have $d_{\mathscr{F}_{i+1}}(v_r) = d_{\mathscr{F}_i}(v_r)$. Therefore $d_{\mathscr{F}_{i+1}}(v_r) = d_{\mathscr{F}_i}(v_r) \approx d_{\mathscr{F}}(\alpha)/g(\alpha)$ for each $v_r \in \Psi_i^{-1}(\alpha)\backslash\{\alpha\}$. This proves (D1)$'$. Let $u_1, \ldots, u_r$ be distinct vertices in V. If $g_{i+1}(u_j) = g_i(u_j)$ for $1 \leqslant j \leqslant r$, then (D2)$'$ is clearly true. Therefore, in order to prove (D2)$'$, without loss of generality we may assume that $g_{i+1}(u_1) = g_i(u_1) - 1$ (so $\alpha = u_1$ and $v_{i+1} \in \Psi_i^{-1}(u_1)$). First, note that for integers a, b we always have $(a - b)\binom{a}{b} = a\binom{a-1}{b} = (b + 1)\binom{a}{b+1}$. If $v_{i+1} \notin U_1$,

we have

$$\frac{m_{\mathscr{F}_{i+1}}(u_1^{a_1}, U_1, \ldots, u_r^{a_r}, U_r)}{\prod_{j=1}^r \binom{g_{i+1}(u_j)}{a_j}} \overset{\text{(B4)}}{\approx} \frac{m_{\mathscr{F}_i}(u_1^{a_1}, U_1, \ldots, u_r^{a_r}, U_r)(g_i(u_1) - a_1)/g_i(u_1)}{\binom{g_i(u_1)-1}{a_1}\prod_{j=2}^r \binom{g_i(u_j)}{a_j}}$$

$$= \frac{m_{\mathscr{F}_i}(u_1^{a_1}, U_1, \ldots, u_r^{a_r}, U_r)(g_i(u_1) - a_1)/g_i(u_1)}{(g_i(u_1) - a_1)/g_i(u_1)\binom{g_i(u_1)}{a_1}\prod_{j=2}^r \binom{g_i(u_j)}{a_j}}$$

$$= \frac{m_{\mathscr{F}_i}(u_1^{a_1}, U_1, \ldots, u_r^{a_r}, U_r)}{\prod_{j=1}^r \binom{g_i(u_j)}{a_j}}$$

$$\overset{\text{(D2)}}{\approx} \frac{m_{\mathscr{F}}(u_1^{m_1}, \ldots, u_r^{m_r})}{\prod_{j=1}^r \binom{g(u_j)}{m_j}}.$$

If $v_{i+1} \in U_1$, we have

$$\frac{m_{\mathscr{F}_{i+1}}(u_1^{a_1}, U_1, \ldots, u_r^{a_r}, U_r)}{\prod_{j=1}^r \binom{g_{i+1}(u_j)}{a_j}} \overset{\text{(B5)}}{\approx} \frac{m_{\mathscr{F}_i}(u_1^{a_1+1}, U_1\backslash\{v_{i+1}\}, \ldots, u_r^{a_r}, U_r)(a_1 + 1)/g_i(u_1)}{\binom{g_i(u_1)-1}{a_1}\prod_{j=2}^r \binom{g_i(u_j)}{a_j}}$$

$$= \frac{m_{\mathscr{F}_i}(u_1^{a_1+1}, U_1\backslash\{v_{i+1}\}, \ldots, u_r^{a_r}, U_r)}{g_i(u_1)/(a_1 + 1)\binom{g_i(u_1)-1}{a_1}\prod_{j=2}^r \binom{g_i(u_j)}{a_j}}$$

$$= \frac{m_{\mathscr{F}_i}(u_1^{a_1+1}, U_1\backslash\{v_{i+1}\}, \ldots, u_r^{a_r}, U_r)}{\binom{g_i(u_1)}{a_1+1}\prod_{j=2}^r \binom{g_i(u_j)}{a_j}}$$

$$\overset{\text{(D2)}}{\approx} \frac{m_{\mathscr{F}}(u_1^{m_1}, \ldots, u_r^{m_r})}{\prod_{j=1}^r \binom{g(u_j)}{m_j}}.$$

This proves (D2)$'$. $\qquad\square$

Let us fix $j \in \{1, \ldots, k\}$. It is enough to replace $\mathscr{F}$ with $\mathscr{F}(j)$, $\mathscr{F}_i$ with $\mathscr{F}_i(j)$, $\mathscr{F}_{i+1}$ with $\mathscr{F}_{i+1}(j)$, and (Bi) with (Ci) for $i = 1, 2, 4, 5$, in the statement and the proof of (D1) and (D2) to obtain companion conditions, say (E1) and (E2) for each color class.

8.5.4 $\mathscr{G}$ satisfies (A1)–(A4)

Recall that $\mathscr{G} = \mathscr{F}_n$ and $g_n(u) = 1$ for every $u \in V$, therefore when $i = n$, (D1) implies (A1). Moreover, if we let $i = n$ in (D2), we have $a_j \in \{0, 1\}$ for $1 \leqslant j \leqslant r$ and thus $\prod_{j=1}^r \binom{g_i(u_j)}{a_j} = \prod_{j=1}^r \binom{1}{a_j} = 1$. This proves (A3). By a similar argument, one can prove (A2) and (A4), and this completes the proof of Theorem 8.2. $\qquad\square$

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8.6 Corollaries

For a matrix A, let A_j denote the jth column of A, and let $s(A)$ denote the sum of all the elements of A. Let $R = [r_1 \ldots r_k]^T$ (or $R^T = [r_i]_{1 \times k}$), $\Lambda = [\lambda_1 \ldots \lambda_m]^T$ and $H = [h_1 \ldots h_m]^T$ be three column vectors with $r_i, \lambda_i \in \mathbb{N}$, and $h_i \in \{1, \ldots, n\}$ for $1 \leqslant i \leqslant m$, such that $h_1 \ldots, h_m$ are distinct. Let ΛK_n^H denote a hypergraph with vertex set V, $|V| = n$, such that there are λ_i edges of size h_i incident with every h_i vertices for $1 \leqslant i \leqslant m$. A hypergraph $\mathcal{G}$ is said to be k-*regular* if every vertex has degree k. A k-*factor* of $\mathcal{G}$ is a k-regular spanning sub-hypergraph of $\mathcal{G}$. An R-*factorization* is a partition (decomposition) $\{F_1, \ldots, F_k\}$ of $E(\mathcal{G})$ in which F_i is an r_i-factor for $1 \leqslant i \leqslant k$. Notice that ΛK_n^H is $\sum_{i=1}^m \lambda_i \binom{n-1}{h_i-1}$-regular. We show that the obvious necessary conditions for the existence of an R-factorization of ΛK_n^H, are also sufficient.

Theorem 8.3. ΛK_n^H *is R-factorizable if and only if* $s(R) = \sum_{i=1}^m \lambda_i \binom{n-1}{h_i-1}$, *and there exists a non-negative integer matrix* $A = [a_{ij}]_{k \times m}$ *such that* $AH = nR$, *and* $s(A_j) = \lambda_j \binom{n}{h_j}$ *for* $1 \leqslant j \leqslant m$.

Proof. To prove the necessity, suppose that ΛK_n^H is R-factorizable. Since each r_i-factor is an r_i-regular spanning sub-hypergraph for $1 \leqslant i \leqslant k$, and ΛK_n^H is $\sum_{i=1}^m \lambda_i \binom{n-1}{h_i-1}$-regular, we must have $s(R) = \sum_{i=1}^k r_i = \sum_{i=1}^m \lambda_i \binom{n-1}{h_i-1}$. Let a_{ij} be the number of edges (counting multiplicities) of size h_j contributing to the ith factor for $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant m$. Since for $1 \leqslant j \leqslant m$, each edge of size h_j contributes h_j to the the sum of the degrees of the vertices in an r_i-factor for $1 \leqslant i \leqslant k$, we must have $\sum_{j=1}^m a_{ij} h_j = nr_i$ for $1 \leqslant i \leqslant k$ and $\sum_{i=1}^k a_{ij} = \lambda_j \binom{n}{h_j}$ for $1 \leqslant j \leqslant m$.

To prove the sufficiency, let $\mathcal{F}$ be a hypergraph consisting of a single vertex v with $m_{\mathcal{F}}(v^{h_j}) = \lambda_j \binom{n}{h_j}$ for $1 \leqslant j \leqslant m$. Note that $\mathcal{F}$ is an amalgamation of ΛK_n^H. Now we color the edges of $\mathcal{F}$ so that $m_{\mathcal{F}(i)}(v^{h_j}) = a_{ij}$ for $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant m$. This can be done, because:

$$\sum_{i=1}^k m_{\mathcal{F}(i)}(v^{h_j}) = \sum_{i=1}^k a_{ij} = \lambda_j \binom{n}{h_j} = m_{\mathcal{F}}(v^{h_j}) \quad \text{for } 1 \leqslant j \leqslant m.$$

Moreover,

$$d_{\mathscr{F}(i)}(v) = \sum_{j=1}^{m} a_{ij}h_j = nr_i \quad \text{for } 1 \leqslant i \leqslant k.$$

Let $g : V(\mathscr{F}) \to \mathbb{N}$ be a function so that $g(v) = n$. Since for $1 \leqslant i \leqslant m$, $h_i \leqslant n$, g is simple. By Theorem 8.2, there exists a simple g-detachment $\mathscr{G}$ of $\mathscr{F}$ with n vertices, say $v_1, \ldots, v_n$ such that by (A2), $d_{\mathscr{G}(i)}(v_j) \approx d_{\mathscr{F}(i)}(v)/g(v) = nr_i/n = r_i$ for $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant n$, and by (A3), for each $U \subset \{v_1, \ldots, v_n\}$ with $|U| = h_j$, $m_{\mathscr{G}}(U) \approx m_{\mathscr{F}}(v^{h_j})/\binom{n}{h_j} = \lambda_j \binom{n}{h_j}/\binom{n}{h_j} = \lambda_j$ for $1 \leqslant j \leqslant m$. Therefore $\mathscr{G} \cong \Lambda K_n^H$, and the ith color class induces an r_i-factor for $1 \leqslant i \leqslant k$. $\qquad\square$

In particular, if $m = 1$, $h := h_1$, $\lambda_1 = 1$, $r := r_1 = \cdots = r_k$, then Theorem 8.3 implies Baranyai's theorem: the complete h-uniform hypergraph K_n^h is r-factorizable if and only if $h \mid rn$ and $r \mid \binom{n-1}{h-1}$.

Now let $h_i \geqslant 2$ for $1 \leqslant i \leqslant m$, and let $\Lambda K_{p_1, \ldots, p_n}^H$ be a hypergraph with vertex partition $\{V_1, \ldots, V_n\}$, $|V_i| = p_i$ for $1 \leqslant i \leqslant n$ such that there are λ_i edges of size h_i incident with every h_i vertices, at most one vertex from each part for $1 \leqslant i \leqslant m$ (so no edge is incident with more than one vertex of a part). If $p_1 = \cdots = p_n := p$, we denote $\Lambda K_{p_1, \ldots, p_n}^H$ by $\Lambda K_{n \times p}^H$.

Theorem 8.4. $\Lambda K_{p_1, \ldots, p_n}^H$ *is R-factorizable if and only if $p_1 = \cdots = p_n := p$, $s(R) = \sum_{i=1}^{m} \lambda_i \binom{n-1}{h_i-1} p^{h_i-1}$, and there exists a non-negative integer matrix $A = [a_{ij}]_{k \times m}$ such that $AH = npR$, and $s(A_j) = \lambda_j \binom{n}{h_j} p^{h_j}$ for $1 \leqslant j \leqslant m$.*

Proof. To prove the necessity, suppose that $\Lambda K_{p_1, \ldots, p_n}^H$ is R-factorizable (so it is regular). Let u and v be two vertices from two different parts, say a^{th} and b^{th} parts, respectively. Since

$d(u) = d(v)$, we have

$$\sum_{1 \leqslant j \leqslant m} \lambda_j \sum_{\substack{1 \leqslant i_1 < \cdots < i_{h_j-1} \leqslant n \\ a \notin \{i_1,\ldots,i_{h_j-1}\}}} p_{i_1} \cdots p_{i_{h_j-1}} = \sum_{1 \leqslant j \leqslant m} \lambda_j \sum_{\substack{1 \leqslant i_1 < \cdots < i_{h_j-1} \leqslant n \\ b \notin \{i_1,\ldots,i_{h_j-1}\}}} p_{i_1} \cdots p_{i_{h_j-1}} \iff$$

$$\sum_{1 \leqslant j \leqslant m} \lambda_j \left(\sum_{\substack{1 \leqslant i_1 < \cdots < i_{h_j-1} \leqslant n \\ a \notin \{i_1,\ldots,i_{h_j-1}\}}} p_{i_1} \cdots p_{i_{h_j-1}} - \sum_{\substack{1 \leqslant i_1 < \cdots < i_{h_j-1} \leqslant n \\ b \notin \{i_1,\ldots,i_{h_j-1}\}}} p_{i_1} \cdots p_{i_{h_j-1}} \right) = 0 \iff$$

$$\sum_{1 \leqslant j \leqslant m} \lambda_j \left(p_b \sum_{1 \leqslant i_1 < \cdots < i_{h_j-2} \leqslant n} p_{i_1} \cdots p_{i_{h_j-2}} - p_a \sum_{1 \leqslant i_1 < \cdots < i_{h_j-2} \leqslant n} p_{i_1} \cdots p_{i_{h_j-2}} \right) = 0 \iff$$

$$(p_b - p_a) \sum_{1 \leqslant j \leqslant m} \lambda_j \sum_{1 \leqslant i_1 < \cdots < i_{h_j-2} \leqslant n} p_{i_1} \cdots p_{i_{h_j-2}} = 0 \iff$$

$$p_b = p_a.$$

Therefore, $p_1 = \cdots = p_n := p$. So $\Lambda K^H_{n \times p}$ is $\sum_{i=1}^m \lambda_i \binom{n-1}{h_i-1} p^{h_i-1}$-regular, and we must have $s(R) = \sum_{i=1}^k r_i = \sum_{i=1}^m \lambda_i \binom{n-1}{h_i-1} p^{h_i-1}$. Moreover, there must exist non-negative integers a_{ij}, $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant m$, such that $\sum_{j=1}^m a_{ij} h_j = npr_i$ for $1 \leqslant i \leqslant k$ and $\sum_{i=1}^k a_{ij} = \lambda_j \binom{n}{h_j} p^{h_j}$ for $1 \leqslant j \leqslant m$. We note that a_{ij} is in fact the number of edges (counting multiplicities) of size h_j contributing to the ith factor.

To prove the sufficiency, let $\Lambda^p = [p^{h_i} \lambda_i]^T_{1 \times m}$, and let $\mathscr{F} = \Lambda^p K^H_n$ with vertex set $V = \{v_1, \ldots, v_n\}$. Notice that $\mathscr{F}$ is an amalgamation of $\Lambda K^H_{n \times p}$. By Theorem 8.3, $\mathscr{F}$ is pR-factorizable. Therefore, we can color the edges of $\mathscr{F}$ so that

$$d_{\mathscr{F}(i)}(v) = pr_i \text{ for } v \in V, 1 \leqslant i \leqslant k.$$

Let $g : V \to \mathbb{N}$ be a function so that $g(v) = p$ for $v \in V$. Since $p \geqslant 1$, g is simple. By Theorem 8.2, there exists a simple g-detachment $\mathscr{G}$ of $\mathscr{F}$ with np vertices, say v_i is detached to $v_{i1}, \ldots, v_{ip}$ for $1 \leqslant i \leqslant n$, such that by (A2), $d_{\mathscr{G}(i)}(v_{ab}) \approx d_{\mathscr{F}(i)}(v_a)/g(v_a) = pr_i/p = r_i$ for $1 \leqslant i \leqslant k$, $1 \leqslant a \leqslant n$, $1 \leqslant b \leqslant p$, and by (A3), $m_{\mathscr{G}}(v_{a_1 b_1}, \ldots, v_{a_{h_j} b_{h_j}}) \approx m_{\mathscr{F}}(v_{a_1}, \ldots, v_{a_{h_j}})/p^{h_j} = p^{h_j} \lambda_j/p^{h_j} = \lambda_j$ for $1 \leqslant j \leqslant m$, $1 \leqslant a_1 < \cdots < a_{h_j} \leqslant n$, $1 \leqslant b_1, \ldots, b_{h_j} \leqslant p$. Therefore $\mathscr{G} \cong \Lambda K^H_{n \times p}$, and the ith color class induces an r_i-factor for $1 \leqslant i \leqslant k$. $\square$

In particular, if $m = 1$, $h := h_1$, $\lambda_1 = 1$, $r := r_1 = \cdots = r_k$, then Theorem 8.4 implies another one of Baranyai's theorems: the complete h-uniform n-partite hypergraph $K_{n \times p}^h$ is r-factorizable if and only if $h \mid npr$ and $r \mid \binom{n-1}{h-1} p^{h-1}$.

Let $J_k^T = [1 \ldots 1]_{1 \times k}$. For two column vectors $Q = [q_1 \ldots q_k]^T$, $R = [r_1 \ldots r_k]^T$, if $q_i \leqslant r_i$ for $1 \leqslant i \leqslant k$, we say that $Q \leqslant R$. For a hypergraph $\mathscr{G}$, a (q, r)-factor is a spanning sub-hypergraph in which

$$q \leqslant d(v) \leqslant r \text{ for each } v \in V(\mathscr{G}).$$

A (Q, R)-factorization is a partition $\{F_1, \ldots, F_k\}$ of $E(\mathscr{G})$ in which F_i is a (q_i, r_i)-factor for $1 \leqslant i \leqslant k$. An almost k-factor of $\mathscr{G}$ is $(k-1, k)$-factor. An almost R-factorization is an $(R - J_k, R)$-factorization. The proof of the following theorems are very similar to those of Theorem 8.3 and 8.4.

Theorem 8.5. ΛK_n^H is (Q, R)-factorizable if and only if $s(Q) \leqslant \sum_{i=1}^m \lambda_i \binom{n-1}{h_i-1} \leqslant s(R)$, and there exists a non-negative integer matrix $A = [a_{ij}]_{k \times m}$ such that $nQ \leqslant AH \leqslant nR$, and $s(A_j) = \lambda_j \binom{n}{h_j}$ for $1 \leqslant j \leqslant m$.

Proof. To prove the necessity, suppose that ΛK_n^H is (Q, R)-factorizable. Since ΛK_n^H is $\sum_{i=1}^m \lambda_i \binom{n-1}{h_i-1}$-regular, we must have $s(Q) = \sum_{i=1}^k q_i \leqslant \sum_{i=1}^m \lambda_i \binom{n-1}{h_i-1} \leqslant \sum_{i=1}^k r_i = s(R)$. Since for $1 \leqslant j \leqslant m$, each edge of size h_j contributes h_j to the the sum of the degrees of the vertices in (q_i, r_i)-factor for $1 \leqslant i \leqslant k$, there must exist non-negative integers a_{ij}, $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant m$, such that $nq_i \leqslant \sum_{j=1}^m a_{ij} h_j \leqslant nr_i$ for $1 \leqslant i \leqslant k$ and $\sum_{i=1}^k a_{ij} = \lambda_j \binom{n}{h_j}$ for $1 \leqslant j \leqslant m$.

To prove the sufficiency, let $\mathscr{F}$ be a hypergraph consisting of a single vertex v with $m_{\mathscr{F}}(v^{h_j}) = \lambda_j \binom{n}{h_j}$ for $1 \leqslant j \leqslant m$. Note that $\mathscr{F}$ is an amalgamation of ΛK_n^H. Now we color the edges of $\mathscr{F}$ so that $m_{\mathscr{F}(i)}(v^{h_j}) = a_{ij}$ for $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant m$. This can be done, because:

$$\sum_{i=1}^k m_{\mathscr{F}(i)}(v^{h_j}) = \sum_{i=1}^k a_{ij} = \lambda_j \binom{n}{h_j} = m_{\mathscr{F}}(v^{h_j}) \quad \text{for } 1 \leqslant j \leqslant m.$$

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Moreover,

$$nq_i \leqslant d_{\mathscr{F}(i)}(v) = \sum_{j=1}^{m} a_{ij} h_j \leqslant nr_i \quad \text{for } 1 \leqslant i \leqslant k.$$

Let $g : V(\mathscr{F}) \to \mathbb{N}$ be a function so that $g(v) = n$. Since for $1 \leqslant i \leqslant m$, $h_i \leqslant n$, g is simple. By Theorem 8.2, there exists a simple g-detachment $\mathscr{G}$ of $\mathscr{F}$ with n vertices, say $v_1, \ldots, v_n$ such that by (A2), $q_i = nq_i/n \leqslant d_{\mathscr{G}(i)}(v_j) \leqslant nr_i/n = r_i$ for $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant n$, and by (A3), for each $U \subset \{v_1, \ldots, v_n\}$ with $|U| = h_j$, $m_{\mathscr{G}}(U) \approx m_{\mathscr{F}}(v^{h_j})/\binom{n}{h_j} = \lambda_j \binom{n}{h_j}/\binom{n}{h_j} = \lambda_j$ for $1 \leqslant j \leqslant m$. Therefore $\mathscr{G} \cong \Lambda K_n^H$, and the ith color class induces a (q_i, r_i)-factor for $1 \leqslant i \leqslant k$. $\qquad\square$

Theorem 8.6. ΛK_n^H is almost R-factorizable if and only if $s(R) - k \leqslant \sum_{i=1}^{m} \lambda_i \binom{n-1}{h_i-1} \leqslant s(R)$, and there exists a non-negative integer matrix $A = [a_{ij}]_{k \times m}$ such that $n(R - J_k) \leqslant AH \leqslant nR$, and $s(A_j) = \lambda_j \binom{n}{h_j}$ for $1 \leqslant j \leqslant m$.

Proof. It is enough to take $Q = R - J_k$ in Theorem 8.5. $\qquad\square$

Theorem 8.7. $\Lambda K_{n \times p}^H$ is (Q, R)-factorizable if and only if $s(Q) \leqslant \sum_{i=1}^{m} \lambda_i \binom{n-1}{h_i-1} p^{h_i-1} \leqslant s(R)$, and there exists a non-negative integer matrix $A = [a_{ij}]_{k \times m}$ such that $npQ \leqslant AH \leqslant npR$, and $s(A_j) = \lambda_j \binom{n}{h_j} p^{h_j}$ for $1 \leqslant j \leqslant m$.

Proof. To prove the necessity, suppose that $\Lambda K_{n \times p}^H$ is (Q, R)-factorizable. Since $\Lambda K_{n \times p}^H$ is $\sum_{i=1}^{m} \lambda_i \binom{n-1}{h_i-1} p^{h_i-1}$-regular, we must have $s(Q) = \sum_{i=1}^{k} q_i \leqslant \sum_{i=1}^{m} \lambda_i \binom{n-1}{h_i-1} p^{h_i-1} \leqslant \sum_{i=1}^{k} r_i = s(R)$. Moreover, there must exist non-negative integers a_{ij}, $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant m$, such that $npq_i \leqslant \sum_{j=1}^{m} a_{ij} h_j \leqslant npr_i$ for $1 \leqslant i \leqslant k$ and $\sum_{i=1}^{k} a_{ij} = \lambda_j \binom{n}{h_j} p^{h_j}$ for $1 \leqslant j \leqslant m$.

To prove the sufficiency, let $\Lambda^p = [p^{h_i} \lambda_i]_{1 \times m}^T$, and let $\mathscr{F} = \Lambda^p K_n^H$ with vertex set $V = \{v_1, \ldots, v_n\}$. Notice that $\mathscr{F}$ is an amalgamation of $\Lambda K_{n \times p}^H$. By Theorem 8.5, $\mathscr{F}$ is (pQ, pR)-factorizable. Therefore, we can color the edges of $\mathscr{F}$ so that

$$pq_i \leqslant d_{\mathscr{F}(i)}(v) \leqslant pr_i \text{ for } v \in V, 1 \leqslant i \leqslant k.$$

Let $g : V \to \mathbb{N}$ be a function so that $g(v) = p$ for $v \in V$. Since $p \geqslant 1$, g is simple. By Theorem 8.2, there exists a simple g-detachment $\mathscr{G}$ of $\mathscr{F}$ with np vertices, say v_i is detached to $v_{i1}, \ldots, v_{ip}$ for $1 \leqslant i \leqslant n$, such that by (A2), $q_i = pq_i/p \leqslant d_{\mathscr{G}(i)}(v_{ab}) \leqslant pr_i/p = r_i$ for $1 \leqslant i \leqslant k$, $1 \leqslant a \leqslant n$, $1 \leqslant b \leqslant p$, and by (A3), $m_{\mathscr{G}}(v_{a_1 b_1}, \ldots, v_{a_{h_j} b_{h_j}}) \approx m_{\mathscr{F}}(v_{a_1}, \ldots, v_{a_{h_j}})/p^{h_j} = p^{h_j}\lambda_j/p^{h_j} = \lambda_j$ for $1 \leqslant j \leqslant m$, $1 \leqslant a_1 < \cdots < a_{h_j} \leqslant n$, $1 \leqslant b_1, \ldots, b_{h_j} \leqslant p$. Therefore $\mathscr{G} \cong \Lambda K_{n \times p}^H$, and the ith color class induces a (p_i, r_i)-factor for $1 \leqslant i \leqslant k$. $\qquad\square$

Theorem 8.8. $\Lambda K_{n \times p}^H$ *is almost R-factorizable if and only if $s(R) - k \leqslant \sum_{i=1}^{m} \lambda_i \binom{n-1}{h_i-1} p^{h_i-1} \leqslant s(R)$, and there exists a non-negative integer matrix $A = [a_{ij}]_{k \times m}$ such that $np(R - J_k) \leqslant AH \leqslant npR$, and $s(A_j) = \lambda_j \binom{n}{h_j} p^{h_j}$ for $1 \leqslant j \leqslant m$.*

Proof. It is enough to take $Q = R - J_k$ in Theorem 8.7. $\qquad\square$

Chapter 9

Connected Baranyai Theorem

9.1   Introduction

Let $K_n^h = (V, \binom{V}{h})$ be the complete h-uniform hypergraph on vertex set V with $|V| = n$. Baranyai showed that K_n^h can be expressed as the union of edge-disjoint r-regular factors if and only if h divides rn and r divides $\binom{n-1}{h-1}$. Using a new proof technique, in this chapter we prove that λK_n^h can be expressed as the union $\mathcal{G}_1 \cup \ldots \cup \mathcal{G}_k$ of k edge-disjoint factors, where for $1 \leqslant i \leqslant k$, $\mathcal{G}_i$ is r_i-regular, if and only if (i) h divides $r_i n$ for $1 \leqslant i \leqslant k$, and (ii) $\sum_{i=1}^k r_i = \lambda \binom{n-1}{h-1}$. Moreover, for any i ($1 \leqslant i \leqslant k$) for which $r_i \geqslant 2$, this new technique allows us to guarantee that $\mathcal{G}_i$ is connected, generalizing Baranyai's theorem, and answering a question by Katona.

A *hypergraph* $\mathcal{G}$ is a pair (V, E) where V is a finite set called the vertex set, E is the edge multiset, where every edge is itself a multi-subset of V. This means that not only can an edge occur multiple times in E, but also each vertex can have multiple occurrences within an edge. The total number of occurrences of a vertex v among all edges of E is called the *degree*, $d_{\mathcal{G}}(v)$ of v in $\mathcal{G}$. For a positive integer r, an r-factor in a hypergraph $\mathcal{G}$ is a spanning r-regular sub-hypergraph, and a partition of the edge set of $\mathcal{G}$ into (disjoint) r-factors is called an *r-factorizaton*. The hypergraph $K_n^h := (V, \binom{V}{h})$ with $|V| = n$ (by $\binom{V}{h}$ we mean the collection of all h-subsets of V) is called a *complete h-uniform* hypergraph. Avoiding trivial cases, we assume that $n > h$. Baranyai proved that:

Theorem 9.1. (Baranyai [15]) K_n^h *is r-factorizable if and only if* $h \mid rn$ *and* $r \mid \binom{n-1}{h-1}$.

It is natural to ask if we can obtain a connected factorization; that is, a factorization in which each factor is a connected hypergraph. Let m be the least common multiple of h and

n, and let $a = m/h$. Define the set of edges

$$\mathcal{K} = \{\{1, \ldots, h\}, \{h+1, \ldots, 2h\}, \ldots, \{(a-1)h+1, (a-1)h+2, \ldots, ah\}\},$$

where the elements of the edges are considered mod n. The families obtained from $\mathcal{K}$ by permuting the elements of the underlying set $\{n\}$ are called *wreaths*. If h divides n, then a wreath is just a partition. Baranyai and Katona conjectured that the edge set of K_n^h can be decomposed into disjoint wreaths [54]. In connection with this conjecture, Katona (private communication) suggested the problem of finding a connected factorization for K_n^h. In this chapter, we solve this problem.

An $(r_1, \ldots, r_k)$-*factorization* of $\mathcal{G}$ is a partition of the edge set of $\mathcal{G}$ into $F_1, \ldots, F_k$ where F_i is an r_i-factor for $1 \leqslant i \leqslant k$. If we replace every edge e of K_n^h by λ copies of e, then we denote the new hypergraph by λK_n^h. In this chapter, the main result is the following theorem:

Theorem 9.2. λK_n^h *is* $(r_1, \ldots, r_k)$-*factorizable if and only if* $h \mid r_i n$ *for* $1 \leqslant i \leqslant k$, *and* $\sum_{i=1}^{k} r_i = \lambda \binom{n-1}{h-1}$. *Moreover, for* $1 \leqslant i \leqslant k$, *if* $r_i \geqslant 2$, *then we can guarantee that the* r_i-*factor is connected.*

While this generalizes Baranyai's result in various ways, we note that the major extension is the guarantee of connectivity for the r-factors when $r \geqslant 2$. In particular if $\lambda = 1$, and $h = r_1 = \cdots = r_k = 2$, Theorem 9.2 implies the classical result of Walecki [64] that the edge set of K_n can be partitioned into Hamiltonian cycles if and only if n is odd. Here we list some other interesting special consequences of Theorem 9.2:

Corollary 9.3. K_n^h *is connected* 2-*factorizable if and only if* $\binom{n-1}{h-1}$ *is even and* $h \mid 2n$.

Corollary 9.4. K_n^h *has a connected* $\frac{h}{\gcd(n,h)}$-*factorization.*

We note that the idea behind the proof of Theorem 9.2 is based on the amalgamation technique [44, 70]. Preliminaries are given in Section 9.2, followed by the proof of Theorem 9.2 in Section 9.3.

We end this section with some notation we need to be able to describe hypergraphs that arise in this setting.

Let $\mathcal{G} = (V, E)$ be a hypergraph with $\alpha \in V$, and let $U = \{u_1, \ldots, u_z\} \subset V \backslash \{\alpha\}$. Recall that each edge is a multi-subset of V. We abbreviate an edge of the form $\{\underbrace{\alpha, \ldots, \alpha}_{p}, u_1, \ldots, u_z\}$ to $\{\alpha^p, u_1, \ldots, u_z\}$. An *h-loop* incident with α is an edge of the form $\{\alpha^h\}$, and $m(\alpha^p, U)$ denotes the multiplicity of an edge of the form $\{\alpha^p\} \cup U$. A *k-edge-coloring* of $\mathcal{G}$ is a mapping $f : E \to C$, where C is a set of k *colors* (often we use $C = \{1, \ldots, k\}$), and the edges of one color form a *color class*. The sub-hypergraph of $\mathcal{G}$ induced by the color class i is denoted by $\mathcal{G}_i$, abbreviate $d_{\mathcal{G}_i}(\alpha)$ to $d_i(\alpha)$ and $m_{\mathcal{G}_i}(\alpha^p, U)$ to $m_i(\alpha^p, U)$.

9.2 Preliminaries

A vertex α in a connected hypergraph $\mathcal{G}$ is a *cut vertex* if there exist two non-trivial sub-hypergraphs I, J of $\mathcal{G}$ such that $I \cup J = \mathcal{G}$, $V(I \cap J) = \alpha$ and $E(I \cap J) = \varnothing$. A non-trivial connected sub-hypergraph W of a connected hypergraph $\mathcal{G}$ is said to be an α-wing of $\mathcal{G}$, if α is not a cut vertex of W and no edge in $E(\mathcal{G}) \backslash E(W)$ is incident with a vertex in $V(W) \backslash \{\alpha\}$. The set of all α-wings of $\mathcal{G}$ is denoted by $\mathscr{W}_\alpha(\mathcal{G})$. Figure 9.1 illustrates an example of a hypergraph and the set of all its α-wings.

If the multiplicity of a vertex α in an edge e is p, we say that α is *incident* with p distinct objects, say $h_1, \ldots, h_p$. We call these objects *hinges*, and we say that e is *incident* with $h_1, \ldots, h_p$. The set of all hinges in $\mathcal{G}$ incident with α is denoted by $H_\mathcal{G}(\alpha)$; so $|H_\mathcal{G}(\alpha)|$ is in fact the degree of α.

Intuitively speaking, an α-*detachment* of $\mathcal{G}$ is a hypergraph obtained by splitting a vertex α into one or more vertices and sharing the incident hinges and edges among the subvertices. That is, in an α-detachment $\mathcal{G}'$ of $\mathcal{G}$ in which we split α into α and β, an edge of the form $\{\alpha^p, u_1, \ldots, u_z\}$ in $\mathcal{G}$ will be of the form $\{\alpha^{p-i}, \beta^i, u_1, \ldots, u_z\}$ in $\mathcal{G}'$ for some i, $0 \leqslant i \leqslant p$. Note that a hypergraph and its detachments have the same hinges. Whenever it is not ambiguous, we use d', m', etc. for degree, multiplicity and other hypergraph parameters in $\mathcal{G}'$. Also, for
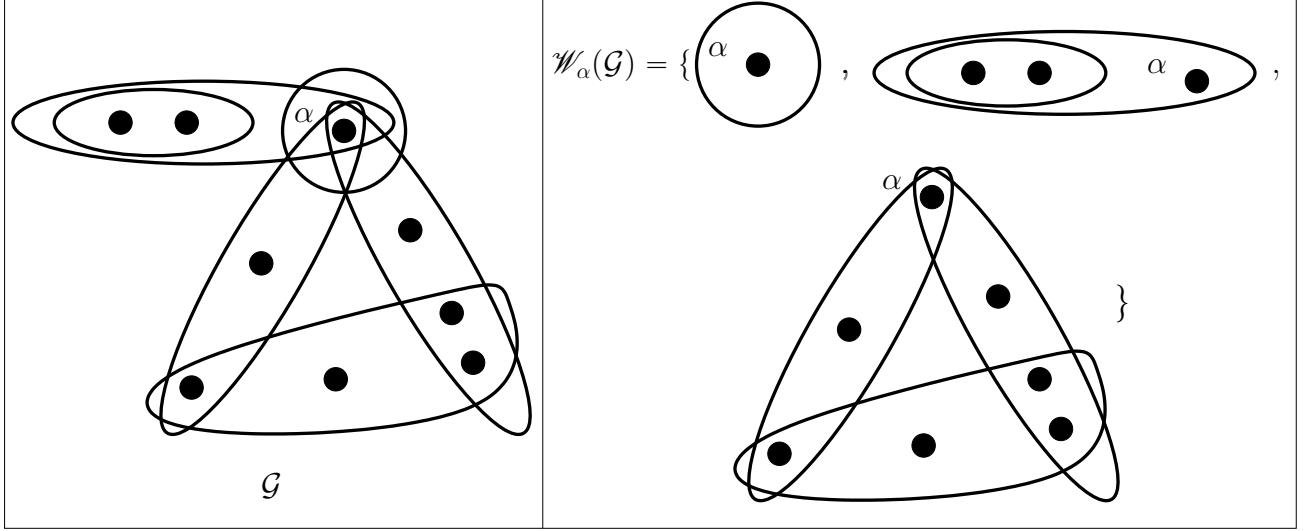
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Figure 9.1: A hypergraph $\mathcal{G}$ and the set of all its α-wings

an α-wing W in $\mathcal{G}$ and an α-detachment $\mathcal{G}'$, let W' denote the sub-hypergraph of $\mathcal{G}'$ whose hinges are the same as those in W.

We shall present three lemmas, all of which follow immediately from definitions.

Lemma 9.5. *Let $\mathcal{G}$ be a connected hypergraph. Let $\mathcal{G}'$ be an α-detachment of $\mathcal{G}$ obtained by splitting a vertex α into two vertices α and β. Then $\mathcal{G}'$ is connected if and only if for some α-wing $W \in \mathscr{W}_\alpha(\mathcal{G})$ with $d_W(\alpha) \geqslant 2$,*

$$1 \leqslant |H_W(\alpha) \cap H_{\mathcal{G}'}(\beta)| < d_W(\alpha).$$

Informally speaking, Lemma 9.5 says that for some α-wing W with $d_W(\alpha) \geqslant 2$, at least one but not all the hinges incident with α in W must be incident with β in $\mathcal{G}'$.

A family $\mathscr{A}$ of sets is *laminar* if, for every pair A, B of sets belonging to $\mathscr{A}$: $A \subset B$, or $B \subset A$, or $A \cap B = \varnothing$.

Let us fix a vertex α of a k-edge-colored hypergraph $\mathcal{G} = (V, E)$. For $1 \leqslant i \leqslant k$, let H_i be the set of hinges each of which is incident with both α and an edge of color i (so $d_i(\alpha) = |H_i|$). For any edge $e \in E$, let H_e be the collection of hinges incident with both α

and e. Clearly, if e is of color i, then $H_e \subset H_i$. For an α-wing W, let $H_W = H_W(\alpha)$. For $1 \leqslant i \leqslant k$, let

$$H^i = \bigcup_{W \in \mathscr{W}_\alpha(\mathcal{G}_i), d_W(\alpha) \geqslant 2} H_W.$$

Lemma 9.6. *Let*

$$
\begin{aligned}
\mathscr{A} \quad &= \quad \{H_1, \ldots, H_k\} \cup \{H_W : W \in \mathscr{W}_\alpha(\mathcal{G}_i), 1 \leqslant i \leqslant k\} \\
&\cup \quad \{H^1, \ldots, H^k\} \cup \{H_e : e \in E\}.
\end{aligned}
$$

Then $\mathscr{A}$ is a laminar family of subsets of $H(\alpha)$.

For each $p \geqslant 1$, and each $U \subset V \backslash \{\alpha\}$, let H_p^U be the set of hinges each of which is incident with both α and an edge of the form $\{\alpha^p\} \cup U$ in $\mathcal{G}$ (so $|H_p^U| = pm(\alpha^p, U)$).

Lemma 9.7. *Let*

$$\mathscr{B} \quad = \quad \{H_p^U : p \geqslant 1, U \subset V \backslash \{\alpha\}\}.$$

Then $\mathscr{B}$ is a laminar family of subsets of $H(\alpha)$.

If x, y are real numbers, $x \approx y$ means $\lfloor y \rfloor \leqslant x \leqslant \lceil y \rceil$. We need the following powerful lemma:

Lemma 9.8. (Nash-Williams [70, Lemma 2]) *If $\mathscr{A}, \mathscr{B}$ are two laminar families of subsets of a finite set S, and n is a positive integer, then there exist a subset A of S such that*

$$|A \cap P| \approx |P|/n \text{ for every } P \in \mathscr{A} \cup \mathscr{B}.$$

9.3   Proof of the Main Theorem

To prove Theorem 9.2, first we look at the obvious necessary conditions:

Lemma 9.9. *If λK_n^h is connected $(r_1, \ldots, r_k)$-factorizable, then*

(i) $r_i \geqslant 2$ *for* $1 \leqslant i \leqslant k$,

(ii) $h \mid r_i n$ *for* $1 \leqslant i \leqslant k$, *and*

(iii) $\sum_{i=1}^{k} r_i = \lambda\binom{n-1}{h-1}$.

Proof. Suppose that λK_n^h is connected $(r_1, \ldots, r_k)$-factorizable. The necessity of (i) is suffi-ciently obvious. Since each edge contributes h to the the sum of the degrees of the vertices in an r_i-factor for $1 \leqslant i \leqslant k$, we must have (ii). Since each r_i-factor is an r_i-regular spanning sub-hypergraph for $1 \leqslant i \leqslant k$, and λK_n^h is $\lambda\binom{n-1}{h-1}$-regular, we must have (iii). $\qquad\square$

In order to get an inductive proof of Theorem 9.2 to work, we actually prove the following seemingly stronger result:

Theorem 9.10. *Let* $n, h, \lambda, k, r_1, \ldots, r_k$ *be positive integers with* $n > h$ *satisfying* (i)–(iii). *For any integer* $1 \leqslant \ell \leqslant n$, *there exists an* ℓ-*vertex* k-*edge-colored hypergraph* $\mathcal{G}$ *with vertex set* V ($\alpha \in V$) *such that*

$$d_i(u) = \begin{cases} r_i(n - \ell + 1) & \text{if } u = \alpha \\ r_i & \text{if } u \neq \alpha \end{cases} \quad \text{for } u \in V, 1 \leqslant i \leqslant k, \tag{9.1}$$

$$m(\alpha^p, U) = \lambda\binom{n - \ell + 1}{p} \text{ for } p \geqslant 0, U \subset V\backslash\{\alpha\} \text{ with } |U| = h - p, \text{ and} \tag{9.2}$$

$$\mathcal{G}_i \text{ is connected if } r_i \geqslant 2, \text{ for } 1 \leqslant i \leqslant k. \tag{9.3}$$

Remark 9.11. Theorem 9.2 follows from Theorem 9.10 in the case where $\ell = n$ as the following argument shows. If $\ell = n$, then conditions (9.1)–(9.3) imply that we have an n-vertex k-edge-colored hypergraph $\mathcal{G}$ in which the ith color class is r_i-regular by (9.1), and connected by (9.3). Moreover, (9.2) implies that for $U \subset V\backslash\{\alpha\}$, (i) $m(U) = \lambda\binom{1}{0} = \lambda$ if $|U| = h$ (when $p = 0$), (ii) $m(\alpha, U) = \lambda\binom{1}{1} = \lambda$ if $|U| = h - 1$ (when $p = 1$), and (iii) $m(\alpha^p, U) = \lambda\binom{1}{p} = 0$ for $p \geqslant 2$, and $|U| = h - p$. Therefore $\mathcal{G} \cong \lambda K_n^h$.

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Proof. The proof is by induction on ℓ. At each step we will assume not only that $\mathcal{G}$ is an ℓ-vertex k-edge-colored hypergraph with vertex set V ($\alpha \in V$) satisfying conditions (9.1)–(9.3), but that $\mathcal{G}$ also satisfies the two additional properties

$$|H_e| \leqslant n - \ell + 1 \text{ for each edge } e \text{ of } \mathcal{G}, \text{ and} \qquad (9.4)$$

$$\text{for } 1 \leqslant i \leqslant k, \text{ if } r_i \geqslant 2, \text{ then } \delta_i = r_i(n - \ell + 1) \qquad (9.5)$$

where for $1 \leqslant i \leqslant k$, $\delta_i = |H^i|$.

First consider the base case when $\ell = 1$. Let $\mathcal{F}$ be a hypergraph with a single vertex α incident with $\lambda \binom{n}{h}$ h-loops; i.e. $m(\alpha^h) = \lambda \binom{n}{h}$. Color the edges of $\mathcal{F}$ such that $m_i(\alpha^h) = r_i n/h$ for $1 \leqslant i \leqslant k$. This is possible since by (ii) $h \mid r_i n$, and by (iii) $\sum_{i=1}^{k} m_i(\alpha^h) = \sum_{i=1}^{k} r_i n/h = n/h \sum_{i=1}^{k} r_i = \lambda n \binom{n-1}{h-1}/h = \lambda \binom{n}{h} = m(\alpha^h)$. Also, note that for $\ell = 1$, the hypergraph $\mathcal{F}$ trivially satisfies (9.4), and since each h-loop is an α-wing, $\mathcal{F}$ also satisfies (9.5). Therefore, $\mathcal{F}$ shows that conditions (9.1)–(9.5) holds for $\ell = 1$.

Now suppose that $1 \leqslant \ell < n$, and that $\mathcal{G}$ satisfies (9.1)–(9.5). The proof is completed by showing that $\mathcal{G}$ has an $(\ell + 1)$-vertex α-detachment $\mathcal{G}'$ with vertex set $V' = V \cup \{\beta\}$ satisfying

$$|H'_e| \leqslant n - \ell \text{ for each edge } e \text{ of } \mathcal{G}', \qquad (9.6)$$

$$d'_i(u) = \begin{cases} r_i(n - \ell) & \text{if } u = \alpha \\ r_i & \text{if } u \neq \alpha \end{cases} \quad \text{for } u \in V', 1 \leqslant i \leqslant k, \qquad (9.7)$$

$$m'(\alpha^p, U) = \lambda \binom{n - \ell}{p} \text{ for } p \geqslant 0, U \subset V' \backslash \{\alpha\} \text{ with } |U| = h - p, \qquad (9.8)$$

$$\mathcal{G}'(i) \text{ is connected if } r_i \geqslant 2, \text{ for } 1 \leqslant i \leqslant k, \text{ and} \qquad (9.9)$$

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for $1 \leqslant i \leqslant k$, if $r_i \geqslant 2$ and if $\ell < n - 1$, then

$$\delta'_i = r_i(n - \ell). \tag{9.10}$$

Let $\mathscr{A}$ and $\mathscr{B}$ be the laminar families in Lemmas 9.6, and 9.7. By Lemma 9.8, there exists a subset A of $H(\alpha)$ such that

$$|A \cap P| \approx |P|/(n - \ell + 1) \text{ for every } P \in \mathscr{A} \cup \mathscr{B}. \tag{9.11}$$

Let $\mathcal{G}'$ be the hypergraph obtained from $\mathcal{G}$ by splitting α into two vertices α and β in such a way that hinges which were incident with α in $\mathcal{G}$ become incident in $\mathcal{G}'$ with α or β according as they do not or do belong to A, respectively. More precisely,

$$H'(\beta) = A, \quad H'(\alpha) = H(\alpha) \backslash A. \tag{9.12}$$

Since $H_i \in \mathscr{A}$ for $1 \leqslant i \leqslant k$, we have

$$
\begin{aligned}
d'_i(\beta) &= |A \cap H_i| \\
&\approx |H_i|/(n - \ell + 1) = d_i(\alpha)/(n - \ell + 1) \\
&= r_i(n - \ell + 1)/(n - \ell + 1) = r_i, \\
d'_i(\alpha) &= d_i(\alpha) - d'_i(\beta) \\
&= r_i(n - \ell + 1) - r_i = r_i(n - \ell),
\end{aligned}
$$

and for $u \notin \{\alpha, \beta\}$, $d'_i(u) = d_i(u) = r_i$. Therefore $\mathcal{G}'$ satisfies (9.7).

Let e be an edge in $\mathcal{G}$ incident with α. Then $H_e \in \mathscr{A}$, and so

$$|A \cap H_e| \approx |H_e|/(n - \ell + 1) \leqslant 1,$$

observing that the last inequality implies from (9.4). This means that either $A \cap H_e = \varnothing$ or $|A \cap H_e| = 1$. Therefore $m'(\beta^q, U) = 0$ for $q \geqslant 2$ and $U \subset V'$. Also, note that if $|H_e| = n - \ell + 1$, then $|A \cap H_e| = 1$ and thus $|H'_e| = n - \ell$, and if $|H_e| < n - \ell + 1$, then $|H'_e| \leqslant |H_e| \leqslant n - \ell$, both cases together proving (9.6).

Since for $p \geqslant 1$, and $U \subset V \backslash \{\alpha\}$, $H_p^U \in \mathscr{B}$, we have

$$
\begin{aligned}
m'(\alpha^{p-1}, \beta, U) &= |A \cap H_p^U| \\
&\approx |H_p^U|/(n - \ell + 1) = pm(\alpha^p, U)/(n - \ell + 1) \\
&= \lambda p \binom{n - \ell + 1}{p}/(n - \ell + 1) = \lambda \binom{n - \ell}{p - 1}, \\
m'(\alpha^p, U) &= m(\alpha^p, U) - m'(\alpha^{p-1}, \beta, U) \\
&= \lambda \binom{n - \ell + 1}{p} - \lambda \binom{n - \ell}{p - 1} = \lambda \binom{n - \ell}{p}.
\end{aligned}
$$

Therefore $\mathcal{G}'$ satisfies (9.8).

Let us fix an i, $1 \leqslant i \leqslant k$ such that $r_i \geqslant 2$. Let W be an α-wing of $\mathcal{G}_i$ with $d_W(\alpha) \geqslant 2$. Then $H_W \in \mathscr{A}$, and so

$$
|A \cap H_W| \approx |H_W|/(n - \ell + 1) = d_W(\alpha)/(n - \ell + 1), \tag{9.13}
$$

which implies that (noting that $n - \ell + 1 \geqslant 2$)

$$
|A \cap H_W| < |H_W|. \tag{9.14}
$$

Moreover,

$$
|A \cap H^i| \approx |H^i|/(n - \ell + 1) = \delta_i/(n - \ell + 1) = r_i \geqslant 2, \tag{9.15}
$$

and therefore there exists an α-wing W in $\mathcal{G}_i$ with $d_W(\alpha) \geqslant 2$, such that $A \cap H_W \neq \varnothing$. Therefore by Lemma 9.5, $\mathcal{G}'_i$ is connected.

Now, suppose that $\ell \leqslant n - 2$, or equivalently that $n - \ell + 1 \geqslant 3$. Since $\delta_i = d_i$, we have that for every $W \in \mathscr{W}_\alpha(\mathcal{G}_i)$, $d_W(\alpha) \geqslant 2$. So there is no α-wing W in $\mathcal{G}_i$ with $d_W(\alpha) = 1$. Let us fix an α-wing W in $\mathcal{G}_i$. There are two cases to consider:

- Case 1: If $|H_W| \geqslant 3$, then since $|A \cap H_W| \approx |H_W|/(n - \ell + 1) \leqslant |H_W|/3$, we have that $d'_{W'}(\alpha) \geqslant 2$, and thus $\delta'_i = d'_i(\alpha) = r_i(n - \ell)$. Note that W' is a sub-hypergraph of some α-wing S in $\mathcal{G}'$ with $d'_S(\alpha) \geqslant 2$.

- Case 2: If $|H_W| = 2$, then $|A \cap H_W| \approx |H_W|/(n - \ell + 1) = 2/(n - \ell + 1) \leqslant 2/3$. So $|A \cap H_W| \in \{0, 1\}$. If $A \cap H_W = \varnothing$, we are done. So let us assume that $|A \cap H_W| = 1$. Recall from (9.15) that $|A \cap H^i| \geqslant 2$. Therefore, there is another α-wing T in $\mathcal{G}_i$ with $|H_T| \geqslant 2$ such that $1 \leqslant |A \cap H_T| < |H_T|$. Therefore, there exists an α-wing S in $\mathcal{G}'$ with $W' \cup T' \subset S$, and $d'_S(\alpha) \geqslant 2$. Thus, in this case also we have $\delta'_i = \delta_i - r_i = r_i(n - \ell)$.

Therefore $\mathcal{G}'$ satisfies (9.10) and the proof is complete. $\qquad\square$

Chapter 10

Polynomial Time Parallelisms

10.1 Introduction

Throughout this chapter, k is a fixed positive integer. Let $P_k(n)$ be the collection of all k-element subsets of an n-set. A parallelism on $P_k(n)$ is an equivalence relation of $P_k(n)$ such that the members of each equivalence class form a partition of the n-set. Each equivalence class is called a *parallel class*, that is a set of n/k k-subsets each of which partitions the n-set. In connection with Kirkman's famous Fifteen Schoolgirls Problem [56], in 1850 Sylvester asked whether it is possible to find a parallelism on $P_k(n)$. Of course, it is necessary that k divides n, and the number of parallel classes would be $\frac{k}{n}\binom{n}{k} = \binom{n-1}{k-1}$. For those readers with (hyper)graph theory background, we note that finding a parallelism on $P_k(n)$ is equivalent to finding a 1-factorization for a complete k-uniform hypergraph on n vertices. Sylvester found a parallelism on $P_3(15)$. Several generalizations of this problem were studied during the last 70 years (see for example [71, 73]), but the general case remained open until 1973, when Baranyai settled this old problem [15]. Baranyai's elegant proof actually yields a method for constructing a parallelism on $P_k(n)$ recursively. However, this approach is not be very efficient, its complexity being exponential ($O(2^n)$) [53, p. 226]. Later Brouwer and Schrijver gave another proof for which the complexity is polynomial in $\binom{n}{k}$, the output size for the problem [25].

In this chapter, using our proof techniques of Chapter 5 and Chapter 8, we give a constructive proof of polynomial time complexity for the existence of a parallelism on $P_k(n)$. All known proofs including the one we shall present here, use a form of network flow; specifically, we use an approach which has been useful in finding detachments of graphs [70]. We note even though our proof is very similar to that of Brouwer and Schrijver [25], it is obtained

independently by simplifying the proofs of Theorem 10.3 and Nash-Williams lemma. For applications of parallelism on $P_k(n)$ in computer science and biology (such as parallel algorithms for tightening inter-atomic distance-bounds required for molecular conformation) see [33, 34, 72]. It is shown in [55] that there are 103000 isomorphic classes of parallelisms on $P_3(9)$.

10.2 Terminology

If x, y are real numbers, then $\lfloor x \rfloor$ and $\lceil x \rceil$ denote the integers such that $x - 1 < \lfloor x \rfloor \leqslant x \leqslant \lceil x \rceil < x + 1$, and $x \approx y$ means $\lfloor y \rfloor \leqslant x \leqslant \lceil y \rceil$. For a multiset A and $u \in A$, let $\mu_A(u)$ denote the multiplicity of u in A, and let $|A| = \sum_{u \in A} \mu_A(u)$. For multisets $A_1, \ldots, A_n$, we define $A = \bigcup_{i=1}^{n} A_i$ so that $\mu_A(u) = \sum_{i=1}^{n} \mu_{A_i}(u)$. We abbreviate $\{\underbrace{u, \ldots, u}_{r}\}$ to $\{u^r\}$; for example $\{u^2, v, w^2\} \cup \{u, w^2\} = \{u^3, v, w^4\}$.

A *circulation* on a digraph D is a mapping f from $E(D)$ to the reals satisfying conservation of flow at every vertex (see [84, chap. 7]). Let $N^-(v)$ and $N^+(v)$ denote the in-neighbor and out-neighbor of the vertex v, respectively. By (v, w) we mean a directed edge from v to w, and we abbreviate $f(\{v, w\})$ to $f(v, w)$. Let f be a circulation on a finite digraph D. Then it is known that there exists an integral circulation g (obtainable by a polytime algorithm) such that $g(e) \approx f(e)$ for every edge e (see for example [70, Lemma 1]).

10.3 Proofs

Theorem 10.1. *If k divides n, then the set of all $\binom{n}{k}$ k-subsets of an n-set may be partitioned into disjoint parallel classes $\mathcal{A}_i$, $i = 1, \ldots, \binom{n-1}{k-1}$.*

In order to get an inductive proof to work, rather than prove Theorem 10.1, we prove the stronger result Theorem 10.2 below. Let $m = n/k$, $M = \binom{n-1}{k-1}$. We use the term (m, k)-split of a set X for a multiset $\mathcal{A}$ of m k-multi-subsets of X whose union contains X. For an

integer ℓ and a set $\mathcal{A}_1, \ldots, \mathcal{A}_M$ of (m, k)-splits of $\{1, \ldots, \ell\}$, let $\mu_i^j = \sum_{\alpha \in \mathcal{A}_j} \mu_\alpha(i)$, and for $0 \leqslant r \leqslant k$ and $S \subset \{2, \ldots, \ell\}$ with $|S| = k - r$, let $\mu_S^r = \sum_{i=1}^M \mu_{\mathcal{A}_i}(S \cup \{1^r\})$.

Theorem 10.2. *For any integer ℓ, $1 \leqslant \ell \leqslant n$, there exist a set*

$$P = \{\mathcal{A}_1, \ldots, \mathcal{A}_M\}$$

of (m, k)-splits of $\{1, \ldots, \ell\}$ such that for $1 \leqslant j \leqslant M$, $\mu_1^j = n - \ell + 1$, $\mu_i^j = 1$ for $2 \leqslant i \leqslant \ell$, and $\mu_S^r = \binom{n - \ell + 1}{r}$ for $0 \leqslant r \leqslant k$ and each $S \subset \{2, \ldots, \ell\}$ with $|S| = k - r$. Moreover, P can be obtained by a polynomial time algorithm.

Proof. We prove our assertion by induction on ℓ. Notice that it is true for $\ell = 1$ by choosing $\mathcal{A}_1 = \cdots = \mathcal{A}_M = \{\{1^k\}^m\}$. Also notice that proving the case $\ell = n$ will prove Theorem 10.1, since $\mu_i^j = 1$ for $1 \leqslant i \leqslant n$ means that each $\mathcal{A}_j$ forms a partition of $\{1, \ldots, n\}$ for $j = 1, \ldots, M$, and $\mu_S^r = \binom{1}{r}$ for $0 \leqslant r \leqslant k$ and each $S \subset \{2, \ldots, n\}$ with $|S| = k - r$ means that every k-subset of the n-set appears exactly once in $\bigcup_{i=1}^M \mathcal{A}_i$ (the cases $r = 1$ and 0 consider subsets of $\{1, \ldots, n\}$ that do and do not contain 1 respectively).

Assume for some value $\ell < n$ that (m, k)-splits $\mathcal{A}_1, \ldots, \mathcal{A}_M$ exist with the required properties. We form a digraph D with vertex multiset $V = \{\sigma, \tau\} \cup \{y_1, \ldots, y_M\} \cup \{w_\alpha : \alpha \in \bigcup_{i=1}^M \mathcal{A}_i, \mu_\alpha(1) > 0\} \cup \{v_S^r : 0 \leqslant r \leqslant k, S \subset \{2, \ldots, \ell\}, |S| = k - r, \mu_S^r > 0\}$ and with a circulation f as follows. (Note that some α may occur several times in $\mathcal{A}_i$, the name w_α may occur on several vertices, so V is a multiset.)

- For $1 \leqslant i \leqslant M$, there is a directed edge from σ to y_i such that $f(\sigma, y_i) = 1$.

- For $1 \leqslant i \leqslant M$ and for each $\alpha \in \mathcal{A}_i$ with $\mu_\alpha(1) > 0$, there is a directed edge from y_i to w_α such that $f(y_i, w_\alpha) = \mu_\alpha(1)/(n - \ell + 1)$.

- For $0 \leqslant r \leqslant k$, and for $S \subset \{2, \ldots, \ell\}$ with $|S| = k - r$, if $\mu_S^r > 0$ then for each $\alpha = S \cup \{1^r\}$ in $\bigcup_{i=1}^M \mathcal{A}_i$, there is a directed edge from w_α to v_S^r such that $f(w_\alpha, v_S^r) = \mu_\alpha(1)/(n - \ell + 1)$, and there is a directed edge from v_S^r to τ such that $f(v_S^r, \tau) = \binom{n-\ell}{r-1}$.

- There is a directed edge from τ to σ such that $f(\tau, \sigma) = M$.

It is straightforward to check that f is a circulation (see Figure 10.1). There is an integer circulation g on D such that $g(e) \approx f(e)$ for each edge e in D. Let us fix an i, $1 \leqslant i \leqslant M$. For each $\alpha \in \mathcal{A}_i$ with $\mu_\alpha(1) > 0$, we have $g(y_i, w_\alpha) \in \{0, 1\}$. More important, since $g(\sigma, y_i) = 1$, there is exactly one α in $\mathcal{A}_i$ such that $g(y_i, w_\alpha) = 1$. Now, we obtain an (m, k)-split $\mathcal{A}'_i$ of the set $\{1, \ldots, \ell + 1\}$ by letting $\mathcal{A}'_i$ be obtained from $\mathcal{A}_i$ by replacing one 1 in $\alpha \in \mathcal{A}_i$ with $\ell + 1$ if $g(y_i, w_\alpha) = 1$. At this point, it is clear that our construction is of polynomial time complexity.

Finally, we show that the (m, k)-splits $\mathcal{A}'_1, \ldots, \mathcal{A}'_M$ satisfy the required properties. We define μ'^j_i and μ'^r_S for $\mathcal{A}'_1, \ldots, \mathcal{A}'_M$ similarly to the way we defined them for $\mathcal{A}_1, \ldots, \mathcal{A}_M$. Obviously, $\mu'^j_i = \mu^j_i = 1$ for $2 \leqslant i \leqslant \ell$, $1 \leqslant j \leqslant M$. Also $\mu'^j_{\ell+1} = 1$, and $\mu'^j_1 = \mu^j_1 - \mu'^j_{\ell+1} = n - \ell$ for $1 \leqslant j \leqslant M$. Moreover, for $0 \leqslant r \leqslant k$, $S \subset \{1, \ldots, \ell+1\}$ with $|S| = k - r$, if $\ell + 1 \in S$ then $\mu'^r_S = g(v^{r+1}_{S \setminus \{\ell+1\}}, \tau) = \binom{n-\ell}{r}$, and if $S \subset \{1, \ldots, \ell\}$ then $\mu'^r_S = \binom{n-\ell+1}{r} - g(v^r_S, \tau) = \binom{n-\ell+1}{r} - \binom{n-\ell}{r-1} = \binom{n-\ell}{r}$. This completes the proof. $\square$

Figure 10.1: Digraph D with circulation f

Chapter 11

Recent Results and Future Directions

In this chapter, I shall summarize my research, the significance of my results, and some motivation for future research. For each topic, I describe the problem with a brief discussion on proof techniques, applications and extensions together with related open problems.

11.1   Amalgamations and Connected Fair Detachments

A *detachment* of a graph H is a graph obtained from H by splitting some or all of its vertices into more than one vertex. If g is a function from $V(H)$ into $\mathbb{N}$, then a *g-detachment* of H is a detachment of H in which each vertex u of H splits into $g(u)$ vertices. H is an *amalgamation* of G if there exists a function ϕ called an *amalgamation function* from $V(G)$ onto $V(H)$ and a bijection $\phi' : E(G) \to E(H)$ such that e joining u and v is in $E(G)$ iff $\phi'(e)$ joining $\phi(u)$ and $\phi(v)$ is in $E(H)$.

A *k-edge-coloring* of G is a mapping $f : E \to C$, where E is the edge set of G and C is a set of k *colors* (we often use $C = \{1, \ldots, k\}$), and the edges of one color form a *color class*. In [5], we proved that for a given edge-colored graph there exists a detachment so that the result is a graph in which the edges are shared among the vertices in ways that are fair with respect to several notions of balance (such as between pairs of vertices, degrees of vertices in both the graph and in each color class, etc.). The connectivity of color classes is also addressed. Applications of this result are addressed in Sections 11.3 and 11.5. Most results in the literature on amalgamations focus on the detachments of amalgamated complete graphs and complete multipartite graphs. Many such results ([44, 48, 58, 61, 74], Theorem 1, Theorem 1, Theorem 3.1, Theorem 2.1 and Theorem 2.1, respectively) follow as

immediate corollaries to our main result in [5], which addresses amalgamations of graphs in general.

11.1.1 Edge-Coloring Techniques

An edge-coloring of a multigraph is (i) *equalized* if the number of edges colored with any two colors differs by at most one, (ii) *balanced* if for each pair of vertices, among the edges joining the pair, the number of edges of each color differs by at most one from the number of edges of each other color, and (iii) *equitable* if, among the edges incident with each vertex, the number of edges of each color differs by at most one from the number of edges of each other color. In [80, 81, 82, 83] de Werra studied balanced equitable edge-colorings of bipartite graphs. The following lemma by de Werra is used to prove the main result in [5]:

Lemma 11.1. *Every bipartite graph has a balanced, equitable and equalized k-edge-coloring $\forall k \in \mathbb{N}$.*

11.2 Fair Detachments of Hypergraphs

A *hypergraph* $\mathcal{G}$ is a pair (V, E) where V is a finite set called the vertex set, E is the edge multiset, where every edge is a multi-subset of V. A *detachment* of a hypergraph is formed by splitting each vertex into one or more subvertices, and sharing the incident edges arbitrarily among the subvertices. Let $\mathcal{F}$ be a hypergraph in which each edge is of size at most 3. In [6], I proved that for a given edge-coloring of $\mathcal{F}$, there exists a detachment $\mathcal{G}$ such that the degree of each vertex and the multiplicity of each edge in $\mathcal{F}$ (and each color class of $\mathcal{F}$) are shared fairly among the subvertices in $\mathcal{G}$ (and each color class of $\mathcal{G}$, respectively).

11.2.1 Laminar Families

A family $\mathcal{A}$ of sets is *laminar* if, for every pair A, B of sets belonging to $\mathcal{A}$, either $A \subset B$, or $B \subset A$, or $A \cap B = \varnothing$. To extend our main result in [5] to hypergraphs [6], I used the following lemma by Nash-Williams [70] (Here $x \approx y$ means $\lfloor y \rfloor \leqslant x \leqslant \lceil y \rceil$):

Lemma 11.2. *If $\mathscr{A}, \mathscr{B}$ are two laminar families of subsets of a finite set S, and $n \in \mathbb{N}$, then there exist a subset A of S such that for every $P \in \mathscr{A} \cup \mathscr{B}$, $|A \cap P| \approx |P|/n$.*

In [8], I generalized the results in [6] to arbitrary hypergraphs. Here $d(v)$ denotes the degree of the vertex v, $\mathcal{G}(j)$ denotes the color class j of $\mathcal{G}$, and $m(u_1^{m_1}, \ldots, u_r^{m_r})$ denotes the multiplicity of an edge of the form

$$\{\underbrace{u_1, \ldots, u_1}_{m_1}, \ldots, \underbrace{u_r, \ldots, u_r}_{m_r}\}.$$

Theorem 11.3. *Let $\mathcal{F}$ be a k-edge-colored hypergraph and let $g : V(\mathcal{F}) \to \mathbb{N}$. Then $\mathcal{F}$ has a fair g-detachment $\mathcal{G}$. That is, there exists a g-detachment $\mathcal{G}$ of $\mathcal{F}$ with amalgamation function $\Psi : V(\mathcal{G}) \to V(\mathcal{F})$ ($\forall v \in V(\mathcal{F})$, $g(v) = |\Psi^{-1}(v)|$) such that:*

(A1) $d_{\mathcal{G}}(v) \approx d_{\mathcal{F}}(u)/g(u)$ *for each $u \in V(\mathcal{F})$ and each $v \in \Psi^{-1}(u)$;*

(A2) $d_{\mathcal{G}(j)}(v) \approx d_{\mathcal{F}(j)}(u)/g(u)$ *for each $u \in V(\mathcal{F})$, each $v \in \Psi^{-1}(u)$ and $1 \leqslant j \leqslant k$;*

(A3) $m_{\mathcal{G}}(U_1, \ldots, U_r) \approx m_{\mathcal{F}}(u_1^{m_1}, \ldots, u_r^{m_r})/\Pi_{i=1}^r \binom{g(u_i)}{m_i}$ *for distinct $u_1, \ldots, u_r \in V(\mathcal{F})$ and $U_i \subset \Psi^{-1}(u_i)$ with $|U_i| = m_i \leqslant g(u_i)$ for $1 \leqslant i \leqslant r$;*

(A4) $m_{\mathcal{G}(j)}(U_1, \ldots, U_r) \approx m_{\mathcal{F}(j)}(u_1^{m_1}, \ldots, u_r^{m_r})/\Pi_{i=1}^r \binom{g(u_i)}{m_i}$ *for distinct $u_1, \ldots, u_r \in V(\mathcal{F})$ and $U_i \subset \Psi^{-1}(u_i)$ with $|U_i| = m_i \leqslant g(u_i)$ for $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant k$.*

Applications of this theorem are discussed in Sections 11.4 and 11.6.

11.3 Edge-Decompositions and Edge-Colorings

An $(r_1, \ldots, r_k)$-*factorization* of a graph G is a partition (decomposition) $\{F_1, \ldots, F_k\}$ of $E(G)$ in which F_i is an r_i-factor (r_i-regular spanning) for $i = 1, \ldots, k$. While the main result in [5] is interesting by itself, it provides a short proof for the following well-known results (see [9]):

- λK_n (λ-fold complete graph) is decomposable into Hamiltonian cycles iff $\lambda(n-1)$ is even.

- λK_n is $(r_1, \ldots, r_k)$-factorizable iff $r_i n$ is even for $1 \leqslant i \leqslant k$, and $\sum_{i=1}^{k} r_i = \lambda(n-1)$. Moreover, each r_i-factor can be guaranteed to be connected if r_i is even.

- $\lambda K_{n_1, \ldots, n_m}$ (λ-fold complete multipartite graph) is Hamiltonian decomposable iff $n_1 = \cdots = n_m := n$, and $\lambda n(m-1)$ is even.

- $\lambda K_{n_1, \ldots, n_m}$ is $(r_1, \ldots, r_k)$-factorizable iff $n_1 = \cdots = n_m := n$, $r_i n m$ is even for $1 \leqslant i \leqslant k$, and $\sum_{i=1}^{k} r_i = \lambda n(m-1)$.

Let $m(u,v)$ denote the number of edges between u and v. Let $K(a_1, \ldots, a_p; \lambda, \mu)$ be a graph with p parts $V_1, \ldots, V_p$, with $|V_i| = a_i$ for $1 \leqslant i \leqslant p$, $m_G(u,v) = \lambda$ for every pair of distinct vertices $u, v \in V_i$ for $1 \leqslant i \leqslant p$, and $m_G(u,v) = \mu$ for each $u \in V_i, v \in V_j$ for $1 \leqslant i < j \leqslant p$. This graph arises naturally in statistical settings [22]. In [5], we found necessary and sufficient conditions for $K(a_1, \ldots, a_p; \lambda, \mu)$ to be decomposable into Hamiltonian cycles.

The Oberwolfach problem asks whether or not it is possible to partition the edge set of K_n, n odd, into isomorphic 2-factors such that each 2-factor consists of a_j cycles of length r_j, $1 \leqslant j \leqslant k$, and $n = \sum_{j=1}^{k} r_j a_j$. In [46] some new solutions to the Oberwolfach problem are given using the amalgamation technique. I am planning to attack the following problem using amalgamations for which I need to obtain a detachment result in which each color class is evenly equitable:

Conjecture 11.4. *(Alspach 1981) If n is odd, $3 \leqslant c_1, \ldots, c_m \leqslant n$, and $\sum_{i=1}^{n} c_i = \binom{n}{2}$, then K_n decomposes into cycles of lengths $c_1, \ldots, c_m$.*

11.4 Hypergraph Edge-Colorings and Baranyai's Theorem

In a mathematics workshop with mn mathematicians in n different areas, each area consisting of m mathematicians, we want to create a collaboration network. For this purpose,

we would like to schedule daily meetings between groups of size three, so that (i) two persons of the same area meet one person of another area, (ii) each person has exactly r meetings each day, and (iii) every two persons of the same area have exactly λ meetings with each person of another area by the end of the workshop. Using hypergraph amalgamations, in [7] I proved a general result regarding factorizations of a family of multipartite hypergraphs, and as a corollary I showed that the above scheduling can be done if: $3 \mid rm$, $2 \mid rnm$ and $r \mid 3\lambda(n-1)\binom{m}{2}$.

Let $\binom{[n]}{h}$ denote the set of all h-subsets of $[n] := \{1, \ldots, n\}$. Let $K_n^h = ([n], \binom{[n]}{h})$. The problem of finding 1-factorizations for K_n^h remained an unsolved problem for 120 years until it was settled by Baranyai (1975) [15]. Since then not much has been done in this area and many problems remain open.

Here we discuss a different approach (amalgamations and detachments) to extend Baranyai's results and to answer various related questions. An immediate corollary of Theorem 11.3 is that the obvious necessary conditions for λK_n^h to be $(r_1, \ldots, r_k)$-factorizable are also sufficient. Let $K_{p_1,\ldots,p_n}^h = (V, E)$ be a hypergraph with vertex partition $\{V_1, \ldots, V_n\}$, $|V_i| = p_i$ for $1 \leqslant i \leqslant n$, and $E = \{e \subset V : |e| = h, |e \cap V_i| \leqslant 1$ for $1 \leqslant i \leqslant n\}$. Another consequence of Theorem 11.3 is that the obvious necessary conditions for $\lambda K_{p_1,\ldots,p_n}^h$ to be $(r_1, \ldots, r_k)$-factorizable are also sufficient.

11.4.1   The Berge-Johnson Problem

For a matrix A, let A_j denote the jth column of A, and let $s(A)$ denote the sum of all the entries of A. Let $R^T = [r_i]_{1 \times k}$, $\Lambda^T = [\lambda_i]_{1 \times m}$ and $H^T = [h_i]_{1 \times m}$ be three vectors with $r_i, \lambda_i \in \mathbb{N}$, and $h_i \in \{1, \ldots, n\}$ for $1 \leqslant i \leqslant m$, such that $h_1 \ldots, h_m$ are distinct.

Let $\Lambda K_{p_1,\ldots,p_n}^H$ be a hypergraph with vertex partition $\{V_1, \ldots, V_n\}$, $|V_i| = p_i$ for $1 \leqslant i \leqslant n$ such that there are λ_i edges of size h_i incident with every h_i vertices, at most one vertex from each part for $1 \leqslant i \leqslant m$ (so no edge is incident with more than one vertex of a part). Here is another interesting corollary of Theorem 11.3:

Theorem 11.5. $\Lambda K^H_{p_1 \dots p_n}$ *is* $(r_1, \dots, r_k)$*-factorizable iff* $p_1 = \cdots = p_n := p$, $s(R) = \sum_{i=1}^m \lambda_i \binom{n-1}{h_i-1} p^{h_i-1}$, *and there exists a non-negative integer matrix* $A = [a_{ij}]_{k \times m}$ *such that* $AH = npR$, *and* $s(A_j) = \lambda_j \binom{n}{h_j} p^{h_j}$ *for* $1 \leqslant j \leqslant m$.

Baranyai [15, 16] solved the case of $h_1 = \cdots = h_m$, $\lambda_1 = \dots, \lambda_m = 1$, $p_1 = \cdots = p_m$, $r_1 = \cdots = r_k$. Berge and Johnson [21], (and later Brouwer and Tijdeman [26], respectively) considered (and solved, respectively) the case of $h_i = i$, $1 \leqslant i \leqslant m$, $p_1 = \cdots = p_m = \lambda_1 = \cdots = \lambda_m = r_1 = \cdots = r_k = 1$.

11.4.2 Baranyai-Katona Conjecture

Let m be the least common multiple of h and n, and let $a = m/h$. Define

$$\mathscr{K} = \{\{1, \dots, h\}, \{h+1, \dots, 2h\}, \dots, \{(a-1)h+1, (a-1)h+2, \dots, ah\}\},$$

where the elements of the sets are considered mod n. The families obtained from $\mathscr{K}$ by permuting the elements of the underlying set $[n]$ are called *wreaths*. If h divides n, then a wreath is just a partition. It was conjectured that K^h_n can be decomposed into disjoint wreaths [54]. In connection with this conjecture, I am currently working on the connectivity of factors [12, 13].

11.4.3 Connected Factorizations

In [12], I solved the following problem which was suggested by Katona:

Theorem 11.6. λK^h_n *is* $(r_1, \dots, r_k)$*-factorizable iff* $h \mid r_i n$ *for* $1 \leqslant i \leqslant k$, *and* $\sum_{i=1}^k r_i = \lambda \binom{n-1}{h-1}$. *Moreover, for* $1 \leqslant i \leqslant k$ *an* r_i*-factor is connected if* $r_i \geqslant 2$.

This can be considered as a connected version of Baranyai's Theorem. In particular if $\lambda = 1$, and $h = r_1 = \cdots = r_k = 2$, this implies the classical result of Walecki that the edge set of K_n can be partitioned into Hamiltonian cycles iff n is odd. A related problem due to Bermond (1978) asked for conditions under which one can decompose K^h_n into Hamiltonian

cycles. I am interested in the following more general problem that relates my work to the results of Nash-Williams [68], and their extensions [18]:

Problem 5. *Find necessary and sufficient conditions for an edge-colored hypergraph $\mathcal{F}$ to have a fair detachment in which each color class is k-edge-connected.*

So far [13], I have been able to solve this problem when all edges of $\mathcal{F}$ are of size at most 3, and $k = 2$. This, in particular, implies another Baranyai-type theorem ($h = 3$) in which each factor is 2-edge-connected.

11.4.4   Kneser Graphs and the Middle Levels Problem

The *Kneser graph $K(n, h)$* has as vertices the h-subsets of $[n]$. Two vertices are adjacent if the corresponding h-subsets are disjoint. It is widely conjectured that all Kneser graphs but the Petersen graph, $K(5, 2)$, have Hamiltonian cycles. Let $n \geqslant 2h + 1$. The *bipartite Kneser Graph $H(n, h)$* has as its partite sets the h- and $(n - h)$-subsets of $[n]$. Two vertices A and B from different partite sets are adjacent if the h-subset A is contained in the $(n - h)$-subset B. It is conjectured that $H(2h + 1, h)$ is Hamiltonian. Using Baranyai's Theorem, partial results to these two conjectures are given in [30, 32]. I am interested in working on these two conjectures.

11.5   Extending Partial Decompositions and Graph Embedding Problems

In this section and the next section I describe the usefulness of amalgamations in solving embedding problems. For example the main result in [5] provides a short proof for the following theorems (see [9]): A k-edge-coloring of K_m can be embedded into (i) a Hamiltonian decomposition of K_{m+n}(Hilton [44]), (ii) an $(r_1, \ldots, r_k)$-factorization of K_{m+n}(Johnson [51]) iff the obvious necessary conditions are satisfied. Embedding Hamiltonian cycles in complete multipartite graphs is considered in [48] but the problem is still open and I am interested in working on it. When $a_1 = \cdots = a_p := a$, let $K(a^{(p)}; \lambda, \mu)$ denote $K(a_1, \ldots, a_p; \lambda, \mu)$. In [10], we asked:

Problem 6. *When can a graph decomposition of $K(a^{(p)}; \lambda, \mu)$ be extended to a Hamiltonian decomposition of $K(a^{(p+r)}; \lambda, \mu)$ for $r > 0$?*

We proved [10]:

Theorem 11.7. *Let $f : E \to C$ be a k-edge coloring for $K(a^{(p)}; \lambda, \mu)$, and let ω_j denote the number of components of color class j. For $1 \leqslant j \leqslant k$, define $s_j \equiv \omega_j \pmod{r}$ with $1 \leqslant s_j \leqslant r$, and suppose $\sum_{j=1}^{k} s_j \geqslant kr - \mu a^2 \binom{r}{2}$. Then f can be embedded into a Hamiltonian decomposition of $K(a^{(p+r)}; \lambda, \mu)$ iff the obvious necessary conditions are satisfied.*

We used this general result to give a complete solution to Problem 6 for all $r \geqslant \frac{\lambda}{\mu a} + \frac{p-1}{a-1}$. We also solved the problem when r is as small as possible in two different senses, namely when $r = 1$ and when $r = \frac{\lambda}{\mu a} - p + 1$ [10].

11.6 Embedding Problems for Hypergraphs

Over 35 years ago, Cameron asked [29]: Under what conditions can partial 1-factorizations of K_n^h be extended to 1-factorizations? In [11] we considered a more general problem for $h = 3$. We proved that

Theorem 11.8. *Suppose that $n > 2m + \lfloor (1 + \sqrt{8m^2 - 16m - 7})/2 \rfloor$. Then an edge-coloring of K_m^3 can be embedded into an r-factorization of K_n^3 iff the obvious necessary conditions are satisfied.*

One can assume that not only the hyperedges of size 3 are colored, but so are all the hyperedges of "pieces" of hypergraphs (i.e. n and $\binom{n}{2}$ copies of the hyperedges in K_m^2 and K_m^1, respectively) that are built up to size 3 when the new vertices are added. In this case we solved the problem completely in [11]:

Theorem 11.9. *An edge-coloring of $K_m^3 \cup nK_m^2 \cup \binom{n}{2} K_m^1$ can be extended to an r-factorization of K_n^3 iff the obvious necessary conditions are satisfied.*

Brouwer, Schrijver and Baranyai [17, 25] studied special cases of Cameron's Problem and conjectured that: A 1-factorization of K_m^h can be extended to a 1-factorization of K_n^h iff h divides both m and n, and $n \geqslant 2m$. Häggkvist and Hellgren settled this conjecutre [40]. The more general question I am interested in working on is the conditions under which one can extend an equitable edge-coloring of K_m^h into a factorization of K_n^h for $n > m$.

11.7 Matroids

I am also interested to study amalgamations and detachments for matroids. Finding companion results for matroid will lead to interesting matroid decomposition and embedding results.

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