

Almost Resolvable Maximum Packings of Complete Bipartite Graphs with 4-Cycles

by

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Abstract

A packing of a complete bipartite graph $K(m,n)$ with 4-cycles is a decomposition of $K(m,n)$ into a collection C of edge-disjoint 4-cycles and a set of unused edges L referred to as the leave. If C is as large as possible (and L is as small as possible), then the packing is a maximum packing. An almost parallel class of a maximum packing is a largest possible set of vertex disjoint 4-cycles. In this paper we establish values of m and n for which a maximum packing of $K(m,n)$ with 4-cycles may be resolved into almost parallel classes with the remaining cycles vertex disjoint.

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List of Abbreviations

APC	Almost Parallel Class
ARMP	Almost Resolvable Maximum Packing
CQG	Commutative Quasigroup
KTS	Kirkman Triple System
NKTS	Nearly Kirkman Triple System
PPC	Partial Parallel Class

Chapter 1

Introduction

1.1 Terminology/Definitions

A *packing* of a complete bipartite graph $K(m,n)$ with 4-cycles is a decomposition of $K(m,n)$ into a collection C of edge-disjoint 4-cycles and a set of unused edges L referred to as the *leave*. If C is as large as possible (and L is as small as possible), then the packing is referred to as a *maximum packing*. An *almost parallel class* of a maximum packing is a largest possible set of vertex disjoint 4-cycles. A maximum packing of $K(m,n)$ with 4-cycles is said to be *almost resolvable* if C can be partitioned into as many almost parallel classes as possible so that the remaining 4-cycles are vertex-disjoint. This set of “leftover” cycles is referred to as a *partial parallel class*.

Example 1.1. Consider the graph of $K(4,4)$ with part $A = \{a_1, a_2, a_3, a_4\}$ and part $B = \{b_1, b_2, b_3, b_4\}$. Figure 1.1 illustrates the maximum packing of $K(4,4)$ with the collection of edge-disjoint 4-cycles $C = \{(a_1, b_2, a_2, b_1), (a_3, b_4, a_4, b_3), (a_1, b_3, a_2, b_4), (a_3, b_1, a_4, b_2)\}$ and the set of unused edges $L = \emptyset$. In this case, C is completely resolved into the parallel classes $\pi_1 = \{(a_1, b_2, a_2, b_1), (a_3, b_4, a_4, b_3)\}$ and $\pi_2 = \{(a_1, b_3, a_2, b_4), (a_3, b_1, a_4, b_2)\}$ which are colored red and green, respectively, in the figure.

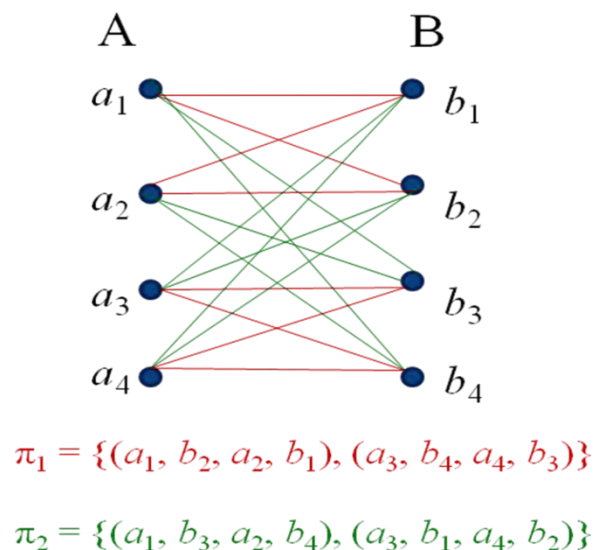


Figure 1.1: Resolved Maximum Packing of $K(4,4)$ with 4-Cycles

Although the maximum packing of $K(4,4)$ with 4-cycles resulted in a complete resolution and an empty leave, those results are not typical for all complete bipartite graphs. The results of related research provide insight to the parameters for C and L for almost resolvable maximum packings of complete bipartite graphs with m and n of various sizes.

1.2. Historical Results and Implications

In [2], Elizabeth J. Billington, Hung-Lin Fu, and C.A. Rodger give the solution for finding a maximum packing for any complete multipartite graph with edge-disjoint 4-cycles. In particular, their work shows that if m and n are both even, $K(m,n)$ will decompose into 4-cycles with an empty leave. But when at least one of m and n is odd, $K(m,n)$ decomposes into 4-cycles with the leave dependent upon the values of m and n . In the case that m is even and n is odd, the leave is a star consisting of m edges. (A similar result would apply if n is odd and m is even, and regardless of whether m or n is larger.) In the case that $m \leq n$ and $m \equiv 1 \pmod{4}$, the leave is a 1-factor on the first $m - 1$ vertices plus a star with $n - (m - 1)$ vertices. (A similar result would

apply if $n \leq m$ and $n \equiv 1 \pmod{4}$.) In the case that $m \leq n$ and $m \equiv 3 \pmod{4}$, the leave is a one-factor on the first $m - 3$ vertices plus two stars, one with 3 edges and one with $n - (m - 3)$ edges, with the two stars sharing one edge. These leaves are pictured in Figure 1.2 below. Of course, these are only examples of the edge arrangements that meet the vertex degree requirements for the minimum leave in each case; many other arrangements are possible. Unless otherwise noted, the proofs in this paper will assume leaves identical to those given here and will refer to them as leaves in “standard position”.

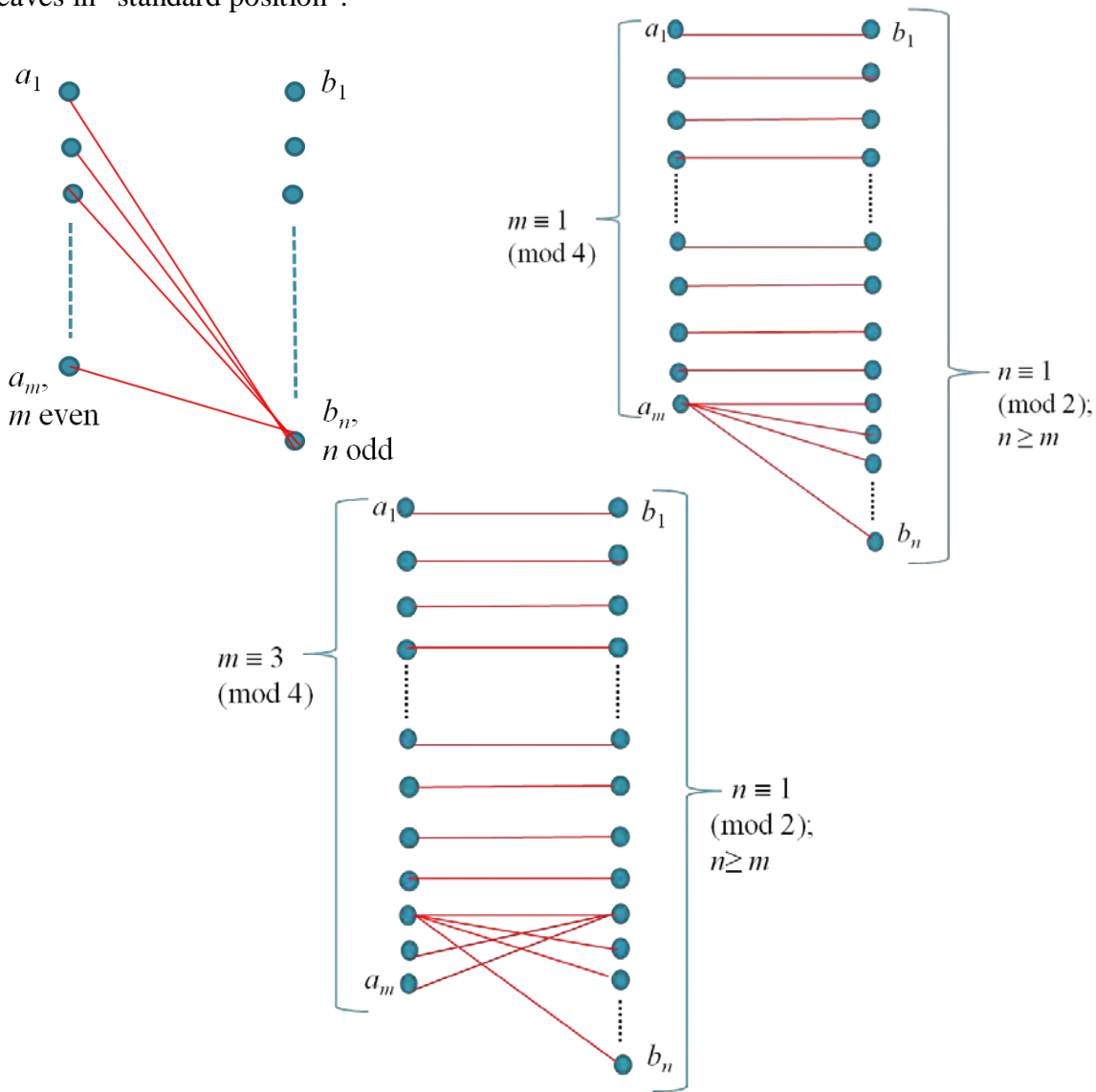


Figure 1.2: Leaves for Maximum Packings of $K(m,n)$; At Least One of m, n Odd

Knowing the leave structures for maximum packings of $K(m,n)$ with 4-cycles allows us to determine the number of cycles per packing, the number of cycles per almost parallel class, and the number of parallel classes, facts that will be useful in proofs later in this paper. For example, when m is even, n is odd, and $m \leq n$, $|L| = m$. Since the total number of edges in $K(m,n)$ is mn , $|C| = \frac{mn-m}{4} = \frac{m(n-1)}{4}$, an integer since both m and $(n-1)$ are even. The number of cycles per parallel class is governed by the size of the smallest part, m and each cycle uses 2 vertices from each part, so the number of cycles per parallel class is $\frac{m}{2}$. Thus, the number of parallel classes will be $\frac{m(n-1)}{4} \div \frac{m}{2} = \frac{m(n-1)}{4} \cdot \frac{2}{m} = \frac{n-1}{2}$ or $\lfloor \frac{n}{2} \rfloor$. There is no partial parallel class. When other conditions are the same, but $n \leq m$, the calculations are (as expected) the same except the number of cycles per almost parallel class and the number of parallel classes reverses.

When both m and n are odd, $m \equiv 1 \pmod{4}$, and $m \leq n$, we have $|L| = n$ and $|C| = \frac{mn-n}{4} = \frac{n(m-1)}{4}$, an integer since $m \equiv 1 \pmod{4}$. This time, since m is odd, the number of cycles per parallel class will be $\frac{m-1}{2}$ or $\lfloor \frac{m}{2} \rfloor$ and the number of parallel classes will be $\frac{n(m-1)}{4} \div \frac{m-1}{2} = \frac{n(m-1)}{4} \cdot \frac{2}{m-1} \equiv \lfloor \frac{n}{2} \rfloor$, given that n is odd. The number of cycles contained in the combined parallel classes can be calculated as $\frac{m-1}{2} \cdot \frac{n-1}{2} = \frac{(m-1)(n-1)}{4}$. Subtracting this from $|C|$ yields the number of cycles in the partial parallel class, $\frac{m-1}{4} = \lfloor \frac{m}{4} \rfloor$.

In like manner, given the specified leave, we can easily calculate that the number of cycles per almost parallel class and the number of almost parallel classes for a maximum packing of $K(m,n)$ with 4-cycles when both m and n are odd, $m \leq n$, and $m \equiv 3 \pmod{4}$ will be $\lfloor \frac{m}{2} \rfloor$ and $\lfloor \frac{n}{2} \rfloor$ respectively, with $\lfloor \frac{m}{4} \rfloor$ cycles remaining for the partial parallel class. This is noteworthy, since it

(along with the other calculation results when at least one of m and n is odd) implies that adding a single vertex to either or both parts of $K(m,n)$ when m and n are both even does not change the number of cycles per almost parallel class nor does it change the number of parallel classes. For the proofs involving the cases when both m and n are odd that are found later in this paper, a summary of these calculation results may be useful and is provided in Table 1.1 below. We assume here and in future work that $m \leq n$.

Number of 4-Cycles C	$\frac{n(m-1)}{4}$ when $m \equiv 1 \pmod{4}$ $\frac{n(m-1)-2}{4}$ when $m \equiv 3 \pmod{4}$
Number of Edges in the Leave L	n when $m \equiv 1 \pmod{4}$ $n + 2$ when $m \equiv 3 \pmod{4}$
Number of 4-Cycles per Almost Parallel Class	$\lfloor \frac{m}{2} \rfloor$
Number of Almost Parallel Classes	$\lfloor \frac{n}{2} \rfloor$
Number of Cycles in the Partial Parallel Class	$\lfloor \frac{m}{4} \rfloor$

Table 1.1: Composition of an Almost Resolvable Maximum Packing of $K(m, n)$;
Both m and n Odd, $m \leq n$

Subsequent research [3] by Billington, I. J. Dejter, C.C. Linder, M. Meszka and C.A. Rodger in 2011 proved that an almost resolvable maximum packing of complete graphs K_n is possible for all n except 9 (which does not exist) and possibly 23, 41, and 57. Later that same year, research [4] by Billington, Hoffman, Lindner and Meszka solved the cases in question and proved that for all $n \neq 9$, there is indeed an almost resolvable maximum packing of K_n with 4-cycles. The question naturally arises, then, as to whether these results can be extended to multipartite graphs. In this paper, we address the bipartite case.

Chapter 2

The Even/Even and Even/Odd Cases

2.1 $K(2m, 2n)$; The Even/Even Case

Theorem 2.1. A completely resolvable maximum packing of 4-cycles exists for $K(2m, 2n)$ for all $m, n \in \mathbb{N}$.

Proof: Let $K(m, n)$ be the complete bipartite graph with parts $A = \{a_1, a_2, a_3, \dots, a_m\}$ and $B = \{b_1, b_2, b_3, \dots, b_n\}$. Without loss of generality, assume $n \geq m$. It is a well-known result (König, 1916 in [6]) that every complete bipartite graph G has a proper $\Delta(G) = n$ -edge-coloring. Assume $K(m, n)$ to be so colored. The edges of $K(m, n)$ of color c_i , $1 \leq i \leq n$, form a color class.

Now consider the graph $K(2m, 2n)$ with partitions $A' = \{a_{11}, a_{12}, a_{21}, a_{22}, a_{31}, a_{32}, \dots, a_{m1}, a_{m2}\}$ and $B' = \{b_{11}, b_{12}, b_{21}, b_{22}, b_{31}, b_{32}, \dots, b_{n1}, b_{n2}\}$ constructed as follows:

If $a_x b_y$, $x \leq m$, $y \leq n$, is an edge of $K(m, n)$ of color c_i , then $(a_{x1}, b_{y1}, a_{x2}, b_{y2})$ is a 4-cycle in $K(2m, 2n)$ with every edge of color c_i , and the set of m cycles of color c_i is an almost parallel class of $K(2m, 2n)$, with the set of n such almost parallel classes comprising a completely resolved maximum packing of $K(2m, 2n)$ with 4-cycles.

Example 2.1 Figure 2.1 illustrates the specific example of transforming the proper 3-edge coloring of $K(2, 3)$ into a completely resolved maximum packing of $K(4, 6)$ with 4-cycles.

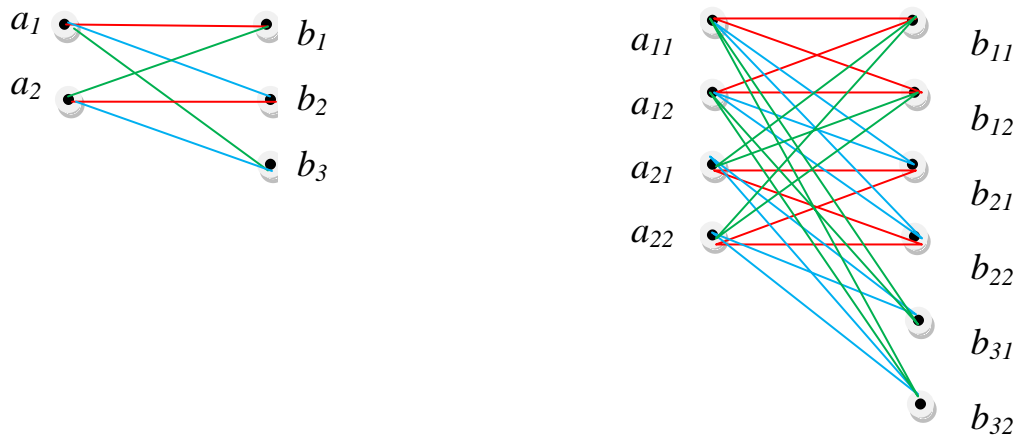


Figure 2.1: Proper Edge Coloring of $K(2, 3)$ and Resulting Resolved Maximum Packing of $K(4, 6)$ with 4-Cycles

2.2 $K(2m, 2n + 1)$; The Even/Odd Case

Theorem 2.2. An almost resolvable maximum packing with 4-cycles exists for $K(2m, 2n+1)$ for all $m, n \in \mathbb{N}$.

Proof: As just demonstrated in Theorem 2.1, a resolvable maximum packing with 4-cycles exists for $K(2m, 2n)$ for all $m, n \in \mathbb{N}$. Such a packing is also an almost resolvable maximum packing of $K(2m, 2n+1)$ since, as shown in the Introduction, the addition of one vertex to one part of a complete bipartite graph with both parts of even order does not change the number of parallel classes nor does it change the number of cycles per parallel class. The new edges that connect the added vertex and the vertices of the opposite part comprise the required minimum leave, a star with m edges.

Chapter 3

The Odd/Odd Cases of Small Order

3.1 $K(3, 2n+1)$

Theorem 3.1. There exists an almost resolvable maximum packing of $K(3, 2n+1)$ with 4-cycles for all $n \in \mathbb{N}$.

Proof. An almost resolvable maximum packing of $K(3, 2n+1)$ with 4-cycles will consist of n almost parallel classes, each containing one 4-cycle. Let $K(3, 2n+1)$ have parts $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, \dots, b_{2n+1}\}$. An ARMP of $K(3, 2n+1)$ can be constructed by assigning the cycle $(a_2, b_{2x}, a_3, b_{2x+1})$ to almost parallel class π_x for $1 \leq x \leq n$, as illustrated by the cycles in color in the graph below. Edges of the leave are denoted by dashed lines.

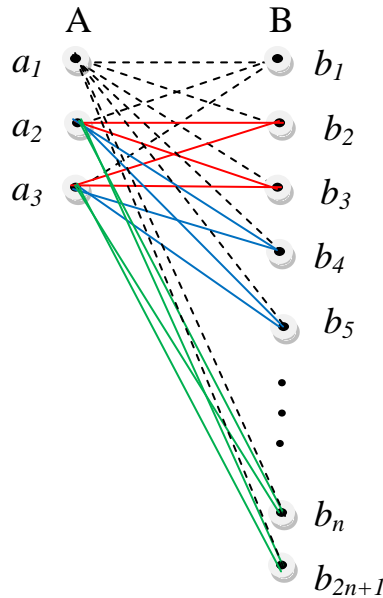


Figure 3.1: ARMP of $K(3, 2n+1)$

3.2 $K(5, 2n+1)$

When $n = 1$, we know that an ARMP of $K(5, 2n + 1)$ exists because we have proven the case for $K(3, 2n + 1)$ for all n . For $n = 2$, we have the following:

Theorem 3.2.1. There exists no almost resolvable maximum packing of $K(5, 5)$ with 4-cycles.

Proof. Suppose there does exist such a packing. It must contain 2 almost parallel classes (APCs) of two 4-cycles each and 1 partial parallel class (PPC) of one 4-cycle, with the leave a 1-factor. Let $A = \{a_1, a_2, a_3, a_4, a_5\}$ and $B = \{b_1, b_2, b_3, b_4, b_5\}$ be parts of $K(5, 5)$ with leave $\{(a_1, b_1), (a_2, b_2), (a_3, b_3), (a_4, b_4), (a_5, b_5)\}$. The first APC, π_1 , of the packing must omit one vertex from part A and one vertex from part B. Either these two vertices are adjacent in the leave or not. Let's consider the latter case first. Without loss of generality, let a_2 and b_3 be the omitted vertices. Then (a_1, b_2, a_4, b_5) and (a_3, b_1, a_5, b_4) , or some combination isomorphic to this one, must comprise the first APC π_1 . See Figure 3.2.1 which shows the leave in dashed lines and the cycles of APC π_1 in solid lines.

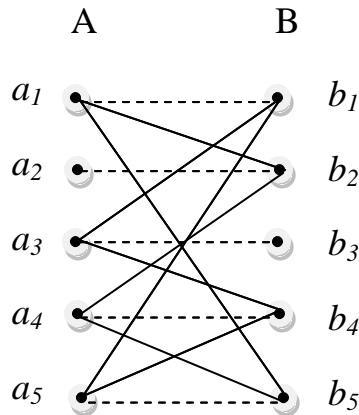


Figure 3.2.1: Trial Construction of ARMP of $K(5, 5)$;
(Vertices Omitted in π_1 Not Adjacent in Leave)

A 4-cycle constructed for the second APC, π_2 cannot use two vertices from part A which were both used in the first APC. Why? Any two such vertices would have either (1) been contained in the same 4-cycle in the first almost parallel class or (2) been contained in different 4-cycles in the first almost parallel class.

In the first case, consider a_1 and a_4 , which cannot both be joined to b_2 or b_5 , since all four vertices were previously joined in the first APC. And a_1 and a_4 cannot both be joined to b_1 or b_4 , since doing so would use edges of the leave. There remains only one available vertex in part B to which a_1 and a_4 can both be joined, insufficient for construction of a 4-cycle. The same situation exists for the combination a_3 and a_5 .

In the second case, consider a_1 and a_3 ; these two vertices cannot both be joined to b_1, b_2, b_4 , or b_5 , since each of b_1, b_2, b_4 , and b_5 was previously joined to either a_1 or a_3 in the first APC. This again leaves only one available vertex in part B to which a_1 and a_3 can both be joined, insufficient for construction of a 4-cycle. The same situation exists for the combinations of a_1 with a_5 , a_3 with a_4 and a_4 with a_5 . Thus, both 4-cycles constructed for π_2 must use the vertex from part A which was not used in π_1 , a contradiction.

Now assume that the two vertices omitted from π_1 are adjacent in the leave. Then (a_1, b_3, a_2, b_4) and (a_3, b_1, a_4, b_2) , or some combination isomorphic to this one, must comprise π_1 . See Figure 3.2.2 that follows.

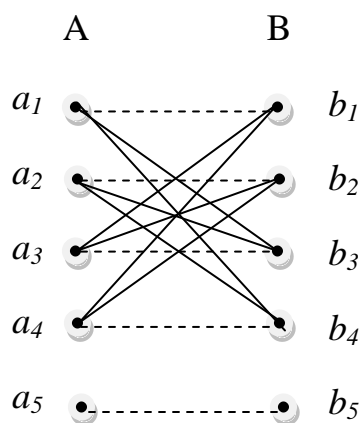


Figure 3.2.2: Trial Construction of ARMP of $K(5, 5)$
 (Vertices Omitted in π_1 Adjacent in Leave)

Again, a 4-cycle constructed for APC π_2 cannot use two vertices from part A which were both used in π_1 . Why? Any two such vertices would have either (1) been contained in the same 4-cycle in the first almost parallel class or (2) been contained in different 4-cycles in the first almost parallel class.

In the first case, consider vertices a_1 and a_2 ; after accounting for the leave and previously used edges of π_1 , there is only one vertex (b_5) in part B to which both of these can be joined. A similar situation exists for vertices a_3 and a_4 .

In the second case, consider vertices a_1 and a_3 ; after accounting for the leave and previously used edges of π_1 , there again is only one vertex (b_5) in part B to which both of these can be joined. A similar situation exists for vertices a_1 and a_4 , for a_2 and a_3 , and for a_2 and a_4 .

Thus, both 4-cycles in π_2 must use the vertex in part A which was omitted in π_1 , again a contradiction. Hence, there is no ARMP of $K(5, 5)$ with 4-cycles. However, for $n \geq 3$, we will see that there does exist an ARMP of $K(5, 2n + 1)$ with 4-cycles.

Theorem 3.2.2. An almost resolvable maximum packing with 4-cycles exists for $K(5, 2n+1)$ for all $n \geq 3$.

Proof. Let $K(5, 7)$ be the complete bipartite graph with parts $A = \{a_1, a_2, a_3, a_4, a_5\}$ and $B = \{b_1, b_2, b_3, \dots, b_7\}$. Then an almost resolvable maximum packing of $K(5, 7)$ is given by the following:

$$\pi_1: \{(a_4, b_7, a_3, b_2), (a_2, b_5, a_1, b_3)\}$$

$$\pi_2: \{(a_2, b_6, a_1, b_7), (a_5, b_1, a_4, b_3)\}$$

$$\pi_3: \{(a_4, b_5, a_3, b_6), (a_5, b_2, a_1, b_4)\}$$

$$\text{PPC: } \{(a_3, b_4, a_2, b_1)\}$$

An almost resolvable maximum packing of $K(5, 9)$ with parts $A = \{a_1, a_2, a_3, a_4, a_5\}$ and $B = \{b_1, b_2, b_3, \dots, b_9\}$ is given by the following:

$$\pi_1: \{(a_2, b_1, a_5, b_3), (a_1, b_2, a_4, b_6)\}$$

$$\pi_2: \{(a_3, b_2, a_5, b_4), (a_1, b_3, a_4, b_5)\}$$

$$\pi_3: \{(a_3, b_1, a_4, b_7), (a_1, b_8, a_2, b_9)\}$$

$$\pi_4: \{(a_1, b_4, a_2, b_7), (a_3, b_8, a_4, b_9)\}$$

$$\text{PPC: } \{(a_2, b_5, a_3, b_6)\}$$

The matrix in Figure 3.2.3 that follows gives an alternative representation of this ARMP of $K(5, 9)$ with 4-cycles. Part A vertices are shown in the first column with part B vertices shown in the first row. Every remaining cell of the matrix represents an edge connecting the part A

vertex and part B vertex of that cell's row and column. The entry of each cell gives the parallel class which would contain the 4-cycle containing that edge. For ease of reading, each parallel class π_i is designated by the number of its subscript (i), while entries of P indicate edges of 4-cycles in the partial parallel class. Cells with entries of L correspond to edges of the leave.

	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9
a_1	L	1	2	4	2	1	4	3	3
a_2	1	L	1	4	P	P	4	3	3
a_3	3	2	L	2	P	P	3	4	4
a_4	3	1	2	L	2	1	3	4	4
a_5	1	2	1	2	L	L	L	L	L

Figure 3.2.3: ARMP of $K(5, 9)$ with 4-cycles

An almost resolvable maximum packing of $K(5, 2n + 1)$ with 4-cycles for any $n \geq 5$ can be formed by extending this packing of $K(5, 9)$. First note that an extension of an ARMP of $K(5, 2n - 1)$ to an ARMP of $K(5, 2n + 1)$ requires the addition of 2 vertices to part B and the addition of one almost parallel class. The number (2) of cycles per APC remains the same, as does the number (1) of cycles in the partial parallel class.

The extension of an ARMP of $K(5, 2n - 1)$ to an ARMP of $K(5, 2n + 1)$ can be accomplished by “shifting” the cycle $(a_1, b_{2n-2}, a_2, b_{2n-1})$ from π_3 into the new APC π_n and replacing it in π_3 with the newly created cycle $(a_1, b_{2n}, a_2, b_{2n+1})$. The new APC π_n is completed with the addition of the second newly created cycle $(a_3, b_{2n}, a_4, b_{2n+1})$. The remaining new edges complete the leave.

Example 3.2. The procedure just described is illustrated in the matrices of Figures 3.2.4 and 3.2.5 that follow, showing the extension of the ARMP of $K(5, 9)$ to an ARMP of $K(5, 11)$ and the extension of an ARMP of $K(5, 11)$ to an ARMP of $K(5, 13)$.

ARMP of $K(5, 9)$ with Intended Shifts for Extension to ARMP of $K(5, 11)$

	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}
a_1	L	1	2	4	2	1	4	3	3		
a_2	1	L	1	4	P	P	4	3	3		
a_3	3	2	L	2	P	P	3	4	4		
a_4	3	1	2	L	2	1	3	4	4		
a_5	1	2	1	2	L	L	L	L	L		

ARMP of $K(5, 11)$ Produced from the Extension of ARMP of $K(5, 9)$

	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}
a_1	L	1	2	4	2	1	4	5	5	3	3
a_2	1	L	1	4	P	P	4	5	5	3	3
a_3	3	2	L	2	P	P	3	4	4	5	5
a_4	3	1	2	L	2	1	3	4	4	5	5
a_5	1	2	1	2	L	L	L	L	L	L	L

Figure 3.2.4: Extension of the ARMP of $K(5, 9)$ to an ARMP of $K(5, 11)$

ARMP of $K(5, 11)$ With Intended Shifts for Extension to ARMP of $K(5, 13)$

	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}
a_1	L	1	2	4	2	1	4	5	5	3	3	→	→
a_2	1	L	1	4	P	P	4	5	5	3	3	→	→
a_3	3	2	L	2	P	P	3	4	4	5	5		
a_4	3	1	2	L	2	1	3	4	4	5	5		
a_5	1	2	1	2	L	L	L	L	L	L	L		

ARMP of $K(5, 13)$ from the Extension of the ARMP of $K(5, 11)$

	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}
a_1	L	1	2	4	2	1	4	5	5	6	6	3	3
a_2	1	L	1	4	P	P	4	5	5	6	6	3	3
a_3	3	2	L	2	P	P	3	4	4	5	5	6	6
a_4	3	1	2	L	2	1	3	4	4	5	5	6	6
a_5	1	2	1	2	L	L	L	L	L	L	L	L	L

Figure 3.2.5: Extension of the ARMP of $K(5, 11)$ to an ARMP of $K(5, 13)$

3.3 $K(7, 2n + 1)$

Theorem 3.3. There exists an almost resolvable maximum packing of $K(7, 2n + 1)$ with 4-cycles for all $n \in \mathbb{N}$.

Proof. We have already shown that there exists an ARMP of $K(7, 2n + 1)$ with 4-cycles for $n = 1$ and $n = 2$ in the proofs of Theorems 3.1 and 3.2.2 For $n = 3$, let $K(7, 7)$ be the complete bipartite graph with parts $A = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$ and $B = \{b_1, b_2, b_3, b_4, b_5, b_6, b_7\}$. Then

an almost resolvable maximum packing of $K(7, 7)$ with 4-cycles is given by the following almost parallel classes and partial parallel class:

$$\pi_1: \{(a_1, b_4, a_2, b_5), (a_6, b_6, a_3, b_7), (a_7, b_1, a_5, b_3)\}$$

$$\pi_2: \{(a_1, b_6, a_2, b_7), (a_6, b_4, a_7, b_2), (a_3, b_5, a_4, b_1)\}$$

$$\pi_3: \{(a_2, b_1, a_6, b_3), (a_3, b_4, a_5, b_2), (a_7, b_6, a_4, b_7)\}$$

$$\text{PPC: } \{(a_1, b_3, a_4, b_2)\}$$

Many thanks to Dr. Mariusz Meszka, who graciously used his computer programming expertise to find this ARMP of $K(7, 7)$.

We will show that an ARMP of $K(7, 2n + 1)$ with 4-cycles for all $n \geq 4$ can be produced from an extension of the ARMP of $K(5, 2n + 1)$. Note that in such an extension, the number of vertices in part A increases by 2, the number of cycles per almost parallel class increases by one, while the number (n) of parallel classes remains the same, as does the number of cycles in the partial parallel class.

An extension of the ARMP of $K(5, 2n + 1)$ to an ARMP of $K(7, 2n + 1)$ can be accomplished using the new 4-cycles created with the addition of edges corresponding to the new vertices a_6 and a_7 . To do so, we must only find a set of n disjoint pairs of vertices in part B that are each not “hit” by a different APC in the ARMP of $K(5, 2n + 1)$ and that do not include vertex b_5 , which will be hit by additional edges of the leave in $K(7, 2n + 1)$. This is in fact always possible given the nature of the provided construction which extends the ARMP of $K(5, 2n - 1)$ to an ARMP of $K(5, 2n + 1)$ by “shifting” a cycle from an existing almost parallel class into the new APC.

The four matrices in Figure 3.3.1 on the following page illustrate the construction of an ARPM of $K(5, 11)$, $K(5, 13)$ and $K(5, 15)$ from the ARMP of $K(5, 9)$ through the extension

process outlined in the proof of Theorem 3.2.2. The numbers in bold print given directly beneath each matrix indicate certain parallel classes (among others) that do not hit the indicated vertices of Part B. In each case, we find a set of n disjoint pairs of vertices in Part B that are each not hit by a different APC.

ARMP of $K(5, 9)$

	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9
a_1	L	1	2	4	2	1	4	3	3
a_2	1	L	1	4	P	P	4	3	3
a_3	3	2	L	2	P	P	3	4	4
a_4	3	1	2	L	2	1	3	4	4
a_5	1	2	1	2	L	L	L	L	L
	4	3	3	1		4	1	2	2

ARMP of $K(5, 11)$

	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}
a_1	L	1	2	4	2	1	4	5	5	3	3
a_2	1	L	1	4	P	P	4	5	5	3	3
a_3	3	2	L	2	P	P	3	4	4	5	5
a_4	3	1	2	L	2	1	3	4	4	5	5
a_5	1	2	1	2	L	L	L	L	L	L	L
	4	5	5	1		4	1	3	3	2	2

ARMP of $K(5, 13)$

	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}
a_1	L	1	2	4	2	1	4	5	5	6	6	3	3
a_2	1	L	1	4	P	P	4	5	5	6	6	3	3
a_3	3	2	L	2	P	P	3	4	4	5	5	6	6
a_4	3	1	2	L	2	1	3	4	4	5	5	6	6
a_5	1	2	1	2	L	L	L	L	L	L	L	L	L
	4	5	5	1		4	1	6	6	3	3	2	2

ARMP of $K(5, 15)$

	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}	b_{14}	b_{15}
a_1	L	1	2	4	1	1	4	5	5	6	6	7	7	3	3
a_2	1	L	1	4	P	P	4	5	5	6	6	7	7	3	3
a_3	3	2	L	2	P	P	3	4	4	5	5	6	6	7	7
a_4	3	1	2	L	2	1	3	4	4	5	5	6	6	7	7
a_5	1	2	1	2	L	L	L	L	L	L	L	L	L	L	L
	4	5	5	1		4	1	6	6	7	7	3	3	2	2

Figure 3.3.1: Extension of an ARMP of $K(5, 9)$ to an ARPM of $K(5, 11)$, $K(5, 13)$ and $K(5, 15)$

Readers will observe that for $n \geq 6$, the pattern established by this construction produces an ARMP of $K(5, 2n + 1)$ with the following characteristics:

1. π_1 will not hit vertices b_4 and b_7
2. π_2 will not hit vertices b_{2n} and b_{2n+1}
3. π_3 will not hit vertices b_{2n-2} and b_{2n-1}
4. π_4 will not hit vertices b_1 and b_6
5. π_5 will not hit vertices b_2 and b_3
6. π_x will not hit vertices b_{2x-4} and b_{2x-3} for all x such that $6 \leq x \leq n$.

We now have precisely the set of n disjoint pairs of vertices in part B (excluding vertex b_5), each pair not hit by a different almost parallel class, that is needed to extend the ARMP of $K(5, 2n + 1)$ to $K(7, 2n + 1)$.

Example 3.3. For confirmation, examples of the ARMPs of $K(7, 9)$, $K(7, 11)$, $K(7, 13)$ and $K(7, 15)$ produced using this extension construction are given in Figure 3.3.2 on the following page.

ARMP of $K(7, 9)$

	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9
a_1	L	1	2	4	2	1	4	3	3
a_2	1	L	1	4	P	P	4	3	3
a_3	3	2	L	2	P	P	3	4	4
a_4	3	1	2	L	2	1	3	4	4
a_5	1	2	1	2	L	L	L	L	L
a_6	4	3	3	1	L	4	1	2	2
a_7	4	3	3	1	L	4	1	2	2

ARMP of $K(7, 11)$

	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}
a_1	L	1	2	4	2	1	4	5	5	3	3
a_2	1	L	1	4	P	P	4	5	5	3	3
a_3	3	2	L	2	P	P	3	4	4	5	5
a_4	3	1	2	L	2	1	3	4	4	5	5
a_5	1	2	1	2	L	L	L	L	L	L	L
a_6	4	5	5	1	L	4	1	3	3	2	2
a_7	4	5	5	1	L	4	1	3	3	2	2

ARMP of $K(7, 13)$

	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}
a_1	L	1	2	4	2	1	4	5	5	6	6	3	3
a_2	1	L	1	4	P	P	4	5	5	6	6	3	3
a_3	3	2	L	2	P	P	3	4	4	5	5	6	6
a_4	3	1	2	L	2	1	3	4	4	5	5	6	6
a_5	1	2	1	2	L	L	L	L	L	L	L	L	L
a_6	4	5	5	1	L	4	1	6	6	3	3	2	2
a_7	4	5	5	1	L	4	1	6	6	3	3	2	2

ARMP of $K(7, 15)$

	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}	b_{14}	b_{15}
a_1	L	1	2	4	1	1	4	5	5	6	6	7	7	3	3
a_2	1	L	1	4	P	P	4	5	5	6	6	7	7	3	3
a_3	3	2	L	2	P	P	3	4	4	5	5	6	6	7	7
a_4	3	1	2	L	2	1	3	4	4	5	5	6	6	7	7
a_5	1	2	1	2	L	L	L	L	L	L	L	L	L	L	L
a_6	4	5	5	1	L	4	1	6	6	7	7	3	3	2	2
a_7	4	5	5	1	L	4	1	6	6	7	7	3	3	2	2

Figure 3.3.2: ARMPs of $K(7, 9)$, $K(7, 11)$, $K(7, 13)$ and $K(7, 15)$ Produced by Extending the ARMPs of $K(5, 9)$, $K(5, 11)$, $K(5, 13)$ and $K(5, 15)$ Given in Figure 3.3.1

Chapter 4

General Odd/Odd Cases: $K(8m + 1, 2n + 1)$ with $n \geq 4$

We now turn our attention to the more general cases where both parts of the complete bipartite graph are odd, and where the part of smaller order is equivalent to 1, 3, 5, or 7 (mod 8). In this and the following three chapters, we will address each of these four categories of the problem by examining the case with the smallest orders first, i.e. $K(9, 9)$, $K(11, 11)$, $K(13, 13)$, and $K(15, 15)$. These small-order cases will frequently serve as the foundation for extensions to larger orders within each category.

4.1 $K(9, 9)$

Lemma 4.1. There exists an almost resolvable maximum packing of $K(9, 9)$ with 4-cycles.

Proof: Let $K(9, 9)$ be the complete bipartite graph with parts $A = \{a_0, a_1, a_2, a_3, a_0', a_1', a_2', a_3', a_\infty\}$ and $B = \{b_0, b_1, b_2, b_3, b_0', b_1', b_2', b_3', b_\infty\}$. The base almost parallel class $\{(a_0, b_0, a_3, b_1'), (a_2, b_1, a_3', b_2'), (a_1, b_3, a_0', b_\infty), (a_2', b_2, a_\infty, b_3')\}$ cycled modulo 4 on subscripts yields four almost parallel classes. With the partial parallel class $\{(a_0', b_0', a_2', b_2'), (a_1', b_1', a_3', b_3')\}$, these four APCs complete an ARMP of $K(9, 9)$ with 4-cycles.

Alternative Proof. Let $K(9, 9)$ have parts $A = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9\}$ and $B = \{b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9\}$. Then the following is an almost resolvable maximum packing of $K(9, 9)$ with 4-cycles:

$$\pi_1: \{(a_2, b_1, a_3, b_6), (a_1, b_2, a_6, b_7), (a_5, b_3, a_9, b_8), (a_4, b_5, a_7, b_9)\}$$

$$\pi_2: \{(a_4, b_1, a_5, b_6), (a_1, b_4, a_2, b_5), (a_8, b_2, a_9, b_7), (a_3, b_8, a_6, b_9)\}$$

$$\pi_3: \{(a_1, b_3, a_4, b_8), (a_3, b_4, a_8, b_5), (a_7, b_1, a_9, b_6), (a_2, b_7, a_5, b_9)\}$$

$$\pi_4: \{(a_3, b_2, a_4, b_7), (a_2, b_3, a_7, b_8), (a_6, b_4, a_9, b_5), (a_1, b_6, a_8, b_9)\}$$

$$\text{PPC: } \{(a_6, b_1, a_8, b_3), (a_5, b_2, a_7, b_4)\}$$

$$\text{Leave: } \{(a_1, b_1), (a_2, b_2), (a_3, b_3), (a_4, b_4), (a_5, b_5), (a_6, b_6), (a_7, b_7), (a_8, b_8), (a_9, b_9)\}$$

The matrix in Figure 4.1 below gives an alternative representation of this ARMP of $K(9, 9)$.

As in previous proofs in the paper, numbered entries of $i = 1, 2, 3,$ or 4 indicate edges of 4-cycles in the corresponding APCs designated π_i and defined above, while entries of P indicate edges of 4-cycles in the PPC. Cells with entries of L correspond to edges of the leave. Matrices of this type are a simple, compact, and visually effective way to represent ARMPs of $K(m, n)$ and we will rely on them extensively in the remaining proofs of this paper.

	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9
a_1	L	1	3	2	2	4	1	3	4
a_2	1	L	4	2	2	1	3	4	3
a_3	1	4	L	3	3	1	4	2	2
a_4	2	4	3	L	1	2	4	3	1
a_5	2	P	1	P	L	2	3	1	3
a_6	P	1	P	4	4	L	1	2	2
a_7	3	P	4	P	1	3	L	4	1
a_8	P	2	P	3	3	4	2	L	4
a_9	3	2	1	4	4	3	2	1	L

Figure 4.1: Matrix Representation of ARMP of $K(9, 9)$

4.2 $K(9, 2n + 1)$

Theorem 4.2. An almost resolvable maximum packing of 4-cycles exists for $K(9, 2n + 1)$ for all $n \in \mathbb{N}$.

Proof: We will show that it is possible to extend the almost resolvable maximum packing of $K(9, 9)$ with 4-cycles to an ARMP of $K(9, 11)$, which can then be extended to an ARMP of $K(9, 13)$, which can in turn be extended to a packing of $K(9, 15)$. From that point forward a general extension formula is developed to extend any ARMP of $K(9, 2n - 1)$ to an ARMP of $K(9, 2n + 1)$.

First, note that any ARMP of $K(9, 2n + 1)$ with 4-cycles will contain 1 more almost parallel class than $K(9, 2n - 1)$ but will contain the same number (4) of cycles per APC and same number (2) of cycles in the PPC. An ARMP of $K(9, 11)$ can be constructed by extending the packing for $K(9, 9)$ in the following manner:

Select one 4-cycle from each of 3 different almost parallel classes in the packing of $K(9, 9)$, ensuring that the selected cycles are vertex disjoint. The set $\{(a_1, b_3, a_4, b_8), (a_2, b_1, a_3, b_6), (a_5, b_2, a_7, b_4)\}$ is one such a selection. Form the union of this set and the set containing the new cycle $(a_6, b_{10}, a_8, b_{11})$ to create the one new APC needed for the packing of $K(9, 11)$. Replace the cycles (a_1, b_3, a_4, b_8) , (a_2, b_1, a_3, b_6) , and (a_5, b_2, a_7, b_4) that were “shifted” from the 3 original parallel classes with the new cycles $(a_1, b_{10}, a_4, b_{11})$, $(a_2, b_{10}, a_3, b_{11})$, and $(a_5, b_{10}, a_7, b_{11})$ respectively to refill those parallel classes. Add the remaining new edges (a_9, b_{10}) and (a_9, b_{11}) to the leave. This can be seen more easily by examining the matrices that correspond to each ARMP, as shown in Figures 4.2.1 and 4.2.2 that follow.

	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}
a_1	L	1	3	2	2	4	1	3	4	→	→
a_2	1	L	4	2	2	1	3	4	3	→	→
a_3	1	4	L	3	3	1	4	2	2	→	→
a_4	2	4	3	L	1	2	4	3	1	→	→
a_5	2	P	1	P	L	2	3	1	3	→	→
a_6	P	1	P	4	4	L	1	2	2		
a_7	3	P	4	P	1	3	L	4	1	→	→
a_8	P	2	P	3	3	4	2	L	4		
a_9	3	2	1	4	4	3	2	1	L		

Figure 4.2.1: ARMP of $K(9, 9)$ With Intended Shifts for Extension to $K(9, 11)$

	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}
a_1	L	1	5	2	2	4	1	5	4	3	3
a_2	5	L	4	2	2	5	3	4	3	1	1
a_3	5	4	L	3	3	5	4	2	2	1	1
a_4	2	4	5	L	1	2	4	5	1	3	3
a_5	2	5	1	5	L	2	3	1	3	P	P
a_6	P	1	P	4	4	L	1	2	2	5	5
a_7	3	5	4	5	1	3	L	4	1	P	P
a_8	P	2	P	3	3	4	2	L	4	5	5
a_9	3	2	1	4	4	3	2	1	L	L	L

Figure 4.2.2: ARMP of $K(9, 11)$ with 4-Cycles

In like manner, we can extend the ARMP of $K(9, 11)$ to an ARMP of $K(9, 13)$, as illustrated by the matrices in Figures 4.2.3 and 4.2.4 that follow.

	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}
a_1	L	1	5	2	2	4	1	5	4	3	3	→	→
a_2	5	L	4	2	2	5	3	4	3	1	1		
a_3	5	4	L	3	3	5	4	2	2	1	1		
a_4	2	4	5	L	1	2	4	5	1	3	3	→	→
a_5	2	5	1	5	L	2	3	1	3	P	P	→	→
a_6	P	1	P	4	4	L	1	2	2	5	5	→	→
a_7	3	5	4	5	1	3	L	4	1	P	P	→	→
a_8	P	2	P	3	3	4	2	L	4	5	5	→	→
a_9	3	2	1	4	4	3	2	1	L	L	L		

Figure 4.2.3: ARMP of $K(9, 11)$ with Intended Extension to $K(9, 13)$

	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}
a_1	L	1	5	2	2	4	1	5	4	6	6	3	3
a_2	5	L	4	2	2	5	3	4	3	1	1	6	6
a_3	5	4	L	3	3	5	4	2	2	1	1	6	6
a_4	2	4	5	L	1	2	4	5	1	6	6	3	3
a_5	2	6	1	6	L	2	3	1	3	P	P	5	5
a_6	6	1	6	4	4	L	1	2	2	5	5	P	P
a_7	3	6	4	6	1	3	L	4	1	P	P	5	5
a_8	6	2	6	3	3	4	2	L	4	5	5	P	P
a_9	3	2	1	4	4	3	2	1	L	L	L	L	L

Figure 4.2.4: ARMP of $K(9,13)$ with 4-Cycles

Again, in like manner, we can extend the ARMP of $K(9, 13)$ with 4-cycles to an ARMP of $K(9, 15)$, as shown in Figures 4.2.5 and 4.2.6 that follow:

	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}	b_{14}	b_{15}
a_1	L	1	5	2	2	4	1	5	4	6	6	3	3	→	→
a_2	5	L	4	2	2	5	3	4	3	1	1	6	6	→	→
a_3	5	4	L	3	3	5	4	2	2	1	1	6	6	→	→
a_4	2	4	5	L	1	2	4	5	1	6	6	3	3	→	→
a_5	2	6	1	6	L	2	3	1	3	P	P	5	5		
a_6	6	1	6	4	4	L	1	2	2	5	5	P	P	→	→
a_7	3	6	4	6	1	3	L	4	1	P	P	5	5		
a_8	6	2	6	3	3	4	2	L	4	5	5	P	P	→	→
a_9	3	2	1	4	4	3	2	1	L	L	L	L	L		

Figure 4.2.5: ARMP of $K(9, 13)$ with Intended Extension to ARMP of $K(9, 15)$

	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}	b_{14}	b_{15}
a_1	L	1	5	2	2	4	1	5	4	6	6	7	7	3	3
a_2	5	L	4	2	2	5	3	4	3	7	7	6	6	1	1
a_3	5	4	L	3	3	5	4	2	2	7	7	6	6	1	1
a_4	2	4	5	L	1	2	4	5	1	6	6	7	7	3	3
a_5	2	6	1	6	L	2	3	1	3	P	P	5	5	7	7
a_6	7	1	7	4	4	L	1	2	2	5	5	P	P	6	6
a_7	3	6	4	6	1	3	L	4	1	P	P	5	5	7	7
a_8	7	2	7	3	3	4	2	L	4	5	5	P	P	6	6
a_9	3	2	1	4	4	3	2	1	L	L	L	L	L	L	L

Figure 4.2.6: ARMP of $K(9, 15)$ with 4-Cycles

We now generalize this extension procedure. For all $n \geq 8$, an ARMP of $K(9, 2n + 1)$ can be obtained from the extension of $K(9, 2n - 1)$ by shifting cycles $(a_1, b_{2n-2}, a_4, b_{2n-1})$, $(a_2, b_{2n-4}, a_3, b_{2n-3})$, and $(a_5, b_{2n-6}, a_7, b_{2n-5})$ to a new APC and adding the new cycle $(a_6, b_{2n}, a_8, b_{2n+1})$ to

complete this class. The shifted cycles $(a_1, b_{2n-2}, a_4, b_{2n-1})$, $(a_2, b_{2n-4}, a_3, b_{2n-3})$, and $(a_5, b_{2n-6}, a_7, b_{2n-5})$ are replaced, respectively, by the new cycles $(a_1, b_{2n}, a_4, b_{2n+1})$, $(a_2, b_{2n}, a_3, b_{2n+1})$, and $(a_5, b_{2n}, a_7, b_{2n+1})$, refilling the three parallel classes that were reduced by the shift. The remaining edges (a_9, b_{2n}) and (a_9, b_{2n+1}) complete the leave.

4.3 $K(8m + 1, 8m + 1)$

Theorem 4.3. There exists an almost resolvable maximum packing of $K(8m + 1, 8m + 1)$ for all m except possibly $m = 2$.

Proof. We have already proven the case for $m = 1$ (Lemma 4.1). We now examine the case for $m \geq 3$. For $1 \leq x \leq m$, let $A_{2x-1} = \{a_{(2x-1,1)}, a_{(2x-1,2)}, a_{(2x-1,3)}, a_{(2x-1,4)}\}$ and let $A_{2x} = \{a_{(2x,1)}, a_{(2x,2)}, a_{(2x,3)}, a_{(2x,4)}\}$. Similarly, let $B_{2x-1} = \{b_{(2x-1,1)}, b_{(2x-1,2)}, b_{(2x-1,3)}, b_{(2x-1,4)}\}$ and let $B_{2x} = \{b_{(2x,1)}, b_{(2x,2)}, b_{(2x,3)}, b_{(2x,4)}\}$. Then define $K_x(9, 9)$ to be the complete bipartite graph with parts $(A_{2x} \cup A_{2x-1} \cup \{a_9\})$ and $(B_{2x} \cup B_{2x-1} \cup \{b_9\})$. Then we may define $K(8m + 1, 8m + 1)$ as the complete bipartite graph with parts $\bigcup_{x=1}^m A_{2x} \cup A_{2x-1} \cup \{a_9\}$ and $\bigcup_{x=1}^m B_{2x} \cup B_{2x-1} \cup \{b_9\}$ as shown in the Figure 4.3.1 that follows.

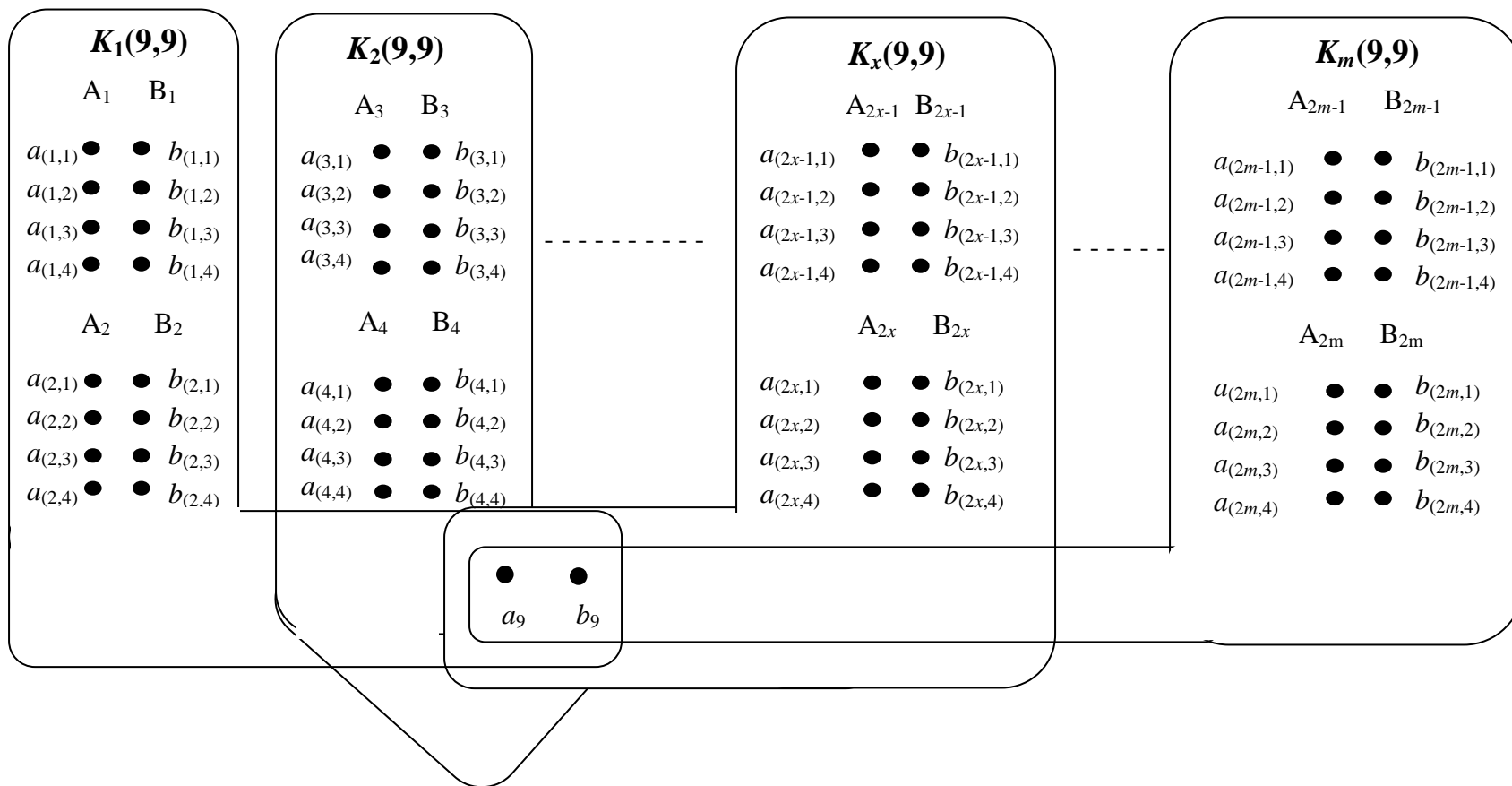


Figure 4.3.1: $K(8m+1, 8m+1)$ Constructed from the Special Union of $K_x(9, 9)$ with Shared Vertices a_9 and b_9

We have already shown that for each x , there exists an ARMP of $K_x(9, 9)$ containing 4 almost parallel classes (call them $\pi_{(x,1)}$, $\pi_{(x,2)}$, $\pi_{(x,3)}$ and $\pi_{(x,4)}$) of 4 cycles each and a partial parallel class $\pi_{(x,P)}$ containing 2 cycles that omit vertices a_9 and b_9 , and with the edge $(a_9, b_9) \in L$.

Let (Q, \circ) be a commutative quasigroup of order $2m$ on the set of symbols $\{1, 2, 3, \dots, 2m\}$ with the set of size 2 holes $H = \{h_x \mid 1 \leq x \leq m\}$. (There exists such a commutative quasigroup since $2m \geq 6$ [8].) Let $h_x = \{y_i \mid 1 \leq i \leq 2\}$ where $y_2 = 2x$ and $y_1 = 2x - 1$. In other words, $H = \{\{1,2\}, \{3,4\}, \{5, 6\} \dots, \{2m - 1, 2m\}\}$. For $y_i \in h_x$, let $K_{(z,w)}(4, 4)$ be the complete bipartite graph with partition $\{A_z, B_w\}$ iff $z \circ w = y_i$. Note that $K_{(z,w)}(4, 4) \neq K_{(w,z)}(4, 4)$ even though $z \circ w = w \circ z$. Thus each $y_i \in h_x$ will be associated with $2(m - 1) = 2m - 2$ distinct $K(4, 4)$.

Recall that every $K(4, 4)$ is completely resolvable to 2 parallel classes of two 4-cycles each (the “even-even” case). For this and later constructions, we will use the parallel classes in Table 4.3 that follows as the standard resolution of $K_{(z,w)}(4, 4)$:

Almost Parallel Classes	Cycles
$\pi_{(z,w,1)}$	$(a_{(z,1)}, b_{(w,1)}, a_{(z,2)}, b_{(w,2)})$ and $(a_{(z,3)}, b_{(w,3)}, a_{(z,4)}, b_{(w,4)})$
$\pi_{(z,w,2)}$	$(a_{(z,1)}, b_{(w,3)}, a_{(z,2)}, b_{(w,4)})$ and $(a_{(z,3)}, b_{(w,1)}, a_{(z,4)}, b_{(w,2)})$

Table 4.3: Standard Resolution of $K_{(z,w)}(4, 4)$

This resolution is pictured in the graph in Figure 4.3.2 below, with $\pi_{(z,w,1)}$ cycles shown in pink and $\pi_{(z,w,2)}$ cycles shown in black:

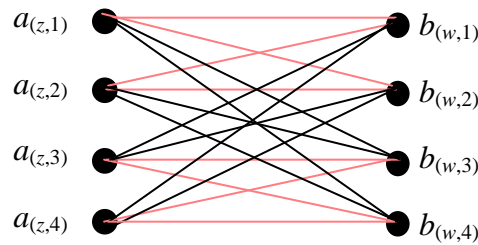


Figure 4.3.2: Standard Resolution of $K_{(z,w)}(4, 4)$

A representation of this resolution in the familiar matrix form, which is more useful for later constructions, is shown below in Figure 4.3.3. Note that for simplicity, the edges of cycles in the almost parallel class $\pi_{(z,w,1)}$ are shown as entries of 1 in pink shaded cells and edges of cycles in the almost parallel class $\pi_{(z,w,2)}$ are shown as entries of 2 in gray shaded cells.

	$b_{(w,1)}$	$b_{(w,2)}$	$b_{(w,3)}$	$b_{(w,4)}$
$a_{(z,1)}$	1	1	2	2
$a_{(z,2)}$	1	1	2	2
$a_{(z,3)}$	2	2	1	1
$a_{(z,4)}$	2	2	1	1

Figure 4.3.3: Standard Resolution of $K_{(z,w)}(4, 4)$ in Matrix Form

For $y_1 \in h_x$, form the union of $\pi_{(x,1)}$ with $\pi_{(z,w,1)}$ for all z, w such that $z \circ w = y_1$. This forms one complete APC (which we name $\pi_{(x,1)}$ for its “parent” APC) containing $4 + 2(2m - 2) = 4m$ cycles for $K(8m + 1, 8m + 1)$. Similarly, form the union of $\pi_{(x,2)}$ and $\pi_{(z,w,2)}$ naming it $\pi_{(x,2)}$, making a second complete APC for $K(8m + 1, 8m + 1)$. Likewise, for $y_2 \in h_x$, $\pi_{(x,3)} \cup \bigcup_{z \circ w = y_2} \pi_{(z,w,1)}$ and $\pi_{(x,4)} \cup \bigcup_{z \circ w = y_2} \pi_{(z,w,2)}$ forms two additional complete APCs ($\pi_{(x,3)}$ and $\pi_{(x,4)}$) for a total of 4 almost parallel classes for $K(8m + 1, 8m + 1)$. Running over all m holes will yield $4m$ APCs of $4m$ cycles each, as required. Finally, $\bigcup_{x=1}^m \pi_{(x,P)}$ forms the necessary partial parallel class with $2m$ 4-cycles.

Example 4.3. As an example, we will construct an ARMP of $K(25, 25)$ which will require 12 APCs of 12 cycles each plus a PPC of 6 cycles. For simplicity of presentation, APCs will be numbered from 1 to 12 using the correspondence $\pi_{(x,i)} \rightarrow 4(x - 1) + i$.

Since $25 = 8(3) + 1$, we will need a commutative quasigroup with holes of size 2 and order $2(3) = 6$. The one shown in the following Figure 3.11 satisfies our need:

o	1	2	3	4	5	6
1	1	2	5	6	3	4
2	2	1	6	5	4	3
3	5	6	3	4	1	2
4	6	5	4	3	2	1
5	3	4	1	2	5	6
6	4	3	2	1	6	5

Figure 4.3.4: Commutative Quasigroup of Order 6

In this example, a copy of the ARMP of $K(9, 9)$ given in the proof of Theorem 4.1 is placed on $K_x(9, 9)$ for $1 \leq x \leq 3$. The resolution for $K_1(9, 9)$ is shown in Figure 4.3.5 that follows.

Using the correspondence established earlier ($\pi_{(x,i)} \rightarrow 4(x-1) + i$), almost parallel class $\pi_{(1,1)}$ is renamed 1, class $\pi_{(1,2)}$ is renamed 2, etc. Note that edges of the cycles for APC 1 are shown in the blue shaded cells with entries of 1, edges of the cycles for APC 2 are shown in the red shaded cells with entries of 2, etc. Edges of the cycles in the partial parallel class $\pi_{(1,P)}$ are shown in the black/white striped cells with entries of P. Edges of the leave are shown in cells on the diagonal and are shaded with a black trellis design.

	$b_{(1,1)}$	$b_{(1,2)}$	$b_{(1,3)}$	$b_{(1,4)}$	$b_{(2,1)}$	$b_{(2,2)}$	$b_{(2,3)}$	$b_{(2,4)}$	b_9
$a_{(1,1)}$	+	3	1	2	2	4	3	1	4
$a_{(1,2)}$	3	+	4	2	2	3	1	4	1
$a_{(1,3)}$	3	4	+	1	1	3	4	2	2
$a_{(1,4)}$	2	4	1	+	3	2	4	1	3
$a_{(2,1)}$	2	P	3	P	+	2	1	3	1
$a_{(2,2)}$	P	3	P	4	4	+	3	2	2
$a_{(2,3)}$	1	P	4	P	3	1	+	4	3
$a_{(2,4)}$	P	2	P	1	1	4	2	+	4
a_9	1	2	3	4	4	1	2	3	+

Figure 4.3.5: The resolution for $K_1(9, 9)$ in the ARMP of $K(25, 25)$

For hole h_1 , we note that $y_1 = 2(1 - 1) + 1 = 1$. In (Q, \circ) , $1 = 3 \circ 5 = 5 \circ 3 = 4 \circ 6 = 6 \circ 4$. The standard resolution is applied to each of the associated $K(4, 4)$ of $K(25, 25)$ and is shown in Figure 4.3.6. Note that edges of the cycles in the parallel classes $\pi_{(3,5,1)}$, $\pi_{(5,3,1)}$, $\pi_{(4,6,1)}$, and $\pi_{(6,4,1)}$ are shown in the pink shaded cells with entries of 1 and the edges of the cycles in the parallel classes $\pi_{(3,5,2)}$, $\pi_{(5,3,2)}$, $\pi_{(4,6,2)}$, and $\pi_{(6,4,2)}$ are shown in the gray shaded cells with entries of 2.

$K_{(3,5)}(4, 4)$				
	$b_{(5,1)}$	$b_{(5,2)}$	$b_{(5,3)}$	$b_{(5,4)}$
$a_{(3,1)}$	1	1	2	2
$a_{(3,2)}$	1	1	2	2
$a_{(3,3)}$	2	2	1	1
$a_{(3,4)}$	2	2	1	1

$K_{(5,3)}(4, 4)$				
	$b_{(3,1)}$	$b_{(3,2)}$	$b_{(3,3)}$	$b_{(3,4)}$
$a_{(5,1)}$	1	1	2	2
$a_{(5,2)}$	1	1	2	2
$a_{(5,3)}$	2	2	1	1
$a_{(5,4)}$	2	2	1	1

$K_{(4,6)}(4, 4)$				
	$b_{(6,1)}$	$b_{(6,2)}$	$b_{(6,3)}$	$b_{(6,4)}$
$a_{(4,1)}$	1	1	2	2
$a_{(4,2)}$	1	1	2	2
$a_{(4,3)}$	2	2	1	1
$a_{(4,4)}$	2	2	1	1

$K_{(6,4)}(4, 4)$				
	$b_{(4,1)}$	$b_{(4,2)}$	$b_{(4,3)}$	$b_{(4,4)}$
$a_{(6,1)}$	1	1	2	2
$a_{(6,2)}$	1	1	2	2
$a_{(6,3)}$	2	2	1	1
$a_{(6,4)}$	2	2	1	1

Figure 4.3.6: Standard Resolution of $K_{(z,w)}(4, 4)$ for $y_1 = 1 \in h_1$ of CQG in Figure 4.3.4

We now form the union of $\pi_{(1,1)}$ with $\pi_{(3,5,1)}$, $\pi_{(5,3,1)}$, $\pi_{(4,6,1)}$, and $\pi_{(6,4,1)}$ to produce an almost parallel class for $K(25, 25)$ which we name $\pi_{(1,1)}$. The construction of the ARMP of $K(25, 25)$ thus far is shown in the matrix of Figure 4.3.7 that follows. $K_1(9, 9)$ is outlined in a thick, black border. Almost parallel class $\pi_{(1,1)}$ is renamed “1” using the correspondence rule we established earlier: $\pi_{(x,i)} \rightarrow 4(x - 1) + i$. Thus edges of cycles in $\pi_{(1,1)}$ are shown in the cells shaded blue with entries of 1.

	$b_{(1,1)}$	$b_{(1,2)}$	$b_{(1,3)}$	$b_{(1,4)}$	$b_{(2,1)}$	$b_{(2,2)}$	$b_{(2,3)}$	$b_{(2,4)}$	$b_{(3,1)}$	$b_{(3,2)}$	$b_{(3,3)}$	$b_{(3,4)}$	$b_{(4,1)}$	$b_{(4,2)}$	$b_{(4,3)}$	$b_{(4,4)}$	$b_{(5,1)}$	$b_{(5,2)}$	$b_{(5,3)}$	$b_{(5,4)}$	$b_{(6,1)}$	$b_{(6,2)}$	$b_{(6,3)}$	$b_{(6,4)}$	b_9	
$a_{(1,1)}$	+	3	1	2	2	4	3	1																	4	
$a_{(1,2)}$	3	+	4	2	2	3	1	4																		1
$a_{(1,3)}$	3	4	+	1	1	3	4	2																		2
$a_{(1,4)}$	2	4	1	+	3	2	4	1																		3
$a_{(2,1)}$	2	P	3	P	+	2	1	3																		1
$a_{(2,2)}$	P	3	P	4	4	+	3	2																		2
$a_{(2,3)}$	1	P	4	P	3	1	+	4																		3
$a_{(2,4)}$	P	2	P	1	1	4	2	+																		4
$a_{(3,1)}$									+								1	1								
$a_{(3,2)}$										+							1	1								
$a_{(3,3)}$											+								1	1						
$a_{(3,4)}$												+							1	1						
$a_{(4,1)}$													+								1	1				
$a_{(4,2)}$														+							1	1				
$a_{(4,3)}$															+								1	1		
$a_{(4,4)}$																+							1	1		
$a_{(5,1)}$									1	1																
$a_{(5,2)}$									1	1																
$a_{(5,3)}$											1	1														
$a_{(5,4)}$											1	1														
$a_{(6,1)}$													1	1												
$a_{(6,2)}$													1	1												
$a_{(6,3)}$															1	1										
$a_{(6,4)}$															1	1										
a_9	1	2	3	4	4	1	2	3																		

Figure 4.3.7: Construction of ARMP of $K(25, 25)$, Adding $K_1(9, 9)$ and Completing π_1

Continuing the construction process, we now form the union of $\pi_{(1,2)}$ with $\pi_{(3,5,2)}$, $\pi_{(5,3,2)}$, $\pi_{(4,6,2)}$, and $\pi_{(6,4,2)}$ to produce an almost parallel class for $K(25, 25)$ which we name $\pi_{(1,2)}$ and then rename “2” using the correspondence rule established earlier: $\pi_{(x,i)} \rightarrow 4(x-1) + i$. Thus edges of cycles in $\pi_{(1,2)}$ are shown in the cells shaded red with entries of 2 in the updated matrix that follows in Figure 4.3.8.

	$b_{(1,1)}$	$b_{(1,2)}$	$b_{(1,3)}$	$b_{(1,4)}$	$b_{(2,1)}$	$b_{(2,2)}$	$b_{(2,3)}$	$b_{(2,4)}$	$b_{(3,1)}$	$b_{(3,2)}$	$b_{(3,3)}$	$b_{(3,4)}$	$b_{(4,1)}$	$b_{(4,2)}$	$b_{(4,3)}$	$b_{(4,4)}$	$b_{(5,1)}$	$b_{(5,2)}$	$b_{(5,3)}$	$b_{(5,4)}$	$b_{(6,1)}$	$b_{(6,2)}$	$b_{(6,3)}$	$b_{(6,4)}$	b_9	
$a_{(1,1)}$	+	3	1	2	2	4	3	1																	4	
$a_{(1,2)}$	3	+	4	2	2	3	1	4																		1
$a_{(1,3)}$	3	4	+	1	1	3	4	2																		2
$a_{(1,4)}$	2	4	1	+	3	2	4	1																		3
$a_{(2,1)}$	2	P	3	P	+	2	1	3																		1
$a_{(2,2)}$	P	3	P	4	4	+	3	2																		2
$a_{(2,3)}$	1	P	4	P	3	1	+	4																		3
$a_{(2,4)}$	P	2	P	1	1	4	2	+																		4
$a_{(3,1)}$									+								1	1	2	2						
$a_{(3,2)}$										+							1	1	2	2						
$a_{(3,3)}$											+						2	2	1	1						
$a_{(3,4)}$												+					2	2	1	1						
$a_{(4,1)}$													+								1	1	2	2		
$a_{(4,2)}$														+							1	1	2	2		
$a_{(4,3)}$															+						2	2	1	1		
$a_{(4,4)}$																+					2	2	1	1		
$a_{(5,1)}$									1	1	2	2					+									
$a_{(5,2)}$									1	1	2	2						+								
$a_{(5,3)}$									2	2	1	1							+							
$a_{(5,4)}$									2	2	1	1									+					
$a_{(6,1)}$													1	1	2	2						+				
$a_{(6,2)}$													1	1	2	2							+			
$a_{(6,3)}$													2	2	1	1								+		
$a_{(6,4)}$													2	2	1	1									+	
a_9	1	2	3	4	4	1	2	3																		+

Figure 4.3.8: Construction of ARMP of $K(25, 25)$, Completing π_2

Continuing the construction, we repeat this process for $y_2 = 2 \in h_1$ by merging $\pi_{(1,3)}$ with $\pi_{(3,6,1)}$, $\pi_{(6,3,1)}$, $\pi_{(5,4,1)}$, and $\pi_{(4,5,1)}$ and by merging $\pi_{(1,3)}$ with $\pi_{(3,6,2)}$, $\pi_{(6,3,2)}$, $\pi_{(5,4,2)}$, and $\pi_{(4,5,2)}$, forming two additional complete APCs, $\pi_{(1,3)}$ and $\pi_{(1,4)}$ for $K(25, 25)$. These classes are renamed using the established correspondence rule and the edges for cycles in each class are shown in the cells numbered 3 (green) and 4 (lavender) in the matrix of Figure 4.3.9 that follows.

	$b_{(1,1)}$	$b_{(1,2)}$	$b_{(1,3)}$	$b_{(1,4)}$	$b_{(2,1)}$	$b_{(2,2)}$	$b_{(2,3)}$	$b_{(2,4)}$	$b_{(3,1)}$	$b_{(3,2)}$	$b_{(3,3)}$	$b_{(3,4)}$	$b_{(4,1)}$	$b_{(4,2)}$	$b_{(4,3)}$	$b_{(4,4)}$	$b_{(5,1)}$	$b_{(5,2)}$	$b_{(5,3)}$	$b_{(5,4)}$	$b_{(6,1)}$	$b_{(6,2)}$	$b_{(6,3)}$	$b_{(6,4)}$	b_9	
$a_{(1,1)}$	+	3	1	2	2	4	3	1																	4	
$a_{(1,2)}$	3	+	4	2	2	3	1	4																		1
$a_{(1,3)}$	3	4	+	1	1	3	4	2																		2
$a_{(1,4)}$	2	4	1	+	3	2	4	1																		3
$a_{(2,1)}$	2	P	3	P	+	2	1	3																		1
$a_{(2,2)}$	P	3	P	4	4	+	3	2																		2
$a_{(2,3)}$	1	P	4	P	3	1	+	4																		3
$a_{(2,4)}$	P	2	P	1	1	4	2	+																		4
$a_{(3,1)}$									+								1	1	2	2	3	3	4	4		
$a_{(3,2)}$										+							1	1	2	2	3	3	4	4		
$a_{(3,3)}$											+						2	2	1	1	4	4	3	3		
$a_{(3,4)}$												+					2	2	1	1	4	4	3	3		
$a_{(4,1)}$													+				3	3	4	4	1	1	2	2		
$a_{(4,2)}$														+			3	3	4	4	1	1	2	2		
$a_{(4,3)}$															+		4	4	3	3	2	2	1	1		
$a_{(4,4)}$																+	4	4	3	3	2	2	1	1		
$a_{(5,1)}$									1	1	2	2	3	3	4	4	+									
$a_{(5,2)}$									1	1	2	2	3	3	4	4		+								
$a_{(5,3)}$									2	2	1	1	4	4	3	3			+							
$a_{(5,4)}$									2	2	1	1	4	4	3	3				+						
$a_{(6,1)}$									3	3	4	4	1	1	2	2					+					
$a_{(6,2)}$									3	3	4	4	1	1	2	2						+				
$a_{(6,3)}$									4	4	3	3	2	2	1	1							+			
$a_{(6,4)}$									4	4	3	3	2	2	1	1								+		
a_9	1	2	3	4	4	1	2	3																		+

Figure 4.3.9: Construction of ARMP of $K(25, 25)$, Completing π_3 and π_4

Finally, placing a copy of the ARMP for $K(9, 9)$ established in the proof of Theorem 4.1 on $K_2(9, 9)$ and $K_3(9, 9)$, and repeating the parallel class union procedures for holes h_2 and h_3 completes the construction of the ARMP of $K(25, 25)$ and is shown in the matrix of Figure 4.3.10 that follows.

	$b_{(1,1)}$	$b_{(1,2)}$	$b_{(1,3)}$	$b_{(1,4)}$	$b_{(2,1)}$	$b_{(2,2)}$	$b_{(2,3)}$	$b_{(2,4)}$	$b_{(3,1)}$	$b_{(3,2)}$	$b_{(3,3)}$	$b_{(3,4)}$	$b_{(4,1)}$	$b_{(4,2)}$	$b_{(4,3)}$	$b_{(4,4)}$	$b_{(5,1)}$	$b_{(5,2)}$	$b_{(5,3)}$	$b_{(5,4)}$	$b_{(6,1)}$	$b_{(6,2)}$	$b_{(6,3)}$	$b_{(6,4)}$	b_9	
$a_{(1,1)}$	+	3	1	2	2	4	3	1	9	9	10	10	11	11	12	12	5	5	6	6	7	7	8	8	4	
$a_{(1,2)}$	3	+	4	2	2	3	1	4	9	9	10	10	11	11	12	12	5	5	6	6	7	7	8	8	1	
$a_{(1,3)}$	3	4	+	1	1	3	4	2	10	10	9	9	12	12	11	11	6	6	5	5	8	8	7	7	2	
$a_{(1,4)}$	2	4	1	+	3	2	4	1	10	10	9	9	12	12	11	11	6	6	5	5	8	8	7	7	3	
$a_{(2,1)}$	2	P	3	P	+	2	1	3	11	11	12	12	9	9	10	10	7	7	8	8	5	5	6	6	1	
$a_{(2,2)}$	P	3	P	4	4	+	3	2	11	11	12	12	9	9	10	10	7	7	8	8	5	5	6	6	2	
$a_{(2,3)}$	1	P	4	P	3	1	+	4	12	12	11	11	10	10	9	9	8	8	7	7	6	6	5	5	3	
$a_{(2,4)}$	P	2	P	1	1	4	2	+	12	12	11	11	10	10	9	9	8	8	7	7	6	6	5	5	4	
$a_{(3,1)}$	9	9	10	10	11	11	12	12	+	7	5	6	6	8	7	5	1	1	2	2	3	3	4	4	8	
$a_{(3,2)}$	9	9	10	10	11	11	12	12	7	+	8	6	6	7	5	8	1	1	2	2	3	3	4	4	5	
$a_{(3,3)}$	10	10	9	9	12	12	11	11	7	8	+	5	5	7	8	6	2	2	1	1	4	4	3	3	6	
$a_{(3,4)}$	10	10	9	9	12	12	11	11	6	8	5	+	7	6	8	5	2	2	1	1	4	4	3	3	7	
$a_{(4,1)}$	11	11	12	12	9	9	10	10	6	P	7	P	+	6	5	7	3	3	4	4	1	1	2	2	5	
$a_{(4,2)}$	11	11	12	12	9	9	10	10	P	7	P	8	+	8	8	7	6	3	3	4	4	1	1	2	2	6
$a_{(4,3)}$	12	12	11	11	10	10	9	9	5	P	8	P	7	+	5	8	4	4	3	3	2	2	1	1	7	
$a_{(4,4)}$	12	12	11	11	10	10	9	9	P	6	P	5	5	8	6	+	4	4	3	3	2	2	1	1	8	
$a_{(5,1)}$	5	5	6	6	7	7	8	8	1	1	2	2	3	3	4	4	+	11	9	10	10	12	11	9	12	
$a_{(5,2)}$	5	5	6	6	7	7	8	8	1	1	2	2	3	3	4	4	11	+	12	10	10	11	9	12	9	
$a_{(5,3)}$	6	6	5	5	8	8	7	7	2	2	1	1	4	4	3	3	11	12	+	9	9	11	12	10	10	
$a_{(5,4)}$	6	6	5	5	8	8	7	7	2	2	1	1	4	4	3	3	10	12	9	+	11	10	12	9	11	
$a_{(6,1)}$	7	7	8	8	5	5	6	6	3	3	4	4	1	1	2	2	10	P	11	P	+	10	9	11	9	
$a_{(6,2)}$	7	7	8	8	5	5	6	6	3	3	4	4	1	1	2	2	P	11	P	12	+	11	10	10		
$a_{(6,3)}$	8	8	7	7	6	6	5	5	4	4	3	3	2	2	1	1	9	P	12	P	11	+	9	12	11	
$a_{(6,4)}$	8	8	7	7	6	6	5	5	4	4	3	3	2	2	1	1	P	10	P	9	9	12	+	10	12	
a_9	1	2	3	4	4	1	2	3	5	6	7	8	8	5	6	7	9	10	11	12	12	9	10	11	+	

Figure 4.3.10: Completion of the Construction of ARMP of $K(25, 25)$

4.4 $K(8m + 1, 8m + 3)$

Theorem 4.4: There exists an almost resolvable maximum packing of $K(8m + 1, 8m + 3)$ for all m except possibly $m = 2$.

Proof. We will extend the ARMP of $K(8m + 1, 8m + 1)$ to an ARMP of $K(8m + 1, 8m + 3)$ using the same procedures used to extend the ARMP of $K(9, 9)$ to $K(9, 11)$.

In the proof of Theorem 4.1, we established an ARMP of $K(9, 9)$ as given by the following matrix (Figure 4.4.1):

	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9
a_1	L	1	3	2	2	4	1	3	4
a_2	1	L	4	2	2	1	3	4	3
a_3	1	4	L	3	3	1	4	2	2
a_4	2	4	3	L	1	2	4	3	1
a_5	2	P	1	P	L	2	3	1	3
a_6	P	1	P	4	4	L	1	2	2
a_7	3	P	4	P	1	3	L	4	1
a_8	P	2	P	3	3	4	2	L	4
a_9	3	2	1	4	4	3	2	1	L

Figure 4.4.1: ARMP of $K(9, 9)$

Recall that an extension of an ARMP of $K(9, 9)$ to an ARMP of $K(9, 11)$ requires the addition of two vertices to the B side of the partition, which in turn introduces exactly one new almost parallel class with the number of cycles (4) per APC remaining the same. This extension as shown in the proof of Theorem 4.2 is accomplished by shifting cycles out of initial APCs and into the new APC and replacing those cycles with new cycles created from the new edges that join vertices in part A to the new vertices in Part B. Such an extension is illustrated in the matrix in Figure 4.4.2 that follows.

	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}
a_1	L	5	3	2	2	4	5	3	4	1	1
a_2	1	L	5	2	2	1	3	5	3	4	4
a_3	1	4	L	5	5	1	4	2	2	3	3
a_4	5	4	3	L	1	5	4	3	1	2	2
a_5	5	P	1	P	L	5	3	1	3	2	2
a_6	P	5	P	4	4	L	5	2	2	1	1
a_7	3	P	5	P	1	3	L	5	1	4	4
a_8	P	2	P	5	5	4	2	L	4	3	3
a_9	3	2	1	4	4	3	2	1	L	L	L

Figure 4.4.2: ARMP of $K(9, 11)$ Constructed from ARMP of $K(9,9)$

The more general case of extending an ARMP of $K(8m + 1, 8m + 1)$ to an ARMP of $K(8m + 1, 8m + 3)$ will use this same extension process.

First, construct an ARMP of $K(8m + 1, 8m + 1)$ using m copies of Figure 4.4.1 above and the construction method given in the proof of Theorem 4.3 (see Figure 4.4.3 on next page). Recall that in this construction, for $1 \leq x \leq m$, copy x of the ARMP of $K(9, 9)$ in Figure 4.4.1 is denoted as $K_x(9, 9)$ and contains almost parallel classes $\pi_{(x,1)}$, $\pi_{(x,2)}$, $\pi_{(x,3)}$ and $\pi_{(x,4)}$. Note that in Figure 4.4.3, we use a slightly different numbering system on the vertex subscripts to correspond more closely with the system used in Figure 4.4.1.

$K(8m + 1, 8m + 1)$

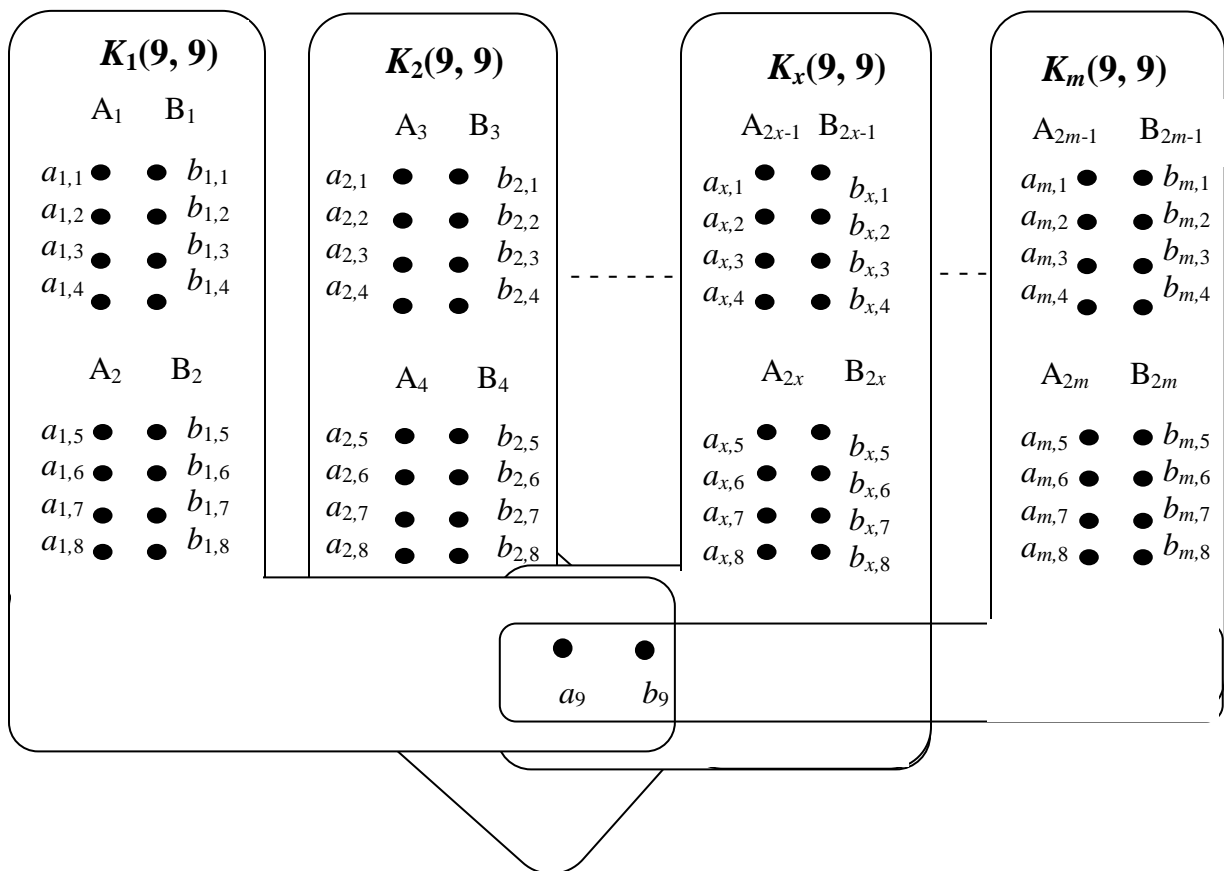


Figure 4.4.3: ARMP of $K(8m + 1, 8m + 1)$

Next, append two new vertices, b_{8m+2} and b_{8m+3} , to $K(8m + 1, 8m + 1)$ and add all associated edges to yield $K(8m + 1, 8m + 3)$. Create a new almost parallel class $\pi_{(x,5)}$ by making the shifts of cycles as shown in Table 4.4.1 from the indicated existing APCs in each $K_x(9, 9)$ to the new APC:

APC	Cycle
$\pi_{(x,1)}$	$(a_{x,1}, b_{x,2}, a_{x,6}, b_{x,7})$
$\pi_{(x,2)}$	$(a_{x,4}, b_{x,1}, a_{x,5}, b_{x,6})$
$\pi_{(x,3)}$	$(a_{x,3}, b_{x,4}, a_{x,8}, b_{x,5})$
$\pi_{(x,4)}$	$(a_{x,2}, b_{x,3}, a_{x,7}, b_{x,8})$

Table 4.4.1: Cycle Shifts from $K(8m + 1, 8m + 1)$ to $K(8m + 3, 8m + 3)$

Finally, replace, respectively, each shifted cycle listed in the table above with new cycles $(a_{x,1}, b_{8m+2}, a_{x,6}, b_{8m+3})$, $(a_{x,4}, b_{8m+2}, a_{x,5}, b_{8m+3})$, $(a_{x,3}, b_{8m+2}, a_{x,4}, b_{8m+3})$ and $(a_{x,2}, b_{8m+2}, a_{x,7}, b_{8m+3})$ to refill parallel classes $\pi_{(x,1)}$, $\pi_{(x,2)}$, $\pi_{(x,3)}$, and $\pi_{(x,4)}$. The result will be a new APC with $4m$ cycles as required. The remaining new edges complete the leave.

Chapter 5

General Odd/Odd Cases: $K(8m + 3, 2n + 1)$ with $n \geq 4m$

5.1 $K(11, 11)$

Lemma 5.1: There exists an ARMP of $K(11, 11)$ with 4-cycles.

Proof. The matrix in Figure 5.1 below illustrates an ARMP of $K(11, 11)$ produced by extending the ARMP of $K(9, 9)$ given in the proof of Lemma 4.1.

	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}
a_1	L	1	5	2	2	4	1	5	4	3	3
a_2	5	L	4	2	2	5	3	4	3	1	1
a_3	5	4	L	3	3	5	4	2	2	1	1
a_4	2	4	5	L	1	2	4	5	1	3	3
a_5	2	5	1	5	L	2	3	1	3	P	P
a_6	P	1	P	4	4	L	1	2	2	5	5
a_7	3	5	4	5	1	3	L	4	1	P	P
a_8	P	2	P	3	3	4	2	L	4	5	5
a_9	3	2	1	4	4	3	2	1	L	L	L
a_{10}	4	3	2	1	5	1	5	3	L	4	2
a_{11}	4	3	2	1	5	1	5	3	L	4	2

Figure 5.1: ARMP of $K(11, 11)$

5.2 $K(11, 2n + 1)$

Theorem 5.2. An almost resolvable maximum packing with 4-cycles exists for $K(11, 2n + 1)$ for all $n \in \mathbb{N}$.

Proof. For $n \leq 5$, we may rely on the proofs of Theorems 3.1, 3.2.1, 3.3, 4.2, and Lemma 5.1.

For $n > 5$, we will show that an ARMP of $K(11, 2n + 1)$ can be produced from an extension of the ARMP of $K(9, 2n + 1)$ in the same way that an ARMP of $K(7, 2n + 1)$ is produced from an

extension of the ARMP of $K(9, 2n + 1)$. Recall that in such an extension, the number of vertices in part A increases by 2, the number of cycles per almost parallel class increases by one, while the number (n) of parallel classes remains the same, as does the number of cycles in the partial parallel class.

An extension of the ARMP of $K(9, 2n + 1)$ to an ARMP of $K(11, 2n + 1)$ can be accomplished using new 4-cycles created from the addition of edges between vertices in part B and the new vertices a_{10} and a_{11} in part A. To do so, we must, as before, find a set of n disjoint pairs of vertices in part B that are each not “hit” by a different APC in the ARMP of $K(9, 2n + 1)$ and that do not include vertex b_9 , which will be hit by additional edges of the leave in $K(11, 2n + 1)$. As we saw in the proof of $K(7, 2n + 1)$, this is always possible given the nature of the construction that extends the ARMP of $K(9, 2n - 1)$ to an ARMP of $K(9, 2n + 1)$ by shifting cycles from existing almost parallel classes into the new APC.

The matrices in Figures 5.2.1 and 5.2.2 on the next two pages depict the almost resolvable maximum packings of $K(9, 2n + 1)$ for $5 \leq n \leq 10$ developed using the extension construction described in the proof of Theorem 4.2. The numbers given in bold print directly beneath each matrix indicate certain parallel classes (among others) that do not hit the indicated vertices of part B. In each illustration, we find a set of n disjoint pairs of vertices in part B that are each not hit by a different APC.

ARMP of $K(9, 11)$

	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}
a_1	L	1	5	2	2	4	1	5	4	3	3
a_2	5	L	4	2	2	5	3	4	3	1	1
a_3	5	4	L	3	3	5	4	2	2	1	1
a_4	2	4	5	L	1	2	4	5	1	3	3
a_5	2	5	1	5	L	2	3	1	3	P	P
a_6	P	1	P	4	4	L	1	2	2	5	5
a_7	3	5	4	5	1	3	L	4	1	P	P
a_8	P	2	P	3	3	4	2	L	4	5	5
a_9	3	2	1	4	4	3	2	1	L	L	L
	4	3	2	1	5	1	5	3		4	2

ARMP of $K(9, 13)$

	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}
a_1	L	1	5	2	2	4	1	5	4	6	6	3	3
a_2	5	L	4	2	2	5	3	4	3	1	1	6	6
a_3	5	4	L	3	3	5	4	2	2	1	1	6	6
a_4	2	4	5	L	1	2	4	5	1	6	6	3	3
a_5	2	6	1	6	L	2	3	1	3	P	P	5	5
a_6	6	1	6	4	4	L	1	2	2	5	5	P	P
a_7	3	6	4	6	1	3	L	4	1	P	P	5	5
a_8	6	2	6	3	3	4	2	L	4	5	5	P	P
a_9	3	2	1	4	4	3	2	1	L	L	L	L	L
	4	5	2	1	5	1	6	6		3	3	4	2

ARMP of $K(9, 15)$

	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}	b_{14}	b_{15}
a_1	L	1	5	2	2	4	1	5	4	6	6	7	7	3	3
a_2	5	L	4	2	2	5	3	4	3	7	7	6	6	1	1
a_3	5	4	L	3	3	5	4	2	2	7	7	6	6	1	1
a_4	2	4	5	L	1	2	4	5	1	6	6	7	7	3	3
a_5	2	6	1	6	L	2	3	1	3	P	P	5	5	7	7
a_6	7	1	7	4	4	L	1	2	2	5	5	P	P	6	6
a_7	3	6	4	6	1	3	L	4	1	P	P	5	5	7	7
a_8	7	2	7	3	3	4	2	L	4	5	5	P	P	6	6
a_9	3	2	1	4	4	3	2	1	L	L	L	L	L	L	L
	4	5	2	1	6	6	7	7		3	3	1	4	5	2

Figure 5.2.1: ARMPs of $K(9, 11)$, $K(9, 13)$, and $K(9, 15)$ with Missing APCs on Part B Vertices

ARMP of $K(9, 17)$

	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}	b_{14}	b_{15}	b_{16}	b_{17}
a_1	L	1	5	2	2	4	1	5	4	6	6	7	7	8	8	3	3
a_2	5	L	4	2	2	5	3	4	3	7	7	8	8	1	1	6	6
a_3	5	4	L	3	3	5	4	2	2	7	7	8	8	1	1	6	6
a_4	2	4	5	L	1	2	4	5	1	6	6	7	7	8	8	3	3
a_5	2	6	1	6	L	2	3	1	3	8	8	5	5	7	7	P	P
a_6	7	1	7	4	4	L	1	2	2	5	5	P	P	6	6	8	8
a_7	3	6	4	6	1	3	L	4	1	8	8	5	5	7	7	P	P
a_8	7	2	7	3	3	4	2	L	4	5	5	P	P	6	6	8	8
a_9	3	2	1	4	4	3	2	1	L	L	L	L	L	L	L	L	L
	1	3	3	1	6	6	8	8		2	2	4	4	5	5	7	7

ARMP of $K(9, 19)$

	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}	b_{14}	b_{15}	b_{16}	b_{17}	b_{18}	b_{19}
a_1	L	1	5	2	2	4	1	5	4	6	6	7	7	8	8	9	9	3	3
a_2	5	L	4	2	2	5	3	4	3	7	7	8	8	9	9	6	6	1	1
a_3	5	4	L	3	3	5	4	2	2	7	7	8	8	9	9	6	6	1	1
a_4	2	4	5	L	1	2	4	5	1	6	6	7	7	8	8	9	9	3	3
a_5	2	6	1	6	L	2	3	1	3	8	8	9	9	7	7	P	P	5	5
a_6	7	1	7	4	4	L	1	2	2	5	5	P	P	6	6	8	8	9	9
a_7	3	6	4	6	1	3	L	4	1	8	8	9	9	7	7	P	P	5	5
a_8	7	2	7	3	3	4	2	L	4	5	5	P	P	6	6	8	8	9	9
a_9	3	2	1	4	4	3	2	1	L	L	L	L	L	L	L	L	L	L	L
	1	3	3	1	6	6	9	9		2	2	4	4	5	5	7	7	8	8

ARMP of $K(9, 21)$

	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}	b_{14}	b_{15}	b_{16}	b_{17}	b_{18}	b_{19}	b_{20}	b_{21}
a_1	L	1	5	2	2	4	1	5	4	6	6	7	7	8	8	9	9	10	10	3	3
a_2	5	L	4	2	2	5	3	4	3	7	7	8	8	9	9	10	10	1	1	6	6
a_3	5	4	L	3	3	5	4	2	2	7	7	8	8	9	9	10	10	1	1	6	6
a_4	2	4	5	L	1	2	4	5	1	6	6	7	7	8	8	9	9	10	10	3	3
a_5	2	6	1	6	L	2	3	1	3	8	8	9	9	10	10	P	P	5	5	7	7
a_6	7	1	7	4	4	L	1	2	2	5	5	P	P	6	6	8	8	9	9	10	10
a_7	3	6	4	6	1	3	L	4	1	8	8	9	9	10	10	P	P	5	5	7	7
a_8	7	2	7	3	3	4	2	L	4	5	5	P	P	6	6	8	8	9	9	10	10
a_9	3	2	1	4	4	3	2	1	L	L	L	L	L	L	L	L	L	L	L	L	L
	1	3	3	1	6	6	10	10		2	2	4	4	5	5	7	7	8	8	9	9

Figure 5.2.2: ARMP of $K(9, 17)$, $K(9, 19)$ and $K(9, 21)$ with Missing APCs on Part B Vertices

The reader will observe in these illustrations and recall from the proof of Theorem 4.2 that this construction was generalized to produce an ARMP of $K(9, 2n + 1)$ from $K(9, 2n - 1)$ only after n reached a value of 8. Nevertheless, the existence of an ARMP of $K(11, 2n + 1)$ for $n \in \{5, 6\}$ is guaranteed by the existence of the identified sets of n disjoint pairs of vertices in part B, each not hit by a different APC in the ARMP of $K(9, 11)$ and $K(9, 13)$. We can extend this guarantee for $n \geq 7$ based on the pattern established by the generalized construction, which produces an ARMP of $K(9, 2n + 1)$ with the following characteristics:

1. π_1 will not hit vertices b_1 and b_4
2. π_2 will not hit vertices b_{10} and b_{11}
3. π_3 will not hit vertices b_2 and b_3
4. π_4 will not hit vertices b_{12} and b_{13}
5. π_5 will not hit vertices b_{14} and b_{15}
6. π_6 will not hit vertices b_5 and b_6
7. π_n will not hit vertices b_7 and b_8
8. π_x will not hit vertices b_{2x+2} and b_{2x+3} for $7 \leq x < n$

Hence, there exists an ARMP of $K(11, 2n + 1)$ for all $n \in \mathbb{N}$.

5.3 $K(8m + 3, 8m + 3)$

Theorem 5.3. There exists an almost resolvable maximum packing of $K(8m + 3, 8m + 3)$ for all m except possibly $m = 2$.

Proof. We will extend the ARMP of $K(8m + 1, 8m + 3)$ to an ARMP of $K(8m + 3, 8m + 3)$ using the same procedures used to extend the ARMP of $K(9, 11)$ to $K(11, 11)$. Recall that in such an extension, no new almost parallel classes are added, but exactly one cycle is added to

each existing APC. The procedure as applied to extending $K(9, 11)$ to $K(11, 11)$ is illustrated by the matrices shown in Figure 5.3.1 and 5.3.2 below. The numbers given in bold print directly below the matrix in Figure 5.3.1 indicate parallel classes that do not hit the corresponding part B vertices and which are used to create the new cycles shown in Figure 5.3.2. Note that the ARMP of $K(9, 11)$ shown in Figure 5.3.1 was produced using the identical cycle shifts given in the construction used to produce the ARMP of $K(8m + 1, 8m + 3)$ from the ARMP of $K(8m + 1, 8m + 1)$.

	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}
a_1	L	5	3	2	2	4	5	3	4	1	1
a_2	1	L	5	2	2	1	3	5	3	4	4
a_3	1	4	L	5	5	1	4	2	2	3	3
a_4	5	4	3	L	1	5	4	3	1	2	2
a_5	5	P	1	P	L	5	3	1	3	2	2
a_6	P	5	P	4	4	L	5	2	2	1	1
a_7	3	P	5	P	1	3	L	5	1	4	4
a_8	P	2	P	5	5	4	2	L	4	3	3
a_9	3	2	1	4	4	3	2	1	L	L	L
	4	1	2	3	3	2	1	4		5	5

Figure 5.3.1: ARMP of $K(9, 11)$ with Missing APCs Identified for Part B Vertices

	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}
a_1	L	5	3	2	2	4	5	3	4	1	1
a_2	1	L	5	2	2	1	3	5	3	4	4
a_3	1	4	L	5	5	1	4	2	2	3	3
a_4	5	4	3	L	1	5	4	3	1	2	2
a_5	5	P	1	P	L	5	3	1	3	2	2
a_6	P	5	P	4	4	L	5	2	2	1	1
a_7	3	P	5	P	1	3	L	5	1	4	4
a_8	P	2	P	5	5	4	2	L	4	3	3
a_9	3	2	1	4	4	3	2	1	L	L	L
a_{10}	4	1	2	3	3	2	1	4	L	5	5
a_{11}	4	1	2	3	3	2	1	4	L	5	5

Figure 5.3.2: ARMP of $K(11, 11)$ Produced from the ARMP of $K(9, 11)$ in Figure 5.3.1

The more general case of extending an ARMP of $K(8m + 1, 8m + 3)$ to an ARMP of $K(8m + 3, 8m + 3)$ will use this same extension process with the same pattern of cycle additions.

First, construct an ARMP of $K(8m + 1, 8m + 3)$ using the construction method given in the proof of Theorem 4.4. For $1 \leq x \leq m$, Let the parallel classes for each $K_x(9,9)$ used in the construction be denoted $\pi_{(x,1)}$, $\pi_{(x,2)}$, $\pi_{(x,3)}$ and $\pi_{(x,4)}$ and let them correspond to the parallel classes 1, 2, 3, and 4, respectively, as shown in Figures 5.3.1 and 5.3.2 above. The new parallel class will be denoted π_5 and will correspond to parallel class 5 shown in Figures 5.3.1 and 5.3.2.

Observe that the extension process that constructs the ARMP of $K(8m + 1, 8m + 3)$ preserves APC “misses”; that is, if an APC does not hit a particular part B vertex in the underlying individual $K_x(9, 9)$, it will also not hit that same vertex in the ARMP of $K(8m + 1, 8m + 3)$. This is ensured by the commutative quasigroup with holes, which governs the selection of cycles that are added to the parallel classes of each $K_x(9, 9)$.

Now add two new vertices, a_{8m+2} and a_{8m+3} to part A of $K(8m + 1, 8m + 3)$, along with the necessary edges to complete $K(8m + 3, 8m + 3)$. Assign new cycles created from these edges to each APC as indicated in Table 5.3 below:

APC ($1 \leq x \leq m$)	Cycle
$\pi_{(x,1)}$	$(a_{8m+2}, b_{x,2}, a_{8m+3}, b_{x,7})$
$\pi_{(x,2)}$	$(a_{8m+2}, b_{x,3}, a_{8m+3}, b_{x,6})$
$\pi_{(x,3)}$	$(a_{8m+2}, b_{x,4}, a_{8m+3}, b_{x,5})$
$\pi_{(x,4)}$	$(a_{8m+2}, b_{x,1}, a_{8m+3}, b_{x,8})$
π_5	$(a_{8m+2}, b_{8m+2}, a_{8m+3}, b_{8m+3})$

Table 5.3: New Cycle/APC Assignments for $K(8m + 3, 8m + 3)$

The result is $4m + 1$ APCs with $4m + 1$ cycles per class as required. The remaining edges complete the leave.

Chapter 6

General Odd/Odd Cases: $K(8m + 5, 8m + 5)$

6.1 General Odd/Odd Cases: $K(8m + 5, 8m + 5)$

Lemma 6.1. An almost resolvable maximum packing of 4-cycles exists for $K(13, 13)$.

Proof. Let $K(13, 13)$ have parts $A = \{a_0, a_1, a_2, \dots, a_5, a_0', a_1', a_2', \dots, a_5', a_\infty\}$ and $B = \{b_0, b_1, b_2, \dots, b_5, b_0', b_1', b_2', \dots, b_5', b_\infty\}$. Then the following base almost parallel class, cycled modulo 6 on subscripts, along with the partial parallel class $\{(a_0', b_0', a_3', b_3'), (a_1', b_1', a_4', b_4'), (a_2', b_2', a_5', b_5')\}$ constitute an almost resolvable maximum packing of $K(13, 13)$ with 4-cycles:

$$\{(a_1, b_4, a_1', b_0'), (a_4, b_2, a_4', b_0'), (a_3, b_3, a_2', b_3'), (a_0, b_1, a_5, b_4'), (a_\infty, b_1', a_3', b_5), (a_0', b_\infty, a_2, b_5')\}$$

By renaming the vertices, we can represent the given ARMP of $K(13, 13)$ using the matrix in Figure 6.1 on the following page. Again, numbered entries indicate edges of 4-cycles in the corresponding almost parallel classes, while entries of P indicate edges of 4-cycles in the partial parallel class. Cells with entries of L correspond to edges of the leave.

	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}
a_1	L	1	2	3	4	5	5	3	4	6	1	6	2
a_2	2	L	5	6	4	1	3	2	4	1	3	5	6
a_3	5	6	L	2	3	1	2	4	6	1	5	4	3
a_4	5	2	3	L	6	4	1	6	1	3	5	2	4
a_5	1	2	6	4	L	3	6	5	3	5	4	2	1
a_6	4	5	6	3	1	L	6	3	2	4	2	1	5
a_7	2	1	P	5	3	P	L	2	5	6	1	6	3
a_8	P	6	5	P	2	4	3	L	6	2	3	5	4
a_9	1	P	3	2	P	6	2	4	L	3	6	4	1
a_{10}	3	5	P	4	6	P	1	6	1	L	4	3	5
a_{11}	P	4	2	P	1	3	4	5	3	5	L	1	2
a_{12}	4	P	1	6	P	5	5	1	2	4	2	L	6
a_{13}	3	4	1	5	2	6	4	1	5	2	6	3	L

Figure 6.1: ARMP of $K(13, 13)$

6.2 $K(8m + 5, 8m + 5)$

Theorem 6.2. There exists an ARMP of $K(8m + 5, 8m + 5)$ with 4-cycles for all $m \leq 4$ (except possibly $m = 2$); there exists an ARMP of $K(8m + 5, 8m + 5)$ for all $m > 4$ whenever there exists a commutative quasigroup of order $2m$ with holes of size 2 that contains 2 complete transversals outside the holes.

Proof: We have already proven that there exists an ARMP of $K(8m + 5, 8m + 5)$ when $m = 1$ (Lemma 6.1). Next, we will construct an ARMP of $K(8m + 5, 8m + 5)$ for $m = 3$ and $m = 4$, then generalize that construction for $m > 4$.

An ARMP of $K(29, 29)$ is constructed by extending the ARMP of $K(25, 25)$ through a series of shifts and replacements of cycles. Note that any ARMP of $K(25, 25)$ contains 12 APCs with 12 cycles per class and a PPC with 6 cycles while an ARMP of $K(29, 29)$ contains 14 APCs with 14 cycles per class and a PPC with 7 cycles. Thus any extension of an ARMP of $K(25, 25)$ to an ARMP of $K(29, 29)$ must introduce 2 new APCs and 2 new cycles per pre-existing APC, plus 1 new cycle for the PPC.

The ARMP of $K(25, 25)$ which will be used to construct the ARMP of $K(29, 29)$ was formed in Example 4.3 using the $K(8m + 1, 8m + 1)$ construction and is based on the commutative quasigroup shown in Figure 6.2.1 below.

o	1	2	3	4	5	6
1	1	2	5	6	3	4
2	2	1	6	5	4	3
3	5	6	3	4	1	2
4	6	5	4	3	2	1
5	3	4	1	2	5	6
6	4	3	2	1	6	5

Figure 6.2.1: Commutative Quasigroup of Order 6 with Holes of Size 2

The familiar matrix representation for the ARMP of $K(25, 25)$ is pictured on the next page in Figure 6.2.2. The cells within striped borders represent cycles targeted for shifting in Phase 1 of the construction.

	$b_{(1,1)}$	$b_{(1,2)}$	$b_{(1,3)}$	$b_{(1,4)}$	$b_{(2,1)}$	$b_{(2,2)}$	$b_{(2,3)}$	$b_{(2,4)}$	$b_{(3,1)}$	$b_{(3,2)}$	$b_{(3,3)}$	$b_{(3,4)}$	$b_{(4,1)}$	$b_{(4,2)}$	$b_{(4,3)}$	$b_{(4,4)}$	$b_{(5,1)}$	$b_{(5,2)}$	$b_{(5,3)}$	$b_{(5,4)}$	$b_{(6,1)}$	$b_{(6,2)}$	$b_{(6,3)}$	$b_{(6,4)}$	b_9
$a_{(1,1)}$	3	1	2	2	4	3	1	9	9	10	10	11	11	12	12	5	5	6	6	7	7	8	8	4	
$a_{(1,2)}$	3	4	2	2	3	1	4	9	9	10	10	11	11	12	12	5	5	6	6	7	7	8	8	1	
$a_{(1,3)}$	3	4	1	1	3	4	2	10	10	9	9	12	12	11	11	6	6	5	5	8	8	7	7	2	
$a_{(1,4)}$	2	4	1	3	2	4	1	10	10	9	9	12	12	11	11	6	6	5	5	8	8	7	7	3	
$a_{(2,1)}$	2	P	3	P	2	1	3	11	11	12	12	9	9	10	10	7	7	8	8	5	5	6	6	1	
$a_{(2,2)}$	P	3	P	4	4	3	2	11	11	12	12	9	9	10	10	7	7	8	8	5	5	6	6	2	
$a_{(2,3)}$	1	P	4	P	3	1	4	12	12	11	11	10	10	9	9	8	8	7	7	6	6	5	5	3	
$a_{(2,4)}$	P	2	P	1	1	4	2	12	12	11	11	10	10	9	9	8	8	7	7	6	6	5	5	4	
$a_{(3,1)}$	9	9	10	10	11	11	10	10	7	5	6	6	8	7	5	1	1	2	2	3	3	4	4	8	
$a_{(3,2)}$	9	9	10	10	11	11	10	10	7	8	6	6	7	5	8	1	1	2	2	3	3	4	4	5	
$a_{(3,3)}$	10	10	9	9	10	11	11	7	8	5	5	7	8	6	2	2	1	1	4	4	3	3	6		
$a_{(3,4)}$	10	10	9	9	10	11	11	6	8	5	5	7	8	5	2	2	1	1	4	4	3	3	7		
$a_{(4,1)}$	11	11	10	10	9	9	10	10	6	P	7	P	6	5	7	3	3	4	4	1	1	2	2	5	
$a_{(4,2)}$	11	11	10	10	9	9	10	10	7	8	8	6	7	6	3	3	4	4	1	1	2	2	6		
$a_{(4,3)}$	10	10	11	11	10	10	9	9	5	P	8	P	7	5	8	4	4	3	3	2	2	1	1	7	
$a_{(4,4)}$	10	10	11	11	10	10	9	9	6	5	5	8	6	8	4	4	3	3	2	2	1	1	8		
$a_{(5,1)}$	5	5	6	6	7	7	8	8	1	1	2	2	3	3	4	4	11	9	10	10	12	11	9	12	
$a_{(5,2)}$	5	5	6	6	7	7	8	8	1	1	2	2	3	3	4	4	11	12	10	10	11	9	12	9	
$a_{(5,3)}$	6	6	5	5	8	8	7	7	2	2	1	1	4	4	3	3	11	12	9	9	11	12	10	10	
$a_{(5,4)}$	6	6	5	5	8	8	7	7	2	2	1	1	4	4	3	3	10	12	9	9	11	10	12	9	11
$a_{(6,1)}$	7	7	8	8	5	5	6	6	3	3	4	4	1	1	2	2	10	P	11	P	10	9	11	9	
$a_{(6,2)}$	7	7	8	8	5	5	6	6	3	3	4	4	1	1	2	2	P	10	P	12	12	11	10	10	
$a_{(6,3)}$	8	8	7	7	6	6	5	5	4	4	3	3	2	2	1	1	9	P	12	P	11	9	12	11	
$a_{(6,4)}$	8	8	7	7	6	6	5	5	4	4	3	3	2	2	1	1	P	10	P	9	9	12	10	12	
a_9	1	2	3	4	4	1	2	3	5	6	7	8	8	5	6	7	9	10	11	12	12	9	10	11	12

Figure 6.2.2: ARMP of $K(25, 25)$ to be Extended to ARMP of $K(29, 29)$
(Cycles targeted for shifting are shown in cells with striped borders)

PHASE 1: The following actions are shown in the matrix of Figure 6.2.3 on the following page:

Four vertices (a_{10} through a_{13}) are appended to part A and 4 vertices (b_{10} through b_{13}) are appended to part B of $K(25, 25)$ to form $K(29, 29)$. The edges (a_{10}, b_{10}) , (a_{11}, b_{11}) , (a_{12}, b_{12}) and (a_{13}, b_{13}) are assigned to the leave as required for any ARMP of $K(8m + 5, 8m + 5)$.

Two cycles from each of the existing 12 APC in the ARMP of $K(25, 25)$ are targeted for shifting. These are shown in cells with striped borders in the matrix on the previous page. The method of selecting these cycles is detailed later in the generalized portion of this proof. For now, it is sufficient to note that the cycles targeted for shifting comprise two disjoint sets of vertex-disjoint cycles, one cycle per APC in each set.

In **Round 1** of the cycle shifts, one of these sets is selected and its cycles are shifted so that each one's B vertices switch to b_{10} and b_{11} ; these cycles are replaced with cycles for the new APC 14.

In **Round 2** of the cycle shifts, the cycles from the other set are shifted so that each one's B vertices switch to b_{12} and b_{13} ; these cycles are replaced with cycles for the new APC 13. These shifts form 12 cycles each for the new APCs 13 and 14. The shifted cycles are shown in cells with striped borders in the matrix on the following page. New cycles for APCs 13 and 14 are shown in magenta and peach colors respectively. At this point, each APC 1 – 14 contains 12 cycles, while the PPC contains 6 cycles.

	$b_{(1,1)}$	$b_{(1,2)}$	$b_{(1,3)}$	$b_{(1,4)}$	$b_{(2,1)}$	$b_{(2,2)}$	$b_{(2,3)}$	$b_{(2,4)}$	$b_{(3,1)}$	$b_{(3,2)}$	$b_{(3,3)}$	$b_{(3,4)}$	$b_{(4,1)}$	$b_{(4,2)}$	$b_{(4,3)}$	$b_{(4,4)}$	$b_{(5,1)}$	$b_{(5,2)}$	$b_{(5,3)}$	$b_{(5,4)}$	$b_{(6,1)}$	$b_{(6,2)}$	$b_{(6,3)}$	$b_{(6,4)}$	b_9	b_{10}	b_{11}	b_{12}	b_{13}	
$a_{(1,1)}$		3	1	2	2	4	3	1	14	14	13	13	11	11	12	12	5	5	6	6	7	7	8	8	4	9	9	10	10	
$a_{(1,2)}$	3		4	2	2	3	1	4	14	14	13	13	11	11	12	12	5	5	6	6	7	7	8	8	1	9	9	10	10	
$a_{(1,3)}$	3	4		1	1	3	4	2	10	10	9	9	14	14	13	13	6	6	5	5	8	8	7	7	2	12	12	11	11	
$a_{(1,4)}$	2	4	1		3	2	4	1	10	10	9	9	14	14	13	13	6	6	5	5	8	8	7	7	3	12	12	11	11	
$a_{(2,1)}$	2	P	3	P		2	1	3	11	11	12	12	13	13	14	14	7	7	8	8	5	5	6	6	1	10	10	9	9	
$a_{(2,2)}$	P	3	P	4	4		3	2	11	11	12	12	13	13	14	14	7	7	8	8	5	5	6	6	2	10	10	9	9	
$a_{(2,3)}$	1	P	4	P	3	1		4	13	13	14	14	10	10	9	9	8	8	7	7	6	6	5	5	3	11	11	12	12	
$a_{(2,4)}$	P	2	P	1	1	4	2		13	13	14	14	10	10	9	9	8	8	7	7	6	6	5	5	4	11	11	12	12	
$a_{(3,1)}$	9	9	10	10	11	11	10	10		7	5	6	6	8	7	5	14	14	13	13	3	3	4	4	8	1	1	2	2	
$a_{(3,2)}$	9	9	10	10	11	11	10	10	7		8	6	6	7	5	8	14	14	13	13	3	3	4	4	5	1	1	2	2	
$a_{(3,3)}$	10	10	9	9	10	10	11	11	7	8		5	5	7	8	6	2	2	1	1	14	14	13	13	6	4	4	3	3	
$a_{(3,4)}$	10	10	9	9	10	10	11	11	6	8	5		7	6	8	5	2	2	1	1	14	14	13	13	7	4	4	3	3	
$a_{(4,1)}$	11	11	10	10	9	9	10	10	6		7			6	5	7	3	3	4	4	13	13	14	14	5	2	2	1	1	
$a_{(4,2)}$	11	11	10	10	9	9	10	10	P	7	P	8	8		7	6	3	3	4	4	13	13	14	14	6	2	2	1	1	
$a_{(4,3)}$	10	10	11	11	10	10	9	9	5	P	8	P	7	5		8	13	13	14	14	2	2	1	1	7	3	3	4	4	
$a_{(4,4)}$	10	10	11	11	10	10	9	9	P	6	P	5	5	8	6		13	13	14	14	2	2	1	1	8	3	3	4	4	
$a_{(5,1)}$	14	14	13	13	7	7	8	8	1	1	2	2	3	3	4	4		11	11	9	10	10	12	11	9	12	5	5	6	6
$a_{(5,2)}$	14	14	13	13	7	7	8	8	1	1	2	2	3	3	4	4	10		12	10	10	11	9	12	9	5	5	6	6	
$a_{(5,3)}$	6	6	5	5	14	14	13	13	2	2	1	1	4	4	3	3	11	12		9	9	11	12	10	10	8	8	7	7	
$a_{(5,4)}$	6	6	5	5	14	14	13	13	2	2	1	1	4	4	3	3	10	12	9		11	10	12	9	11	8	8	7	7	
$a_{(6,1)}$	7	7	8	8	13	13	14	14	3	3	4	4	1	1	2	2	10	P	11	P		10	9	11	9	6	6	5	5	
$a_{(6,2)}$	7	7	8	8	13	13	14	14	3	3	4	4	1	1	2	2	P	11	P	12	12		11	10	10	6	6	5	5	
$a_{(6,3)}$	13	13	14	14	6	6	5	5	4	4	3	3	2	2	1	1	9	P	12	P	11	9		12	11	7	7	8	8	
$a_{(6,4)}$	13	13	14	14	6	6	5	5	4	4	3	3	2	2	1	1	P	10	P	9	9	12	10		12	7	7	8	8	
a_9	1	2	3	4	4	1	2	3	5	6	7	8	8	5	6	7	9	10	11	12	12	9	10	11						
a_{10}																														
a_{11}																														
a_{12}																														
a_{13}																														

Figure 6.2.3: Phase 2 of the Extension of an ARMP of $K(25, 25)$ to an ARMP of $K(29, 29)$
 (Shifted cycles shown in cells with striped border; Cycles for new APCs 13 and 14 shown in peach and magenta shaded cells)

PHASE 2: We must still add two cycles to each of the 2 new APCs and to each of the 12 pre-existing APCs, plus add one cycle to the PPC. The addition of cycles to APCs 1 – 12 will be easy; every cycle whose B vertices were shifted to b_{10} and b_{11} or to b_{12} and b_{13} vacates corresponding pairs of B vertices that are now unused by the APCs of those cycles. Therefore we can form two new cycles for every APC 1 – 12 by using those vacated B vertices and the newly appended A vertices a_{10} through a_{13} . However, this relegates the formation of two more cycles for each of APCs 13 and 14 and one more cycle for the PPC to the block of edges between a_9 through a_{13} and b_9 through b_{13} , a task which is complicated by the position of the leave edges. In fact, we must solve this problem first if all other tasks are to be accomplished.

Placing two cycles each for APCs 13 and 14 as well as one cycle for the PPC in the block just described is, in fact, impossible, as one quickly discovers upon attempting it. However, we can get close. The cycle $(a_{11}, b_{10}, a_{12}, b_{13})$ can be assigned to the PPC, the cycle $(a_9, b_{10}, a_{13}, b_{11})$ can be assigned to APC 13 and the cycle $(a_9, b_{12}, a_{10}, b_{13})$ can be assigned to APC 14. Furthermore, we can get “half” of a cycle for APC 13 from edges (a_{10}, b_9) and (a_{12}, b_9) and “half” of a cycle for APC 14 from edges (a_{11}, b_9) and (a_{13}, b_9) . These newly established whole and half cycles are shown in the matrix of Figure 6.2.4 on the following page in the block outlined in striped border.

	$b_{(1,1)}$	$b_{(1,2)}$	$b_{(1,3)}$	$b_{(1,4)}$	$b_{(2,1)}$	$b_{(2,2)}$	$b_{(2,3)}$	$b_{(2,4)}$	$b_{(3,1)}$	$b_{(3,2)}$	$b_{(3,3)}$	$b_{(3,4)}$	$b_{(4,1)}$	$b_{(4,2)}$	$b_{(4,3)}$	$b_{(4,4)}$	$b_{(5,1)}$	$b_{(5,2)}$	$b_{(5,3)}$	$b_{(5,4)}$	$b_{(6,1)}$	$b_{(6,2)}$	$b_{(6,3)}$	$b_{(6,4)}$	b_9	b_{10}	b_{11}	b_{12}	b_{13}	
$a_{(1,1)}$		3	1	2	2	4	3	1	14	14	13	13	11	11	12	12	5	5	6	6	7	7	8	8	4	9	9	10	10	
$a_{(1,2)}$	3		4	2	2	3	1	4	14	14	13	13	11	11	12	12	5	5	6	6	7	7	8	8	1	9	9	10	10	
$a_{(1,3)}$	3	4		1	1	3	4	2	10	10	9	9	14	14	13	13	6	6	5	5	8	8	7	7	2	12	12	11	11	
$a_{(1,4)}$	2	4	1		3	2	4	1	10	10	9	9	14	14	13	13	6	6	5	5	8	8	7	7	3	12	12	11	11	
$a_{(2,1)}$	2	P	3	P		2	1	3	11	11	12	12	13	13	14	14	7	7	8	8	5	5	6	6	1	10	10	9	9	
$a_{(2,2)}$	P	3	P	4	4		3	2	11	11	12	12	13	13	14	14	7	7	8	8	5	5	6	6	2	10	10	9	9	
$a_{(2,3)}$	1	P	4	P	3	1		4	13	13	14	14	10	10	9	9	8	8	7	7	6	6	5	5	3	11	11	12	12	
$a_{(2,4)}$	P	2	P	1	1	4	2		13	13	14	14	10	10	9	9	8	8	7	7	6	6	5	5	4	11	11	12	12	
$a_{(3,1)}$	9	9	10	10	11	11	12	12		7	5	6	6	8	7	5	14	14	13	13	3	3	4	4	8	1	1	2	2	
$a_{(3,2)}$	9	9	10	10	11	11	12	12	7		8	6	6	7	5	8	14	14	13	13	3	3	4	4	5	1	1	2	2	
$a_{(3,3)}$	10	10	9	9	12	12	11	11	7	8		5	5	7	8	6	2	2	1	1	14	14	13	13	6	4	4	3	3	
$a_{(3,4)}$	10	10	9	9	12	12	11	11	6	8	5		7	6	8	5	2	2	1	1	14	14	13	13	7	4	4	3	3	
$a_{(4,1)}$	11	11	12	12	9	9	10	10	6	P	7	P		6	5	7	3	3	4	4	13	13	14	14	5	2	2	1	1	
$a_{(4,2)}$	11	11	12	12	9	9	10	10	P	7	P	8	8		7	6	3	3	4	4	13	13	14	14	6	2	2	1	1	
$a_{(4,3)}$	12	12	11	11	10	10	9	9	5	P	8	P	7	5		8	13	13	14	14	2	2	1	1	7	3	3	4	4	
$a_{(4,4)}$	12	12	11	11	10	10	9	9	P	6	P	5	5	8	6		13	13	14	14	2	2	1	1	8	3	3	4	4	
$a_{(5,1)}$	14	14	13	13	7	7	8	8	1	1	2	2	3	3	4	4		11	9	10	10	12	11	9	12	5	5	6	6	
$a_{(5,2)}$	14	14	13	13	7	7	8	8	1	1	2	2	3	3	4	4		11	12	10	10	11	9	12	9	5	5	6	6	
$a_{(5,3)}$	6	6	5	5	14	14	13	13	2	2	1	1	4	4	3	3		11	12	9	9	11	12	10	10	8	8	7	7	
$a_{(5,4)}$	6	6	5	5	14	14	13	13	2	2	1	1	4	4	3	3		10	12	9		11	10	12	9	11	8	8	7	7
$a_{(6,1)}$	7	7	8	8	13	13	14	14	3	3	4	4	1	1	2	2		10	P	11	P		10	9	11	9	6	6	5	5
$a_{(6,2)}$	7	7	8	8	13	13	14	14	3	3	4	4	1	1	2	2		P	11	P	12	12		11	10	10	6	6	5	5
$a_{(6,3)}$	13	13	14	14	6	6	5	5	4	4	3	3	2	2	1	1		9	P	12	P	11	9		12	11	7	7	8	8
$a_{(6,4)}$	13	13	14	14	6	6	5	5	4	4	3	3	2	2	1	1		P	10	P	9	9	12	10		12	7	7	8	8
a_9	1	2	3	4	4	1	2	3	5	6	7	8	8	5	6	7	9	10	11	12	12	9	10	11		13	13	14	14	
a_{10}																										13			14	14
a_{11}																										14	P			P
a_{12}																										13	P			P
a_{13}																										14	13	13		

Figure 6.2.4: Phase 2 of the Extension of an ARMP of $K(25, 25)$ to an ARMP of $K(29, 29)$
(Newly established whole and half cycles for new APCs 13 and 14 shown in cells with striped border)

PHASE 3: As evident by the matrix in Figure 6.2.4 on the previous page, we still need the other half of a cycle for each of APCs 13 and 14. But we also still need to add two cycles to APCs 1 through 12, and we take care of that next.

Each cycle that was shifted in Phase 1 vacated two part B vertices which, along with the newly appended A vertices a_{10} through a_{13} , gives rise to a new cycle. For example, the cycle $(a_{(1,1)}, b_{(3,1)}, a_{(1,2)}, b_{(3,2)})$ was previously assigned to APC 9, but was reassigned to APC 14 in the shifts of Phase 1. We can establish a new cycle for APC 9 as $(a_{10}, b_{(3,1)}, a_{12}, b_{(3,2)})$. Similarly, the cycle $(a_{(2,1)}, b_{(4,1)}, a_{(2,2)}, b_{(4,2)})$ was previously assigned to APC 9, but was reassigned to APC 13 in the shifts of Phase 1. So we can establish another new cycle for APC 9 as $(a_{11}, b_{(4,1)}, a_{13}, b_{(4,2)})$. In like manner we create two new cycles for each of the other 11 old APCs, with the results shown in the matrix on the next page. The newly created cycles are shown in the block of cells with striped border in the matrix of Figure 6.2.5 on the following page.

	$b_{(1,1)}$	$b_{(1,2)}$	$b_{(1,3)}$	$b_{(1,4)}$	$b_{(2,1)}$	$b_{(2,2)}$	$b_{(2,3)}$	$b_{(2,4)}$	$b_{(3,1)}$	$b_{(3,2)}$	$b_{(3,3)}$	$b_{(3,4)}$	$b_{(4,1)}$	$b_{(4,2)}$	$b_{(4,3)}$	$b_{(4,4)}$	$b_{(5,1)}$	$b_{(5,2)}$	$b_{(5,3)}$	$b_{(5,4)}$	$b_{(6,1)}$	$b_{(6,2)}$	$b_{(6,3)}$	$b_{(6,4)}$	b_9	b_{10}	b_{11}	b_{12}	b_{13}
$a_{(1,1)}$		3	1	2	2	4	3	1	14	14	13	13	11	11	12	12	5	5	6	6	7	7	8	8	4	9	9	10	10
$a_{(1,2)}$	3		4	2	2	3	1	4	14	14	13	13	11	11	12	12	5	5	6	6	7	7	8	8	1	9	9	10	10
$a_{(1,3)}$	3	4		1	1	3	4	2	10	10	9	9	14	14	13	13	6	6	5	5	8	8	7	7	2	12	12	11	11
$a_{(1,4)}$	2	4	1		3	2	4	1	10	10	9	9	14	14	13	13	6	6	5	5	8	8	7	7	3	12	12	11	11
$a_{(2,1)}$	2	P	3	P		2	1	3	11	11	12	12	13	13	14	14	7	7	8	8	5	5	6	6	1	10	10	9	9
$a_{(2,2)}$	P	3	P	4	4		3	2	11	11	12	12	13	13	14	14	7	7	8	8	5	5	6	6	2	10	10	9	9
$a_{(2,3)}$	1	P	4	P	3	1		4	13	13	14	14	10	10	9	9	8	8	7	7	6	6	5	5	3	11	11	12	12
$a_{(2,4)}$	P	2	P	1	1	4	2		13	13	14	14	10	10	9	9	8	8	7	7	6	6	5	5	4	11	11	12	12
$a_{(3,1)}$	9	9	10	10	11	11	12	12		7	5	6	6	8	7	5	14	14	13	13	3	3	4	4	8	1	1	2	2
$a_{(3,2)}$	9	9	10	10	11	11	12	12	7		8	6	6	7	5	8	14	14	13	13	3	3	4	4	5	1	1	2	2
$a_{(3,3)}$	10	10	9	9	12	12	11	11	7	8		5	5	7	8	6	2	2	1	1	14	14	13	13	6	4	4	3	3
$a_{(3,4)}$	10	10	9	9	12	12	11	11	6	8	5		7	6	8	5	2	2	1	1	14	14	13	13	7	4	4	3	3
$a_{(4,1)}$	11	11	12	12	9	9	10	10	6	P	7	P		6	5	7	3	3	4	4	13	13	14	14	5	2	2	1	1
$a_{(4,2)}$	11	11	12	12	9	9	10	10	P	7	P	8	8		7	6	3	3	4	4	13	13	14	14	6	2	2	1	1
$a_{(4,3)}$	12	12	11	11	10	10	9	9	5	P	8	P	7	5		8	13	13	14	14	2	2	1	1	7	3	3	4	4
$a_{(4,4)}$	12	12	11	11	10	10	9	9	P	6	P	5	5	8	6		13	13	14	14	2	2	1	1	8	3	3	4	4
$a_{(5,1)}$	14	14	13	13	7	7	8	8	1	1	2	2	3	3	4	4		11	9	10	10	12	11	9	12	5	5	6	6
$a_{(5,2)}$	14	14	13	13	7	7	8	8	1	1	2	2	3	3	4	4		11	12	10	10	11	9	12	9	5	5	6	6
$a_{(5,3)}$	6	6	5	5	14	14	13	13	2	2	1	1	4	4	3	3	11	12		9	9	11	12	10	10	8	8	7	7
$a_{(5,4)}$	6	6	5	5	14	14	13	13	2	2	1	1	4	4	3	3	10	12	9		11	10	12	9	11	8	8	7	7
$a_{(6,1)}$	7	7	8	8	13	13	14	14	3	3	4	4	1	1	2	2	10	P	11	P		10	9	11	9	6	6	5	5
$a_{(6,2)}$	7	7	8	8	13	13	14	14	3	3	4	4	1	1	2	2	P	11	P	12	12		11	10	10	6	6	5	5
$a_{(6,3)}$	13	13	14	14	6	6	5	5	4	4	3	3	2	2	1	1	9	P	12	P	11	9		12	11	7	7	8	8
$a_{(6,4)}$	13	13	14	14	6	6	5	5	4	4	3	3	2	2	1	1	P	10	P	9	9	12	10		12	7	7	8	8
a_9	1	2	3	4	4	1	2	3	5	6	7	8	8	5	6	7	9	10	11	12	12	9	10	11		13	13	14	14
a_{10}	5	5	7	7	8	8	6	6	9	9	10	10	12	12	10	10	1	1	3	3	4	4	2	2	13			14	14
a_{11}	8	8	6	6	5	5	7	7	12	12	10	10	9	9	11	11	4	4	2	2	1	1	3	3	14	P			P
a_{12}	5	5	7	7	8	8	6	6	9	9	10	10	12	12	10	10	1	1	3	3	4	4	2	2	13	P			P
a_{14}	8	8	6	6	5	5	7	7	12	12	10	10	9	9	11	11	4	4	2	2	1	1	3	3	14	13	13		

Figure 6.2.5: Phase 3 of the Extension of an ARMP of $K(25, 25)$ to an ARMP of $K(29, 29)$
(Replacement cycles for previously shifted cycles shown in cells with striped border)

Phase 4: In this last stage of the construction, we solve the problem of creating the other half of the cycles needed for the new APCs 13 and 14. This is actually quite simple, given that the edges (a_{10}, b_{11}) , (a_{12}, b_{11}) , (a_{11}, b_{12}) , and (a_{13}, b_{12}) are unassigned as yet, with vertex b_{11} not hit by APC 14 and vertex b_{12} not hit by APC 13. We select a single part B vertex from the set $b_{(1,1)}$ through $b_{(6,4)}$. Any vertex in this set will suffice; for this example we select $b_{(4,1)}$. We then make the following shifts:

1. Edges $(a_{10}, b_{(4,1)})$ and $(a_{12}, b_{(4,1)})$ in the cycle $(a_{10}, b_{(4,1)}, a_{10}, b_{(4,2)})$ of APC 12 are reassigned to APC 13 and previously unassigned edges (a_{10}, b_{11}) and (a_{12}, b_{11}) are now assigned to APC 12 to replace them.
2. Edges $(a_{11}, b_{(4,1)})$ and $(a_{13}, b_{(4,1)})$ in the cycle $(a_{11}, b_{(4,1)}, a_{13}, b_{(4,2)})$ of APC 9 are reassigned to APC 14 and the previously unassigned edges (a_{11}, b_{12}) and (a_{13}, b_{12}) are now assigned to APC 9 to replace them.
3. Edges $(a_{(1,3)}, b_{(4,1)})$ and $(a_{(1,4)}, b_{(4,1)})$ that were previously assigned to a cycle in APC 14 are reassigned to APC 12 while edges $(a_{(1,3)}, b_{11})$ and $(a_{(1,4)}, b_{11})$ previously assigned to a cycle in APC 12 are reassigned to APC 14.
4. Edges $(a_{(2,1)}, b_{(4,1)})$ and $(a_{(2,2)}, b_{(4,1)})$ that were previously assigned to a cycle in APC 13 are reassigned to APC 9 while edges $(a_{(2,1)}, b_{12})$ and $(a_{(2,2)}, b_{12})$ previously assigned to a cycle in APC 9 are reassigned to APC 13.

The shifts made in steps 1 – 4 above complete the last cycles for APCs 13 and 14 and preserve the correct number of cycles for all other APCs. The completed ARMP of $K(29, 29)$ is shown in the matrix of Figure 6.2.6 on the following page. The cells shown in striped border indicate the shifted edges.

	$b_{(1,1)}$	$b_{(1,2)}$	$b_{(1,3)}$	$b_{(1,4)}$	$b_{(2,1)}$	$b_{(2,2)}$	$b_{(2,3)}$	$b_{(2,4)}$	$b_{(3,1)}$	$b_{(3,2)}$	$b_{(3,3)}$	$b_{(3,4)}$	$b_{(4,1)}$	$b_{(4,2)}$	$b_{(4,3)}$	$b_{(4,4)}$	$b_{(5,1)}$	$b_{(5,2)}$	$b_{(5,3)}$	$b_{(5,4)}$	$b_{(6,1)}$	$b_{(6,2)}$	$b_{(6,3)}$	$b_{(6,4)}$	b_9	b_{10}	b_{11}	b_{12}	b_{13}	
$a_{(1,1)}$		3	1	2	2	4	3	1	14	14	13	13	11	11	12	12	5	5	6	6	7	7	8	8	4	9	9	10	10	
$a_{(1,2)}$	3		4	2	2	3	1	4	14	14	13	13	11	11	12	12	5	5	6	6	7	7	8	8	1	9	9	10	10	
$a_{(1,3)}$	3	4		1	1	3	4	2	10	10	9	9	12	14	13	13	6	6	5	5	8	8	7	7	2	12	14	11	11	
$a_{(1,4)}$	2	4	1		3	2	4	1	10	10	9	9	12	14	13	13	6	6	5	5	8	8	7	7	3	12	14	11	11	
$a_{(2,1)}$	2	P	3	P		2	1	3	11	11	12	12	9	13	14	14	7	7	8	8	5	5	6	6	1	10	10	13	9	
$a_{(2,2)}$	P	3	P	4	4		3	2	11	11	12	12	9	13	14	14	7	7	8	8	5	5	6	6	2	10	10	13	9	
$a_{(2,3)}$	1	P	4	P	3	1		4	13	13	14	14	10	10	9	9	8	8	7	7	6	6	5	5	3	11	11	12	12	
$a_{(2,4)}$	P	2	P	1	1	4	2		13	13	14	14	10	10	9	9	8	8	7	7	6	6	5	5	4	11	11	12	12	
$a_{(3,1)}$	9	9	10	10	11	11	12	12		7	5	6	6	8	7	5	14	14	13	13	3	3	4	4	8	1	1	2	2	
$a_{(3,2)}$	9	9	10	10	11	11	12	12	7		8	6	6	7	5	8	14	14	13	13	3	3	4	4	5	1	1	2	2	
$a_{(3,3)}$	10	10	9	9	12	12	11	11	7	8		5	5	7	8	6	2	2	1	1	14	14	13	13	6	4	4	3	3	
$a_{(3,4)}$	10	10	9	9	12	12	11	11	6	8	5		7	6	8	5	2	2	1	1	14	14	13	13	7	4	4	3	3	
$a_{(4,1)}$	11	11	12	12	9	9	10	10	6	P	7	P		6	5	7	3	3	4	4	13	13	14	14	5	2	2	1	1	
$a_{(4,2)}$	11	11	12	12	9	9	10	10	P	7	P	8	8		7	6	3	3	4	4	13	13	14	14	6	2	2	1	1	
$a_{(4,3)}$	12	12	11	11	10	10	9	9	5	P	8	P	7	5		8	13	13	14	14	2	2	1	1	7	3	3	4	4	
$a_{(4,4)}$	12	12	11	11	10	10	9	9	P	6	P	5	5	8	6		13	13	14	14	2	2	1	1	8	3	3	4	4	
$a_{(5,1)}$	14	14	13	13	7	7	8	8	1	1	2	2	3	3	4	4		11	9	10	10	12	11	9	12	5	5	6	6	
$a_{(5,2)}$	14	14	13	13	7	7	8	8	1	1	2	2	3	3	4	4		11	12	10	10	11	9	12	9	5	5	6	6	
$a_{(5,3)}$	6	6	5	5	14	14	13	13	2	2	1	1	4	4	3	3		11	12	9	9	11	12	10	10	8	8	7	7	
$a_{(5,4)}$	6	6	5	5	14	14	13	13	2	2	1	1	4	4	3	3		10	12	9		11	10	12	9	11	8	8	7	7
$a_{(6,1)}$	7	7	8	8	13	13	14	14	3	3	4	4	1	1	2	2		10	P	11	P	+	10	9	11	9	6	6	5	5
$a_{(6,2)}$	7	7	8	8	13	13	14	14	3	3	4	4	1	1	2	2		P	11	P	12	12		11	10	10	6	6	5	5
$a_{(6,3)}$	13	13	14	14	6	6	5	5	4	4	3	3	2	2	1	1		9	P	12	P	11	9		12	11	7	7	8	8
$a_{(6,4)}$	13	13	14	14	6	6	5	5	4	4	3	3	2	2	1	1		P	10	P	9	9	12	10		12	7	7	8	8
a_9	1	2	3	4	4	1	2	3	5	6	7	8	8	5	6	7		9	10	11	12	12	9	10	11	+	13	13	14	14
a_{10}	5	5	7	7	8	8	6	6	9	9	11	11	13	12	10	10		1	1	3	3	4	4	2	2	13	+	12	14	14
a_{11}	8	8	6	6	5	5	7	7	12	12	10	10	14	9	11	11		4	4	2	2	1	1	3	3	14	P	+	9	P
a_{12}	5	5	7	7	8	8	6	6	9	9	11	11	13	12	10	10		1	1	3	3	4	4	2	2	13	P	12	+	P
a_{14}	8	8	6	6	5	5	7	7	12	12	10	10	14	9	11	11		4	4	2	2	1	1	3	3	14	13	13	9	+

Figure 6.2.6: Phase 4 (Completion) of the Extension of an ARMP of $K(25, 25)$ to an ARMP of $K(29, 29)$
(Shifted and added half-cycles shown in cells with striped borders)

As illustrated in the extension of an ARMP of $K(25, 25)$ to an ARMP of $K(29, 29)$, the construction of an ARMP of $K(8m + 5, 8m + 5)$ from an ARMP of $K(8m + 1, 8m + 1)$ depends on being able to identify two disjoint sets of cycles to be shifted, each set containing exactly 1 cycle per APC from the ARMP of $K(8m + 1, 8m + 1)$ such that the cycles within each set are vertex-disjoint. As we will show in the generalized proof, this is possible whenever the CQG used for the construction of the ARM of $K(8m + 1, 8m + 1)$ contains a pair of disjoint complete transversals in the cells outside the holes.

Although the CQG used in the extension of an ARMP of $K(25, 25)$ to an ARMP of $K(29, 29)$ does not possess this feature, the particular arrangement of elements in the cells outside of its holes made it possible to identify the two necessary disjoint sets of vertex-disjoint cycles used for shifting. We now prove the theorem for $m = 4$ by extending an ARMP of $K(33,33)$ constructed using a CQG that does possess that feature to an ARMP of $K(37, 37)$. The CQG used is pictured below in Figure 6.2.7. The elements of the two transversals selected for use in the construction are shown in cells shaded pink and green, respectively. The cycles targeted for shifting will correspond directly to the elements of the indicated transversals.

o	1	2	3	4	5	6	7	8
1	1	2	5	6	7	8	3	4
2	2	1	8	7	3	4	6	5
3	5	8	3	4	2	7	1	6
4	6	7	4	3	8	1	5	2
5	7	3	2	8	5	6	4	1
6	8	4	7	1	6	5	2	3
7	3	6	1	5	4	2	7	8
8	4	5	6	2	1	3	8	7

Figure 6.2.7: Commutative Quasigroup of Order 8 with Holes of Size 2 and 2 Transversals Outside the Holes

The ARMP of $K(33,33)$ created using the CGQ of Figure 6.2.7 and the $K(8m + 1, 8m + 1)$ construction is shown on the next page in Figure 6.2.8 with cycles targeted for shifting shown in striped border. The extension to an ARMP of $K(37, 37)$ will be shown in phases with abbreviated explanations.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	
1	+	3	1	2	2	4	3	1	9	9	10	10	11	11	12	12	13	13	14	14	15	15	16	16	5	5	6	6	7	7	8	8	4	
2	3	+	4	2	2	3	1	4	9	9	10	10	11	11	12	12	13	13	14	14	15	15	16	16	5	5	6	6	7	7	8	8	1	
3	3	4	+	1	1	3	4	2	10	10	9	9	12	12	11	11	14	14	13	13	16	16	15	15	6	6	5	5	8	8	7	7	2	
4	2	4	1	+	3	2	4	1	10	10	9	9	12	12	11	11	14	14	13	13	16	16	15	15	6	6	5	5	8	8	7	7	3	
5	2	P	3	P	+	2	1	3	15	15	16	16	13	13	14	14	5	5	6	6	7	7	8	8	11	11	12	12	9	9	10	10	1	
6	1	3	+	4	4	+	3	2	15	15	16	16	13	13	14	14	5	5	6	6	7	7	8	8	11	11	12	12	9	9	10	10	2	
7	1	+	4	+	3	1	+	4	16	16	15	15	14	14	13	13	6	6	5	5	8	8	7	7	12	12	11	11	10	10	9	9	3	
8	P	2	P	1	1	4	2	+	16	16	15	15	14	14	13	13	6	6	5	5	8	8	7	7	12	12	11	11	10	10	9	9	4	
9	9	9	10	10	15	15	16	16	+	7	5	6	6	8	7	5	3	3	4	4	13	13	14	14	1	1	2	2	11	11	12	12	8	
10	9	9	10	10	15	15	16	16	7	+	8	6	6	7	5	8	3	3	4	4	13	13	14	14	1	1	2	2	11	11	12	12	5	
11	10	10	9	9	16	16	15	15	7	8	+	5	5	7	8	6	4	4	3	3	14	14	13	13	2	2	1	1	12	12	11	11	6	
12	10	10	9	9	16	16	15	15	6	8	5	+	7	6	8	5	4	4	3	3	14	14	13	13	2	2	1	1	12	12	11	11	7	
13	11	11	12	12	13	13	14	14	6	P	7	P	+	6	5	7	15	15	16	16	1	1	2	2	9	9	10	10	3	3	4	4	5	
14	11	11	12	12	13	13	14	14	P	7	P	8	8	+	7	6	15	15	16	16	1	1	2	2	9	9	10	10	3	3	4	4	6	
15	12	12	11	11	14	14	13	13	5	P	8	P	7	5	+	8	16	16	15	15	2	2	1	1	10	10	9	9	4	4	3	3	7	
16	12	12	11	11	14	14	13	13	P	6	P	5	5	8	6	+	16	16	15	15	2	2	1	1	10	10	9	9	4	4	3	3	8	
17	13	13	14	14	5	5	6	6	3	3	4	4	15	15	16	16	+	12	10	10	11	9	12	7	7	8	8	1	1	2	2	12		
18	13	13	14	14	5	5	6	6	3	3	4	4	15	15	16	16	11	+	12	10	10	11	9	12	7	7	8	8	1	1	2	2	9	
19	14	14	13	13	6	6	5	5	4	4	3	3	16	16	15	15	11	12	+	9	9	11	12	10	8	8	7	7	2	2	1	1	10	
20	14	14	13	13	6	6	5	5	4	4	3	3	16	16	15	15	10	12	9	+	11	10	12	9	8	8	7	7	2	2	1	1	11	
21	15	15	16	16	7	7	8	8	13	13	14	14	1	1	2	2	10	P	11	P	12	12	+	9	11	3	3	4	4	5	5	6	6	9
22	15	15	16	16	7	7	8	8	13	13	14	14	1	1	2	2	P	11	P	12	12	+	11	10	3	3	4	4	5	5	6	6	10	
23	16	16	15	15	8	8	7	7	14	14	13	13	2	2	1	1	9	P	12	P	11	+	12	4	4	3	3	6	6	5	5	+	12	
24	16	16	15	15	8	8	7	7	14	14	13	13	2	2	1	1	P	10	P	9	9	12	+	10	4	4	3	3	6	6	5	5	12	
25	5	5	6	6	11	11	12	12	1	1	2	2	9	9	10	10	7	7	8	8	3	3	4	4	+	15	13	14	14	16	15	13	16	
26	5	5	6	6	11	11	12	12	1	1	2	2	9	9	10	10	7	7	8	8	3	3	4	4	15	+	14	14	15	13	16	13	13	
27	6	6	5	5	12	12	11	11	2	2	1	1	10	10	9	9	8	8	7	7	4	4	3	3	15	16	+	13	13	15	16	14	14	
28	6	6	5	5	12	12	11	11	2	2	1	1	10	10	9	9	8	8	7	7	4	4	3	3	14	16	13	+	15	14	16	13	15	
29	7	7	8	8	9	9	10	10	11	11	12	12	3	3	4	4	1	1	2	2	5	5	6	6	14	P	15	P	+	14	13	15	13	
30	7	7	8	8	9	9	10	10	11	11	12	12	3	3	4	4	1	1	2	2	5	5	6	6	+	15	+	16	16	+	15	14	14	
31	8	8	7	7	10	10	9	9	12	12	11	11	4	4	3	3	2	2	1	1	6	6	5	5	13	P	16	P	15	13	+	16	15	
32	8	8	7	7	10	10	9	9	12	12	11	11	4	4	3	3	2	2	1	1	6	6	5	5	P	14	P	13	13	16	14	+	16	
33	1	2	3	4	4	1	2	3	5	6	7	8	8	5	6	7	9	10	11	12	12	9	10	11	13	14	15	16	16	13	14	15	+	

Figure 6.2.8: ARMP of $K(33, 33)$ to be Extended to ARMP of $K(37, 37)$

(Cycles targeted for shifting are selected in correspondence with the elements of the transversals of the commutative quasigroup with holes shown in Figure 6.2.7 and are highlighted in cells with striped border.)

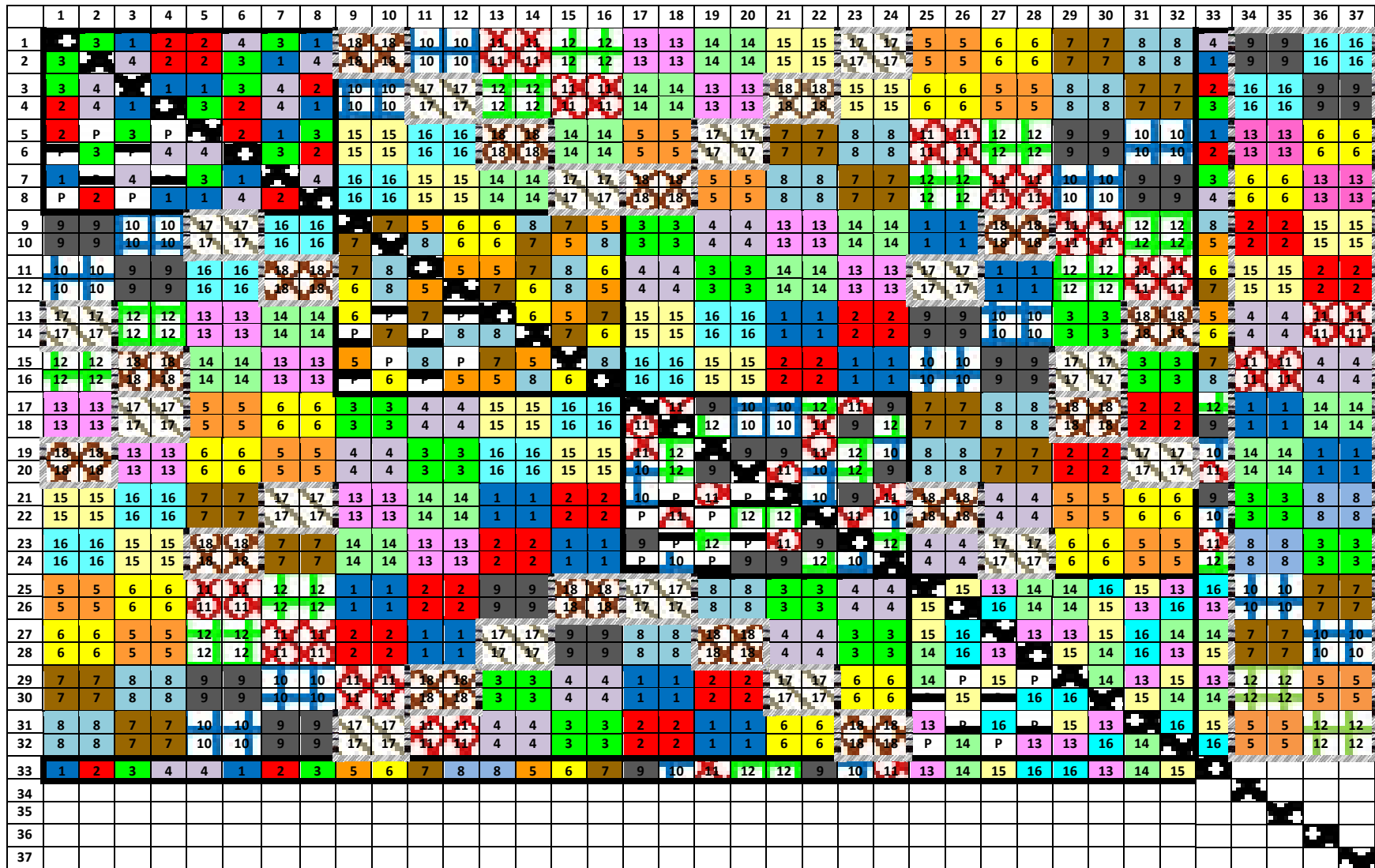


Figure 6.2.9: Phase 1 of the Extension of an ARMP of $K(33,33)$ to an ARMP of $K(37,37)$

(4 vertices are appended to part B and 4 vertices are appended to part A. Edges of the leaf are assigned. Cycles targeted in APCs 1 – 16 are shifted and replaced with cycles in the new APCs 17 and 18 (shown in cells with striped borders)).

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	
1			3	1	2	2	4	3	1	18	18	10	10	11	11	12	12	13	13	14	14	15	15	17	17	5	5	6	6	7	7	8	8	4	9	9	16	16
2	3		4	2	2	3	1	4		8	8	10	10	11	11	12	12	13	13	14	14	15	15	17	17	5	5	6	6	7	7	8	8	1	9	9	16	16
3	3	4		1	1	3	4	2		10	10	17	17	12	12	11	11	14	14	13	13	18	18	15	15	6	6	5	5	8	8	7	7	2	16	16	9	9
4	2	4	1		3	2	4	1	10	10	17	17	12	12	11	11	14	14	13	13	18	18	15	15	6	6	5	5	8	8	7	7	3	16	16	9	9	
5	2	P	3	P		2	1	3	15	15	16	16	18	18	14	14	5	5	17	17	7	7	8	8	11	11	12	12	9	9	10	10	1	13	13	6	6	
6	1	3	1		4	4		3	2	15	15	16	16	18	18	14	14	5	5	17	17	7	7	8	8	11	11	12	12	9	9	10	10	2	13	13	6	6
7	1	P	4	P	3	1		4		16	16	15	15	14	14	17	17	18	18	5	5	8	8	7	7	12	12	11	11	10	10	9	9	3	6	6	13	13
8	P	2	P	1	1	4	2			16	16	15	15	14	14	17	17	18	18	5	5	8	8	7	7	12	12	11	11	10	10	9	9	4	6	6	13	13
9	9	9	10	10	17	17	16	16		7	5	6	6	8	7	5	3	3	4	4	13	13	14	14	1	1	16	16	11	11	12	12	8	2	2	15	15	
10	9	9	10	10	17	17	16	16	7		8	6	6	7	5	8	3	3	4	4	13	13	14	14	1	1	16	16	11	11	12	12	5	2	2	15	15	
11	10	10	9	9	16	16	18	18	7	8		5	5	7	8	6	4	4	3	3	14	14	13	13	17	17	1	1	12	12	11	11	6	15	15	2	2	
12	10	10	9	9	16	16	18	18	6	8	5		7	6	8	5	4	4	3	3	14	14	13	13	17	17	1	1	12	12	11	11	7	15	15	2	2	
13	17	17	12	12	13	13	14	14	6		7		6	5	7	15	15	16	16	1	1	2	2	9	9	10	10	3	3	18	18	5	4	4	11	11		
14	17	17	12	12	13	13	14	14	P	7	P	8	8		7	6	15	15	16	16	1	1	2	2	9	9	10	10	3	3	18	18	6	4	4	11	11	
15	12	12	18	18	14	14	13	13	5	P	8	P	7	5		8	16	16	15	15	2	2	1	1	10	10	9	9	17	17	17	17	3	7	11	11	4	4
16	12	12	18	18	14	14	13	13		6	1	5	5	8	6		16	16	15	15	2	2	1	1	10	10	9	9	17	17	17	17	3	8		4	4	
17	13	13	17	17	5	5	6	6	3	3	4	4	15	15	16	16		9	10	10	12	11	9	7	7	8	8	18	18	2	2	12	1	1	14	14		
18	13	13	17	17	5	5	6	6	3	3	4	4	15	15	16	16	11		12	10	10	11	9	12	7	7	8	8	18	18	2	2	9	1	1	14	14	
19	18	18	13	13	6	6	5	5	4	4	3	3	16	16	15	15		12		9	9	11	12	10	8	8	7	7	2	2	17	17	10	14	14	1	1	
20	18	18	13	13	6	6	5	5	4	4	3	3	16	16	15	15	10	12		9		11	10	12	9	8	8	7	7	2	2	17	17	14	14	1	1	
21	15	15	16	16	7	7		17	17	13	13	14	14	1	1	2	2	10	P	11	P		10	9	11	18	18	4	4	5	5	6	6	9	3	3	8	8
22	15	15	16	16	7	7		17	17	13	13	14	14	1	1	2	2	P	11	P	12	12		10	18	18	4	4	5	5	6	6	10	3	3	8	8	
23	16	16	15	15	18	18	7	7	14	14	13	13	2	2	1	1	9		12		11	9		12	4	4	17	17	6	6	5	5	11	8	8	3	3	
24	16	16	15	15	18	18	7	7	14	14	13	13	2	2	1	1	P	10	P	9	9	12	10		4	4	17	17	6	6	5	5	12	8	8	3	3	
25	5	5	6	6	1	1	12	12	1	1	2	2	9	9	18	18	17	17	8	8	3	3	4	4		15	13	14	14	16	15	13	16	10	10	7	7	
26	5	5	6	6	1	1	12	12	1	1	2	2	9	9	18	18	17	17	8	8	3	3	4	4	15		16	14	14	15	13	16	13	10	10	7	7	
27	6	6	5	5	12	12		11	2	2	1	1	17	17	9	9	8	8	18	18	4	4	3	3	15	16		13	13	15	16	14	14	7	7	10	10	
28	6	6	5	5	12	12		11	2	2	1	1	17	17	9	9	8	8	18	18	4	4	3	3	14	16	13		15	14	16	13	15	7	7	10	10	
29	7	7	8	8	9	9	10	10		18	18	3	3	4	4	1	1	2	2	17	17	6	6	14	P	15	P		14	13	15	13	12	12	5	5		
30	7	7	8	8	9	9	10	10		18	18	3	3	4	4	1	1	2	2	17	17	6	6		15		16	16		15	14	14	12	12	5	5		
31	8	8	7	7	10	10	9	9	17	17	1	1	4	4	3	3	2	2	1	1	6	6	18	18	13	P	16	P	15	13		16	15	5	5	12	12	
32	8	8	7	7	10	10	9	9	17	17	1	1	4	4	3	3	2	2	1	1	6	6	18	18	P	14	P	13	13	16	14		16	5	5	12	12	
33	1	2	3	4	4	1	2	3	5	6	7	8	8	5	6	7	9	10	11	12	12	9	10	11	13	14	15	16	16	13	14	15		17	17	18	18	
34																																			17		18	18
35																																			18	P		P
36																																			17	P		P
37																																			18	17	17	

Figure 6.2.10: Phase 2 of Extension of ARMP of $K(33, 33)$ to an ARMP of $K(37, 37)$
 (One and one-half cycle each is added to APCs 17 and 18 and one cycle is added to the PPC as shown in the cells with striped border.)

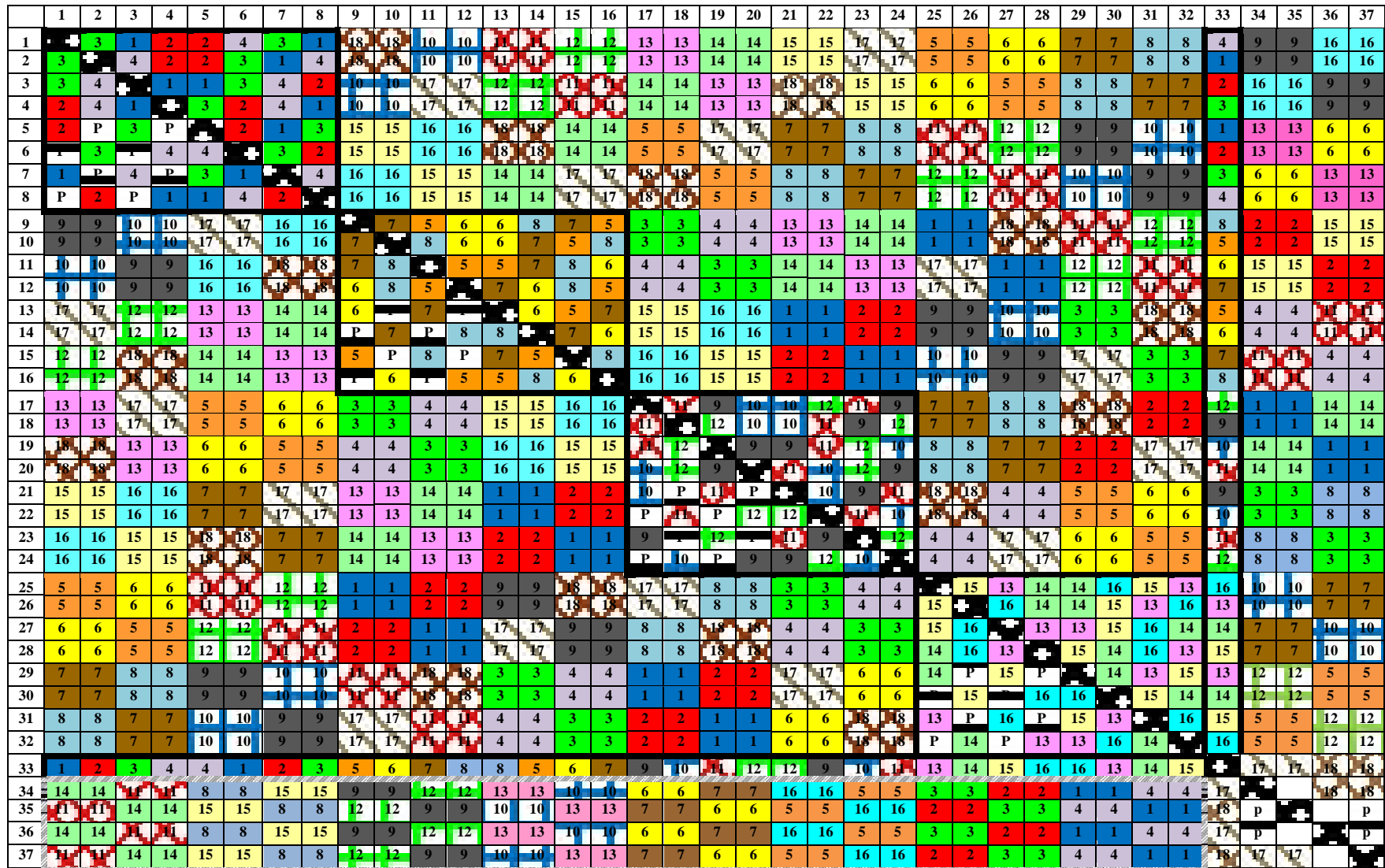


Figure 6.2.11: Phase 3 of Extension of ARMP of $K(33, 33)$ to an ARMP of $K(37, 37)$

(Two new cycles are added to each APC 1 through 16 (shown in cells in striped border) using newly appended part A vertices and part B vertices vacated by the corresponding APCs in the cycle shifts of Phase 1.)

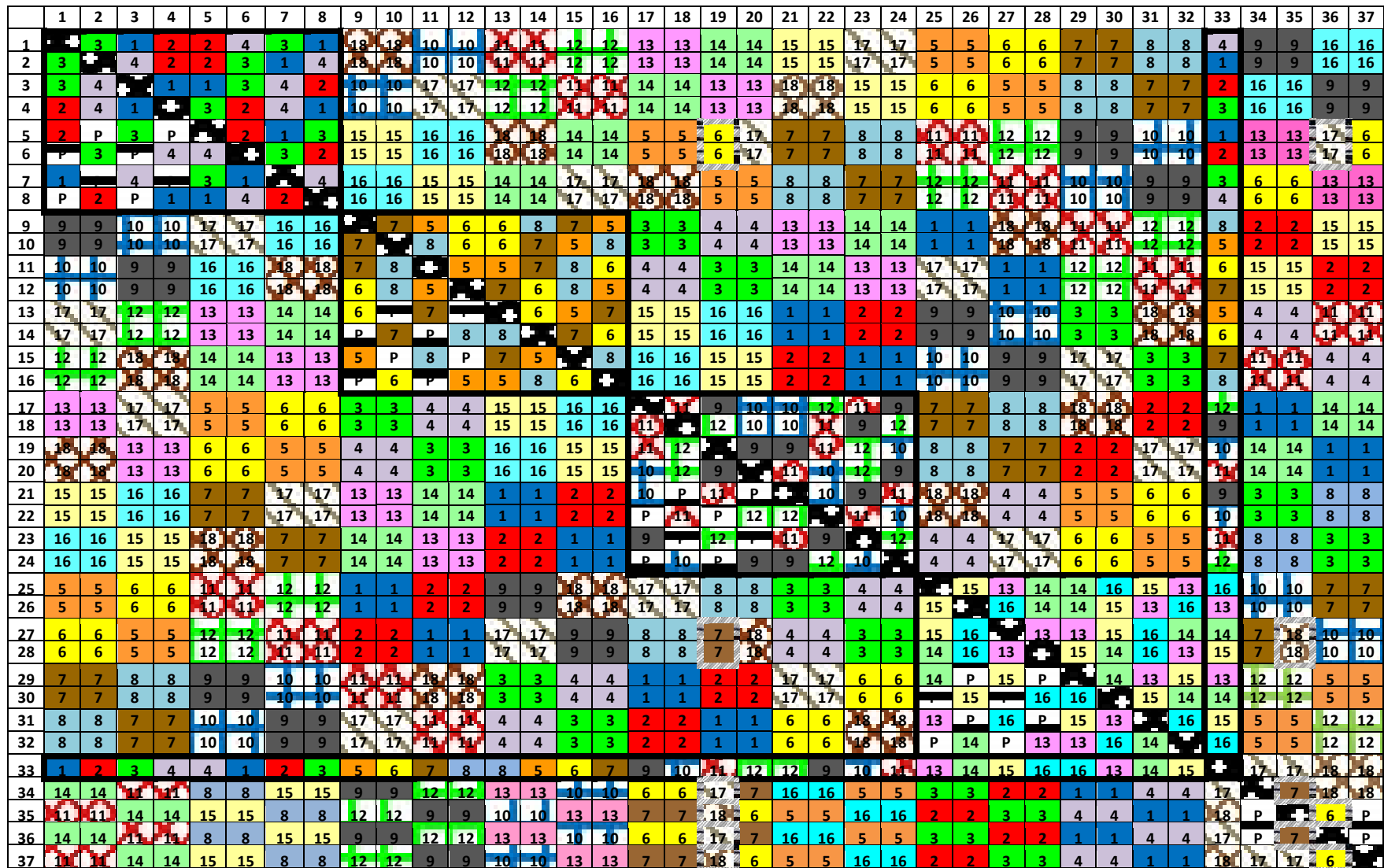


Figure 6.2.12: Phase 4 of Extension of ARMP of $K(33, 33)$ to an ARMP of $K(37, 37)$

(The second new cycle is completed for APC 17 and APC 18 using “half-cycles” formed in Phase 2 and class shifts on individual edges as shown in cells in striped borders. The extension and resolution is complete.)

We now generalize the procedure illustrated in the previous two examples for values of $m > 3$. We must begin by constructing an ARMP of $K(8m + 1, 8m + 1)$ using a commutative quasigroup (Q, \circ) order $2m$ on the set of symbols $\{1, 2, 3, \dots, 2m\}$ with set of holes $\{(1,2), (3,4), (5,6), \dots, (2m-1, 2m)\}$ of size 2 and containing a pair of disjoint complete transversals, T_1 and T_2 , outside the holes. (Constructions for such quasigroups will be given in Chapter 8.) For $i \in \{1, 2\}$, let the elements $1, 2, 3, \dots, 2m$ of T_i be represented as $1_i, 2_i, 3_i, \dots, 2m_i$ respectively. Append 4 vertices ($a_{10}, a_{11}, a_{12},$ and a_{13}) to part A and 4 vertices ($b_{10}, b_{11}, b_{12},$ and b_{13}) to part B of $K(8m + 1, 8m + 1)$ along with the associated edges necessary to create $K(8m + 5, 8m + 5)$.

Phase 1: Target particular cycles of the ARMP of $K(8m + 1, 8m + 1)$ for shifting and replacement according to the following selection procedures:

Round 1: In (Q, \circ) , locate 1_1 and determine z and w such that $z \circ w = 1_1$. In the corresponding $K_{(z,w)}(4, 4)$ in the ARMP of $K(8m + 1, 8m + 1)$, select one of the cycles from APC 1: $(a_{(z,1)}, b_{(w,1)}, a_{(z,2)}, b_{(w,2)})$ or $(a_{(z,3)}, b_{(w,3)}, a_{(z,4)}, b_{(w,4)})$. It does not matter which of the two cycles is selected; the other will be used in the next round). It's position within this $K(4,4)$, however, should be noted. We will denote the cycle positions as quadrant I, II, III, or IV according to the standard quadrant positions of the Cartesian Coordinate System. Shown below in Figure 6.2.13 is the example of selecting the cycle $(a_{(z,1)}, b_{(w,1)}, a_{(z,2)}, b_{(w,2)})$ for APC 1 in quadrant II (see cells with striped border).

	$b_{(w,1)}$	$b_{(w,2)}$	$b_{(w,3)}$	$b_{(w,4)}$
$a_{(z,1)}$	1	1	2	2
$a_{(z,2)}$	1	1	2	2
$a_{(z,3)}$	2	2	1	1
$a_{(z,4)}$	2	2	1	1

Figure 6.2.13: Round 1, Phase 1, Selecting a Cycle from APC 1

Next locate 1_2 in (Q, \circ) and determine h and j such that $h \circ j = 1_2$. In the corresponding $K_{(h,j)}(4, 4)$ in the ARMP of $K(8m + 1, 8m + 1)$, select one of the cycles from APC 2: $(a_{(h,3)}, b_{(j,1)}, a_{(h,4)}, b_{(j,2)})$ or $(a_{(h,1)}, b_{(j,3)}, a_{(h,2)}, b_{(j,4)})$. It does not matter which of the two cycles is selected; the other will be used in the next round). It's position within this $K(4,4)$ should be noted, however (as quadrant I, II, III, or IV of the matrix). Shown below in Figure 6.2.14 is the example of selecting the cycle $(a_{(h,1)}, b_{(j,3)}, a_{(h,2)}, b_{(j,4)})$ in quadrant I.

	$b_{(j,1)}$	$b_{(j,2)}$	$b_{(j,3)}$	$b_{(j,4)}$
$a_{(h,1)}$	1	1	2	2
$a_{(h,2)}$	1	1	2	2
$a_{(h,3)}$	2	2	1	1
$a_{(h,4)}$	2	2	1	1

Figure 6.2.14: Phase 1, Round 1, Selecting a Cycle from APC 2

We are assured that $z \neq h$ and $w \neq j$ because $1_1 = 1_2$ and hence must occur in different columns and different rows of (Q, \circ) .

Next, locate 2_1 in (Q, \circ) and determine c and d such that $c \circ d = 2_1$. In the corresponding $K_{(c,d)}(4, 4)$ in the ARMP of $K(8m + 1, 8m + 1)$, select one of the cycles from APC 3: $(a_{(c,1)}, b_{(d,1)}, a_{(c,2)}, b_{(d,2)})$ or $(a_{(c,3)}, b_{(d,3)}, a_{(c,4)}, b_{(d,4)})$. This time, it does matter which cycle is selected; two different elements of a CQG, each in a different transversal, can occur in the same row or same column of that CQG.

Specifically, since $2_1 \neq 1_2$ and $2_1 \in T_1$ while $1_2 \in T_2$, it is possible that 2_1 occurs in the same row or column as 1_2 in (Q, \circ) . Since we want our targeted cycles in Round 1 to be vertex-disjoint, we must ensure that we choose a cycle from APC 3 in $K_{(c,d)}(4, 4)$ that

shares no vertices with the two cycles previously selected. Fortunately, we can do this since we will have two cycles from which to choose. We illustrate this situation as follows:

Suppose that 2_1 occurs in the same row as 1_2 , so that $c = h$ and we have in Figure 6.2.15 the following two associated $K(4, 4)$, which show the cycle already selected for APC 2 in striped border.

	$b_{(j,1)}$	$b_{(j,2)}$	$b_{(j,3)}$	$b_{(j,4)}$
$a_{(h,1)}$	1	1	2	2
$a_{(h,2)}$	1	1	2	2
$a_{(h,3)}$	2	2	1	1
$a_{(h,4)}$	2	2	1	1

	$b_{(d,1)}$	$b_{(d,2)}$	$b_{(d,3)}$	$b_{(d,4)}$
$a_{(h,1)}$	3	3	4	4
$a_{(h,2)}$	3	3	4	4
$a_{(h,3)}$	4	4	3	3
$a_{(h,4)}$	4	4	3	3

Figure 6.2.15: Phase 1, Round 1, Selecting a Cycle from APC 3

We cannot select the cycle $(a_{(h,1)}, b_{(d,1)}, a_{(h,2)}, b_{(d,2)})$ for APC 3 since it shares the vertices $a_{(h,1)}$ and $a_{(h,2)}$ with the cycle already chosen for APC 2. So we simply select the cycle $(a_{(h,3)}, b_{(d,3)}, a_{(h,4)}, b_{(d,4)})$ for APC 3 in $K_{(c,d)}(4, 4)$ which shares no vertices with the cycle selected for APC 2. This selection is pictured below in Figure 6.2.16 and the reader will note that the two selected cycles appear in different “rows” and “columns” of the overall matrix representing $K(8m + 5, 8m + 5)$.

	$b_{(j,1)}$	$b_{(j,2)}$	$b_{(j,3)}$	$b_{(j,4)}$
$a_{(h,1)}$	1	1	2	2
$a_{(h,2)}$	1	1	2	2
$a_{(h,3)}$	2	2	1	1
$a_{(h,4)}$	2	2	1	1

	$b_{(d,1)}$	$b_{(d,2)}$	$b_{(d,3)}$	$b_{(d,4)}$
$a_{(h,1)}$	3	3	4	4
$a_{(h,2)}$	3	3	4	4
$a_{(h,3)}$	4	4	3	3
$a_{(h,4)}$	4	4	3	3

Figure 6.2.16: Phase 1, Round 1, Selecting a Cycle from APC 3 with No Conflict

Similarly, suppose that 2_1 occurs in the same column as 1_2 , so that $d = j$ and we have in Figure 6.2.17 the following two associated $K(4,4)$, which show the cycle already selected for APC 2 in striped border

	$b_{(j,1)}$	$b_{(j,2)}$	$b_{(j,3)}$	$b_{(j,4)}$
$a_{(h,1)}$	1	1	2	2
$a_{(h,2)}$	1	1	2	2
$a_{(h,3)}$	2	2	1	1
$a_{(h,4)}$	2	2	1	1

	$b_{(j,1)}$	$b_{(j,2)}$	$b_{(j,3)}$	$b_{(j,4)}$
$a_{(c,1)}$	3	3	4	4
$a_{(c,2)}$	3	3	4	4
$a_{(c,3)}$	4	4	3	3
$a_{(c,4)}$	4	4	3	3

Figure 6.2.17: Phase 1, Round 1, Selecting a Cycle from APC 3, 2nd Possible Conflict

We cannot select the cycle $(a_{(c,3)}, b_{(j,3)}, a_{(c,4)}, b_{(j,4)})$ for APC 3 since it would share vertices with the cycle $(a_{(h,1)}, b_{(j,3)}, a_{(h,2)}, b_{(j,4)})$ already selected from APC 2. So we simply select the cycle $(a_{(c,1)}, b_{(j,1)}, a_{(c,2)}, b_{(j,2)})$ for APC 3 in $K_{(c,d)}(4,4)$ which shares no vertices with the cycle selected for APC 2. This selection is pictured in Figure 6.2.18 that follows and the reader will note that the two selected cycles appear in different “rows” and “columns” of the overall matrix representing $K(8m + 5, 8m + 5)$.

	$b_{(j,1)}$	$b_{(j,2)}$	$b_{(j,3)}$	$b_{(j,4)}$
$a_{(h,1)}$	1	1	2	2
$a_{(h,2)}$	1	1	2	2
$a_{(h,3)}$	2	2	1	1
$a_{(h,4)}$	2	2	1	1

	$b_{(j,1)}$	$b_{(j,2)}$	$b_{(j,3)}$	$b_{(j,4)}$
$a_{(c,1)}$	3	3	4	4
$a_{(c,2)}$	3	3	4	4
$a_{(c,3)}$	4	4	3	3
$a_{(c,4)}$	4	4	3	3

Figure 6.2.18: Phase 1, Round 1, Selecting a Cycle from APC 3, Avoiding 2nd Conflict

Next locate 2_2 in (Q, \circ) and determine e and f such that $e \circ f = 2_2$. In the corresponding $K_{(e,f)}(4, 4)$ in the ARMP of $K(8m+1, 8m+1)$, select one of the cycles from APC 4: $(a_{(e,1)}, b_{(f,1)}, a_{(e,2)}, b_{(f,2)})$ or $(a_{(e,3)}, b_{(f,3)}, a_{(e,4)}, b_{(f,4)})$. Again this time, it does matter which cycle is selected; since $2_2 \neq 1_1$ and $2_2 \in T_2$ while $1_1 \in T_2$, it is possible that 2_2 is in the same row or column in (Q, \circ) as 1_1 . Fortunately, at this early point in our selection procedure, it is not possible for 2_2 to be simultaneously in the same column of one element and in the same row of another element; this can only happen after the 4th cycle selection. So we can handle the cycle selection for APC 4 in the same manner we handled the cycle selection for APC 3.

From this point forward, cycle selections will be made in the same manner; locate the corresponding $K(4,4)$ for x_i and select a cycle from APC $2x - 1$ when $i = 1$ and from APC $2x$ when $i = 2$. In other words, select a cycle from the odd numbered parallel class of each $K(4,4)$ for elements of T_1 and select a cycle from the even numbered parallel class

of each $K(4,4)$ for elements of T_2 , taking care to select a cycle that is vertex disjoint from every cycle previously selected. The ability to do this is guaranteed, as we explain in detail in the following paragraphs:

Every row and column of (Q, \circ) contains one element from each of T_1 and T_2 . Consequently, after the 4th cycle selection, it is possible that for a particular parallel class in a particular $K(4, 4)$, we cannot select either of the available cycles in their position because a previously selected cycle already uses the same part B vertices and another previously selected cycle already uses the same part A vertices. For example, suppose that we need to select a cycle from APC 10 for 5_2 , which is in the same row of (Q, \circ) as 4_1 and in the same column as 3_1 and assume that the previous cycle selections for 4_1 and 3_1 are both quadrant IV positions as shown in the matrices of Figure 6.2.19 below:

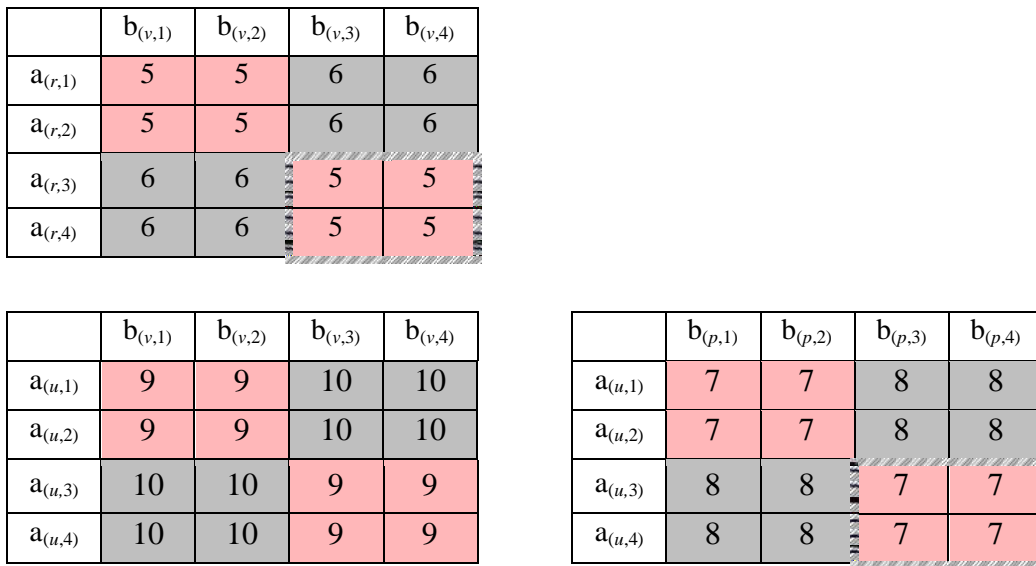


Figure 6.2.19: Phase 1, Round 1, Possible Conflict with Cycle Selections after 4th Choice

The choice of a quadrant III cycle for APC 10 is not vertex-disjoint from the APC 7 cycle selection and the choice of the quadrant I cycle for APC 10 is not vertex-disjoint

from the APC 5 cycle selection. However, there is a simple solution – simply switch the parallel class assignments for the cycles in $K_{(u,v)}(4,4)$. This is possible since the initial class assignments were arbitrary and since the switch will not affect the integrity of the ARMP of $K(8m + 1, 8m + 1)$. The switch is shown in the matrices of Figure 6.2.20 that follows. Notice that we can now easily select a cycle from APC 10 (quadrant II) that does not conflict with the selections for APC 5 and APC 7.

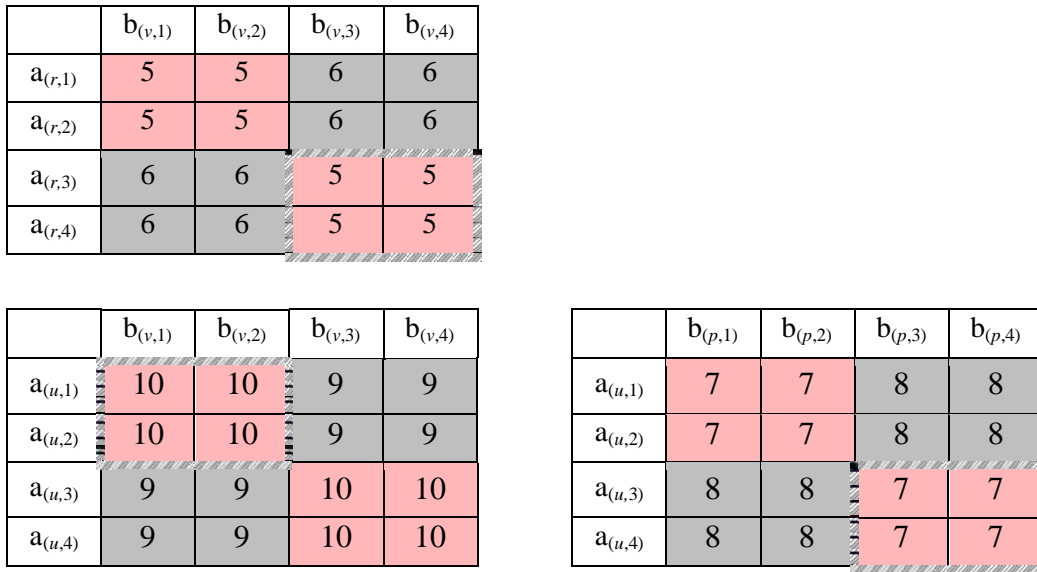


Figure 6.2.20: Phase 1, Round 1, APC Switch Resolves Cycle Selection Conflict

Continuing in this manner we are able to identify $4m$ vertex-disjoint cycles, one from each almost parallel class, in the ARMP of $K(8m + 1, 8m + 1)$.

Round 2: For every cycle selected in Round 1, identify the cycle “diagonal” from it in the matrix representation of its associated $K(4, 4)$. (Two cycles are diagonal if they appear in quadrants I and III or in quadrants II and IV.) For example, if cycle $(a_{(u,3)}, b_{(p,3)}, a_{(u,4)}, b_{(p,4)})$ from APC 7 is chosen in Round 1 (as shown in quadrant IV of the matrix on the right in Figure 6.2.20 above), then the cycle $(a_{(u,1)}, b_{(p,1)}, a_{(u,2)}, b_{(p,2)})$ in quadrant II

will be selected from APC 7 in Round 2. The $4m$ cycles thus selected, one from each almost parallel class, constitute a second vertex-disjoint set of cycles in the ARMP of $K(8m + 1, 8m + 1)$ that we target for shifting.

Make the cycle shifts as follows: Change the part B vertices of any cycle chosen in Round 1 to vertices b_{10} and b_{11} . Replace these shifted cycles with cycles from APC $4m+2$. Change the part B vertices of any cycle chosen in Round 2 with vertices b_{12} and b_{13} . Replace these shifted cycles with cycles from APC $4m + 1$. At this point, we will have established $4m$ cycles for each of the two new APCs. Assign the edges (a_{10}, b_{10}) , (a_{11}, b_{11}) , (a_{12}, b_{12}) , and (a_{13}, b_{13}) to the leave.

Phase 2: Add one new cycle $(a_{11}, b_{10}, a_{12}, b_{13})$ to the PPC. Also, add one new cycle $(a_9, b_{10}, a_{13}, b_{11})$ to the new APC $4m + 1$ and add one new cycle $(a_9, b_{12}, a_{10}, b_{13})$ to the new APC $4m + 2$. In addition, form half of a second new cycle for APC $4m + 1$ from the edges (a_{10}, b_9) and (a_{12}, b_9) and form half of a second new cycle for APC $4m + 2$ from the edges (a_{11}, b_9) and (a_{13}, b_9) . This leaves the edges (a_{10}, b_{11}) , (a_{12}, b_{11}) , (a_{11}, b_{12}) , and (a_{13}, b_{12}) free to be filled in Phase 4.

Phase 3: Add two new cycles to each of the original $4m$ APCs according to the following rule: For $u, v \in \{1, 2, 3, \dots, 2m\}$, for $i, j \in \{1, 2, 3, 4\}$, and for $1 \leq k \leq 4m$, if the cycle $(a_{(u, i)}, b_{(v, j)}, a_{(u, i+1)}, b_{(v, j+1)})$ in APC k was shifted in Phase 1 so that its B vertices switched to b_{10} and b_{11} , then add the cycle $(a_{10}, b_{(v, j)}, a_{12}, b_{(v, j+1)})$ to APC k . If, instead, this same cycle in APC k was shifted in Phase 1 so that its B vertices switched to b_{12} and b_{13} , then add the cycle $(a_{11}, b_{(v, j)}, a_{13}, b_{(v, j+1)})$ to APC k .

Phase 4: Again, for $t, u, v \in \{1, 2, 3, \dots, 2m\}$, for $h, i, j \in \{1, 2, 3, 4\}$, and for $k, l \in \{1, 2, 3, \dots, 4m\}$, select any part B vertex $b(u, i)$ and make the following shifts:

1. Edges $(a_{10}, b_{(u,i)})$ and $(a_{12}, b_{(u,i)})$ are assigned to a cycle in some APC k ; reassign these edges to APC $4m+1$. When joined with the edges (a_{10}, b_9) and (a_{12}, b_9) that comprised a half-cycle in Phase 2, these edges will complete the second new cycle for APC $4m+1$. To replace the two edges now missing in APC k , assign (a_{10}, b_{11}) and (a_{12}, b_{11}) to APC k . This action temporarily “overloads” vertex $b_{(u,i)}$ with edges from APC $4m + 1$ and vertex b_{11} with edges from APC k . This will be corrected in step 3 below.
2. Edges $(a_{11}, b_{(u,i)})$ and $(a_{13}, b_{(u,i)})$ are assigned to a cycle in some APC l ; reassign these edges to APC $4m+2$. When joined with the edges (a_{11}, b_9) and (a_{13}, b_9) that comprised a half-cycle in Phase 2, these edges will complete the second new cycle for APC $4m + 2$. To replace the two edges now missing in APC l , assign (a_{11}, b_{12}) and (a_{13}, b_{12}) to APC l . This action temporarily “overloads” vertex $b_{(u,i)}$ with edges from APC $4m + 2$ and vertex b_{12} with edges from APC l . This will be corrected in step 4 below.
3. For some v , the edges $(a_{(v,j)}, b_{(u,i)})$ and $(a_{(v,j+1)}, b_{(u,i)})$ were reassigned to APC k in Phase 1, while edges $(a_{(v,j)}, b_{11})$ and $(a_{(v,j+1)}, b_{11})$ were assigned to APC $4m + 2$. Switch the class assignments of these two pairs of edges. This relieves the overloads on vertices $b_{(u,i)}$ and b_{11} caused by actions in step 1 above.
4. For some t , the edges $(a_{(t,h)}, b_{(u,i)})$ and $(a_{(t,h+1)}, b_{(u,i)})$ were reassigned to APC l in Phase 1, while edges $(a_{(t,h)}, b_{12})$ and $(a_{(t,h+1)}, b_{12})$ were assigned to APC $4m + 1$. Switch the class assignments of these two pairs of edges. This relieves the overloads on vertices $b_{(u,i)}$ and b_{12} caused by actions in step 2 above.

The edges of $K(8m + 5, 8m + 5)$ are now resolved into $4m + 2$ APCs of $4m + 2$ cycles each and one PPC with $2m + 1$ cycles, as required for an ARMP of 4-cycles.

Chapter 7

General Odd/Odd Cases: $K(8m + 7, 2n + 1)$ with $n \geq 4$

7.1 $K(15, 15)$

Lemma 7.1. There exists an almost resolvable maximum packing of $K(15, 15)$ with 4-cycles.

Proof. Extending the ARMP of $K(13, 13)$ in the proof of Lemma 6.1 by shifting and replacing cycles gives the ARMP of $K(13, 15)$ shown in Figure 7.1.1 below.

	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}	b_{14}	b_{15}
a_1	L	7	2	3	4	5	5	3	4	6	7	6	2	1	1
a_2	2	L	7	6	4	1	3	2	4	1	3	7	6	5	5
a_3	5	6	L	7	3	1	7	4	6	1	5	4	3	2	2
a_4	5	2	3	L	6	4	1	6	1	3	5	2	4	7	7
a_5	1	2	6	4	L	7	6	5	7	5	4	2	1	3	3
a_6	7	5	6	3	1	L	6	3	2	7	2	1	5	4	4
a_7	2	7	P	5	3	P	L	2	5	6	7	6	3	1	1
a_8	P	6	7	P	2	4	3	L	6	2	3	7	4	5	5
a_9	1	P	3	7	P	6	7	4	L	3	6	4	1	2	2
a_{10}	3	5	P	4	6	P	1	6	1	L	4	3	5	7	7
a_{11}	P	4	2	P	1	7	4	5	7	5	L	1	2	3	3
a_{12}	7	P	1	6	P	5	5	1	2	7	2	L	6	4	4
a_{13}	3	4	1	5	2	6	4	1	5	2	6	3	L	L	L

Figure 7.1.1: ARMP of $K(13, 15)$

The ARMP of $K(13, 15)$ shown in Figure 7.1.1 can be extended to an ARMP of $K(15, 15)$ by adding one cycle to each existing APC using edges from the two new vertices a_{14} and a_{15} and part B vertices b_1 through b_{13} as given in the matrix of Figure 7.1.2 below:

	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}	b_{14}	b_{15}
a_1	L	7	2	3	4	5	5	3	4	6	7	6	2	1	1
a_2	2	L	7	6	4	1	3	2	4	1	3	7	6	5	5
a_3	5	6	L	7	3	1	7	4	6	1	5	4	3	2	2
a_4	5	2	3	L	6	4	1	6	1	3	5	2	4	7	7
a_5	1	2	6	4	L	7	6	5	7	5	4	2	1	3	3
a_6	7	5	6	3	1	L	6	3	2	7	2	1	5	4	4
a_7	2	7	P	5	3	P	L	2	5	6	7	6	3	1	1
a_8	P	6	7	P	2	4	3	L	6	2	3	7	4	5	5
a_9	1	P	3	7	P	6	7	4	L	3	6	4	1	2	2
a_{10}	3	5	P	4	6	P	1	6	1	L	4	3	5	7	7
a_{11}	P	4	2	P	1	7	4	5	7	5	L	1	2	3	3
a_{12}	7	P	1	6	P	5	5	1	2	7	2	L	6	4	4
a_{13}	3	4	1	5	2	6	4	1	5	2	6	3	L	L	L
a_{14}	4	1	5	2	7	3	2	7	3	4	1	5	L	6	6
a_{15}	4	1	5	2	7	3	2	7	3	4	1	5	L	6	6

Figure 7.1.2: ARMP of $K(15, 15)$ with 4-Cycles

7.2 $K(8m + 7, 8m + 7)$

Theorem 7.2. There exists an ARMP of $K(8m + 7, 8m + 7)$ with 4-cycles for all $m \leq 4$ (except possibly $m = 2$); there exists an ARMP of $K(8m + 7, 8m + 7)$ for all $m > 4$ whenever there exists a commutative quasigroup of order $2m$ with holes of size 2 that contains 2 complete transversals outside the holes.

Proof. We have already proven the case for $m = 1$ in Lemma 7.1. For $m = 3$ and $m = 4$, we will construct an ARMP of $K(31, 31)$ and $K(39, 39)$, then generalize that construction for $m > 4$, following essentially the same lines as the proof for $K(8m + 5, 8m + 5)$.

An ARMP of $K(31, 31)$ is constructed by extending the ARMP of $K(25, 25)$ through a series of shifts and replacements of cycles. While an ARMP of $K(25, 25)$ contains 12 APCs with 12 cycles per class and a PPC with 6 cycles, an ARMP of $K(31, 31)$ must contain 15 APCs with 15 cycles per class and a PPC with 7 cycles. Thus any extension of an ARMP of $K(25, 25)$ to an ARMP of $K(31, 31)$ must introduce 3 new APCs and 3 new cycles per pre-existing APC, plus 1 new cycle for the PPC.

The ARMP of $K(25, 25)$ which will be used to construct the ARMP of $K(31, 31)$ was formed in Example 4.3 using the $K(8m + 1, 8m + 1)$ construction and is shown in its familiar matrix representation on the next page in Figure 7.2.1. The cells within striped borders contain the cycles which will be shifted in Phase 1 of the 4-phase construction. For space-saving and readability, the part A and part B vertices are labeled simply 1 through 25. Abbreviated explanations of actions taken in each phase of the construction are given on the same page as the corresponding matrix.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1	+	3	1	2	2	4	3	1	9	9	10	10	11	11	12	12	5	5	6	6	7	7	8	8	4
2	3	+	4	2	2	3	1	4	9	9	10	10	11	11	12	12	5	5	6	6	7	7	8	8	1
3	3	4	+	1	1	3	4	2	10	10	9	9	12	12	11	11	6	6	5	5	8	8	7	7	2
4	2	4	1	+	3	2	4	1	10	10	9	9	12	12	11	11	6	6	5	5	8	8	7	7	3
5	2	P	3	P	+	2	1	3	11	11	12	12	9	9	10	10	7	7	8	8	5	5	6	6	1
6	P	3	P	4	4	+	3	2	11	11	12	12	9	9	10	10	7	7	8	8	5	5	6	6	2
7	1	P	4	P	3	1	+	4	12	12	11	11	10	10	9	9	8	8	7	7	6	6	5	5	3
8	P	2	P	1	1	4	2	+	12	12	11	11	10	10	9	9	8	8	7	7	6	6	5	5	4
9	9	9	10	10	11	11	10	10	+	7	5	6	6	8	7	5	1	1	2	2	3	3	4	4	8
10	9	9	10	10	11	11	10	10	7	+	8	6	6	7	5	8	1	1	2	2	3	3	4	4	5
11	10	10	9	9	10	110	11	11	7	8	+	5	5	7	8	6	2	2	1	1	4	4	3	3	6
12	10	10	9	9	10	110	11	11	6	8	5	+	7	6	8	5	2	2	1	1	4	4	3	3	7
13	11	11	10	10	9	9	10	10	6	P	7	P	+	6	5	7	3	3	4	4	1	1	2	2	5
14	11	11	10	10	9	9	10	10	P	7	P	8	8	+	7	6	3	3	4	4	1	1	2	2	6
15	10	10	11	11	10	10	9	9	5	P	8	P	7	5	+	8	4	4	3	3	2	2	1	1	7
16	10	10	11	11	10	10	9	9	6	+	5	5	8	6	+	4	4	3	3	2	2	1	1	8	
17	5	5	6	6	7	7	8	8	1	1	2	2	3	3	4	4	+	11	9	10	10	12	11	9	12
18	5	5	6	6	7	7	8	8	1	1	2	2	3	3	4	4	11	+	12	10	10	11	9	12	9
19	6	6	5	5	8	8	7	7	2	2	1	1	4	4	3	3	11	12	+	9	9	11	12	10	10
20	6	6	5	5	8	8	7	7	2	2	1	1	4	4	3	3	10	12	9	+	11	10	12	9	11
21	7	7	8	8	5	5	6	6	3	3	4	4	1	1	2	2	10	P	11	P	+	10	9	11	9
22	7	7	8	8	5	5	6	6	3	3	4	4	1	1	2	2	P	11	P	12	12	+	11	10	10
23	8	8	7	7	6	6	5	5	4	4	3	3	2	2	1	1	9	P	12	P	11	9	+	12	11
24	8	8	7	7	6	6	5	5	4	4	3	3	2	2	1	1	P	10	P	9	9	12	10	+	12
25	1	2	3	4	4	1	2	3	5	6	7	8	8	5	6	7	9	10	11	12	12	9	10	11	+

Figure 7.2.1: ARMP of $K(25, 25)$ to be Extended to an ARMP of $K(31, 31)$

(Cycles targeted for shifting and replacement are indicated in cells with striped borders)

Phase 1: Figure 7.2.2 on the next page shows 6 new vertices appended to each of parts A and B of $K(25, 25)$ to Form $K(31, 31)$. Cycles targeted for shifting have their part B vertices changed to the newly appended B vertices and are shown in the cells with striped border. These cycles are replaced with cycles from the three new APCs 13 (colored magenta), 14 (colored peach), and 15 (colored pale yellow). Edges are added to the leave as required and are shown in cells shaded with black trellis design.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31		
1		3	1	2	2	4	3	1	14	14	13	13	15	15	12	12	5	5	6	6	7	7	8	8	4	9	9	10	10	11	11		
2	3		4	2	2	3	1	4	14	14	13	13	15	15	12	12	5	5	6	6	7	7	8	8	1	9	9	10	10	11	11		
3	3	4		1	1	3	4	2	15	15	9	9	14	14	13	13	6	6	5	5	8	8	7	7	2	12	12	11	11	10	10		
4	2	4	1		3	2	4	1	15	15	9	9	14	14	13	13	6	6	5	5	8	8	7	7	3	12	12	11	11	10	10		
5	2	P	3	P		2	1	3	11	11	15	15	13	13	14	14	7	7	8	8	5	5	6	6	1	10	10	9	9	12	12		
6	P	3	P	4	4		3	2	11	11	15	15	13	13	14	14	7	7	8	8	5	5	6	6	2	10	10	9	9	12	12		
7	1	P	4	P	3	1		4	13	13	14	14	10	10	15	15	8	8	7	7	6	6	5	5	3	11	11	12	12	9	9		
8	P	2	P	1	1	4	2		13	13	14	14	10	10	15	15	8	8	7	7	6	6	5	5	4	11	11	12	12	9	9		
9	9	9	10	10	11	11	10	10		7	5	6	6	8	7	5	14	14	13	13	15	15	4	4	8	1	1	2	2	3	3		
10	9	9	10	10	11	11	10	10	7		8	6	6	7	5	8	14	14	13	13	15	15	4	4	5	1	1	2	2	3	3		
11	10	10	9	9	10	10	11	11	7	8		5	5	7	8	6	15	15	1	1	14	14	13	13	6	4	4	3	3	2	2		
12	10	10	9	9	10	10	11	11	6	8	5		7	6	8	5	15	15	1	1	14	14	13	13	7	4	4	3	3	2	2		
13	11	11	10	10	9	9	10	10	6	P	7	P		6	5	7	3	3	15	15	13	13	14	14	5	2	2	1	1	4	4		
14	11	11	10	10	9	9	10	10	P	7	P	8	8		7	6	3	3	15	15	13	13	14	14	6	2	2	1	1	4	4		
15	10	10	11	11	10	10	9	9	5	P	8	P	7	5		8	13	13	14	14	2	2	15	15	7	3	3	4	4	1	1		
16	10	10	11	11	10	10	9	9	P	6	P	5	5	8	6		13	13	14	14	2	2	15	15	8	3	3	4	4	1	1		
17	14	14	13	13	15	15	8	8	1	1	2	2	3	3	4	4		11	9	10	10	12	11	9	12	5	5	6	6	7	7		
18	14	14	13	13	15	15	8	8	1	1	2	2	3	3	4	4	11		12	10	10	11	9	12	9	5	5	6	6	7	7		
19	15	15	5	5	14	14	13	13	2	2	1	1	4	4	3	3	11	12		9	9	11	12	10	10	8	8	7	7	6	6		
20	15	15	5	5	14	14	13	13	2	2	1	1	4	4	3	3	10	12	9		11	10	12	9	11	8	8	7	7	6	6		
21	7	7	15	15	13	13	14	14	3	3	4	4	1	1	2	2	10	P	11	P		10	9	11	9	6	6	5	5	8	8		
22	7	7	15	15	13	13	14	14	3	3	4	4	1	1	2	2	P	11	P	12	12		11	10	10	6	6	5	5	8	8		
23	13	13	14	14	6	6	15	15	4	4	3	3	2	2	1	1	9	P	12	P	11	9		12	7	7	8	8	5	5			
24	13	13	14	14	6	6	15	15	4	4	3	3	2	2	1	1	P	10	P	9	9	12	10		12	7	7	8	8	5	5		
25	1	2	3	4	4	1	2	3	5	6	7	8	8	5	6	7	9	10	11	12	12	9	10	11									
26																																	
27																																	
28																																	
29																																	
30																																	
31																																	

Figure 7.2.2: Extension of ARMP of $K(25, 25)$ to ARMP of $K(31, 31)$: Phase 1

Phase 2: In Figure 7.2.3 on the next page, two and one-half cycles are added to APCs 13 and 14, three new cycles are added to APC 15, and one new cycle is added to the PPC as indicated within the block of cells shown in striped border.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
1	3	1	2	2	4	3	1	14	14	13	13	15	15	12	12	5	5	6	6	7	7	8	8	4	9	9	10	10	11	11	
2	3	4	2	2	3	1	4	14	14	13	13	15	15	12	12	5	5	6	6	7	7	8	8	1	9	9	10	10	11	11	
3	3	4	1	1	3	4	2	15	15	9	9	14	14	13	13	6	6	5	5	8	8	7	7	2	12	12	11	11	10	10	
4	2	4	1	3	2	4	1	15	15	9	9	14	14	13	13	6	6	5	5	8	8	7	7	3	12	12	11	11	10	10	
5	2	P	3	P	2	1	3	11	11	15	15	13	13	14	14	7	7	8	8	5	5	6	6	1	10	10	9	9	12	12	
6	P	3	P	4	4	3	2	11	11	15	15	13	13	14	14	7	7	8	8	5	5	6	6	2	10	10	9	9	12	12	
7	1	P	4	P	3	1	4	13	13	14	14	10	10	15	15	8	8	7	7	6	6	5	5	3	11	11	12	12	9	9	
8	P	2	P	1	1	4	2	13	13	14	14	10	10	15	15	8	8	7	7	6	6	5	5	4	11	11	12	12	9	9	
9	9	9	10	10	11	11	10	10	7	5	6	6	8	7	5	14	14	13	13	15	15	4	4	8	1	1	2	2	3	3	
10	9	9	10	10	11	11	10	10	7	8	6	6	7	5	8	14	14	13	13	15	15	4	4	5	1	1	2	2	3	3	
11	10	10	9	9	10	11	11	7	8	5	5	7	8	6	15	15	1	1	14	14	13	13	6	4	4	3	3	2	2		
12	10	10	9	9	10	11	11	6	8	5	5	7	6	8	5	15	15	1	1	14	14	13	13	7	4	4	3	3	2	2	
13	11	11	10	10	9	9	10	10	6	P	7	P	6	5	7	3	3	15	15	13	13	14	14	5	2	2	1	1	4	4	
14	11	11	10	10	9	9	10	10	P	7	P	8	8	7	6	3	3	15	15	13	13	14	14	6	2	2	1	1	4	4	
15	10	10	11	11	10	10	9	9	5	P	8	P	7	5	8	13	13	14	14	2	2	15	15	7	3	3	4	4	1	1	
16	10	10	11	11	10	10	9	9	P	6	P	5	5	8	6	13	13	14	14	2	2	15	15	8	3	3	4	4	1	1	
17	14	14	13	13	15	15	8	8	1	1	2	2	3	3	4	4	11	11	9	10	10	12	11	9	12	5	5	6	6	7	7
18	14	14	13	13	15	15	8	8	1	1	2	2	3	3	4	4	11	11	12	10	10	11	9	12	9	5	5	6	6	7	7
19	15	15	5	5	14	14	13	13	2	2	1	1	4	4	3	3	11	12	9	9	11	12	10	10	8	8	7	7	6	6	
20	15	15	5	5	14	14	13	13	2	2	1	1	4	4	3	3	10	12	9	11	10	12	9	11	8	8	7	7	6	6	
21	7	7	15	15	13	13	14	14	3	3	4	4	1	1	2	2	10	P	11	P	10	9	11	9	6	6	5	5	8	8	
22	7	7	15	15	13	13	14	14	3	3	4	4	1	1	2	2	P	11	P	12	12	11	10	10	6	6	5	5	8	8	
23	13	13	14	14	6	6	15	15	4	4	3	3	2	2	1	1	9	P	12	P	11	9	12	11	7	7	8	8	5	5	
24	13	13	14	14	6	6	15	15	4	4	3	3	2	2	1	1	P	10	P	9	9	12	10	12	7	7	8	8	5	5	
25	1	2	3	4	4	1	2	3	5	6	7	8	8	5	6	7	9	10	11	12	12	9	10	11	13	13	14	14	15	15	
26																									13	13	14	14	15	15	
27																									14	15	15	15	13	13	
28																									13	15	15	15	14	14	
29																									14	13	13	15	14	14	
30																									15	P	15	P	13	13	
31																									15	P	15	P	14	14	

Figure 7.2.3: Extension of ARMP of $K(25, 25)$ to ARMP of $K(31, 31)$: Phase 2

Phase 3: In Figure 7.2.4 on the following page, three new cycles are added to each of APCs 1 through 12 using the newly appended A vertices 26 through 31 and the B vertices 1 through 24 that were vacated by the cycle shifts in Phase 1. These new cycles are shown in the block of cells within the striped border.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
1	3	1	2	2	4	3	1	14	14	13	13	15	15	12	12	5	5	6	6	7	7	8	8	4	9	9	10	10	11	11	
2	3	4	2	2	3	1	4	14	14	13	13	15	15	12	12	5	5	6	6	7	7	8	8	1	9	9	10	10	11	11	
3	3	4	1	1	3	4	2	15	15	9	9	14	14	13	13	6	6	5	5	8	8	7	7	2	12	12	11	11	10	10	
4	2	4	1	3	2	4	1	15	15	9	9	14	14	13	13	6	6	5	5	8	8	7	7	3	12	12	11	11	10	10	
5	2	P	3	P	2	1	3	11	11	15	15	13	13	14	14	7	7	8	8	5	5	6	6	1	10	10	9	9	12	12	
6	P	3	P	4	4	3	2	14	14	15	15	13	13	14	14	7	7	8	8	5	5	6	6	2	10	10	9	9	12	12	
7	1	P	4	P	3	1	4	13	13	14	14	10	10	15	15	8	8	7	7	6	6	5	5	3	11	11	12	12	9	9	
8	P	2	P	1	1	4	2	13	13	14	14	10	10	15	15	8	8	7	7	6	6	5	5	4	11	11	12	12	9	9	
9	9	9	10	10	11	11	10	10	7	5	6	6	8	7	5	14	14	13	13	15	15	4	4	8	1	1	2	2	3	3	
10	9	9	10	10	11	11	10	10	7	8	6	6	7	5	8	14	14	13	13	15	15	4	4	5	1	1	2	2	3	3	
11	10	10	9	9	10	10	11	11	7	8	5	5	7	8	6	15	15	1	1	14	14	13	13	6	4	4	3	3	2	2	
12	10	10	9	9	10	10	11	11	6	8	5	5	7	6	8	5	15	15	1	1	14	14	13	13	7	4	4	3	3	2	2
13	11	11	10	10	9	9	10	10	6	P	7	P	6	5	7	3	3	15	15	13	13	14	14	5	2	2	1	1	4	4	
14	11	11	10	10	9	9	10	10	P	7	P	8	8	7	6	3	3	15	15	13	13	14	14	6	2	2	1	1	4	4	
15	10	10	11	11	10	10	9	9	5	P	8	P	7	5	8	13	13	14	14	2	2	15	15	7	3	3	4	4	1	1	
16	10	10	11	11	10	10	9	9	P	6	P	5	5	8	6	13	13	14	14	2	2	15	15	8	3	3	4	4	1	1	
17	14	14	13	13	15	15	8	8	1	1	2	2	3	3	4	4	11	9	10	10	12	11	9	12	5	5	6	6	7	7	
18	14	14	13	13	15	15	8	8	1	1	2	2	3	3	4	4	11	12	10	10	11	9	12	9	5	5	6	6	7	7	
19	15	15	5	5	14	14	13	13	2	2	1	1	4	4	3	3	11	12	9	9	11	12	10	10	8	8	7	7	6	6	
20	15	15	5	5	14	14	13	13	2	2	1	1	4	4	3	3	10	12	9	11	10	12	9	11	8	8	7	7	6	6	
21	7	7	15	15	13	13	14	14	3	3	4	4	1	1	2	2	10	P	11	P	10	9	11	9	6	6	5	5	8	8	
22	7	7	15	15	13	13	14	14	3	3	4	4	1	1	2	2	P	11	P	12	12	11	10	10	6	6	5	5	8	8	
23	13	13	14	14	6	6	15	15	4	4	3	3	2	2	1	1	9	P	12	P	11	9	12	11	7	7	8	8	5	5	
24	13	13	14	14	6	6	15	15	4	4	3	3	2	2	1	1	P	10	P	9	9	12	10	12	7	7	8	8	5	5	
25	1	2	3	4	4	1	2	3	5	6	7	8	8	5	6	7	9	10	11	12	12	9	10	11	13	13	14	14	15	15	
26	5	5	7	7	8	8	6	6	9	9	11	11	12	12	10	10	1	1	3	3	4	4	2	2	13	14	14	15	15	15	
27	8	8	6	6	5	5	7	7	12	12	10	10	9	9	11	11	4	4	2	2	1	1	3	3	14	15	15	15	13	13	
28	5	5	7	7	8	8	6	6	9	9	11	11	12	12	10	10	1	1	3	3	4	4	2	2	13	15	15	15	14	14	
29	8	8	6	6	5	5	7	7	12	12	10	10	9	9	11	11	4	4	2	2	1	1	3	3	14	13	13	15	14	14	
30	6	6	8	8	7	7	5	5	10	10	12	12	11	11	9	9	2	2	4	4	3	3	1	1	15	15	15	15	13	13	
31	6	6	8	8	7	7	5	5	10	10	12	12	11	11	9	9	2	2	4	4	3	3	1	1	15	15	15	15	14	14	

Figure 7.2.4: Extension of ARMP of $K(25, 25)$ to ARMP of $K(31, 31)$: Phase 3

Phase 4: In Figure 7.2.5 on the following page, one-half of a cycle is added to each of APC 13 and 14 by shifting the class assignments on edges previously assigned to cycles in APCs 6 and 7 and utilizing the previously unassigned edges (a_{26}, b_{27}) , (a_{28}, b_{27}) , (a_{27}, b_{28}) and (a_{29}, b_{28}) , all shown in cells with striped borders. The extension and resolution are now complete.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
1		3	1	2	2	4	3	1	14	14	13	13	15	15	12	12	5	5	6	6	7	7	8	8	4	9	9	10	10	11	11
2	3		4	2	2	3	1	4	14	14	13	13	15	15	12	12	5	5	6	6	7	7	8	8	1	9	9	10	10	11	11
3	3	4		1	1	3	4	2	15	15	9	9	14	14	13	13	6	6	5	5	8	8	7	7	2	12	12	11	11	10	10
4	2	4	1		3	2	4	1	15	15	9	9	14	14	13	13	6	6	5	5	8	8	7	7	3	12	12	14	14	10	10
5	2	P	3	P		2	1	3	11	11	15	15	13	13	14	14	7	7	8	8	5	5	6	6	1	10	10	9	9	12	12
6		3		4	4		3	2	11	11	15	15	13	13	14	14	7	7	8	8	5	5	6	6	2	10	10	9	9	12	12
7	1	P	4	P	3	1		4	13	13	14	14	10	10	15	15	8	8	7	7	6	6	5	5	3	11	11	12	12	9	9
8	P	2	P	1	1	4	2		13	13	14	14	10	10	15	15	8	8	7	7	6	6	5	5	4	11	11	12	12	9	9
9	9	9	10	10	11	11	10	10		7	5	6	6	8	7	5	14	14	13	13	15	15	4	4	8	1	1	2	2	3	3
10	9	9	10	10	11	11	10	10	7		8	6	6	7	5	8	14	14	13	13	15	15	4	4	5	1	1	2	2	3	3
11	10	10	9	9	10	11	11	11	7	8		5	5	7	8	6	15	15	1	1	14	14	13	13	6	4	4	3	3	2	2
12	10	10	9	9	10	11	11	11	6	8	5		7	6	8	5	15	15	1	1	14	14	13	13	7	4	4	3	3	2	2
13	11	11	10	10	9	9	10	10	6		7		6	5	7	3	3	15	15	13	13	14	14	5	2	2	1	1	4	4	
14	11	11	10	10	9	9	10	10	P	7	P	8	8		7	6	3	3	15	15	13	13	14	14	6	2	2	1	1	4	4
15	10	10	11	11	10	10	9	9	5	P	8	P	7	5		8	13	13	14	14	2	2	15	15	7	3	3	4	4	1	1
16	10	10	11	11	10	10	9	9	P	6	P	5	5	8	6		13	13	14	14	2	2	15	15	8	3	3	4	4	1	1
17	14	14	6	13	15	15	8	8	1	1	2	2	3	3	4	4		11	9	10	10	12	11	9	12	5	5	13	6	7	7
18	14	14	6	13	15	15	8	8	1	1	2	2	3	3	4	4	11		12	10	10	11	9	12	9	5	5	13	6	7	7
19	15	15	5	5	14	14	13	13	2	2	1	1	4	4	3	3	11	12		9	9	11	12	10	10	8	8	7	7	6	6
20	15	15	5	5	14	14	13	13	2	2	1	1	4	4	3	3	10	12	9		11	10	12	9	11	8	8	7	7	6	6
21	7	7	15	15	13	13	14	14	3	3	4	4	1	1	2	2	10	P	11	P		10	9	11	9	6	6	5	5	8	8
22	7	7	15	15	13	13	14	14	3	3	4	4	1	1	2	2	P	11	P	12	12		11	10	10	6	6	5	5	8	8
23	13	13	7	14	6	6	15	15	4	4	3	3	2	2	1	1	9	P	12	P	11	9		12	11	7	14	8	8	5	5
24	13	13	7	14	6	6	15	15	4	4	3	3	2	2	1	1	P	10	P	9	9	12	10		12	7	14	8	8	5	5
25	1	2	3	4	4	1	2	3	5	6	7	8	8	5	6	7	9	10	11	12	12	9	10	11		13	13	14	14	15	15
26	5	5	13	7	8	8	6	6	9	9	11	11	12	12	10	10	1	1	3	3	4	4	2	2	13		7	14	14	15	15
27	8	8	14	6	5	5	7	7	12	12	10	10	9	9	11	11	4	4	2	2	1	1	3	3	14	15		6	15	13	13
28	5	5	13	7	8	8	6	6	9	9	11	11	12	12	10	10	1	1	3	3	4	4	2	2	13	15	7		15	14	14
29	8	8	14	6	5	5	7	7	12	12	10	10	9	9	11	11	4	4	2	2	1	1	3	3	14	13	13	6		13	13
30	6	6	8	8	7	7	5	5	10	10	12	12	11	11	9	9	2	2	4	4	3	3	1	1	15	P	15	P		13	13
31	6	6	8	8	7	7	5	5	10	10	12	12	11	11	9	9	2	2	4	4	3	3	1	1	15	P	15	P		14	14

Figure 7.2.5: Extension of ARMP of $K(25, 25)$ to ARMP of $K(31, 31)$: Phase 4

As in the proof of $K(8m + 5, 8m + 5)$, an extension of an ARMP of $K(8m + 1, 8m + 1)$ to an ARMP of $K(8m + 7, 8m + 7)$ depends on being able to identify (this time) three disjoint sets of cycles, each set containing 1 cycle per APC such that the cycles within each set are vertex-disjoint. This is possible whenever the CQG used for the construction contains a pair of disjoint transversals outside the holes. Although the CQG used in the preceding example does not possess this feature, the particular arrangement of elements in the cells outside of its holes made it possible to identify the three necessary disjoint sets of vertex-disjoint cycles used for shifting. We now prove the theorem for $m = 4$ by extending an ARMP of $K(33, 33)$ constructed using a CQG that does possess that feature to an ARMP of $K(39, 39)$. The CQG used is pictured below in Figure 7.2.6. The elements of the two transversals selected for use in the construction are shown in cells shaded pink and green, respectively. The cycles targeted for shifting will correspond directly to the elements of the indicated transversals.

o	1	2	3	4	5	6	7	8
1	1	2	5	6	7	8	3	4
2	2	1	8	7	3	4	6	5
3	5	8	3	4	2	7	1	6
4	6	7	4	3	8	1	5	2
5	7	3	2	8	5	6	4	1
6	8	4	7	1	6	5	2	3
7	3	6	1	5	4	2	7	8
8	4	5	6	2	1	3	8	7

Figure 7.2.6: Commutative Quasigroup of Order 8, Holes of Size 2, and 2 Complete Transversals Outside Holes

The ARMP of $K(33, 33)$ created using this CGQ and the $K(8m + 1, 8m + 1)$ construction is shown in Figure 7.2.7 on the next page with cycles targeted for shifting shown in striped border. The extension to an ARMP of $K(39, 39)$ will be shown in phases with abbreviated explanations.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33
1	+	3	1	2	2	4	3	1	9	9	10	10	11	11	12	12	13	13	14	14	15	15	16	16	5	5	6	6	7	7	8	8	4
2	3	+	4	2	2	3	1	4	9	9	10	10	11	11	12	12	13	13	14	14	15	15	16	16	5	5	6	6	7	7	8	8	1
3	3	4	+	1	1	3	4	2	10	10	9	9	12	12	11	11	14	14	13	13	16	16	15	15	6	6	5	5	8	8	7	7	2
4	2	4	1	+	3	2	4	1	10	10	9	9	12	12	11	11	14	14	13	13	16	16	15	15	6	6	5	5	8	8	7	7	3
5	2	P	3	P	+	2	1	3	15	15	16	16	13	13	14	14	5	5	6	6	7	7	8	8	11	11	12	12	9	9	10	10	1
6	P	3	P	4	4	+	3	2	15	15	16	16	13	13	14	14	5	5	6	6	7	7	8	8	11	11	12	12	9	9	10	10	2
7	1	P	4	P	3	1	+	4	16	16	15	15	14	14	13	13	6	6	5	5	8	8	7	7	12	12	11	11	10	10	9	9	3
8	P	2	P	1	1	4	2	+	16	16	15	15	14	14	13	13	6	6	5	5	8	8	7	7	12	12	11	11	10	10	9	9	4
9	9	9	10	10	15	15	16	16	+	7	5	6	6	8	7	5	3	3	4	4	13	13	14	14	1	1	2	2	11	11	12	12	8
10	9	9	10	10	15	15	16	16	7	+	8	6	6	7	5	8	3	3	4	4	13	13	14	14	1	1	2	2	11	11	12	12	5
11	10	10	9	9	16	16	15	15	7	8	+	5	5	7	8	6	4	4	3	3	14	14	13	13	2	2	1	1	12	12	11	11	6
12	10	10	9	9	16	16	15	15	6	8	5	+	7	6	8	5	4	4	3	3	14	14	13	13	2	2	1	1	12	12	11	11	7
13	11	11	12	12	13	13	14	14	6	P	7	P	+	6	5	7	15	15	16	16	1	1	2	2	9	9	10	10	3	3	4	4	5
14	11	11	12	12	13	13	14	14	P	7	P	8	8	+	7	6	15	15	16	16	1	1	2	2	9	9	10	10	3	3	4	4	6
15	12	12	11	11	14	14	13	13	5	P	8	P	7	5	+	8	16	16	15	15	2	2	1	1	10	10	9	9	4	4	3	3	7
16	12	12	11	11	14	14	13	13	P	6	8	5	5	8	6	+	16	16	15	15	2	2	1	1	10	10	9	9	4	4	3	3	8
17	13	13	14	14	5	5	6	6	3	3	4	4	15	15	16	16	+	11	9	10	10	12	11	9	7	7	8	8	1	1	2	2	12
18	13	13	14	14	5	5	6	6	3	3	4	4	15	15	16	16	11	+	12	10	10	11	9	12	7	7	8	8	1	1	2	2	9
19	14	14	13	13	6	6	5	5	4	4	3	3	16	16	15	15	11	12	+	9	9	11	12	10	8	8	7	7	2	2	1	1	10
20	14	14	13	13	6	6	5	5	4	4	3	3	16	16	15	15	10	12	9	+	11	10	12	9	8	8	7	7	2	2	1	1	11
21	15	15	16	16	7	7	8	8	13	13	14	14	1	1	2	2	10	P	11	P	+	10	9	11	3	3	4	4	5	5	6	6	9
22	15	15	16	16	7	7	8	8	13	13	14	14	1	1	2	2	P	11	P	12	12	+	11	10	3	3	4	4	5	5	6	6	10
23	16	16	15	15	8	8	7	7	14	14	13	13	2	2	1	1	9	P	12	P	11	9	+	12	4	4	3	3	6	6	5	5	11
24	16	16	15	15	8	8	7	7	14	14	13	13	2	2	1	1	P	10	P	9	9	12	10	+	4	4	3	3	6	6	5	5	12
25	5	5	6	6	11	11	12	12	1	1	2	2	9	9	10	10	7	7	8	8	3	3	4	4	+	15	13	14	14	16	15	13	16
26	5	5	6	6	11	11	12	12	1	1	2	2	9	9	10	10	7	7	8	8	3	3	4	4	15	+	16	14	14	15	13	16	13
27	6	6	5	5	12	12	11	11	2	2	1	1	10	10	9	9	8	8	7	7	4	4	3	3	15	16	+	13	13	15	16	14	14
28	6	6	5	5	12	12	11	11	2	2	1	1	10	10	9	9	8	8	7	7	4	4	3	3	14	16	13	+	15	14	16	13	15
29	7	7	8	8	9	9	10	10	11	11	12	12	3	3	4	4	1	1	2	2	5	5	6	6	14	P	15	P	+	14	13	15	13
30	7	7	8	8	9	9	10	10	11	11	12	12	3	3	4	4	1	1	2	2	5	5	6	6	P	15	P	16	16	+	15	14	14
31	8	8	7	7	10	10	9	9	12	12	11	11	4	4	3	3	2	2	1	1	6	6	5	5	13	P	16	P	15	13	+	16	15
32	8	8	7	7	10	10	9	9	12	12	11	11	4	4	3	3	2	2	1	1	6	6	5	5	P	14	P	13	13	16	14	+	16
33	1	2	3	4	4	1	2	3	5	6	7	8	8	5	6	7	9	10	11	12	12	9	10	11	13	14	15	16	16	13	14	15	+

Figure 7.2.7: ARMP of $K(33, 33)$ with Intended Cycle Shifts for an Extension to an ARMP of $K(39, 39)$

Phase 1: In Figure 7.2.8 on the following page, 6 vertices are appended each to part A and B. Edges of the leave are assigned. Cycles targeted in APCs 1 – 16 are shifted and replaced with cycles in the new APCs 17, 18 and 19 (all shown in cells with striped borders).

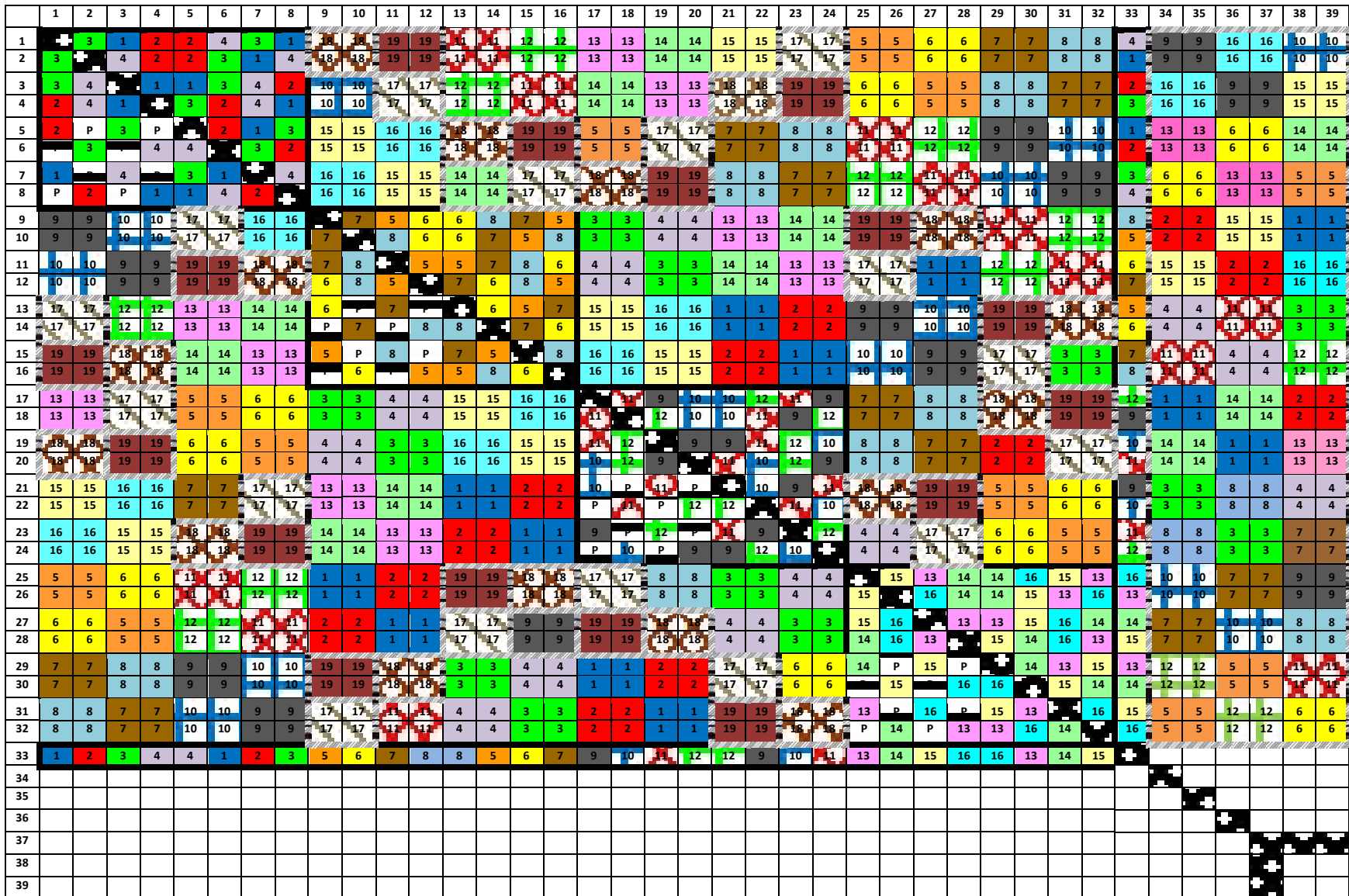


Figure 7.2.8: Extension of ARMP of $K(33, 33)$ to an ARMP of $K(39, 39)$: Phase 1

Phase 2: In Figure 7.2.9 on the following page, two and one-half cycles are added to APCs 17, and 18, three cycles are added to APC 19, and one cycle is added to the PPC as shown in the cells with striped border.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	
1	+	3	1	2	2	4	3	1	18	18	19	19	11	11	12	12	13	13	14	14	15	15	17	17	5	5	6	6	7	7	8	8	4	9	9	16	16	10	10	
2	3	+	4	2	2	3	1	4	18	18	19	19	11	11	12	12	13	13	14	14	15	15	17	17	5	5	6	6	7	7	8	8	1	9	9	16	16	10	10	
3	3	4	+	1	1	3	4	2	10	10	17	17	12	12	11	11	14	14	13	13	18	18	19	19	6	6	5	5	8	8	7	7	2	16	16	9	9	15	15	
4	2	4	1	+	3	2	4	1	10	10	17	17	12	12	11	11	14	14	13	13	18	18	19	19	6	6	5	5	8	8	7	7	3	16	16	9	9	15	15	
5	2	P	3	P	+	2	1	3	15	15	16	16	18	18	19	19	5	5	17	17	7	7	8	8	11	11	12	12	9	9	10	10	1	13	13	6	6	14	14	
6	P	3	P	+	4	4	3	2	15	15	16	16	18	18	19	19	5	5	17	17	7	7	8	8	11	11	12	12	9	9	10	10	2	13	13	6	6	14	14	
7	1	P	4	P	+	3	1	4	16	16	15	15	14	14	17	17	18	18	19	19	8	8	7	7	12	12	11	11	10	10	9	9	3	6	6	13	13	5	5	
8	P	2	P	+	1	1	4	2	16	16	15	15	14	14	17	17	18	18	19	19	8	8	7	7	12	12	11	11	10	10	9	9	4	6	6	13	13	5	5	
9	9	9	10	10	17	17	16	16	+	7	5	6	6	8	7	5	3	3	4	4	13	13	14	14	19	19	18	18	11	11	12	12	8	2	2	15	15	1	1	
10	9	9	10	10	17	17	16	16	7	+	8	6	6	7	5	8	3	3	4	4	13	13	14	14	19	19	18	18	11	11	12	12	5	2	2	15	15	1	1	
11	10	10	9	9	19	19	18	18	7	8	+	5	5	7	8	6	4	4	3	3	14	14	13	13	17	17	1	1	12	12	11	11	6	15	15	2	2	16	16	
12	10	10	9	9	19	19	18	18	6	8	5	+	7	6	8	5	4	4	3	3	14	14	13	13	17	17	1	1	12	12	11	11	7	15	15	2	2	16	16	
13	17	17	12	12	13	13	14	14	6	+	7	+	6	5	7	15	15	16	16	1	1	2	2	9	9	10	10	19	19	18	18	5	4	4	11	11	3	3		
14	17	17	12	12	13	13	14	14	P	7	P	8	8	+	7	6	15	15	16	16	1	1	2	2	9	9	10	10	19	19	18	18	6	4	4	11	11	3	3	
15	19	19	18	18	14	14	13	13	5	P	8	P	7	5	+	8	16	16	15	15	2	2	1	1	10	10	9	9	17	17	3	3	7	11	11	4	4	12	12	
16	19	19	18	18	14	14	13	13	P	6	P	5	5	8	6	+	16	16	15	15	2	2	1	1	10	10	9	9	17	17	3	3	8	11	11	4	4	12	12	
17	13	13	17	17	5	5	6	6	3	3	4	4	15	15	16	16	+	9	10	10	11	9	7	7	8	8	18	18	19	19	12	1	1	14	14	2	2			
18	13	13	17	17	5	5	6	6	3	3	4	4	15	15	16	16	11	+	12	10	10	11	9	12	7	7	8	8	18	18	19	19	9	1	1	14	14	2	2	
19	18	18	19	19	6	6	5	5	4	4	3	3	16	16	15	15	11	12	+	9	9	14	12	10	8	8	7	7	2	2	17	17	10	14	14	1	1	13	13	
20	18	18	19	19	6	6	5	5	4	4	3	3	16	16	15	15	10	12	+	9	+	10	12	9	8	8	7	7	2	2	17	17	10	14	14	1	1	13	13	
21	15	15	16	16	7	7	17	17	13	13	14	14	1	1	2	2	10	P	11	P	+	10	9	11	18	18	19	19	5	5	6	6	9	3	3	8	8	4	4	
22	15	15	16	16	7	7	17	17	13	13	14	14	1	1	2	2	P	11	P	12	12	+	10	18	18	19	19	5	5	6	6	10	3	3	8	8	4	4		
23	16	16	15	15	18	18	19	19	14	14	13	13	2	2	1	1	9	+	12	+	11	9	+	12	4	4	17	17	6	6	5	5	11	8	8	3	3	7	7	
24	16	16	15	15	18	18	19	19	14	14	13	13	2	2	1	1	P	10	P	9	9	12	10	+	4	4	17	17	6	6	5	5	12	8	8	3	3	7	7	
25	5	5	6	6	11	11	12	12	1	1	2	2	19	19	18	18	17	17	8	8	3	3	4	4	+	15	13	14	14	16	15	13	16	10	10	7	7	9	9	
26	5	5	6	6	11	11	12	12	1	1	2	2	19	19	18	18	17	17	8	8	3	3	4	4	15	+	16	14	14	15	13	16	13	10	10	7	7	9	9	
27	6	6	5	5	12	12	11	11	2	2	1	1	17	17	9	9	19	19	18	18	4	4	3	3	15	16	+	13	13	15	16	14	14	7	7	10	10	8	8	
28	6	6	5	5	12	12	11	11	2	2	1	1	17	17	9	9	19	19	18	18	4	4	3	3	14	16	13	+	15	14	16	13	15	7	7	10	10	8	8	
29	7	7	8	8	9	9	10	10	19	19	18	18	3	3	4	4	1	1	2	2	17	17	6	6	14	P	15	P	+	14	13	15	13	12	12	5	5	11	11	
30	7	7	8	8	9	9	10	10	19	19	18	18	3	3	4	4	1	1	2	2	17	17	6	6	+	15	+	16	16	+	15	14	14	12	12	5	5	11	11	
31	8	8	7	7	10	10	9	9	17	17	18	18	4	4	3	3	2	2	1	1	19	19	18	18	13	P	16	P	15	13	+	16	15	5	5	12	12	6	6	
32	8	8	7	7	10	10	9	9	17	17	18	18	4	4	3	3	2	2	1	1	19	19	18	18	P	14	P	13	13	16	14	+	16	5	5	12	12	6	6	
33	1	2	3	4	4	1	2	3	5	6	7	8	8	5	6	7	9	10	11	12	12	9	10	11	13	14	15	16	16	13	14	15	+	17	17	18	18	19	19	
34																																								
35																																								
36																																								
37																																								
38																																								
39																																								

Figure 7.2.9: Extension of ARMP of $K(33, 33)$ to an ARMP of $K(39, 39)$: Phase 2

Phase 3: In figure 7.2.10 on the next page, 3 new cycles are added each to APC 1 through 16 (shown in cells in striped border) using newly appended part A vertices and part B vertices vacated by the corresponding APCs in the cycle shifts of Phase 1.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	
1		3	1	2	2	4	3	1	18	18	19	19	11	11	12	12	13	13	14	14	15	15	17	17	5	5	6	6	7	7	8	8	4	9	9	16	16	10	10	
2	3		4	2	2	3	1	4	18	18	19	19	11	11	12	12	13	13	14	14	15	15	17	17	5	5	6	6	7	7	8	8	1	9	9	16	16	10	10	
3	3	4		1	1	3	4	2	10	10	17	17	12	12	11	11	14	14	13	13	18	18	19	19	6	6	5	5	8	8	7	7	2	16	16	9	9	15	15	
4	2	4	1		3	2	4	1	10	10	17	17	12	12	11	11	14	14	13	13	18	18	19	19	6	6	5	5	8	8	7	7	3	16	16	9	9	15	15	
5	2	P	3	P		2	1	3	15	15	16	16	18	18	19	19	5	5	17	17	7	7	8	8	11	11	12	12	9	9	10	10	1	13	13	6	6	14	14	
6	P	3	P		4	4	3	2	15	15	16	16	18	18	19	19	5	5	17	17	7	7	8	8	11	11	12	12	9	9	10	10	2	13	13	6	6	14	14	
7	1	P	4	P		3	1		4	16	16	15	15	14	14	17	17	18	18	19	19	8	8	7	7	12	12	11	11	10	10	9	9	3	6	6	13	13	5	5
8	P	2	P		1	1	4	2		16	16	15	15	14	14	17	17	18	18	19	19	8	8	7	7	12	12	11	11	10	10	9	9	4	6	6	13	13	5	5
9	9	9	10	10	17	17	16	16		7	5	6	6	8	7	5	3	3	4	4	13	13	14	14	19	19	18	18	11	11	12	12	8	2	2	15	15	1	1	
10	9	9	10	10	17	17	16	16	7		8	6	6	7	5	8	3	3	4	4	13	13	14	14	19	19	18	18	11	11	12	12	5	2	2	15	15	1	1	
11	10	10	9	9	19	19	18	18	7	8		5	5	7	8	6	4	4	3	3	14	14	13	13	17	17	1	1	12	12	11	11	6	15	15	2	2	16	16	
12	10	10	9	9	19	19	18	18	6	8	5		7	6	8	5	4	4	3	3	14	14	13	13	17	17	1	1	12	12	11	11	7	15	15	2	2	16	16	
13	17	17	12	12	13	13	14	14	6		7		6	5	7	15	15	16	16	1	1	2	2	9	9	10	10	19	19	18	18	5	4	4	11	11	3	3		
14	17	17	12	12	13	13	14	14	P	7	P	8	8		7	6	15	15	16	16	1	1	2	2	9	9	10	10	19	19	18	18	6	4	4	11	11	3	3	
15	19	19	18	18	14	14	13	13	5	P	8	P	7	5		8	16	16	15	15	2	2	1	1	10	10	9	9	17	17	3	3	7	11	11	4	4	12	12	
16	19	19	18	18	14	14	13	13	P	6	P	5	5	8	6		16	16	15	15	2	2	1	1	10	10	9	9	17	17	3	3	8	11	11	4	4	12	12	
17	13	13	17	17	5	5	6	6	3	3	4	4	15	15	16	16		9	10	10	12	12	11	9	7	7	8	8	18	18	19	19	12	1	1	14	14	2	2	
18	13	13	17	17	5	5	6	6	3	3	4	4	15	15	16	16	11		12	10	10	11	9	12	7	7	8	8	18	18	19	19	9	1	1	14	14	2	2	
19	18	18	19	19	6	6	5	5	4	4	3	3	16	16	15	15	11	12		9	9	14	12	10	8	8	7	7	2	2	17	17	10	14	14	1	1	13	13	
20	18	18	19	19	6	6	5	5	4	4	3	3	16	16	15	15	10	12	9		1	10	12	9	8	8	7	7	2	2	17	17	10	14	14	1	1	13	13	
21	15	15	16	16	7	7	17	17	13	13	14	14	1	1	2	2	10	P	11	P		10	9	11	18	18	19	19	5	5	6	6	9	3	3	8	8	4	4	
22	15	15	16	16	7	7	17	17	13	13	14	14	1	1	2	2	P	11	P	12	12		10	18	18	19	19	5	5	6	6	10	3	3	8	8	4	4		
23	16	16	15	15	18	18	19	19	14	14	13	13	2	2	1	1	9		12		11	9	12	4	4	17	17	6	6	5	5	11	8	8	3	3	7	7		
24	16	16	15	15	18	18	19	19	14	14	13	13	2	2	1	1	P	10	P	9	9	12	10		4	4	17	17	6	6	5	5	12	8	8	3	3	7	7	
25	5	5	6	6	11	11	12	12	1	1	2	2	19	19	18	18	17	17	8	8	3	3	4	4		15	13	14	14	16	15	13	16	10	10	7	7	9	9	
26	5	5	6	6	11	11	12	12	1	1	2	2	19	19	18	18	17	17	8	8	3	3	4	4	15		16	14	14	15	13	16	13	10	10	7	7	9	9	
27	6	6	5	5	12	12	11	11	2	2	1	1	17	17	9	9	19	19	18	18	4	4	3	3	15	16		13	13	15	16	14	14	7	7	10	10	8	8	
28	6	6	5	5	12	12	11	11	2	2	1	1	17	17	9	9	19	19	18	18	4	4	3	3	14	16	13		15	14	16	13	15	7	7	10	10	8	8	
29	7	7	8	8	9	9	10	10	19	19	18	18	3	3	4	4	1	1	2	2	17	17	6	6	14	P	15	P		14	13	15	13	12	12	5	5	11	11	
30	7	7	8	8	9	9	10	10	19	19	18	18	3	3	4	4	1	1	2	2	17	17	6	6		15		16	16		15	14	14	12	12	5	5	11	11	
31	8	8	7	7	10	10	9	9	17	17	18	18	4	4	3	3	2	2	1	1	19	19	18	18	13	P	16	P	15	13		16	15	5	5	12	12	6	6	
32	8	8	7	7	10	10	9	9	17	17	18	18	4	4	3	3	2	2	1	1	19	19	18	18	P	14	P	13	13	16	14		16	5	5	12	12	6	6	
33	1	2	3	4	4	1	2	3	5	6	7	8	8	5	6	7	9	10	11	12	12	9	10	11	13	14	15	16	16	13	14	15		17	17	18	18	19	19	
34	14	14	11	11	8	8	15	15	9	9	12	12	13	13	10	10	6	6	7	7	16	16	5	5	3	3	2	2	1	1	4	4		17		18	18	19	19	
35	11	11	14	14	15	15	8	8	12	12	9	9	10	10	13	13	7	7	6	6	5	5	16	16	2	2	3	3	4	4	1	1		18	19		19	17	17	
36	14	14	11	11	8	8	15	15	9	9	12	12	13	13	10	10	6	6	7	7	16	16	5	5	3	3	2	2	1	1	4	4		17	19		19	18	18	
37	14	14	14	14	15	15	8	8	12	12	9	9	10	10	13	13	7	7	6	6	5	5	16	16	2	2	3	3	4	4	1	1		18	17	17		19	18	18
38	12	12	13	13	16	16	7	7	11	11	10	10	9	9	14	14	8	8	5	5	6	6	15	15	1	1	4	4	3	3	2	2		19	P	19	P		17	17
39	12	12	13	13	16	16	7	7	11	11	10	10	9	9	14	14	8	8	5	5	6	6	15	15	1	1	4	4	3	3	2	2		19	P	19	P		18	18

Figure 7.2.10: Extension of ARMP of $K(33,33)$ to an ARMP of $K(39,39)$: Phase 3

Phase 4: In Figure 7.2.11 on the following page, the third new cycles are completed for APC 17 and APC 18 using “half-cycles” formed in Phase 2 and class shifts on individual edges as shown in cells in striped borders. The extension and resolution is now complete.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39				
1	+	3	1	2	2	4	3	1	18	18	19	19	11	11	12	12	13	13	14	14	15	15	17	17	5	5	6	6	7	7	8	8	4	9	9	16	16	10	10				
2	3	+	4	2	2	3	1	4	18	18	19	19	11	11	12	12	13	13	14	14	15	15	17	17	5	5	6	6	7	7	8	8	1	9	9	16	16	10	10				
3	3	4	+	1	1	3	4	2	10	10	17	17	12	12	11	11	14	14	13	13	18	18	19	19	6	6	5	5	8	8	7	7	2	16	16	9	9	15	15				
4	2	4	1	+	3	2	4	1	10	10	17	17	12	12	11	11	14	14	13	13	18	18	19	19	6	6	5	5	8	8	7	7	3	16	16	9	9	15	15				
5	2	P	3	P	+	2	1	3	15	15	16	16	18	18	19	19	5	5	6	6	17	17	7	7	8	8	11	11	12	12	9	9	10	10	1	13	13	17	6	14	14		
6	P	3	P	4	4	+	3	2	15	15	16	16	18	18	19	19	5	5	6	6	17	17	7	7	8	8	11	11	12	12	9	9	10	10	2	13	13	17	6	14	14		
7	1	+	4	+	3	1	4	4	16	16	15	15	14	14	17	17	18	18	19	19	8	8	7	7	12	12	11	11	10	10	9	9	3	6	6	13	13	5	5	5	5		
8	P	2	P	1	1	4	2	4	16	16	15	15	14	14	17	17	18	18	19	19	8	8	7	7	12	12	11	11	10	10	9	9	4	6	6	13	13	5	5	5	5		
9	9	9	10	10	17	17	16	16	+	7	5	6	6	8	7	5	3	3	4	4	13	13	14	14	19	19	18	18	11	11	12	12	8	2	2	15	15	1	1	1	1		
10	9	9	10	10	17	17	16	16	7	+	8	6	6	7	5	8	3	3	4	4	13	13	14	14	19	19	18	18	11	11	12	12	5	2	2	15	15	1	1	1	1		
11	10	10	9	9	19	19	18	18	7	8	+	5	5	7	8	6	4	4	3	3	14	14	13	13	17	17	1	1	12	12	11	11	6	15	15	2	2	16	16				
12	10	10	9	9	19	19	18	18	6	8	5	+	7	6	8	5	4	4	3	3	14	14	13	13	17	17	1	1	12	12	11	11	7	15	15	2	2	16	16				
13	17	17	12	12	13	13	14	14	6	+	7	+	6	5	7	15	15	16	16	1	1	2	2	9	9	10	10	19	19	18	18	5	4	4	11	11	3	3	3	3			
14	17	17	12	12	13	13	14	14	P	7	P	8	8	+	7	6	15	15	16	16	1	1	2	2	9	9	10	10	19	19	18	18	6	4	4	11	11	3	3				
15	19	19	18	18	14	14	13	13	5	P	8	P	7	5	+	8	16	16	15	15	2	2	1	1	10	10	9	9	17	17	3	3	7	11	11	4	4	12	12				
16	19	19	18	18	14	14	13	13	P	6	P	5	5	8	6	+	16	16	15	15	2	2	1	1	10	10	9	9	17	17	3	3	8	11	11	4	4	12	12				
17	13	13	17	17	5	5	6	6	3	3	4	4	15	15	16	16	+	9	10	10	12	12	11	9	7	7	8	8	18	18	19	19	12	1	1	14	14	2	2	2	2		
18	13	13	17	17	5	5	6	6	3	3	4	4	15	15	16	16	11	+	12	10	10	11	9	12	7	7	8	8	18	18	19	19	9	1	1	14	14	2	2	2	2		
19	18	18	19	19	6	6	5	5	4	4	3	3	16	16	15	15	11	12	+	9	9	14	12	10	8	8	7	7	2	2	17	17	10	14	14	1	1	13	13				
20	18	18	19	19	6	6	5	5	4	4	3	3	16	16	15	15	10	12	9	+	1	10	12	9	8	8	7	7	2	2	17	17	10	14	14	1	1	13	13				
21	15	15	16	16	7	7	17	17	13	13	14	14	1	1	2	2	10	P	11	P	+	10	9	11	18	18	19	19	5	5	6	6	9	3	3	8	8	4	4	4	4		
22	15	15	16	16	7	7	17	17	13	13	14	14	1	1	2	2	P	11	P	12	12	+	10	18	18	19	19	5	5	6	6	10	3	3	8	8	4	4	4	4			
23	16	16	15	15	18	18	19	19	14	14	13	13	2	2	1	1	9	+	12	+	11	9	+	12	4	4	17	17	6	6	5	5	11	8	8	3	3	7	7	7	7		
24	16	16	15	15	18	18	19	19	14	14	13	13	2	2	1	1	P	10	P	9	9	12	10	+	4	4	17	17	6	6	5	5	12	8	8	3	3	7	7	7	7		
25	5	5	6	6	11	11	12	12	1	1	2	2	19	19	18	18	17	17	8	8	3	3	4	4	+	15	13	14	14	16	15	13	16	10	10	7	7	9	9	9	9		
26	5	5	6	6	11	11	12	12	1	1	2	2	19	19	18	18	17	17	8	8	3	3	4	4	15	+	16	14	14	15	13	16	13	10	10	7	7	9	9	9	9		
27	6	6	5	5	12	12	11	11	2	2	1	1	17	17	9	9	19	19	7	18	4	4	3	3	15	16	+	13	13	15	16	14	7	18	10	10	8	8	8	8			
28	6	6	5	5	12	12	11	11	2	2	1	1	17	17	9	9	19	19	7	18	4	4	3	3	14	16	13	+	15	14	16	13	15	7	18	10	10	8	8	8	8		
29	7	7	8	8	9	9	10	10	19	19	18	18	3	3	4	4	1	1	2	2	17	17	6	6	14	P	15	P	+	14	13	15	13	12	12	5	5	11	11	11	11		
30	7	7	8	8	9	9	10	10	19	19	18	18	3	3	4	4	1	1	2	2	17	17	6	6	+	15	+	16	16	+	15	14	14	12	12	5	5	11	11	11	11		
31	8	8	7	7	10	10	9	9	17	17	18	18	4	4	3	3	2	2	1	1	19	19	18	18	13	P	16	P	15	13	+	16	15	5	5	12	12	6	6	6	6		
32	8	8	7	7	10	10	9	9	17	17	18	18	4	4	3	3	2	2	1	1	19	19	18	18	P	14	P	13	13	16	14	+	16	5	5	12	12	6	6	6	6		
33	1	2	3	4	4	1	2	3	5	6	7	8	8	5	6	7	9	10	11	12	12	9	10	11	13	14	15	16	16	13	14	15	+	17	17	18	18	19	19	19	19		
34	14	14	11	11	8	8	15	15	9	9	12	12	13	13	10	10	6	6	17	7	16	16	5	5	3	3	2	2	1	1	4	4	17	+	7	18	18	19	19	19	19		
35	11	11	14	14	15	15	8	8	12	12	9	9	10	10	13	13	7	7	18	6	5	5	16	16	2	2	3	3	4	4	1	1	18	19	+	6	19	17	17	17	17		
36	14	14	11	11	8	8	15	15	9	9	12	12	13	13	10	10	6	6	17	7	16	16	5	5	3	3	2	2	1	1	4	4	17	19	+	7	19	18	18	18	18		
37	14	14	11	11	8	8	15	15	8	8	12	12	9	9	10	10	13	13	7	7	18	6	5	5	16	16	2	2	3	3	4	4	1	1	17	17	+	6	19	18	18	18	18
38	12	12	13	13	16	16	7	7	11	11	10	10	9	9	14	14	8	8	5	5	6	6	15	15	1	1	4	4	3	3	2	2	19	P	19	P	+	17	17	17	17		
39	12	12	13	13	16	16	7	7	11	11	10	10	9	9	14	14	8	8	5	5	6	6	15	15	1	1	4	4	3	3	2	2	19	P	19	P	+	18	18	18	18		

Figure 7.2.11: Extension of ARMP of $K(33, 33)$ to an ARMP of $K(39, 39)$: Phase 4

We now generalize the procedures illustrated in the constructions of the ARMPs of $K(31, 31)$ and $K(39, 39)$ to construct an ARMP of $K(8m + 7, 8m + 7)$ for values of $m > 4$.

As illustrated in these two previous constructions, the extension of an ARMP of $K(8m + 1, 8m + 1)$ to an ARMP of $K(8m + 7, 8m + 7)$ depends on the identification of *three* disjoint sets of cycles to be shifted, each set containing exactly 1 cycle per APC from the ARMP of $K(8m + 1, 8m + 1)$ such that the cycles within each set are vertex-disjoint. Again, we will show that this is possible whenever the CQG used for the construction of the ARMP of $K(8m + 1, 8m + 1)$ contains a pair of disjoint complete transversals in the cells outside the holes. (Constructions for such quasigroups are addressed in Chapter 8.)

So we again begin by constructing an ARMP of $K(8m + 1, 8m + 1)$ using a commutative quasigroup (Q, \circ) with holes of size 2 and order $2m$ containing a pair of disjoint complete transversals, T_1 and T_2 , outside the holes. For $i \in \{1, 2\}$, let the elements $1, 2, 3, \dots, 2m$ of T_i be represented as $1_i, 2_i, 3_i, \dots, 2m_i$ respectively. Label the part A vertices of $K(8m + 1, 8m + 1)$ as $a_{(1,1)}, a_{(1,2)}, a_{(1,3)}, a_{(1,4)}, a_{(2,1)}, a_{(2,2)}, a_{(2,3)}, a_{(2,4)}, \dots, a_{(m,1)}, a_{(m,2)}, a_{(m,3)}, a_{(m,4)}$ and a_9 . Use similar labeling for the part B vertices. Append 6 vertices ($a_{10}, a_{11}, a_{12}, a_{13}, a_{14}$, and a_{15}) to part A and 6 vertices ($b_{10}, b_{11}, b_{12}, b_{13}, b_{14}$, and b_{15}) to part B of $K(8m + 1, 8m + 1)$ along with the associated edges necessary to create $K(8m + 5, 8m + 5)$.

Phase 1: Target particular cycles of the ARMP of $K(8m + 1, 8m + 1)$ for shifting and replacement according to the same procedures shown in Rounds 1 and 2 of the proof of $K(8m + 5, 8m + 5)$. At this point two cycles for each of APCs 1 through $4m$ will have been selected, with their positions in each associated $K(4, 4)$ in either quadrant II and quadrant IV or in quadrant I and quadrant III. The third cycle targeted for shifting from each APC should be selected using the following rules:

1. If the cycle chosen for a particular APC in Round 1 was positioned in quadrant II or IV of a given $K(4, 4)$, the third cycle selected should be the cycle immediately following that cycle in a *clockwise* rotation through the quadrants of that $K(4, 4)$.
2. If the cycle chosen for a particular APC in Round 1 was positioned in quadrant I or III of a given $K(4, 4)$, the third cycle selected should be the cycle immediately following that cycle in a *counterclockwise* rotation through the quadrants of that $K(4, 4)$.

Make the cycle shifts as follows: Change the part B vertices of any cycle chosen in Round 1 to vertices b_{10} and b_{11} . Replace these shifted cycles with cycles from APC $4m + 2$. Change the part B vertices of any cycle chosen in Round 2 with vertices b_{12} and b_{13} . Replace these shifted cycles with cycles from APC $4m + 1$. Change the part B vertices of any cycle chosen in Round 3 to vertices b_{14} and b_{15} . Replace these shifted cycles with cycles from APC $4m + 3$. At this point, we will have established $4m$ cycles for each of the three new APCs. Assign the edges (a_{10}, b_{10}) , (a_{11}, b_{11}) , (a_{12}, b_{12}) , (a_{13}, b_{13}) , (a_{13}, b_{14}) , (a_{13}, b_{15}) , (a_{14}, b_{13}) , and (a_{15}, b_{13}) to the leave.

Phase 2: Add one new cycle $(a_{14}, b_{10}, a_{15}, b_{12})$ to the PPC. Add two new cycles $(a_9, b_{10}, a_{13}, b_{11})$ and $(a_{11}, b_{14}, a_{14}, b_{15})$ to APC $4m + 1$ and add two new cycles $(a_9, b_{12}, a_{10}, b_{13})$ and $(a_{12}, b_{14}, a_{15}, b_{15})$ to APC $4m + 2$. In addition, form half of a third new cycle for APC $4m + 1$ from the edges (a_{10}, b_9) and (a_{12}, b_9) and form half of a third new cycle for APC $4m + 2$ from the edges (a_{11}, b_9) and (a_{13}, b_9) . Add three new cycles $(a_9, b_{14}, a_{10}, b_{15})$, $(a_{11}, b_{10}, a_{12}, b_{13})$, and $(a_{14}, b_9, a_{15}, b_{11})$ to APC $4m + 3$. This leaves the edges (a_{10}, b_{11}) , (a_{12}, b_{11}) , (a_{11}, b_{12}) , and (a_{13}, b_{12}) free to be assigned in Phase 4.

Phase 3: Three new cycles are added to each of the original $4m$ APCs according to the following rule: For $u, v \in \{1, 2, 3, \dots, 2m\}$, for $i, j \in \{1, 2, 3, 4\}$, and for $1 \leq k \leq 4m$, if the cycle $(a_{(u,i)}, b_{(v,j)}, a_{(u, i+1)}, b_{(v, j+1)})$ in APC k was shifted in Phase 1 so that its B vertices switched to b_{10}

and b_{11} , then the cycle $(a_{10}, b_{(v,j)}, a_{12}, b_{(v,j+1)})$ is added to APC k . If, instead, this same cycle in APC k was shifted in Phase 1 so that its B vertices switched to b_{12} and b_{13} , then the cycle $(a_{11}, b_{(v,j)}, a_{13}, b_{(v,j+1)})$ is added to APC k . If this cycle was shifted in Phase 1 so that its B vertices switched to b_{14} and b_{15} , then the cycle $(a_{14}, b_{(v,j)}, a_{15}, b_{(v,j+1)})$ is added to APC k .

Phase 4: Again, for $t, u, v \in \{1, 2, 3, \dots, 2m\}$, for $h, i, j \in \{1, 2, 3, 4\}$, and for $k, l \in \{1, 2, 3, \dots, 4m\}$, select any part B vertex $b_{(u,i)}$ and make the following shifts:

1. Edges $(a_{10}, b_{(u,i)})$ and $(a_{12}, b_{(u,i)})$ are assigned to a cycle in some APC k ; reassign these edges to APC $4m+1$. When joined with the edges (a_{10}, b_9) and (a_{12}, b_9) that comprised a half-cycle in Phase 2, these edges will complete the second new cycle for APC $4m + 1$. To replace the two edges now missing in APC k , assign (a_{10}, b_{11}) and (a_{12}, b_{11}) to APC k . This action temporarily “overloads” vertex $b_{(u,i)}$ with edges from APC $4m + 1$ and vertex b_{11} with edges from APC k . This will be corrected in step 3 below.
2. Edges $(a_{11}, b_{(u,i)})$ and $(a_{13}, b_{(u,i)})$ are assigned to a cycle in some APC l ; reassign these edges to APC $4m + 2$. When joined with the edges (a_{11}, b_9) and (a_{13}, b_9) that comprised a half-cycle in Phase 2, these edges will complete the second new cycle for APC $4m + 2$. To replace the two edges now missing in APC l , assign (a_{11}, b_{12}) and (a_{13}, b_{12}) to APC l . This action temporarily “overloads” vertex $b_{(u,i)}$ with edges from APC $4m + 2$ and vertex b_{12} with edges from APC l . This will be corrected in step 4 below.
3. For some v , the edges $(a_{(v,j)}, b_{(u,i)})$ and $(a_{(v,j+1)}, b_{(u,i)})$ were reassigned to APC k in Phase 1, while edges $(a_{(v,j)}, b_{11})$ and $(a_{(v,j+1)}, b_{11})$ were assigned to APC $4m + 2$. Switch the class assignments of these two pairs of edges. This relieves the overloads on vertices $b_{(u,i)}$ and b_{11} caused by actions in step 1 above.

4. For some t , the edges $(a_{(t,h)}, b_{(u,i)})$ and $(a_{(t,h+1)}, b_{(u,i)})$ were reassigned to APC l in Phase 1, while edges $(a_{(t,h)}, b_{12})$ and $(a_{(t,h+1)}, b_{12})$ were assigned to APC $4m + 1$.

Switch the class assignments of these two pairs of edges. This relieves the overloads on vertices $b_{(u,i)}$ and b_{12} caused by actions in step 2 above.

The edges of $K(8m + 7, 8m + 7)$ are now resolved into $4m + 3$ APCs of $4m + 3$ cycles each and one PPC with $2m + 1$ cycles, as required for an ARMP of 4-cycles.

Chapter 8

Commutative Quasigroup Constructions and Conclusions for the General Odd/Odd Cases

In this chapter, we exhibit constructions which produce the quasigroups that are necessary for the extensions of an ARMP of $K(8m + 1, 8m + 1)$ to an ARMP of $K(8m + 5, 8m + 5)$ and $K(8m + 7, 8m + 7)$ that were discussed in Chapters 6 and 7. Recall that these quasigroups must be commutative of order $2m$ ($m \geq 3$) with holes of size 2 and must contain 2 complete transversals outside the holes. We divide these constructions into three cases according to the value of $m \pmod{3}$. The fundamental ideas behind each construction were devised by Dr. Curt Lindner, and are greatly appreciated.

8.1 $m \equiv 0 \pmod{3}$; The $6k$ Construction

We have already exhibited a commutative quasigroup of order 6 that, although it does not contain transversals outside the holes, nevertheless meets our needs for $m = 3$ (see Figure 6.2.1). Thus we need only focus on $m \geq 6$.

Theorem 8.1.1. When $m = 3k$, there exists a commutative quasigroup of order $2m$ with holes of size 2 and containing 2 complete transversals outside the holes for $m \geq 6$.

Proof. For $m = 3k$, we seek a commutative quasigroup (Q, \circ) of order $2m = 6k$ ($k > 1$) with holes of size 2 and two complete transversals outside the hole. To construct (Q, \circ) we will use a Steiner triple system (V, S) of order $6k+1$, vertex set $V = \{ \alpha, 1, 2, 3, \dots, 6k \}$, block set S , and having the special characteristic that among the triples of S we find a parallel class π of $V \setminus \{ \alpha \}$.

Steiner triple systems of this type are known to exist; for $k = 2$, we exhibit such a system in Example 8.1 that follows this proof.

For $k \geq 3$, we may use a *nearly Kirkman triple system* (NKTS). A nearly Kirkman triple system exists when the removal of a single point and all triples containing that point from a Steiner triple system of order $6k + 1$ leaves $3k - 1$ almost parallel classes of $2k$ triples each. Nearly Kirkman triple systems were shown to exist for $k = (3a + 1)(2b - 1)$ for all $a, b > 0$ in 1974 by Kotzig and Rosa [7]. In 1977, Baker and Wilson [1] proved all remaining cases for $k \geq 3$ except for 14, 17, and 29. The cases of 17 and 29 were shown to exist in [5] by A. E. Brouwer in 1978, while Reese and Stinson proved in [9] the existence for the last remaining case, $k = 14$ in 1987.

The construction begins by renaming the symbols of V so that $\{1,2,\alpha\}, \{3,4,\alpha\}, \{5,6,\alpha\}, \dots, \{6k-1, 6k, \alpha\}$ are triples of S . Let $H = \{\{a,b\} \mid \{a, b, \alpha\} \in S\}$ so that H is the set of holes of (Q, \circ) . For every triple $\{a, b, c\}$ of S not containing α , define $a \circ b = b \circ a = c$, $a \circ c = c \circ a = b$, and $b \circ c = c \circ b = a$. Then (Q, \circ) is a commutative quasigroup of order $6k$.

Now consider any triple $\{a, b, c\}$ in π . Since π is a parallel class, all triples are disjoint and any $x \in V$ that is not in $\{a, b, c\}$ occurs in a different row and column than either a, b , or c .

Furthermore, the elements in $\{a, b, c\}$ are always located so that pairwise they occur in different columns and different rows. For example, since $a \circ b = c$, the element c occurs in row a and column b . But since $c \circ a = b$, element b occurs in row c and column a . And since $b \circ c = a$, the element a occurs in row b and column c . Thus we are able to identify a set of $6k$ unique elements in (Q, \circ) , one in each column and each row, i.e. a complete transversal. Since (Q, \circ) is commutative, a second transversal is determined by the reflection of the first transversal across the diagonal.

Example 8.1. As an example for this case, we will exhibit the construction of a CQG of order $12 = 6k$ with holes of size 2. The construction begins with a Steiner triple system (V, S) of order $6k + 1 = 13$ on the vertex set $V = \{1, 2, 3, \dots, 12, 13\}$ which is given below. Note the parallel class on $V \setminus \{13\}$ shown in red type.

{1, 5, 9}	{2, 6, 10}	{3, 5, 13}	{4, 6, 13}	{7, 9, 13}	{8, 10, 13}
{11, 1, 13}	{12, 2, 13}	{1, 2, 7}	{1, 3, 6}	{1, 4, 8}	{2, 3, 8}
{2, 4, 5}	{3, 4, 7}	{5, 6, 11}	{5, 7, 10}	{5, 8, 12}	{6, 7, 12}
{6, 8, 9}	{7, 8, 11}	{3, 9, 10}	{2, 9, 11}	{4, 9, 12}	{4, 10, 11}
{1, 10, 12}	{3, 11, 12}				

We now use the triples that contain vertex 13 to rename the other vertices so that $\{1, 2, 13\}$, $\{3, 4, 13\}$, $\{5, 6, 13\}$, etc., are triples of S :

$\{3, 5, 13\} \rightarrow \{1, 2, 13\}$
$\{4, 6, 13\} \rightarrow \{3, 4, 13\}$
$\{7, 9, 13\} \rightarrow \{5, 6, 13\}$
$\{8, 10, 13\} \rightarrow \{7, 8, 13\}$
$\{11, 1, 13\} \rightarrow \{9, 10, 13\}$
$\{12, 2, 13\} \rightarrow \{11, 12, 13\}$

Using the convention just established to rename the vertices of V , we obtain the following complete Steiner triple system (V, S) which contains the parallel class π on $V \setminus \{13\}$ as shown in red type:

{1, 2, 13}	{3, 4, 13}	{5, 6, 13}	{7, 8, 13}	{9, 10, 13}	{11, 12, 13}
{10, 2, 6}	{12, 4, 8}	{10, 12, 5}	{10, 1, 4}	{10, 3, 7}	{12, 1, 7}
{12, 3, 2}	{1, 3, 5}	{2, 4, 9}	{2, 5, 8}	{2, 7, 11}	{4, 5, 11}
{4, 7, 6}	{5, 7, 9}	{6, 8, 1}	{6, 9, 12}	{6, 11, 3}	{8, 9, 3}
{8, 11, 10}	{9, 11, 1}				

Now we define the set of holes for (Q, \circ) to be the set $H = \{\{x, y\} \mid \{x, y, 13\} \in S\}$. And for each triple $\{a, b, c\}$ that does not contain the vertex 13, we define $a \circ b = b \circ a = c$, $a \circ c = c \circ a = b$, and $b \circ c = c \circ b = a$. From these definitions, we construct the CQG (Q, \circ) shown in Figure 8.1 that follows:

o	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	5	10	3	8	12	6	11	4	9	7
2	2	1	12	9	8	10	11	5	4	6	7	3
3	5	12	3	4	1	11	10	9	8	7	6	2
4	10	9	4	3	11	7	6	12	2	1	5	8
5	3	8	1	11	5	6	9	2	7	12	4	10
6	8	10	11	7	6	5	4	1	12	2	3	9
7	12	11	10	6	9	4	7	8	5	3	2	1
8	6	5	9	12	2	1	8	7	3	11	10	4
9	11	4	8	2	7	12	5	3	9	10	1	6
10	4	6	7	1	12	2	3	11	10	9	8	5
11	9	7	6	5	4	3	2	10	1	8	11	12
12	7	3	2	8	10	9	1	4	6	5	12	11

Figure 8.1: CQG of Order 12 with Holes of Size 2 and 2 Transversals Outside Holes

Note that the CQG in Figure 8.1 above does indeed contain two complete transversals outside the holes (shown in the cells shaded pink and green). The elements of these transversals correspond exactly to the triples contained in the parallel class π .

It is interesting to note that when using a nearly Kirkman triple system in this construction, each of the system's $3k - 1$ parallel classes yields 2 complete, disjoint transversals outside the holes; i.e. a total of $6k - 2$ complete, disjoint transversals partitioning the cells outside the holes. So we have the following:

Corollary 8.1.2. When $m = 3k$, there exists a commutative quasigroup of order $2m$ with holes of size 2 and containing $6k - 2$ transversals outside the holes for all $k \geq 3$.

8.2 $m \equiv 1 \pmod{3}$; The $6k + 2$ Construction

Theorem 8.2. When $m = 3k + 1$, there exists a commutative quasigroup of order $2m$ with holes of size 2 and containing 2 complete transversals outside the holes for all $m \geq 4$.

Proof. When $m \equiv 1 \pmod{3}$, we have $2m = 2(3k + 1) = 6k + 2$. When $k = 1$, $2m = 8$ and we have already exhibited a CQG of order 8 with holes of size 2 that contains 2 transversals outside the holes (see Figure 7.2.6 in the proof of Theorem 7.2).

When $k = 2$, the following quasigroup (Figure 8.2.1) of order 14 is indeed commutative and contains 2 transversals outside the holes, as indicated by the cells shaded pink and green respectively.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	1	2	9	10	3	4	11	12	5	6	13	14	8	7
2	2	1	10	9	4	3	12	11	6	5	14	13	7	8
3	9	10	3	4	11	12	5	6	13	14	7	8	1	2
4	10	9	4	3	12	11	6	5	14	13	8	7	2	1
5	3	4	11	12	5	6	13	14	7	8	1	2	9	10
6	4	3	12	11	6	5	14	13	8	7	2	1	10	9
7	11	12	5	6	13	14	7	8	1	2	9	10	3	4
8	12	11	6	5	14	13	8	7	2	1	10	9	4	3
9	5	6	13	14	7	8	1	2	9	10	3	4	11	12
10	6	5	14	13	8	7	2	1	10	9	4	3	12	11
11	13	14	7	8	1	2	9	10	3	4	11	12	5	6
12	14	13	8	7	2	1	10	9	4	3	12	11	6	5
13	8	7	1	2	9	10	3	4	11	12	5	6	13	14
14	7	8	2	1	10	9	4	3	12	11	6	5	14	13

Figure 8.2.1: CQG of Order 14 with Holes of Size 2 and 2 Transversals Outside the Holes

For $m = 3k + 1$, $k \geq 3$, we need the resolved Steiner triple system (i.e., *Kirkman triple system*) of order 9 shown in Table 8.2.1 using vertex set $V_1 = \{\infty, 1, 2, 3, 4, \dots, 8\}$ and parallel classes π_1 through π_4 :

Parallel Classes ► Triples ▼	π_1	π_2	π_3	π_4
t_1	$\{\infty, 1, 2\}$	$\{\infty, 3, 4\}$	$\{\infty, 5, 6\}$	$\{\infty, 7, 8\}$
t_2	$\{3, 5, 7\}$	$\{1, 5, 8\}$	$\{1, 7, 4\}$	$\{1, 3, 6\}$
t_3	$\{4, 8, 6\}$	$\{2, 7, 6\}$	$\{2, 3, 8\}$	$\{2, 5, 4\}$

Table 8.2.1: Kirkman Triple System of Order 9

The name $KTS_1(9)$ will be assigned to the Kirkman triple system of Table 8.2.1 and it will be used to generate $k - 1$ additional Kirkman triple systems using the correspondences in Table 8.2.2 that follows for $1 \leq n \leq k$:

KTS _n (9): $V_n = \{\infty, 1, 2, 6n - 3, 6n - 2, 6n - 1, 6n, 6n + 1, 6n + 2\}$				
Parallel Classes ► Triples ▼	π_1	π_2	π_3	π_4
t_1	$\{\infty, 1, 2\}$	$\{\infty, 6n - 3, 6n - 2\}$	$\{\infty, 6n - 1, 6n\}$	$\{\infty, 6n + 1, 6n + 2\}$
t_2	$\{6n - 3, 6n - 1, 6n + 1\}$	$\{1, 6n - 1, 6n + 2\}$	$\{1, 6n + 1, 6n - 2\}$	$\{1, 6n - 3, 6n\}$
t_3	$\{6n - 2, 6n + 2, 6n\}$	$\{2, 6n + 1, 6n\}$	$\{2, 6n - 3, 6n + 2\}$	$\{2, 6n - 1, 6n - 2\}$

Table 8.2.2: Correspondences to Generate Additional Kirkman Triple Systems

Each $KTS_n(9)$ can be used to define a CQG (Q_n, \circ_n) of order 8 as follows: Define the set of holes H_n to be the set of pairs (x, y) that results from deleting the point ∞ from each triple t_i in each parallel class of $KTS_n(9)$. For every triple $\{a, b, c\}$ not containing ∞ , define $a \circ b = b \circ a = c$, $a \circ c = c \circ a = b$, and $b \circ c = c \circ b = a$. Then (Q_n, \circ_n) is a commutative quasigroup of order 8. Note that all CQGs thus defined have in common the elements 1 and 2 and hence the hole $(1, 2)$. As in the proof of Theorem 8.1, we can show that each parallel class gives rise to two complete transversals outside the holes. In this case, however, the fact that the triples are entirely resolved into 4 parallel classes means we have 8 complete transversals outside the holes; i.e., the cells outside the holes are partitioned into transversals.

Each individual CQG (Q_n, \circ_n) of order 8 will serve as a piece of the larger CQG of order $6k + 2$ whose construction is our goal. We will build this larger CQG by “stacking” k iterations of (Q_n, \circ_n) as shown in Figure 8.2.2 that follows:

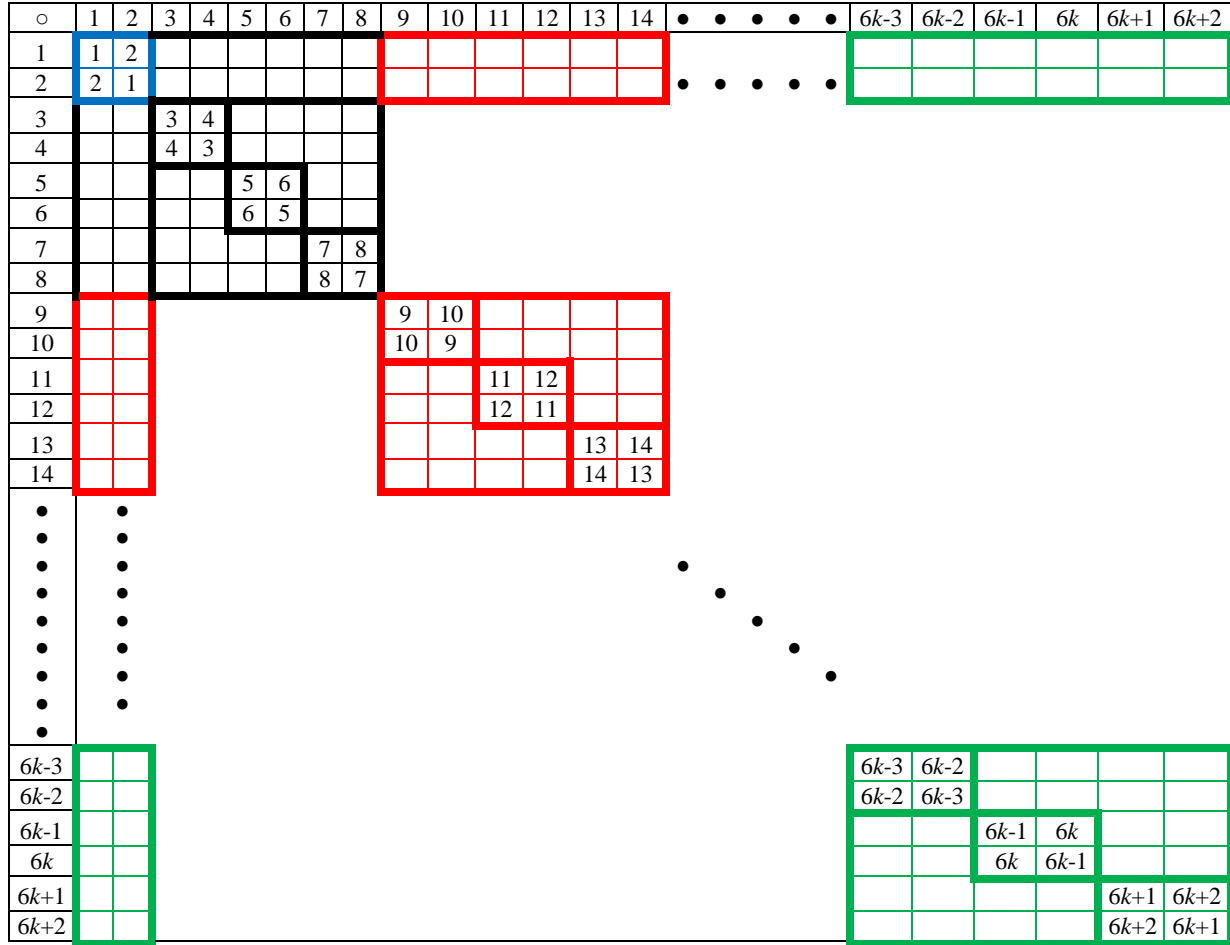


Figure 8.2.2: Constructing a CQG of Order $6k+2$ Using Stacked CQGs of Order 6

In Figure 8.2.2 above, the union of the cells outlined in blue with the cells outlined in black comprises the commutative quasigroup (Q_1, \circ_1) of order 8. The union of the cells outlined in blue with the cells outlined in red comprises the commutative quasigroup (Q_2, \circ_2) of order 8, and the stacking would continue in similar manner until we reach the cells outlined in green, which when joined with the cells outlined in blue form (Q_k, \circ_k) . At this point we only have a partial commutative quasigroup of order $6k + 2$ with holes. We must still fill in the cells outside the stacked CQGs of order 8. The number of these unfilled cells can be easily computed as $36k^2 - 36k$.

To accomplish this final task, we need two additional tools. First we need an array A using the symbols 1 through $6k + 2$ arranged on 3 levels with the holes of each (Q_n, \circ_n) arranged in 2-element pairs grouped vertically, as shown in Figure 8.2.3 that follows. The example of the holes of Q_1 is outlined in red.

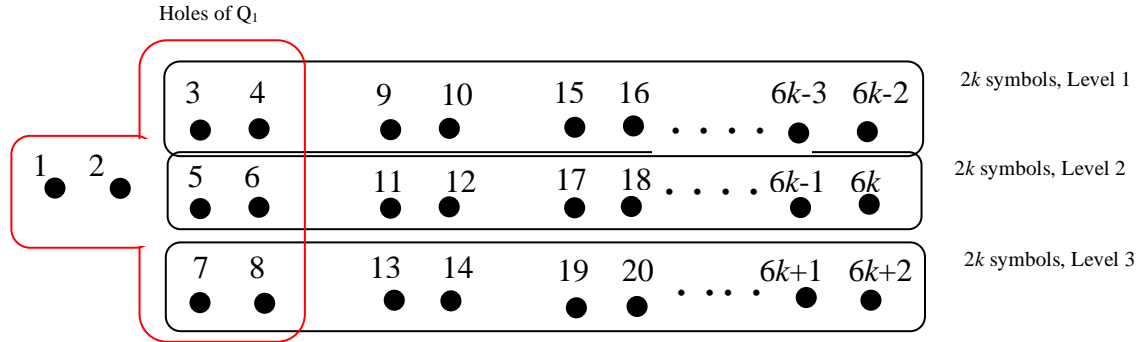


Figure 8.2.3: Array A

Elements of Array A on Level 1 will be designated with a subscript of 1, those on Level 2 with a subscript of 2, etc. Elements in the same column will be designated by the element that heads that column. So for example, the triple $\{a_1, b_2, c_3\}$ would stand for the triple $\{3, 11, 19\}$ in the array A of Figure 8.2.3 if $a = 3$, $b = 9$, and $c = 15$, while the triple $\{a_2, b_2, c_2\}$ would stand for $\{5, 11, 17\}$ given the same values for a , b , and c .

The second tool needed is a commutative quasigroup (Q_A, \circ_A) of order $2k$ with holes of size 2. (Such quasigroups are widely known to exist for all $k \geq 3$ [8].) We rename the symbols of (Q_A, \circ_A) so that we have the set of holes $H_A = \{(3,4), (9,10), (15, 16), \dots, (6k-3, 6k-2)\}$, in other words, the set composed of the holes on Level 1 in the array A. We select every distinct triple $\{a, b, c\}$ with $a \neq b \neq c$ such that $a \circ_A b = c$ in (Q_A, \circ_A) . For verification purposes, we compute the number of such triples as follows:

1. The total number of distinct triples possible is equal to the number of 3-element subsets that can be formed from a set of $2k$ elements, which is $\binom{2k}{3}$.
2. Discard any triples that contain two elements constituting a hole in Q_A . Since there are k holes and each hole can be matched with any one of the remaining $2k - 2$ elements, this amounts to $k(2k - 2)$ triples.
3. To eliminate further duplications, we divide the difference of (1) and (2) above by the number of ways any given pair of elements not constituting a hole can be matched with 1 other element that is both different from the two given elements and forms no hole with either given element. This number is $2k - 4$, since the restrictions eliminate 4 elements from the total $2k$ elements that could have been chosen to match the given pair.
4. Putting steps (1) – (3) from above together, we have $\left[\frac{\left(\binom{2k}{3} - k(2k-2) \right)}{2k-4} \right]$ triples which simplifies to $\frac{2k(k-1)}{3}$.

Now each triple obtained from (Q_A, \circ_A) gives rise to 9 triples from the array A of Figure 8.2.2 above:

$$\begin{array}{lll}
 \{a_1, b_1, c_1\} & \{a_2, b_2, c_2\} & \{a_3, b_3, c_3\} \\
 \{a_1, b_2, c_3\} & \{b_1, a_2, c_3\} & \{c_1, a_2, b_3\} \\
 \{a_1, b_3, c_2\} & \{b_1, a_3, c_2\} & \{c_1, a_3, b_2\}
 \end{array}$$

To fill the remaining cells of (Q, \circ) outside the stacked CQGs of order 8, for $x, y, z \in \{1, 2, 3\}$

we simply define $a_x \circ b_y = b_y \circ a_x = c_z$, $a_x \circ c_z = c_z \circ a_x = b_y$, and $b_y \circ c_z = c_z \circ b_y = a_x$ for each of the 9 triples $\{a_x, b_y, c_z\}$ defined above. Each such triple fills 6 of the unfilled cells of (Q, \circ) .

Thus we will fill $\frac{2k(k-1)}{3} \cdot 9 \cdot 6 = 36k^2 - 36k$ cells, the exact number of cells we need to fill.

Finally, we must verify the existence of two complete transversals outside the holes of (Q, \circ) . We have already established that the CQGs of order 8 stacked to form (Q, \circ) each contain a full set of complete transversals outside the holes. We select any two such transversals from (Q_1, \circ_1) call them $T_{1,1}$ and $T_{1,2}$. Since each (Q_n, \circ_n) is generated from a Kirkman triple system that was in turn generated from $KTS_1(9)$ (which produced (Q_1, \circ_1)), each (Q_n, \circ_n) is merely a duplication of (Q_1, \circ_1) with all vertices except 1 and 2 renamed. We must be careful in the selection of transversals from the remaining (Q_n, \circ_n) to avoid duplicating the elements 1 and 2 that are part of any transversal selected in (Q_1, \circ_1) . Fortunately, we do not need a complete transversal from any (Q_n, \circ_n) other than (Q_1, \circ_1) ; we need only two sets of 6 elements from each (Q_n, \circ_n) , each set forming a partial transversal avoiding the elements 1 and 2. These can be readily identified by row and column as:

Partial transversal $T_{n,1}$:

$$(6n - 3, 6n + 1), (6n - 2, 6n + 2), (6n - 1, 6n - 3), \\ (6n, 6n - 2), (6n + 1, 6n - 1), (6n + 2, 6n)$$

Partial transversal $T_{n,2}$ (reflection of partial transversal 1 across the diagonal):

$$(6n - 3, 6n - 1), (6n - 2, 6n), (6n - 1, 6n + 1), (6n, 6n + 2), \\ (6n + 1, 6n - 3), (6n + 2, 6n - 2)$$

Then $\bigcup_1^n T_{n,1}$ forms a complete transversal for (Q, \circ) , as does $\bigcup_1^n T_{n,2}$.

Example 8.2. As an example for Theorem 8.2, we will construct a CQG of order 20 with holes of size 2 in which the cells outside the holes contain 2 complete transversals. Starting with $KTS_1(9)$ we generate two additional Kirkman triple systems, all sharing the vertices 1 and 2 as shown in Figure 8.2.4 that follows:

KTS₁(9)			
$\{\infty, 1, 2\}$	$\{\infty, 3, 4\}$	$\{\infty, 5, 6\}$	$\{\infty, 7, 8\}$
$\{3, 5, 7\}$	$\{1, 5, 8\}$	$\{1, 7, 4\}$	$\{1, 3, 6\}$
$\{4, 8, 6\}$	$\{2, 7, 6\}$	$\{2, 3, 8\}$	$\{2, 5, 4\}$
KTS₂(9)			
$\{\infty, 1, 2\}$	$\{\infty, 9, 10\}$	$\{\infty, 11, 12\}$	$\{\infty, 13, 14\}$
$\{9, 11, 13\}$	$\{1, 11, 14\}$	$\{1, 13, 10\}$	$\{1, 9, 12\}$
$\{10, 14, 12\}$	$\{2, 13, 12\}$	$\{2, 9, 14\}$	$\{2, 11, 10\}$
KTS₃(9)			
$\{\infty, 1, 2\}$	$\{\infty, 15, 16\}$	$\{\infty, 17, 18\}$	$\{\infty, 19, 20\}$
$\{15, 17, 19\}$	$\{1, 17, 20\}$	$\{1, 19, 16\}$	$\{1, 15, 18\}$
$\{16, 20, 18\}$	$\{2, 19, 18\}$	$\{2, 15, 20\}$	$\{2, 17, 16\}$

Figure 8.2.4: Generated Kirkman Triple Systems

Each of the Kirkman triple systems shown in Figure 8.2.4 above is used to produce one of the CQGs of order 8 with holes of size 2 with the cells outside the holes resolved into 8 complete transversals. These are then stacked to partially fill the cells of a CQG of order 20 as shown in the following Figure 8.2.5:

o	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	2	6	7	8	3	4	5	12	13	14	9	10	11	18	19	20	15	16	17
2	2	1	8	5	4	7	6	3	14	11	12	13	12	9	20	17	16	19	18	15
3	6	8	3	4	7	1	5	2												
4	7	5	4	3	2	8	1	6												
5	8	4	7	2	5	6	3	1												
6	3	7	1	8	6	5	2	4												
7	4	6	5	1	3	2	7	8												
8	5	3	2	6	1	4	8	7												
9	12	14							9	10	13	1	11	2						
10	13	11							10	9	2	14	1	12						
11	14	10							13	2	11	12	9	1						
12	9	13							1	14	12	11	2	10						
13	10	12							11	1	9	2	13	14						
14	11	9							2	12	1	10	14	13						
15	18	20													15	16	19	1	17	2
16	19	17													16	15	2	20	1	18
17	20	16													19	2	17	18	15	1
18	15	19													1	20	18	17	2	16
19	16	18													17	1	15	2	19	20
20	17	15													2	18	1	16	20	19

Figure 8.2.5: Stacked CQGs (Order 8) to Construct CQG (Order 20)

To fill the remaining cells of this CQG, we will use the array in Figure 8.2.6 and the CQG of order 6 in Figure 8.2.7 that follow:

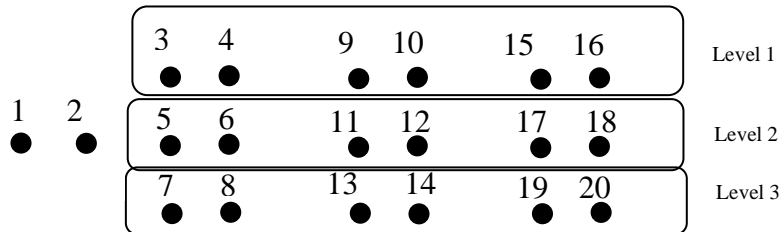


Figure 8.2.6: Array A for Constructing CQG of Order 20

\circ_A	3	4	9	10	15	16
3	3	4	15	16	9	10
4	4	3	16	15	10	9
9	15	16	9	10	3	4
10	16	15	10	9	4	3
15	9	10	3	4	15	16
16	10	9	4	3	16	15

Figure 8.2.7: CQG of Order 6 for Construction of CQG of Order 20

Note that we have renamed the elements of the CQG above so that the holes are precisely the holes on Level 1 in the array above. The distinct triples obtained from this CQG are:

$$\{3, 9, 15\} \quad \{3, 10, 16\} \quad \{4, 9, 16\} \quad \{4, 10, 15\}$$

Using the array given in Figure 8.2.6, each of these triples gives rise to 8 additional triples for a total of 36 triples which will be used to complete the unfilled cells of our CQG:

$$\begin{array}{cccc}
\{3, 9, 15\} & \{3, 10, 16\} & \{4, 9, 16\} & \{4, 10, 15\} \\
\{3, 11, 19\} & \{3, 12, 20\} & \{4, 11, 20\} & \{4, 12, 19\} \\
\{3, 13, 17\} & \{3, 14, 18\} & \{4, 13, 18\} & \{4, 14, 17\} \\
\{15, 11, 7\} & \{10, 5, 20\} & \{9, 18, 8\} & \{10, 6, 19\} \\
\{15, 13, 5\} & \{10, 7, 18\} & \{9, 20, 6\} & \{10, 8, 17\} \\
\{9, 5, 19\} & \{16, 5, 14\} & \{16, 6, 13\} & \{15, 6, 14\} \\
\{9, 7, 17\} & \{16, 7, 12\} & \{16, 8, 11\} & \{15, 8, 12\} \\
\{5, 11, 17\} & \{7, 13, 19\} & \{6, 11, 18\} & \{6, 12, 17\} \\
\{5, 12, 18\} & \{7, 14, 20\} & \{8, 13, 20\} & \{8, 14, 19\}
\end{array}$$

The completion of the CQG using these triples is shown in Figure 8.2.8 that follows. Observe the two transversals outside the holes in cells shaded pink and green, selected according to the exact procedures prescribed in the general proof of Theorem 8.2

(Q, ◦)

◦	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	2	6	7	8	3	4	5	12	13	14	9	10	11	18	19	20	15	16	17
2	2	1	8	5	4	7	6	3	14	11	12	13	12	9	20	17	16	19	18	15
3	6	8	3	4	7	1	5	2	15	16	19	20	17	18	9	10	13	14	11	12
4	7	5	4	3	2	8	1	6	16	15	20	19	18	17	10	9	14	13	12	11
5	8	4	7	2	5	6	3	1	19	20	17	18	15	16	13	14	11	12	9	10
6	3	7	1	8	6	5	2	4	20	19	18	17	16	15	14	13	12	11	10	9
7	4	6	5	1	3	2	7	8	17	18	15	16	19	20	11	12	9	10	13	14
8	5	3	2	6	1	4	8	7	18	17	16	15	20	19	12	11	10	9	14	13
9	12	14	15	16	19	20	17	18	9	10	13	1	11	2	3	4	7	8	5	6
10	13	11	16	15	20	19	18	17	10	9	2	14	1	12	4	3	8	7	6	5
11	14	10	19	20	17	18	15	16	13	2	11	12	9	1	7	8	5	6	3	4
12	9	13	20	19	18	17	16	15	1	14	12	11	2	10	8	7	6	5	4	3
13	10	12	17	18	15	16	19	20	11	1	9	2	13	14	5	6	3	4	7	8
14	11	9	18	17	16	15	20	19	2	12	1	10	14	13	6	5	4	3	8	7
15	18	20	9	10	13	14	11	12	3	4	7	8	5	6	15	16	19	1	17	2
16	19	17	10	9	14	13	12	11	4	3	8	7	6	5	16	15	2	20	1	18
17	20	16	13	14	11	12	9	10	7	8	5	6	3	4	19	2	17	18	15	1
18	15	19	14	13	12	11	10	9	8	7	6	5	4	3	1	20	18	17	2	16
19	16	18	11	12	9	10	13	14	5	6	3	4	7	8	17	1	15	2	19	20
20	17	15	12	11	10	9	14	13	6	5	4	3	8	7	2	18	1	16	20	19

Figure 8.2.8: Completed CQG of Order 20 with Holes Size 2 and Identified Transversals

8.3 $m \equiv 2 \pmod{3}$; The $6k + 4$ Construction

Theorem 8.3. When $m = 3k + 2$, there exists a commutative quasigroup of order $2m$ with holes of size 2 and containing 2 complete transversals outside the holes for all $m \geq 5$ except possibly $m = 8$.

Proof. For $m = 3k + 2$, we seek to construct a CQG of order $2m = 6k + 4$ ($k = 1$ and $k \geq 3$) with holes of size 2 and, of course, with the cells outside the holes containing two transversals. For $k = 1$, the quasigroup (Q_1, \circ_1) of order 10 shown in Figure 8.3.1 below meets these criteria, and is, in fact, the first tool needed for the construction when $k \geq 4$. Transversals outside the holes are

shaded in specific colors. In this particular case, the transversals partition the cells outside the transversals.

\circ_1	1	2	3	4	5	6	7	8	9	10
1	1	2	7	8	3	4	9	10	5	6
2	2	1	8	7	4	3	10	9	6	5
3	7	8	3	4	9	10	5	6	1	2
4	8	7	4	3	10	9	6	5	2	1
5	3	4	9	10	5	6	1	2	7	8
6	4	3	10	9	6	5	2	1	8	7
7	9	10	5	6	1	2	7	8	4	3
8	10	9	6	5	2	1	8	7	3	4
9	5	6	1	2	7	8	4	3	9	10
10	6	5	2	1	8	7	3	4	10	9

Figure 8.3.1: CQG of Order 10, Holes of Size 2, and Transversals Outside the Holes

For $k = 3$, the CQG of order 22 shown in Figure 8.3.2 that follows meets the criteria of having 2 transversals outside the holes, which are shown in cells shaded pink and green respectively.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
1	1	2	13	14	3	4	15	16	5	6	17	18	7	8	19	20	9	10	21	22	11	12
2	2	1	14	13	4	3	16	15	6	5	18	17	8	7	20	19	10	9	22	21	12	11
3	13	14	3	4	15	16	5	6	17	18	7	8	19	20	9	10	21	22	11	12	2	1
4	14	13	4	3	16	15	6	5	18	17	8	7	20	19	10	9	22	21	12	11	1	2
5	3	4	15	16	5	6	17	18	7	8	19	20	9	10	21	22	11	12	1	2	13	14
6	4	3	16	15	6	5	18	17	8	7	20	19	10	9	22	21	12	11	2	1	14	13
7	15	16	5	6	17	18	7	8	19	20	9	10	21	22	11	12	1	2	13	14	3	4
8	16	15	6	5	18	17	8	7	20	19	10	9	22	21	12	11	2	1	14	13	4	3
9	5	6	17	18	7	8	19	20	9	10	21	22	11	12	1	2	13	14	3	4	15	16
10	6	5	18	17	8	7	20	19	10	9	22	21	12	11	2	1	14	13	4	3	16	15
11	17	18	7	8	19	20	9	10	21	22	11	12	1	2	13	14	3	4	15	16	5	6
12	18	17	8	7	20	19	10	9	22	21	12	11	2	1	14	13	4	3	16	15	6	5
13	7	8	19	20	9	10	21	22	11	12	1	2	13	14	3	4	15	16	5	6	17	18
14	8	7	20	19	10	9	22	21	12	11	2	1	14	13	4	3	16	15	6	5	18	17
15	19	20	9	10	21	22	11	12	1	2	13	14	3	4	15	16	5	6	17	18	7	8
16	20	19	10	9	22	21	12	11	2	1	14	13	4	3	16	15	6	5	18	17	8	7
17	9	10	21	22	11	12	1	2	13	14	3	4	15	16	5	6	17	18	7	8	19	20
18	10	9	22	21	12	11	2	1	14	13	4	3	16	15	6	5	18	17	8	7	20	19
19	21	22	11	12	1	2	13	14	3	4	15	16	5	6	17	18	7	8	19	20	9	10
20	22	21	12	11	2	1	14	13	4	3	16	15	6	5	18	17	8	7	20	19	10	9
21	11	12	2	1	13	14	3	4	15	16	5	6	17	18	7	8	19	20	9	10	21	22
22	12	11	1	2	14	13	4	3	16	15	6	5	18	17	8	7	20	19	10	9	22	21

Figure 8.3.2: CQG of Order 22 with Holes of Size 2 and 2 Transversals Outside Holes

As previously indicated, for the construction of CQGs of order $6k + 4$ where $k \geq 4$, we will need to use the CQG of order 10 shown before in Figure 8.3.1, plus we will need a CQG (Q_2, \circ_2) of order 10 with one hole of size 4 and three holes of size 2 on the set $\{1, 2, 3, 4, 11, 12, 13, 14, 15, 16\}$ as shown in Figure 8.3.3 that follows. No transversals outside the holes will be needed for this CQG.

○	1	2	3	4	11	12	13	14	15	16
1	1	2	3	4	13	14	15	16	11	12
2	2	3	4	1	14	13	16	15	12	11
3	3	4	1	2	15	16	11	12	13	14
4	4	1	2	3	16	15	12	11	14	13
11	13	14	15	16	11	12	1	2	3	4
12	14	13	16	15	12	11	4	3	1	2
13	15	16	11	12	1	4	13	14	2	3
14	16	15	12	11	4	1	14	13	3	2
15	11	12	13	14	3	1	2	4	15	16
16	12	11	14	13	4	2	3	1	16	15

Figure 8.3.3: CQG of Order 10, One Hole Size 4, 3 Holes Size 2, No Transversals

We now begin building the CQG (Q, \circ) of order $6k + 4$ by stacking (Q_1, \circ_1) , (Q_2, \circ_2) , and $k - 2$ “copies” of (Q_2, \circ_2) where each copy of (Q_2, \circ_2) shares the symbols 1 through 4 but has symbols 11 through 16 renamed mod 6; i.e., for $2 \leq i \leq k$, the set of symbols for (Q_i, \circ_i) is $\{1, 2, 3, 4, 6k - 1, 6k, 6k + 1, 6k + 2, 6k + 3, 6k + 4\}$. Although these copies of (Q_2, \circ_2) share symbols 1 – 4, the cells shaded in blue in Figure 8.3.4 below are NOT shared. These cells would correspond to the hole of size 4 in (Q_2, \circ_2) , which is simply discarded for the purposes of the overall construction. The construction to this point is shown in Figure 8.3.4 that follows.

○	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	•	•	•	$6k-1$	$6k$	$6k+1$	$6k+2$	$6k+3$	$6k+4$
1	1	2	7	8	3	4	9	10	5	6	13	14	15	16	11	12	•	•	•	$6k+1$	$6k+2$	$6k+3$	$6k+4$	$6k-1$	$6k$
2	2	1	8	7	4	3	10	9	6	5	14	13	16	15	12	11	•	•	•	$6k+2$	$6k+1$	$6k+4$	$6k+3$	$6k$	$6k-1$
3	7	8	3	4	9	10	5	6	1	2	15	16	11	12	13	14				$6k+3$	$6k+4$	$6k-1$	$6k$	$6k+1$	$6k+2$
4	8	7	4	3	10	9	6	5	2	1	16	15	12	11	14	13				$6k+4$	$6k+3$	$6k$	$6k-1$	$6k+2$	$6k+1$
5	3	4	9	10	5	6	1	2	7	8															
6	4	3	10	9	6	5	2	1	8	7															
7	9	10	5	6	1	2	7	8	4	3															
8	10	9	6	5	2	1	8	7	3	4															
9	5	6	1	2	7	8	4	3	9	10															
10	6	5	2	1	8	7	3	4	10	9															
11	13	14	15	16							11	12	1	2	3	4									
12	14	13	16	15							12	11	4	3	1	2									
13	15	16	11	12							1	4	13	14	2	3									
14	16	15	12	11							4	1	14	13	3	2									
15	11	12	13	14							3	1	2	4	15	16									
16	12	11	14	13							4	2	3	1	16	15									
•																									
•																									
•																									
•																									
$6k-1$	$6k+1$	$6k+2$	$6k+3$	$6k+4$																$6k-1$	$6k$	1	2	3	4
$6k$	$6k+2$	$6k+1$	$6k+4$	$6k+3$																$6k$	$6k-1$	4	3	1	2
$6k+1$	$6k+3$	$6k+4$	$6k-1$	$6k$																1	4	$6k+1$	$6k+2$	2	3
$6k+2$	$6k+4$	$6k+3$	$6k$	$6k-1$																4	1	$6k+2$	$6k+1$	3	2
$6k+3$	$6k-1$	$6k$	$6k+1$	$6k+2$																3	1	2	4	$6k+3$	$6k+4$
$6k+4$	$6k$	$6k-1$	$6k+2$	$6k+1$																4	2	3	1	$6k+4$	$6k+3$

Figure 8.3.4: Stacked CQGs of Order 10 in the Construction of a CQG of Order $6k + 4$

We now focus on filling the cells outside the stacked CQGs, readily computed to number $36k^2 - 36k$. For this we will need a quasigroup (Q_A, \circ_A) (not necessarily commutative) of order $2k$ with holes of size 2 that has an orthogonal mate. These exist for $k \geq 4$ [10]. We will also need the array A of our $6k + 4$ symbols shown in Figure 8.3.5 that follows:

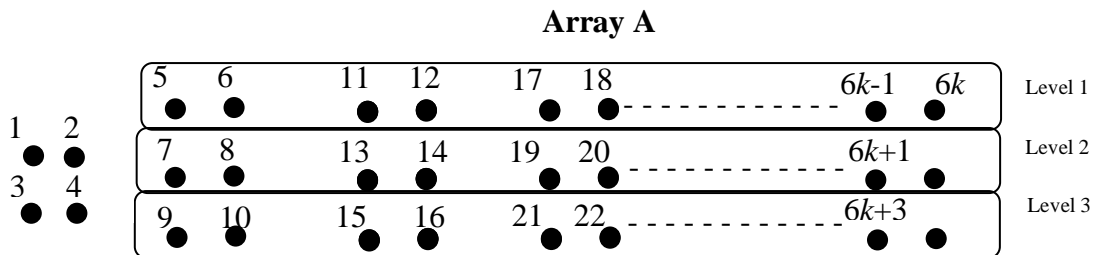


Figure 8.3.5: Array A for the $6k + 4$ Construction for CQGs

We rename the symbols of the quasigroup (Q_A, \circ_A) so that the holes correspond to the holes of each of the stacked CQGs as arranged on level 3 of array A in Figure 8.3.4 and rename the headline and sideline to correspond to the holes on Level 2 and Level 1 respectively of array A, as the framework in Figure 8.3.6 that follows suggests. We should point out that although we shall still refer to the structure (Q_A, \circ_A) as a quasigroup after this renaming, technically speaking, we no longer have a quasigroup, since $x \circ_A y = z$ is true only for $x \in \{5, 6, 11, 17, 18, \dots, 6k - 1, 6k\}$, $y \in \{7, 8, 13, 14, 19, 20, \dots, 6k + 1, 6k + 2\}$ and $z \in \{9, 10, 15, 16, 21, 22, \dots, 6k + 3, 6k + 4\}$, i.e., mutually disjoint sets of symbols from which each of x , y , and z originate.

(Q_A, \circ_A)

\circ_A	7	8	13	14	19	20	...	$6k+1$	$6k+2$
5	9	10							
6	10	9							
11			15	16					
12			16	15					
17					21	22			
18					22	21			
.									
.									
.									
.									
$6k-1$								$6k+3$	$6k+4$
$6k$								$6k+4$	$6k+3$

Figure 8.3.6: Framework for the CQG of Order $2k$ with Orthogonal Mate, Symbols Renamed

For every triple (a, b, c) such that $a \circ_A b = c$ in the quasigroup shown in Figure 8.3.6, we define:

$$\begin{aligned}
 a \circ b &= b \circ a = c \\
 a \circ c &= c \circ a = b \\
 c \circ b &= b \circ c = a
 \end{aligned}$$

thereby determining the entry for 6 cells outside the stacked CQGs in the commutative quasigroup (Q, \circ) of order $6k + 4$ that we seek to construct. Since there are $4k^2 - 4k$ such triples in the quasigroup (Q_A, \circ_A) , we are able to fill $6(4k^2 - 4k) = 24k^2 - 24k$ of the $36k^2 - 36k$ previously unfilled cells. To complete the remaining unfilled cells, we will need a commutative quasigroup of order $2k$ with holes of size 2, which we know exists for all $k \geq 3$ [8]. We use this CQG to form 3 new CQGs (Q_I, \circ_I) , (Q_{II}, \circ_{II}) , and (Q_{III}, \circ_{III}) by renaming the symbols so that the holes, headlines, and sidelines correspond to the holes of Levels 1, 2, and 3 respectively in the array A of Figure 8.3.5, as suggested by the frameworks illustrated in Figure 8.3.7 that follows.

(Q_I, \circ_I)

\circ_I	5	6	11	12	17	18	...	$6k-1$	$6k$
5	5	6							
6	6	5							
11			11	12					
12			12	11					
17					17	18			
18					18	17			
.									
$6k-1$								$6k-1$	$6k$
$6k$								$6k$	$6k-1$

(Q_{II}, \circ_{II})

\circ_{II}	7	8	13	14	19	20	...	$6k+1$	$6k+2$
7	7	8							
8	8	7							
13			13	14					
14			14	13					
19					19	20			
20					20	19			
.									
$6k+1$								$6k+1$	$6k+2$
$6k+2$								$6k+2$	$6k+1$

(Q_{III}, \circ_{III})

\circ_{III}	9	10	15	16	21	22	...	$6k+3$	$6k+4$
9	9	10							
10	10	9							
15			15	16					
16			16	15					
21					21	22			
22					22	21			
.									
$6k+3$								$6k+3$	$6k+4$
$6k+4$								$6k+4$	$6k+3$

Figure 8.3.7: Frameworks for CQGs (Q_I, \circ_I) , (Q_{II}, \circ_{II}) , and (Q_{III}, \circ_{III}) in the $6k + 4$ Construction

For every ordered triple (a, b, c) such that $a \circ_I b = c$ in (Q_I, \circ_I) above, we will similarly define $a \circ b = c$ in the commutative quasigroup (Q, \circ) of order $6k + 4$ that we seek to construct. This rule is repeated for triples of (Q_{II}, \circ_{II}) , and (Q_{III}, \circ_{III}) . Since each of (Q_I, \circ_I) , (Q_{II}, \circ_{II}) , and

(Q_{III}, \circ_{III}) produces $4k^2 - 4k$ triples, we will fill $3(4k^2 - 4k) = 12k^2 - 12k$ cells of (Q, \circ) , the exact number that remain to be filled.

The task still remains to exhibit 2 complete transversals outside the holes of (Q, \circ) . Recall that the cells outside the holes of (Q_I, \circ_I) were partitioned into transversals. Select any one of those transversals, call it T_1 . The entries of T_1 constitute a partial transversal of (Q, \circ) on the rows and columns 1 through 10.

Recall also that (Q_A, \circ_A) has an orthogonal mate which we will denote $(Q_{A'}, \circ_{A'})$. In $(Q_{A'}, \circ_{A'})$, consider the set of ordered pairs (a, b) such that $a \circ_{A'} b = 9$. The same set of ordered pairs applied to (Q_A, \circ_A) yields a partial transversal $T_{A,9}$ of (Q_A, \circ_A) on the set of rows $\{11, 12, 17, 18, \dots, 6k - 1, 6k\}$ and the set of columns $\{13, 14, 19, 20, \dots, 6k + 1, 6k + 2\}$ and on the set of symbols $\{15, 16, 21, 22, \dots, 6k + 3, 6k + 4\}$. Furthermore the set of corresponding triples $\{(a, b, c), (b, c, a), (c, a, b) \mid (a, b) \in T_{A,9} \text{ and } a \circ_A b = c\}$ identifies the row, column, and entry for the cells of a partial transversal, call it T_9 , of (Q, \circ) on the set of rows $\{11, 12, 13, \dots, 6k + 4\}$, the set of columns $\{11, 12, 13, \dots, 6k + 4\}$, and the set of symbols $\{11, 12, 13, \dots, 6k + 4\}$. Then $T_1 \cup T_9$ is a complete transversal of (Q, \circ) outside the holes.

Now select any other transversal of (Q_I, \circ_I) , call it T_2 and determine the set of ordered pairs (a, b) such that $a \circ_{A'} b = 10$ in $(Q_{A'}, \circ_{A'})$. The same set of ordered pairs applied to (Q_A, \circ_A) yields a partial transversal $T_{A,10}$ of (Q_A, \circ_A) on the set of rows $\{11, 12, 17, 18, \dots, 6k - 1, 6k\}$ and the set of columns $\{13, 14, 19, 20, \dots, 6k + 1, 6k + 2\}$ and on the set of symbols $\{15, 16, 21, 22, \dots, 6k + 3, 6k + 4\}$. Furthermore the set of corresponding triples $\{(a, b, c), (b, c, a), (c, a, b) \mid (a, b) \in T_{A,10} \text{ and } a \circ_A b = c\}$ identifies the row, column, and entry for the cells of a second partial transversal, call it T_{10} , of (Q, \circ) on the set of rows $\{11, 12, 13, \dots, 6k + 4\}$, the set of

columns $\{11, 12, 13, \dots, 6k + 4\}$, and the set of symbols $\{11, 12, 13, \dots, 6k + 4\}$. Then $T_2 \cup T_{10}$

is a second complete transversal of (Q, \circ) outside the holes.

Example 8.3.1. We will give an example of the $6k + 4$ construction of Theorem 8.3 by building a CQG of order 28 where $k = 4$. We begin by stacking (Q_1, \circ_1) , (Q_2, \circ_2) , and two additional “copies” of (Q_2, \circ_2) with all symbols other than 1, 2, 3, and 4 renamed mod 6, as illustrated in Figure 8.3.8 that follows.

o	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
1	1	2	7	8	3	4	9	10	5	6	13	14	15	16	11	12	19	20	21	22	17	18	25	26	27	28	23	24
2	2	1	8	7	4	3	10	9	6	5	14	13	16	15	12	11	20	19	22	21	18	17	26	25	28	27	24	23
3	7	8	3	4	9	10	5	6	1	2	15	16	11	12	13	14	21	22	17	18	19	20	27	28	23	24	25	26
4	8	7	4	3	10	9	6	5	2	1	16	15	12	11	14	13	22	21	18	17	20	19	28	27	24	23	26	25
5	3	4	9	10	5	6	1	2	7	8																		
6	4	3	10	9	6	5	2	1	8	7																		
7	9	10	5	6	1	2	7	8	4	3																		
8	10	9	6	5	2	1	8	7	3	4																		
9	5	6	1	2	7	8	4	3	9	10																		
10	6	5	2	1	8	7	3	4	10	9																		
11	13	14	15	16							11	12	1	2	3	4												
12	14	13	16	15							12	11	4	3	1	2												
13	15	16	11	12							1	4	13	14	2	3												
14	16	15	12	11							4	1	14	13	3	2												
15	11	12	13	14							3	1	2	4	15	16												
16	12	11	14	13							4	2	3	1	16	15												
17	19	20	21	22													17	18	1	2	3	4						
18	20	19	22	21													18	17	4	3	1	2						
19	21	22	17	18													1	4	19	20	2	3						
20	22	21	18	17													2	3	20	19	4	1						
21	17	18	19	20													3	1	2	4	21	22						
22	18	17	20	19													4	2	3	1	22	21						
23	25	26	27	28																			23	24	1	2	3	4
24	26	25	28	27																			24	23	4	3	1	2
25	27	28	23	24																			1	4	25	26	2	3
26	28	27	24	23																			2	3	26	25	4	1
27	23	24	25	26																			3	1	2	4	27	28
28	24	23	26	25																			4	2	3	1	28	27

Figure 8.3.8: Stacked CQGs of order 10 in the Construction of a CQG of Order 28

This leaves 432 cells unfilled. To begin filling the cells outside the stacked CQGs, we will use the array A and the associated quasigroup (Q_A, \circ_A) of order $2k = 8$ with holes of size 2 that has an orthogonal mate, $(Q_A', \circ_{A'})$, both given in Figure 8.3.9 that follows. We have already renamed the symbols of (Q_A, \circ_A) to correspond to the holes on Levels 1, 2, and 3 of the array A .

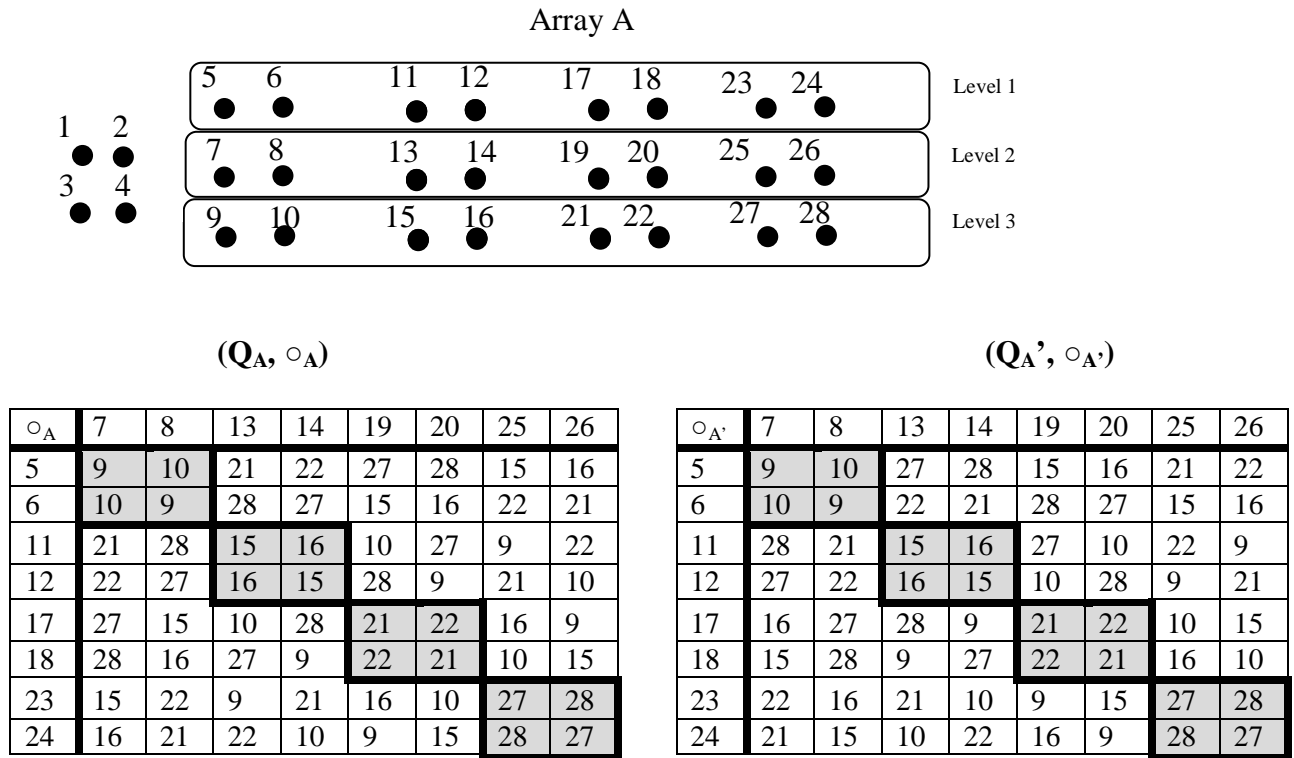


Figure 8.3.9: Array and CQGs with Orthogonal Mates for Construction of CQG of Order 28

For every triple (a, b, c) such that $a \circ b = c$ in the quasigroup (Q_A, \circ_A) of Figure 8.3.9, we define:

$$\begin{aligned}
 a \circ b &= b \circ a = c \\
 a \circ c &= c \circ a = b \\
 c \circ b &= b \circ c = a
 \end{aligned}$$

thereby determining 6 entries in (Q, \circ) for each of the 48 entries in (Q_A, \circ_A) . We now fill those 288 cells, as the updated version of (Q, \circ) shows in Figure 8.3.10 that follows.

○	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
1	1	2	7	8	3	4	9	10	5	6	13	14	15	16	11	12	19	20	21	22	17	18	25	26	27	28	23	24
2	2	1	8	7	4	3	10	9	6	5	14	13	16	15	12	11	20	19	22	21	18	17	26	25	28	27	24	23
3	7	8	3	4	9	10	5	6	1	2	15	16	11	12	13	14	21	22	17	18	19	20	27	28	23	24	25	26
4	8	7	4	3	10	9	6	5	2	1	16	15	12	11	14	13	22	21	18	17	20	19	28	27	24	23	26	25
5	3	4	9	10	5	6	1	2	7	8			21	22	25	26			27	28	13	14			15	16	19	20
6	4	3	10	9	6	5	2	1	8	7			28	27	19	20			15	16	26	25			22	21	14	13
7	9	10	5	6	1	2	7	8	4	3	21	22			23	24	27	28			11	12	15	16			17	18
8	10	9	6	5	2	1	8	7	3	4	28	27			17	18	15	16			24	23	22	21			12	11
9	5	6	1	2	7	8	4	3	9	10	25	20	23	18			26	14	24	12			13	19	11	17		
10	6	5	2	1	8	7	3	4	10	9	19	26	17	24			13	25	11	23			20	14	18	12		
11	13	14	15	16			21	28	25	19	11	12	1	2	3	4			10	27	7	26			9	22	20	8
12	14	13	16	15			22	27	20	26	12	11	4	3	1	2			28	9	25	7			21	10	8	19
13	15	16	11	12	21	28			23	17	1	4	13	14	2	3	10	27			5	24	9	22			18	6
14	16	15	12	11	22	27			18	24	4	1	14	13	3	2	28	9			23	5	21	10			6	17
15	11	12	13	14	25	19	23	17			3	1	2	4	15	16	8	26	6	24			7	20	5	18		
16	12	11	14	13	26	20	24	18			4	2	3	1	16	15	25	8	23	6			19	7	17	5		
17	19	20	21	22			27	15	26	13			10	28	8	25	17	18	1	2	3	4			16	9	7	14
18	20	19	22	21			28	16	14	25			27	9	26	8	18	17	4	3	1	2			10	15	13	7
19	21	22	17	18	27	15			24	11	10	28			6	23	1	4	19	20	2	3	16	9			5	12
20	22	21	18	17	28	16			12	23	27	9			24	6	2	3	20	19	4	1	10	15			11	5
21	17	18	19	20	13	26	11	24			7	25	5	23			3	1	2	4	21	22	14	8	12	6		
22	18	17	20	19	14	25	12	23			26	7	24	5			4	2	3	1	22	21	8	13	6	11		
23	25	26	27	28			15	22	13	20			9	21	7	19			16	10	14	8	23	24	1	2	3	4
24	26	25	28	27			16	21	19	14			22	10	20	7			9	15	8	13	24	23	4	3	1	2
25	27	28	23	24	15	22			11	18	9	21			5	17	16	10			12	6	1	4	25	26	2	3
26	28	27	24	23	16	21			17	12	22	10			18	5	9	15			6	11	2	3	26	25	4	1
27	23	24	25	26	19	14	17	12			20	8	18	6			7	13	5	11			3	1	2	4	27	28
28	24	23	26	25	20	13	18	11			8	19	6	17			14	7	12	5			4	2	3	1	28	27

Figure 8.3.10: Construction of CQG of Order 28 (144 Cells Yet Unfilled)

To complete the remaining unfilled cells, we will use a commutative quasigroup of order $2k = 8$ with holes of size 2, which we in turn use to form 3 new CQGs (Q_I, \circ_I) , (Q_{II}, \circ_{II}) , and (Q_{III}, \circ_{III}) by renaming the symbols so that the holes, headlines, and sidelines correspond to the holes of Levels 1, 2, and 3 respectively in the array A. These three CQGs are shown in Figure 8.3.11 that follows:

\circ_I	5	6	11	12	17	18	23	24
5	5	6	17	18	23	24	11	12
6	6	5	24	23	11	12	18	17
11	17	24	11	12	6	23	5	18
12	18	23	12	11	24	5	17	6
17	23	11	6	24	17	18	12	5
18	24	12	23	5	18	17	6	11
23	11	18	5	17	12	6	23	24
24	12	17	18	6	5	11	24	23

\circ_{II}	7	8	13	14	19	20	25	26
7	7	8	19	20	25	26	13	14
8	8	7	26	25	13	24	20	19
13	19	26	13	14	8	25	7	20
14	20	25	14	13	26	7	19	8
19	25	13	8	26	19	20	14	7
20	26	14	25	7	20	19	8	13
25	13	20	7	19	14	8	25	26
26	14	19	20	8	7	13	26	25

\circ_{III}	9	10	15	16	21	22	27	28
9	9	10	21	22	27	28	15	16
10	10	9	28	27	15	16	22	21
15	21	28	15	16	10	27	9	22
16	22	27	16	15	28	9	21	10
21	27	15	10	28	21	22	16	9
22	28	16	27	9	22	21	10	15
27	15	22	9	21	16	10	27	28
28	16	21	22	10	9	15	28	27

Figure 8.3.11: CQGs (Q_I, \circ_I) , (Q_{II}, \circ_{II}) , and (Q_{III}, \circ_{III}) for Construction of CQG of order 28

For every ordered triple (a, b, c) such that $a \circ_I b = c$ in (Q_I, \circ_I) , $a \circ_{II} b = c$ in (Q_{II}, \circ_{II}) , or $a \circ_{III} b = c$ in (Q_{III}, \circ_{III}) above we will similarly define $a \circ b = c$ in the commutative quasigroup (Q, \circ) , and thus fill the remaining $48(3) = 144$ cells, as shown in the completed quasigroup that follows in Figure 8.3.12:

o	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
1	1	2	7	8	3	4	9	10	5	6	13	14	15	16	11	12	19	20	21	22	17	18	25	26	27	28	23	24
2	2	1	8	7	4	3	10	9	6	5	14	13	16	15	12	11	20	19	22	21	18	17	26	25	28	27	24	23
3	7	8	3	4	9	10	5	6	1	2	15	16	11	12	13	14	21	22	17	18	19	20	27	28	23	24	25	26
4	8	7	4	3	10	9	6	5	2	1	16	15	12	11	14	13	22	21	18	17	20	19	28	27	24	23	26	25
5	3	4	9	10	5	6	1	2	7	8	17	18	21	22	25	26	23	24	27	28	13	14	11	12	15	16	19	20
6	4	3	10	9	6	5	2	1	8	7	24	23	28	27	19	20	11	12	15	16	26	25	18	17	22	21	14	13
7	9	10	5	6	1	2	7	8	4	3	21	22	19	20	23	24	27	28	25	26	11	12	15	16	13	14	17	18
8	10	9	6	5	2	1	8	7	3	4	28	27	26	25	17	18	15	16	13	14	24	23	22	21	20	19	12	11
9	5	6	1	2	7	8	4	3	9	10	25	20	23	18	21	22	26	14	24	12	27	28	13	19	11	17	15	16
10	6	5	2	1	8	7	3	4	10	9	19	26	17	24	28	27	13	25	11	23	15	16	20	14	18	12	22	21
11	13	14	15	16	17	24	21	28	25	19	11	12	1	2	3	4	6	23	10	27	7	26	5	18	9	22	20	8
12	14	13	16	15	18	23	22	27	20	26	12	11	4	3	1	2	24	5	28	9	25	7	17	6	21	10	8	19
13	15	16	11	12	21	28	19	26	23	17	1	4	13	14	2	3	10	27	8	25	5	24	9	22	7	20	18	6
14	16	15	12	11	22	27	20	25	18	24	4	1	14	13	3	2	28	9	26	7	23	5	21	10	19	8	6	17
15	11	12	13	14	25	19	23	17	21	28	3	1	2	4	15	16	8	26	6	24	10	27	7	20	5	18	9	22
16	12	11	14	13	26	20	24	18	22	27	4	2	3	1	16	15	25	8	23	6	28	9	19	7	17	5	21	10
17	19	20	21	22	23	11	27	15	26	13	6	24	10	28	8	25	17	18	1	2	3	4	12	5	16	9	7	14
18	20	19	22	21	24	12	28	16	14	25	23	5	27	9	26	8	18	17	4	3	1	2	6	11	10	15	13	7
19	21	22	17	18	27	15	25	13	24	11	10	28	8	26	6	23	1	4	19	20	2	3	16	9	14	7	5	12
20	22	21	18	17	28	16	26	14	12	23	27	9	25	7	24	6	2	3	20	19	4	1	10	15	8	13	11	5
21	17	18	19	20	13	26	11	24	27	15	7	25	5	23	10	28	3	1	2	4	21	22	14	8	12	6	16	9
22	18	17	20	19	14	25	12	23	28	16	26	7	24	5	27	9	4	2	3	1	22	21	8	13	6	11	10	15
23	25	26	27	28	11	18	15	22	13	20	5	17	9	21	7	19	12	6	16	10	14	8	23	24	1	2	3	4
24	26	25	28	27	12	17	16	21	19	14	18	6	22	10	20	7	5	11	9	15	8	13	24	23	4	3	1	2
25	27	28	23	24	15	22	13	20	11	18	9	21	7	19	5	17	16	10	14	8	12	6	1	4	25	26	2	3
26	28	27	24	23	16	21	14	19	17	12	22	10	20	8	18	5	9	15	7	13	6	11	2	3	26	25	4	1
27	23	24	25	26	19	14	17	12	15	22	20	8	18	6	9	21	7	13	5	11	16	10	3	1	2	4	27	28
28	24	23	26	25	20	13	18	11	16	21	8	19	6	17	22	10	14	7	12	5	9	15	4	2	3	1	28	27

Figure 8.3.12: Completed CQG of Order 28

Finally, we identify two distinct complete transversals outside the holes of our CQG. We select two transversals from (Q_1, \circ_1) , as shown in the cells shaded blue and red in Figure 8.3.12 above. Then we use the entries of 9 and 10 in the orthogonal mate $(Q_A', \circ_{A'})$ of (Q_A, \circ_A) to find the partial transversals in (Q_A, \circ_A) shown in Figure 8.3.13 that follows:

(Q_A, \circ_A)								
\circ_A	7	8	13	14	19	20	25	26
5	9	10	21	22	27	28	15	16
6	10	9	28	27	15	16	22	21
11	21	28	15	16	10	27	9	22
12	22	27	16	15	28	9	21	10
17	27	15	10	28	21	22	16	9
18	28	16	27	9	22	21	10	15
23	15	22	9	21	16	10	27	28
24	16	21	22	10	9	15	28	27

$(Q_{A'}, \circ_{A'})$								
$\circ_{A'}$	7	8	13	14	19	20	25	26
5	9	10	27	28	15	16	21	22
6	10	9	22	21	28	27	15	16
11	28	21	15	16	27	10	22	9
12	27	22	16	15	10	28	9	21
17	16	27	28	9	21	22	10	15
18	15	28	9	27	22	21	16	10
23	22	16	21	10	9	15	27	28
24	21	15	10	22	16	9	28	27

Figure 8.3.13: CQG of Order 8 with Orthogonal Mate for Construction of CQG of Order 28

Each cell shaded red or blue represents a unique ordered triple (a, b, c) . Cyclic reordering of each of these triples from (Q_A, \circ_A) maintains disjoint rows, columns and entries and results in the following two collections of ordered triples $\{(a, b, c), (b, c, a), (c, a, b)\}$:

- (11, 26, 22), (26, 22, 11), (22, 11, 26)**
- (12, 25, 21), (25, 21, 12), (21, 12, 25)**
- (17, 14, 28), (14, 28, 17), (28, 17, 14)**
- (18, 13, 27), (13, 27, 18), (27, 18, 13)**
- (23, 19, 16), (19, 16, 23), (16, 23, 19)**
- (24, 20, 15), (20, 15, 24), (15, 24, 20)**

- (11, 20, 27), (20, 27, 11), (27, 11, 20)**
- (12, 19, 28), (19, 28, 12), (28, 12, 19)**
- (17, 25, 16), (25, 16, 17), (16, 17, 25)**
- (18, 26, 15), (26, 15, 18), (15, 18, 26)**
- (23, 14, 21), (14, 21, 23), (21, 23, 14)**
- (24, 13, 22), (13, 22, 24), (22, 24, 13)**

When applied to (Q, \circ) , the triples of each set above identify the row, column and entry of the cells of one of two partial transversals of (Q, \circ) on the set of rows $\{11, 12, 13, \dots, 28\}$, the set of columns $\{11, 12, 13, \dots, 28\}$, and the set of symbols $\{11, 12, 13, \dots, 28\}$. When joined with the previously chosen transversals from (Q_1, \circ_1) , they form two complete transversals of (Q, \circ) outside the holes as shown in the cells shaded red and blue in the final view of (Q, \circ) that follows in Figure 8.3.14:

(Q, ◦)

◦	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
1	1	2	7	8	3	4	9	10	5	6	13	14	15	16	11	12	19	20	21	22	17	18	25	26	27	28	23	24
2	2	1	8	7	4	3	10	9	6	5	14	13	16	15	12	11	20	19	22	21	18	17	26	25	28	27	24	23
3	7	8	3	4	9	10	5	6	1	2	15	16	11	12	13	14	21	22	17	18	19	20	27	28	23	24	25	26
4	8	7	4	3	10	9	6	5	2	1	16	15	12	11	14	13	22	21	18	17	20	19	28	27	24	23	26	25
5	3	4	9	10	5	6	1	2	7	8	17	18	21	22	25	26	23	24	27	28	13	14	11	12	15	16	19	20
6	4	3	10	9	6	5	2	1	8	7	24	23	28	27	19	20	11	12	15	16	26	25	18	17	22	21	14	13
7	9	10	5	6	1	2	7	8	4	3	21	22	19	20	23	24	27	28	25	26	11	12	15	16	13	14	17	18
8	10	9	6	5	2	1	8	7	3	4	28	27	26	25	17	18	15	16	13	14	24	23	22	21	20	19	12	11
9	5	6	1	2	7	8	4	3	9	10	25	20	23	18	21	22	26	14	24	12	27	28	13	19	11	17	15	16
10	6	5	2	1	8	7	3	4	10	9	19	26	17	24	28	27	13	25	11	23	15	16	20	14	18	12	22	21
11	13	14	15	16	17	24	21	28	25	19	11	12	1	2	3	4	6	23	10	27	7	26	5	18	9	22	20	8
12	14	13	16	15	18	23	22	27	20	26	12	11	4	3	1	2	24	5	28	9	25	7	17	6	21	10	8	19
13	15	16	11	12	21	28	19	26	23	17	1	4	13	14	2	3	10	27	8	25	5	24	9	22	7	20	18	6
14	16	15	12	11	22	27	20	25	18	24	4	1	14	13	3	2	28	9	26	7	23	5	21	10	19	8	6	17
15	11	12	13	14	25	19	23	17	21	28	3	1	2	4	15	16	8	26	6	24	10	27	7	20	5	18	9	22
16	12	11	14	13	26	20	24	18	22	27	4	2	3	1	16	15	25	8	23	6	28	9	19	7	17	5	21	10
17	19	20	21	22	23	11	27	15	26	13	6	24	10	28	8	25	17	18	1	2	3	4	12	5	16	9	7	14
18	20	19	22	21	24	12	28	16	14	25	23	5	27	9	26	8	18	17	4	3	1	2	6	11	10	15	13	7
19	21	22	17	18	27	15	25	13	24	11	10	28	8	26	6	23	1	4	19	20	2	3	16	9	14	7	5	12
20	22	21	18	17	28	16	26	14	12	23	27	9	25	7	24	6	2	3	20	19	4	1	10	15	8	13	11	5
21	17	18	19	20	13	26	11	24	27	15	7	25	5	23	10	28	3	1	2	4	21	22	14	8	12	6	16	9
22	18	17	20	19	14	25	12	23	28	16	26	7	24	5	27	9	4	2	3	1	22	21	8	13	6	11	10	15
23	25	26	27	28	11	18	15	22	13	20	5	17	9	21	7	19	12	6	16	10	14	8	23	24	1	2	3	4
24	26	25	28	27	12	17	16	21	19	14	18	6	22	10	20	7	5	11	9	15	8	13	24	23	4	3	1	2
25	27	28	23	24	15	22	13	20	11	18	9	21	7	19	5	17	16	10	14	8	12	6	1	4	25	26	2	3
26	28	27	24	23	16	21	14	19	17	12	22	10	20	8	18	5	9	15	7	13	6	11	2	3	26	25	4	1
27	23	24	25	26	19	14	17	12	15	22	20	8	18	6	9	21	7	13	5	11	16	10	3	1	2	4	27	28
28	24	23	26	25	20	13	18	11	16	21	8	19	6	17	22	10	14	7	12	5	9	15	4	2	3	1	28	27

Figure 8.3.14: Completed CQG of Order 28, Holes of Size 2, with 2 Transversals Outside Holes

8.4 Conclusions for the General Odd/Odd Cases (mod 8)

The constructions for commutative quasigroups given in the previous sections of this chapter allow us to now show for certain what was speculated in Theorems 6.2 and 7.2 regarding the existence of ARMPs of $K(8m + 5, 8m + 5)$ and $K(8m + 7, 8m + 7)$.

In Theorem 6.2, we proved that there exists an ARMP of $K(8m + 5, 8m + 5)$ with 4-cycles for all $m \leq 4$ (except possibly $m = 2$) and that there exists an ARMP of $K(8m + 5, 8m + 5)$ for all $m > 4$ whenever there exists a commutative quasigroup of order $2m$ with holes of size 2 that contains 2 complete transversals outside the holes. Additionally, we proved in Theorems 8.1.1, 8.2, and 8.3 that such quasigroups do exist except for possibly $m = 8$. Thus we have:

Theorem 8.4.1. There exists an ARMP of $K(8m + 5, 8m + 5)$ with 4-cycles for all m except possibly 2 and 8.

Similarly, in Theorem 7.2 we showed that there exists an ARMP of $K(8m + 7, 8m + 7)$ with 4-cycles for all $m \leq 4$ (except possibly $m = 2$) and that there exists an ARMP of $K(8m + 7, 8m + 7)$ for all $m > 4$ whenever there exists a commutative quasigroup of order $2m$ with holes of size 2 that contains 2 complete transversals outside the holes. So again, along with Theorems 8.1.1, 8.2, and 8.3 which provide the needed CQGs, we have:

Theorem 8.4.2. There exists an ARMP of $K(8m + 7, 8m + 7)$ with 4-cycles for all m except possibly 2 and 8.

Chapter 9

Other Results

9.1 $K(2m + 1, 2n + 1)$ and $K(2m + 1, 2t + 1)$ for $t \geq n \geq m$

Theorem 9.1. Let $n \geq m$. If an almost resolvable maximum packing with 4-cycles exists for $K(2m + 1, 2n + 1)$ for some $m, n \in \mathbb{N}$, then an almost resolvable maximum packing with 4-cycles exists for $K(2m + 1, 2t + 1)$ for all $t \geq (m + n)$.

Proof: First note that $2t + 1$ can be written as $(2n + 1) + 2(t - n)$ and that the partition of $K(2m + 1, 2t + 1)$ can be represented graphically as the union of $K(2m + 1, 2n + 1)$ and $K(2m + 1, 2(t - n))$ as illustrated in Figure 9.1 on the following page.

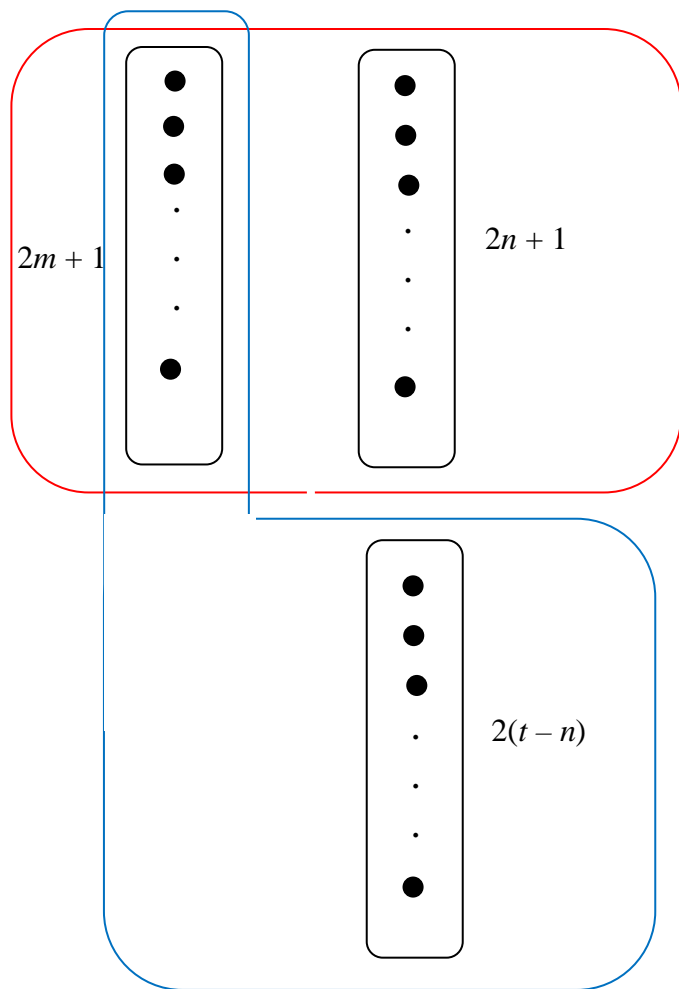


Figure 9.1: $K(2m+1, 2t+1)$

Assume first that $2m+1 \equiv 1 \pmod{4}$. Then the leave of any ARMP of $K(2m+1, 2t+1)$ must contain $2t+1$ edges. An ARMP of $K(2m+1, 2n+1)$ is assumed and must contain a leave with $2n+1$ edges. From Theorem 2.2 (the even/odd case), we know that an ARMP of $K(2m+1, 2(t-n))$ exists and must contain a leave with $2(t-n)$ edges. The union of these two almost resolvable maximum packings results in a leave with $(2n+1) + 2(t-n) = 2t+1$ edges, as required for an ARMP of $K(2m+1, 2t+1)$.

Now assume that $2m + 1 \equiv 3 \pmod{4}$. Then the leave of any ARMP of $K(2m + 1, 2t + 1)$ must contain $2t + 3$ edges. Again, an ARMP of $K(2m + 1, 2n + 1)$ is given and must contain a leave with $2n + 3$ edges. We also know that an ARMP of $K(2m + 1, 2(t - n))$ is possible and must contain a leave with $2(t - n)$ edges. The union of these two almost resolvable maximum packings results in a leave with $(2n + 3) + 2(t - n) = 2t + 3$ edges, as required for an ARMP of $K(2m + 1, 2t + 1)$.

9.2 $K(4m + 1, 4m + 1)$ and $K(4mn + 1, 4mn + 1)$

Theorem 9.2.1. If there exists an almost resolvable maximum packing of $K(4m + 1, 4m + 1)$ for some $m \in \mathbb{N}$, then there exists an almost resolvable maximum packing of $K(4mn + 1, 4mn + 1)$ for all $n \geq 3$.

Proof. This proof generalizes the construction given in the proof for $K(8m + 1, 8m + 1)$.

Divide $K(4mn + 1, 4mn + 1)$ into n copies of $K(4m + 1, 4m + 1)$ with all copies sharing the same single vertex per part, as shown in the diagram of Figure 9.2 on the next page.

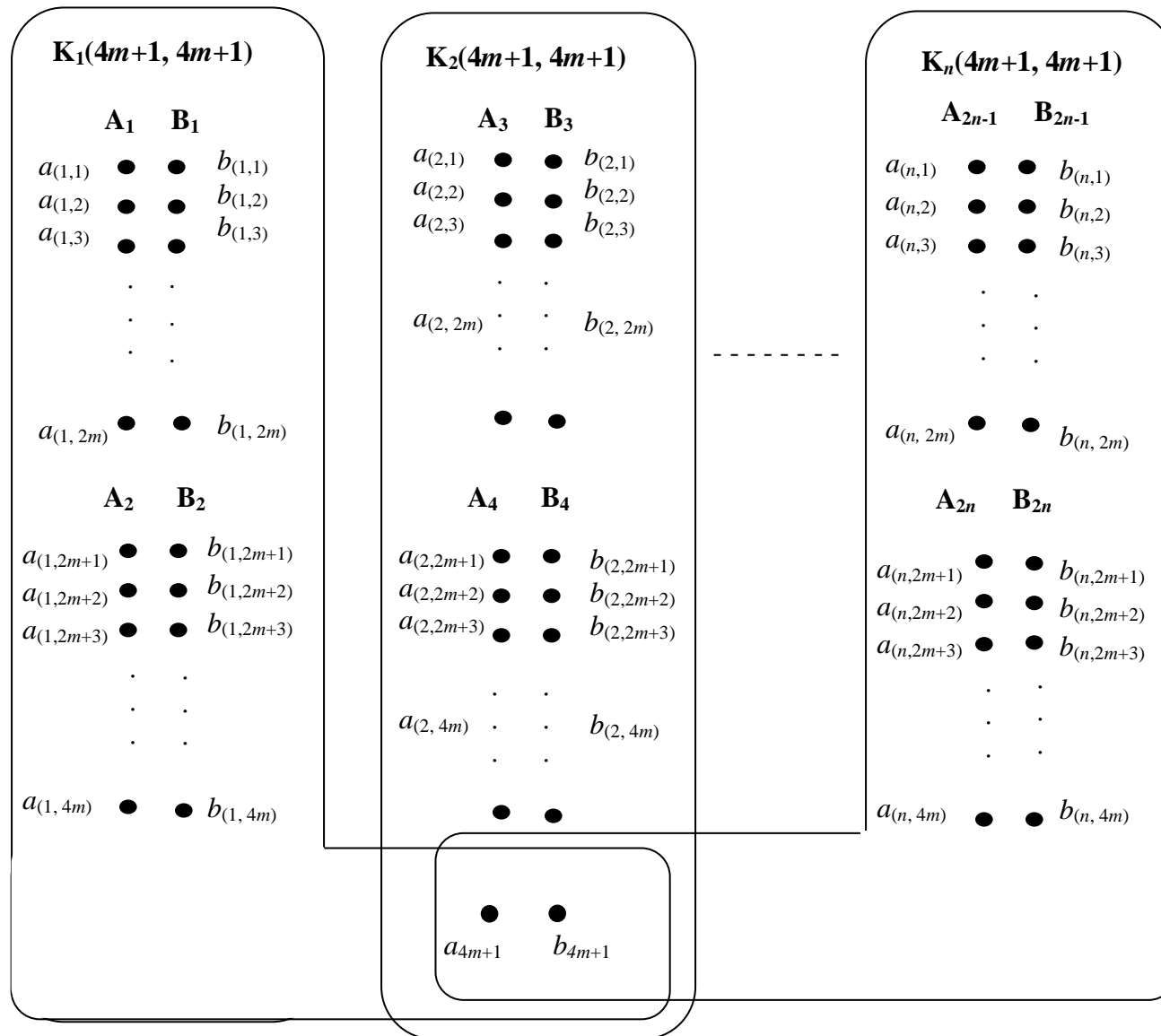


Figure 9.2: $K(4mn + 1, 4mn + 1)$

We are assuming that for each x , there exists an ARMP of $K_x(4m + 1, 4m + 1)$ containing $2m$ almost parallel classes (call them $\pi_{(x,1)}, \pi_{(x,2)}, \pi_{(x,3)} \dots \pi_{(x,2m)}$) of m cycles each and a partial parallel class $\pi_{(x,P)}$ containing m cycles that omit the shared vertices, with the leave of each containing the edge connecting the two shared vertices. Let (Q, \circ) be a commutative quasigroup of order $2n$ on the set of symbols $\{1, 2, 3, \dots, 2n\}$ with set of size 2 holes $H = \{h_x \mid 1 \leq x \leq n\}$ such that $y \in h_x$ iff $y = 2x$ or $y = 2x - 1$. (This is possible since $2n \geq 6$.) For each $y \in h_x$, let $K_{(z,w)}(2m, 2m)$ be the complete bipartite graph with partition $\{A_z, B_w\}$ iff $z \circ w = y$. Note that $K_{(z,w)}(2m, 2m) \neq K_{(w,z)}(2m, 2m)$ even though $z \circ w = w \circ z$. Thus each $y \in h_x$ will be associated with $2n - 2$ distinct $K(2m, 2m)$.

Recall that every $K(2m, 2m)$ is completely resolvable into m almost parallel classes of m 4-cycles each (the even-even case). For $y \in h_x$, form the union of $\pi_{(x,1)}$ with one APC from each of the associated $K(2m, 2m)$ to form one complete APC containing $2m + m(2n - 2) = 2m + 2mn - 2m = 2mn$ cycles for $K(4mn + 1, 4mn + 1)$. Repeat, using $\pi_{(x,2)}$ through $\pi_{(x,m)}$ and a different APC from each of the associated $K(2m, 2m)$ for each repetition. Since each hole contains 2 elements, this can again be repeated using $\pi_{(x,m+1)}$ through $\pi_{(x,2m)}$ for a total of $2m$ complete APCs of $K(4mn + 1, 4mn + 1)$. Running over all n holes will yield $2mn$ APCs of $2mn$ cycles each, as required. Finally, $\bigcup_{P=1}^m \pi_{(x,P)}$ forms a complete partial parallel class with mn 4-cycles, also as required.

9.3 $K(4m + 1, 4n + 1)$ and $K(4mq + 1, 4nq + 1)$

Theorem 9.3.1 If there exists an almost resolvable maximum packing of $K(4m + 1, 4n + 1)$ for some $m, n \in \mathbb{N}$, then there exists an almost resolvable maximum packing of $K(4mq + 1, 4nq + 1)$ for all $q \geq 3$.

Proof. This proof once again generalizes the construction given in the proof for $K(8m + 1, 8m + 1)$.

Without loss of generality, let $m \leq n$. For $1 \leq x \leq q$, let $A_{2x-1} = \{a_{(x,1)}, a_{(x,2)}, a_{(x,3)}, \dots, a_{(x,2m)}\}$ and let $A_{2x} = \{a_{(x,2m+1)}, a_{(x,2m+2)}, a_{(x,2m+3)}, \dots, a_{(x,4m)}\}$. Similarly, let $B_{2x-1} = \{b_{(x,1)}, b_{(x,2)}, b_{(x,3)}, \dots, b_{(x,2n)}\}$ and let $B_{2x} = \{b_{(x,2m+1)}, b_{(x,2m+2)}, b_{(x,2m+3)}, \dots, b_{(x,4n)}\}$. Then define $K_x(4m+1, 4n+1)$ to be the complete bipartite graph with parts $(A_{2x} \cup A_{2x-1} \cup \{a_{4m+1}\}) = \{a_{(x,1)}, a_{(x,2)}, a_{(x,3)}, \dots, a_{(x,4m)}, a_{4m+1}\}$ and $(B_{2x} \cup B_{2x-1} \cup \{b_{4n+1}\}) = \{b_{(x,1)}, b_{(x,2)}, b_{(x,3)}, \dots, b_{(x,4n)}, b_{4n+1}\}$. Then we may define $K(4mq + 1, 4nq + 1)$ as the complete bipartite graph with parts $\bigcup_{x=1}^q A_{2x} \cup A_{2x-1} \cup \{a_{4m+1}\}$ and $\bigcup_{x=1}^q B_{2x} \cup B_{2x-1} \cup \{b_{4n+1}\}$. See Figure 9.3 on the following page.

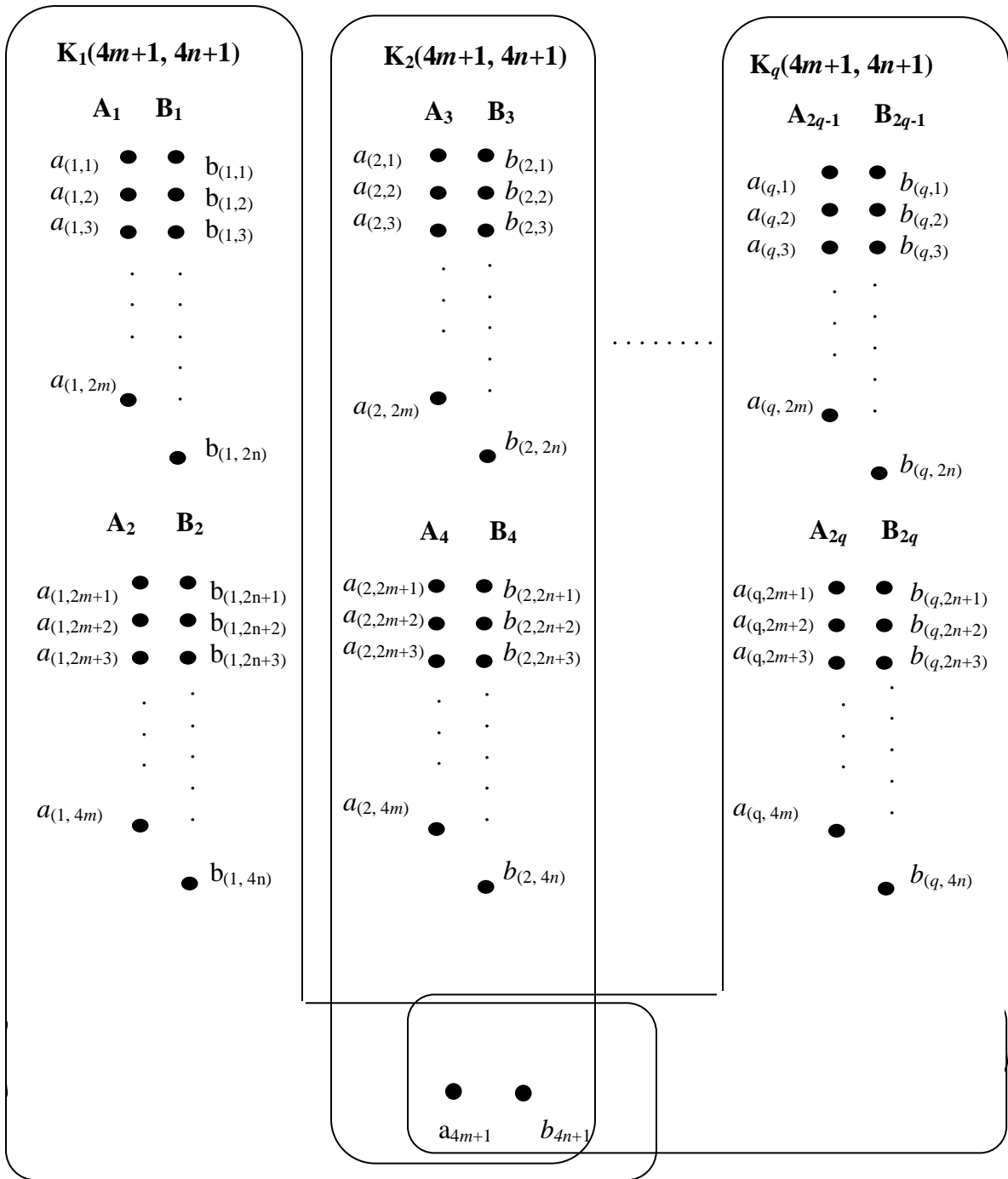


Figure 9.3: $K(4mq + 1, 4nq + 1)$

Note that for an almost resolvable maximum packing of $K(4mq + 1, 4nq + 1)$ with 4-cycles, we will need $2nq$ APC with $2mq$ cycles each and 1 PPC with mq cycles.

Assume that for $1 \leq x \leq q$ there exists an ARMP of $K_x(4m + 1, 4n + 1)$ containing $2n$ almost parallel classes (call them $\pi_{(x,1)}, \pi_{(x,2)}, \pi_{(x,3)} \dots \pi_{(x,2n)}$) of $2m$ cycles each and a partial parallel class $\pi_{(x,P)}$ containing m cycles that upon relabeling of vertices, if necessary, omit vertices a_{4m+1} and b_{4m+1} . Let (Q, \circ) be a commutative quasigroup of order $2q$ on the set of symbols $\{1, 2, 3, \dots, 2q\}$ with holes h_x ($1 \leq x \leq q$) of size 2 such that $y \in h_x$ iff $y = 2x$ or $y = 2x - 1$. (This is possible since $2q \geq 6$.) For each $y \in h_x$, let $K_{(z,w)}(2m, 2n)$ be the complete bipartite graph with partition $\{A_z, B_w\}$ iff $z \circ w = y$. Note that $K_{(z,w)}(2m, 2n) \neq K_{(w,z)}(2m, 2n)$ even though $z \circ w = w \circ z$. Thus each $y \in h_x$ will be associated with $2q - 2$ distinct $K(2m, 2n)$.

Recall that every $K(2m, 2n)$ is completely resolvable into n almost parallel classes of m 4-cycles each (the even-even case). For $y \in h_x$, form the union of $\pi_{(x,1)}$ with one APC from each of the associated $K(2m, 2n)$ to form one complete APC containing $2m + m(2q - 2) = 2m + 2mq - 2m = 2mq$ cycles for $K(4mq + 1, 4nq + 1)$. Repeat, using $\pi_{(x,2)}$ through $\pi_{(x,n)}$ and a different APC from each of the associated $K(2m, 2n)$ for each repetition. Since each hole contains 2 elements, this can again be repeated using $\pi_{(x,n+1)}$ through $\pi_{(x,2n)}$ for a total of $2n$ complete APCs of $K(4mq + 1, 4nq + 1)$. Running over all q holes will yield $2nq$ APCs of $2mq$ cycles each, as required. Finally, $\bigcup_{P=1}^q \pi_{(x,P)}$ forms a complete partial parallel class with mq 4-cycles, also as required.

Corollary 9.3.2. There exists an ARMP of $K(12q + 1, 12q + 1)$ for all q except possibly 2.

Proof. We have already proven the case for $q = 1$ (Lemma 6.1). Plus, we have just shown in Theorem 9.3.1 that if there exists an ARMP of $K(4m + 1, 4n + 1)$ for some $m, n \in \mathbb{N}$, then there exists an ARMP of $K(4mq + 1, 4nq + 1)$ for all $q \geq 3$. The almost resolvable maximum packing

of $K(13, 13)$ in Lemma 6.1 proves that there exists an ARMP of $K(4m + 1, 4n + 1)$ for $m, n = 3$.

It follows that there exists an ARMP of $K(12q + 1, 12q + 1)$ for all $q \geq 3$.

9.4 $K(12m + 3, 12m + 3)$

Theorem 9.4. There exists an almost resolvable maximum packing of $K(12m + 3, 12m + 3)$ for all m except possibly $m = 2$.

Proof: We have already proven the case for $m = 1$ in Lemma 7.1. However a different

construction of the ARMP of $K(15, 15)$ will be useful for this proof and is provided below in

Figure 9.4.1:

	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}	b_{14}	b_{15}
a_1	7	1	2	6	3	6	7	L	3	5	1	2	5	4	4
a_2	1	7	2	3	1	4	3	7	L	4	6	2	6	5	5
a_3	5	2	7	3	4	2	3	4	7	L	5	1	1	6	6
a_4	3	6	3	7	4	5	2	4	5	7	L	6	2	1	1
a_5	6	4	1	4	7	5	1	3	5	6	7	L	3	2	2
a_6	6	1	5	2	5	7	L	2	4	6	1	7	4	3	3
a_7	L	7	4	6	3	6	P	7	3	P	4	1	1	5	5
a_8	1	L	7	5	1	4	2	P	7	4	P	5	2	6	6
a_9	5	2	L	7	6	2	6	3	P	7	5	P	3	1	1
a_{10}	3	6	3	L	7	1	P	1	4	P	7	6	4	2	2
a_{11}	2	4	1	4	L	7	1	P	2	5	P	7	5	3	3
a_{12}	7	3	5	2	5	L	7	2	P	3	6	P	6	4	4
a_{13}	2	3	4	5	6	1	6	1	2	3	4	5	L	L	L
a_{14}	4	5	6	1	2	3	5	6	1	2	3	4	L	7	7
a_{15}	4	5	6	1	2	3	5	6	1	2	3	4	L	7	7

Figure 9.4.1: Alternative ARMP of $K(15, 15)$

Notice that although the leave in this construction (shown in cells shaded gray) is not in the standard position adopted for other proofs in this paper, it nevertheless meets the vertex degree requirements for a minimum leave of a complete bipartite graph of its type. Also observe that the partial parallel class (shown in cells shaded green) is totally confined to vertices a_7 through

a_{12} and b_7 through b_{12} , and the almost parallel class 7 contains the cycle $(a_{14}, b_{14}, a_{15}, b_{15})$ and hits neither vertex a_{13} nor b_{13} .

We now examine the case for $m \geq 3$. For $1 \leq x \leq m$, let $A_{2x-1} = \{a_{x,1}, a_{x,2}, a_{x,3}, a_{x,4}, a_{x,5}, a_{x,6}\}$ and let $A_{2x} = \{a_{x,7}, a_{x,8}, a_{x,9}, a_{x,10}, a_{x,11}, a_{x,12}\}$. Similarly, let $B_{2x-1} = \{b_{x,1}, b_{x,2}, b_{x,3}, b_{x,4}, b_{x,5}, b_{x,6}\}$ and let $B_{2x} = \{b_{x,7}, b_{x,8}, b_{x,9}, b_{x,10}, b_{x,11}, b_{x,12}\}$. Then define $K_x(15, 15)$ to be the complete bipartite graph with parts $(A_{2x} \cup A_{2x-1} \cup \{a_{13}, a_{14}, a_{15}\})$ and $(B_{2x} \cup B_{2x-1} \cup \{b_{13}, b_{14}, b_{15}\})$. Then we may define $K(12m + 3, 12m + 3)$ as the complete bipartite graph with parts $\bigcup_{x=1}^m A_{2x} \cup A_{2x-1} \cup \{a_{13}, a_{14}, a_{15}\}$ and $\bigcup_{x=1}^m B_{2x} \cup B_{2x-1} \cup \{b_{13}, b_{14}, b_{15}\}$. See the diagram in figure 9.4.2 on the following page.

$K(12m+3, 12m+3)$

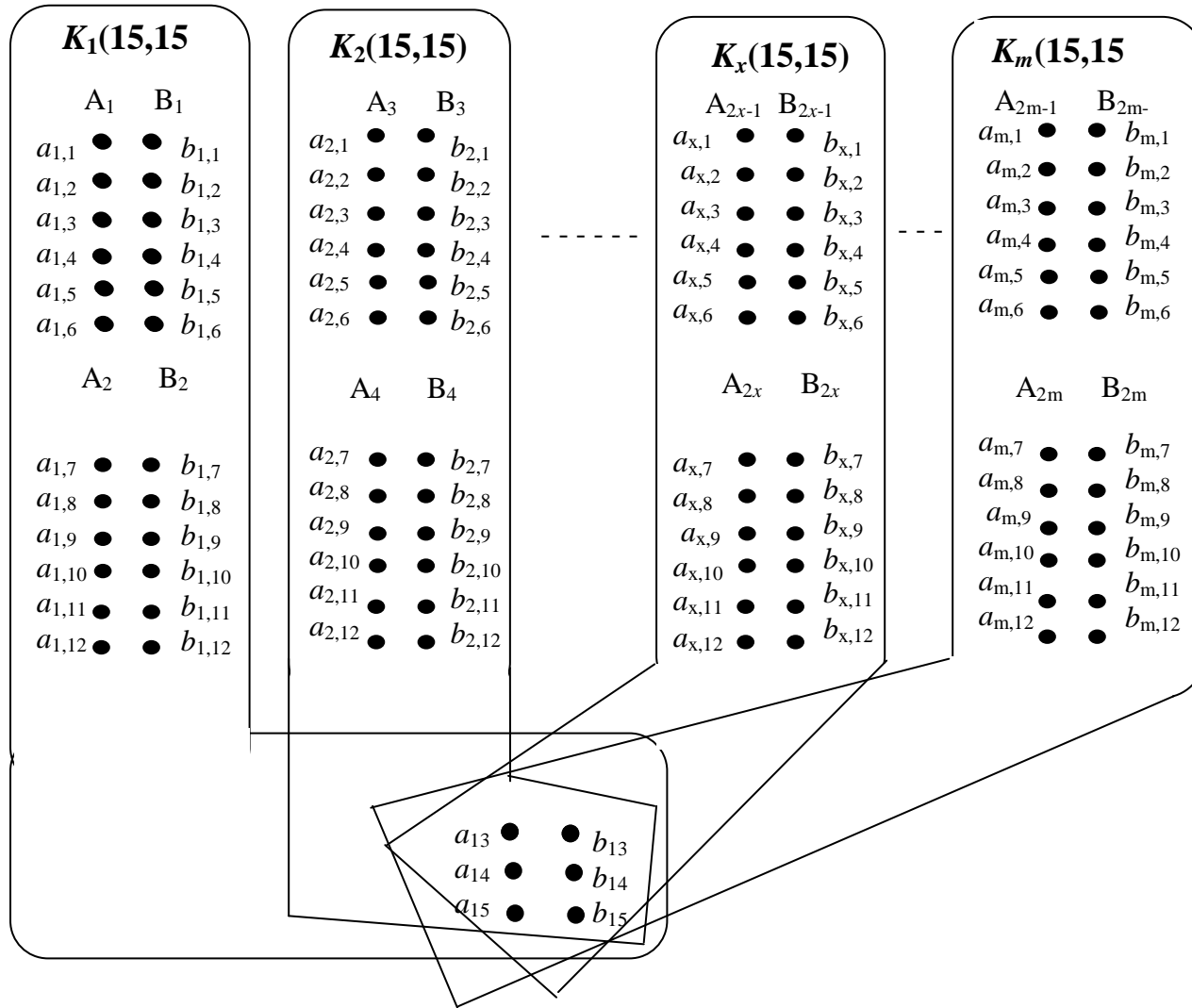


Figure 9.4.2: $K(12m + 3, 12m + 3)$

Place a copy of the ARMP of $K(15, 15)$ given in Figure 9.4.1 on each $K_x(15, 15)$ and denote the almost parallel classes for each copy as $\pi_{(x,1)}, \pi_{(x,2)}, \pi_{(x,3)} \dots, \pi_{(x,7)}$ and the partial parallel class for each copy as $\pi_{(x,P)}$.

Let (Q, \circ) be a commutative quasigroup of order $2m$ on the set of symbols $\{1, 2, 3, \dots, 2m\}$ and having holes h_x ($1 \leq x \leq m$) of size 2 such that $y \in h_x$ iff $y = 2x$ or $y = 2x - 1$. (This is possible since $2m \geq 6$.) For each $y \in h_x$, let $K_{(z,w)}(6, 6)$ be the complete bipartite graph with partition $\{A_z, B_w\}$ iff $z \circ w = y$. Note that $K_{(z,w)}(6,6) \neq K_{(w,z)}(6, 6)$ even though $z \circ w = w \circ z$. Thus each $y \in h_x$ will be associated with $2(m - 1) = 2m - 2$ distinct $K(6, 6)$.

Recall that every $K(6, 6)$ is completely resolvable into 3 almost parallel classes of three 4-cycles each (the even-even case). For $y \in h_x$, form the union of $\pi_{(x,1)}$ with one APC from each of the associated $K(6, 6)$ to form one complete APC containing $7 + 3(2m - 2) = 6m + 1$ cycles for $K(12m + 3, 12m + 3)$. Repeat, using $\pi_{(x,2)}$ and one of the two remaining APC from each of the associated $K(6, 6)$ and repeat a third time using $\pi_{(x,3)}$ and the last unused APC from each associated $K(6, 6)$. Since each hole contains 2 elements, this can be repeated again using $\pi_{(x,4)}, \pi_{(x,5)}$ and $\pi_{(x,6)}$ for a total of 6 complete APCs of $K(12m + 3, 12m + 3)$. Running over all m holes will yield $6m$ APCs of $6m + 1$ cycles each. In addition, $\bigcup_{P=1}^m \pi_{(x,P)}$ forms a complete partial parallel class with $3m$ 4-cycles, as required. One APC remains to be constructed.

Recall that in each $K_x(15, 15)$, within the set of shared vertices the almost parallel class 7 hits neither a_{13} nor b_{13} and does contain the cycle $(a_{14}, b_{14}, a_{15}, b_{15})$. Therefore, we may form the union of $\pi_{(1,7)}$ with $\pi_{(x,7)} \setminus (a_{14}, b_{14}, a_{15}, b_{15})$ for $2 \leq x \leq m$. This yields $7 + 6(m - 1) = 6m + 1$ disjoint cycles, as required, for completion of the final almost parallel class.

9.5 $K(4mq + 3, 4nq + 3)$

Theorem 9.5. For some m, n , with $m \leq n$, let $K(4m+3, 4n+3)$ be the complete bipartite graph with parts $A = \{a_1, a_2, a_3, \dots, a_{4m+1}, a_{4m+2}, a_{4m+3}\}$ and $B = \{b_1, b_2, b_3, \dots, b_{4n+1}, b_{4n+2}, b_{4n+3}\}$. If there exists an almost resolvable maximum packing of $K(4m + 3, 4n + 3)$ with an almost parallel class that, with leave in the standard position adopted for proofs in this paper, (1) misses both a_{4m+1} and b_{4m+1} and (2) contains the cycle $(a_{4m+2}, b_u, a_{4m+3}, b_v)$ for some u, v such that $4n + 3 \geq v > u > 4m + 1$, then there exists an almost resolvable maximum packing of $K(4mq + 3, 4nq + 3)$ for all $q \geq 3$.

Proof: This proof generalizes the construction given in the proof for $K(12m + 3, 12m + 3)$.

Assume the premises of the hypothesis are true. For ease of notation, rename the vertex b_u as b_{4m+2} and rename the vertex b_v as b_{4m+3} . Then define $K(4mq + 3, 4nq + 3)$ as the complete bipartite graph with parts $\bigcup_{x=1}^q A_{2x} \cup A_{2x-1} \cup \{a_{4m+1}, a_{4m+2}, a_{4m+3}\}$ and $\bigcup_{x=1}^q B_{2x} \cup B_{2x-1} \cup \{b_{4n+1}, b_{4n+2}, b_{4n+3}\}$ as shown in Figure 9.5 on following page. Place a copy of the ARMP described in the hypothesis on each $K_x(4m + 3, 4n + 3)$. Denote the almost parallel classes for each copy as $\pi_{(x,1)}, \pi_{(x,2)}, \pi_{(x,3)} \dots, \pi_{(x,2n+1)}$ and the partial parallel class for each copy as $\pi_{(x,P)}$. By relabeling the almost parallel classes, we may assume that the APC which meets conditions (1) and (2) of the hypothesis is $\pi_{(x,2n+1)}$.

$K(4mq+3, 4nq+3)$

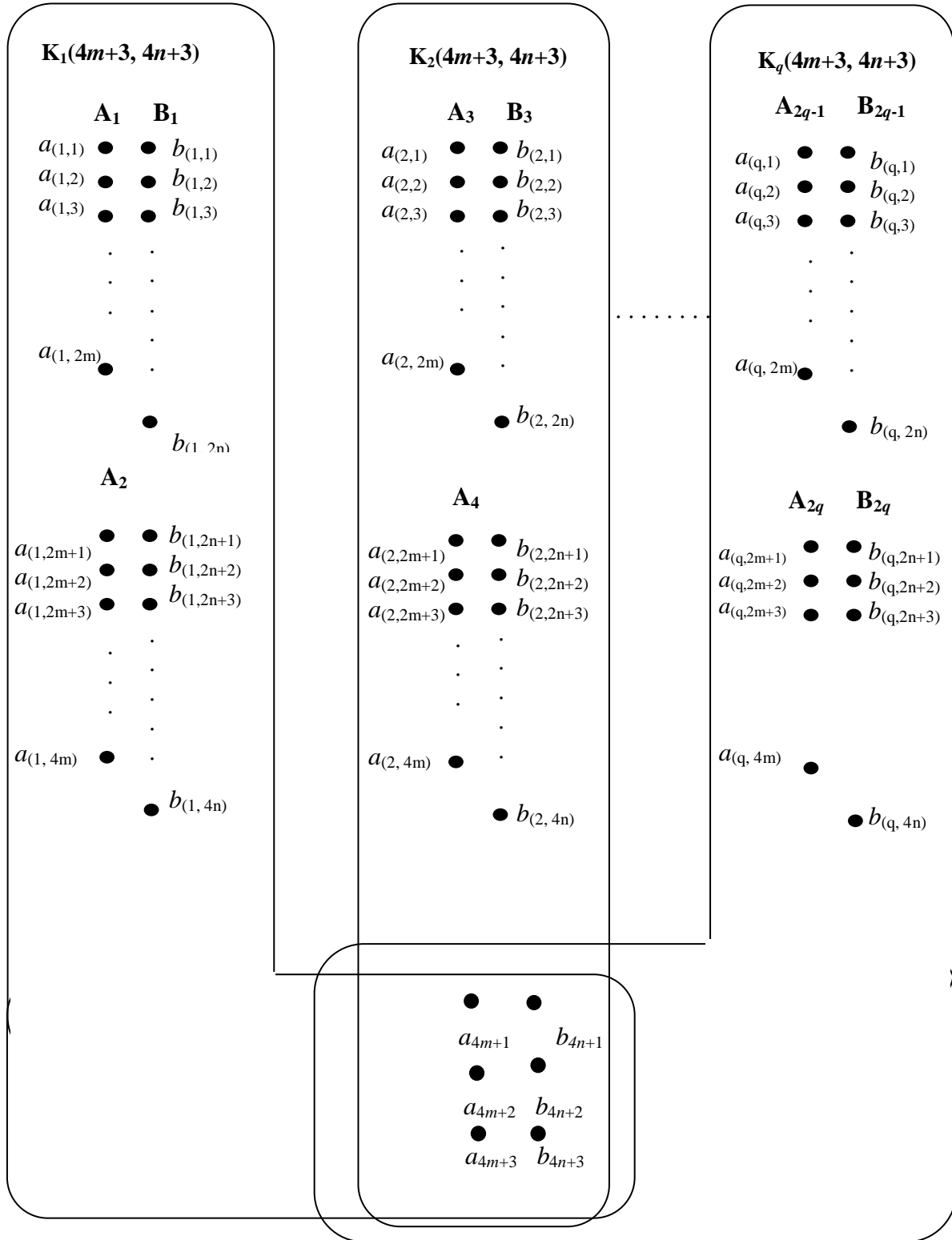


Figure 9.5: $K(4mq + 3, 4nq + 3)$

Note that for an almost resolvable maximum packing of $K(4mq+3, 4nq+3)$ with 4-cycles, we will need $2nq+1$ APCs with $2mq+1$ cycles each and 1 PPC with mq cycles.

Let (Q, \circ) be a commutative quasigroup of order $2q$ on the set of symbols $\{1, 2, 3, \dots, 2q\}$ with holes h_x of size 2 ($1 \leq x \leq q$) such that $y \in h_x$ iff $y = 2x$ or $y = 2x - 1$. (This is possible since $2q \geq 6$.) For each $y \in h_x$, let $K_{(z,w)}(2m, 2n)$ be the complete bipartite graph with partition $\{A_z, B_w\}$ iff $z \circ w = y$. Note that $K_{(z,w)}(2m, 2n) \neq K_{(w,z)}(2m, 2n)$ even though $z \circ w = w \circ z$. Thus each $y \in h_x$ will be associated with $2q - 2$ distinct $K(2m, 2n)$.

Recall that every $K(2m, 2n)$ is completely resolvable into n almost parallel classes of m 4-cycles each (the even-even case). For $y \in h_x$, form the union of $\pi_{(x,1)}$ with one APC from each of the associated $K(2m, 2n)$ to form one complete APC containing $2m+1 + m(2q - 2) = 2m + 1 + 2mq - 2m = 2mq+1$ cycles for $K(4mq+1, 4nq+1)$. Repeat, using $\pi_{(x,2)}$ through $\pi_{(x,n)}$ and a different APC from each of the associated $K(2m, 2n)$ for each repetition. Since each hole contains 2 elements, this can again be repeated using $\pi_{(x,n+1)}$ through $\pi_{(x,2n)}$ for a total of $2n$ complete APCs of $K(4mq+1, 4nq+1)$. Running over all q holes will yield $2nq$ APCs of $2mq+1$ cycles each. In addition, $\bigcup_{P=1}^q \pi_{(x,P)}$ forms a complete partial parallel class with mq 4-cycles, as required. One APC remains to be constructed.

Recall that for each x , we assume that the APC $\pi_{(x,2n+1)}$ meets conditions (1) and (2) of the hypothesis. Consequently, we may form the union of $\pi_{(1,2n+1)}$ with $\pi_{(x,2n+1)} \setminus (a_{4m+2}, b_{4n+2}, a_{4m+3}, b_{4n+3})$ for $2 \leq x \leq q$, yielding $2m + 1 + 2m(q - 1) = 2mq + 1$ disjoint cycles, as required for completion of the final almost parallel class.

Chapter 10

Summary

The results of this paper can be summarized by stating that an almost resolvable maximum packing with 4-cycles exists for the complete bipartite graphs with specified parameters as given in the following Table 10.1:

Complete Bipartite Graph	Spectrum
$K(2m, 2n)$	all $m, n \in \mathbb{N}$
$K(2m, 2n + 1)$	all $m, n \in \mathbb{N}$
$K(8m + 1, 8m + 1)$ and $K(8m + 3, 8m + 3)$	all $m \in \mathbb{N}$ except possibly 2
$K(8m + 5, 8m + 5)$ and $K(8m + 7, 8m + 7)$	all $m \in \mathbb{N}$ except possibly 2 and 8
$K(m, 2n + 1)$	$m \in \{3, 5, 7, 9, 11\}$; all $n \in \mathbb{N}$ except for $K(5, 5)$, for which no ARMP of 4-cycles exists
$K(2m + 1, 2t + 1)$	all $t \geq (n + m)$ whenever there exists an ARMP of $K(2m + 1, 2n + 1)$ for some $m, n \in \mathbb{N}$
$K(4mq + 1, 4nq + 1)$	all $q \geq 3$ whenever there exists an ARMP of $K(4m + 1, 4n + 1)$ for some $m, n \in \mathbb{N}$
$K(12m + 1, 12m + 1)$	all m except possibly 2
$K(12m + 3, 12m + 3)$	all m except possibly 2
$K(4mq + 3, 4nq + 3)$	all $q \geq 3$ whenever there exists an ARMP of $K(4m + 1, 4n + 1)$ for some $m, n \in \mathbb{N}$ under the conditions of Theorem 9.5

Table 10.1: Summary of Results

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