Security, (F,I)-security, and Ultra-security in Graphs

by

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Abstract

Let $G = (V, E)$ be a graph and $S \subseteq V$. The notion of security in graphs was first presented by Brigham et al [3]. A set S is secure if every attack on S is defendable. The cardinality of a smallest secure set of G is the security number of G.

We give several new definitions of security. We show that some of these new definitions are equivalent to the definition given by Brigham et al, while others are not. In these new situations, we find necessary and sufficient conditions for security. Various Hall-type theorems are used in these proofs. We also define analogues of the security number and find them for various classes of graphs.

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Chapter 1

Introduction

All graphs in this paper are finite and simple. For a graph $G = (V, E)$ and $v \in V$ we will follow convention by letting $N(v) = \{u \mid uv \in E\}$, and $N[v] = \{v\} \cup N(v)$. For $S \subseteq V$, $N(S) = \cup_{v \in S} N(v)$ and $N[S] = N(S) \cup S$. If not clear by the context, the graph G may be indicated by the subscript: N_G. The foundation for the study of security in graphs was laid by Brigham, Dutton, and Hedetniemi in 2007 [3]. Security is a variation of the topic of alliances in graphs, which is a just a few years older [13]. Given a graph $G = (V, E)$, they define an *attack* on $S = \{s_1, s_2, ..., s_k\} \subseteq V$ to be a collection of pairwise disjoint sets $A = \{A_1, A_2, ..., A_k\}$ for which $A_i \subseteq N[s_i] - S$, $1 \leq i \leq k$. A *defense* of S is a collection of pairwise disjoint sets $D = \{D_1, D_2, ..., D_k\}$ such that $D_i \subseteq N[s_i] \cap S$, $1 \leq i \leq k$. An attack is *defendable* if there is a defense D such that $|D_i| \geq |A_i|$ for $1 \leq i \leq k$. In this setting, each vertex in $N[S] - S$ can attack only one of its neighbors in S, and each vertex in S can defend itself or one of its neighbors in S. The set S is defined to be *secure* if for every attack A on S there exists a defense of D of S such that $|D_i| \geq |A_i|$ for $1 \leq i \leq k$. Brigham et al show that a set S is secure if and only if for every $X \subseteq S$, $|N[X] \cap S| \geq |N[X] - S|$. They also define the security number of G, denoted $s(G)$, to be the cardinality of a smallest secure set in G. Some results and unsolved questions about the security number can be found in [3, 4, 6, 12]. In the subsequent chapters we explore varying definitions of security, and their relationship to the original definition, and to each other.

Chapter 2

Integer and Fractional Security

We consider the cases in which a vertex may send out attack or defense to as many appropriate vertices as it wants, so long as the total amount of attack or defense from that vertex sums to at most one. This chapter is joint work with Dr. Garth Isaak and Dr. Peter Johnson and appears in [10].

2.1 Definitons

Let $G = (V, E)$ be a graph and $S \subseteq V$. An *attack* on S is a function $A{:}(V - S) \times S \to [0, 1]$ such that $A(u, v) = 0$ if $uv \notin E$ and for $u \in V - S, \sum_{v \in N(u) \cap S} A(u, v) \leq 1$. A *defense* of S is a function $D{:}S \times S \to [0, 1]$ such that $D(u, v) = 0$ if $u \neq v$ and $uv \notin E$ and for $u \in S, \sum_{v \in N[u] \cap S} D(u, v) \leq 1$. Suppose that A is an attack on S and D is a defense of S. For $u \in S$ let $D^*(u) = \sum_{v \in N[u] \cap S} D(v, u)$ and $A^*(u) = \sum_{v \in N(u) - S} A(v, u)$. An attack A is *defendable* if there exists a defense D such that for each $u \in S$, $D^*(u) \geq A^*(u)$. The set S is *secure* if every attack on S is defendable.

The definition of attack and defense given by Brigham, et al in [3] corresponds to the case $A(u, v) \in \{0, 1\}$ for all $u \in (V - S), v \in S$ and $D(u, v) \in \{0, 1\}$ for all $u, v \in S$. We will refer to these as *integer attack* and *integer defense* respectively. This naturally leads to four scenarios:

a) an integer attack against an integer defense, (I,I);

b) an integer attack against a defense, (I,F);

c) an attack against an integer defense, (F,I);

d) an attack against a defense, (F,F).

(The letter F has been chosen to suggest the word "fractional".) For each situation there is a corresponding definition of security. For instance, a set S is (I,F)-secure if every integer attack is defendable. We now explore relationships among the different kinds of security.

2.2 A New Proof of Theorem BDH

Lemma. *If $S \subseteq V$ is secure in G in any of the four senses, then for each set $X \subseteq S$, $|N[X] \cap S| \geq |N[X] - S|$.*

Proof. Suppose that, for some $X \subseteq S$, $|N[X] \cap S| < |N[X] - S|$. Make an integer attack A on S by letting each vertex in $|N[X] - S|$ attack any of its neighbors in X with its whole unit of attack. Let every other vertex in $N[S] - S$ attack one of its neighbors in S, or none; it does not matter. Then for any defense D of S,

$$\begin{aligned}\sum_{x \in X} D^*(x) &= \sum_{x \in X} \sum_{u \in N[x] \cap S} D(u,x) = \sum_{u \in N[X] \cap S} \sum_{x \in X} D(u,x) \\ &\leq \sum_{u \in N[X] \cap S} 1 = |N[X] \cap S| < |N[X] - S| \\ &= \sum_{x \in X} A^*(x).\end{aligned}$$

Therefore, for any such D, there must be some $x \in X$ such that $D^*(x) < A^*(x)$. Thus A is not defendable, and so S is not secure. $\square$

Brigham et al [3] gave a necessary and sufficient condition for a set to be (I,I)-secure. We shall refer to this as the BDH Theorem. As an aside, we give a short proof of this result using Hall's Theorem:

Theorem HRHV ([8, 14, 9]). *Suppose $P_1, ..., P_n$ are sets and $k_1, ..., k_n$ are non-negative integers. There exist pairwise disjoint sets $D_1, ..., D_n$ such that $D_i \subseteq P_i$ and $|D_i| = k_i$ for*

$1 \leq i \leq n$ if, and only if, for each $J \subseteq \{1, ..., n\}$, $|\cup_{j \in J} P_j| \geq \sum_{j \in J} k_j$.

Theorem BDH. *A set $S \subseteq V$ is (I,I)-secure if and only if $|N[X] \cap S| \geq |N[X] - S|$ for all $X \subseteq S$.*

Proof. The necessity of the condition follows from the Lemma. Let $G = (V, E)$ be a graph. Let $S = \{s_1, s_2, ..., s_n\} \subseteq V$ be a set such that $|N[X] \cap S| \geq |N[X] - S|$ for all $X \subseteq S$. Let $A = \{A_1, ..., A_n\}$ be an integer attack on S, by the original definition in [3]. Define $P_i = N[s_i] \cap S$ for $1 \leq i \leq n$. Thus P_i is the set of potential defenders of s_i. Let $k_i = |A_i|$ for $1 \leq i \leq n$; k_i is the number of attackers of s_i. For any $J \subseteq \{1, 2, ..., n\}$, let $X_J = \{s_j \mid j \in J\}$. Then we have $\sum_{j \in J} k_j = \sum_{j \in J} |A_j| \leq |N[X_J] - S| \leq |N[X_J] \cap S| = \left|\bigcup_{j \in J} P_j\right|$. By Theorem HRHV we can find $D_i \subseteq P_i$ for $1 \leq i \leq n$ such that the D_i are pairwise disjoint and $|D_i| = k_i$. Thus $D = \{D_1, ..., D_n\}$ is an integer defense that thwarts the attack, and S is secure. $\square$

As remarked in [3], Theorem BDH shows that the problem of deciding whether or not $S \subseteq V$ is (I,I)-secure is in co-NP: S is not (I,I)-secure if and only if there is a certificate proving that it is not, a set $X \subseteq S$ such that $|N[X] \cap S| < |N[X] - S|$. Reportedly, Dutton [5] has recently shown that the problem is co-NP-complete.

2.3 The Equivalence of (I,I)-security, (I,F)-security, and (F,F)-security

In order to prove the next result, we need the following analogue of Hall's Theorem due to Bollobás and Varopoulos [1].

Theorem BV. *Suppose that (X, μ) is an atomless measure space, $M_1, ..., M_n$ are subsets of X of finite measure, and $r_1, ..., r_n$ are non-negative real numbers. There exist pairwise disjoint sets $C_1, ..., C_n$ such that $C_i \subseteq M_i$ and $\mu(C_i) = r_i, 1 \leq i \leq n$, if, and only if, for each*

$J \subseteq \{1, ..., n\}$ we have $\mu(\bigcup_{j \in J} M_j) \geq \sum_{j \in J} r_j$.

Theorem. *Let $G = (V, E)$ be a graph and $S \subseteq V$. Then (a) S is (I,I)-secure $\Leftrightarrow$ (b) S is (I,F)-secure $\Leftrightarrow$ (c) S is (F,F)-secure.*

Proof. We will show (c) $\Rightarrow$ (b) $\Rightarrow$ (a) $\Rightarrow$ (c) . If S is (F,F)-secure then all attacks on S are defendable, so all integer attacks on S are defendable. Thus S is (I,F)-secure. Now suppose S is (I,F)-secure. By the Lemma, for any $X \subseteq S$ it must be that $|N[X] \cap S| \geq |N[X] - S|$. Therefore, by the BDH Theorem, S is also (I,I)-secure.

Let S be (I,I)-secure. Then $|N[X] \cap S| \geq |N[X] - S|$ for all $X \subseteq S$. Let A:$(N(S) - S) \times S \rightarrow$ [0,1] be an attack on S. (We restrict the domain of the attackers to $N(S) - S$ because all vertices in $V - N(S)$ will have no attack.) Recall that for $v \in S$, $A^*(v) = \sum_{u \in N(v) - S} A(u, v)$. Let $\{I(v) | v \in S\}$ be an indexed family of pairwise disjoint unit intervals in the real numbers. These are the defense reservoirs of the vertices of S. For $v \in S$ let $M(v) = \bigcup_{w \in N[v] \cap S} I(w)$. This is the total defense available to v. So we have another indexed family $\{M(v) | v \in S\}$. Let λ denote the Lebesgue measure. To achieve a successful fractional defense against the attack A, it suffices to find an indexed family $\{C(v) | v \in S\}$ of pairwise disjoint Lebesgue measurable sets such that for all $v \in S$, $C(v) \subseteq M(v)$ and $\lambda(C(v)) = A^*(v)$. If such a family is found, define a defense D:$S \times S \rightarrow$[0,1] by $D(w, v) = \lambda(I(w) \cap C(v))$.

Then for v $\in S$ we would have

$$\begin{aligned}
D^*(v) &= \sum_{w \in S} D(w, v) \\
&= \sum_{w \in S} \lambda(I(w) \cap C(v)) \\
&= \lambda(\bigcup_{w \in S} (I(w) \cap C(v))) \qquad \text{[the intervals } I(w), w \in S \text{ are pairwise disjoint]} \\
&\geq \ \lambda(M(v) \cap C(v)) = \lambda(C(v)) = A^*(v).
\end{aligned}$$

Also for each $w \in S$,

$$\sum_{v \in S} D(w,v) = \sum_{v \in S} \lambda(I(w) \cap C(v))$$
$$= \lambda(\bigcup_{v \in S} (I(w) \cap C(v)) \leq \lambda(I(w)) = 1.$$

So D is a defense of S, and it defends against A.

Now we will show that $|N[X] \cap S| \geq |N[X] - S|$ for all $X \subseteq S$ implies the existence of such a family $\{C(v) \mid v \in S\}$. By Theorem BV, it is sufficient to show that for all $X \subseteq S$, $\sum_{v \in X} A^*(v) \leq \lambda(\bigcup_{v \in X} M(v))$. Suppose $X \subseteq S$. We have

$$\begin{aligned}
\lambda(\bigcup_{v \in X} M(v)) &= \lambda(\bigcup_{v \in X} (\bigcup_{w \in N[v] \cap S} I(w))) \\
&= \lambda(\bigcup_{w \in N[X] \cap S} I(w)) = \sum_{w \in N[X] \cap S} \lambda(I(w)) \\
&= \sum_{w \in N[X] \cap S} 1 = |N[X] \cap S| \geq |N[X] - S| \\
&\geq \sum_{u \in N[X]-S} \sum_{v \in S} A(u,v) \geq \sum_{u \in N[X]-S} \sum_{v \in X} A(u,v) \\
&= \sum_{v \in X} \sum_{u \in N[X]-S} A(u,v) = \sum_{v \in X} A^*(v). \quad \square
\end{aligned}$$

This leaves the question of how (F,I)-security relates to (I,I)-security. Clearly, (F,I)-security implies (I,I)-security, but we will show the converse does not hold.

Example. In [3] it is seen that any $\lceil \frac{n}{2} \rceil$ vertices of K_n form an (I,I)-secure set; however, $n-1$ vertices are needed to be (F,I)-secure. Let $S = \{s_1, ..., s_k\} \subseteq V(K_n)$ such that $k \leq n-2$. Then $|V(K_n) - S| \geq 2$. Let $v_1, v_2 \in V(K_n) - S$. Let $A(v_1, s_i) = \frac{1}{|S|}$ for $1 \leq i \leq k$ and $A(v_2, s_1) = 1$ and $A(v_2, s_i) = 0$ for $2 \leq i \leq k$. A successful integer defense of this attack requires $|S| + 1$ defenders, so S is not (F,I)-defendable. So, in order for S to be (F,I)-secure,

$|S| \geq n-1$. Any set S such that $|S| = n-1$ is (F,I)-secure.

So an (F,I)-secure set has a greater security than a set that is only (I,I)-secure. We give a necessary and sufficient condition, in the spirit of Theorem BDH, for (F,I)-security in Chapter 3. It may be possible to obtain results similar to Theorem BDH and the main theorem of this chapter when attack and defense capabilities are extended to general values, and are not necessarily constant from vertex to vertex.

Chapter 3

(F,I)-security

Let $G = (V, E)$ be a graph. We now give a necessary and sufficient condition for a set $S \subseteq V$ to be (F,I)-secure. The *(F,I)-security number* of G, $s_{(\mathrm{F,I})}(G)$, is the cardinality of a smallest (F,I)-secure set. The values of the (F,I)-security number for various classes of graphs is determined.

3.1 Bipartite Graph Lemma

We develop a lemma about bipartite graphs for use in the proof of the Main Theorem. This lemma is also used in determing the (F,I)-security number of various families of graphs, including complete multipartite graphs. In the setting of (F,I)-security, the vertices that are attacking can fractionalize, but defending vertices cannot. Thus, in order for S to be (F,I)-secure, for any attack A there must be an integer defense D such that $D^*(s) \geq \lceil A^*(s) \rceil$ for all $s \in S$. In this case, we may as well suppose that every $s \in S$ contributes 1 to the sum $\sum_{s \in S} D^*(s)$, and so we have $|S| = \sum_{s \in S} D^*(s) \geq \sum_{s \in S} \lceil A^*(s) \rceil$. Given an attack A on a set S in an (F,I)-security setting, the *total effective attack* is $\sum_{s \in S} \lceil A^*(s) \rceil$.

Bipartite Graph Lemma. *Let G be a connected bipartite graph with bipartition (X, Y). Among (F,I)-attacks on Y, the maximum total effective attack possible is $|X| + |Y| - 1$.*

3.1.1 A Proof Using the Max-Flow Min Cut Theorem

Lemma 1. *Let G be a complete bipartite graph with bipartition (X, Y). Among (F,I)-attacks on Y, the maximum total effective attack possible is $|X| + |Y| - 1$.*

Proof. Let $X = \{x_1, ..., x_m\}$ and $Y = \{y_1, ..., y_n\}$. Then we are looking for the largest possible value of $\sum_1^n \lceil a_i \rceil$ where $a_1, ..., a_n$ are non-negative real numbers that sum to m. We have

$$\begin{aligned} \lceil a_i \rceil &< a_i + 1,\ \ i = 1, ..., n \\ \Rightarrow \sum_1^n \lceil a_i \rceil &< \sum_1^n (a_i + 1) = m + n \\ \Rightarrow \sum_1^n \lceil a_i \rceil &\leq m + n - 1. \end{aligned}$$

To achieve the bound in this inequality, let $A(x_1, y_j) = \frac{1}{n}$ for $1 \leq j \leq n$; and let $A(x_i, y_1) = 1$ for $2 \leq i \leq m$. $\square$

Corollary 1. *Let G be a bipartite graph with bipartition (X, Y). For any (F,I)-attack on Y, the total effective attack is less than or equal to $|X| + |Y| - 1$.*

The proof of Lemma 1 gives a solution to a problem proposed in [11]. Next we use the Max-Flow Min-Cut Theorem to show that this upper bound can be achieved when G is a tree.

Lemma 2. *Let $G = (V, E)$ be a tree with bipartition (X, Y). Define $f{:}V \to [0, \infty)$ by $f(v) = 1$ for $v \in X$ and $f(v) = d_G(v) - \frac{|Y|-1}{|Y|}$ for $v \in Y$. Then there exists a function $wt{:}E \to [0, \infty)$ such that for all $v \in V$,*

$$f(v) = \sum_{\substack{e \in E \\ e \text{ incident to } v}} wt(e).$$

Proof. Create a digraph D from G by directing every edge from X to Y. Add a vertex s and for each $x \in X$, form an arc directed from s to x. Similarly, add a vertex t and for each $y \in Y$, form an arc directed from y to t. Let $\mathcal{A}[s, X]$ be the set of arcs with one end at s

and one end in X, $\mathcal{A}[Y,t]$ be the set of arcs with one end in Y and one at t, and $\mathcal{A}[X,Y]$ denote the arcs with one end in X and one end in Y. From this digraph, form a network by designating s the source and t the sink (see Figure 3.1). Define the capacity function c as follows: $c(a) = 1$ if $a \in \mathcal{A}[s,X]$, $c(a) = d_G(y) - \frac{|Y|-1}{|Y|}$ if $a \in \mathcal{A}[Y,t]$, and $c(a) = \infty$ otherwise. Note that $\sum_{a\in\mathcal{A}[Y,t]} c(a) = |X|$, because G is a tree.

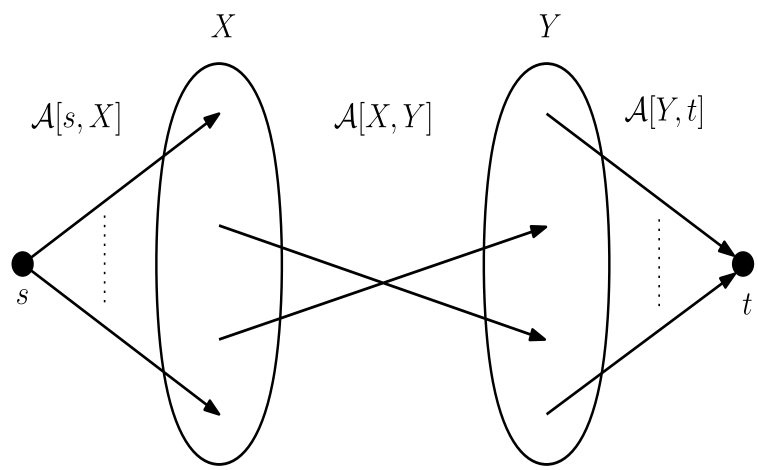

Figure 3.1: The network formed from the bipartite graph G.

If we can find a flow in this network of value $|X|$, then the desired function wt on $E(G)$ can be found by assigning $wt(e), e \in E(G)$, to be equal to the flow of the corresponding arc. To show we can find a flow of value $|X|$, we will show that the minimum cut has capacity $|X|$. The capacity of a minimum cut is clearly less than or equal to $|X|$, because we can form a cut by taking all the arcs of $\mathcal{A}[s,X]$ (or all the arcs of $\mathcal{A}[Y,t]$). Now we will show that every other cut has a capacity bigger than $|X|$.

Any cut including an arc of $\mathcal{A}[X,Y]$ will have infinite capacity. The only other cuts to check must include some, but not all, arcs of $\mathcal{A}[s,X]$ and some, but not all, arcs of $\mathcal{A}[Y,t]$. Let K be a cut that has at least one arc of $\mathcal{A}[s,X]$ and at least one arc of $\mathcal{A}[Y,t]$. Let $X_0 = \{x | \, x \in X \text{ and } (s,x) \notin \mathrm{K}\}$ and $Y_0 = \{y | \, y \in Y \text{ and } (y,t) \notin \mathrm{K}\}$. By assumption about K, $|X_0| \geq 1$ and $|Y_0| \geq 1$. Note that there are no arcs between X_0 and Y_0; otherwise K would not be a cut. The capacity of the cut K is $|X - X_0| + \sum_{y\in Y-Y_0} \left(d_G(y) - \frac{|Y|-1}{|Y|} \right)$.

Since each edge of G has one end in X and one end in Y, $\sum_{y\in Y-Y_0} d_G(y)$ is the number of edges of G with one end in $Y-Y_0$. G has $|X|+|Y|-1$ edges. Each edge has an end in $Y-Y_0$ or in Y_0. An edge with an end in Y_0 must have its other end in $X-X_0$, as noted above. Looking at the forest induced in G by $(X-X_0)\cup Y_0$, the most edges it can have is $|X-X_0|+|Y_0|-1$. Thus, $\sum_{y\in Y-Y_0} d_G(y) \geq (|X|+|Y|-1)-(|X-X_0|+|Y_0|-1) = |X|-|X-X_0|+|Y-Y_0|$. So we have

$$
\begin{aligned}
c(K) &= |X-X_0| + \sum_{y\in Y-Y_0}\left(d_G(y) - \frac{|Y|-1}{|Y|}\right)\\
&\geq |X-X_0| + |X| - |X-X_0| + |Y-Y_0| - \frac{|Y|-1}{|Y|}|Y-Y_0|\\
&= |X| + |Y-Y_0| - |Y-Y_0| + \frac{|Y-Y_0|}{|Y|}\\
&= |X| + \frac{|Y-Y_0|}{|Y|}\\
&> |X|. \quad \square
\end{aligned}
$$

Let $G=(V,E)$ be a connected bipartite graph with bipartition (X,Y). By Corollary 1, $|X|+|Y|-1$ is an upper bound for the total effective attack on Y. Also, G has a spanning tree, and thus can achieve the bound of $|X|+|Y|-1$ by Lemma 2. One attack that achieves this bound corresponds naturally to the function wt.

3.1.2 A Proof by Induction

Here we prove a more general result that implies the Bipartite Graph Lemma. Let $f{:}V\to[0,\infty)$. A function $wt{:}E\to[0,\infty)$ *represents* f if for all $v\in V$,

$$f(v) = \sum_{\substack{e\in E\\ \text{e incident to v}}} wt(e).$$

Lemma 3. *Let $G = (V, E)$ be a tree and $f : V \to [0, \infty)$. Let (X, Y) be a bipartition of G. If for all S such that $S \subseteq X$ or $S \subseteq Y$, $f(S) = \sum_{s \in S} f(v) \leq f(N_G(S))$, then there exists a function $wt : E \to [0, \infty)$ that represents f.*

Proof. Let (*) be the condition that for all S such that $S \subseteq X$ or $S \subseteq Y$, $f(S) = \sum_{s \in S} f(v) \leq f(N_G(S))$, and assume (*) holds. We will proceed by induction on $|X| + |Y| \geq 2$. First let $|X| + |Y| = 2$. Then $V = \{u, v\}$ and $E = \{uv\}$. $f(v) \leq f(u)$ and $f(u) \leq f(v)$, so $f(v) = f(u)$. Define $wt(uv) = f(u)$. Now suppose that $|X| + |Y| > 2$. Let u be a leaf in G. Without loss of generality, $u \in X$. Let v be the neighbor of u. By (*), $f(u) \leq f(v)$. Define $\widetilde{f}$ on $G - u$ by $\widetilde{f}(v) = f(v) - f(u)$ and $\widetilde{f}(w) = f(w)$ for all other $w \in V(G - u)$. We will show that the tree $G - u$ and the function $\widetilde{f}$ satisfy (*). If $S \subseteq X - u$ and $v \notin N_G(S) = N_{G-u}(S)$, then $\widetilde{f}(S) = f(S) \leq f(N_G(S)) = \widetilde{f}(N_{G-u}(S))$. If $v \in N_G(S)$, then $\widetilde{f}(S) + f(u) = f(S \cup \{u\}) \leq f(N_G(S \cup \{u\})) = f(N_{G-u}(S)) = \widetilde{f}(N_{G-u}(S)) + f(u)$. So we get that $\widetilde{f}(S) \leq \widetilde{f}(N_{G-u}(S))$.

Now suppose $S \subseteq Y$. If $v \notin S$, then $\widetilde{f}(S) = f(S) \leq f(N_G(S)) = \widetilde{f}(N_{G-u}(S))$. If $v \in S$ then $\widetilde{f}(S) = f(S) - f(u) \leq f(N_G(S)) - f(u) = f(N_{G-u}(S)) = \widetilde{f}(N_{G-u}(S))$. So $G - u$ with $\widetilde{f}$ satisfies (*). So there is a $\widetilde{wt} : E(G - u) \to [0, \infty)$ representing $\widetilde{f}$ by the induction hypothesis. Extend $\widetilde{wt}$ to $E(G)$ by putting weight $f(u)$ on the edge uv. Call this extension wt. It is straight forward to see that wt represents f. $\square$

Lemma 4. *If G is a tree with bipartition (X, Y) then there exists an attack on Y by X with a total effective attack of $|X| + |Y| - 1$.*

Proof. Define $f(v) = 1$ for $v \in X$ and $f(v) = d_G(v) - \frac{y-1}{y}$ for $v \in Y$ where $|Y| = y$. We will show f can be represented by a function wt by applying Lemma 2. First suppose that $S \subseteq Y$. For any graph H let $c(H)$ denote the number of connected components of H. Let

$H(S)$ be the subgraph of G induced by $S \cup N_G(S)$. Note that $H(S)$ is a forest. Then

$$\begin{aligned}
f(S) &= \left(\sum_{v \in S} d_G(v) \right) - \frac{y-1}{y}|S| \\
&= |E(H(S))| - \frac{y-1}{y}|S| \\
&= |V(H(S))| - c(H(S)) - \frac{y-1}{y}|S| \\
&= |S| + |N_G(S)| - c(H(S)) - \frac{y-1}{y}|S| \\
&= |N_G(S))| + \frac{1}{y}|S| - c(H(S)) \\
&\leq |N_G(S))| + 1 - c(H(S)) \\
&\leq |N_G(S)| \\
&= f(N_G(S)).
\end{aligned}$$

Now suppose that $S \subseteq X$. We want to show the following inequality:

$$\begin{aligned}
f(S) &= |S| \\
&\leq f(N_G(S) \\
&= \left(\sum_{v \in N_G(S)} d_G(v) \right) - \frac{y-1}{y}|N_G(S)| \\
&= |E(H(N_G(S)))| - \frac{y-1}{y}|N_G(S)| \\
&= |N_G(S)| + |N_G(N_G(S))| - c(H(N_G(S))) - \frac{y-1}{y}|N_G(S)| \\
&= |N_G(N_G(S))| + \frac{1}{y}|N_G(S)| - c(H(N_G(S))).
\end{aligned}$$

If $N_G(S) = Y$, then $c(H(N_G(S))) = 1$ and $\frac{1}{y}|N_G(S)| = \frac{1}{y}|Y| = 1$. So we need $|S| \leq |N_G(N_G(S))|$. This holds because $S \subseteq N_G(N_G(S))$.

Otherwise, $|N_G(S)| < |Y|$ so $N_G(S) \subsetneq Y$. It suffices to show that $|S| + c(H(N_G(S))) \leq |N_G(N_G(S))|$. Let $u \in Y$ such that $u \notin N_G(S)$. We will see that every component of

$H(N_G(S))$ contains a vertex in $N_G(N_G(S)) - S$ that is not in any other component. If a component C of $H(N_G(S))$ did not have any vertices in $N_G(N_G(S)) - S$, then it would not be connected in G to u, a contradiction. So C must have a vertex in $N_G(N_G(S)) - S$, and it is not in any other component because components do not share vertices. So we have $|S| + c(H(N_G(S))) \leq |S| + |N_G(N_G(S)) - S| = |N_G(N_G(S))|$.

So we can find a function wt to represent f. Define an attack A by $A(u, v) = wt(uv)$ for every $u \in X$ and $v \in Y$. The total effective attack is then

$$\sum_{v \in Y} \lceil f(v) \rceil = \sum_{v \in Y} \left\lceil d_G(v) - \frac{y-1}{y} \right\rceil = \sum_{v \in Y} d_G(v) = |E(G)| = |X| + |Y| - 1. \quad \square$$

The Bipartite Graph Lemma follows from Lemma 1 and Lemma 4.

3.2 Main Theorem

Given a graph $G = (V, E)$ and a set $S \subseteq V$, the set of edges with one end in set $V - S$ and one end in S induces a graph whose components are connected bipartite graphs. By looking at the total effective attack possible from the attackers in each component, we can develop a necessary and sufficient condition for (F,I)-security.

Let $G = (V, E)$ be a graph and $S = \{s_1, s_2, ..., s_k\} \subseteq V$. For $X \subseteq S$, let G^X denote the subgraph of G whose vertex set is $X \cup (N[X] - S)$ and whose edge set is $E[X, N[X] - S]$, the set of edges of G with one end in X and the other in $N[X] - S$. Let $C_1, C_2, ..., C_t$ be the components of G^X. Let $X_i = V(C_i) \cap X$ for $1 \leq i \leq t$.

Each C_i is a connected bipartite graph with bipartition $(X_i, N_G[X_i] - S)$. So the maximum total effective attack from $N_G[X_i] - S$ to X_i is $|X_i| + |N_G[X_i] - S| - 1$. Since X_i, $1 \leq i \leq t$, are pairwise disjoint, the most attack that can be sent from $V - S$ to X is $\sum_1^t (|X_i| + |N_G[X_i] - S| - 1) = |X| + |N_G[X] - S| - c(G^X)$, where $c(G^X)$ denotes the number of components of G^X.

In order for S to be (F,I)-secure, it is thus necessary that for all $X \subseteq S$, $|N_G[X] \cap S| \geq |X| + |N_G[X] - S| - c(G^X)$. The Main Theorem states that this condition is also sufficient:

Main Theorem. *Let $G = (V, E)$ be a graph and $S \subseteq V$. Then S is (F,I)-secure if, and only if, $|N_G[X] \cap S| \geq |X| + |N_G(X) - S| - c(G^X)$ for all $X \subseteq S$.*

The use of Theorem HRHV to prove the sufficiency of this condition is similar to the use of Theorem HRHV in Chapter 2.2 and Chapter 4.1.

Theorem HRHV ([8, 14, 9]). *Suppose $P_1, ..., P_n$ are sets and $k_1, ..., k_n$ are non-negative integers. There exist pairwise disjoint sets $D_1, ..., D_n$ such that $D_i \subseteq P_i$ and $|D_i| = k_i$ for $1 \leq i \leq n$ if, and only if, for each $J \subseteq \{1, ..., n\}$, $|\cup_{j \in J} P_j| \geq \sum_{j \in J} k_j$.*

Proof of Main Theorem. Necessity has already been proven. Let $G = (V, E)$ be a graph and let $S = \{s_1, s_2, ..., s_n\} \subseteq V$ be such that for all $X \subseteq S$, $|N_G[X] \cap S| \geq |X| + |N_G(X) - S| - c(G^X)$. Let A be any attack on S, and $P_i = N[s_i] \cap S$ for $1 \leq i \leq n$. Let $k_i = \lceil A^*(s_i) \rceil$ for $1 \leq i \leq k$. Then P_i is the set of potential defenders of s_i, and k_i is the number of defenders of s_i needed for a successful defense of A. For any $J \subseteq \{1, ..., n\}$ let $X_J = \cup_{j \in J} \{s_j\}$. Then we have $\sum_{j \in J} k_j \leq |X_J| + |N_G(X_J) - S| - c(G^{X_J}) \leq |N_G[X_J] \cap S| = |\cup_{j \in J}(N_G[s_j] \cap S)| = |\cup_{j \in J} P_j|$. By Theorem HRHV we can find pairwise disjoint $D_i \subseteq P_i$, $1 \leq i \leq n$, such that $|D_i| = k_i$ for $1 \leq i \leq n$. Then $D = \{D_1, ..., D_n\}$ is a succesful defense of A. $\square$

3.3 (F,I)-security Numbers of Various Graphs

Recall that for any graph G, $s(G) \leq s_{(\mathrm{F,I})}(G)$. Given two graphs G and H, we denote their cartesian product by $G \square H$ and their join by $G \vee H$. The path on n vertices is denoted by P_n. Let $F_n = K_1 \vee P_n$ and $W_n = K_1 \vee C_n$. As observed in [3], it is clear that a minimum secure set is connected; that is, a minimum secure set induces a connected subgraph of the graph in which it is a minimum secure set. Likewise, a minimum (F,I)-secure set is connected.

Proposition. *Let $G = (V, E)$ be a graph.*

1) $s_{(F,I)}(G) = 1$ *if and only if* $\delta(G) \leq 1$.

2)$s_{(F,I)}(G) = 2$ *if and only if* $\delta(G) = 2$ *and there exists* $uv \in E$ *such that* $d(u) = d(v) = 2$.

3) $s_{(F,I)}(P_n \square P_m) = \min\{m, n, 4\}$.

4) $\min\{m, 2n, 6\} \leq s_{(F,I)}(C_m \square P_n) \leq \min\{m, 2n, 8\}$.

5) $s_{(F,I)}(F_n) = 1 + \lceil \frac{n}{2} \rceil$, $n \geq 2$.

6) $s_{(F,I)}(W_n) = 1 + \lceil \frac{n+1}{2} \rceil$, $n \geq 3$.

Proof. 1) A vertex of degree zero or one is (F,I)-secure, but any vertex of greater degree is not.

2) By the first result, $\delta(G) > 1$ is necessary for $s_{(\mathrm{F,I})}(G) = 2$. Let $S = \{u, v\}$ be (F,I)-secure. Since a minimum (F,I)-secure set is connected, $uv \in E$. In order to be (F,I)-secure, S must also be secure. So $|N[S] - S| \leq 2$, which forces $d(u) \leq 3$ and $d(v) \leq 3$. If $d(u) = 3$ or $d(v) = 3$, then $c(G^S) = 1$, $|N[S] - S| = 2$, and $|S| + |N[S] - S| - c(G^S) = 3 > |S|$. So by the Main Theorem, S is not (F,I)-secure. Thus $d(u) = d(v) = 2$ is necessary. If $S = \{u, v\}$, $uv \in E$, and $d(u) = d(v) = 2$, then the defense where u defends itself and v defends itself is successful against any attack.

3) In [3] it is shown that $s(P_n \square P_m) = \min\{m, n, 3\}$. For the first case, suppose $\min\{m, n\} \leq 3$. Then $s_{(\mathrm{F,I})}(P_n \square P_m) \geq s(P_n \square P_m) = \min\{m, n, 3\} = \min\{m, n\}$. The n vertices that make up the end vertices of the P_m paths form an (F,I)-secure set, and the m vertices that make up the end vertices of the P_n paths form an (F,I)-secure set. So $s_{(\mathrm{F,I})}(P_n \square P_m) = \min\{m, n\}$. Now suppose that $\min\{m, n\} \geq 4$. Then $s_{(\mathrm{F,I})}(P_n \square P_m) \geq s(P_n \square P_m) = \min\{m, n, 3\} = 3$. If S consists of a corner and its two neighbors, let $S = \{s_1, s_2, s_3\}$, $N[S] - S = \{v_1, v_2, v_3\}$

and $\{s_1s_2, s_1v_1, s_1v_2, s_3s_2, s_3v_2, s_3v_3\}$ be a subset of the edges (see Figure 3.2). Then the attack A defined by $A(v_1, s_1) = 1, A(v_2, s_1) = 0.5, A(v_2, s_3) = 0.5$, and $A(v_3, s_3) = 1$ is not defendable. Any other S such that $|S| = 3$ satisfies $|S| < |N[S] - S|$ and is not secure, much less (F,I)-secure. Four vertices that induce a 4-cycle and include a corner vertex forms an (F,I)-secure set. So in this case, $s_{(\mathrm{F,I})}(P_n \Box P_m) = 4$.

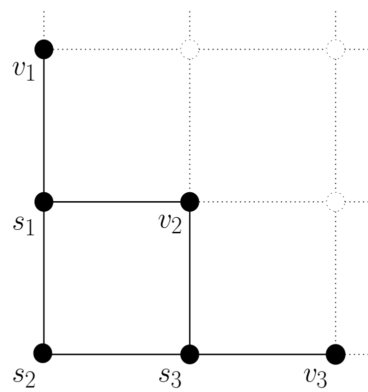

Figure 3.2: The set $S = \{s_1, s_2, s_3\}$ is not (F,I)-secure.

4) Kozawa et al [12] show that $s(C_m \Box P_n) = \min\{m, 2n, 6\}$. This settled a conjecture of Brigham et al [3]. This gives the lower bound for the (F,I)-security number. Two consecutive copies of P_n form an (F,I)-secure set, as does an m-cycle consisting of end vertices of the paths. If $m \geq 4$ and $n \geq 2$, taking the end vertex of a path and its neighbor in the path, in four consecutive paths, gives an (F,I)-secure set of eight vertices.

5) Suppose S is a minimum (F,I)-secure set and that $V(K_1) \notin S$. The subgraph G^S is connected. By the Main Theorem, it is then necessary that $1 = c(G^S) \geq |N(S) - S|$. So $S = V(P_n)$ and $|S| = n$. If $V(K_1) \subseteq S$, let $S' = V(P_n) \cap S$; then $|S'| \leq n - 1$ (otherwise $S' = V(P_n)$ and $S = V(F_n)$). Every vertex in $V(P_n) - S'$ can attack $V(K_1)$ and there is at least one $u \in V(P_n) - S'$ such that u has a neighbor in S'. Define an attack by letting u send a half unit of attack to $V(K_1)$ and a half unit of attack to its neighbor in S'. Let

every other vertex of $V(P_n) - S'$ attack $V(K_1)$ with its whole unit of attack. To be defended successfully, this attack requires $|S| = 1 + |S'| \geq |V(P_n) - S'| + 1 = n - |S'| + 1$. This implies that $|S'| \geq \lceil \frac{n}{2} \rceil$ and thus $|S| \geq 1 + \lceil \frac{n}{2} \rceil$. Next we construct an (F,I)-secure set S, such that $|S| = 1 + \lceil \frac{n}{2} \rceil$. Let S be the set containing $V(K_1)$ and the first $\lceil \frac{n}{2} \rceil$ vertices of $V(P_n)$. Only two vertices can be attacked: $V(K_1)$ and some $s \in V(P_n) \cap S$. Let s defend itself, and have all other vertices of S defend the $V(K_1)$. Since s has exactly one neighbor in $V(P_n) - S$, it has sufficient defense. Likewise, $V(K_1)$ has a total defense of $\lceil \frac{n}{2} \rceil$, and the attack there is at most $\lfloor \frac{n}{2} \rfloor$. This defense is successful against any attack on S.

6) The proof is similar to the proof of 5). If a set $S \subseteq V$ does not include the $V(K_1)$, then $V(K_1)$ can attack every vertex in the set S. In order for S to be (F,I)-secure, it can have no other attackers. So $S = V(C_n)$ and $|S| = n$. If an (F,I)-secure set $S \subseteq V$ includes $V(K_1)$, let $S' = S \cap V(C_n)$. If $|S'| \geq n - 1$, then $|S| \geq n$, and we already can find an (F,I)-secure set of size n. If $|S'| \leq 1$, then $|S| \leq 2$ and S is not (F,I)-secure by inspection. (In fact, for $n \geq 4$, S is not secure.) So let $2 \leq |S'| \leq n - 2$. The graph G^S is a connected bipartite graph. Since every vertex in $V(C_n) - S'$ is adjacent to $V(K_1)$, one part of the bipartition has size $|V(C_n) - S'| = n - |S'|$. The other part of the bipartition must have at least three vertices, because it is $V(K_1) \cup S'$. By the Bipartite Graph Lemma, in order for S to be (F,I)-secure, it is required that $|S| = 1 + |S'| \geq n - |S'| + 3 - 1$. Isolating $|S'|$ yields $|S'| \geq \lceil \frac{n+1}{2} \rceil$. So $|S| \geq 1 + \lceil \frac{n+1}{2} \rceil$.

An (F,I)-secure set of size $1+\lceil \frac{n+1}{2} \rceil$ can be found by taking $V(K_1)$ and $\lceil \frac{n+1}{2} \rceil$ consecutive vertices of the $V(C_n)$. Exactly three vertices can be attacked, $V(K_1)$, and some $u, v \in S'$; u and v each have exactly one neighbor in $V(C_n) - S'$. Let u and v defend themselves, and have every other vertex of S defend $V(K_1)$. Then u and v clearly have sufficient defense, while $V(K_1)$ has a total defense of $\lceil \frac{n-1}{2} \rceil$ and at most the attack at $V(K_1)$ is $\lfloor \frac{n-1}{2} \rfloor$. So this defense is successful against any attack on S. $\square$

3.3.1 Complete Multipartite Graphs

Next we find the (F,I)-security number of complete multipartite graphs. We begin with a corollary of the Bipartite Graph Lemma.

Corollary 2. *Let G be a complete bipartite graph with bipartition (X, Y). If $S = X \cup Y_1$, where $|Y_1| = \left\lfloor \frac{|Y|}{2} \right\rfloor$ and $Y_1 \subseteq Y$, then S is (F,I)-secure.*

Proof. Let A be an attack on S. Let $X_A = \{x \mid x \in X \text{ and } A^*(x) > 0\}$. By applying the Bipartite Graph Lemma to the subgraph induced by $X_A \cup (Y - S)$, the maximum total effective attack possible is then

$$|X_A| + |Y - S| - 1 = |X_A| + \left\lceil \frac{|Y|}{2} \right\rceil - 1.$$

Construct a defense D as follows. Let every vertex in X_A defend itself. This leaves $\left\lceil \frac{|Y|}{2} \right\rceil - 1$ effective attack remaining, but $|Y_1| = \left\lfloor \frac{|Y|}{2} \right\rfloor \geq \left\lceil \frac{|Y|}{2} \right\rceil - 1$. Since G is complete, the vertices of Y_1 can send their units of defense wherever they are necessary to finish constructing D. □

We will employ Corollary 2 in the following proof of Theorem 1, below. By inspection $s_{(\mathrm{F,I})}(K_{2,2}) = 2$.

Theorem 1. *If $2 \leq n_1 \leq n_2$ and $n_2 \neq 2$, then $s_{(F,I)}(K_{n_1,n_2}) = n_1 + \left\lfloor \frac{n_2}{2} \right\rfloor$.*

Proof. Let the bipartition of $G = K_{n_1,n_2}$ be (X, Y) where $|X| = n_1$ and $|Y| = n_2$, and let $S \subseteq X \cup Y$ be (F,I)-secure. First assume $S \cap Y = \emptyset$. Then $S \subseteq X$, $S \cup Y$ induces a complete bipartite graph, and there exists an attack on S by Y with a total effective attack of $|Y| + |S| - 1$. Since $|Y| \geq 2$ the value of this total effective attack is strictly greater than $|S|$, contradicting the assumption that S is (F,I)-secure. Thus we conclude that $S \cap Y \neq \emptyset$.

By a similar argument it follows that $S \cap X \neq \emptyset$. An (F,I)-secure set $S \subseteq V(K_{n_1,n_2})$ must therefore contain vertices of both X and Y. So let $S \cap X \neq \emptyset$, $S \cap Y \neq \emptyset$, and let $x = |X \cap S|$ and $y = |Y \cap S|$, so that $x + y = |S|$. Also note that $x \geq 1$ and $y \geq 1$. There are four possible scenarios:

1) $x = |X|$ and $y = |Y|$. In this case S is (F,I)-secure because $S = V(K_{n_1,n_2})$.

2) $1 \leq x < |X|$ and $1 \leq y < |Y|$. $(X \cap S) \cup (Y - S)$ induces a complete bipartite graph, so there exists an attack from $Y - S$ to $X \cap S$ with total effective attack $|Y - S| + |X \cap S| - 1 = n_2 - y + x - 1$. Likewise, $(Y \cap S) \cup (X - S)$ induces a complete bipartite graph, so there exists an attack from $X - S$ to $Y \cap S$ with total effective attack $|X - S| + |Y \cap S| - 1 = n_1 - x + y - 1$. Since these two complete bipartite graphs are disjoint, there exists an attack from $V(K_{n_1,n_2}) - S$ to S with total effective attack $(n_2 - y + x - 1) + (n_1 - x + y - 1) = n_1 + n_2 - 2$. So $|S| \geq n_1 + n_2 - 2$. Taking $x = |X| - 1 = n_1 - 1$ and $y = |Y| - 1 = n_2 - 1$ provides an (F,I)-secure set of this size. The integer defense where each vertex of S defends itself is successful against any attack.

3) $x = |X|$ and $1 \leq y < |Y|$. Only vertices in X can be attacked. There is a complete bipartite graph induced by $X \cup (Y - S)$. So there is an attack of X by $Y - S$ with total effective attack $|X| + |Y - S| - 1 = x + n_2 - y - 1 = n_1 + n_2 - y - 1$. In order for S to be (F,I)-secure, it is necessary that $|S| = x + y = n_1 + y \geq n_1 + n_2 - y - 1$. Thus $y \geq \left\lceil \frac{n_2 - 1}{2} \right\rceil = \left\lfloor \frac{n_2}{2} \right\rfloor$. So $|S| = x + y \geq n_1 + \left\lfloor \frac{n_2}{2} \right\rfloor$. On the other hand, any set S with all the vertices of X and $\left\lfloor \frac{n_2}{2} \right\rfloor$ vertices of Y is (F,I)-secure, by Corollary 2.

4) $1 \leq x < |X|$ and $y = |Y|$. Simlar to case 3), the smallest (F,I)-secure set in this case is of size $n_2 + \left\lfloor \frac{n_1}{2} \right\rfloor$.

So the $s_{(\mathrm{F,I})}(K_{n_1,n_2}) = \min\{n_1+n_2, n_1+n_2-2, n_1+\left\lfloor\frac{n_2}{2}\right\rfloor, n_2+\left\lfloor\frac{n_1}{2}\right\rfloor\}$. The minimum is $n_1+\left\lfloor\frac{n_2}{2}\right\rfloor$ unless $n_1=n_2=2$. So if $n_2 \neq 2$, $s_{(\mathrm{F,I})}(K_{n_1,n_2}) = n_1+\left\lfloor\frac{n_2}{2}\right\rfloor$. $\square$

Theorem 2. *Let $k \geq 3$ and $1 \leq n_1 \leq n_2 \leq ... \leq n_k$. Then $s_{(F,I)}(K_{n_1,n_2,...,n_k}) = n_1+n_2+...+n_{k-1}+\left\lfloor\frac{n_k}{2}\right\rfloor$.*

Proof: Let $\{X_1, X_2, ..., X_k\}$ be the partition of $K_{n_1,n_2,...,n_k}$ so that $|X_i| = n_i$ for $1 \leq i \leq k$. Let $V = V(K_{n_1,n_2,...,n_k})$ and $\emptyset \neq S \subseteq V$. If $|\{i : (V-S)\cap X_i \neq \emptyset\}| \geq 2$, then the graph induced by the edges with one end in S and one end in $V-S$ induces a connected bipartite graph with bipartition $(S, V-S)$. So $V-S$ can attack S with a total effective attack of $|V-S|+|S|-1 \geq 2+|S|-1 = |S|+1$. So S cannot be (F,I)-secure. Thus $V-S=\emptyset$ or $|\{i : (V-S)\cap X_i \neq \emptyset\}| = 1$. If $V-S=\emptyset$, then $S=V$. This is not a smallest (F,I)-secure set, because any set S with $|S| = |V|-1$ is (F,I)-secure.

So we must have $|\{i : (V-S)\cap X_i \neq \emptyset\}| = 1$. Let X_α be such that $(V-S)\cap X_\alpha \neq \emptyset$ and $(V-S)\cap X_j = \emptyset$ for all $j \neq \alpha$. Let $x = |S \cap X_\alpha|$. Then $|V-S| = |(V-S)\cap X_\alpha| = n_\alpha - x$. There are no edges between $S\cap X_\alpha$ and $V-S$ because both are contained in X_α. So the complete bipartite graph induced by the edges with one end in $V-S$ and one end in S has bipartition $(S-X_\alpha, V-S)$. There is an attack from $V-S$ to $S-X_\alpha$ with total effective attack $|V-S|+|S-X_\alpha|-1 = n_\alpha - x + |S| - x - 1$. In order for S to be (F,I)-secure, it is necessary that $|S| \geq n_\alpha - 2x - 1 + |S|$ from which it follows $x \geq \left\lceil\frac{n_\alpha-1}{2}\right\rceil = \left\lfloor\frac{n_\alpha}{2}\right\rfloor$. We can find an (F,I)-secure set of size $\sum_{j\neq\alpha} n_j + \left\lfloor\frac{n_\alpha}{2}\right\rfloor$ by choosing any $\left\lfloor\frac{n_\alpha}{2}\right\rfloor$ vertices of X_α along with $\cup_{j\neq\alpha} X_j$. This set S is (F,I)-secure by Corollary 2, because S is (F,I)-secure in the complete bipartite graph induced by edges with one end in X_α and one end in $V-X_\alpha$. The other edges in the multipartite graph do not allow for any new attack possibilities. Therefore,

$$s_{(\mathrm{F,I})}(K_{n_1,n_2,...,n_k}) = \min_{1\leq i\leq k}\left\{\sum_{j\neq i} n_j + \left\lfloor\frac{n_i}{2}\right\rfloor\right\} = \sum_{j=1}^{k-1} n_j + \left\lfloor\frac{n_k}{2}\right\rfloor. \quad \square$$

Chapter 4

Ultra-security

In this chapter, we will look only at integer attacks and integer defenses. Recall that an integer attack and an integer defense are equivalent to the definitions given by Brigham et al [3] and mentioned in the Introduction. In this chapter, we will use the notation of Brigham et al. That is, for a graph $G = (V, E)$ and $S = \{s_1, ..., s_k\} \subseteq V$, an integer attack is a collection of pairwise disjoint sets $A = \{A_1, ..., A_k\}$ such that $A_i \subseteq N[s_i] - S$ for $1 \leq i \leq k$. An integer defense is a collection of pairwise disjoint sets $D = \{D_1, ..., D_k\}$ such that $D_i \subseteq N[s_i] \cap S$ for $1 \leq i \leq k$; an integer defense D of S such that $|D_i| \geq |A_i|$ for $1 \leq i \leq k$ is a *successful* integer defense of A.

For a graph $G = (V, E)$, this chapter looks at a reversal of the usual situation. This course of inquiry was suggested by Dr. Chris Rodger. Usually we ask: For each integer attack on S, does their exist a successful integer defense of S? Now we will ask the following question: Is there one integer defense that will successfully defend against any integer attack? A set $S \subseteq V$ is *ultra-secure* if there exists an integer defense D of S, such that for any integer attack A on S, D is a successful integer defense of A. Note that if $S \subseteq V$ is ultra-secure, then S is secure.

4.1 A Necessary and Sufficient Condition

The following Hall-type theorem is used in the proof of the Theorem after it.

Theorem HRHV ([8, 14, 9]). *Suppose $P_1, P_2, ..., P_n$ are sets and $k_1, k_2, ..., k_n$ are non-negative integers. There exist pairwise disjoint sets $D_1, ..., D_n$ such that $D_i \subseteq P_i$ and $|D_i| = k_i$*

for $1 \leq i \leq n$ *if, and only if, for each* $I \subseteq \{1, ..., n\}$, $|\cup_{i \in I} P_i| \geq \sum_{i \in I} k_i$.

Theorem HRHV is used in [10] to give an alternative proof of the necessary and sufficient condtion for security due to Brigham et al [3]. Its use in the proof of the following Theorem is similar.

Theorem. *A set* $S \subseteq V$ *is ultra-secure if and only if* $|N[X] \cap S| \geq \sum_{x \in X} |N[x] - S|$ *for all* $X \subseteq S$.

Proof. Let $S = \{s_1, s_2, ..., s_k\}$ be ultra-secure, and $X \subseteq S$. Suppose that $D = \{D_1, ..., D_k\}$ is a successful integer defense of every integer attack on S. Let $I = \{\, i \,|\, 1 \leq i \leq k \text{ and } s_i \in X\}$. For each $i \in I$ there exists an integer attack A on S with $|A_i| = |N[s_i] - S|$. If S is ultra-secure, then $|D_i| \geq |N[s_i] - S|$ for all $i \in I$. Since $D_i \cap D_j = \emptyset$ for $i \neq j$ and $D_i \subseteq N[s_i] \cap S \subseteq N[X] \cap S$ for $i \in I$, it is necessary that $|N[X] \cap S| \geq \sum_{i \in I} |D_i| \geq \sum_{i \in I} |N[s_i] - S| = \sum_{x \in X} |N[x] - S|$.

To show sufficiency, let $S = \{s_1, s_2, ..., s_n\} \subseteq V$ such that for all $X \subseteq S$, $|N[X] \cap S| \geq \sum_{x \in X} |N[x] - S|$. Let $k_i = |N[s_i] - S|$ and $P_i = N[s_i] \cap S$, $1 \leq i \leq n$. For $1 \leq i \leq n$, k_i represents the number of vertices that can attack s_i, and P_i is the set of potential defenders of s_i. For $I \subseteq \{1, 2, ...n\}$, letting $X = \{s_i | \, i \in I\}$, we have $\sum_{i \in I} k_i = \sum_{i \in I} |N[s_i] - S| = \sum_{x \in X} |N[x] - S| \leq |N[X] \cap S| = |\cup_{i \in I} P_i|$. By Theorem HRHV, for $1 \leq i \leq n$, there exist pairwise disjoint $D_i \subseteq P_i$ such that $|D_i| = k_i$. For any integer attack $A = \{A_1, A_2, ..., A_n\}$ on S, $|A_i| \leq k_i$ for $1 \leq i \leq n$. It follows that $D = \{D_1, D_2, ..., D_n\}$ is a successful integer defense of any integer attack A on S. $\square$

4.2 The Ultra-security Number

The *security number* $s(G)$ of a graph G is the cardinality of a smallest secure set in G. Some results on the security number can be found in [3, 4, 6, 12]. Given a graph G, the *ultra-security number*, denoted $s^u(G)$, is the cardinality of a smallest ultra-secure set in G. Note that for any graph G, $s(G) \leq s^u(G)$. Comments by Dr. Amin Bahmanian led to improvements in the statements of parts 1) and 2) in the following result.

Proposition. *Let $G = (V, E)$ be a graph.*

1) $s^u(G) = 1$ *if and only if* $\delta(G) \leq 1$.

2) $s^u(G) = 2$ *if and only if* $\delta(G) = 2$ *and there exits* $uv \in E$ *such that* $d(u) = d(v) = 2$.

3) $s^u(K_n) = n - 1$, *if* $n \geq 2$.

4) $s^u(P_n \Box P_m) = \min\{m, n, 4\}$.

5) $s^u(K_{m,n}) = \min\left\{m + n - 2, m + \left\lceil \frac{m(n-1)}{m+1} \right\rceil\right\}, 2 \leq m \leq n$.

6)*Let* $k \geq 3$ *and* $1 \leq n_1 \leq ... \leq n_k$. *Let* $n = \sum_{i=1}^{k} n_i$. *Then* $s^u(K_{n_1,...,n_k}) = n - \left\lfloor \frac{n}{n-n_k+1} \right\rfloor$.

Proof. 1) A single vertex of degree zero or one is ultra-secure, but any vertex with degree greater than one is not.

2) For $s^u(G) > 1$, $\delta(G) \geq 2$ is required. Let $S \subseteq V$ with $|S| = 2$. If S contains two nonadjacent vertices, it is not ultra-secure. In order for a set $S = \{u, v\}$ with $uv \in E$ to be ultra-secure, $|N[u]-S|+|N[v]-S| \leq 2$. Note that $|N[u]-S| \geq 1$ and $|N[v]-S| \geq 1$, so that $|N[u]-S| = 1$ and $|N[v]-S| = 1$ is required. Furthermore, $|N(u) \cap S| = |N(v) \cap S| = 1$ and thus $d(u) = d(v) = 2$ is necessary. If $S = \{u, v\}$, $uv \in E$, and $d(u) = d(v) = 2$, then the integer defense where u defends itself and v defends itself is successful against any integer attack.

3) Suppose that $S \subseteq V$ is ultra-secure and $1 \leq |S| \leq n - 2$. Then for $x \in S$, $|N[x] - S| \geq 2$. It is necessary that $|N[S] \cap S| \geq \sum_{x \in S} |N[x]-S|$, but $|N[S] \cap S| = |S|$ and $\sum_{x \in S} |N[x]-S| \geq \sum_{x \in S} 2 = 2|S|$. So S is not ultra-secure. Any S with $|S| = n - 1$ is ultra-secure; D is found by letting each vertex of S defend itself.

4) In [3] it is shown that $s(P_n \Box P_m) = \min\{m, n, 3\}$. First suppose that $\min\{m, n\} \leq 3$. Then we have $s^u(P_n \Box P_m) \geq s(P_n \Box P_m) = \min\{m, n, 3\} = \min\{m, n\}$. The m vertices of an end column, and the n vertices of an end row are ultra-secure, so $s^u(P_n \Box P_m) \leq \min\{m, n\}$, and thus $s^u(P_n \Box P_m) = \min\{m, n\}$. Now suppose that $\min\{m, n\} \geq 4$. Then $s^u(P_n \Box P_m) \geq s(P_n \Box P_m) = \min\{m, n, 3\} = 3$. Let S be a set consisting of a corner and its two neighbors. Then $\sum_{s \in S} |N[s] - S| = 4$, and $|S| = 3$. So three vertices of a corner do not form an ultra-secure set. Any other set S with $|S| = 3$ satisfies $|S| < |N[X] - S|$ and is not secure, much less ultra-secure. Four vertices that induce a C_4 and contain a corner vertex form an ultra-secure set. Thus when $\min\{m, n\} \geq 4$, $s^u(P_n \Box P_m) = 4$. Combining the two cases we get the desired result: $s^u(P_n \Box P_m) = \min\{m, n, 4\}$.

5) Let (Y, Z) be the bipartition of $V(K_{m,n})$ such that $|Y| = m$ and $|Z| = n$. Let $S \subseteq V(K_{m,n})$. For S to be ultra-secure, by the Theorem, it is necessary that $|S| \geq \sum_{x \in S} |N[x] - S|$. Clearly $|S| = |S \cap Y| + |S \cap Z|$. On the other hand,

$$\begin{aligned}\sum_{x \in S} |N[x] - S| &= \sum_{x \in S \cap Y} |N[x] - S| + \sum_{x \in S \cap Z} |N[x] - S| \\ &= \sum_{x \in S \cap Y} |Z - S| + \sum_{x \in S \cap Z} |Y - S| \\ &= |S \cap Y||Z - S| + |S \cap Z||Y - S|.\end{aligned}$$

So in order to be ultra-secure, S must satisfy the inequality:

$$|S \cap Y| + |S \cap Z| \geq |S \cap Y||Z - S| + |S \cap Z||Y - S|. \tag{4.1}$$

If $S \subseteq Y$, then $|Z - S| = |Z| \geq 2$, because $2 \leq m \leq n$, and the inequality (4.1) fails. Likewise, if $S \subseteq Z$, the inequality (4.1) will fail. So for S to be ultra-secure, it is necessary that $S \cap Y \neq \emptyset$ and $S \cap Z \neq \emptyset$. We will consider three cases.

First assume that $Y - S \neq \emptyset$ and $Z - S \neq \emptyset$. Inequality (4.1) implies $0 \geq (|Z-S|-1)|S \cap Y| + (|Y-S|-1)|S \cap Z|$. Since $|S \cap Y| \geq 1$ and $|S \cap Z| \geq 1$, we have $|Z - S| \leq 1$ and $|Y - S| \leq 1$. So if there are attackers in both parts of the bipartition, there may be at most one attacker from each part. Letting $|S \cap Y| = m-1$ and $|S \cap Z| = n-1$, S is ultra-secure and $|S| = m+n-2$. The integer defense is given by letting each vertex of S defend itself.

For the second case, assume $Y - S = \emptyset$ and $Z - S \neq \emptyset$. In this case, the inequality (1) becomes $|Y| + |S \cap Z| \geq |Y||Z - S|$ which is equivalent to $m + |S \cap Z| \geq m(n - |S \cap Z|)$. Isolating $|S \cap Z|$ yields

$$|S \cap Z| \geq \left\lceil \frac{m(n-1)}{m+1} \right\rceil .$$

Let S contain all m vertices of Y and $\left\lceil \frac{m(n-1)}{m+1} \right\rceil$ vertices of Z. We will show this set satisfies the condition of the Theorem. For any $v \in Z \cap S$, $|N[v] - S| = 0$, so we can assume that $X \subseteq S \cap Y = Y$. For $X \subseteq Y$,

$$|N[X] \cap S| = |X| + \left\lceil \frac{m(n-1)}{m+1} \right\rceil \text{ and}$$
$$\sum_{x \in X} |N[x] - S| = |X| \left(n - \left\lceil \frac{m(n-1)}{m+1} \right\rceil \right).$$

So S is secure if and only if for all $X \subseteq Y$,

$$|X| + \left\lceil \frac{m(n-1)}{m+1} \right\rceil \geq |X| \left(n - \left\lceil \frac{m(n-1)}{m+1} \right\rceil \right).$$

This inequality reduces to

$$\left\lceil \frac{m(n-1)}{m+1} \right\rceil \geq \frac{|X|(n-1)}{|X|+1}. \tag{4.2}$$

Since $m = |Y| \geq |X|$ it follows that

$$\frac{m}{m+1} \geq \frac{|X|}{|X|+1}.$$

This shows that (2) holds and we have an ultra-secure set S with $|S| = m + \left\lceil \frac{m(n-1)}{m+1} \right\rceil$.

For the final case, assume $Y - S \neq \emptyset$ and $Z - S = \emptyset$. Following a similar argument to the second case, S must satisfy $|S| \geq n + \left\lceil \frac{n(m-1)}{n+1} \right\rceil$. Since $n \geq m$ it follows that $\frac{m-1}{n+1} < 1$ and thus

$$\left\lceil \frac{n}{n+1}(m-1) \right\rceil = \left\lceil (m-1) - \frac{m-1}{n+1} \right\rceil = m-1.$$

So in this case, a minimum ultra-secure set S has $|S| \geq n + m - 1$.

Thus $s^u(K_{m,n}) = \min\{m+n-2, m + \lceil \frac{m}{m+1}(n-1) \rceil\}$. For $2 \leq m \leq n$,

$$\begin{aligned}
m+n-2 \leq m + \left\lceil \frac{m}{m+1}(n-1) \right\rceil \\
\Leftrightarrow \quad n-2 \leq \left\lceil (n-1) - \frac{n-1}{m+1} \right\rceil \\
\Leftrightarrow \quad \frac{n-1}{m+1} < 2 \\
\Leftrightarrow \quad n \leq 2m+2.
\end{aligned}$$

So we have, for $2 \leq m \leq n$,

$$s^u(K_{m,n}) = \begin{cases} m+n-2 & \text{if } n \leq 2m+2 \\ m + \left\lceil \frac{m(n-1)}{m+1} \right\rceil & \text{otherwise} \end{cases}$$

as an alternative statement for $s^u(K_{m,n})$. Comparing the two expressions in the reverse order gives $m + \left\lceil \frac{m(n-1)}{m+1} \right\rceil \leq m+n-2$ if $m+2 \leq n$. So we have $m+n-2 = m + \left\lceil \frac{m(n-1)}{m+1} \right\rceil$ if and only if $m+2 \leq n \leq 2m+2$.

6)Let $X_1, ..., X_k$ be the partition of $K_{n_1,...,n_k}$, so that $|X_i| = n_i$ for $1 \leq i \leq k$. Let $V = V(K_{n_1,...,n_k})$ and let $\emptyset \neq S \subseteq V$. If $|\{i | X_i \cap (V-S) \neq \emptyset\}| \geq 2$, then it is easy to see that $|N[x] - S| \geq 1$ for all $x \in S$. Further, because $k \geq 3$, it follows that $|N[y] - S| \geq 2$ for some $y \in S$. Then $|S| < \sum_{x \in S} |N[x] - S|$ and S is not ultra-secure.

Now suppose that S is ultra-secure and $|\{i | X_i \cap (V-S) \neq \emptyset\}| = 0$. Then $S = V$, but this is not a minimum ultra-secure set, because any set of $n-1$ vertices is ultra-secure. So let

$|\{i|X_i \cap (V-S) \neq \emptyset\}| = 1$. Let $\alpha \in \{1, ..., n\}$ such that $X_\alpha \cap (V-S) \neq \emptyset$. Let $r = |S \cap X_\alpha|$. In order for S to be ultra-secure, we must have $n - n_\alpha + r = |S| \geq \sum_{x \in S} |N[x] - S| = \sum_{x \in (S - X_\alpha)} (n_\alpha - r) = (n - n_\alpha)(n_\alpha - r)$. Solving for r yields $r \geq \left\lceil n_\alpha - \frac{n}{n-n_\alpha+1} \right\rceil$. Thus $|S| = n - n_\alpha + r \geq n - n_\alpha + \left\lceil n_\alpha - \frac{n}{n-n_\alpha+1} \right\rceil = n - \left\lfloor \frac{n}{n-n_\alpha+1} \right\rfloor$. We will find an ultra-secure set with $\min\{n - \left\lfloor \frac{n}{n-n_i+1} \right\rfloor |1 \leq i \leq k\} = n - \left\lfloor \frac{n}{n-n_k+1} \right\rfloor$ vertices.

Let S be the set consisting of the vertices $V - X_k$ and $\left\lceil n_k - \frac{n}{n-n_k+1} \right\rceil$ of X_k. If $n_k \leq \left\lceil \frac{n}{2} \right\rceil$, then $n - \left\lfloor \frac{n}{n-n_k+1} \right\rfloor = n-1$, and S is ultra-secure. If $n_k \geq \left\lceil \frac{n}{2} \right\rceil + 1$, then consider the complete bipartite graph induced by the edges with one end in X_k and the other end in $V - X_k$. One part of the bipartition has size $n - n_k$, and the other part has size n_k. In the proof of 5), it was shown that a set consisting of all m vertices of the smaller part and $\left\lceil \frac{m(p-1)}{m+1} \right\rceil$ of the larger part forms an ultra-secure set. The set S does contain all $n - n_k$ vertices of $V - X_k$, and $\left\lceil n_k - \frac{n}{n-n_k+1} \right\rceil = \left\lceil \frac{(n-n_k)(n_k-1)}{n-n_k+1} \right\rceil$ vertices of X_k. So S is ultra-secure in the complete bipartite graph. Thus S is also ultra-secure in the complete multipartite graph, because the additional edges offer no new attack possiblities. $\square$

Chapter 5

Relationships Between Security, (F,I)-security, and Ultra-security

At this point, we have seen that for a graph G, $s(G) \leq s_{(\mathrm{F,I})}(G)$ and $s(G) \leq s^u(G)$. In this chapter, we explore further relationships between these three types of security.

5.1 Ultra-security Implies (F,I)-security

Let $G = (V, E)$ be a finite, simple graph. Let $S \subseteq V$. We compare (F,I)-security to ultra-security. An equivalent formulation of ultra-security requires each vertex of $N(S) - S$ to send one unit of attack along each edge it has into S. Note that in ultra-security the attacks are integer attacks, and the defenses are integer defenses.

Lemma 1. *Let G=(V,E) be a graph, and $S \subseteq V$. Then $S \subseteq V$ is ultra-secure if and only if there exists an integer defense D such that for all $v \in S$, $D^*(v) \geq |N[v] - S|$.*

Proof. Suppose D is an integer defense such that for all $v \in S$, $D^*(v) \geq |N[v] - S|$. In any attack A, $A^*(v) \leq |N[v] - S|$ and so for all $v \in V$, $A^*(v) \leq |N[v] - S| \leq D^*(v)$. Thus D is successful against A and S is ultra-secure.

If S is ultra-secure, then there exists a defense D that is a successful defense of any attack on S. For each $v \in S$, there exists an attack such that $A^*(v) = |N[v] - S|$. Since D is a successful defense of all attacks, $D^*(v) \geq |N[v] - S|$. $\square$

Proposition 1. *Let $G = (V, E)$ be a graph and $S \subseteq V$. If S is ultra-secure, then S is (F,I)-secure.*

Proof. Let $S \subseteq V$ be an ultra-secure set, and A a (fractional) attack on S. Note that for $u \in N(S) - S$ and $v \in S$, $A(u, v) \leq 1$. So then $A^*(v) = \sum_{u \in N[v]-S} A(u, v) \leq \sum_{u \in N[v]-S} 1 = |N[v] - S|$. Due to Lemma 1, because S is ultra-secure, there is an integer defense D such that for all $v \in S$, $D^*(v) \geq |N[v] - S| \geq A^*(v)$, and thus S is also (F, I)-secure. $\square$

Note that this shows $s_{(\mathrm{F,I})}(G) \leq s^u(G)$. In the proofs of the values of $s_{(\mathrm{F,I})}(F_n)$, $s_{(\mathrm{F,I})}(W_n)$, and $s_{(\mathrm{F,I})}(C_m \Box P_n)$ in Chapter 3.3, it is easy to see that the minimum (F,I)-secure sets constructed are also ultra-secure.

Proposition 2.

1) $s^u(F_n) = 1 + \left\lceil \frac{n}{2} \right\rceil, n \geq 2.$

2) $s^u(W_n) = 1 + \left\lceil \frac{n+1}{2} \right\rceil$, $n \geq 3$.

3) $\min\{m, 2n, 6\} \leq s^u(C_m \Box P_n) \leq \min\{m, 2n, 8\}$.

Now we look at one condition which will guarantee an (F,I)-secure set is also ultra-secure. Recall that given a graph $G = (V, E)$ and $X \subseteq S \subseteq V$, we define G^X to be the subgraph of G whose vertex set is $X \cup (N[X] - S)$ and whose edge set is the set of all edges with one end in X and one end in $N[X] - S$.

Proposition 3. *If $S \subseteq V$ is (F,I)-secure and G^S is a forest, then S is ultra-secure.*

Proof. If G^S is a forest, then for $X \subseteq S$, G^X is a forest because $E(G^X) \subseteq E(G^S)$. Since G^X is a forest $|E(G^X)| = |V(G^X)| - c(G^X) = |X| + |N_G[X] - S| - c(G^X)$. On the other hand, recalling G^X is bipartite with bipartition $(X, N_G[X] - S)$, $|E(G^X)| = \sum_{x \in X} |N_G[x] - S|$. Since S is (F,I)-secure, we have, for all $X \subseteq S$, $|N_G[X] \cap S| \geq |X| + |N_G[X] - S| - c(G^X) = |E(G^X)| = \sum_{x \in X} |N_G[x] - S|$. So S is also ultra-secure. $\square$

5.2 Relating Security, (F,I)-security, and Ultra-security

Let $G = (V, E)$ be a graph. By Proposition 1, we now have $s(G) \leq s_{(\mathrm{F,I})}(G) \leq s^u(G)$. Below are the necessary and sufficient conditions for the three kinds of security:

$$S \text{ is secure} \Leftrightarrow \text{for all } X \subseteq S, |N[X] \cap S| \geq |N[X] - S|;$$
$$S \text{ is (F,I)-secure} \Leftrightarrow \text{for all } X \subseteq S, |N[X] \cap S| \geq |X| + |N[X] - S| - c(G^X);$$
$$S \text{ is ultra-secure} \Leftrightarrow \text{for all } X \subseteq S, |N[X] \cap S| \geq \sum_{x \in X} |N[x] - S|.$$

Note that for any $X \subseteq S$, $|N[X] - S| \leq \sum_{x \in X} |N[x] - S|$. If $S \subseteq V$ is secure and for all $X \subseteq S$, $|N[X] - S| = \sum_{x \in X} |N[x] - S|$, then S is also ultra-secure, and therefore (F,I)-secure. Also note that $|N[X] - S| = \sum_{x \in X} |N[x] - S|$ for all $X \subseteq S$ if, and only if, for each pair $s_1, s_2 \in S$, $s_1 \neq s_2$, $N(s_1) \cap N(s_2) - S = \emptyset$; that is, any two vertices of S have no common neighbor outside of S. So potential attackers do not have any choice as to where they can send their attack.

Proposition 4. *Let $G = (V, E)$ be a graph and $S \subseteq V$ be secure. If for distinct $s_1, s_2 \in S$, $N(s_1) \cap N(s_2) - S = \emptyset$, then S is (F,I)-secure and ultra-secure.*

Corollary. *Let $G = (V, E)$ be a graph. If there exists a minimum secure set $S \subseteq V$ such that for distinct $s_1, s_2 \in S$, $N(s_1) \cap N(s_2) - S = \emptyset$, then $s(G) = s_{(F,I)}(G) = s^u(G)$.*

However; there are infinitely many graphs where the condition in Proposition 4 fails for every minimum secure set, and yet $s(G) = s_{(\mathrm{F,I})}(G) = s^u(G)$ still holds. The smallest example is K_3.

Because $s(K_n) = \left\lceil \frac{n}{2} \right\rceil$, $s_{(\mathrm{F,I})}(K_n) = n - 1$, and $s^u(K_n) = n - 1$, we see it is possible to have a graph G where $s(G) < s_{(\mathrm{F,I})}(G) = s^u(G)$. Likewise, when $n \geq 5$, $s(K_{n,n}) = n$, $s_{(\mathrm{F,I})}(K_{n,n}) = n + \left\lfloor \frac{n}{2} \right\rfloor$, and $s^u(K_{n,n}) = 2n - 2$, so $K_{n,n}$ is an example of a graph G such that

$s(G) < s_{(\mathrm{F,I})}(G) < s^u(G)$.

Example. We will show the graph G in Figure 5.1 satisfies $s(G) = s_{(\mathrm{F,I})}(G) = 3$ and $s^u(G) = 4$. There is no vertex of degree one, and no subset of size two is secure, so $s(G) \geq 3$. The subset $\{a, b, f\}$ is a secure set. Applying the Main Theorem of Chapter 3 shows that $\{a, b, f\}$ is also (F,I)-secure. No subset of size three is ultra-secure: there are 14 subsets of size three that induce a connected subgraph in G, and each fails the necessary and sufficient condition for ultra-security. The set $\{b, c, e, f\}$ is ultra secure. The defense where each vertex defends itself is a successful defense against any attack.

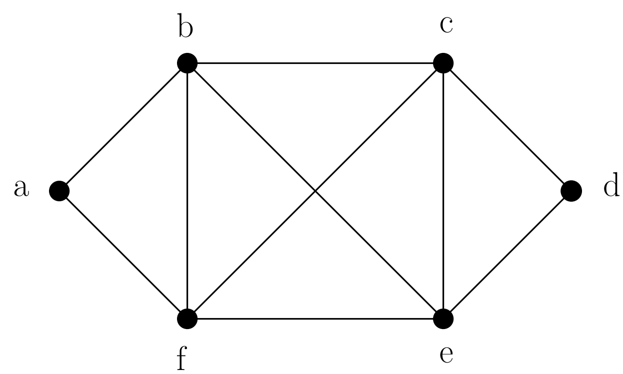

Figure 5.1: A graph G such that $s(G) = s_{(\mathrm{F,I})}(G) < s^u(G)$.

5.2.1 An Infinite Family of Graphs

So far, we have seen examples of graphs G such that $s(G) = s_{(\mathrm{F,I})}(G) = s^u(G) \in \{1, 2\}$.

Proposition 5. *For each positive integer n, there exists a graph G such that $s(G) = s_{(F,I)}(G) = s^u(G) = n$.*

Proof. Clearly $s(P_n) = s_{(\mathrm{F,I})}(P_n) = s^u(P_n) = 1$ and $s(C_n) = s_{(\mathrm{F,I})}(C_n) = s^u(C_n) = 2$. For $n = 3$, the graph $G = (V, E)$ in Figure 5.2 satisfies $s(G) = s_{(\mathrm{F,I})}(G) = s^u(G) = 3$. The set

$S = \{s_1, s_2, s_3\}$ is an ultra-secure set of order three, and no secure set of order one or two exists.

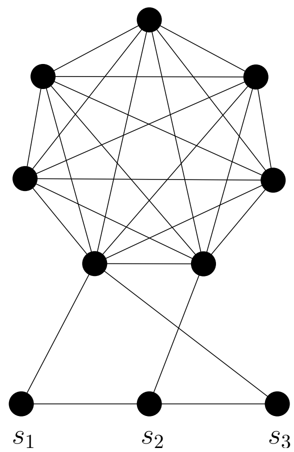

Figure 5.2: A graph such that $s(G) = s_{(\mathrm{F,I})}(G) = s^u(G) = 3$.

For $n \geq 4$, we generalize the graph in the figure. Let $G = (V, E)$ be such that $|V| = 3n+1$. Let the vertices of the graph be $\{s_1, ..., s_n\} \cup \{v_1, ..., v_{2n+1}\}$. Let $E = \{s_i s_{i+1} | 1 \leq i \leq n-1\} \cup \{v_j v_k | j \neq k\} \cup \{v_i s_i | 1 \leq i \leq n-1\} \cup \{v_1 s_n\}$. Then $\{v_1, ..., v_{2n+1}\}$ induces a K_{2n+1} and $\{s_1, ..., s_n\}$ induces a P_n. Letting each s_i defend itself shows that $\{s_1, ..., s_n\}$ is ultra-secure with size n. So we have $s(G) \leq s_{(\mathrm{F,I})}(G) \leq s^u(G) \leq n$. We will now show $s(G) \geq n$, and the result follows. Let $S \subseteq V$ be secure. If $v_j \in S$ for some j then $|S| \geq n+1$, because $|N[v_j] \cap S| \geq |N[v_j] - S|$ and $|N[v_j]| \geq 2n+1$. So we can find no smaller secure set containing any v_j. So if there is a smaller secure set, it has to be a subset of $\{s_1, ..., s_n\}$. A minimum secure set is connected [3], meaning the subgraph induced by the minimum secure set is connected. So let T be a subset of $\{s_1, ...s_n\}$ such that $|T| \leq n-1$, and T induces a connected subgraph. Then $|(N[T]-T) \cap \{v_1, ..., v_{2n+1}\}| = |T|$ and $|(N[T]-T) \cap \{s_1, ..., s_n\}| \geq 1$, so that $|T| < |T| + 1 \leq |N[T] - T|$, and T is not secure. $\square$

Note that in the proof above, the vertices of $V - \{s_1, ..., s_n\}$ do not have to induce a K_{2n+1}. If $|V| > 3n+1$, so that $|V - \{s_1, ..., s_n\}| > 2n+1$, it is sufficient for the vertices of $V - \{s_1, ..., s_n\}$ to induce any graph H with $\delta(H) \geq 2n$.

Chapter 6

On the Security Number of Regular Bipartite Graphs

6.1 d-regular Bipartite Graphs with $s(G) = d$

It has been shown that for a graph G, $s(G) \geq \left\lceil \frac{\delta+1}{2} \right\rceil$ [6]. If G is d-regular, then $s(G) \geq \left\lceil \frac{d+1}{2} \right\rceil$. The complete graph K_n, achieves this bound, as it is regular of degree $n-1$ and $s(K_n) = \left\lceil \frac{n}{2} \right\rceil$. When $n \geq 4$, $\left\lceil \frac{n}{2} \right\rceil < n-1$. So if G is a d-regular graph, it is possible that $s(G) < d$. If we require G to be bipartite, in addition to being regular, then the security number cannot be less than the degree.

Proposition 1. *Suppose G=(V,E) is a regular bipartite graph with degree d. Then $s(G) \geq d$.*

Proof. If $d = 0$ or $d = 1$, $s(G) = 1$. Let $d \geq 2$. Then $s(G) \geq 2$ and a secure set S must contain vertices u, v such that $uv \in E$. The vertices u and v must be in different parts of the bipartition, and thus $|N[\{u,v\}]| = 2d$. If S is secure, then by Theorem BDH $|N[\{u,v\}]| = 2d = |N[\{u,v\}] - S| + |N[\{u,v\}] \cap S| \leq 2|N[\{u,v\}] \cap S|$, so $|N[\{u,v\}] \cap S| \geq d$. Thus we have $|S| \geq |N[\{u,v\}] \cap S| \geq d$. $\square$

We now characterize the structure of d-regular, bipartite graphs with $s(G) = d$.

Theorem 1. *Suppose $G = (V,E)$ is a regular, bipartite graph with degree $d \geq 2$ and $s(G) = d$. Let S be a secure set with $|S| = d$. Then we can label the bipartition (Y,Z) in such a way that $S = A \cup B$, $A \subseteq Y$, $B \subseteq Z$, $|A| = \left\lceil \frac{d}{2} \right\rceil$, and $|B| = \left\lfloor \frac{d}{2} \right\rfloor$. Furthermore, for $u, v \in A$, $B \subseteq N(u) = N(v)$, and for $w, z \in B$, $A \subseteq N(w) = N(z)$. Conversely, if G has a bipartition (Y,Z) with $A \subseteq Y$ and $B \subseteq Z$ satisfying the preceding conditions, then*

$S = A \cup B$ *is a secure set in* G *with* $|S| = d$.

Proof. Name the bipartition (Y, Z) in such a way that $|S \cap Y| \geq |S \cap Z|$. Let $A = S \cap Y$ and $B = S \cap Z$. Suppose $|A| \geq \lceil \frac{d}{2} \rceil + 1$. Then $|B| \leq \lfloor \frac{d}{2} \rfloor - 1$. Let $u \in A$. Then $|N(u) - S| = d - |N(u) \cap B| \geq d - (\lfloor \frac{d}{2} \rfloor - 1) = \lceil \frac{d}{2} \rceil + 1 > \lfloor \frac{d}{2} \rfloor \geq |N[u] \cap S|$, which contradicts S being secure. Therefore $|A| \leq \lceil \frac{d}{2} \rceil$. Since $|A| \geq |B|$ we must have $|A| = \lceil \frac{d}{2} \rceil$, and thus $|B| = \lfloor \frac{d}{2} \rfloor$. Now $d = |S| \geq |N[S] - S|$. For $u \in A$, $|N(u) - S| \geq d - |B| = \lceil \frac{d}{2} \rceil$. Likewise, for $w \in B$, $|N(w) - S| \geq d - |A| = \lfloor \frac{d}{2} \rfloor$. Since $(N(u) - S) \cap (N(w) - S) = \emptyset$, we have $|N[S] - S| \geq |N(u) - S| + |N(w) - S| \geq d$. So we must have $|N[S] - S| = d$, which means that $|N(u) - S| = \lceil \frac{d}{2} \rceil$, $|N(v) - S| = \lfloor \frac{d}{2} \rfloor$, $N[S] - S = (N(u) \cup N(w)) - S$, $N(u) \cap S = B$, and $N(w) \cap S = A$. So for any $v \in A$, $v \neq u$, $N(v) - S \subseteq N(u) - S$, and $N(v) \cap S \subseteq B$. Since $N(u) = d$ and $|N(u) - S| + |B| = d$, we must have $N(v) = N(u)$. Similarly, for any $z \in B$, $z \neq w$, we have $N(z) = N(w)$. Now suppose that G has a bipartition (Y, Z) with $A \subseteq Y$ and $B \subseteq Z$ satisfying the preceding properties. By Theorem BDH, $S = A \cup B$ is secure in G if for all $X \subseteq S$, $|N[X] \cap S| \geq |N[X] - S|$. We examine three cases. First, if $\emptyset \neq X \subseteq A$, then $|N[X] \cap S| = |X| + |B| = |X| + \lfloor \frac{d}{2} \rfloor \geq \lceil \frac{d}{2} \rceil = |N[A] - S| \geq |N[X] - S|$. If $\emptyset \neq X \subseteq B$, then $|N[X] \cap S| = |X| + |A| = |X| + \lceil \frac{d}{2} \rceil > \lfloor \frac{d}{2} \rfloor = |N[B] - S| \geq |N[X] - S|$. Lastly, if $X \cap A \neq \emptyset$ and $X \cap B \neq \emptyset$, then $|N[X] \cap S| = |A| + |B| = d = |N[S] - S| \geq |N[X] - S|$. So $S = A \cup B$ is secure and $|S| = d$. $\square$

Figure 6.1, below, uses the notation of Theorem 1. In the case G is 4-regular and bipartite, Figure 6.1 shows the subgraph of G induced by the set of edges with at least one end in the secure set $S = A \cup B$.

For any regular, bipartite graph the number of vertices in one part of the bipartition must equal the number of vertices in the other part. So the total number of vertices must be even. The d-regular, bipartite graph with the fewest vertices is $K_{d,d}$, and $s(K_{d,d}) = d$.

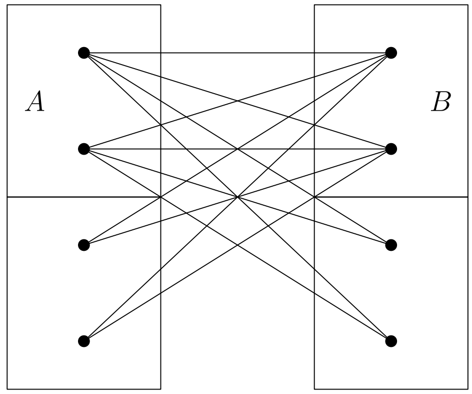

Figure 6.1: $S = A \cup B$ is secure in a 4-regular, bipartite graph.

Lemma 1. *There does not exist a d-regular, bipartite graph $G = (V, E)$ such that $s(G) = d$ and $2d < |V| < 3d$.*

Proof. We proceed by contradiction. Suppose that $G = (V, E)$ is a d-regular, bipartite graph such that $s(G) = d$ and $2d < |V| < 3d$. We can label the bipartition (Y, Z) in such a way that there is a secure set $S = A \cup B$ as in Theorem 1. Letting $A' = N(B) - A$ and $B' = N(A) - B$, it follows that $|A \cup A'| = |B \cup B'| = d$. Since $d < |Z|$ there exists $v \in Z - (B \cup B')$ such that $|N(v)| = d$ and $N(v) \subseteq Y - A$. Since $|Y| < \frac{3d}{2}$ and $|A| = \lceil \frac{d}{2} \rceil$, it follows that $|N(v)| \leq |Y - A| < \frac{3d}{2} - \lceil \frac{d}{2} \rceil \leq d$, contradicting $|N(v)| = d$. $\square$

Lemma 2. *If $d > 0$ is even, there exists a d-regular, bipartite graph $G = (V, E)$ such that $s(G) = d$ and $|V| = 3d$. If d is odd, there exists a d-regular, bipartite graph $G = (V, E)$ such that $s(G) = d$ and $|V| = 3d + 1$.*

Proof. First suppose that d is even. Let V be a set such that $|V| = 3d$. Partition the V into two sets, Y and Z such that $|Y| = |Z| = \frac{3d}{2}$. Partition vertices of Y into three sets A, A', A'' such that $|A| = |A'| = |A''| = \frac{d}{2}$. Partition the vertices of Z into three sets B, B', B'' such that $|B| = |B'| = |B''| = \frac{d}{2}$. Define the edge set

$$E = \{uv|\, u \in A, v \in B \cup B'\} \cup \{uv|\, u \in A', v \in B \cup B''\} \cup \{uv|\, u \in A'', v \in B' \cup B''\}.$$

Letting $G = (V, E)$ and $S = A \cup B$, S satisifes the conditions of Theorem 1 and is secure in G.

Now suppose that d is odd. Let V be a set such that $|V| = 3d + 1$. Partition the V into two sets, Y and Z such that $|Y| = |Z| = \frac{3d+1}{2}$. Partition the vertices of Y into three sets A, A', A'' such that $|A| = |A''| = \frac{d+1}{2}$ and $|A'| = \frac{d-1}{2}$. Partition the vertices of Z into three sets B, B', B'' such that $|B| = \frac{d-1}{2}$ and $|B'| = |B''| = \frac{d+1}{2}$. Let $r = \frac{d+1}{2}$, $A'' = \{a_1, ..., a_r\}$ and $B' = \{b_1, ..., b_r\}$. Let $F = \{a_i b_i|\, 1 \le i \le r\}$. Define the edge set $E = (\{uv|\, u \in A, v \in B \cup B'\} \cup \{uv|\, u \in A', v \in B \cup B''\} \cup \{uv|\, u \in A'', v \in B' \cup B''\}) - F$. Letting $G = (V, E)$ and $S = A \cup B$, S satisfies the conditions of Theorem 1 and is secure in G. $\square$

Theorem 2. *There exists a d-regular, bipartite graph $G = (V, E)$ such that $s(G) = d$, if and only if $|V| = 2d$, or $|V| \ge 3d$ and $|V|$ is even.*

Proof. Let $G = (V, E)$ be a d-regular, bipartite graph such that $s(G) = d$. The discussion preceding Lemma 1 shows that $|V|$ must be even and that $K_{d,d}$ satisfies $s(G) = d$ and $|V| = 2d$. So if G is not isomorphic to $K_{d,d}$, then, by Lemma 1, $|V| \ge 3d$. We proceed by induction, Lemma 2 providing the base when d is even or d is odd.

Let $|V| \ge 3d + 2$ and $|V|$ even. Let $H = (U, F)$ be a d-regular, bipartite graph with $|U| = |V| - 2$ satisfying $s(H) = d$. As in Theorem 1, we can write the bipartition of H as (Y, Z) and find a secure set $S = A \cup B$. Construct the graph $G = (V, E)$ as follows. Add a vertex u to Y and a vertex v to Z. We now find a set of d independent edges in H, using Hall's Theorem.

Case 1: d is even. Let $Y' = (V(H) - S) \cap Y$ and $Z' = (V(H) - S) \cap Z$. Since $|U| = |V| - 2 \geq 3d$, we have $|Y| = |Z| \geq \frac{3d}{2}$ so that $|Y'| = |Z'| = |Y| - \frac{d}{2} \geq d$. Also, Y' has $\frac{d}{2}$ vertices of degree $\frac{d}{2}$ in $H - S$, and every other vertex has degree d. Let $\mathcal{D} \subseteq Y'$ such that $|\mathcal{D}| = d$. Examine the graph Γ induced by $\mathcal{D} \cup Z'$. Let $X \subseteq \mathcal{D}$. If X has a vertex of degree d, then $|X| \leq d \leq N_\Gamma(X)$. If X does not have a vertex of degree d, then each vertex in X has degree $\frac{d}{2}$, and $|X| \leq \frac{d}{2} \leq N_\Gamma(X)$. So by Hall's Theorem, there must be a matching in Γ that saturates $\mathcal{D}$. Let M be this set of d independent edges, which are also independent in $H - S$ and H.

Case 2: d is odd. Let $Y' = (V(H) - S) \cap Y$ and $Z' = (V(H) - S) \cap Z$. Since $|U| = |V| - 2 \geq 3d + 1$, we have $|Y| = |Z| \geq \frac{3d+1}{2}$, so that $|Y'| = |Y| - |Y \cap S| \geq \frac{3d+1}{2} - \frac{d+1}{2} = d$ and $|Z'| = |Z| - |Z \cap S| \geq \frac{3d+1}{2} - \frac{d-1}{2} = d + 1$. In $H - S$, Y' has $\frac{d-1}{2}$ vertices of degree $\frac{d+1}{2}$, and every other vertex of Y' has degree d. Let $\mathcal{D} \subseteq Y'$ such that $|\mathcal{D}| = d$. Examine the graph Γ induced by $\mathcal{D} \cup Z'$. Let $X \subseteq \mathcal{D}$. If X has a vertex of degree d, then $|X| \leq d \leq N_\Gamma(X)$. If X does not have a vertex of degree d, then each vertex in X has degree $\frac{d+1}{2}$, and $|X| \leq \frac{d-1}{2} < \frac{d+1}{2} \leq N_\Gamma(X)$. So by Hall's Theorem, there must be a matching in Γ that saturates $\mathcal{D}$. Let M be this set of d independent edges, which are also independent in $H - S$ and H.

Next we decribe how to obtain the edge set E from F. For each $w_1 w_2 \in M$, $w_1 \in Y - S$ and $w_2 \in Z - S$, delete $w_1 w_2$ from F, and replace it by two edges: $w_1 v$ and $w_2 u$. Thus the degree of every vertex of $(Y - S) \cup (Z - S)$ is not changed, and the degree of u and of v is d. Then $G = (V, E)$ is a d-regular, bipartite graph on $|V|$ vertices. But the neighbors of vertices in S in G are the same as in H, so S is secure in $G = (V, E)$. $\square$

Proposition 2. *Let $G = (V, E)$ be a d-regular, bipartite graph such that $s(G) = d$. If $|V| \in \{2d, 3d, 3d + 1\}$ then the graph G is unique up to isomorphism.*

Proof. Let S be a secure set in $G = (V, E)$ of order d. By Theorem 1, we can lable the bipartition of G (Y, Z) so that there is a secure set $S = A \cup B$ where A and B have the properties listed. Let $A' = N(B) - A$ and $B' = N(A) - B$. Note that $|A'| = \left\lfloor \frac{d}{2} \right\rfloor$ and $|B'| = \left\lceil \frac{d}{2} \right\rceil$.

Suppose $|V| = 2d$. In this case, $Y = A \cup A'$, $Z = B \cup B'$, $|Y| = d$, and $|Z| = d$. Since every vertex has degree d, the graph must be isomorphic to $K_{d,d}$.

Now suppose that $|V| = 3d$. In this case, d must be even. Let $A'' = Y - (A \cup A')$ and $B'' = Z - (B \cup B')$. Note that $|A| = |A'| = |A''| = |B| = |B'| = |B''| = \frac{d}{2}$. Let $F = \{uv \,|\, u \in A, v \in B \cup B'\} \cup \{uv \,|\, u \in B, v \in A'\}$. By Theorem 1, $F \subseteq E$. Note that every vertex of $A \cup B$ is incident with d edges of F. So no vertex in A is adjacent to a vertex in B'', and no vertex in B is adjacent to a vertex in A''. In order for each vertex of A'' to have degree d, it must be adjacent to every vertex in $B' \cup B''$. Likewise, every vertex of B'' must be adjacent to every vertex of $A' \cup A''$. This forces $E = \{uv|\, u \in A, v \in B \cup B'\} \cup \{uv|\, u \in A', v \in B \cup B''\} \cup \{uv|\, u \in A'', v \in B' \cup B''\}$.

Lastly, suppose $|V| = 3d + 1$. In this case, d must be odd. Again, let $A'' = Y - (A \cup A')$ and $B'' = Z - (B \cup B')$. Note that $|A| = |A''| = |B'| = |B''| = \frac{d+1}{2}$ and $|B| = |A'| = \frac{d-1}{2}$. Let $F_1 = \{uv \,|\, u \in A, v \in B \cup B'\} \cup \{uv \,|\, u \in B, v \in A'\}$. By Theorem 1, $F_1 \subseteq E$. Every vertex of $A \cup B$ is incident with d edges in F_1. So no vertex of B'' is adjacent to any vertex of A. Likewise, no vertex of A'' is adjacent to any vertex of B. In order to have degree d, every vertex of B'' must be adjacent to every vertex of $A' \cup A''$. It remains to describe the edges between vertices of A'' and vertices of B'. Since G contains no edges between vertices of A' and vertices of B', and every vertex of A'' is adjacent to every vertex of B'', the subgraph H of G induced by $A'' \cup B'$ is regular of degree $\frac{d-1}{2}$. Since H is bipartite it is therefore obtained from $K_{\frac{d+1}{2}, \frac{d+1}{2}}$ by removing a perfect matching. Clearly the different versions of G obtained by removing different perfect matchings in H are isomorphic. $\square$

6.2 On the Security Number of Q_n

The n-cube, denoted Q_n, can be defined inductively. Let $Q_0 \cong K_1$; then $Q_n \cong Q_{n-1} \Box K_2$. Note that $|V(Q_n)| = 2^n$ and $|E(Q_n)| = n2^{n-1}$. Also note that Q_n is an n-regular bipartite graph. By examination, $s(Q_0) = s(Q_1) = 1$, and $s(Q_2) = 2$. If the number of vertices of a subgraph of Q_n is known, Graham's Density Lemma gives an upper bound on the number of edges that subgraph may contain. It is used in the proofs of the results after it.

Graham's Density Lemma ([7, 2]). *Let G be a subgraph of Q_n. Then $|E(G)| \leq \frac{1}{2}|V(G)|\log_2 |V(G)|$, with equality if and only if G is isomorphic to Q_m, for some $m \in \{0, 1, ..., n\}$.*

Proposition 3. *For $n \geq 1$, $s^u(Q_n) = 2^{n-1}$, and the only minimum ultra-secure sets of vertices induce subgraphs isomorphic to Q_{n-1}.*

Proof. Let $S \subseteq V(Q_n)$ be ultra-secure. Since Q_n is n-regular, the sum of the degrees of the vertices of S is $n|S|$. Applying the Theorem of Chapter 4.1 for ultra-security with $X = S$, we have $|S| \geq \sum_{x \in S} |N[x] - S|$. In other words, the number of edges with one end in S and one end in $N[S] - S$ can be at most $|S|$. Let G be the subgraph of Q_n induced by S. Then $\sum_{v \in S} d_G(v) \geq n|S| - |S|$, and thus $|E(G)| \geq \frac{1}{2}|S|(n-1)$. By Graham's Lemma, $\frac{1}{2}|S|(n-1) \leq \frac{1}{2}|S|\log_2 |S|$, and so $2^{n-1} \leq |S|$. If $|S| = 2^{n-1}$, then $|E(G)| = 2^{n-2}(n-1)$ and we have equality in Graham's Density Lemma. Thus G is isomorphic to Q_{n-1}. Then for $x \in S$, $|N[x] - S| = 1$. Letting each vertex of S defend itself, we see that S is ultra-secure. Therefore, $s^u(G) = 2^{n-1}$ and any minimum ultra-secure set of Q_n must induce a subgraph isormorphic to Q_{n-1}. $\Box$

Proposition 4. *For $n \geq 1$, $2^{\lfloor \frac{n}{2} \rfloor} \leq s(Q_n)$.*

Proof. Let $S \subseteq V(Q_n)$ be secure. Let G be the subgraph of Q_n induced by S. Then for each $v \in S$, $1 + d_G(v) = |N[v] \cap S| \geq \frac{1}{2}|N[v]| = \frac{1}{2}(n+1)$. Thus $d_G(v) \geq \left\lceil \frac{n-1}{2} \right\rceil = \left\lfloor \frac{n}{2} \right\rfloor$. So G must have at least $\frac{1}{2}|S| \left\lfloor \frac{n}{2} \right\rfloor$ edges. Applying Graham's Lemma, $\frac{1}{2}|S| \left\lfloor \frac{n}{2} \right\rfloor \leq \frac{1}{2}|S| \log_2 |S|$, which yields $2^{\left\lfloor \frac{n}{2} \right\rfloor} \leq |S|$. $\square$

Corollary. *For $n \geq 1$, $2^{\left\lfloor \frac{n}{2} \right\rfloor} \leq s(Q_n) \leq s_{(F,I)}(Q_n) \leq 2^{n-1}$.*

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Appendices

Appendix A

An Alternative Proof that (I,I)-security Implies (F,F)-security

The notation here is as in Chapter 2. This proof predates the more elegant proof included in Chapter 2. It essentially follows the proof of Theorem BDH by Brigham et al [3].

Theorem. *Let $G = (V, E)$ be a graph and $S \subseteq V$. If S is (I,I)-secure, then S is (F,F)-secure.*

Let $S \subseteq S' \subseteq V$. Let A be an attack on S' such that $A^*(v) = 0$ for all $v \in S' - S$. Let $\mathscr{A} = \{s \in S' | \, A^*(s) > D^*(s)\}$. Note that if $s \in S' - S$, then $s \notin \mathscr{A}$, because $A^*(s) = 0 \leq D^*(s)$. A *best defense* of S is a defense D such that $\sum_{s \in \mathscr{A}} A^*(s) - D^*(s)$ is minimized. Assume in these best defenses, that every defending vertex sends out its entire unit of defense; that is, for $v \in S'$, and any best defense D, $\sum_{u \in N[v] \cap S} D(v, u) = 1$. An attack A is defendable if there is a defense D for which $\sum_{s \in \mathscr{A}} A^*(s) - D^*(s) = 0$; that is, $\mathscr{A} = \emptyset$.

If D is a best defense, another best defense D' can be found through one of two possible transformations. For $i \in S'$ such that $D^*(i) > A^*(i)$ and $j \in S'$,

1) If $D(j, i) > 0$, define $\gamma_1(i, j) = \min\{D(j, i), D^*(i) - A^*(i)\}$. Reduce $D(j, i)$ by $\gamma_1(i, j)$ and increase $D(j, j)$ by $\gamma_1(i, j)$.

2) If $D(i, i) > 0$ and $D(j, i) = 0$, define $\gamma_2(i) = \min\{D(i, i), D^*(i) - A^*(i)\}$. Reduce $D(i, i)$ by $\gamma_2(i)$ and increase $D(i, j)$ by $\gamma_2(i)$.

(Note that for both transformations, j must satisfy $D^*(j) \geq A^*(j)$, or D would not be a best defense.)

Given a best defense D, let $\mathscr{D}$ be the set consisting of D and every defense that can be derived from D by repeated applications of transformations 1) and 2). Note that if $D^*(u) \geq A^*(u)$ for some $D \in \mathscr{D}$, then $D^*(u) \geq A^*(u)$ for all $D \in \mathscr{D}$. Let $V^+ \subseteq S'$ be

$V^+ = \{v \in S' | D^*(v) > A^*(v) \text{ for some } D \in \mathscr{D}\}$. Let $X = S' - V^+$.

Lemma 1. *Let $x \in X$, $v \in S'$. Then $D(v,x) = D'(v,x)$ and $D(x,v) = D'(x,v)$, for all defenses D' in $\mathscr{D}$.*

Proof. Let $\Delta, \Delta' \in \mathscr{D}$ such that Δ' is obtained from Δ by one of the transformations. If $x \in X$ is involved in transformation 1) or 2), then it must play the role of j, because $i \in V^+$, and $x \notin V^+$. So after the transformation $(\Delta')^*(x) > \Delta^*(x)$. Note that $\Delta^*(x) \leq A^*(x)$ because $x \notin V^+$. If $\Delta^*(x) = A^*(x)$, then after the transformation, $(\Delta')^*(x) > A^*(x)$, and $x \in V^+$, a contradiction. If $\Delta^*(x) < A^*(x)$, then after the transformation, $A^*(x) - \Delta^*(x) > A^*(x) - (\Delta')^*(x)$, contradicting that Δ is a best defense. $\Box$

Lemma 2. *Let $\Delta \in \mathscr{D}$. Every $s \in N[X] \cap S'$ satisfies $\Delta(s,v) = 0$ for $v \in V^+ = S' - X$.*

Proof. Let $s \in X$. Suppose that $\Delta(s,v) > 0$, for some $v \in V^+$. Since $v \in V^+$, there is a $\Delta' \in \mathscr{D}$ such that $(\Delta')^*(v) > A(v)$. Applying transformation 1) with $i = v$ and $j = s$ would result in a new defense Δ'' with $\Delta''(s,s) > \Delta'(s,s)$, contradicting Lemma 1.

Now let $s \in (N[X] \cap S') - X$. Note that $s \in V^+$. Suppose that $\Delta(s,y) > 0$ for some $y \in V^+$. Now s must have a neighbor $x \in X$ and by the first part of this proof, $\Delta(x,s) = 0$. We can assume $\Delta^*(y) > A^*(y)$ ($y \in V^+$, so $\Delta^*(y) \geq A^*(y)$. If $\Delta^*(y) = A^*(y)$, a series of transformations will result in a a defense D' such that $(D')^*(y) > A^*(y)$. The vertex y does not play the role of i, because i already satisfies $\Delta^*(i) > A^*(i)$. If $y = j$ and $s = i$, then in either transformation, s still sends defense to y afterwards). Applying transformation 1) with $j = s$ and $i = y$ results in a defense Δ' such that $(\Delta')^*(s) > A^*(s)$. (We started with $\Delta^*(s) \geq A^*(s)$.) Now apply transformation 2) with $j = x$ and $i = s$. This results in a defense Δ'' where $\Delta''(s,x) > \Delta'(s,x)$, contradicting Lemma 1. $\Box$

Suppose that S' is not defendable from an attack A, where $A^*(v) = 0$ for all $v \in S' - S$. Let V^+ and X be as above and let $X' = X \cap S$. If $v \in X - X'$, then $A^*(v) = 0$ and so $D^*(v) = 0$, because $v \notin V^+$. So by Lemma 2, for any $s \in N[X] \cap S'$, $v \in N[s] - X'$, $D(s, v) = 0$. In other words, all the defense power of $N[X] \cap S'$ is sent into X'. We have

$$
\begin{aligned}
|N[X] \cap S'| = \sum_{x \in X'} D^*(x) < \sum_{x \in X} A^*(x) &= \sum_{x \in X} \sum_{v \in N[X]-S'} A(v, x) \\
&= \sum_{v \in N[X]-S'} \sum_{x \in X} A(v, x) \leq \sum_{v \in N[X]-S'} 1 = |N[X] - S'|.
\end{aligned}
$$

Thus there is an $X \subseteq S'$ such that $|N[X] \cap S'| < |N[X] - S'|$. Lemma 3, below, states this result.

Lemma 3 *Let $G = (V, E)$ be a graph and $S \subseteq S' \subseteq V$. Let A be an attack on S', such that for $v \in S' - S$, $A^*(v) = 0$. If there is no successful defense of A, then there is an $X \subseteq S$ such that $|N[X] \cap S'| < |N[X] - S'|$.*

Letting $S = S'$ in Lemma 3, shows that if S is not (F,F)-secure, then there exists an $X \subseteq S$, such that $|N[X] \cap S| < |N[X] - S|$, and the Theorem follows.

Appendix B

A Table of Security Numbers

G	$s(G)$	$s_{(\mathrm{F,I})}(G)$	$s^u(G)$
P_n	1	1	1
C_n	2	2	2
F_n $n \geq 2$	2	$1 + \left\lceil \frac{n}{2} \right\rceil$	$1 + \left\lceil \frac{n}{2} \right\rceil$
W_n $n \geq 4$	3	$1 + \left\lceil \frac{n+1}{2} \right\rceil$	$1 + \left\lceil \frac{n+1}{2} \right\rceil$
$P_m \square P_n$	$\min\{m, n, 3\}$	$\min\{m, n, 4\}$	$\min\{m, n, 4\}$
K_n $n \geq 2$	$\left\lceil \frac{n}{2} \right\rceil$	$n - 1$	$n - 1$
$K_{m,n}$ $2 \leq m \leq n$, $n \neq 2$	$\left\lceil \frac{m+n}{2} \right\rceil$	$m + \left\lfloor \frac{n}{2} \right\rfloor$	$\min\{m + n - 2, m + \left\lceil \frac{m(n-1)}{m+1} \right\rceil\}$
$K_{n_1,\ldots,n_k}$ $k \geq 3$, $n = \sum_{i=1}^{k} n_i$, $n_k = \max_{1 \leq i \leq k}\{n_i\}$	$\left\lceil \frac{n}{2} \right\rceil$	$n - \left\lceil \frac{n_k}{2} \right\rceil$	$n - \left\lfloor \frac{n}{n - n_k + 1} \right\rfloor$

Table B.1: Security Numbers