Maximal Sets of Hamilton Cycles in Complete Multipartite Graphs

by

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Abstract

A set of S edge-disjoint hamilton cycles in a graph G is said to be *maximal* if the hamilton cycles in S form a subgraph of G such that $G - E(S)$ has no hamilton cycle. The set of integers m for which a graph G contains a maximal set of m edge-disjoint hamilton cycles has previously been determined whenever G is a complete graph, a complete bipartite graph, and in many cases when G is a complete multipartite graph. In this dissertation, some of the remaining open cases regarding complete multipartite graphs will be resolved.

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Chapter 1

Introduction

1.1 Definitions

The *complete multipartite graph*, K_n^p is the graph with np vertices that have been partitioned into p parts of size n such that an edge exists between vertices u and v if and only if u and v are in different parts. A *hamilton cycle* in a graph G is a spanning cycle of G. If S is a set of hamilton cycles, then let $G(S)$ be the graph induced by the edges in cycles of S. We denote the edges in this graph by $E(G(S))$ or $E(S)$. The set S is maximal in G if $G - E(S)$ has no hamilton cycle.

1.2 History

Considerable research has come before this dissertation to find maximal sets of hamilton cycles in certain graphs. Hoffman, Rodger, and Rosa [7] found that there exists a maximal set S of m edge-disjoint hamilton cycles in K_n if and only if $m \in \left\{ \left\lfloor \frac{n+3}{4} \right\rfloor, \left\lfloor \frac{n+3}{4} \right\rfloor + 1, ..., \left\lfloor \frac{n-1}{2} \right\rfloor \right\}$. It was later shown by Bryant, El-Zanati, and Rodger [1] that there exists a maximal set S of m edge-disjoint hamilton cycles in $K_{n,n}$ if and only if $\frac{n}{4} < m \leq \frac{n}{2}$. Daven, MacDougall and Rodger [3] extended the results to complete multipartite graphs, showing that there exists a maximal set of m hamilton cycles in K_n^p if and only if

(a) $\left\lceil \frac{n(p-1)}{4} \right\rceil \leq m \leq \left\lfloor \frac{n(p-1)}{2} \right\rfloor$ and

(b) $m > \frac{n(p-1)}{4}$ if

1. n is odd and $p \equiv 1 \pmod 4$, or

2. $p = 2$, or

3. $n = 1$,

except possibly for the undecided case when $n \geq 3$ is odd, p is odd and $m \leq \frac{(n+1)(p-1)-2}{4}$.

Jarrell and Rodger [8] solved these open cases when $n \geq 5$, and removed all but the smallest possible exceptional values when $n = 3$, showing that a maximal set of hamilton cycles of size m exists when $n = 3$ and $\left\lceil \frac{n(p-1)}{4} \right\rceil + 1 \leq m \leq \left\lfloor \frac{(n+1)(p-1)-2}{4} \right\rfloor$, with strict inequality for the lower bound of m when $p \equiv 1 \pmod 4$. Together, these results mean that for each odd value of p, exactly one value of m remains in doubt (namely $\left\lceil \frac{n(p-1)}{4} \right\rceil + 1$ for $p \equiv 1 \pmod 4$ and $\left\lceil \frac{n(p-1)}{4} \right\rceil$ for $p \equiv 3 \pmod 4$) and even that is only in doubt in the case when $n = 3$. Naturally, each remaining case becomes more and more difficult. Indeed, for some time it was unclear whether the remaining values would be orders of maximal sets of hamilton cycles. To summarize, these existing results in the literature can be combined to produce the following theorem:

Theorem 1.1 *There exists a maximal set of m hamilton cycles in K_n^p if and only if*

1. $\left\lceil \frac{n(p-1)}{4} \right\rceil \leq m \leq \left\lfloor \frac{n(p-1)}{2} \right\rfloor$ *and*
2. $m > \frac{n(p-1)}{4}$ *if*
   (a) n *is odd and* $p \equiv 1 \pmod 4)$, *or*
   (b) $p = 2$, *or*
   (c) $n = 1$,

except possibly when $n = 3$, and $m = \left\lceil \frac{n(p-1)}{4} \right\rceil + 1$ and $p \equiv 1 \pmod 4$ or $m = \left\lceil \frac{n(p-1)}{4} \right\rceil$ and $p \equiv 3 \pmod 4$.

For this dissertation, the goal is to clear up most of these last remaining cases, namely those multipartite graphs where $n = 3$ and p is odd. We do so by splitting this into two cases, one in which $n = 3$ and $p \equiv 1 \pmod 4$, and the other in which $n = 3$ and $p \equiv 3$

(mod 4). The former case is completely resolved in Chapter 2, while the latter is resolved in Chapter 3 when $p \equiv 7 \pmod 8$ with two possible exceptions.

The work of this dissertation culminates in the following state of knowledge:

Theorem 1.2 *There exists a maximal set of m hamilton cycles in K_n^p if and only if*

1. $\left\lceil \frac{n(p-1)}{4} \right\rceil \leq m \leq \left\lfloor \frac{n(p-1)}{2} \right\rfloor$ *and*
2. $m > \frac{n(p-1)}{4}$ *if*
   (a) n *is odd and* $p \equiv 1 \pmod 4$), *or*
   (b) $p = 2$, *or*
   (c) $n = 1$,

except possibly when $n = 3$, $m = \left\lceil \frac{n(p-1)}{4} \right\rceil$ and either $p \equiv 3 \pmod 8$, $p \geq 19$ or $p \in \{15, 23\}$.

1.3 The Technique of Amalgamations

The approach used to prove the theorems of this dissertation is that of amalgamations. An *amalgamation* of a graph G is a graph H formed by a homomorphism $f : V(G) \to V(H)$. So for each $v \in V(H)$, the vertices of $f^{-1}(v)$ can be thought of as being amalgamated into the single vertex v in H; for each $v \in V(H)$, $\eta(v) = |f^{-1}(v)|$ is known as the amalgamation number of v. G is said to be a *disentanglement* of H.

In most of our proofs, an amalgamated graph is constructed in which each color class is connected and each vertex v is incident with $2\eta(v)$ edges of each color, thus looking like what would be obtained by amalgamating a graph in which each color class is a hamilton cycle. For our purposes, the two following results will be essential. The first result describes properties of a graph formed by amalgamating K_n. The second will be used to show that the amalgamated graph we construct can be disentangled ("pulled apart", if you will) to form a subgraph of K_3^p that has a hamilton decomposition.

Lemma 1.1 *[10] Let $G \cong K_n$ be an l-edge-colored graph, and let $f : V(G) \to V(H)$ be an amalgamating function with amalgamation numbers given by the function $\eta : V(H) \to \mathbb{N}$. Then H satisfies the following conditions for any vertices $w, v \in V(H)$:*

1. *$d(w) = \eta(w)(n-1)$,*

2. *$m(w, v) = \eta(w)\eta(v)$ if $w \neq v$*

3. *w is incident with $\binom{\eta(w)}{2}$ loops, and*

4. *$d_{H(i)}(w) = \sum_{u \in f^{-1}(w)} d_{G(i)}(u)$ for $1 \leq i \leq l$,*

where $m(w, v)$ is the number of edges between vertex w and vertex v,

Once our graph satisfies the conditions of Lemma 1.1, the following theorem allows us to disentangle the graph in such a way that we preserve connectivity and evenly divide the edge ends among the $\eta(u)$ disentangled vertices in each color class.

Theorem 1.3 *[10] Let H be an l-edge-colored graph satisfying conditions (1)-(3) of Lemma 1.1 for the function $\eta : V(H) \to N$. Then there exists a disentanglement G of H that satisfies*

1. *$G \cong K_n$,*

2. *for any $z \in V(H), |d_{G(i)}(v) - d_{G(i)}(u)| \leq 1$ for $1 \leq i \leq l$ and all $u, v \in f^{-1}(z)$,*

3. *if $\frac{d_{H(i)}(z)}{\eta(z)}$ is an even integer for all $z \in V(H)$, then $\omega(G(i)) = \omega(H(i))$.*

where $\omega(G)$ denotes the number of components for the graph G.

Another important result that is invaluable in the main proof is the following theorem proved by Hilton [6]. A k-edge-coloring of G is said to be *evenly equitable* if $|d_i(v) - d_j(v)| \leq 2$ for $1 \leq i < j \leq k$ and $d_i(v)$ is even for $1 \leq i \leq k$, where $d_i(v)$ is the degree of v in the subgraph induced by the edges colored i.

Theorem 1.4 *[6] For each $k \geq 1$, each finite Eulerian graph has an evenly equitable edge-coloring with k colors.*

The use of the previous lemma and theorems and the technique of amalgamations have allowed for much more efficient and streamlined proofs in these types of edge coloring problems.

Chapter 2

The $p \equiv 1 \pmod 4$ Case

2.1 Introduction

Our aim in this chapter is to solve half of the remaining open cases in Theorem 1.1; specifically the case when $n = 3$, $m = \left\lceil \frac{n(p-1)}{4} \right\rceil + 1$ and $p \equiv 1 \,(\mathrm{mod}\, 4)$.

In the following, edge-colorings are used to represent the hamilton cycles, so let $G(i)$ denote the subgraph of G induced by the edges colored i.

2.2 $p \equiv 1 \pmod 4$

Theorem 2.1 *For the complete multipartite graph K_n^p, let $p = 4x+1$ for some integer $x \geq 2$ and let $n = 3$. Then there exists a maximal set of $m = \left\lceil \frac{3(p-1)}{4} \right\rceil + 1 = 3x + 1$ edge disjoint hamilton cycles in K_n^p.*

Proof We define the hamilton cycles on the vertex set $\mathbb{Z}_p \times \mathbb{Z}_3$ in which the parts are $P_i = \{i\} \times \mathbb{Z}_3$ for each $i \in \mathbb{Z}_p$. As we look at this problem, it is helpful to think of the parts of the graph arranged in p vertical columns with three vertices in each column; so each part has a top, a middle, and a bottom vertex as shown in Figure 2.1. Our goal is to choose the edges for our set S of hamilton cycles wisely so that we ensure that our set is maximal. In each case S is shown to be maximal because $K_n^p - E(S) = G(S)$ has a cut vertex. We do this by splitting $V(G)$ into 3 sections. We denote by G_1 the subgraph induced by vertices in the first $2x$ parts (part 0 to part $2x-1$) together with the top vertex of the center part, which we call u. The subgraph induced by vertices in the last $2x$ parts (part $2x+1$ to part $4x$) together with the bottom vertex of the center part, which we will call w, is denoted by

G_2. Finally, the middle vertex of the center part will be called v. The vertex v will serve as a cut vertex in $G(S)$.

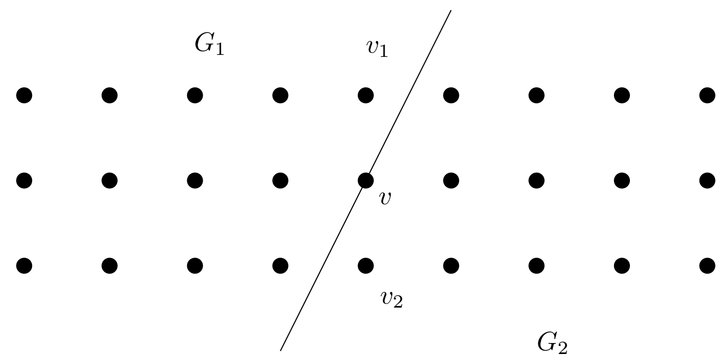

Figure 2.1: View of K_3^{4x+1}

The edges we choose to make our set of hamilton cycles fall into the following three types and are pictured in Figure 2.2:

Type 1: All edges in K_3^p that join vertices in G_1 to vertices in G_2 occur in $E(S)$.

Type 2: Precisely $2m$ edges joining vertices in $V(G_1 \bigcup G_2)$ to v occur in $E(S)$. (Approximately half of these edges are incident with vertices in G_1, while the others are incident with vertices in G_2.)

Type 3: Certain edges between two vertices in G_1 or two vertices in G_2 are finally chosen to make $G(S)$ $2m$-regular.

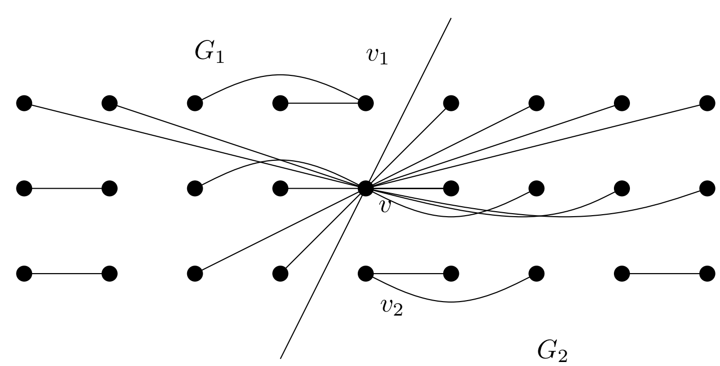

Figure 2.2: Type 2 and 3 Edges

As we select Type 2 edges, we note that v is not yet adjacent to any other vertices. Thus to produce $m = 3x + 1$ hamilton cycles, we need the degree of v to be $2(3x + 1) = 6x + 2$. If m is even, then $3x + 1$ of the edges are chosen to join vertex v to vertices in G_1, while the other $3x + 1$ are chosen to join vertex v and vertices in G_2. If m is odd, then $3x$ of the edges join vertex v and vertices in G_1, while the other $3x + 2$ join vertex v and vertices in G_2. For the Type 3 edges, we carefully pick edges between two vertices in G_1 or two vertices in G_2 so that we build each vertex to degree $6x + 2$. It turns out that the edges of Types 2 and 3 need not be precisely chosen if we use amalgamations to produce G. Instead we define H and let Theorem 1.3 produce G.

The method used to construct the hamilton cycles is that of amalgamations. This technique has been used successfully in this setting (see [8] for example). The amalgamation used here is the graph homomorphism $f : V(G) \rightarrow V(H) = (\mathbb{Z}_p \setminus \{2x\}) \bigcup (\{2x\} \times \mathbb{Z}_3)$ defined as follows. For each $i \in \mathbb{Z}_p \setminus \{2x\}$ and for each $j \in \mathbb{Z}_3$, let $f((i, j)) = i$, and for each $j \in \mathbb{Z}_3$ let $f(2x, j) = (2x, j) = u, v,$ or w if $j = 0, 1,$ or 2 respectively. So, except for the part containing v, f amalgamates the vertices in each part into a single vertex in H with amalgamation number 3. The vertices in the part containing v are not amalgamated by f, so each vertex z has amalgamation number $\eta(z) = 1$. So we will require that $d_H(i) = 6m = 6(3x + 1)$ for each $i \in \mathbb{Z}_p \setminus \{2x\}$ and $d_H(i) = 2m = 2(3x + 1)$ for each $i \in \{u, v, w\}$.

The subgraph B of H induced by the edges joining vertices in $\mathbb{Z}_{2x}$ to vertices in $\mathbb{Z}_{4x+1} \setminus \mathbb{Z}_{2x+1}$ is isomorphic to $9K_{2x,2x}$. Let $\varepsilon = 1$ or 0 if m is odd or even respectively. Join v to vertices in $\mathbb{Z}_{x+1-\varepsilon}$ and $\mathbb{Z}_{2x} \setminus \mathbb{Z}_{x+1-\varepsilon}$ with 2 and 1 edges respectively, and join v to vertices in $\mathbb{Z}_{3x+1+\varepsilon} \setminus \mathbb{Z}_{2x+1}$ and $\mathbb{Z}_{4x+1} \setminus \mathbb{Z}_{3x+1+\varepsilon}$ with 2 and 1 edges respectively; these produce Type 2 edges in G. Pair the vertices in $\mathbb{Z}_{2x} \setminus \mathbb{Z}_{x+1-\varepsilon}$ and pair the vertices in $\mathbb{Z}_{4x+1} \setminus \mathbb{Z}_{3x+1+\varepsilon}$, and join each such pair with an edge; these produce the Type 3 edges in G. (Notice that each set has an even size by definition of ε.)

Color the edges of H as follows. Since there exists a hamilton decomposition of $K_{2x,2x}$, the edges of B can be partitioned into $9x$ sets, each of which induces a hamilton cycle of H.

Let B_0, ..., B_{3x} be $3x+1$ of these $9x$ sets, and color the edges in B_i with i for each $i \in \mathbb{Z}_m$. Let $H_1 = H - \bigcup_{i\in\mathbb{Z}_{3x+1}} B_i$. Then $d_{H_1}(i) = 4m$ and $d_{H_1}(i,j) = 2m$.

We now give the subgraph H_1 of H an evenly equitable edge coloring with the $3x+1$ colors in $\mathbb{Z}_{3x+1}$. Such a coloring exists by Theorem 1.4. Thus in H_1 each color appears 4 times at each vertex z with $\eta(z) = 3$ and twice at each vertex z where $\eta(z) = 1$. So in H each color now appears 6 times at each vertex where $\eta = 3$ and once where $\eta = 1$. We are now assured that our color classes are connected and that each color appears on the appropriate number of edges, namely $2\eta(z)$, at each vertex z.

The aim now is to disentangle our graph so that we can pick out our maximal set of hamilton cycles. To be able to apply Lemma 1.1 we still must add more edges to H to form H^+ so that H^+ satisfies properties (1-4) of Lemma 1.1 (i.e. so that it is an amalgamation of K_{3p}). So add edges to H so that between each pair of vertices x and y there are: exactly nine edges if $|\{x,y\} \cap \{u,v,w\}| = 0$; exactly three edges if $|\{x,y\} \cap \{u,v,w\}| = 1$; and exactly one edge if $|\{x,y\} \cap \{u,v,w\}| = 2$. Finally, add three loops to each vertex not in $\{u,v,w\}$. All these additional edges and loops are colored 0. It is straightforward to check that H^+ satisfies properties (1-4) of Lemma 1.1, so we can now apply Theorem 1.3 to H^+ to produce G^+, and edge-colored copy of K_{3p}. Removing all edges in G^+ corresponding to loops in H^+ produces K_n^p, and then removing all remaining edges colored 0 produces G. Each color class in G is 2-regular by property (2) of Theorem 1.3, and is connected by property (3), so is a hamilton cycle. Removing the edges in these hamilton cycles from K_n^p in particular means that all Type 1 edges are removed, so produces a graph in which v is a cut vertex (it is actually the graph induced by the edges (not loops) colored 0 in G^+). So the required maximal set of hamilton cycles has been produced. ∎

So this chapter culminates in the following state of knowledge:

Theorem 2.2 *There exists a maximal set of m hamilton cycles in K_n^p if and only if*

1. $\left\lceil \frac{n(p-1)}{4} \right\rceil \leq m \leq \left\lfloor \frac{n(p-1)}{2} \right\rfloor$ *and*

2. $m > \frac{n(p-1)}{4}$ *if*

   (a) n *is odd and* $p \equiv 1 \pmod 4)$*, or*

   (b) $p = 2$*, or*

   (c) $n = 1$*,*

except possibly when $n = 3$*,* $p \equiv 3 \pmod 4$*, and* $m = \left\lceil \frac{n(p-1)}{4} \right\rceil$.

Chapter 3

The $p \equiv 3 \pmod 4$ Case

3.1 Introduction

In this chapter, we will resolve some of the cases in which $n = 3$ and $p \equiv 3 \pmod 4$. We begin in Section 3.2 by solving the two smallest cases individually, namely when $p = 7$ and when $p = 11$. Section 3.3 uses the solution for $p = 7$ outlined in Section 3.2 to provide two approaches when $p \equiv 7 \pmod 8$. One approach is based on a conjecture that, while unproven in general, does provide a solution for the specific case of K_3^{39}. The other approach uses theorems of Heinrich, Lindner, Rodger, and Burling (see [5] and [2]) that provide a solution when $n = 3$, $p = 7 + 8\alpha$, and $\alpha \geq 3$.

3.2 Small Cases $p = 7$ and $p = 11$

Theorem 3.1 *There exists a maximal set of $m = 5$ edge disjoint hamilton cycles in K_3^7.*

Proof We begin with the complete multipartite graph K_3^7, which we'll refer to as G throughout the proof. We view this complete multipartite graph as seven columns (parts named 0 to 6) and three rows (named 0, 1, and 2). Each vertex is then denoted as an ordered pair (i, j) where $i \in \{0, 1, 2, 3, 4, 5, 6\}$ (the part) and $j \in \{0, 1, 2\}$ (the row). We divide the vertices of the graph into three sections as follows:

$$L_0 = \{(0,0), (0,1), (0,2), (1,0), (1,1), (2,0), (2,1), (3,0), (4,0), (5,0)\},$$

$$R_0 = \{(1,2), (2,2), (3,2), (4,1), (4,2), (5,1), (5,2), (6,0), (6,1), (6,2)\}, \text{ and}$$

$$v = (3,1).$$

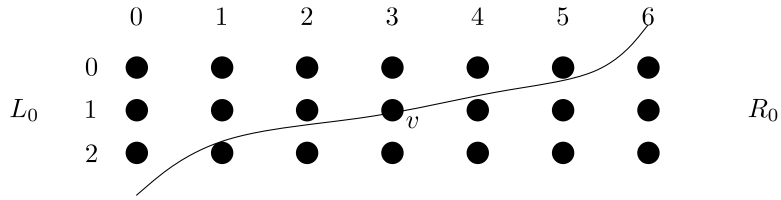

Figure 3.1: Setup of three sections with edge cut depicted

The proof is driven by carefully choosing edges to include in the set of hamilton cycles, S, so that v is a cut vertex in $G - E(S)$. These edges in our hamilton cycles fall into three categories: all edges between vertices in section L_0 and vertices in section R_0; certain edges between v and vertices in section L_0; and certain edges between v and vertices in section R_0 (see Figure 3.2). Our plan is to make our edge set 10-regular considering only these types of edges; it will be shown that there exists a hamilton decomposition of the subgraph induced by these edges. The sections L_0 and R_0 are named to reflect the fact that the edges in S join vertices on the left of the line in Figure 3.2 to vertices on the right, and the subscript 0 is added in anticipation of the generalization presented in 3.3.

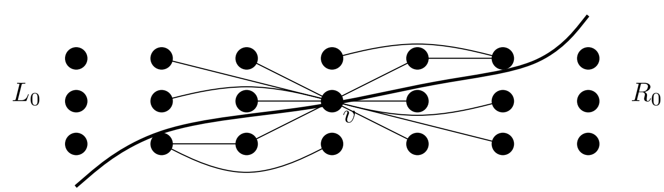

Figure 3.2: Special edges in $E(S)$

Next, we use the technique of amalgamations described in Chapter 1. Our amalgamation function $f : V(G[E(S)]) \to V(H)$ is defined as follows:

$$f(i,j) = \begin{cases} (0,0) & \text{if } i = 0, \\ (i,0) & \text{if } 1 \le i \le 2, j \le 1, \\ (i,0) & \text{if } 3 \le i \le 5, j = 0, \\ (i,1) & \text{if } 1 \le i \le 3, j = 2, \\ (i,1) & \text{if } 4 \le i \le 5, j \ge 1, \\ (6,1) & \text{if } i = 6, \end{cases}$$

and since it plays a special role, we define $f(3,1) = v$. So, $V(H)$ now looks like Figure 3.3.

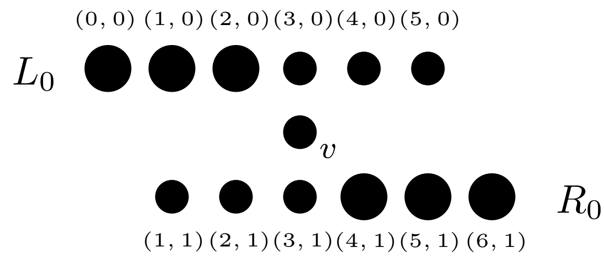

Figure 3.3: The Amalgamated H of G; the size of the vertex u represents the value of $f(u)$.

The critical part in the proof is to color the edges in H so that for $1 \le i \le 5$ (i) for all $u \in V(H)$, $d_{H(i)}(u) = 2f(u)$ and (ii) each color class $H(i)$ is connected. These conditions will allow us to disentangle H so that each color class is a hamilton cycle by properties 2 and 3 respectively of Theorem 1.3.

Our coloring is as follows:

First, each of the five paths below are colored in a different color, namely colors 1-5.

$P_1 = [(3,0),(6,1),(2,0),(5,1),(1,0),(4,1),(0,0),(3,1)]$

$P_2 = [(2,0),(1,1),(0,0),(3,1),(1,0),(2,1),(3,0),(5,1),(4,0),(6,1),(5,0),(4,1)]$

$P_3 = [(1,0),(4,1),(3,0),(6,1),(5,0),(1,1),(4,0),(2,1),(0,0),(3,1),(2,0),(5,1)]$

$P_4 = [(1,0),(4,1),(5,0),(2,1),(0,0),(1,1),(3,0),(6,1),(4,0),(3,1),(2,0),(5,1)]$

$P_5 = [(2,0),(1,1),(0,0),(2,1),(1,0),(3,1),(5,0),(6,1),(4,0),(5,1),(3,0),(4,1)]$

Each of these paths is then joined to a path created from the edges in Figure 3.2, namely

$Q_1 = [(3,1),(1,1),(2,1),v,(4,0),(5,0),(3,0)]$

$Q_2 = [(4,1),v,(2,0)]$

$Q_3 = [(5,1),v,(1,0)]$

$Q_4 = [(5,1),v,(1,0)]$

$Q_5 = [(4,1),v,(2,0)]$

This creates the following cycles $C_i = P_i \cup Q_i$:

$C_1 = ((3,0),(6,1),(2,0),(5,1),(1,0),(4,1),(0,0),(3,1),(1,1),(2,1),v,(4,0),(5,0))$

$C_2 = ((2,0),(1,1),(0,0),(3,1),(1,0),(2,1),(3,0),(5,1),(4,0),(6,1),(5,0),(4,1),v)$

$C_3 = ((1,0),(4,1),(3,0),(6,1),(5,0),(1,1),(4,0),(2,1),(0,0),(3,1),(2,0),(5,1),v)$

$C_4 = ((1,0),(4,1),(5,0),(2,1),(0,0),(1,1),(3,0),(6,1),(4,0),(3,1),(2,0),(5,1),v)$

$C_5 = ((2,0),(1,1),(0,0),(2,1),(1,0),(3,1),(5,0),(6,1),(4,0),(5,1),(3,0),(4,1),v)$

Illustrations of these paths are given in Figure 3.4.

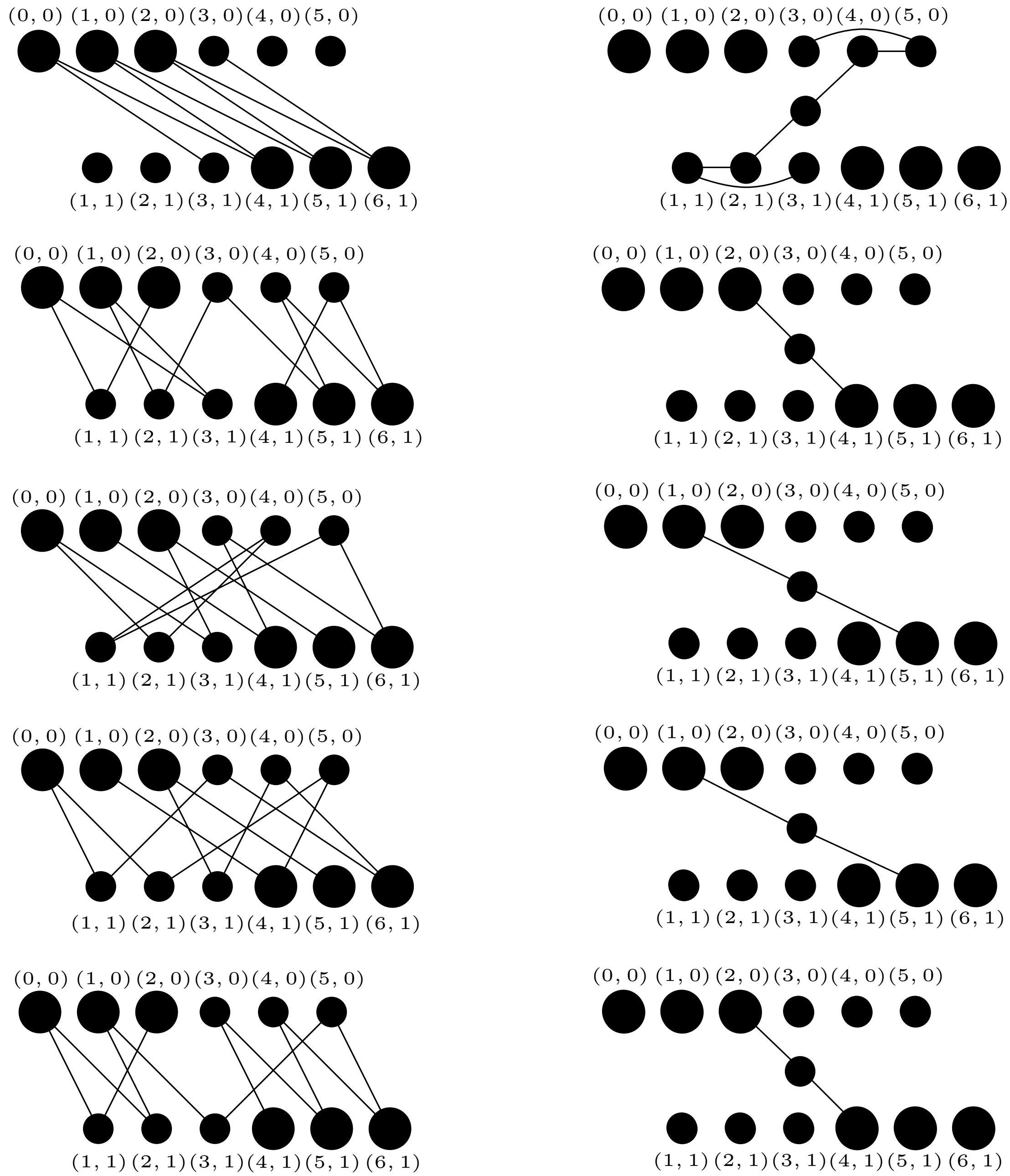

Figure 3.4: $C_i = P_i \cup Q_i$

We have now colored most of the edges in $E(S)$. Left to color are the edges between $\{(1,0),(2,0),(3,0)\} \in L_0$ and $\{(4,1),(5,1),(6,1)\} \in R_0$ that were not used in the cycles denoted by C_i above. Each vertex has degree ten, so we will give these edges an evenly equitable edge coloring with five colors, namely 1-5. Now each of our five colors appears at each vertex u a total $2f(u)$ more times and each color class is connected.

The next goal is to disentangle the graph H. To be able to apply Lemma 1.1, we need to use the same technique as in the $p \equiv 1 \pmod 4$ case which required adding edges to our graph H so that it is the amalgamation of the complete graph K_{3p}. All of these additional edges and loops are colored 0. We call this new graph H^+ and note that it now satisfies the conditions of Lemma 1.1. Thus we apply Theorem 1.3 to H^+ to produce G^+, which is an edge-colored copy of K_{3p}. We now remove all edges in G^+ colored 0. Using Theorem 1.3, we see that each color class is 2-regular by *(i)* and connected by *(ii)*, which implies that each color class is a hamilton cycle. We now consider $G - E(S)$. Note that all edges between vertices in L_0 and R_0 were in $E(S)$, so we have that $G - E(S)$ has cut vertex v. So our set of hamilton cycles is maximal. ■

Theorem 3.2 *There exists a maximal set of $m = 8$ edge disjoint hamilton cycles in K_3^{11}.*

Proof We begin with the complete multipartite graph K_3^{11}, which we'll refer to as G throughout the proof. This construction will be very similar to that of K_3^7 from Section 3.1. We view this complete multipartite graph as eleven columns (parts named 0 to 10) and three rows (named 0, 1, and 2). Each vertex is then denoted as an ordered pair (i, j) where $i \in \{0,1,2,3,4,5,6,7,8,9,10\}$ (the part) and $j \in \{0,1,2\}$ (the row). We divide the vertices of the graph into three sections as follows:

$$L_0 = \{(0,0),(0,1),(0,2),(1,0),(1,1),(2,0),(2,1),(3,0),(3,1),(4,0),(4,1),(5,0),(6,0),(7,0),(8,0),(9,0)\},$$

$R_0 = \{(1,2),(2,2),(3,2),(4,2),(5,2),(6,0),(6,1),(7,0),(7,1),(8,0),(8,1),(9,0),(9,1),$
$(10,0),(10,1),(10,2)\}$, and

$v = (5,1)$.

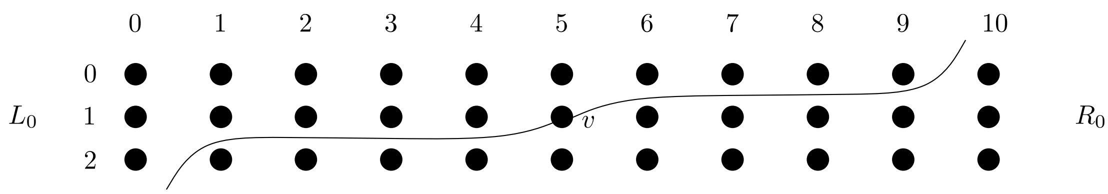

Figure 3.5: Setup of three sections with edge cut depicted

The proof is driven by carefully choosing edges to include in the set of hamilton cycles, S, so that v is a cut vertex in $G - E(S)$. These edges in our hamilton cycles fall into three categories: all edges between vertices in section L_0 and vertices in section R_0; certain edges between v and vertices in section L_0; and certain edges between v and vertices in section R_0 (see Figure 3.6). Our plan is to make our edge set 16-regular considering only these types of edges; it will be shown that there exists a hamilton decomposition of the subgraph induced by these edges. The sections L_0 and R_0 are named to reflect the fact that the edges in S join vertices on the left of the line in Figure 3.5 to vertices on the right, and the subscript 0 is added in anticipation of a general solution similar to when $p \equiv 7 \pmod 8$.

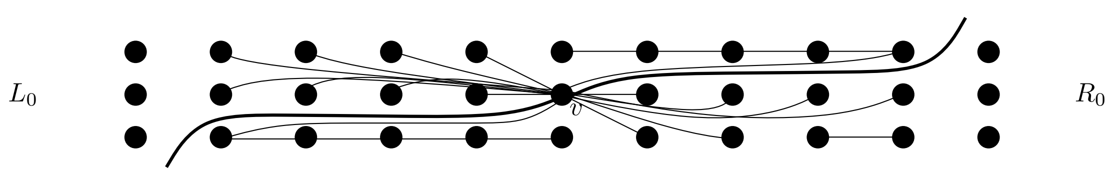

Figure 3.6: Special edges in $E(S)$

Next, we use the technique of amalgamations described in Chapter 1. Our amalgamation function $f : V(G[E(S)]) \to V(H)$ is defined as follows:

$$f(i,j) = \begin{cases} (0,0) & \text{if } i = 0, \\ (i,0) & \text{if } 1 \le i \le 4, j \le 1, \\ (i,0) & \text{if } 5 \le i \le 9, j = 0, \\ (i,1) & \text{if } 1 \le i \le 5, j = 2, \\ (i,1) & \text{if } 6 \le i \le 9, j \ge 1, \\ (10,1) & \text{if } i = 10, \end{cases}$$

and since it plays a special role, we define $f(5,1) = v$. So, $V(H)$ now looks like Figure 3.7.

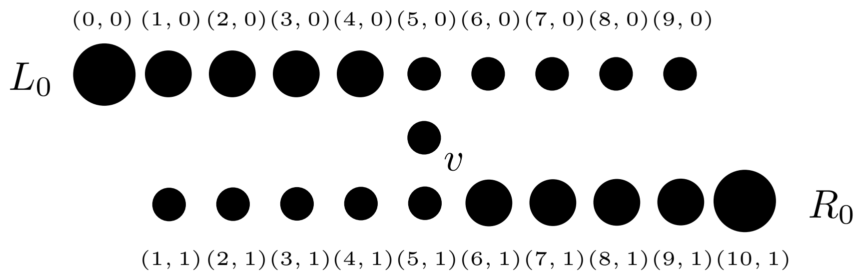

Figure 3.7: The Amalgamated H of G; the size of the vertex u represents the value of $f(u)$.

The critical part in the proof is to color the edges in H so that for $1 \le i \le 8$ (i) for all $u \in V(H)$, $d_{H(i)}(u) = 2f(u)$ and (ii) each color class $H(i)$ is connected. These conditions will allow us to disentangle H so that each color class is a hamilton cycle by properties 2 and 3 respectively of Theorem 1.3.

Our coloring is as follows:

First, each of the eight paths below are colored in a different color, namely colors 1-8.

$P_1 = [(1,0), (4,1), (5,0), (3,1), (6,0), (2,1), (0,0), (1,1), (2,0), (5,1), (3,0), (6,1), (4,0), (7,1), (8,0), (10,1), (9,0), (8,1), (7,0), (9,1)]$

$P_2 = [(2,0), (4,1), (0,0), (3,1), (1,0), (5,1), (4,0), (1,1), (3,0), (2,1), (5,0), (9,1), (6,0), (10,1), (7,0), (6,1), (8,0), (7,1), (9,0), (8,1)]$

$P_3 = [(3,0), (1,1), (4,0), (2,1), (0,0), (3,1), (1,0), (4,1), (2,0), (6,1), (5,0), (8,1), (7,0), (9,1), (8,0), (5,1), (9,0), (10,1), (6,0), (7,1)]$

$P_4 = [(4,0), (3,1), (2,0), (5,1), (0,0), (1,1), (5,0), (7,1), (6,0), (8,1), (3,0), (9,1), (1,0),$

$(2,1),(7,0),(10,1),(8,0),(4,1),(9,0),(6,1)]$

$P_5 = [(4,0),(5,1),(3,0),(4,1),(0,0),(2,1),(8,0),(3,1),(9,0),(1,1),(7,0),(10,1),(5,0),$
$(9,1),(6,0),(8,1),(2,0),(7,1),(1,0),(6,1)]$

$P_6 = [(3,0),(8,1),(5,0),(10,1),(6,0),(1,1),(8,0),(6,1),(9,0),(2,1),(1,0),(5,1),(0,0),$
$(4,1),(7,0),(3,1),(2,0),(9,1),(4,0),(7,1)]$

$P_7 = [(1,0),(8,1),(2,0),(1,1),(0,0),(3,1),(4,0),(2,1),(3,0),(4,1),(6,0),(5,1),(7,0),$
$(6,1),(5,0),(7,1),(9,0),(10,1),(8,0),(9,1)]$

$P_8 = [(5,0),(10,1),(4,0),(9,1),(3,0),(8,1),(2,0),(7,1),(1,0),(6,1),(0,0),(5,1)]$

Each of these paths is then joined to a path created from the edges in Figure 3.6, namely

$Q_1 = [(9,1),v,(1,0)]$

$Q_2 = [(8,1),v,(2,0)]$

$Q_3 = [(7,1),v,(3,0)]$

$Q_4 = [(6,1),v,(4,0)]$

$Q_5 = [(6,1),v,(4,0)]$

$Q_6 = [(7,1),v,(3,0)]$

$Q_7 = [(9,1),v,(1,0)]$

$Q_8 = [(5,1),(4,1),(3,1),(2,1),(1,1),v,(9,0),(8,0),(7,0),(6,0),(5,0)]$

This creates the following cycles $C_i = P_i \cup Q_i$:

$C_1 = ((1,0),(4,1),(5,0),(3,1),(6,0),(2,1),(0,0),(1,1),(2,0),(5,1),(3,0),(6,1),(4,0),$
$(7,1),(8,0),(10,1),(9,0),(8,1),(7,0),(9,1),v)$

$C_2 = ((2,0),(4,1),(0,0),(3,1),(1,0),(5,1),(4,0),(1,1),(3,0),(2,1),(5,0),(9,1),(6,0),$
$(10,1),(7,0),(6,1),(8,0),(7,1),(9,0),(8,1),v)$

$C_3 = ((3,0),(1,1),(4,0),(2,1),(0,0),(3,1),(1,0),(4,1),(2,0),(6,1),(5,0),(8,1),(7,0),$
$(9,1),(8,0),(5,1),(9,0),(10,1),(6,0),(7,1),v)$

$C_4 = ((4,0),(3,1),(2,0),(5,1),(0,0),(1,1),(5,0),(7,1),(6,0),(8,1),(3,0),(9,1),(1,0),$

$(2,1),(7,0),(10,1),(8,0),(4,1),(9,0),(6,1),v)$

$C_5 = ((4,0),(5,1),(3,0),(4,1),(0,0),(2,1),(8,0),(3,1),(9,0),(1,1),(7,0),(10,1),(5,0),$
$(9,1),(6,0),(8,1),(2,0),(7,1),(1,0),(6,1),v)$

$C_6 = ((3,0),(8,1),(5,0),(10,1),(6,0),(1,1),(8,0),(6,1),(9,0),(2,1),(1,0),(5,1),(0,0),$
$(4,1),(7,0),(3,1),(2,0),(9,1),(4,0),(7,1),v)$

$C_7 = ((1,0),(8,1),(2,0),(1,1),(0,0),(3,1),(4,0),(2,1),(3,0),(4,1),(6,0),(5,1),(7,0),$
$(6,1),(5,0),(7,1),(9,0),(10,1),(8,0),(9,1),v)$

$C_8 = ((5,0),(10,1),(4,0),(9,1),(3,0),(8,1),(2,0),(7,1),(1,0),(6,1),(0,0),(5,1),(4,1),$
$(3,1),(2,1),(1,1),v,(9,0),(8,0),(7,0),(6,0))$

Illustrations of these paths are given in Figures 3.8 and 3.9.

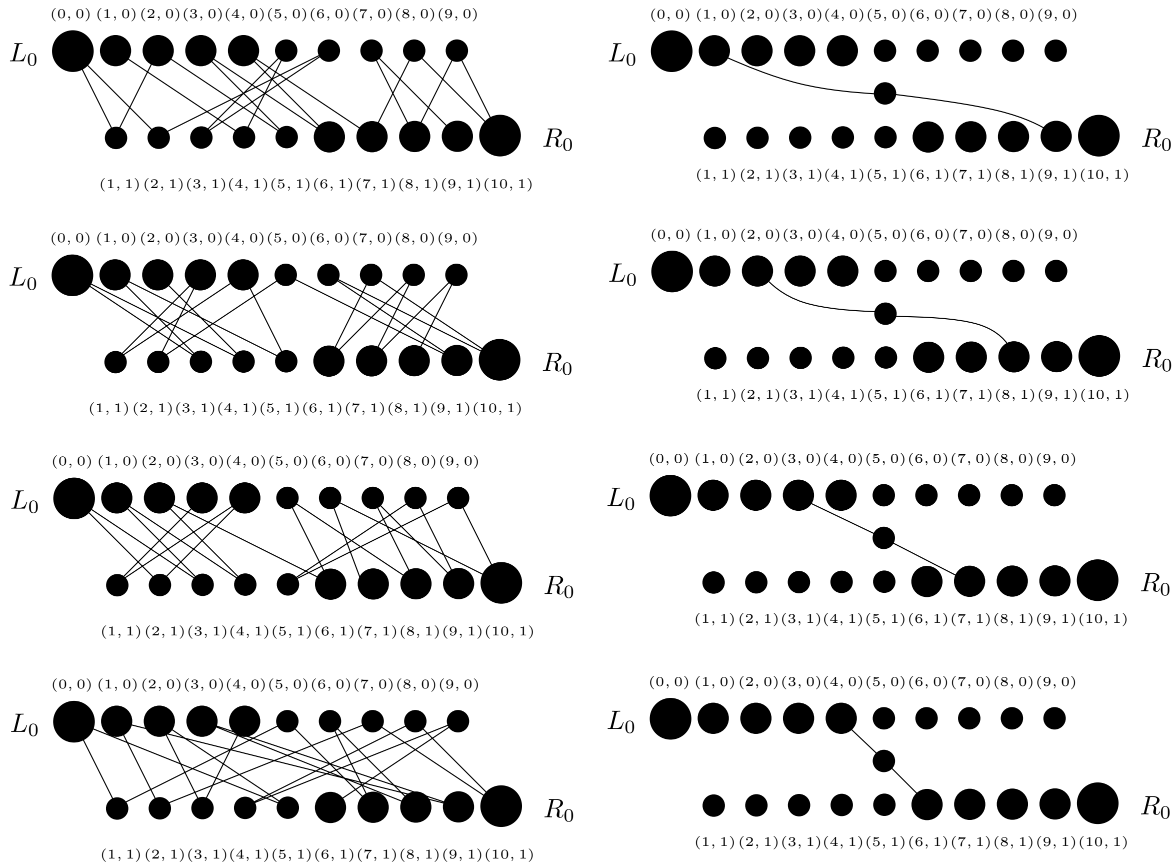

Figure 3.8: $C_i = P_i \cup Q_i$; cycles 1-4

We have now colored most of the edges in $E(S)$. Left to color are the edges between $\{(0,0),(1,0),(2,0),(3,0),(4,0)\} \in L_0$ and $\{(6,1),(7,1),(8,1),(9,1),(10,1)\} \in R_0$ that were not used in the cycles denoted by C_i above. Each vertex has degree sixteen, so we will give these edges an evenly equitable edge coloring with eight colors, namely 1-8. Now each of our eight colors appears at each vertex u a total $2f(u)$ more times and each color class is connected.

The next goal is to disentangle the graph H. To be able to apply Lemma 1.1, we need to use the same technique as in the $p \equiv 1 \pmod 4$ case which required adding edges to our graph H so that it is the amalgamation of the complete graph K_{3p}. All of these additional edges and loops are colored 0. We call this new graph H^+ and note that it now satisfies the conditions of Lemma 1.1. Thus we apply Theorem 1.3 to H^+ to produce G^+, which is an

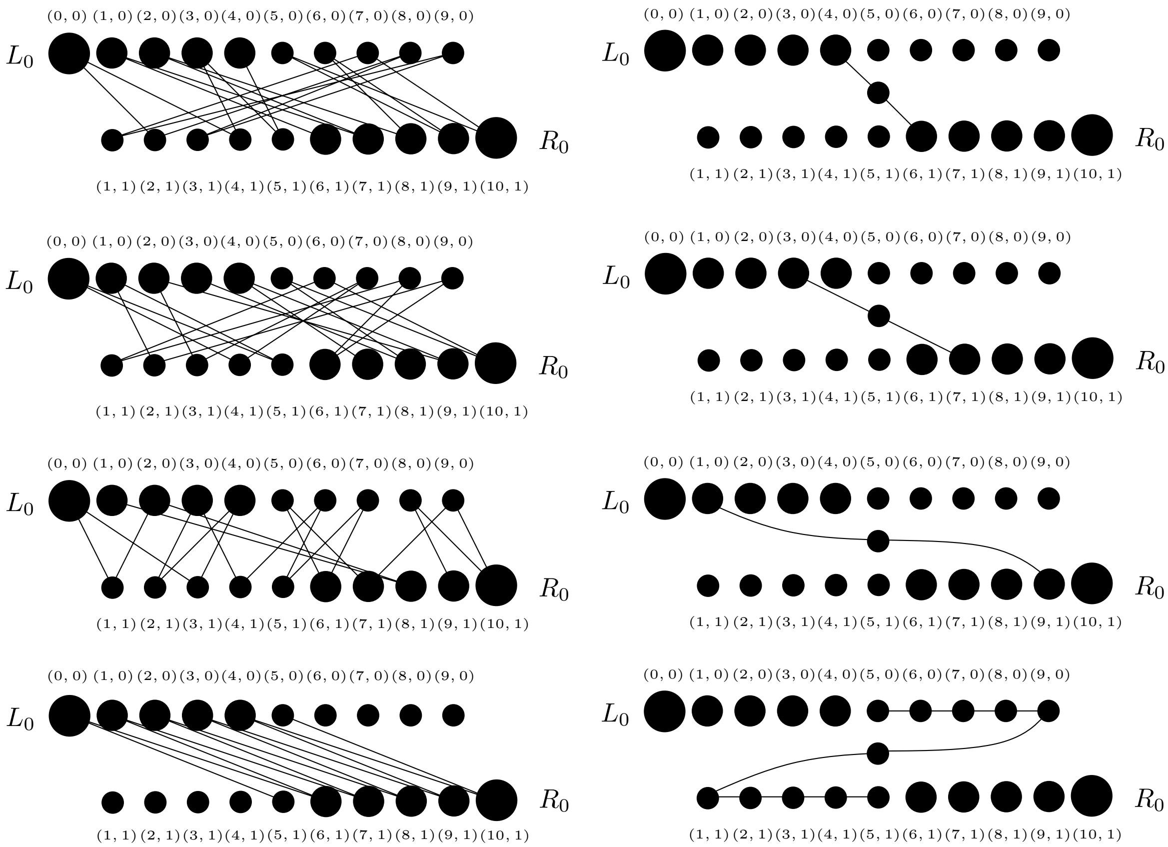

Figure 3.9: $C_i = P_i \cup Q_i$; cycles 5-8

edge-colored copy of K_{3p}. We now remove all edges in G^+ colored 0. Using Theorem 1.3, we see that each color class is 2-regular by *(i)* and connected by *(ii)*, which implies that each color class is a hamilton cycle. We now consider $G - E(S)$. Note that all edges between vertices in L_0 and R_0 were in $E(S)$, so we have that $G - E(S)$ has cut vertex v. So our set of hamilton cycles is maximal. ■

3.3 $p \equiv 7 \pmod 8$, $p \geq 31$

As mentioned in the introduction to this chapter, we present two approaches to the general case when $n = 3$ and $p \equiv 7 \pmod 8$. We begin by stating the following unproven conjecture.

Conjecture 3.1 *Let $z \geq 5$. There exist 1-factorizations $\mathcal{F} = \{F_i \mid i \in \mathbb{Z}_z\}$ and $\mathcal{G} = \{G_i \mid i \in \mathbb{Z}_z\}$ of $K_{z,z}$ with vertex set $\mathbb{Z}_z \times \mathbb{Z}_2$ such that for all $i \in \mathbb{Z}_z$,*

1. *$\{(0,0),(i,1)\}$ and $\{(0,1),(i,0)\}$ are in F_i if $i \neq 0$, and $\{(i,0),(i,1)\}$ is in F_0,*

2. *$\{(i,0),(i,1)\}$ is in G_i, and*

3. *$F_i \cup G_i$ is connected if $i \geq 1$.*

Conditions (1-3) of Conjecture 3.1 cannot be satisfied if $z \leq 4$.

Lemma 3.1 *Conjecture 3.1 holds for $z = 5$.*

Proof Let $\mathcal{F}$ be

$$F_0 = \{\{(0,0),(0,1)\},\{(1,0),(1,1)\},\{(2,0),(2,1)\},\{(3,0),(3,1)\},\{(4,0),(4,1)\}\}$$

$$F_1 = \{\{(0,0),(1,1)\},\{(1,0),(0,1)\},\{(2,0),(3,1)\},\{(3,0),(4,1)\},\{(4,0),(3,1)\}\}$$

$$F_2 = \{\{(0,0),(2,1)\},\{(1,0),(4,1)\},\{(2,0),(0,1)\},\{(3,0),(1,1)\},\{(4,0),(3,1)\}\}$$

$$F_3 = \{\{(0,0),(3,1)\},\{(1,0),(2,1)\},\{(2,0),(4,1)\},\{(3,0),(0,1)\},\{(4,0),(1,1)\}\}$$

$$F_4 = \{\{(0,0),(4,1)\},\{(1,0),(3,1)\},\{(2,0),(1,1)\},\{(3,0),(2,1)\},\{(4,0),(0,1)\}\}$$

Let $\mathcal{G}$ be

$$G_0 = \{\{(0,0),(0,1)\},\{(1,0),(2,1)\},\{(2,0),(1,1)\},\{(3,0),(4,1)\},\{(4,0),(3,1)\}\}$$

$$G_1 = \{\{(0,0),(3,1)\},\{(1,0),(1,1)\},\{(2,0),(4,1)\},\{(3,0),(2,1)\},\{(4,0),(0,1)\}\}$$

$$G_2 = \{\{(0,0),(4,1)\},\{(1,0),(3,1)\},\{(2,0),(2,1)\},\{(3,0),(0,1)\},\{(4,0),(1,1)\}\}$$

$$G_3 = \{\{(0,0),(1,1)\},\{(1,0),(4,1)\},\{(2,0),(0,1)\},\{(3,0),(3,1)\},\{(4,0),(2,1)\}\}$$

$$G_4 = \{\{(0,0),(2,1)\},\{(1,0),(0,1)\},\{(2,0),(3,1)\},\{(3,0),(1,1)\},\{(4,0),(4,1)\}\}$$

Then we have hamilton cycles induced by the following sets of edges:

$$\begin{aligned} F_1 \cup G_1 = \ & \{\{(0,0),(1,1)\},\{(1,0),(1,1)\},\{(1,0),(0,1)\},\{(4,0),(0,1)\},\{(4,0),(2,1)\}, \\ & \{(3,0),(2,1)\},\{(3,0),(4,1)\},\{(2,0),(4,1)\},\{(2,0),(3,1)\},\{(0,0),(3,1)\}\} \end{aligned}$$

$$\begin{aligned} F_2 \cup G_2 = \ & \{\{(0,0),(2,1)\},\{(2,0),(2,1)\},\{(2,0),(0,1)\},\{(3,0),(0,1)\},\{(3,0),(1,1)\}, \\ & \{(4,0),(1,1)\},\{(4,0),(3,1)\},\{(1,0),(3,1)\},\{(1,0),(4,1)\},\{(0,0),(4,1)\}\} \end{aligned}$$

$$\begin{aligned} F_3 \cup G_3 = \ & \{\{(0,0),(3,1)\},\{(3,0),(3,1)\},\{(3,0),(0,1)\},\{(2,0),(0,1)\},\{(2,0),(4,1)\}, \\ & \{(1,0),(4,1)\},\{(1,0),(2,1)\},\{(4,0),(2,1)\},\{(4,0),(1,1)\},\{(0,0),(1,1)\}\} \end{aligned}$$

$$\begin{aligned} F_4 \cup G_4 = \ & \{\{(0,0),(4,1)\},\{(4,0),(4,1)\},\{(4,0),(0,1)\},\{(1,0),(0,1)\},\{(1,0),(3,1)\}, \\ & \{(2,0),(3,1)\},\{(2,0),(1,1)\},\{(3,0),(1,1)\},\{(3,0),(2,1)\},\{(0,0),(2,1)\}\} \end{aligned}$$

$\mathcal{F}$, $\mathcal{G}$, and $F_i \cup G_i, 1 \leq i \leq 4$ are pictured in Figures 3.10, 3.11, and 3.12. The bold edges represent those required by Conditions (1-2) in Conjecture 3.1.

■

In proving Theorem 3.4, we can use Conjecture 3.1, but could also make use of the following result.

Let $2K_n$ denote the multigraph on n vertices in which each pair of vertices is joined by exactly two edges. An *i-factor* of a graph G is a spanning subgraph of G that is regular of

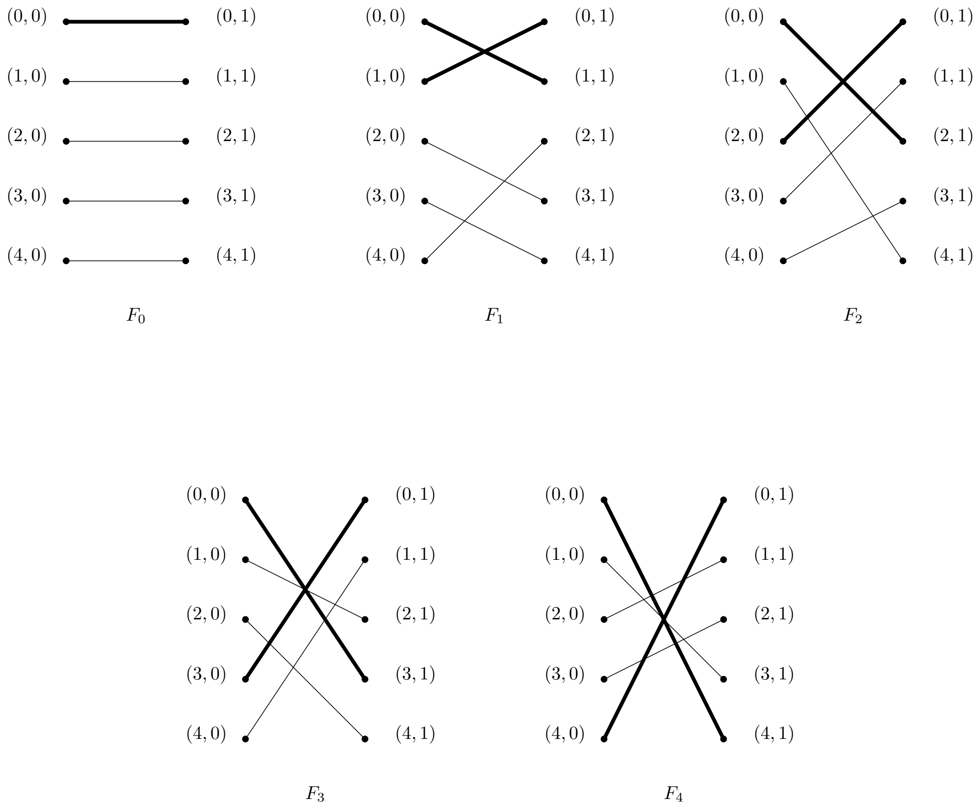

Figure 3.10: $\mathcal{F} = \{F_i \mid i \in \mathbb{Z}_5\}$

degree i. An *m-cycle decomposition* of a graph G is a collection of edge-disjoint m-cycles which partition the edge set $E(G)$. An m-cycle decomposition $C(m)$ is *resolvable* if the m-cycles in $C(m)$ can be partitioned into 2-factors of G. A subgraph X of a graph G is an *almost parallel class* if for some vertex v, X is a 2-factor of $G - v$. In this case v is called the *deficiency* of the almost parallel class and is denoted by $d(X)$. An m-cycle decomposition $C(m)$ is *almost resolvable* if $C(m)$ can be partitioned into almost parallel classes.

From results of Heinrich, Lindner, Rodger, and Burling, we have the following theorem.

Theorem 3.3 *[2, 5] For all $m \geq 3$, there exists an almost resolvable m-cycle system of $2K_n$ if and only if $n \equiv 1 \pmod{m}$.*

Theorem 3.4 *Let $\alpha \geq 3$, let $n = 3$, and let $p = 7 + 8\alpha$. There exists a maximal set of $m = \left\lceil \frac{3(p-1)}{4} \right\rceil = 3x + 2$ edge disjoint hamilton cycles in K_n^p.*

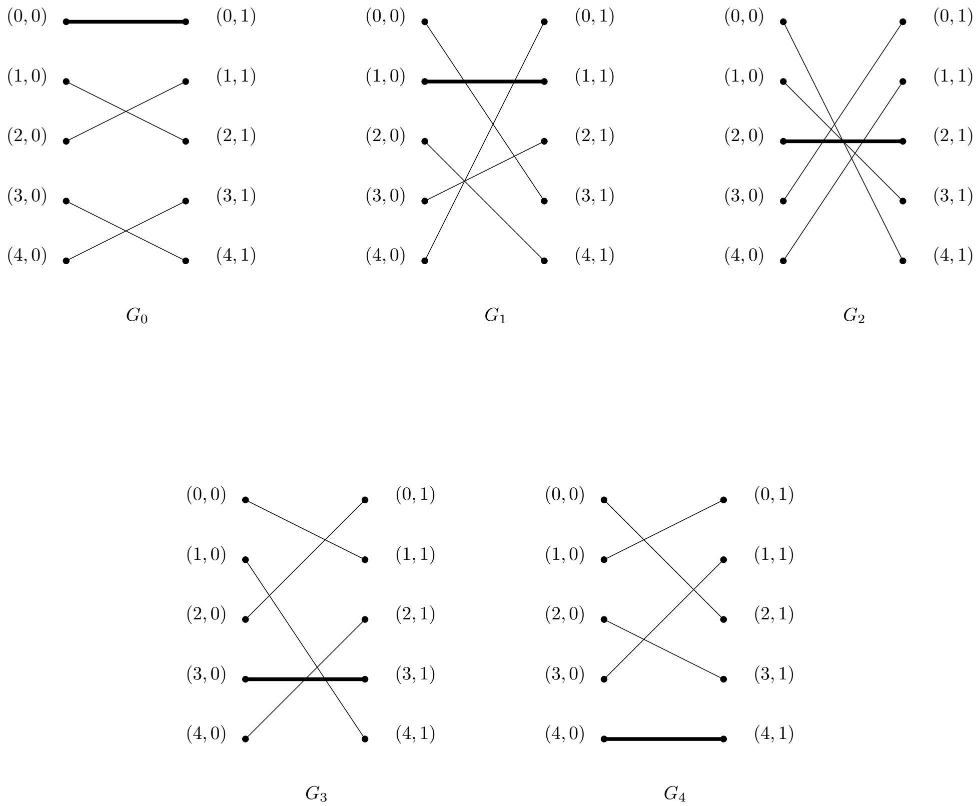

Figure 3.11: $\mathcal{G} = \{G_i \mid i \in \mathbb{Z}_5\}$

Proof The case when $p = 7$ is settled in Theorem 3.1, so we now use our construction for K_3^7 to produce a maximal set S of hamilton cycles when $p = 7 + 8\alpha$. Recall that in producing a maximal set of hamilton cycles for K_3^7, we viewed our complete multipartite graph as having seven columns of three vertices each. We then split this graph into two sections denoted L_0 and R_0 and a single vertex v. As we generalize this case, we start with the aforementioned seven parts and add more in groups of eight. We will visualize this with four parts on each side of the original seven parts. The eight new parts are viewed as eight columns (parts named 0 to 7) and three rows (named 0, 1, and 2). We follow the same naming convention as before, except we include a third coordinate. The original vertices in K_3^7 are now named $(i, j, 0)$ instead of (i, j) as before. Each new vertex is then denoted as an ordered triple (i, j, x) where $i \in \{0, 1, 2, 3, 4, 5, 6, 7\}$ (the part), $j \in \{0, 1, 2\}$ (the row) and $x \in \mathbb{Z}_{\alpha+1} \setminus \{0\}$. These vertices are then grouped into two new sections L_x and R_x, $x \in \mathbb{Z}_{\alpha+1}$ as follows

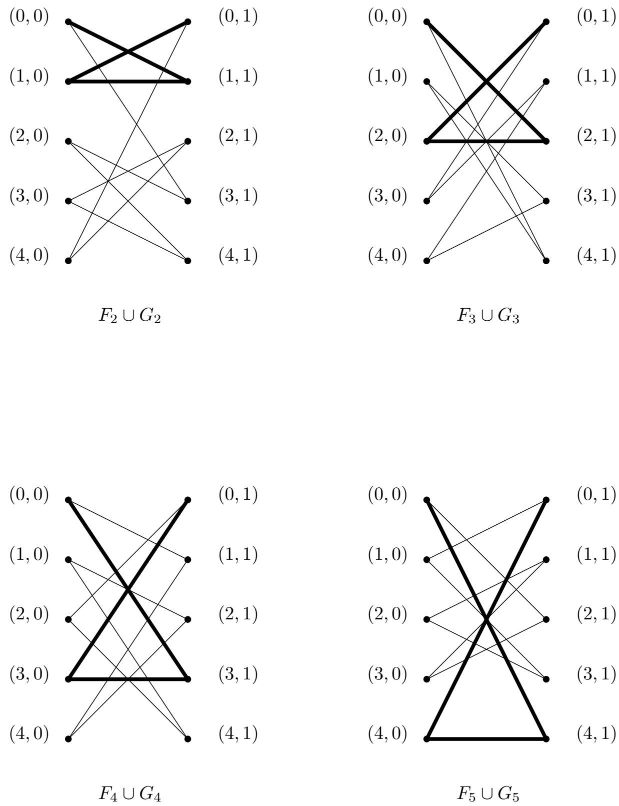

Figure 3.12: $F_i \cup G_i, 1 \leq i \leq 4$

$$L_x = \{(0,0,x),(0,1,x),(0,2,x),(0,3,x),(1,0,x),(1,1,x),(1,2,x),(1,3,x),\\ (2,0,x),(2,1,x),(2,2,x),(2,3,x)\}, \text{ and}$$

$$R_x = \{(0,4,x),(0,5,x),(0,6,x),(0,6,x),(1,4,x),(1,5,x),(1,6,x),(1,7,x),\\ (2,4,x),(2,5,x),(2,6,x),(2,7,x)\}$$

So, our vertex set is $\{\mathbb{Z}_7 \times \mathbb{Z}_3\} \bigcup \{\mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_{\alpha+1} \setminus \{0\}\}$. Thus, for each $x \in \mathbb{Z}_{\alpha+1} \setminus \{0\}$, $L_x = \{\mathbb{Z}_4 \times \mathbb{Z}_3 \times \{x\}\} \setminus \{(2,2,x),(3,2,x)\} \bigcup \{(4,0,x),(5,0,x)\}$ and $R_x = \{\mathbb{Z}_4 \times \mathbb{Z}_3 \times \{x\}\} \setminus L_x$. Therefore, $V(K_n^p) = L \bigcup R \bigcup \{v\}$, where $v = (3,1,0)$, $L = \bigcup_{x \in \mathbb{Z}_\alpha} L_x$, and $R = \bigcup_{x \in \mathbb{Z}_\alpha} R_x$. This can be viewed in Figure 3.13.

As before, our aim is to produce a set of colored edges $E(S)$ in $G = K_n^p$ that form $m = \frac{3p-1}{4} = 6x + 5$ hamilton cycles such that in the complement, $G - E(S)$, v is a cut vertex ensuring that our set is indeed maximal. The edges chosen are as follows:

Type 1: All edges that join a vertex in L to a vertex in R.

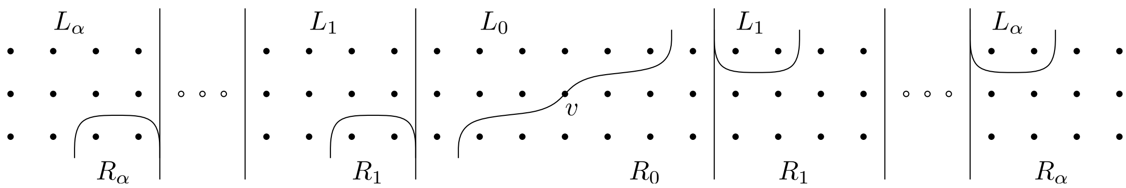

Figure 3.13: Set up of $K_3^{7+8\alpha}$

Type 2: All special edges from the $p = 7$ case. (These were denoted as Type 2 and Type 3 edges previously.)

Type 3: Specific edges incident with v, along with edges among vertices of L_x and edges among vertices of R_x, namely $\{\{v, (i, j, x)\} \mid i \in \{2, 3\}, j \in \{0, 1\}, 1 \leq x \leq \alpha\} \cup \{\{v, (i, j, x)\} \mid i \in \{4, 5\}, j \in \{1, 2\}, 1 \leq x \leq \alpha\} \cup \{\{v, (i, j, x)\} \mid i \in \{2, 3\}, j = 2, 1 \leq x \leq \alpha\} \cup \{\{v, (i, j, x)\} \mid i \in \{4, 5\}, j = 0, 1 \leq x \leq \alpha\} \cup \{\{(2, 2, x), (3, 2, x)\}, \{(4, 0, x), (5, 0, x)\}\} \mid 1 \leq x \leq \alpha\}$. (These are shown in Figure 3.14.)

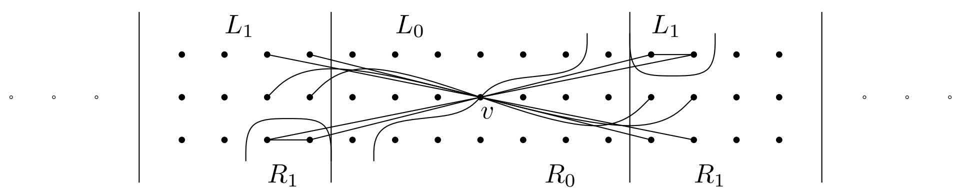

Figure 3.14: Type 3 Edges

It remains to show that these edges induce a graph which has a hamilton decomposition. To do so, the technique of amalgamations will be used to aid in the proof. The amalgamation function $f : V(G[E(S)]) \to V(H)$ is defined as follows and is pictured in Figure 3.15. For all vertices in $L_0 \cup R_0 \cup v$, $f(i, j, 0) = f(i, j)$, where $f(i, j)$ is defined in the proof of Theorem 3.1. If $x \in \{1, ..., \alpha\}$, then

$$f(i,j,x) = \begin{cases} (i,0,x) & \text{if } i \in \{0,1\}, \\ (i,0,x) & \text{if } i \in \{2,3\} \text{ and } j \in 0,1, \\ (i,0,x) & \text{if } i \in \{4,5\} \text{ and } j = 0, \\ (i,2,x) & \text{if } i \in \{2,3\} \text{ and } j = 2, \\ (i,2,x) & \text{if } i \in \{4,5\} \text{ and } j \in \{1,2\}, \\ (i,2,x) & \text{if } i \in \{6,7\}. \end{cases}$$

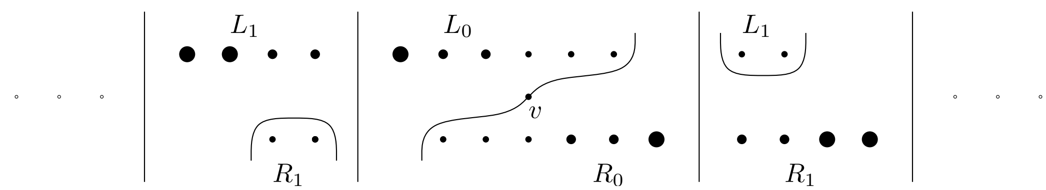

Figure 3.15: The Amalgamated Graph H; the size of the vertex indicated its amalgamation number

It is helpful now to shift our view of this amalgamated graph to where the L_i's and R_i's each form a column. Our special vertex v is then pictured above these columns in the center (see Figure 3.16). Vertices in the same row have the same amalgamation number.

Now, we must color our edge set so that conditions (1-4) of Lemma 1.1 are satisfied in order that Theorem 1.3 can be applied. This is done by coloring the graph in small pieces and then connecting them afterward. These small pieces consist of the trails, denoted $T_1(x), T_2(x), ..., T_6(x)$, and hamilton cycles on the vertices of L_i and R_j, denoted $H(i,j,c)$. Recall from Chapter 1 that trails are denoted using brackets and cycles are denoted using parentheses. These pieces are defined as follows:

- **Hamilton cycles on the vertices in L_0 and R_0:** Use the same construction of five hamilton cycles in the proof of Theorem 3.1. We now rename these $H(0,0,c)$, $c \in \mathbb{Z}_5$ and picture them in Figure 3.17.

- **Eulerian Trails on the vertices in L_0, R_x, L_x and R_0, $1 \leq x \leq \alpha$:** The edges of these trails are defined below. Each trail is defined so that (1) it spans the vertices

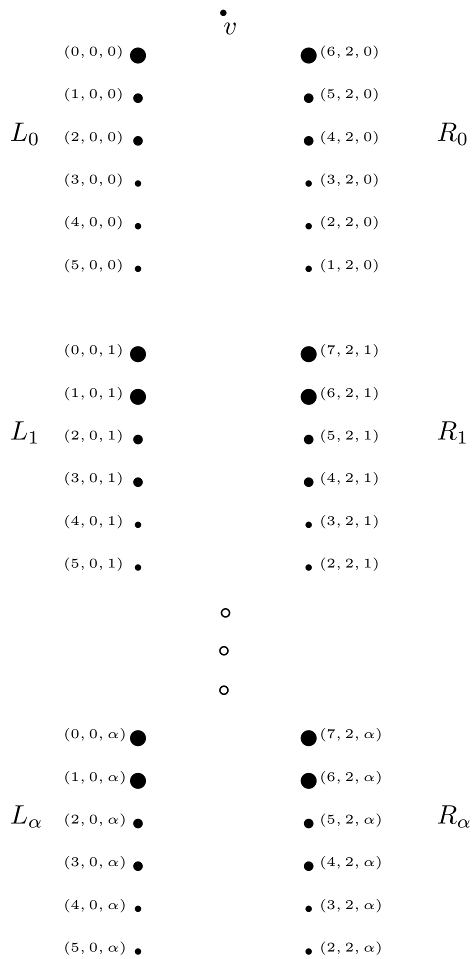

Figure 3.16: Column View of Amalgamated Graph

$\{v\} \cup L_0 \cup R_0 \cup L_x \cup R_x$ (where $x \in \mathbb{Z}_\alpha \setminus \{0\}$), and (2) each vertex has degree 2 except for $(0,0,x)$, $(1,0,x)$, $(6,2,x)$, and $(7,2,x)$, which each have degree 4. We begin defining our trails with a subtrail $T_i'(x)$.

For each $x \in \{1, ..., \alpha\}$, let

$$\begin{aligned} T_0'(x) = \ & [(2,2,x),(0,0,0),(6,2,x),(2,0,0),(6,2,x),(5,0,0),(7,2,x), \\ & (3,0,0),(5,2,x),(1,0,0),(4,2,x),(4,0,0),(7,2,x)] \end{aligned}$$

$$\begin{aligned} T_1'(x) = \ & [(2,2,x),(0,0,0),(6,2,x),(2,0,0),(7,2,x),(3,0,0),(3,2,x) \\ & (5,0,0),(4,2,x),(1,0,0),(5,2,x),(4,0,0),(7,2,x)] \end{aligned}$$

$$\begin{aligned} T_2'(x) = \ & [(4,2,x),(3,0,0),(6,2,x),(2,0,0),(2,2,x),(0,0,0),(7,2,x), \\ & (4,0,0),(3,2,x),(1,0,0),(5,2,x),(5,0,0),(7,2,x)] \end{aligned}$$

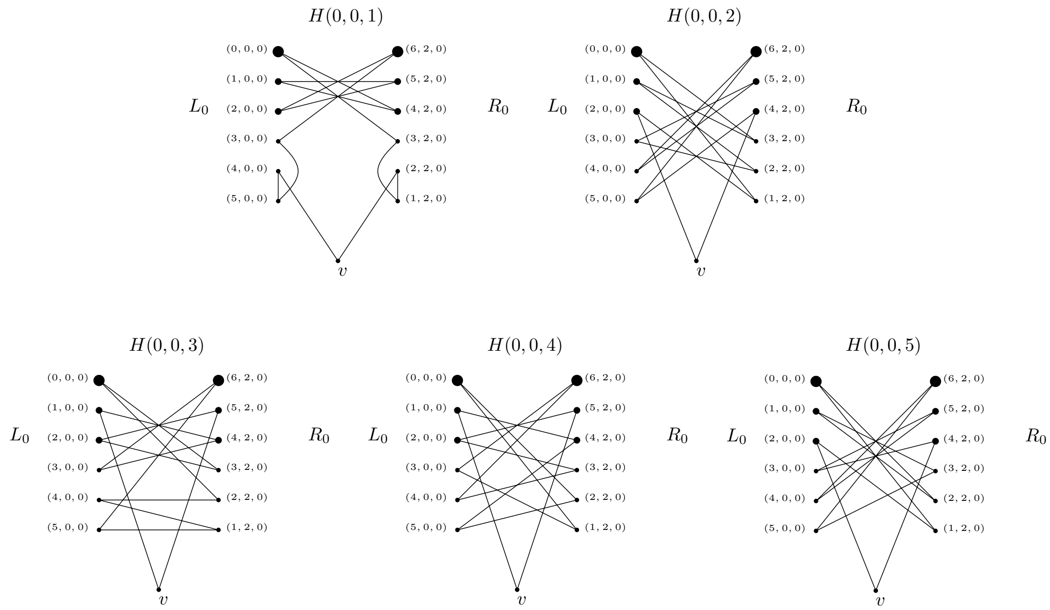

Figure 3.17: Hamilton Cycles $H(0, 0, c)$, $1 \leq c \leq 5$

$$T_3'(x) = [(4,2,x),(5,0,0),(2,2,x),(2,0,0),(7,2,x),(0,0,0),(3,2,x),$$
$$(1,0,0),(5,2,x),(4,0,0),(6,2,x),(3,0,0),(6,2,x)]$$

$$T_4'(x) = [(5,2,x),(3,0,0),(4,2,x),(1,0,0),(2,2,x),(4,0,0),(6,2,x),$$
$$(5,0,0),(7,2,x),(2,0,0),(3,2,x),(0,0,0),(6,2,x)]$$

$$T_5'(x) = [(5,2,x),(5,0,0),(6,2,x),(4,0,0),(4,2,x),(1,0,0),(2,2,x),$$
$$(3,0,0),(7,2,x),(2,0,0),(3,2,x),(0,0,0),(7,2,x)]$$

Now, for each trail $T'(x)$, let $g(T'(x))$ be the trail formed by replacing each vertex $u \in V(T'(x))$ with the vertex $g(u)$, where g is defined by:

$$g(i,0,x) = \begin{cases} (7-i,2,x) & \text{if } i \in \mathbb{Z}_8 \text{ and } x \in \mathbb{Z}_\alpha \setminus \{0\}, \\ (6-i,2,x) & \text{if } i \in \mathbb{Z}_7 \text{ and } x = 0, \end{cases}$$

and g^2 is the identity map on V.

Define the required trails as follows:

$$T_0(x) = T_0'(x) \cup g(T_0'(x)) \cup [(5,0,x),(4,0,x),v,(3,2,x),(2,2,x)] \cup [(0,0,x),(7,2,x)]$$

$$T_1(x) = T_1'(x) \cup g(T_1'(x)) \cup [(5,0,x),v,(2,2,x)] \cup [(1,0,x),(6,2,x),(1,0,x)] \cup [(0,0,x),(7,2,x)]$$

$$T_2(x) = T_2'(x) \cup g(T_2'(x)) \cup [(3,0,x),v,(4,2,x)] \cup [(0,0,x),(6,2,x),(1,0,x),(7,2,x)]$$

$$T_3(x) = T_3'(x) \cup g(T_3'(x)) \cup [(3,0,x),v,(4,2,x)] \cup [(1,0,x),(7,2,x),(0,0,x),(6,2,x)]$$

$$T_4(x) = T_4'(x) \cup g(T_4'(x)) \cup [(2,0,x),v,(5,2,x)] \cup [(1,0,x),(7,2,x),(0,0,x),(6,2,x)]$$

$$T_5(x) = T_5'(x) \cup g(T_5'(x)) \cup [(2,0,x),v,(5,2,x)] \cup [(0,0,x),(6,2,x),(1,0,x),(7,2,x)]$$

- **Hamilton cycles on the vertices in L_x and R_y, for $1 \leq x, y \leq \alpha$:** These cycles are defined below and pictured in Figure 3.19.

$$\begin{aligned} H(i,j,0) = {} & ((0,0,x),(2,2,y),(5,0,x),(7,2,y),(4,0,x),(6,2,y),(3,0,x), \\ & (5,2,y),(2,0,x),(4,2,y),(1,0,x),(3,2,y)) \end{aligned}$$

$$\begin{aligned} H(i,j,1) = {} & ((0,0,x),(3,2,y),(1,0,x),(2,2,y),(2,0,x),(7,2,y),(3,0,x), \\ & (6,2,y),(5,0,x),(5,2,y),(4,0,x),(4,2,y)) \end{aligned}$$

$$\begin{aligned} H(i,j,2) = {} & ((0,0,x),(2,2,y),(4,0,x),(3,2,y),(3,0,x),(7,2,y),(2,0,x), \\ & (6,2,y),(5,0,x),(4,2,y),(1,0,x),(5,2,y)) \end{aligned}$$

$$\begin{aligned} H(i,j,3) = {} & ((0,0,x),(4,2,y),(5,0,x),(3,2,y),(2,0,x),(6,2,y),(4,0,x), \\ & (7,2,y),(3,0,x),(2,2,y),(1,0,x),(5,2,y)) \end{aligned}$$

$$\begin{aligned} H(i,j,4) = {} & ((0,0,x),(2,2,y),(1,0,x),(4,2,y),(4,0,x),(7,2,y),(5,0,x), \\ & (5,2,y),(3,0,x),(6,2,y),(2,0,x),(3,2,y)) \end{aligned}$$

$$\begin{aligned} H(i,j,5) = {} & ((0,0,x),(4,2,y),(1,0,x),(3,2,y),(3,0,x),(2,2,y),(2,0,x), \\ & (7,2,y),(5,0,x),(6,2,y),(4,0,x),(5,2,y)) \end{aligned}$$

- **Hamilton cycles on the vertices in L_x and R_x for $1 \leq x \leq \alpha$:** These cycles are defined below and shown in Figure 3.20.

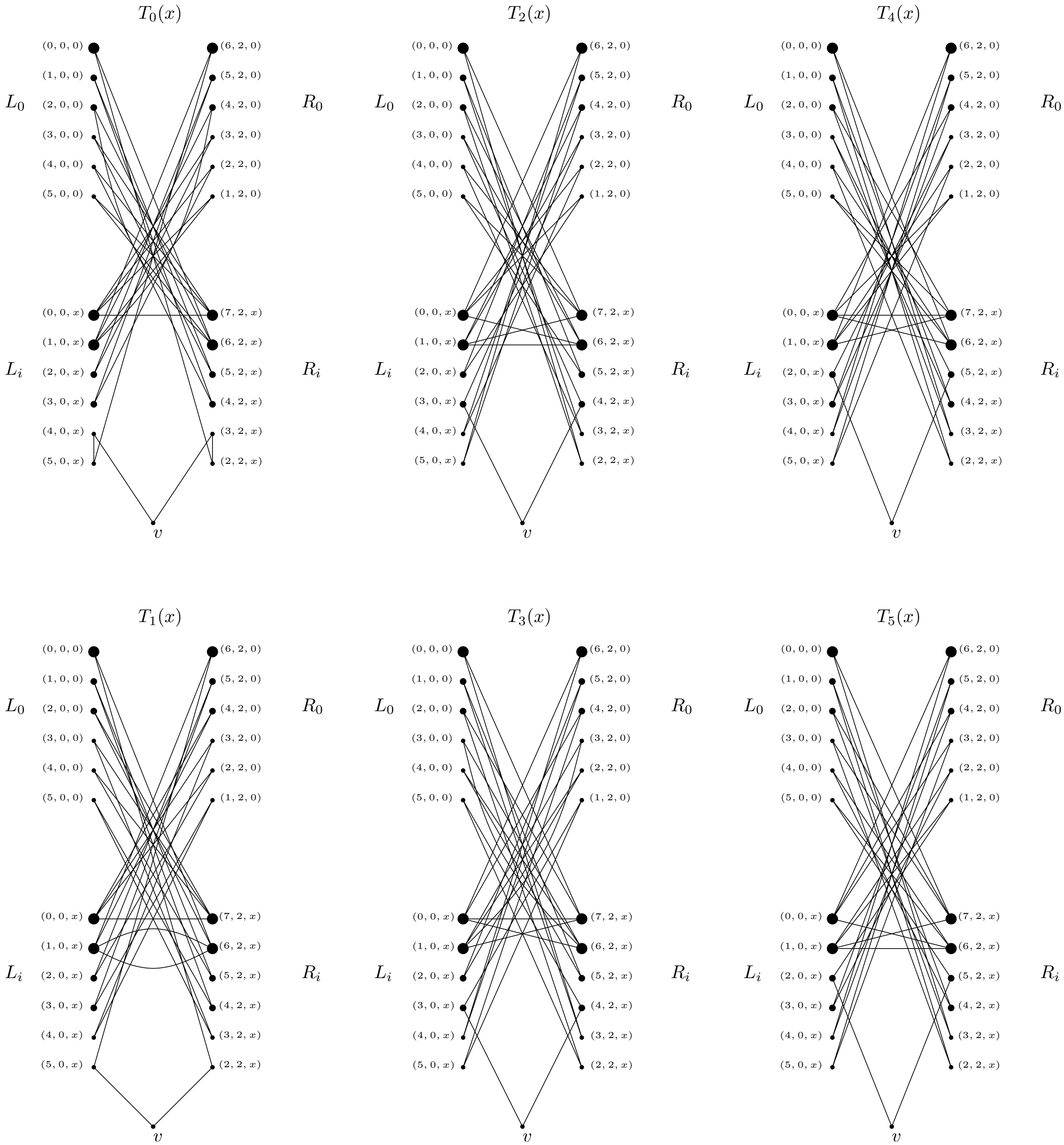

Figure 3.18: Coloring $T_c(x)$ with $6x + c$

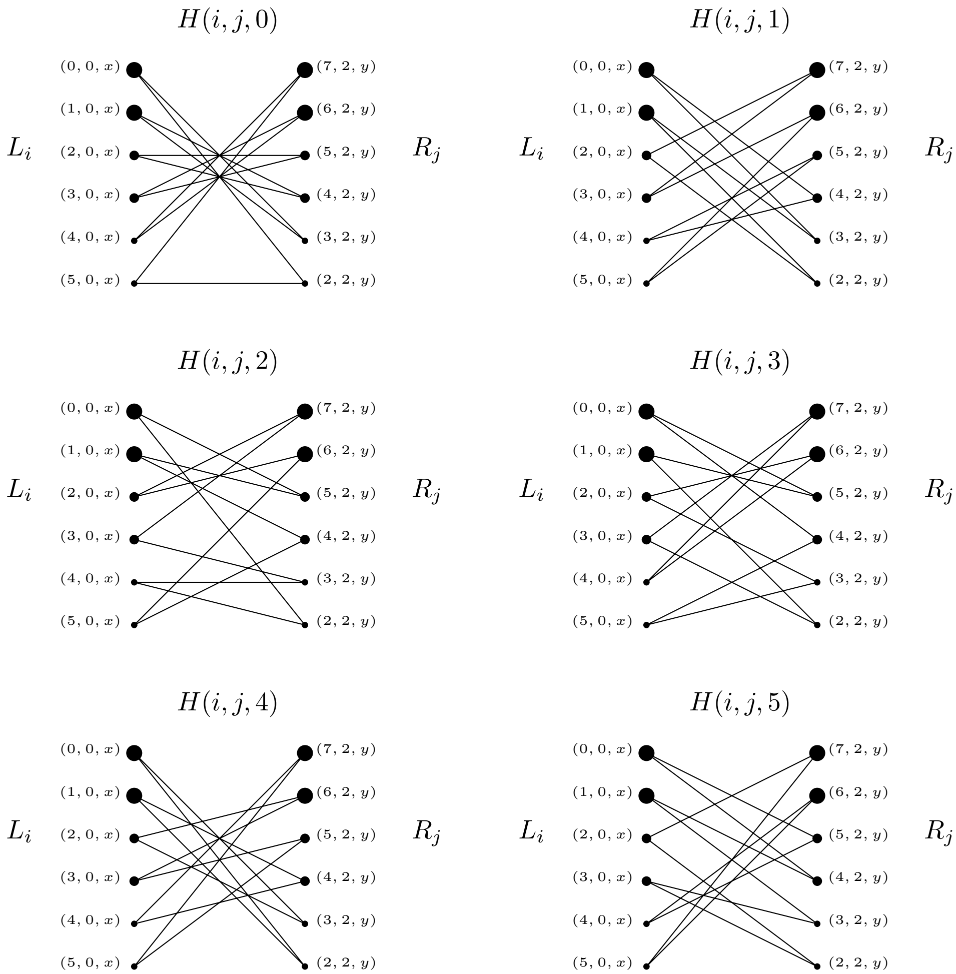

Figure 3.19: Coloring $H(i, j, c)$ with $6x + c$

$$H(i,i,1) = ((0,0,x),(5,2,x),(1,0,x),(3,2,x),(2,0,x),(7,2,x),(4,0,x),$$
$$(6,2,x),(5,0,x),(4,2,x),(3,0,x),(2,2,x))$$

$$H(i,i,2) = ((0,0,x),(3,2,x),(4,0,x),(5,2,x),(3,0,x),(6,2,x),(2,0,x),$$
$$(7,2,x),(5,0,x),(4,2,x),(1,0,x),(2,2,x))$$

$$H(i,i,3) = ((0,0,x),(4,2,x),(2,0,x),(5,2,x),(1,0,x),(2,2,x),(4,0,x),$$
$$(6,2,x),(3,0,x),(7,2,x),(5,0,x),(3,2,x))$$

$$H(i,i,4) = ((0,0,x),(4,2,x),(3,0,x),(5,2,x),(4,0,x),(7,2,x),(5,0,x),$$
$$(6,2,x),(2,0,x),(3,2,x),(1,0,x),(2,2,x))$$

$$H(i,i,5) = ((0,0,x),(4,2,x),(2,0,x),(7,2,x),(4,0,x),(6,2,x),(5,0,x),$$
$$(2,2,x),(3,0,x),(5,2,x),(1,0,x),(3,2,x))$$

Let $\mathcal{F}$ and $\mathcal{G}$ be 1-factorizations of $K_{\alpha+1,\alpha+1}$ satisfying Conditions (1-3) of Conjecture 3.1. Associate the 1-factor F_x (for $1 \leq x \leq \alpha$) with the colors $6x$, $6x+1$, $6x+2$, $6x+3$, $6x+4$, and $6x+5$ as follows. (Note that F_0 will be associated with only the colors 1, 2, 3, 4, and 5. These five colors correspond the the construction of K_3^7, where there is no color 0.) For each edge $\{(a,0),(b,1)\}$, $a,b \neq 0$, in F_x, $x > 0$, the hamilton cycles $H(a,b,c)$, for each $c \in \mathbb{Z}_6$ are colored $6x+c$. For each of the edges $\{(a,0),(0,1)\}$ and $\{(0,0),(b,1)\}$ in F_x, we use the trails $T_0(x), T_1(x), ..., T_5(x)$ given in Figure 3.18, coloring T_c with $6x+c$. For F_0 we have that all edges are of the form $\{(a,0),(a,1)\}$. For $a \neq 0$, we color the cycles of Figure 3.20, denoted by $H(i,i,c)$, with colors 1, 2, 3, 4, and 5. Edge $\{(0,0),(0,1)\}$ corresponds to the 5 hamilton cycles constructed for K_3^7 that are colored as in the proof of Theorem 3.1.

At this point, we note that almost every vertex is incident with exactly 2 edges of each color. The only exceptions are all the vertices (i,j,x) with amalgamation number 3 and $x \geq 1$, namely $(0,0,x)$, $(1,0,x)$, $(7,2,x)$, and $(6,2,x)$, for all $x \in \{1,2,...,\alpha\}$; these exceptional vertices are incident with 4 edges of colors $6x$, $6x+1$, $6x+2$, $6x+3$, $6x+4$, and $6x+5$ and with 2 edges of all other colors.

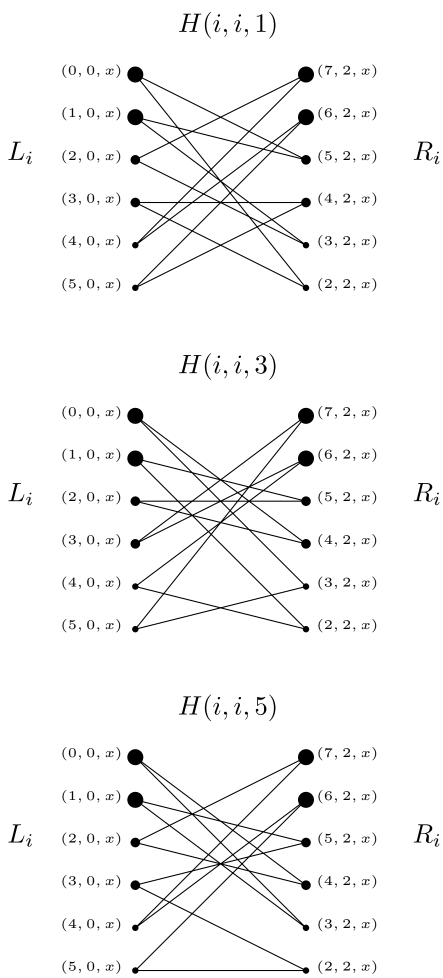

Figure 3.20: Coloring $H(i,i,c)$ with $6x+c$

It is also important to note that the cycles and trails used to color the edges so far purposely did not use many edges between vertices with amalgamation number 3. Namely, there are no edges between vertices in $\{(0,0,x),(1,0,x)\}$ and vertices in $\{(7,2,y),(6,2,y)\}$ for $1 \leq x,y \leq \alpha$, $x \neq y$. For $1 \leq x \leq \alpha$, there are precisely 4 (of the 9) edges joining vertices in $\{(0,0,x),(1,0,x)\}$ to vertices in $\{(7,2,x),(6,2,x)\}$ colored so far. For $1 \leq x \leq \alpha$, there are exactly 3 edges used between $(0,0,0)$ and $(7,2,x)$, $(0,0,0)$ and $(6,2,x)$, $(6,2,0)$ and $(0,0,x)$, and $(6,2,0)$ and $(1,0,x)$. There are no edges used between the vertex $(1,0,0)$ (which has amalgamation number two) and vertices in $\{(7,2,x),(6,2,x)\}$ for $1 \leq x \leq \alpha$ and, symmetrically, no edges used between $(5,2,0)$ (which also has amalgamation number two) and vertices in $\{(0,0,x),(1,0,x)\}$ for $1 \leq x \leq \alpha$. We have done this to allow room to connect our color classes. Most importantly, there are at least six edges left between vertices mentioned above, with the exception that there are only five left between vertices in $\{(0,0,x),(1,0,x)\}$ and vertices in $\{(7,2,x),(6,2,x)\}$ for $1 \leq x \leq \alpha$. Tables 3.1, 3.2, 3.3, and 3.4 summarize this information. Each cell gives the number of edges used between the vertices given in the heading of the row and column. Each cell is further divided into a 2×3 table, with cell $(1,1)$ corresponding to the hamilton cycle or eulerian trail colored $6x$, cell $(1,2)$ corresponding to the hamilton cycle or eulerian trail colored $6x+1$, etc.

We now connect our color classes. To do so, we either use Conditions (2-3) of Conjecture 3.1 or we use results from [5] and [2], so we present each in turn.

Using Conjecture 3.1 to connect the color classes

By Condition (2) of Conjecture 3.1, the rainbow one factor appears as the “horizontal” edges. For $1 \leq x \leq \alpha$ and for each edge $\{(a,0),(b,1)\} \in G_x$ with $a \neq b$ (so by Condition 2 of Conjecture 3.1, $a \neq x$), color six copies of the 4-cycle $((0,0,i),(7,2,j),(1,0,i),(6,2,j))$, the cth copy being colored with $6x+c$ where $c \in \mathbb{Z}_6$. Note that this boosts the degree of the vertices involved to four in each color class. (We leave out edges where $a=b$ both because vertices in L_i and R_i are already connected in those color classes by the construction of $T_c(x)$, and because the vertices $(0,0,x),(1,0,x),(7,2,x),$ and $(6,2,x)$ for $1 \leq x \leq \alpha$ are

<table>
<tr><th></th><th colspan="3">(7,2,y)
$\eta = 3$</th><th colspan="3">(6,2,y)
$\eta = 3$</th><th colspan="3">(5,2,y)
$\eta = 2$</th><th colspan="3">(4,2,y)
$\eta = 2$</th><th colspan="3">(3,2,y)
$\eta = 1$</th><th colspan="3">(2,2,y)
$\eta = 1$</th></tr>
<tr><td>(0,0,x)</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td>1</td><td></td><td>1</td><td></td><td>1</td><td>1</td><td></td><td>1</td><td></td><td>1</td></tr>
<tr><td>$\eta = 3$</td><td></td><td></td><td></td><td></td><td></td><td></td><td>1</td><td></td><td>1</td><td>1</td><td></td><td>1</td><td></td><td>1</td><td></td><td></td><td>1</td><td></td></tr>
<tr><td>(1,0,x)</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td>1</td><td>1</td><td></td><td>1</td><td>1</td><td>1</td><td></td><td></td><td>1</td><td></td></tr>
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<tr><td>$\eta = 2$</td><td></td><td></td><td>1</td><td>1</td><td>1</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td>1</td><td>1</td><td></td><td></td><td></td><td>1</td></tr>
<tr><td>(3,0,x)</td><td></td><td>1</td><td>1</td><td>1</td><td>1</td><td></td><td>1</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td>1</td><td></td><td></td><td></td></tr>
<tr><td>$\eta = 2$</td><td>1</td><td></td><td></td><td></td><td>1</td><td></td><td></td><td>1</td><td></td><td></td><td></td><td></td><td></td><td></td><td>1</td><td>1</td><td></td><td>1</td></tr>
<tr><td>(4,0,x)</td><td>1</td><td></td><td></td><td>1</td><td></td><td></td><td></td><td>1</td><td></td><td></td><td>1</td><td></td><td></td><td></td><td>1</td><td></td><td></td><td>1</td></tr>
<tr><td>$\eta = 1$</td><td>1</td><td>1</td><td></td><td>1</td><td></td><td>1</td><td></td><td></td><td>1</td><td></td><td>1</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>(5,0,x)</td><td>1</td><td></td><td></td><td></td><td>1</td><td>1</td><td></td><td>1</td><td></td><td></td><td></td><td>1</td><td></td><td></td><td></td><td>1</td><td></td><td></td></tr>
<tr><td>$\eta = 1$</td><td></td><td>1</td><td>1</td><td></td><td></td><td>1</td><td></td><td>1</td><td></td><td>1</td><td></td><td></td><td>1</td><td></td><td></td><td></td><td></td><td></td></tr>
</table>

Table 3.1: Number of Edges used in $H(i, j, c)$ with $0 \leq c \leq 5$, broken down by color class

<table>
<tr><th></th><th colspan="3">(7,2,x)
$\eta = 3$</th><th colspan="3">(6,2,x)
$\eta = 3$</th><th colspan="3">(5,2,x)
$\eta = 2$</th><th colspan="3">(4,2,x)
$\eta = 2$</th><th colspan="3">(3,2,x)
$\eta = 1$</th><th colspan="3">(2,2,x)
$\eta = 1$</th></tr>
<tr><td>(0,0,x)</td><td></td><td></td><td></td><td></td><td></td><td></td><td>1</td><td></td><td></td><td></td><td></td><td>1</td><td></td><td>1</td><td>1</td><td>1</td><td>1</td><td></td></tr>
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<tr><td>(1,0,x)</td><td></td><td></td><td></td><td></td><td></td><td></td><td>1</td><td></td><td>1</td><td></td><td>1</td><td></td><td>1</td><td></td><td></td><td></td><td>1</td><td>1</td></tr>
<tr><td>$\eta = 3$</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td>1</td><td></td><td></td><td></td><td></td><td>1</td><td>1</td><td></td><td>1</td><td></td><td></td></tr>
<tr><td>(2,0,x)</td><td>1</td><td>1</td><td></td><td></td><td>1</td><td></td><td></td><td></td><td>1</td><td></td><td></td><td>1</td><td>1</td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>$\eta = 2$</td><td></td><td>1</td><td></td><td>1</td><td></td><td></td><td></td><td></td><td></td><td></td><td>1</td><td></td><td>1</td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>(3,0,x)</td><td></td><td></td><td>1</td><td></td><td>1</td><td>1</td><td></td><td>1</td><td></td><td>1</td><td></td><td></td><td></td><td></td><td></td><td>1</td><td></td><td></td></tr>
<tr><td>$\eta = 2$</td><td></td><td></td><td></td><td></td><td></td><td></td><td>1</td><td>1</td><td></td><td>1</td><td></td><td></td><td></td><td></td><td></td><td></td><td>1</td><td></td></tr>
<tr><td>(4,0,x)</td><td>1</td><td></td><td></td><td>1</td><td></td><td>1</td><td></td><td>1</td><td></td><td></td><td></td><td></td><td></td><td>1</td><td></td><td></td><td></td><td>1</td></tr>
<tr><td>$\eta = 1$</td><td>1</td><td>1</td><td></td><td></td><td>1</td><td></td><td>1</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>(5,0,x)</td><td></td><td>1</td><td>1</td><td>1</td><td></td><td></td><td></td><td></td><td></td><td>1</td><td>1</td><td></td><td></td><td></td><td>1</td><td></td><td></td><td></td></tr>
<tr><td>$\eta = 1$</td><td>1</td><td></td><td></td><td>1</td><td>1</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td>1</td><td></td></tr>
</table>

Table 3.2: Number of Edges used in $H(i, i, c)$ with $1 \leq c \leq 5$, broken down by color class

	(7,2,x) $\eta=3$			(6,2,x) $\eta=3$			(5,2,x) $\eta=2$			(4,2,x) $\eta=2$			(3,2,x) $\eta=1$			(2,2,x) $\eta=1$		
(0,0,0)			1	1	1											1	1	1
$\eta=3$	1		1		1								1	1	1			
(1,0,0)							1	1	1	1	1				1			
$\eta=2$							1				1	1	1				1	1
(2,0,0)		1		2	1	1												1
$\eta=2$	1	1	1											1	1	1		
(3,0,0)	1	1				1	1					1		1				
$\eta=1$			1	2				1			1							1
(4,0,0)	1	1	1					1		1					1			
$\eta=1$				1	1	1	1					1					1	
(5,0,0)	1		1	1					1		1			1				
$\eta=1$		1			1	1			1	1						1		

Table 3.3: Number of Edges used in $T_c(x)$ with $0 \le c \le 5$, broken down by color class

	(6,2,0) $\eta=3$			(5,2,0) $\eta=2$			(4,2,0) $\eta=2$			(3,2,0) $\eta=1$			(2,2,0) $\eta=1$			(1,2,0) $\eta=1$		
(0,0,0)							1			1	1	1			1		1	
$\eta=3$													1	1		1	1	
(1,0,0)				1			1		1		1			1				
$\eta=2$							1				1			1				
(2,0,0)	1			1		1						1					1	
$\eta=2$				1						1							1	
(3,0,0)	1		1		1				1					1				
$\eta=1$	1				1			1								1		
(4,0,0)		1			1										1			1
$\eta=1$	1	1			1					1								
(5,0,0)		1	1					1										1
$\eta=1$		1					1				1		1					

Table 3.4: Number of Edges used in $H(0,0,c)$ with $1 \le c \le 5$, broken down by color class

are already boosted to degree four in each color class.) By Condition (3) of Conjecture 3.1, $F_i \cup G_i$ forms a hamilton cycle for each $i \in \mathbb{Z}_\alpha \setminus \{0\}$, so all color classes except for colors 1, 2, 3, 4, and 5 are now connected.

We connect color classes 1,2,3,4, and 5 using the following cycles.

First suppose that α is even. Color three copies of each of the four cycles $(e_{\epsilon,j,1}, e_{\epsilon,j,2}, ..., e_{\epsilon,j,\alpha})$ for $\epsilon, j \in \mathbb{Z}_2$, the d^{th} copy of each cycle being colored with d for $1 \leq d \leq 3$, where

$$e_{\epsilon,j,i} = \begin{cases} (\epsilon, 0, i) & \text{if } i \text{ is odd and } j = 0, \\ (7-\epsilon, 2, i) & \text{if } i \text{ is even and } j = 0, \\ (7-\epsilon, 2, i) & \text{if } i \text{ is odd and } j = 1, \\ (\epsilon, 0, i) & \text{if } i \text{ is even and } j = 1. \end{cases}$$

Color two copies of each of the four cycles $(e_{\epsilon,j,1}, e_{\epsilon,j,2}, ..., e_{\epsilon,j,\alpha})$ for $\epsilon, j \in \mathbb{Z}_2$, the d^{th} copy of each cycle being colored with $d+3$ for $1 \leq d \leq 2$, where

$$e_{\epsilon,j,i} = \begin{cases} (\epsilon, 0, i) & \text{if } i \text{ is odd and } j = 0, \\ (6+\epsilon, 2, i) & \text{if } i \text{ is even and } j = 0, \\ (6+\epsilon, 2, i) & \text{if } i \text{ is odd and } j = 1, \\ (\epsilon, 0, i) & \text{if } i \text{ is even and } j = 1. \end{cases}$$

Next suppose that α is odd. Color three copies of each of the two cycles $(e_{\epsilon,1}, e_{\epsilon,2}, ..., e_{\epsilon,2\alpha})$ for $\epsilon \in \mathbb{Z}_2$, the d^{th} copy of each cycle being colored with d for $1 \leq d \leq 3$, where

$$e_{\epsilon,j,i} = \begin{cases} (\epsilon, 0, i') & \text{if } i \text{ is odd and } j = 0, \\ (7-\epsilon, 2, i') & \text{if } i \text{ is even and } j = 0, \end{cases}$$

where i' is defined by:

$$i' = \begin{cases} i & \text{if } i \leq \alpha, \\ i - \alpha & \text{if } i > \alpha. \end{cases}$$

Color two copies of each of the four cycles $(e_{\epsilon,1}, e_{\epsilon,2}, ..., e_{\epsilon,\alpha})$ for $\epsilon \in \mathbb{Z}_2$, the d^{th} copy of each cycle being colored with $d+3$ for $1 \leq d \leq 2$, where

$$e_{\epsilon,i} = \begin{cases} (\epsilon, 0, i') & \text{if } i \text{ is odd and } j = 0, \\ (6+\epsilon, 2, i') & \text{if } i \text{ is even and } j = 0, \end{cases}$$

where i' is defined by:

$$i' = \begin{cases} i & \text{if } i \leq \alpha, \\ i - \alpha & \text{if } i > \alpha. \end{cases}$$

Using Theorem 3.3 to connect the color classes

For our purposes, we need an almost resolvable α-cycle decomposition $\mathcal{X} = (X_0, ..., X_\alpha)$ of $2K_{\alpha+1}$ on the vertex set $\mathbb{Z}_{\alpha+1}$; by Theorem 3.3, this decomposition exists for $\alpha \geq 3$. We connect our color classes using these $\alpha + 1$ α-cycles.

First suppose that α is even. For any $X = (x_1, ..., x_\alpha) \in \mathcal{X}$, define four cycles $E(\epsilon, j, X = (x_1, ..., x_\alpha), \text{straight}) = (e_{\epsilon,j,1}, e_{\epsilon,j,2}, ..., e_{\epsilon,j,\alpha})$ for $\epsilon, j \in \mathbb{Z}_2$, where for $1 \leq i \leq \alpha$

$$e_{\epsilon,j,i} = \begin{cases} (\epsilon, 0, x_i) & \text{if } i \text{ is odd and } j = 0, \\ (7-\epsilon, 2, x_i) & \text{if } i \text{ is even and } j = 0, \\ (7-\epsilon, 2, x_i) & \text{if } i \text{ is odd and } j = 1, \\ (\epsilon, 0, x_i) & \text{if } i \text{ is even and } j = 1. \end{cases}$$

For any $X = (x_1, ..., x_\alpha) \in \mathcal{X}$ define four cycles $E(\epsilon, j, X = (x_1, ..., x_\alpha), \text{crossed}) = (e_{\epsilon,j,1}, e_{\epsilon,j,2}, ..., e_{\epsilon,j,\alpha})$ for $\epsilon, j \in \mathbb{Z}_2$, where for $1 \leq i \leq \alpha$

$$e_{\epsilon,j,i} = \begin{cases} (\epsilon, 0, x_i) & \text{if } i \text{ is odd and } j = 0, \\ (6+\epsilon, 2, x_i) & \text{if } i \text{ is even and } j = 0, \\ (6+\epsilon, 2, x_i) & \text{if } i \text{ is odd and } j = 1, \\ (\epsilon, 0, x_i) & \text{if } i \text{ is even and } j = 1. \end{cases}$$

Define $C(X, \text{straight}) = E(0,0,X,\text{straight}) \cup E(1,0,X,\text{straight}) \cup E(0,1,X,\text{straight}) \cup E(1,1,X,\text{straight})$ and $C(X,\text{crossed}) = E(0,0,X,\text{crossed}) \cup E(1,0,X,\text{crossed}) \cup E(0,1,X,\text{crossed}) \cup E(1,1,X,\text{crossed})$. Note that $C(X,\text{straight})$ and $C(X,\text{crossed})$ are each 2-factors on the vertex set $\{(0,0,x),(1,0,x),(7,2,x),(6,2,x) \mid x \in V(X)\} \setminus \{(0,0,d(X)), (1,0,d(X)),(7,2,d(X)),(6,2,d(X))\}$. For each $X \in \mathcal{X}$, if $d(X) \neq 0$ then color three copies of $C(X,\text{straight})$, the d^{th} copy being colored with $6d(X)+d$ for $0 \le d \le 2$. If $d(X) = 0$, then just color two copies of $C(X,\text{straight})$, one with color 1 and one with color 2. For each $X \in \mathcal{X}$, color three copies of $C(X,\text{crossed})$, the d^{th} copy being colored with $6d(X)+d$ for $3 \le d \le 5$.

Next suppose that α is odd. For any $X = (x_1, ..., x_\alpha) \in \mathcal{X}$ define two cycles $E(\epsilon, X = (x_1,...,x_\alpha), \text{straight}) = (e_{\epsilon,1}, e_{\epsilon,2}, ..., e_{\epsilon,2\alpha})$ for $\epsilon \in \mathbb{Z}_2$, where for $1 \le i \le 2\alpha$

$$e_{\epsilon,j,i} = \begin{cases} (\epsilon, 0, x_{i'}) & \text{if } i \text{ is odd,} \\ (7-\epsilon, 2, x_{i'}) & \text{if } i \text{ is even,} \end{cases}$$

where i' is defined by:

$$i' = \begin{cases} i & \text{if } i \le \alpha, \\ i - \alpha & \text{if } i > \alpha. \end{cases}$$

For any $X = (x_1,...,x_\alpha) \in \mathcal{X}$ define two cycles $E(\epsilon, X = (x_1,...,x_\alpha), \text{crossed}) = (e_{\epsilon,1}, e_{\epsilon,2}, ..., e_{\epsilon,\alpha})$ for $\epsilon \in \mathbb{Z}_2$, where for $1 \le i \le 2\alpha$

$$e_{\epsilon,i} = \begin{cases} (\epsilon, 0, x_{i'}) & \text{if } i \text{ is odd,} \\ (6+\epsilon, 2, x_{i'}) & \text{if } i \text{ is even,} \end{cases}$$

where i' is defined by:

$$i' = \begin{cases} i & \text{if } i \le \alpha, \\ i - \alpha & \text{if } i > \alpha. \end{cases}$$

Define $C(X,\text{straight}) = E(0,X,\text{straight}) \cup E(1,X,\text{straight})$ and $C(X,\text{crossed}) = E(0,X,\text{crossed}) \cup E(1,X,\text{crossed})$. Note that $C(X,\text{straight})$ and $C(X,\text{crossed})$ are each

2-factors on the vertex set $\{(0,0,x),(1,0,x),(7,2,x),(6,2,x) \mid x \in V(X)\} \setminus \{(0,0,d(X)),$ $(1,0,d(X)),(7,2,d(X)),(6,2,d(X))\}$. For each $X \in \mathcal{X}$, if $d(X) \neq 0$ then color three copies of $C(X,\text{straight})$, the d^{th} copy being colored with $6d(X)+d$ for $0 \leq d \leq 2$. If $d(X)=0$, then just color two copies of $C(X,\text{straight})$, one with color 1 and one with color 2. For each $X \in \mathcal{X}$, color three copies of $C(X,\text{crossed})$, the d^{th} copy being colored with $6d(X)+d$ for $3 \leq d \leq 5$.

Continuing for both Conjecture 3.1 and Using Theorem 3.3

Color classes $1,2,3,4$, and 5 now consist of two components, with the vertices $(i,j,0)$ inducing one of the two components. Each of the color classes $6,...,6\alpha+5$ is connected because for each $X \in \mathcal{X}$: (i) X spans all vertices except $d(X) \in \mathbb{Z}_{\alpha+1}$, and (ii) by (1) of the definition of $\mathcal{F}$ and $T_c(d(X))$ for $0 \leq c \leq 5$, the vertices $(i',j',d(X))$ are joined to vertices in $(i,j,0)$ by edges colored $6d(X)+c$.

For each vertex $u \in V(H)$, if $u \notin T = \{(0,0,0),(1,0,0),(6,2,0),(5,2,0)\}$ then $u = (i,j,x)$ is incident with the same number of edges of each color as of each other color (namely 4 if $(i,j) \in \{(0,0),\ (1,0),\ (6,2),\ (7,2)\}$ and 2 otherwise). If $u \in T$ then it has degree 4 in each color class except for colors 1,2,3,4, and 5. in which it has degree two. In order for the use of the evenly equitable edge-coloring in the next paragraph to work, it is critical that the degrees of these vertices in each of the five color classes be raised to 4, except possibly for one pair of vertices in one color class. Since vertices $(1,0,0)$ and $(5,2,0)$ have amalgamation number two, there are only four edges between them in H, so we cannot simply use five C_4's as we did when connecting our color classes. (In fact, we have already used one of the edges between $(1,0,0)$ and $(5,2,0)$ in $H(0,0,1)$, so we cannot even place four 4-cycles there.) To boost the degree of these vertices in the first five color classes, take three copies of the 4-cycle $((0,0,0),(6,2,0),(1,0,0),(5,2,0))$ and color them using colors 1,2, and 3. Take the 2-cycles $((0,0,0),(5,2,0))$ and $((1,0,0),(6,2,0))$ and color them using color 4. Finally, take the 2-cycle $((0,0,0),(6,2,0))$ and color it with color 5. So, the vertices $(0,0,0)$ and $(6,2,0)$ are now incident with four edges of each color c, $1 \leq c \leq 6\alpha+5$. Vertices $(1,0,0)$ and

$(5,2,0)$, however, are each incident with four of every color, except only two edges of color 5.

Define G to be the subgraph induced by uncolored edges of S. Note that this is precisely the edges in S not used in the trails or cycles described previously. The degree of each vertex in $E(S)$ is divisible by $2(6\alpha+5)$, except for vertices $(1,0,0)$ and $(5,2,0)$ which have degree 2 (mod $2(6\alpha+5)$). Apply an evenly equitable edge-coloring with the colors $1,2,...,6\alpha+5$ to the edges in G. This edge-coloring has the property that at each vertex each color appears on the same number of edges as each other color, except that one color appears twice more than each other color at the vertex $(1,0,0)$ and $(5,2,0)$ since G is bipartite, necessarily the color appearing twice more at those vertices is the same, so name this color 5. It is also important to note that this edge coloring connects the color classes 1, 2, 3, 4, and 5, which were previously in two components. Let G_0 be the subgraph induced by the vertices of $L_0 \cup R_0$ in G. Note that the vertex $(0,0,0)$ has degree ten in G_0, so the evenly equitable edge coloring will produce at most 5 colors on edges to vertices in $L_0 \cup R_0$. So, at least $6\alpha+5-5=6\alpha$ colors must be on edges joining $(0,0,0)$ to vertices not in $L_0 \cup R_0$. Since $a \geq 4$, $6\alpha > 5$. Name five of these 6α colors $1,2,3,4$, and 5. The component previously induced by the vertices of $L_0 \cup R_0$ is now connected with the second component spanning the rest of the graph.

Now, we must disentangle the graph. We will use the same method as with the other cases. To be able to apply Lemma 1.1, we need to add edges to our graph H so that it is the amalgamation of the complete graph K_{3p}. All of these additional edges and loops are colored 0. We call this new graph H^+ and note that it now satisfies the conditions of Lemma 1.1. Thus we apply Theorem 1.3 to H^+ to produce G^+, which is an edge-colored copy of K_{3p}. We now remove all edges in G^+ colored 0; the resulting graph is K_3^p. Using Theorem 1.3, we see that each color class is 2-regular by *(i)* and connected by *(ii)*, which implies that each color class is a hamilton cycle. We now consider $G-E(S)$. Note that all edges between vertices in L_0 and R_0 were in $E(S)$, so we have that $G - E(S)$ has cut vertex v. So our set of hamilton cycles is maximal. $\blacksquare$

So this chapter culminates in the following state of knowledge:

Theorem 3.5 *There exists a maximal set of m hamilton cycles in K_n^p if and only if*

1. $\left\lceil \frac{n(p-1)}{4} \right\rceil \le m \le \left\lfloor \frac{n(p-1)}{2} \right\rfloor$ *and*

2. $m > \frac{n(p-1)}{4}$ *if*

   (a) *n is odd and $p \equiv 1 \pmod 4$), or*

   (b) *$p = 2$, or*

   (c) *$n = 1$,*

except possibly when $n = 3$, $m = \left\lceil \frac{n(p-1)}{4} \right\rceil$ and either $p \equiv 3 \pmod 8$, $p \geq 19$ or $p \in \{15, 23\}$.

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