

Nim on Graphs

by

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Abstract

Winning strategies for the Game of Nim on Graphs are discussed. Graphs considered are distinguished from those previously studied in that they may have loops. Winning strategies are found for graphs that have loops and whose links form paths, cycles, trees, or monocyclic graphs. Furthermore, strategies are developed to approach games played on graphs that contain an induced tree with loops.

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Chapter 1

Introduction

Before introducing our main topic, the game of Nim on G where G is a multi-graph, we may wish to provide the reader with some basic terminology we will be using to describe the problem and provide some history of the problem in general.

1.1 Background

We assume that the reader is familiar with most basic terminology concerning graphs. However, it is worth noting that for a graph, G , we may let context differentiate between the vertex set and edge set of the graph, writing, for example, $v \in G$ should it be clear that v is a vertex. However, in situations for which there may be some ambiguity, we will denote the vertex set as $V(G)$ and edge set as $E(G)$. We denote the degree of a vertex, v , as $d(v)$ and find reason, on occasion, to distinguish between links of $E(G)$, those edges ending in two distinct vertices, and loops, those incident to a single vertex.

Nim is a two player game in which players alternate turns removing objects, often called “stones” from distinct heaps. On a players turn, the player chooses a heap and removes at least one, but as many as all, objects from that heap. Under conventional play, the last player able to remove an object is the winner; that is, a player without the ability to play loses. There are European references to nim from as early as the beginning of the 16th century. It is suspected that the game is much older, but its origins are unknown. The name, nim, and a complete strategy for the game were developed by Charles Bouton in 1901.

Nim found a place of particular importance in game theory when Sprague and Grundy independently, in 1935 and 1939, respectively, developed a theory showing the equivalence of an entire class of two player games to nim heaps. Should the sum of games be played

in the normal convention, moves and strategy would correspond to a game of nim played on their equivalent heaps. The size of the corresponding heap is called the *Grundy number*, or *nimber*, of the game. Games equivalent to heaps are those that are sequential, have 2 players, are *impartial* games of *perfect information*, and have no infinite lines of play.

A game is *impartial* if, at a given position, the available moves depend solely on the position and are not determined by the player whose move it is. In such a game, the only difference between two players is that one player goes first. We herein denote the first player P_1 and the second player P_2 .

A game is one of *perfect information* if, at every position of the game, all players know all of the possible moves or combinations of moves that would be known at the end of the game. For example, chess is a game of perfect information; each player knows all combinations of possible moves. Poker is not a game of perfect information.

Given a position in a game, *options* of that position are positions of the game to which the player may move.

A game of the type considered by Sprague and Grundy falls into one of two outcome classes. Either it is a won game for the first player to move under optimal play (called a *p-position*) or it is a loss for player one regardless of strategy under optimal play (called a *0-position*). Here the p- and 0- distinctions are derived from the Grundy number of the game as heaps with a positive or zero size respectfully (the first being a P_1 win by removing all the objects, the second a P_1 loss having no options).

The key component to strategy in nim is the representation of the size of each heap as a binary number and summing the digits mod2, neglecting to “carry” from one digit to another. This sum of two numbers, x and y , in such a way is called the *nim-sum* of x and y , and is denoted $x \oplus y$.

A game of nim with k nim heaps of sizes n_1, n_2, \dots, n_k respectively is a 0-position if $n_1 \oplus n_2 \oplus \dots \oplus n_k$ is 0 and a p-position otherwise. If $n_i = 0$ for every i , $n_1 \oplus n_2 \oplus \dots \oplus n_k$ is 0. If $n_1 \oplus n_2 \oplus \dots \oplus n_k$ is 0 then any option (the reduction of any n_i) will have a

positive nim-sum. And if $n_1 \oplus n_2 \oplus \dots \oplus n_k$ is positive then there is an option for which $n_1 \oplus n_2 \oplus \dots \oplus n_k$ is 0. Namely, if the j^{th} digit of $n_1 \oplus n_2 \oplus \dots \oplus n_k$ is the left-most 1, then there is at least one pile whose size, n_α , has a 1 in the j^{th} digit, and that heap may be reduced to $n_\alpha \oplus (n_1 \oplus n_2 \oplus \dots \oplus n_k)$.

For example, consider the game of nim with heaps of size 3, 7, and 6. $3 \oplus 7 \oplus 6$ is digit addition mod two: $011 \oplus 111 \oplus 110 = 010$. Each heap has a 1 in the 2nd digit of its binary representation, so reducing 3 to $011 \oplus 010 = 001 = 1$ giving heaps of size 1, 7, and 6 whose nim-sum is 0. Other winning options include reducing the heap of size 7 to 5 or the heap of size 6 to 4.

1.2 History of Nim on Graphs

A variation of the game of nim played on a graph, G , was first introduced by Fukuyama in [3] and extended in [4]. Fukuyama therein determines winning strategies for paths and odd cycles, as well as Grundy numbers for paths, cycles, trees, and certain bipartite graphs. It is important to note that while the Grundy number of a game informs strategy (should the Grundy number of every option of a position be known, perfect play dictates a move to a 0-position), analyzing every option of a position may be overly cumbersome. We are then interested in developing methods by which to recognize whether a position is p- or 0- and to develop an associated strategy that is perhaps less cumbersome.

Erickson in 2009 developed a strategy for even cycles, [2]. In the same paper, she is able to develop a winning strategy for games played on the complete graph where all heaps are of size 1.

While Fukuyama mentions an equivalence between games on graphs with multiple edges and simple graphs, neither he, nor Erickson, consider strategy or the Grundy number for games played on graphs with loops. Ongoing research seemingly focuses exclusively on simple graphs. We here consider graphs with loops.

1.3 Outline

In Chapter 2, we will begin with a rigorous definition of Nim on Graphs, and discuss how the classic game of nim may be thought of as a special case of Nim on Graphs. We will here, importantly, show that all games on multigraphs are equivalent to games with no multiple edges or multiple loops. Winning strategies for simple cases of Nim on G , paths and cycles, will be given. Since later constructions may depend on strategy or classification of these cases, an easy reference figure caps the chapter.

In Chapter 3, games of Nim on Trees are discussed. We find that the structure of trees allows for a recursive reduction of Nim on G to games of Nim on smaller graphs, and we conclude with several theorems extending strategies for nim on trees, allowing for analysis of games played on graphs with induced subgraphs of trees. These latter theorems are put to use with the analysis of games played on mono-cyclic graphs in Chapter 4.

Chapter 5 sees a brisk discussion of areas for future research.

Chapter 2

The Game of Nim on Graphs

2.1 Introduction to the Game of Nim

Here we will define the game of nim on a graph, G , simplify the case in general by showing that only a certain class of graph need be considered, and examine winning strategies on paths or cycles with loops in order to get a sense of how loops in particular impact the game.

Let G be a multi-graph with loops, edge set E and vertex set V . Positions of the game of Nim on G are ordered pairs, (v, ω) with $v \in V$, $\omega : E \rightarrow \mathbb{Z}^+$, the set of non-negative integers. The value $\omega(e)$ is called the weight of e . A position (v', ω') is an option of (v, ω) if there is an $e \in E$ incident with both v and v' , $\omega(e) > \omega'(e) \geq 0$, and $\omega'(f) = \omega(f)$ for all $f \in E, f \neq e$. A player in a position without options is the loser (normal play convention).

An example of game played on a multigraph is demonstrated in Figure 2.1. The vertex in each position is denoted using a Δ . In the example, P_2 has lost since his final position has no options; $\omega(e) = 0$ for every edge, e , incident with Δ .

We classify each position in a game of Nim on G as being either a *0-position* or a *p-position* as follows:

- A position is a 0-position if it has no options.
- A position is a p-position if and only if it has at least one option that is a 0-position.
- A position is a 0-position if all of its options are p-positions.

Since from any position, a game must eventually terminate, this does in fact give a classification.

We note that the traditional game of Nim on k piles, each with p_i objects, $1 \leq i \leq k$, is equivalent to two different formulations of Nim on G :

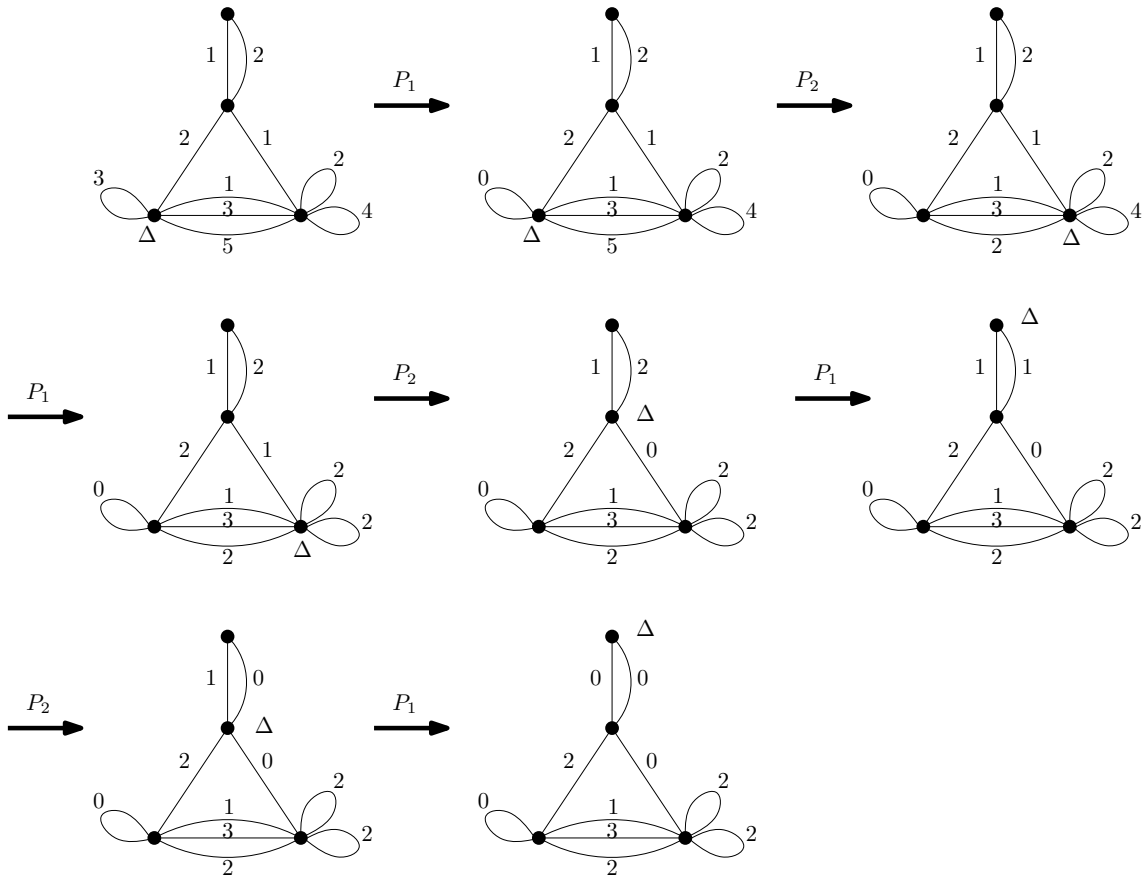


Figure 2.1: An example of Nim on G played on a multi-graph

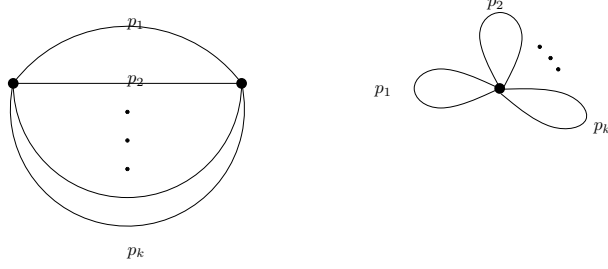


Figure 2.2: Games of Nim on G equivalent to traditional nim games

1. G is the graph with two vertices, one of which is v , and k links, e_1, e_2, \dots, e_k . The position is (v, ω) where $\omega(e_i) = p_i$, $1 \leq i \leq k$

2. G is the graph with a single vertex, v , and k loops, l_1, l_2, \dots, l_k . The position is (v, ω) where $\omega(l_i) = p_i$, $1 \leq i \leq k$.

The Grundy numbers for such positions are $\omega(l_1) \oplus \omega(l_2) \oplus \dots \oplus \omega(l_k) = \Gamma$ (or, respectively, the nimsum of the $\omega(e_i)$), and for each position, there is an option for which $\omega'(l_1) \oplus \omega'(l_2) \oplus \dots \oplus \omega'(l_k) = n$ for each non-negative integer $n < \Gamma$.

Since we know winning strategies for such positions, and the graphs associated (multiple links and loops) are the basic building blocks of larger graphs, it is natural to attempt to use this relationship to simplify these multiple piles (for us, edges) down to a single pile (edge).

Theorem 2.1. *Let G be a graph, and (v, ω) be a position in a game of Nim on G . G may have multiple loops or multiple links. Then there exists a graph, H , on the vertex set of G in which each vertex is incident with at most one loop, each pair of vertices are incident with at most one shared link, and a position in a game of Nim on H , $(v, \bar{\omega})$, such that a winning strategy in Nim on H corresponds to a winning strategy in Nim on G .*

Proof. Construct H as follows: if $v \in V$ is incident with loops l_1, l_2, \dots, l_m in G , then there is a single loop, l , incident with v in H with $\bar{\omega}(l) = \omega(l_1) \oplus \omega(l_2) \oplus \dots \oplus \omega(l_m)$. Similarly if links e_1, e_2, \dots, e_n are each incident with v and v' in G , then there is a single edge, e , in H incident with v and v' with $\bar{\omega}(e) = e_1 \oplus e_2 \oplus \dots \oplus e_n$.

Suppose $(v, \bar{\omega})$ is a p-position in Nim on H with option $(v', \bar{\omega}')$, a 0-position, and let e be the link incident with both v and v' in H , corresponding to e_1, e_2, \dots, e_n in G , where $\bar{\omega}'(e) < \bar{\omega}(e)$. Then there is an option of (v, ω) in Nim on G with $\omega'(e_1) \oplus \omega'(e_2) \oplus \dots \oplus \omega'(e_n) = \bar{\omega}'(e)$. Similarly, for l , a loop incident with v in H corresponding with the collection of loops incident with v in G , there is an option of (v, ω) in Nim on G with $\omega'(l_1) \oplus \omega'(l_2) \oplus \dots \oplus \omega'(l_m) = \bar{\omega}'(l)$. Should $(v', \bar{\omega}')$ be a position with no options, $\bar{\omega}'(f) = 0$ for every edge, f , in H incident with v' (respectively v should the option reduce the weight of a loop). However, it may be the case that for the corresponding option in Nim on G , (v', ω') , $\omega'(f_i) > 0$ for some $i, 1 \leq i \leq d(v')$.

In such a position, suppose (v'', ω'') is an option of (v', ω') , and $\omega''(f_1) < \omega'(f_1)$. Then, for $(v'', \bar{\omega}'')$, the corresponding position in Nim on H , $\bar{\omega}''(f) > 0$ and f is the only edge incident with v'' with non-zero weight. It is clear that in Nim on H removing all of the weight from f is a winning move. A corresponding option exists in Nim on G , and will exist for any future like-options which must be finite since each reduces the finite weight on the edges of G incident with v'' . A winning strategy for Nim on H can thusly be translated into a winning strategy for Nim on G , and we may in the future only consider G in which each vertex is incident with at most one loop and each pair of vertices are incident with at most one shared link. □

Consider the game of Nim on G for the graph, vertex, and weight function as demonstrated in figure 2.1. We may reduce each of the occasions of a multiple edge with a single edge and each occasion of multiple loops with a single loop by adjusting the weight function appropriately. We may discern a strategy for such a position and in doing so appropriate a strategy for our initial game with the small addition of reducing the nim sum of weights back to 0 should P_2 play across such an edge (as they do in our example). The game on a simple graph with at most one loop incident with each vertex and its parallel are demonstrated here in Figure 2.3.

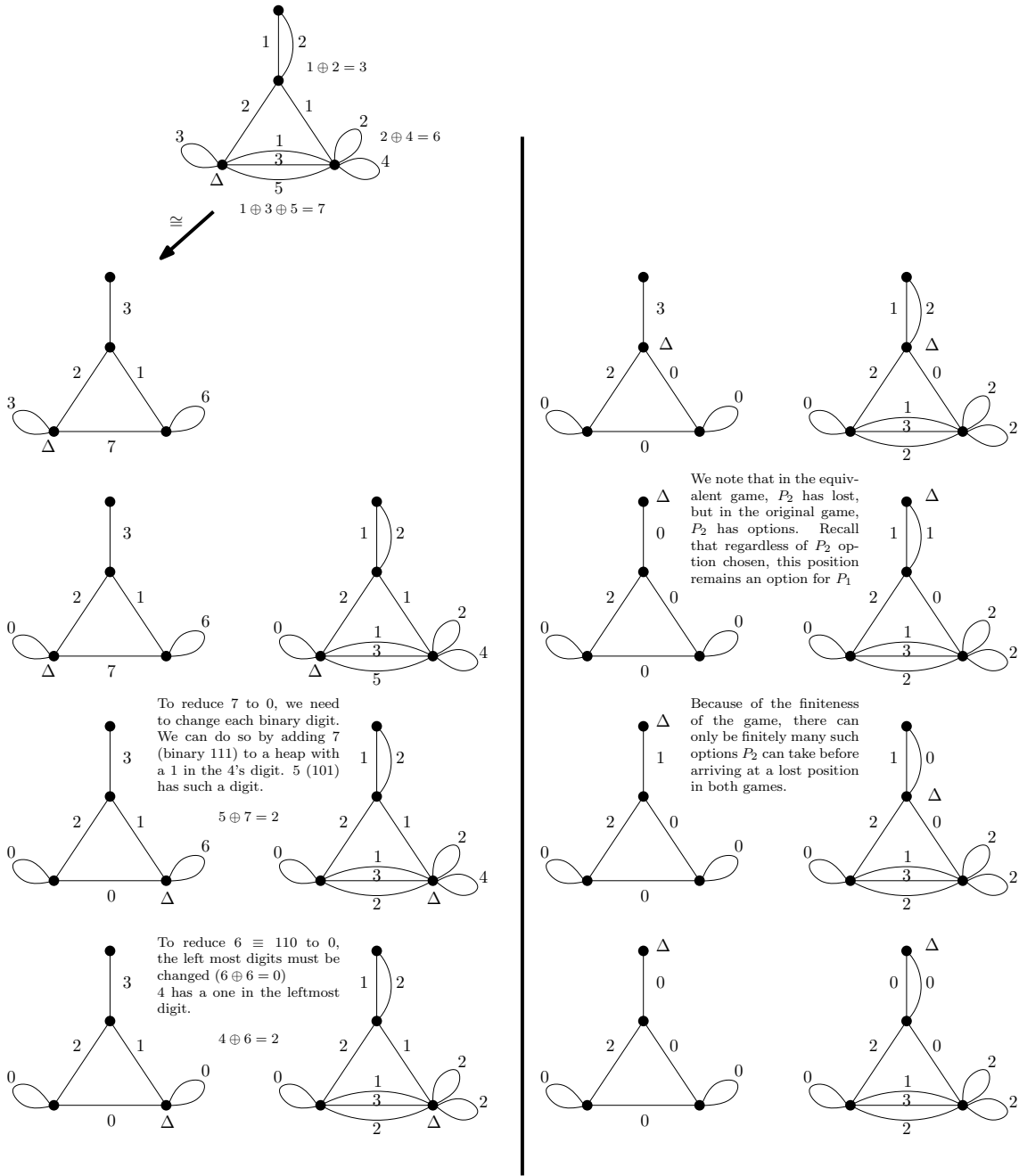


Figure 2.3: Gameplay given by an equivalent graph without multiple edges or loops

2.2 Paths With Loops

In order to get an idea of some simple strategies that arise in Nim on Graphs, it is helpful to examine the simple case of a path with loops at some of the vertices. We'll consider Nim on G , where G is a graph with vertex set $\{v_i | 1 \leq i \leq n\}$, and edge set $\{e_j | 1 \leq j \leq n-1\} \cup \{l_k | 1 \leq k \leq n\}$ where e_i is incident with v_i and v_{i+1} and l_i is a loop incident with v_i . We can assume that there is a loop at each vertex in the general case since if $\omega(l_i) = 0$, play progresses as if no such loop exists. We will assume, however, that for a starting position, $\omega(e_i) > 0$ for all $i, 1 \leq i \leq n-1$

Let us first consider the position (v_1, ω) , and let $\omega(l_i) > 0$ for $1 \leq i \leq m$. If P_1 chooses option (v_1, ω') , where $\omega'(l_1) = 0$, P_2 may choose an option (v_2, ω'') . Should $\omega''(e_1) > 0$, P_1 may simply move the position back to v_1 , reduce the weight on e_1 to 0, and P_2 has lost. $\omega''(e_1) = 0$ arrives at a position equivalent to $(v_2, \omega|_{G \setminus v_1})$ in Nim on $G \setminus v_1$. This position was also an option of (v_1, ω) . If a p-position, P_1 can implement the above strategy to reach it. If a 0-position, P_1 can choose this option from the starting position, thus (v_1, ω) is a p-position with strategy determined by the path induced by $\{v_i | 2 \leq i \leq n\}$.

If $\omega(l_i) = 0$ for $1 \leq i \leq m$, all options of (v_1, ω') for which $\omega'(e_1) > 0$ are p-positions (as outlined above) and need not be considered. So we may assume that $\omega'(e_1) = 0$. The resultant position is a game of Nim on a path. If $m = 1$, then (v_2, ω') is a p-position equivalent to the above case, and (v_1, ω) is a 0-position. The question of whether (v_1, ω) is a p-position or a 0-position resides entirely on the parity of m . Since perfect play removes all weight with each option taken, (v_1, ω) is a p-position for even m and a 0-position for odd m .

But what of strategies for games of Nim on G where G is a path with loops but the starting position is not (v_1, ω) , but rather (v_i, ω) for some $i, 1 < i < n$? If $\omega(l_i) = 0$, (v_i, ω) has options (v_{i-1}, ω_{i-1}) and (v_{i+1}, ω_{i+1}) where $\omega_{i-1}(e_{i-1}) = 0$ and $\omega_{i+1}(e_i) = 0$. Each of these positions is easily analyzed as either a p-position or a 0-position as above. The options effectively delete links incident with v_i and place P_2 in a position on a path with loops on $i-1$ vertices or $n-i$ vertices respectively with the position on a terminal vertex of the path.

Should either of the options be 0-positions, P_1 has a clear choice. Should both (v_{i-1}, ω_{i-1}) and (v_{i+1}, ω_{i+1}) be p-positions, (v, ω) is a 0-position. P_1 has no 0-position options, (v', ω') for which $\omega'(e_i) > 0$ (similarly for $\omega'(e_{i-1}) > 0$), since P_2 would then have the option of reducing the weight on e_i to zero, moving the position back to v_i , a position, which as previously analyzed, would be a 0-position.

If $\omega(l_i) > 0$, then (v_i, ω) is a p-position. As before, if either of (v_{i-1}, ω_{i-1}) or (v_{i+1}, ω_{i+1}) are 0-positions it is clear. If both are p-positions, (v_i, ω') where $\omega'(l_i) = 0$ is an option of (v_i, ω) and a 0-position by the above argument.

The important points that we might draw from this case are:

1. (v_1, ω) is a p-position in Nim on G if $\omega(l_1) > 0$. The existence of weight on a loop at the starting position gives P_1 a great deal of power. P_1 may stall by reducing the weight on l_1 to 0 and force P_2 to move to v_2 or may reduce the weight on e_1 to 0 placing P_2 on v_2 depending on whether $v_2, \omega|_{G \setminus \{v_1\}}$ in Nim on $G \setminus \{v_1\}$ is a p-position or a 0-position respectively.

2. The winning strategy for a game of Nim on G , where G is a path with loops, depends only on the existence of weight on an edge, not on the amount of weight on an edge. If $\omega(e) = 1$ for every $e \in G$, p-positions and 0-positions are preserved for initial positions based on their vertex.

3. Our most natural approach is to analyze each of the options of (v, ω) . However, this can be cumbersome. There may be a great many options, but we might pay particular attention to those that reduce the weight of an edge to 0. These correspond to nim on the subgraphs of G missing an edge incident with v .

2.3 Cycles with Loops

Since we now have an easy way to categorize games of nim on paths with loops as p-positions or 0-positions and developed strategy for these cases, it is natural to look at the case of cycles with loops since options for these games will include options equivalent to

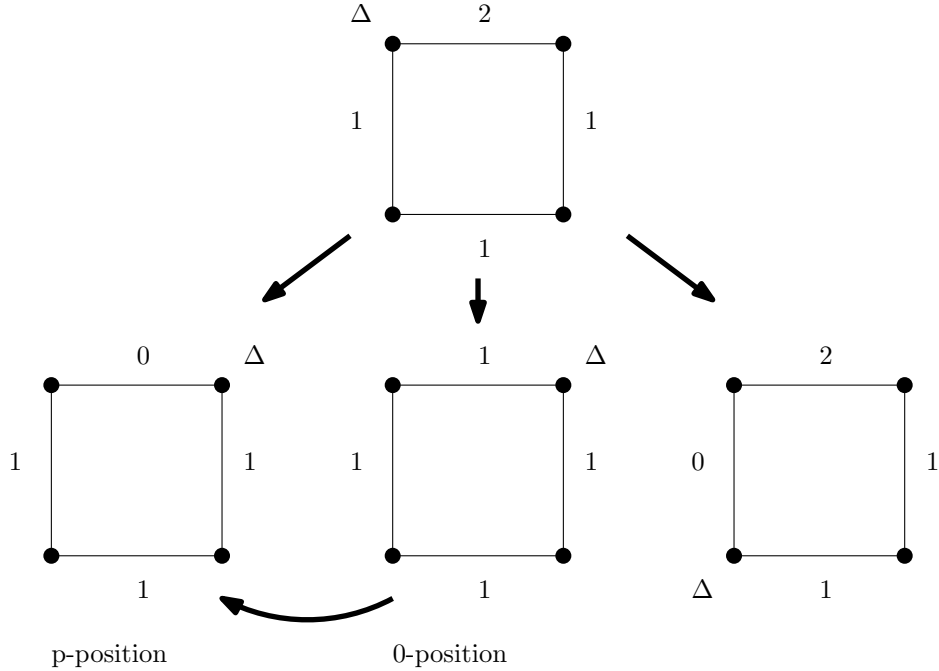


Figure 2.4: Nim on C_4 in which strategy depends on ω

games on paths with loops. We'll here consider G to be a cycle on n vertices with vertex set $\{v_0, v_1, v_2, \dots, v_{n-1}\}$ and edge set $\{e_0, e_1, e_2, \dots, e_{n-1}\} \cup \{l_0, l_1, l_2, \dots, l_{n-1}\}$ where e_i is a link incident with v_i and $v_{i+1 \bmod n}$, and l_j is a loop incident with v_j . Without loss of generality, we will assume the starting position is (v_0, ω) .

It is instructive here to note that while the amount of weight on the edges of a graph had no import on strategy in the case of paths with loops, the weight may well be important to the classification of positions of Nim on G if G is a cycle. Consider the case when $n = 4$, $\omega(e_0) = 2$, $\omega(e_i) = 1$ for $i \neq 0$, and $\omega(l_j) = 0$ for all j .

(v_1, ω') where $\omega'(e_0) = 0$ is an option of (v_0, ω) , but is a p-position as it is equivalent to a loopless path of length three with a position at a terminal vertex. However, should $\omega'(e_0) = 1$, P_2 's only options will be such 0-position paths. Thus (v_0, ω) is a p-position, but P_1 's winning strategy involves reducing the weight of an edge to a non-zero value.

Strategy for cases for which $\omega(l_i) = 0$ for all i are discussed in detail in [3] (odd cycles) and [2] (even cycles). These cases are here considered as special cases of cycles with loops, and we have arrived at the associated strategies independently.

We first consider cycles for which $\omega(l_0) = 0$ but $\omega(l_i) > 0$ for some $i \neq 0$. If we can easily classify these positions, we can use those that are 0-positions to help classify positions for which $\omega(l_0) > 0$ with these 0-positions are options. For the sake of simplicity "vertices counter-clockwise (CCW) of v_0 " will refer to the vertices $v_{n-1}, v_{n-2}, \dots, v_1, v_0$, in that order, while the "vertices clockwise (CW) of v_0 " will refer to $v_1, v_2, \dots, v_{n-1}, v_0$, in that order.

Let $\omega(l_{k+1}) > 0$ and $\omega(l_i) = 0$ for $0 \leq i < k$. Similarly, let $\omega(l_{n-m-1}) > 0$ and $\omega(l_j) = 0$ for $j \pmod n \geq n - m$. If either of k or m are odd then (v_1, ω'_{cw}) (or respectively (v_{n-1}, ω'_{ccw})) where $\omega'_{cw}(e_0) = 0$ (resp. $\omega'_{ccw}(e_{n-1}) = 0$) is an option of (v_0, ω) and is a 0-position, being equivalent to a path with loops that is a 0-position. If both k and m are even, the same options are p-positions. However, options in which $\omega'_{cw}(e_0) > 0$ or $\omega'_{ccw}(e_{n-1}) > 0$ are also p-positions since (v_0, ω'') where $\omega''(e_0) = 0$ or $\omega''(e_{n-1}) = 0$ would be 0-position options. Even k and m must then imply a 0-position.

If $\omega(l_0) > 0$, even k and m would imply a p-position with option (v_0, ω') where $\omega'(l_0) = 0$, a 0-position. If either of k or m are odd and $\omega(l_0) > 0$ and $\omega(l_i) > 0$ for some i , $1 \leq i \leq n - 1$, then, as before, (v_1, ω'_{cw}) (respectively (v_{n-1}, ω'_{ccw})) where $\omega'_{cw}(e_0) = 0$ (resp. $\omega'_{ccw}(e_{n-1}) = 0$) is an option of (v_0, ω) and is a 0-position, being equivalent to a path with loops that is a 0-position. The loop at v_0 does not change the parity of the number of loops for which $\omega(l_i) = 0$ preceding a loop with positive weight on the path that is an option of the initial position.

We are left then with cases for which $\omega(l_i) = 0$ for all $i > 0$. If n is odd and $\omega(l_0) = 0$ options for which $\omega(e_0)$ or $\omega(e_{n-1})$ are zero are 0-positions since they are equivalent to paths for which the initial position is on a terminal vertex and the path begins with an odd number of loops for which the weight is zero. Nim on Odd Cycles for which $\omega(l_i) = 0$ and $\omega(e_i) > 0$

for all i are P_1 wins. Likewise if n is even and $\omega(l_0) > 0$, options for which $\omega(e_0)$ or $\omega(e_{n-1})$ are 0-positions and (v_0, ω) is a p-position.

For odd n , if $\omega(l_0) > 0$, similar options (where one of $\omega'(e_0)$ or $\omega'(e_{n-1})$ are 0) are p-positions equivalent to paths. Options for which $\omega'(e_0) > 0$ or $\omega'(e_{n-1}) > 0$ are p-positions for cycles as outlined above. Reducing the weight on l_0 to zero is also an option, but a p-position. Thus, if $\omega(l_0) = 1$, the only options of (v_0, ω) are p-positions, and (v_0, ω) is a 0-position. Should $\omega(l_0) > 1$, reducing the weight on l_0 to one is a 0-position option available to P_1 . In this case, the weight on the loop, l_0 is of consequence, but only in so much that it is either one or greater than one.

We now consider the case for which n is even and $\omega(l_i) = 0$ for all i . For any position, (v, ω) in which $\omega(e_i) > 0$ for all i , all options for which $\omega'(e_i) = 0$ for some i are equivalent to a game of nim on a path with loops with initial position at a terminal vertex and with an odd number of vertices sequential from the initial position without loops and are thus p-positions. For P_1 , any hope of finding a 0-position option resides in reducing the weight of an edge incident with v_0 . Since edges for which $\omega(e_i) = 1$ may only be reduced to a weight of zero, effectively "breaking" the cycle and placing the subsequent player in a p-position, we may be particularly interested in positions at vertices incident with edges with weight one, and those positions with options that include such vertices.

We may then classify any positions, (v_0, ω) , for which $\omega(e_0) > 1$ and $\omega(e_{n-1}) > 1$ and for which $\omega(e_1)$ or $\omega(e_{n-2})$ is one, as p-positions, since, (v_1, ω') , where $\omega'(e_0) = 1$, or respectively (v_{n-1}, ω') where $\omega'(e_{n-1}) = 1$, are options and 0-positions.

Suppose $\omega(e_i) = 1$ for some i , $0 \leq i \leq n - 1$. Let m be the smallest value of i for which $\omega(e_i) = 1$ and M be the largest. Assuming optimal play, we know that any position, (v_i, ω) , where $i = m - 1$ or $i = M + 2$ are p-positions. We need not consider positions or options (v_i, ω) for which $m - 1 \leq i \leq M + 2$ since we are assuming best play. We may then disregard weight on any of the edges in $\{e_i | m + 1 \leq i \leq M - 1\}$. The game is then reduced to play on a path, but one in which, under optimal play, weight is not reduced to zero, edges

for which $\omega(e) = 1$ will not be crossed should another option be available. If either of m or $n - M$ are odd, P_1 has a winning strategy by choosing option (v_1, ω') where $\omega'(e_0) = 1$ or respectively (v_{n-1}, ω') where $\omega'(e_{n-1}) = 1$. P_2 is then forced to break the cycle, a p-position for P_1 , or to reduce the weight on e_1 (respectively e_{n-2}). If $\omega''(e_1) > 1$ then (v_1, ω''') where $\omega'''(e_1) = 1$ is an option and a 0-position. If $\omega'''(e_1) = 1$, P_1 may reduce the weight on e_2 to one, and play will continue in a like manner until P_2 is in the 0-position, $(v_m, \bar{\omega})$, where $\bar{\omega}(e_{m-1}) = \bar{\omega}(e_m) = 1$.

2.4 A Summary of Strategies So Far

We find that it may be helpful to have a visual reference for winning positions and strategies for the games so far discussed. Many are easily analyzed at first glance, and thus, a diagram representing p-positions and their related 0-options, here discussed, may be of use.

In the associated diagrams, the deletion of an edge in a move is associated with the removal of all weight on that edge. In the case of cycles, we restrict ourselves to defined option of changing the weight of an edge to zero. As before, the vertex of the initial position is denoted with a Δ .

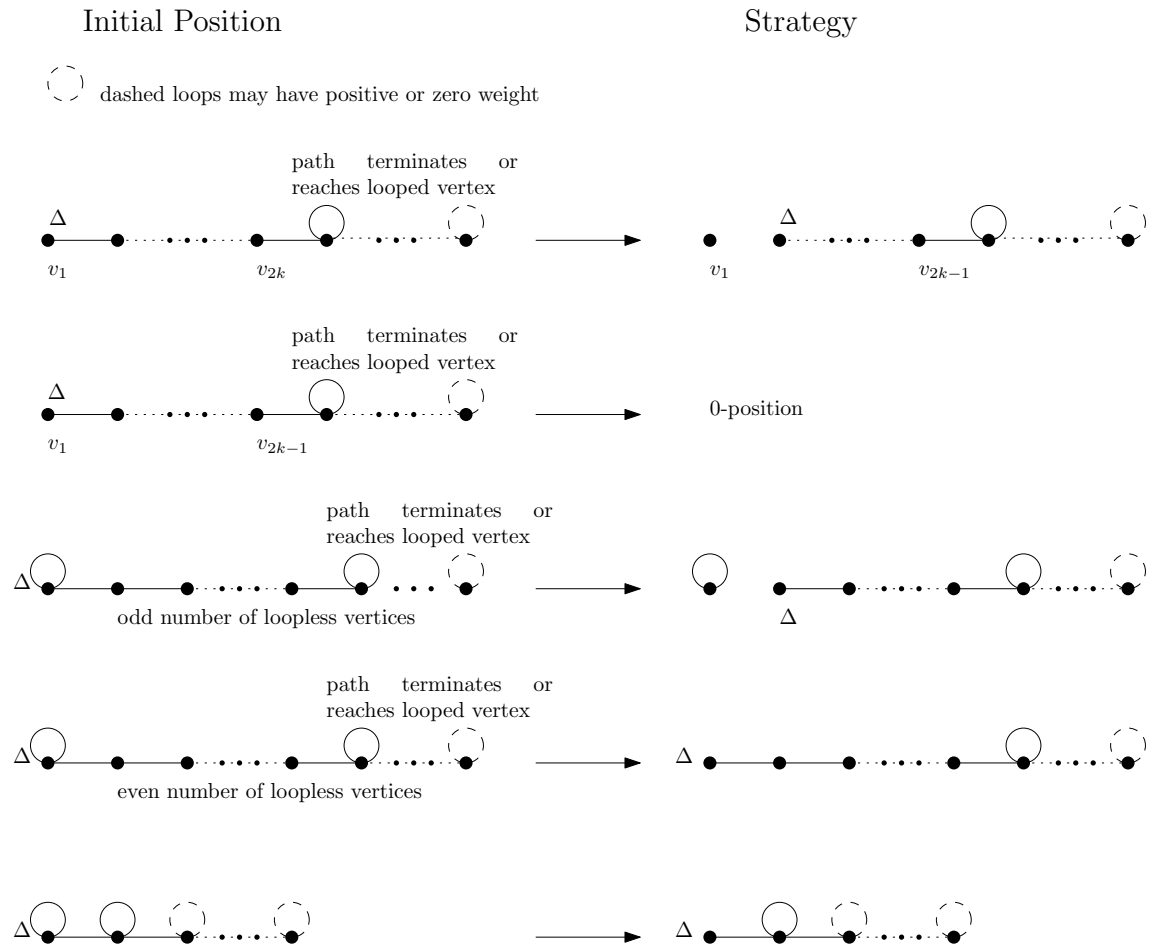


Figure 2.5: Strategies for games of Nim on paths with loops

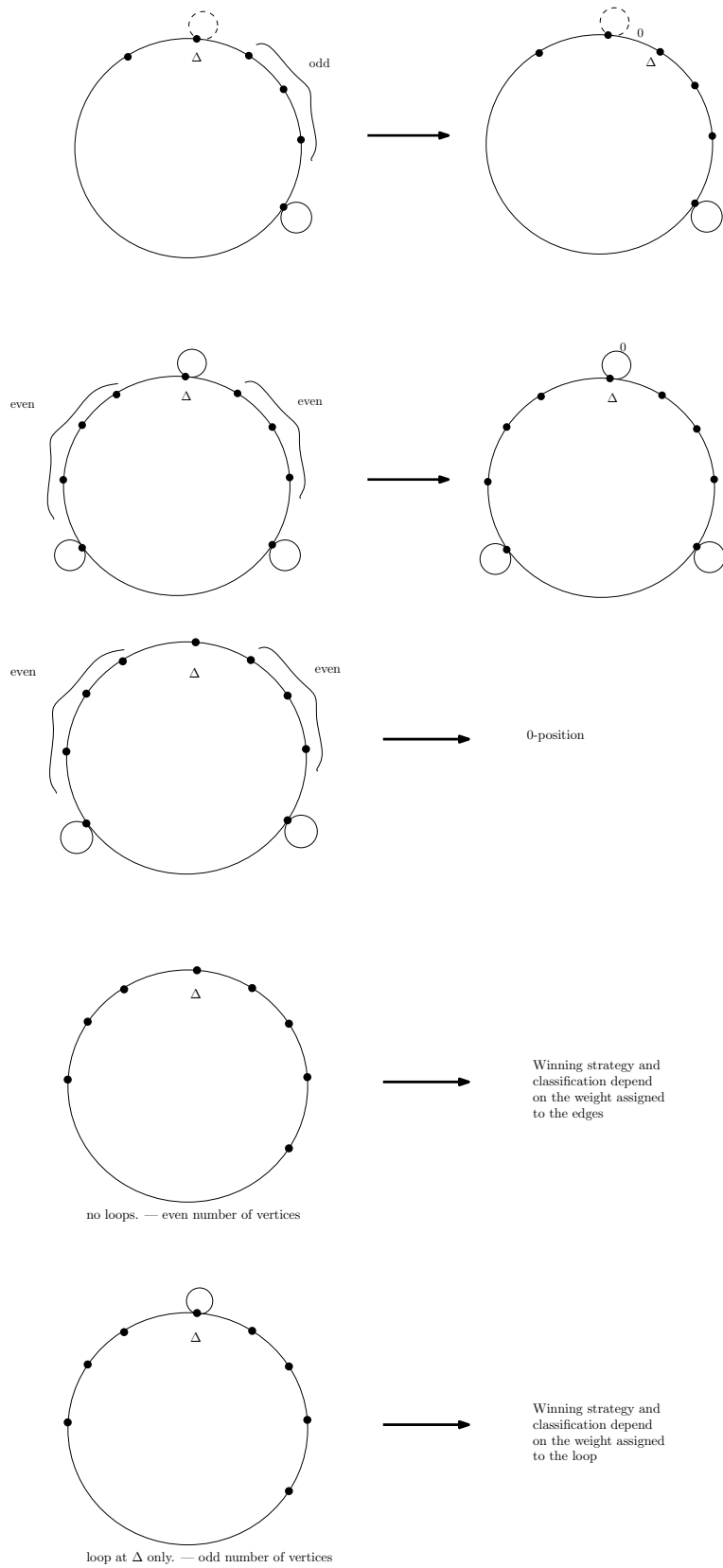


Figure 2.6: Strategies for Games of Nim on cycles with loops

Chapter 3
Trees with Loops

3.1 Does Weight Matter?

As we saw in Chapter 2, the existence of a cycle in a graph has the potential to complicate the Game of Nim played on that graph, introducing situations in which the weight assigned to an edge alters whether a position is a p-position or 0-position. Since trees have no cycles, we may suspect that their structure might make for a simpler analysis. Fukuyama addresses Nim on Trees in [4] and finds the Grundy numbers completely for such games. We'll here consider trees that have loops at some vertices. The case for which the weight on these loops is 0, that is the case for which Fukuyama previously found winning strategies, is of course included as a specific instance of Nim on such graphs.

Theorem 3.1. *For games of Nim on G with initial position (Δ, ω) in which G is a tree with loops, the classification of (Δ, ω) as a p-position or 0-position, is independent of whether $\omega(e) = 1$ or $\omega(e) > 0$ for any $e \in G$.*

Proof. Without loss of generality, we assume that $\omega(e) \geq 1$ for any $e \in G$. Consider the components of $G \setminus \{\Delta\}$, $\{T_i | 1 \leq i \leq n\}$, and the vertices adjacent to Δ , $\{v_i | 1 \leq i \leq n\}$, where $v_i \in T_i$ and Δ and v_i are adjacent by edge e_i . In Nim on T_i , each of $(v_i, \omega|_{T_i})$ are p-positions or 0-positions. If any such games are 0-positions, say $(v_k, \omega|_{T_k})$, then (v_k, ω') where $\omega'(e_k) = 0$ is a 0-position and option of (Δ, ω) . (Δ, ω) is then a p-position.

If there is no loop incident with Δ and each of $(v_i, \omega|_{T_i})$ are p-positions, any option, (v_i, ω') , of (Δ, ω) for which $\omega'(e_i) = 0$ is a p-position and is not a perfect play option for P_1 . If P_1 chooses an option, (v_k, ω') where $\omega'(e_k) > 0$, P_2 has the option (Δ, ω'') where $\omega''(e_k) = 0$. If P_1 continues to avoid options that reduce the weight on an edge incident with Δ to zero,

this strategy may be repeated until P_1 is in the position $(\Delta, \bar{\omega})$ where $\bar{\omega}(e_i) = 0$ for all i which is a 0-position as it has no options. Thus, if there is not a loop incident with Δ , (Δ, ω) is a 0-position. If there is a loop incident with Δ , then reducing the weight on the loop to zero places P_2 in a 0-position and (Δ, ω) is a p-position.

The strategy for the initial move in Nim on a Tree does not depend on the weight of the edges, only the existence of weight. The same argument holds for any future positions in the game, and thus winning strategy and the classification as a p- or 0-position is independent of whether $\omega(e) = 1$ or $\omega(e) > 0$ for any $e \in G$. \square

3.2 Winning Strategies

So, we know that additional weight on any edge with positive weight will not change whether a position is won or lost, and we know that any position, (Δ, ω) for which there is a loop at Δ , is a p-position. We are not, however, directed to a winning strategy by this information since analysis of the games played on each of the T_i isn't provided beyond a recursion of the above argument which seems prohibitively time consuming.

We have noted that if P_2 is in a 0-position and removes weight from an edge without removing all of the weight, the P_1 may remove the remaining weight from the edge and place P_2 back into a 0-position. By assuming such a strategy, if (Δ, ω) is the initial position, for any P_1 position, $(v_k, \bar{\omega})$, we may assume that $\bar{\omega}(e_i) = 0$ for every e_i , $1 \leq i \leq k$, in the path $\Delta e_1 v_1 e_2 \dots v_{k-1} e_k v_k$. If $\bar{\omega}(e_k) > 0$, then the previous strategy offers the 0-position $(v_{k-1}, \bar{\omega}')$ where $\bar{\omega}'(e_{k-1}) = 0$. This fact allows us to develop a recursive algorithm by which we may reduce the game of Nim on G where G is a tree with loops to that of a game of Nim on a star.

Such an algorithm is advantageous since the classification of a game played on a star where the initial vertex is the center is easily deduced:

- If there is a loop incident with Δ , (Δ, ω) is a p-position.

- If there is a leaf that is not incident with a loop (or for which the weight of the loop is 0), (Δ, ω) is a p-position.

- If there is no loop incident with Δ , and each leaf is incident with a loop with positive weight, (Δ, ω) is a 0-position.

The winning strategy for p-positions in such games is also easily identified. If there is no loop with positive weight incident with some v_i , remove the weight on that edge; if there is a loop with positive weight on every vertex of G , remove the weight on the loop incident with Δ).

Theorem 3.2. *Let G be a tree with loops, where G is not a star, and let the initial position of a game of Nim on G be at $\Delta \in V(G)$. Then there is a graph, G^- , where:*

- 1) $V(G^-) \subset V(G)$

- 2) $V(G) \setminus V(G^-)$ is comprised of leaves of G

- 3) $E(G^-) \setminus \{l_i | 1 \leq i \leq n\} \subset E(G)$ where $\{l_i | 1 \leq i \leq n\}$ is a (potentially empty) set of loops incident with a subset of leaves in G^-

- 4) If (Δ, ω^-) is a p-position in Nim on G^- , then (Δ, ω) is a p-position in Nim on G where $\omega^-(e) = \omega(e)$ for every $e \in E(G^-) \cap E(G)$ and $\omega^-(l_i) > 0$ for all i , $1 \leq i \leq n$.

Proof. Since G is a tree, and G is not a star, there exists at least one vertex $v \in G$, $v \neq \Delta$, such that the neighbor set of v , $\{v_i | 1 \leq i \leq k\}$, consists of at least $k - 1$ leaves. That is, at most one of the k neighbors of v is incident with more than one link. If play were to progress with perfect play following the earlier outlined strategy countering options for which a player does not remove all weight from an edge, we may assume that should a position $(v, \bar{\omega})$ be reached, $\bar{\omega}(e_i) = 0$ for every edge, e_i , in the path from Δ to v .

Any position in Nim on G at v will be equivalent to a game of Nim on a star with center v and leaves, the leaves of v in G . Thus, the position may be labeled p- or 0- according to the clear strategy outlined above. Should $(v, \bar{\omega})$ be a p-position, we can delete the leaves of G and assign a loop at v with positive weight without changing the classification of (Δ, ω) as a p-position or 0-position. Likewise, if $(v, \bar{\omega})$ is a 0-position, the leaves may be deleted

and for any loop, l , incident with v , $\bar{\omega}$ may be redefined so that $\bar{\omega}(l) = 0$, ensuring that the one possible option at v under the employed strategy is a 0-position. By replacing the star graph with a graph that is also a p-position (respectively 0-position) we do not change the classification of the initial position as the classification of any options at subsequent vertices have not changed. \square

The algorithm outlined here may be used recursively to reduce a game of Nim on a Tree to the game of Nim on a Star centered at Δ at which point (Δ, ω) may be easily classified as a p-position or a 0-position as outlined above in the strategy on stars.

Since every option of a game of nim on a tree with loops is also played on a tree with loops, we can use the property that positions at a vertex for which a loop is incident are p-positions together with the inconsequence of weight to better the algorithm by which G is truncated. If $v \neq \Delta$ is a vertex incident with a loop for which the weight is positive, every vertex u for which v is in the path from Δ to u may be deleted without changing the classification of (Δ, ω) .

However, the advantage of the previous algorithm is that each step of the recursion also outlines optimal strategy for P_1 at each step, provided that (Δ, ω) is a p-position or for P_2 should it be a 0-position. Strategy is easily determined by considering the star pruned at each iteration. An example of the pruning algorithm as well as the associated strategy for P_1 is demonstrated in Figure 3.1.

3.3 Extensions to Graphs with Trees as Induced Subgraphs

A similar argument may be made for games of nim on graphs with a "tree-like" structure. Given that a graph, G satisfies certain properties that allow us to incorporate such an argument, and knowing strategies for games of Nim played on specific subgraphs of G , we would be able to either classify positions in Nim on G as p- or 0-, or alternatively, we would be able to consider a game of nim played on a graph similar to G but with fewer edges.

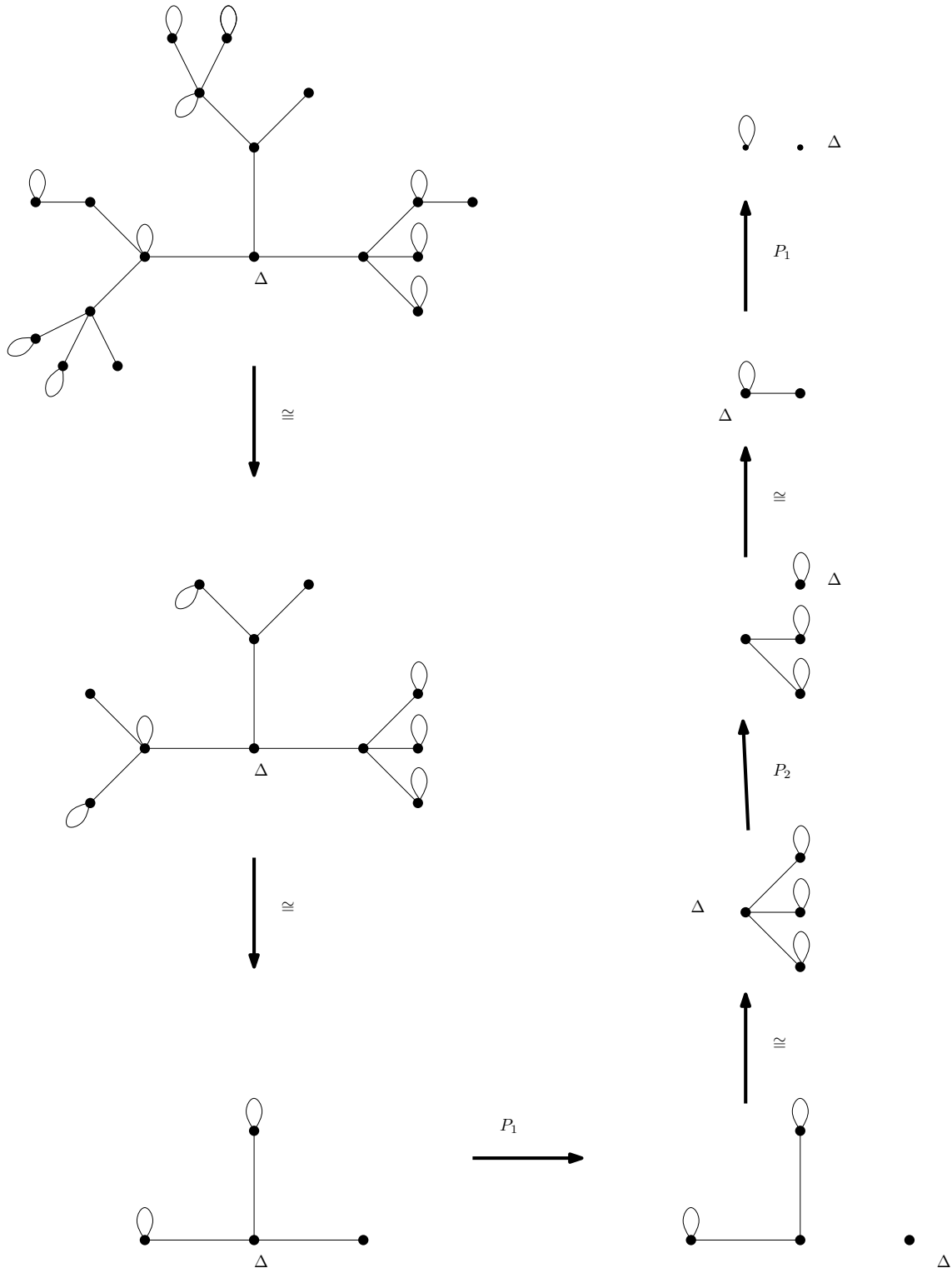


Figure 3.1: Pruning leaves of a tree with loops and associated strategy

Theorem 3.3. Consider a game of nim on a graph, G , with position (Δ, ω) , for which the vertices of a graph, G , may be organized into sets, the induced graphs of which are labeled G_i , $1 \leq i \leq n$, and T where:

- 1) T is a tree with loops
- 2) $V(G_i) \cap V(T) = \{v_i\}$, for every i , where v_i is a leaf in T
- 3) If a link of G is incident with vertices u and v , $u \in G_i$, and $v \in G_j$, then $i \neq j$

Then (Δ, ω_G) , a position in a game of Nim on G with $\Delta \in T$, is a p-position (respectfully 0-position) if (Δ, ω_{T^+}) is a p-position (respectfully 0-position) in a game of Nim on T^+ for a graph T^+ where:

- $E(T) \subset E(T^+)$
- $E(T^+) \setminus E(T) = \{l_i\}$, a set of loops incident with the vertices, v_i
- $\omega_G(e) = \omega_{T^+}(e)$ for every $e \in T$, and $\omega_{T^+}(l_i) \in \{0, 1\}$ for every i

Proof. Since $G \setminus \{\Delta\}$ is a graph of components equal to the number of links in G incident with Δ , each component represents an option of (Δ, ω_G) at a neighbor of v , each is a p-position or a 0-position, and the argument presented in Theorem 3.1 applies to the position in a similiary way: if (Δ, ω_G) is a p-position, P_1 has a wining strategy that does not depend on the weight of the edges incident with Δ , and likewise for P_2 if (Δ, ω_G) is a 0-position. The argument applies for all positions on vertices in $T \setminus \{v_i\}$, and so as gameplay progresses, we may assume that for a future position, $(v, \bar{\omega})$, $\bar{\omega}(e) = 0$ for all links, e , in the path from Δ to v , where $v \in T$. A position $(v_i, \bar{\omega})$ would then only have options on a game equivalent to Nim on G_i .

Just as in Theorem 3.2 we can trade such star subgraphs for positions sharing a p- or 0-classification, we here construct T^+ by deleting the vertices of G_i , for each i . If there is no loop incident with v_i and $(v_i, \omega|_{G_i})$ is a p-position in Nim on G_i , then add a loop, l_i incident with v_i , to G_i . Define ω_{T^+} such that $\omega_{T^+}(l_i) = 1$ for every i and $\omega_{T^+}(e) = \omega(e)$ for every $e \in T \cap T^+$.

We note the amount of weight placed on l_i is inconsequential since the underlying graph is a tree with loops and that the addition of the loop can not affect play for positions at vertices elsewhere in T^+ since we implement an optimal strategy as outlined above which prevents play from returning to a vertex of T , a tree, and since the vertex sets of the various G_i are mutually exclusive. \square

We can see then that if a graph G has a tree with loops as an induced subgraph, nice properties that were of aid in our analysis of trees with loops can be extended to Nim on G . It is worth noting, however, that in order to determine the strategy for play from initial position (Δ, ω) , strategy on the subgraphs, G_i , must be known, and that while there may be tree structures which allow us to categorize (Δ, ω) as p- or 0- , in general a classification of $(v_i, \omega|_{G_i})$ as p- or 0- in Nim on G_i may need to be known in order to arrive at such a classification of (Δ, ω) .

However, provided that such strategies *are* known, the recursive algorithm outlined in Theorem 3.2 may be utilized to provide strategy for P_1 at each step in order to win the game provided (Δ, ω) is a p-position (or P_2 should it be a 0-position).

The game of Nim on G in such a case would not be solved completely since Theorem 3.3 requires that Δ be a vertex in T , the tree portion of G . We might suspect that the induced tree with loops would still be a helpful structure. Indeed, we can "prune" edges from T as in Theorem 3.2, but knowing strategies for the induced subgraph including Δ in such a case would not necessarily be helpful as we will see.

Theorem 3.4. *Let Γ be a graph with vertices Δ and v , (v may equal Δ), for which the vertices of Γ may be organized into two sets with induced subgraphs T and G with:*

- 1) T a tree with loops
- 2) $V(T) \cap V(G) = \{v\}$
- 3) v is a leaf in T

Then there is a graph, G^+ , $G \subset G^+$, for which a game of Nim on Γ with initial position, (Δ, ω) , $\Delta \in G$, is equivalent to a game Nim on G^+ with position (Δ, ω_{G^+}) where:

- G^+ is G with the addition of $d_T(v)$ pendant edges incident with v which may have loops incident with the neighbors of v . These loops are not necessarily elements of $E(\Gamma)$.

- $\omega(e) = \omega_{G^+}(e)$ for every $e \in E(G)$, and $\omega_{G^+}(l) = 1$ for every loop, $l \in E(G^+) \setminus E(G)$

Proof. We construct G^+ by pruning the leaves of T according to the algorithm outlined in Theorem 3.2 treating v as the vertex for the initial position in a game of Nim on T . We must then show that if (Δ, ω) is a p-position (respectively 0-position) in Nim on Γ then (Δ, ω_{G^+}) shares the classification in Nim on G^+ , when G^+ is so defined.

If $v_t \in T$, all trails from Δ to v_t include v , thus, a position (v_t, ω^*) can only be reached after a position at v . We will first then examine positions at v , since if the classification of such positions are retained then so is the classification of (Δ, ω) .

Let $\{t_i | 1 \leq i \leq d_T(v)\}$ denote the vertices incident with the pendant edges in G^+ adjacent to v and suppose that (v, ω_v) is a p-position in Nim on Γ . If there is a t_i which is not incident with a loop in G^+ , then (t_i, ω'_v) where $\omega'_v(vt_i) = 0$ is clearly a 0-position option in Nim on G^+ , but is also a 0-option in Nim on Γ as defined in the algorithm from 3.2. If there is a loop at each t_i in G^+ then no option for which $\omega'_v(vt_i) = 0$ is a 0-position in Nim on G^+ or in the corresponding Nim on Γ . The winning option in Nim on Γ at v can be mirrored in Nim on G^+ and the must be a winning option that is not at some t_i since P_2 could move back to v , and since it was ostensibly from a 0-position P_1 would again be at a p-position at v . This argument hold for any future positions at v and thus (v, ω_v) must also be a p-position in Nim on G^+ .

If (v, ω_v) is a 0-position in Nim on Γ then each of the t_i will be incident with a loop. All gameplay in Nim on Γ mirrored in Nim on G^+ that progress on the vertices of G will not change the classification. Options for which $\omega'_v(vt_i) = 0$ are clearly p-positions, and if $\omega'_v(vt_i) > 0$, P_2 can reduce the weight of the edge vt_i to zero for each i and the strategy on the vertices of G may be followed. □

As we see in the next chapter, these generalizations of the strategy for trees allow us to approach nim on graphs which are composed of subgraphs with which we have experience - namely, trees and cycles.

Chapter 4

Mono-cyclic Graphs

While Theorems 3.3 and 3.4 allow us to reduce the classification of a position in Nim on G to classifying games on relatively simplified graphs, in both cases, we are limited in that we must know strategies or outcomes for either certain induced subgraphs of G , or strategies for induced subgraphs plus a collection of pendant edges.

We would like to apply these theorems to classes of graph for which such strategies are known or can be deduced, which leads us to consider mono-cyclic graphs. For our purposes, it is helpful to picture such graphs as cycles with trees extending from some selection of vertices on the cycle. As before, loops are allowed in the edge set.

4.1 Strategy for Mono-cyclic Graphs

It is sufficient to consider positions, (Δ, ω) , for which Δ is on the cycle in G . If such positions may be classified as p-positions or 0-positions and a strategy for such games is known, then we may apply Theorem 3.3 in order to classify and determine strategy for positions in which Δ is not on the cycle.

Let G be a mono-cyclic graph, and consider position (Δ, ω) in a game of Nim on G , where Δ is on the cycle in G . We may apply Theorem 3.4 to equate G to a graph in which the trees extending from the vertices in the cycle of G are limited to pendant edges that may have loops. Thus we assume that G is a cycle with a collection of pendant edges and loops. We will let $\{v_i | 1 \leq i \leq n - 1\}$ denote the non- Δ vertices of the cycle in G , labeled in order from Δ . Furthermore, we will often herein use the convention of referring to the edge incident with the vertices u and v as simply, uv .

Case 1: Δ is adjacent to a pendant vertex, u , which itself is not incident with a loop.

(u, ω') where $\omega'(\Delta u) = 0$ is a 0-position and an option of (Δ, ω) . Thus (Δ, ω) is a p-position with winning strategy reducing the weight on edge Δu to zero and, if necessary, letting play progress on the tree associated with the pendant edge.

Case 2: There exist at least two vertices in $\{v_i | 1 \leq i \leq n - 1\}$ adjacent to pendant vertices without loops.

Let v_α and v_β , $\alpha < \beta$, be the vertices adjacent to loopless pendant vertices, u_α and u_β respectively, for which no v_i is adjacent to a loopless pendant vertex when $i < \alpha$ or $i > \beta$. Positions $(v_\alpha, \omega_\alpha)$ and (v_β, ω_β) are p-positions since each has a zero option, reducing the weight of the edge $v_\alpha u_\alpha$ (respectively $v_\beta u_\beta$) to 0. Positions (v_i, ω_i) where $\alpha < i < \beta$ are only reachable after positions at either v_α or v_β . If we employ the strategy of reducing the weight of $v_\alpha u_\alpha$ or $v_\beta u_\beta$, and assume optimal play, then such positions will not be reached and will affect neither the strategy nor classification of (Δ, ω) .

Deleting the vertices $\{v_i | \alpha + 1 \leq i \leq \beta - 1\}$ results in a game played on a tree for which classification and strategy may be deduced using Theorem 3.2.

Case 3: There is exactly one vertex in $\{v_i | 1 \leq i \leq n - 1\}$ adjacent to a pendant vertex without loops.

Let v_α be the vertex adjacent to a loopless pendant vertex, u_α . There is not a winning strategy that reduces the weight of a pendant edge that is not $v_\alpha u_\alpha$ since options reducing the weight of such edges to zero are clearly p-positions (equivalent to a vertex with a loop), and if the weight of such edges is positive the alternate player may move back onto the cycle removing the remainder of the weight. Also, as before, the position $(v_\alpha, \omega_\alpha)$ is a p-position, and we will suppose a strategy for which the option $(u_\alpha, \omega'_\alpha)$, where $\omega'_\alpha(v_\alpha u_\alpha) = 0$ is chosen.

All ideal play is then played on the cycle itself. If either of the paths $T_1 = \Delta v_1 v_2 \dots v_\alpha u_\alpha$ or $T_2 = \Delta v_{n-1} v_{n-2} \dots v_\alpha u_\alpha$ with their incident loops are paths for which $(\Delta, \omega|_{T_j})$ is a p-position in Nim on T_j , then (Δ, ω) is a p-position and the strategy for the p- game, say T_1 (without loss of generality), can be employed. It is clear if in T_1 the winning strategy is to reduce the weight of Δv_1 to zero, since positions at vertices $\{v_k\}_k = \alpha + 1^{n-1}$ need not be considered by our supposition that the option $(u_\alpha, \omega'_\alpha)$, where $\omega'_\alpha(v_\alpha u_\alpha) = 0$ is chosen when possible.

If the winning strategy indicates the reduction of the weight on the loop at Δ to zero, the position is exactly that for which $(\Delta, \omega|_{T_j})$ is a 0-position in both of Nim on T_j . Showing that such a position indicates that (Δ, ω) is a 0-position in Nim on G then completes the case.

If $(\Delta, \omega|_{T_j})$ is a 0-position in both of Nim on T_j , Δ is not incident with a loop, and, without loss of generality, we may assume that all options of (Δ, ω) reduce the weight of Δv_1 . If $\omega'(\Delta v_1) > 0$, the resulting position has a path T_k , similarly defined as above, which is a p-position. Specifically, the path will consist of an odd number of loopless vertices starting at v_1 before the path terminates or a looped vertex is reached. However, if $\omega'(\Delta v_1) = 0$, the vertices $\{v_k | \alpha + 1 \leq k \leq n - 1\}$ may be deleted as argued above and the path with loops on which play would progress would be a p-position. All options of (Δ, ω) are thus p-positions, and (Δ, ω) must be a 0-position. Strategy for P_2 in such a game, assuming again P_1 chooses an option at v_1 , is to choose an option at v_2 removing all weight from edge $v_1 v_2$.

Case 4: There are no vertices in the cyclic subgraph of G adjacent with a pendant vertex with a loop.

As discussed in Case 3, no winning strategy will necessitate play that reduces the weight of a pendant edge whose pendant vertex is incident with a loop. Play in this case, therefore, may be thought of as entirely restricted to the subgraph induced by $\{\Delta\} \cup \{v_i | 1 \leq i \leq n - 1\}$. The position is p- or 0- according to the classification outlined in Chapter 2's discussion of

nim on cycles with loops. Strategy is likewise defined. When this strategy culminates, there may be options on a pendant edge with a loop. However, such a position is a 0-position and strategy mirrors play for other pendant edges considered in this case.

Chapter 5

Possibilities for Future Research

Perhaps the most glaring open question left at the conclusion of this work is, "For games of Nim on G where G is a graph discussed herein, determine the Grundy number for an initial position, (Δ, ω) ?" One may recall that Grundy numbers for paths and cycles without loops are found completely in [4].

The use of these Grundy numbers is interesting in a couple of different ways. In terms of strategy, if the Grundy numbers for all of the options of (Δ, ω) could be found, P_1 would know that any option for which the Grundy number was zero would be perfect play. Induced subgraphs of trees are themselves trees, so a finding Grundy numbers for games on trees with loops would allow for such an analysis at any position in the game.

More broadly, the ability to find the Grundy number for a position and its options would provide a strategy for the sum of these games. If all games considered were Nim on Graphs, we could imagine a graph with k components, each component with a game piece placed at a vertex. On a player's turn (s)he would choose a component and make a play with the corresponding game piece in a game of nim on that component. This game would be equivalent to a game of nim with k heaps, with each heap corresponding to a component, the Grundy number for which would give the number of objects in the heap.

The introduction of other game pieces in Nim on G may lead us to consider games on a connected graph, G in which each player is given a distinct game piece, or in which a player were allowed to choose from a selection of game pieces and then make a move on his/her turn. One easily sees that such a variation of Nim on G complicates analysis of the game greatly.

As for other graphs to consider for analysis, I have begun consideration of games played on a theta graph with loops. The theta graph seems a natural problem to approach since its induced subgraphs are trees or mono-cyclic graphs. Since strategies for games played on these graphs are known, Theorems 3.3 and 3.4 are of some use. However, the ability for gamplay to proceed onto cyclic subgraphs at multiple vertices, as well as the fact that such graphs have three induced cycles (on which weight assigned by ω may affect strategy and classification), necessitate a more careful analysis.

It would be interesting to find criteria for G that would indicate whether the amount of positive weight on an edge would be significant.

Erickson has approached the complete graph, developed a winning strategy in general when $\omega(e) = 1$ for every $e \in K_n$, and for small values of n shown that the amount of positive weight placed on any edge does not change the classification of (Δ, ω) as a p-position. Her proposed strategy for the " $\omega(e) = 1 \forall e \in K_n$ " case is easily extended to a case where loops are added at every vertex. To my knowledge, however, no work has yet been done on the complete graph where every vertex is also incident with a loop for $n \geq 4$ and ω is unrestricted.

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