Decomposition of Complete Tri-Partite Graphs into 5-Cycles

by

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A thesis submitted to the Graduate Faculty of
Auburn University
in partial fulfillment of the
requirements for the Degree of
Master of Science

Auburn, Alabama
May 5, 2013

Keywords: tri-partite, 5-cycle

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Abstract

E. S. Mahmoodian gave necessary conditions for when the edges of a complete tripartite graph can be partitioned so that each set in the partition induces a 5-cycle. Since then, he, N. J. Cavenagh, E. J. Billington, S. Alipour, and E. Mollaahmadi have found several cases where these necessary conditions are also sufficient. We continue this investigation by considering further cases for complete tripartite graphs.

Acknowledgments

I would like to thank my parents for all of their love and support.

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History

A graph, G, is said to be decomposable into a subgraph, H, if the edges of G can be partitioned into sets, each of which induces a copy of H. A bipartite graph is a graph in which the vertices can be partitioned into two groups such that no edges exist between any pair of vertices in the same group, but there may be an edge between a pair of vertices in different groups. A complete bipartite graph is a bipartite graph in which every possible edge exists; if the parts have sizes r and s then it is denoted by $K_{r,s}$. Similarly, a tripartite graph is a graph in which the vertices can be partitioned into three groups where no edges exist between any pair of vertices in the same group, but there may be an edge between a pair of vertices in different groups; if all such edges are in the graph then it is called complete, and is denoted by $K_{r,s,t}$, where r, s, and t are the sizes of each part. A cycle is a walk in a graph that starts and ends at the same vertex, but otherwise has no repeated vertices.

In 1981, D. Sotteau[6] developed necessary and sufficient conditions for decomposing complete bipartite graphs into cycles of any fixed even length. More recently, in 1995 E. S. Mahmoodian and M. Mirzakhani[4] developed necessary conditions for decomposing complete tripartite graphs into 5-cycles, and conjectured that these conditions are also sufficient. Since then several papers have been written trying to prove that these necessary conditions are also sufficient. Many cases towards showing that these conditions are sufficient have been settled, while several others are still open for investigation. Necessary conditions for the existence of a 5-cycle system of a tripartite graph $K_{r,s,t}$ follow.

Theorem 1.1 *[4] Let $r \leq s \leq t$. If there exists a decomposition of $K_{r,s,t}$ into 5-cycles then the following conditions are satisfied:*

(i) $r, s,$ *and* t *are either all even or all odd;*

(ii) $5 \mid rs + rt + st$; *and*

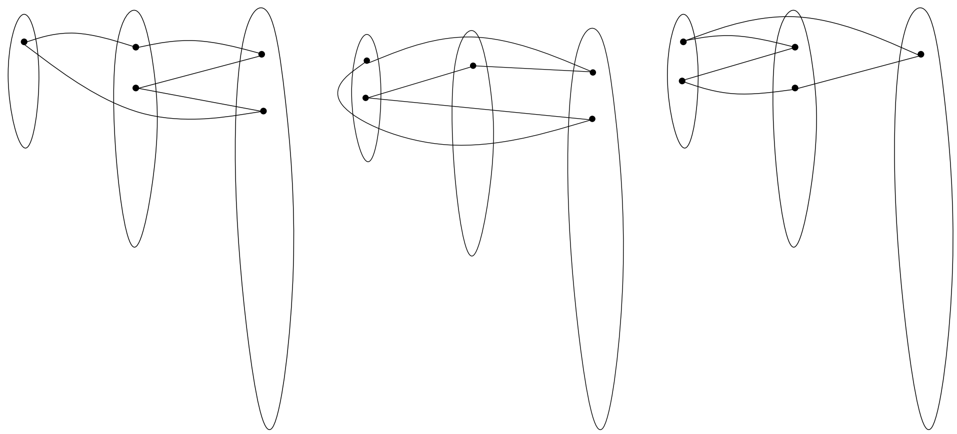
(iii) $t \leq \left(\frac{4rs}{(r+s)}\right)$.

These three conditions are necessary, as the following arguments show.

- (i) is necessary because clearly all vertices must have even degree since each vertex is incident with an even number of edges in each cycle. If only one part has odd size then the degrees of the vertices in the other parts will be odd, and if only one part has even size then the degrees of the vertices in the other parts will be odd.

- (ii) is necessary because the 5-cycles partition the edges of $K_{r,s,t}$, so the number of edges $rs + rt + st$ must be a multiple of 5.

- Now, it is a little less obvious why (iii) is necessary. The basic idea here is that there cannot be too many vertices in the largest part in order for there to be a chance of using up all of the edges that are incident with vertices in this part. To prove (iii), Mahmoodian found three sets of inequalities that must be satisfied for a 5-cycle system to exist, one for each part of the graph. These inequalities were determined by the three types of 5-cycles that can exist, as pictured in Figure 1.1.



(a) Type 1 (b) Type 2 (c) Type 3

Figure 1.1: Types of 5-cycles

It is clear that in each 5-cycle in $K_{r,s,t}$ there must be exactly one edge or exactly three edges between vertices in any two of the three parts. We know that the number of 5-cycles is precisely $|E(K_{r,s,t})|/5 = (rs+st+rt)/5$, so the number of edges between any two parts must be greater than or equal to this number, and one third of the number of edges between any two parts must be less than or equal to this number. So, we get two inequalities for each pair of parts of the graph. For the parts of size r and s these inequalities are $rs/3 \leq (rs + st + rt)/5$ and $(rs + st + rt)/5 \leq rs$. Simply replace rs with st and then with rt to obtain the inequalities for the other two pairs of parts. By solving for each part size the following inequalities can be obtained:

$$\frac{2st}{3(s+t)} \leq r \leq \frac{4st}{(s+t)}$$
$$\frac{2rt}{3(r+t)} \leq s \leq \frac{4rt}{(r+t)}$$
$$\frac{2rs}{3(r+s)} \leq t \leq \frac{4rs}{(r+s)}$$

The last of these three inequalities contains condition (iii). Notices that since t is the largest of these part sizes, requiring that $t \leq \dfrac{4rs}{(r+s)}$ forces the other bounds on the part sizes to hold. To see this, rearrange $t \leq \dfrac{4rs}{(r+s)}$ to obtain $r \geq \dfrac{st}{4s-t} \geq \dfrac{2st}{3(s+t)}$ since

$t \geq s$ by assumption and similarly, $s \geq \frac{rt}{4r-t} \geq \frac{2rt}{3(r+t)}$ since $t \geq r$ by assumption. To see the implication that $s \leq \frac{4rt}{(r+t)}$ we start with $s \leq t$

$$
\begin{aligned}
4r^2 s + 4rst &\leq 4r^2 t + 4rst \\
\frac{4rs}{(r+s)} &\leq \frac{4rt}{(r+t)} \\
s \leq t \leq \frac{4rs}{(r+s)} &\leq \frac{4rt}{(r+t)}
\end{aligned}
$$

So, $s \leq \frac{4rt}{(r+t)}$, and similarly $r \leq t \leq \frac{4rs}{(r+s)} \leq \frac{4st}{(s+t)}$. Similar to the first implication we can see that $r \leq \frac{4st}{(s+t)}$ implies that $t \geq \frac{2rs}{3(r+s)}$.

Mahmoodian goes on to conjecture that the necessary conditions given in Theorem 1.1 are also sufficient. He then proves some very useful corollaries about when these conditions are sufficient. First, if $K_{r,s,t}$ admits a 5-cycle decomposition, then so does $K_{ar,as,at}$ for each positive integer a. Next, $K_{r,r,r}$ admits a 5-cycle decomposition if and only if $5 \mid r$. He also shows that for any positive integer n, $K_{2n,2n,4n}$ and $K_{n,3n,3n}$ admit 5-cycle decompositions.

The main result of Mahmoodian's first paper on decomposing complete tripartite graphs into 5-cycles was his theorem that $K_{r,s,t}$ admits a 5-cycle decomposition if two of the parts have the same size and the necessary conditions are satisfied except possibly when the two parts have size divisible by 5 but the third does not.

Theorem 1.2 *[4] Suppose that at least two parts in a complete tripartite graph G have the same number of vertices, say $G = K_{r,r,s}$. And suppose that $K_{r,r,s}$ satisfies all three necessary conditions given in Theorem 1.1. Then $K_{r,r,s}$ has a 5-cycle decomposition except possibly when r is a multiple of 5 but s is not.*

N.J. Cavenagh and E.J. Billington[2] introduced a new way to represent a complete tripartite graph by extending the idea of a latin square. An $r \times s$ latin rectangle is an $r \times s$ array, each cell being filled with one of n different symbols, with each symbol occurring at most once in each row and at most once in each column. In [2] $K_{r,s,t}$ is represented by starting with an $r \times s$ latin rectangle R, the entries of R being elements of the set $T = \{1, 2, \ldots, t\}$, and finishes the representation by adding $t-s$ entries at the end of each row and $t-r$ entries at the end of each column so that each element of the set $T = \{1, 2, \ldots, t\}$ appears exactly once in every row and exactly once in every column. (See Figure 1.2 for an example)

1	2	3	4	5	6	7
2	3	6	7	4	1	5
3	5	7	6	2	1	4
4	6	1	3	7	2	5
5	7	2	4	1	3	6
6	1	4	1	3		
7	4	5	5	6		

Figure 1.2: Latin Representation of $K_{5,5,7}$

In this Latin Representation of $K_{r,s,t}$ the numbers inside the $r \times s$ latin rectangle represent a 3-cycle in $K_{r,s,t}$ and the numbers outside of the latin rectangle represent single edges in $K_{r,s,t}$ in the following way. Let the vertex set of $K_{r,s,t}$ be $\{(v, w) \mid v \geq 1$, and $v \leq r, s$, or t if $w = 1, 2$ or 3 respectively$\}$. The cell (i, j) in R containing symbol k corresponds to the 3-cycle $((i, 1), (j, 2), (k, 3))$ in $K_{r,s,t}$. Now, the edges that are not in the 3-cycles are either of the form $\{(i, 1), (k, 3)\}$, which is represented by k being an entry in row i outside of R, or $\{(j, 2), (k, 3)\}$, which is represented by k being an entry in column j outside of R.

Cavenagh uses this latin representation to implement a technique known as a trade.

Definition 1.1 *[3] Let M be a Latin Representation of the complete tripartite graph $K_{r,s,t}$. A **trade** is a set of entries in M for which the corresponding 3-cycles and edges in $K_{r,s,t}$ form a graph which can be decomposed into 5-cycles.*

For example, it is possible to form a trade using two entries corresponding to 3-cycles and four entries corresponding to edges. As the name suggests, these two 3-cycles and the four single edges contain ten edges which can be rearranged into two 5-cycles.

Cavenagh goes on to describe two different types of 5-cycle trades. A trade of Type 1 uses both 3-cycles and single edges, and in fact always uses exactly twice as many single edges, or entries from outside the latin rectangle, as 3-cycles, or entries from inside the latin rectangle. So, the different trades of Type 1 either use two 3-cycles and four single edges to obtain two 5-cycles or use three 3-cycles and six single edges to obtain three 5-cycles. A trade of Type 2 uses only 3-cycles, or entries from inside the latin rectangle. So, the different trades of Type 2 use five 3-cycles to obtain three 5-cycles.

Cavenagh then uses this method to show that the necessary conditions that Mahmoodian developed in [4] are also sufficient for several new cases. He starts by finishing the case where two of the partite sets have the same size by considering the situation when the part sizes that are the same are divisible by 5 and the third part size is not, which was the exception Mahmoodian had in [4]. He goes on to complete the case when r and s are both divisible by 10.

In [3] Cavenagh continues to use this method of trades described in [2] to complete the proof that the necessary conditions given by Mahmoodian in [4] are sufficient for $K_{r,s,t}$ when either two of the partite sets have the same size, or in the case where all partite sets have even size. He considered four cases to complete this, one of which he had completed in [2]. These four cases are:

(A) Either $r \equiv t \equiv 0$ (modulo 10) and s is not divisible by 10 or $s \equiv t \equiv 0$ (modulo 10) and neither s nor t are divisible by 10.

(B) Either $s \equiv t \equiv 0$ (modulo 10) and r is not divisible by 10 or $r \equiv t$ (modulo 10) and neither r nor t are divisible by 10.

(C) $r \equiv s$ (modulo 10) and neither r nor s is divisible by 10.

(D) $r \equiv s \equiv 0$ (modulo 10).

E. J. Billington and N. J. Cavenagh introduce a new approach to this problem in [1] by embedding smaller, previously known decompositions into larger decompositions. They start by finding 5-cycle decompositions of subgraphs of $K_{5,5,5}$, and then using this process to embed

$K_{2r,2s,2t}$ into $K_{R,S,T}$ where $R > 2r$, $S > 2s$, and $T > 2t$, by first finding a decomposition of $K_{R,S,T} \backslash E(K_{2r,2s,2t})$ into 5-cycles. Using the two following definitions they give the subsequent theorem for this decomposition.

Definition 1.2 *[1] A triple $\{x, y, z\}$ is said to be good if there exists a subgraph $G_{x,y,z}$ of $K_{5,5,5}$ and there exist subsets X_{ab}, X_{bc}, X_{ca} of $\mathbb{Z}_5$ with $|X_{ab}| = x$, $|X_{bc}| = y$ and $|X_{ca}| = z$ such that $G_{x,y,z}$ contains only the edges of difference $d \in X_{ab}$ between V_a and V_b, the edges of difference $d \in X_{bc}$ between V_b and V_c, and the edges of difference $d \in X_{ca}$ between V_c and V_a and there exists a decomposition of $G_{x,y,z}$ into 5-cycles.*

Definition 1.3 *[1] We say that a positive integer D is permissible with respect to r, s and t if there exist $x_1, x_2, \ldots, x_D$; $y_1, y_2, \ldots, y_D$ and $z_1, z_2, \ldots, z_D$, each between 0 and 5 inclusive, with the following properties:*

- *for each $1 \leq i \leq D$, the triple $\{5 - x, 5 - y, 5 - z\}$ is good;*
- $\sum_{i=1}^{D} x_i = \alpha$;
- $\sum_{i=1}^{D} y_i = \beta$;
- $\sum_{i=1}^{D} z_i = \gamma$.

where $\alpha = r + s - t$, $\beta = r - s + t$, and $\gamma = -r + s + t$ and a

Theorem 1.3 *[1] Let D be an odd integer that is permissible with respect to r, s and t. Let $d = 5D$, $R = 2r + d$, $S = 2s + d$, and $T = 2t + d$. Then there exists a decomposition of $K_{R,S,T} \setminus K_{2r,2s,2t}$ into 5-cycles.*

Billington and Cavenagh go on to give several lemmas describing some values of D that are permissible, one over-arching and two with specific restrictions on r, s, and t. They conclude this paper by considering the case when R, S, and T have similar sizes, meaning that R, S, and T are all odd with $R < S < T$ and $T \leq \kappa R$, where κ is a constant. A bound for this κ is given in the following theorem:

Theorem 1.4 *[1] Let* $\kappa = -\frac{95}{16} + \frac{3}{16}\sqrt{1401} \approx 1.0806$. *Let R, S, and T be odd integers such that $RS + ST + RT$ is divisible by* 5 *and* $100 < R < S < T \leq \kappa R$. *Then $K_{R,S,T}$ has a decomposition into* 5*-cycles.*

In [5] Mahmoodian introduces a new type of trade:

Definition 1.4 *[5] Let M be a Latin representation of $K_{r,s,t}$. A new trade is made of n triangles and $2n$ edges which can be decomposed into n 5-cycles. It has n entries from a row of Latin rectangle M and $2n$ entries from M_1 and/or M_2 (See Figure 1.3).*

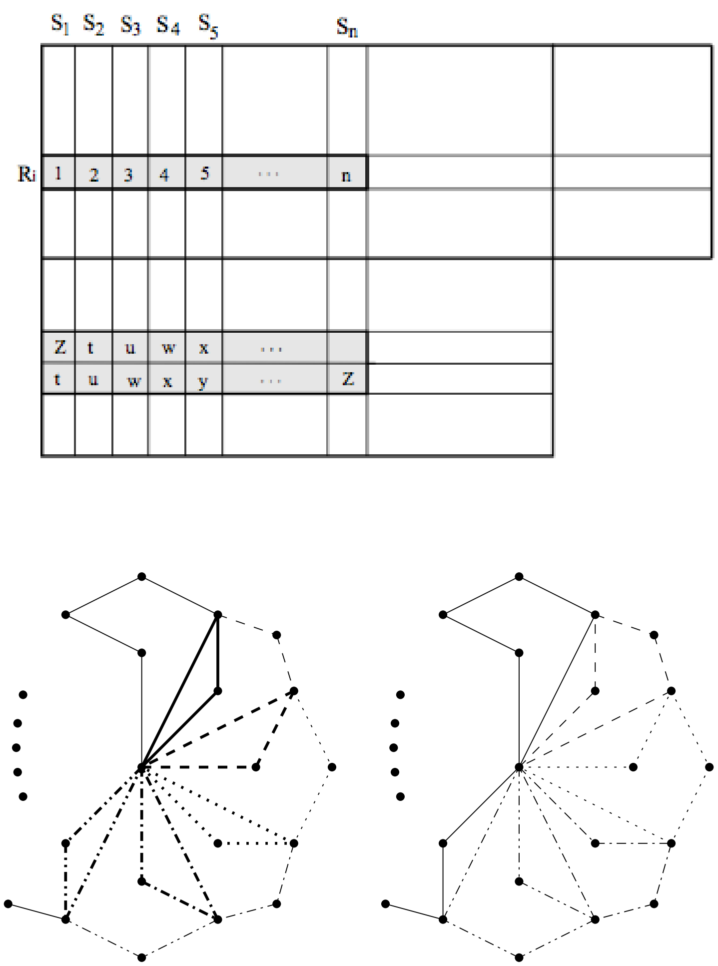


Figure 1.3: A trade with n triangles and $2n$ edges

Mahmoodian uses this new trade together with the old trades to decompose $K_{7,17,19}$, $K_{11,15,25}$, $K_{13,15,25}$, $K_{15,17,25}$, and $K_{15,19,25}$ into 5-cycles. He uses these trades and the Proposition 1.1 to prove the following three theorems which all improve the bound on κ given in [1].

Proposition 1.1 Let $K_{r,s,t}$ be a complete tripartite graph which satisfies the conditions of Theorem 1.1 with r, s, and t all being odd, and let $r \equiv r' \pmod{10}$, $s \equiv s' \pmod{10}$, and $t \equiv t' \pmod{10}$, where $1 \leq r', s', t' \leq 9$. Then the multi set $\{r', s', t'\}$ is equal to one of the following multisets:

$\{1,1,7\}$, $\{1,3,3\}$, $\{3,9,9\}$, $\{7,7,9\}$, $\{1,5,5\}$, $\{3,5,5\}$, $\{5,5,5\}$, $\{5,5,7\}$, $\{5,5,9\}$

Theorem 1.5 *[5] Let $K_{r,s,t}$ be a complete tripartite graph such that*

(i) r, s, and t are all odd and divisible by 5; and

(ii) $75 \leq r \leq s \leq t \leq \frac{5}{3}r - 50$,

Then the necessary conditions of Theorem 1.1 are sufficient.

Theorem 1.6 *[5] Let $K_{r,s,t}$ be a complete tripartite graph such that*

(i) r, s, and t are odd and exactly one of them is not divisible by 5; and

(ii) $86 \leq r \leq s \leq t \leq \frac{5}{3}r - 57$,

Then the necessary conditions of Theorem 1.1 are sufficient.

Theorem 1.7 *[5] Let $K_{r,s,t}$ be a complete tripartite graph such that*

(i) r, s, and t are odd and none of them is divisible by 5; and

(ii) $96 \leq r \leq s \leq t \leq \frac{5}{3}r - 46$,

Then the necessary conditions of Theorem 1.1 are sufficient.

To this point, many cases have been proved in an effort to show that the necessary conditions developed by Mahmoodian in [4] are also sufficient. These are (assuming that the necessary conditions are satisfied):

- $K_{r,r,r}$ admits a 5-cycle decomposition if and only if $5|r$;
- If $K_{r,s,t}$ admits a 5-cycle decomposition, then so does $K_{ar,as,at}$;
- $K_{r,r,s}$ and $K_{r,s,s}$ admit 5-cycle decompositions;
- $K_{r,s,t}$ admits a 5-cycle decomposition if r, s, and t are all even;
- $K_{r,s,t}$ admits a 5-cycle decomposition if $100 < R < S < T \leq \kappa R$;
- $K_{7,17,19}$, $K_{11,15,25}$, $K_{13,15,25}$, $K_{15,17,25}$, and $K_{15,19,25}$ admit 5-cycle decompositions;
- $K_{r,s,t}$ admits a 5-cycle decomposition if:
  - r, s, and t are all odd and divisible by 5 and $75 \leq r \leq s \leq t \leq \frac{5}{3}r - 50$;
  - r, s, and t are odd and exactly one of them is not divisible by 5 and $86 \leq r \leq s \leq t \leq \frac{5}{3}r - 57$; or
  - r, s, and t are odd and none of them are divisible by 5 and $96 \leq r \leq s \leq t \leq \frac{5}{3}r - 46$.

Chapter 2

A New Approach to Decomposing Complete Tripartite Graphs

In this chapter we will introduce a new approach to decomposing complete tripartite graphs. The majority of these graphs that have not been solved are complete tripartite graphs, $K_{r,s,t}$, with small part sizes. When beginning to work on this topic, the $K_{7,17,19}$ case was the smallest such graph. Since then, Mahmoodian has completed this case, finding a decomposition for the graph. However, here a new approach is presented for decomposing this graph, and hopefully also graphs similar to it.

In [5], Mahmoodian gives a decomposition of $K_{7,17,19}$ using trades. Using the trades described by Cavenagh, in [2], and his new trade from [5], both of which are described in the previous chapter, Mahmoodian describes a set of trades on the latin representation of $K_{7,17,19}$ (See Figure 1 in the Appendices). In this representation there are two numbers in each cell, the top number describes the trade of which the corresponding edge or 3-cycle is a part, while the bottom number is the edge or 3-cycle that is in the graph.

The approach that I take to decompose this graph into 5-cycles relies upon edge differences. An edge difference here is defined to be $j - i$ where $j \in \mathbb{Z}_t$ is the vertex in the larger part with which the edge is incident, and $i \in \mathbb{Z}_s$ is the vertex in the smaller part with which the edge is incident. We use differences to find a 3-path between the two largest parts that can subsequently be joined to a pair of edges from a vertex in the smallest part to form a 5-cycle; we can then cycle, or shift the 5-cycle down through the edges of the graph to get as many 5-cycles as possible, stopping just before the process would use an edge for a second time if we were to continue the process. The following lemma shows that this is possible if the part sizes are relatively prime to each other.

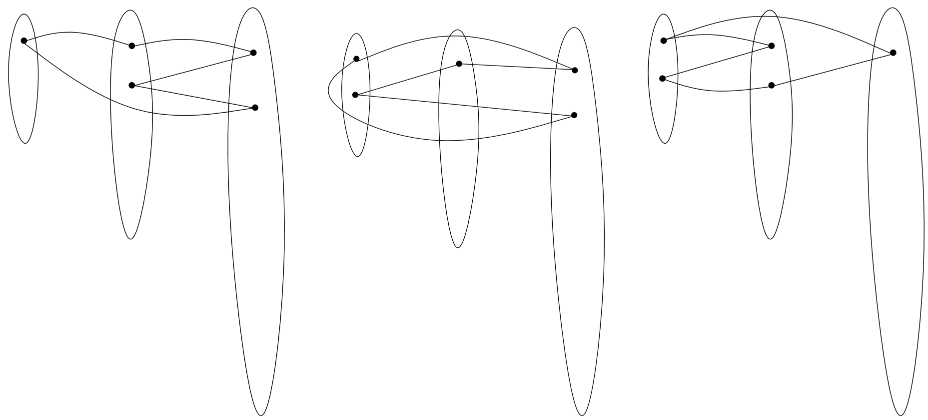
Lemma 2.1 *Consider the complete bipartite graph* $G = K_{x,y}$ *with* $V(G) = \{(i,1),(j,2) \mid i \in \mathbb{Z}_x, j \in \mathbb{Z}_y\}$ *with* $x < y$. *If* $\gcd(x,y) = 1$, *that is* x *and* y *are relatively prime to each other, then* $E(G) = \{((i,1),(i,2)) \mid i \in \mathbb{Z}_{xy}\}$, *reducing the first component modulo* x *or* y *if the second component is* 1 *or* 2 *respectively.*

Proof. Assume that we have a bipartite graph with part sizes x and y with $x < y$ and $\gcd(x,y) = 1$, hence x and y are relatively prime to each other. Start with the edge $\{(1,1),(1,2)\}$, which has edge difference 0, and cycle this edge through the graph, thus the second edge will be $\{(2,1),(2,2)\}$, and so on. This process will use exactly x edges of difference 0 before moving to the next difference of $0 + x = x$. This will use x more edges and the next difference will be $x + x = 2x$ reduced modulo y since the largest difference is $y - 1$. So, the differences used are $0, x, 2x \pmod{y}, 3x \pmod{y}, \ldots y-1$. Since x and y are relatively prime, $ix \pmod{y} \neq jx \pmod{y}$ for $0 \leq i < j \leq y$. So, this process produces x edges of each of y distinct differences. So, all xy edges are used exactly once.

□

Using this lemma we can see that we can find a 3-path that will cycle through the edges between the two larger parts of the tripartite graph to get a large number of the 3-paths that we can then make into 5-cycles.

We first need to see how many of each type of 5-cycle we need. As a reminder there are three types of 5-cycles (see Figure 2.1).



(a) Type 1 (b) Type 2 (c) Type 3

Figure 2.1: Types of 5-cycles

We can see that each type uses a specific number of edges between each pair of parts of the graph. Let the parts of $K_{r,s,t}$ be R, S, and T with $|R| = r$, $|S| = s$, and $|T| = t$. Type 1 uses one edge between parts R and S, one edge between parts R and T, and three edges between parts S and T. Type 2 uses one edge between parts R and S, three edges between parts R and T, and one edge between parts S and T. Type 3 uses three edges between parts R and S, one edge between parts R and T, and one edge between parts S and T. So, we can make a linear system of equations connecting the number of edges that each type of 5-cycle uses with the number of edges between each pair of parts of the graph. This system of equations is:

$$x + y + 3z = rs$$

$$x + 3y + z = rt$$

$$3x + y + z = st$$

where x, y, and z are the number of 5-cycles of Types 1, 2, and 3, respectively.

For example, the $K_{7,17,19}$ case has $rs = 119$, $rt = 133$, and $st = 323$. Substituting rs, rt, and st with these numbers and solving this system of equations shows that we need 104 5-cycles of Type 1, 9 5-cycles of Type 2, and 2 5-cycles of Type 3.

We want to use the approach suggested by Lemma 2.1 to define as many 5-cycles as possible. Removing these 5-cycles from $K_{7,17,19}$ will give us what is called the leave of the graph, (i.e., the graph induced by the edges that remain after removing all edges of the 5-cycles just defined). We know that we need 104 5-cycles of Type 1, so we start with those. Using Lemma 2.1 define as many 5-cycles as possible by carefully choosing a 3-path between parts S and T along with a pair of edges adjacent with the ends of the 3-paths and a vertex in part R, then cycling the resulting 5-cycle through the graph. It is possible to define $s \cdot \left\lfloor \frac{t}{3} \right\rfloor$ such 5-cycles in $K_{r,s,t}$ using this approach. We get this many because each of these 5-cycles takes three edges from between S and T, one edge from between R and S, one edge from between R and T and $t \leq 3r$. If this is as many or more of this type of 5-cycle that we need, then great. However, if we still need 5-cycles of this type we need a way to get these. I introduce a new trade that will take one 5-cycle along with seven edges and produce two new 5-cycles, along with two edges left over, in order to increase $s \cdot \left\lfloor \frac{t}{3} \right\rfloor = 17 \cdot \left\lfloor \frac{19}{3} \right\rfloor = 102$ to the required 104 5-cycles of Type 1.

2.0.1 A New Trade

With this new trade we can take an existing 5-cycle along with seven edges from the leave to get two new 5-cycles with two edges from the original 5-cycle left over. The edges between parts S and T in the leave of the graph will be single edges and 2-paths because there cannot be any more 3-paths and each vertex in part S will have exactly one edge incident with it in the leave. So, we can take a 2-path and a single edge and find the 5-cycle that contains the 3-path that connects these two components, noting that we now have a 6-path that we can then break into two 3-paths and form two new 5-cycles by finding two pairs of edges that are each incident with one vertex in part R and form a 2-path with ends at the ends of the 3-paths. It may be necessary to swap the vertex in part R to which the new endpoints are adjacent to in order for the trade to work. Note that this will always result in one more 5-cycle than was originally defined; thus if we need only one new 5-cycle we need only use the trade once (see Figure 2.2 for an example).

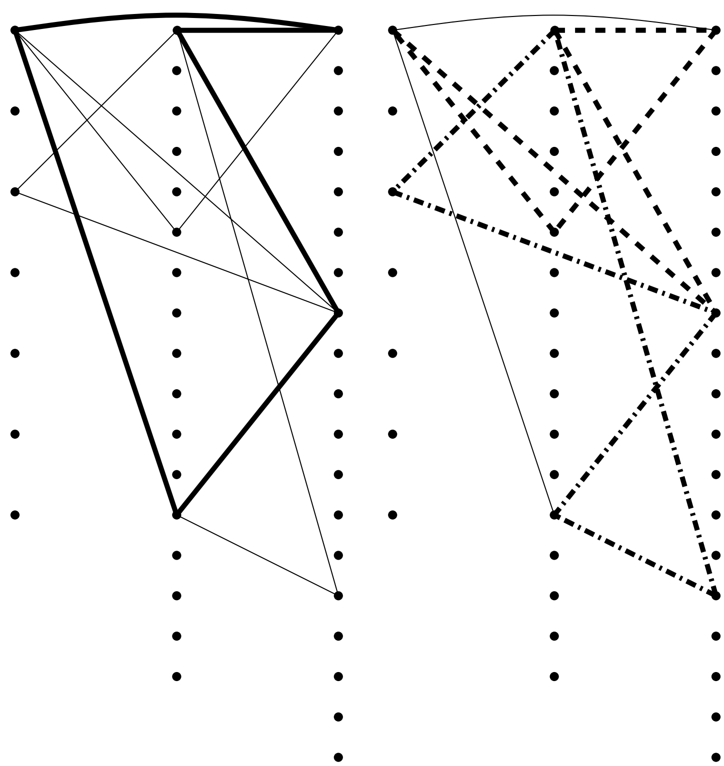


Figure 2.2: Example of the New Trade

After using the new trade we will have exactly the right number of 5-cycles of Type 1. At this point we can look at the leave of the graph and find as many 5-cycles of the other two types as we can. This will possibly leave some edges, the number of which will be some multiple of 5, to be dealt with.

We deal with these edges by making an edge swap. An edge swap here is done by taking an edge from an existing 5-cycle and placing it into a new 5-cycle, thus removing a 4-path from the leave and replacing it with a new 4-path, which will begin and end with the same vertices as the removed 4-path, but is incident with three new vertices in the leave. This gives us a slightly different leave that we can then check for additional 5-cycles. We continue this process until the leave is empty, thus every edge of $K_{r,s,t}$ is part of a 5-cycle.

This process of swapping, or removing an edge from one 5-cycle and creating another, actually describes a trade where we take some number of 5-cycles along with the edges of the leave, which will be a multiple of 5, and are able to decompose this new graph into 5-cycles.

2.0.2 Decomposition of $K_{7,17,19}$ using this new approach

Now we will show this process using the complete tripartite graph $K_{7,17,19}$. First, note that the known necessary conditions for a 5-cycle system of $K_{r,s,t}$ to exist are satisfied. First of all, all three parts sizes are odd, thus the degree of each vertex is even. Secondly, the number of edges is divisible by 5: $|E(K_{7,17,19})| = rs + st + rt = 7 \cdot 17 + 7 \cdot 19 + 17 \cdot 19 = 575$. Lastly, the inequality $t \leq \left(\frac{4rs}{(r+s)}\right)$ holds, since

$$\left(\frac{4 \cdot 7 \cdot 17}{(7+17)}\right) = \frac{476}{24} = 19.833 > 19 = t.$$

Now we see how many of each type of 5-cycle we need. Using the system of linear equations described earlier we take the part sizes $r = 7$, $s = 17$, and $t = 19$ and we get that we need 104 5-cycles of Type 1, 9 5-cycles of Type 2, and 2 5-cycles of Type 3.

We start by taking as many 5-cycles as we can, so we find a 5-cycle that we can cycle through the edges of the graph by adding 1 to each vertex incident with edges of the 5-cycle to get the next 5-cycle. So, we begin by finding a 3-path between parts S and T that will cycle through all the edges of the graph between parts S and T and find two further edges, one incident with one of the ends of this 3-path and the vertex $(1,1)$ and the other edge incident with the other end of the 3-path and the vertex $(1,1)$. In order to define as many 5-cycles as possible in this way, we start with the 3-path $P = ((1,3),(1,2),(8,3),(13,2))$, noting that the edge differences here are 0, 7, and 14. The edge differences here are important because the difference between s and t is 2, so we pick differences that alternate in parity, hence the even-odd-even arrangement. So, the first 5-cycle is $((1,1),(1,3),(1,2),(8,3),(13,2))$, the second 5-cycle will be $((2,1),(2,3),(2,2),(9,3),(14,2))$. When an edge joining vertices in parts S and T is cycled, the difference of the resulting edge is unchanged unless the vertex $(16, 2)$ is cycled to the vertex $(1, 2)$, in which case the edge difference will decrease by $t - s = 19 - 17 = 2$. Note here that looking at the differences 0, 7, and 14 it takes six full cycles for the difference 0 to shift down to 7 (namely 0 to 17 to 15 to 13 to 11 to 9 to 7) , six full cycles for the difference 7 to shift down to 14 (namely 7 to 5 to 3 to 1 to 18 to 16 to 14), and yet again six full cycles for the difference 14 to shift down to 0 (namely 14 to 12 to 10 to 8 to 6 to 4 to 2 to 0). Thus we get $6 \cdot 17 = 102$ 5-cycles of Type 1 this way. Note that $6 = \left\lfloor \frac{t}{3} \right\rfloor$, so this is precisely $s \cdot \left\lfloor \frac{t}{3} \right\rfloor$.

After we take these 5-cycles out of $K_{7,17,19}$ we are left with the leave, L_1, in Figure 2.3.

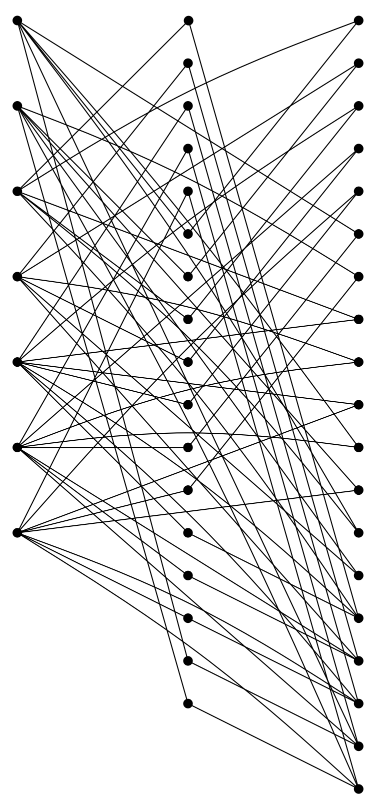


Figure 2.3: Leave L_1

Now, we know that we need 104 5-cycles of type 1, so we implement the new trade described earlier, performing it twice to get two more 5-cycles. For the first implementation we pick the 5-cycle $((1,1), (1,3), (1,2), (8,3), (13,2))$ and the edges $\{(1,3), (6,2)\}$, $\{(1,2), (15,3)\}$, $\{(15,3), (13,2)\}$, $\{(6,2), (1,1)\}$, $\{(1,1), (8,3)\}$, $\{(3,1), (8,3)\}$ and $\{(3,1), (1,2)\}$, though in order to use the edge $\{(1,1), (8,3)\}$ we must take the 5-cycle $((1,1), (3,2), (15,3), (8,2), (8,3))$ and replace it with the 5-cycle $((5,1), (3,2), (15,3), (8,2), (8,3))$. With these edges we can get two 5-cycles instead of just the one 5-cycle that we started with; these 5-cycles are $((1,1), (6,2), (1,3), (1,2), (8,3))$ and $((3,1), (1,2), (15,3), (13,2), (8,3))$ leaving the edges $\{(1,1), (1,3)\}$ and $\{(1,1), (13,2)\}$. For the second implementation we pick the 5-cycle $((2,1), (2,3), (2,2), (9,3), (14,2))$ and the edges $\{(2,3), (7,2)\}$, $\{(2,2), (16,3)\}$, $\{(16,3), (14,2)\}$, $\{(7,2), (2,1)\}$, $\{(2,1), (9,3)\}$, $\{(4,1), (9,3)\}$ and $\{(4,1), (2,2)\}$, though in order to use the edge $\{(2,1), (9,3)\}$ we must take the 5-cycle $((2,1), (4,2), (16,3), (9,2), (9,3))$ and replace it with the 5-cycle $((6,1), (4,2), (16,3), (9,2), (9,3))$. With these edges we can get two 5-cycles instead of just the one 5-cycle that we started with; these 5-cycles are $((2,1), (7,2), (2,3), (2,2), (9,3))$ and $((4,1), (2,2), (16,3), (14,2), (9,3))$ leaving the edges $\{(2,1), (2,3)\}$ and $\{(2,1), (14,2)\}$. See Figure 2.4 and 2.5 for a picture of this trade.

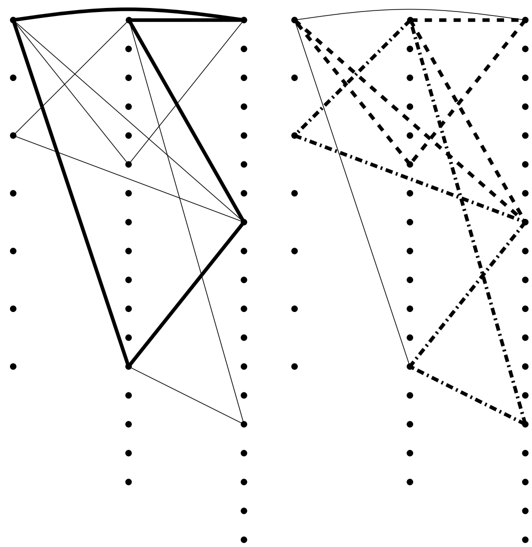


Figure 2.4: First Implementation of this New Trade

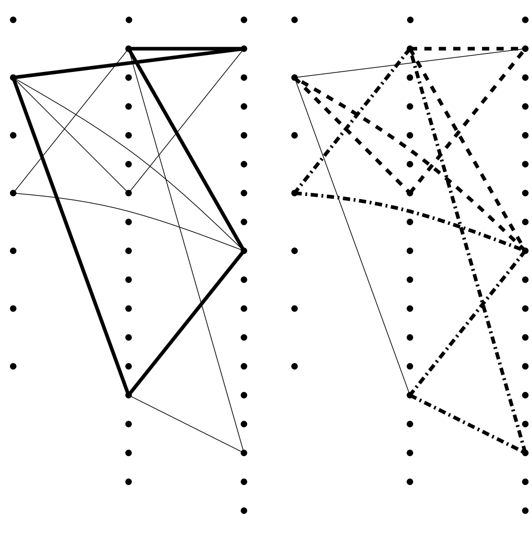


Figure 2.5: Second Implementation of this New Trade

After we implement these trades we have 104 5-cycles of Type 1 and a new leave, L_2 (see Figure 2.6).

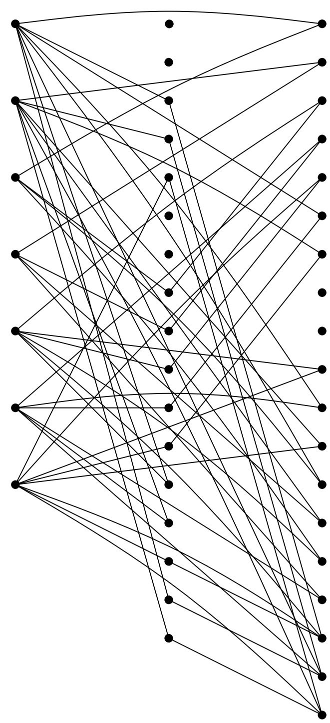


Figure 2.6: Leave L_2

From this leave we will find as many of the other two types of 5-cycles as we can. We can get both of the 5-cycles of Type 3 that we need, these 5-cycles are $((1,1), (13,2), (5,1), (17,3), (3,2))$, and $((2,1), (14,2), (6,1), (18,3), (4,2))$, which we remove to get a new leave, L_3 (see Figure 2.7).

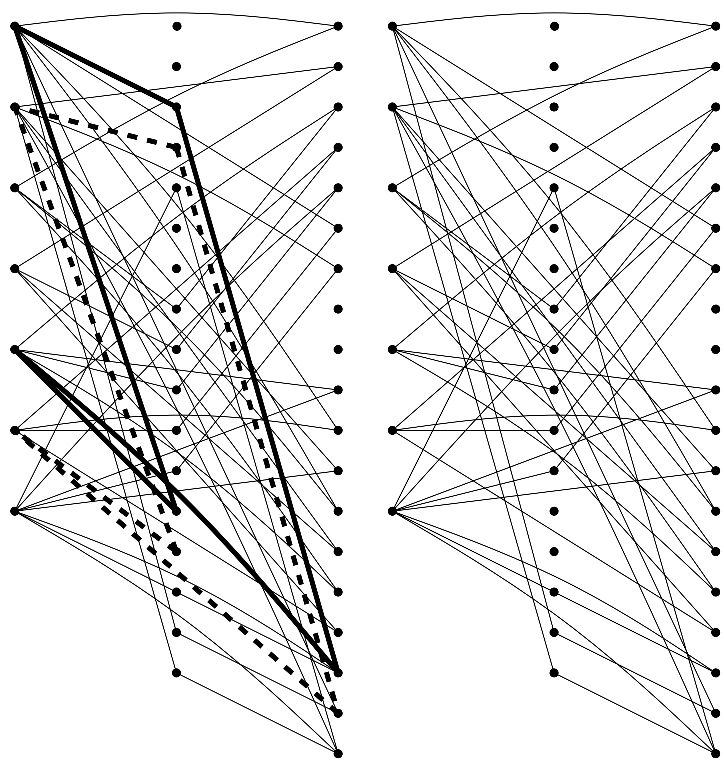


Figure 2.7: Type 3 5-cycles removed and Leave L_3

Now we start looking for 5-cycles of Type 2, we can see that there are five of these 5-cycles in this leave. These 5-cycles are ((5, 1), (10, 3), (7, 1), (5, 3), (10, 2)), ((6, 1), (11, 3), (1, 1), (6, 3), (11, 2)), (7, 1), (12, 3), (2, 1), (7, 3), (12, 2)), ((3, 1), (15, 3), (5, 1), (3, 3), (8, 2)), and ((4, 1), (16, 3), (6, 1), (4, 3), (9, 2)), which we remove to get a new leave, L_4 (see Figure 2.8).

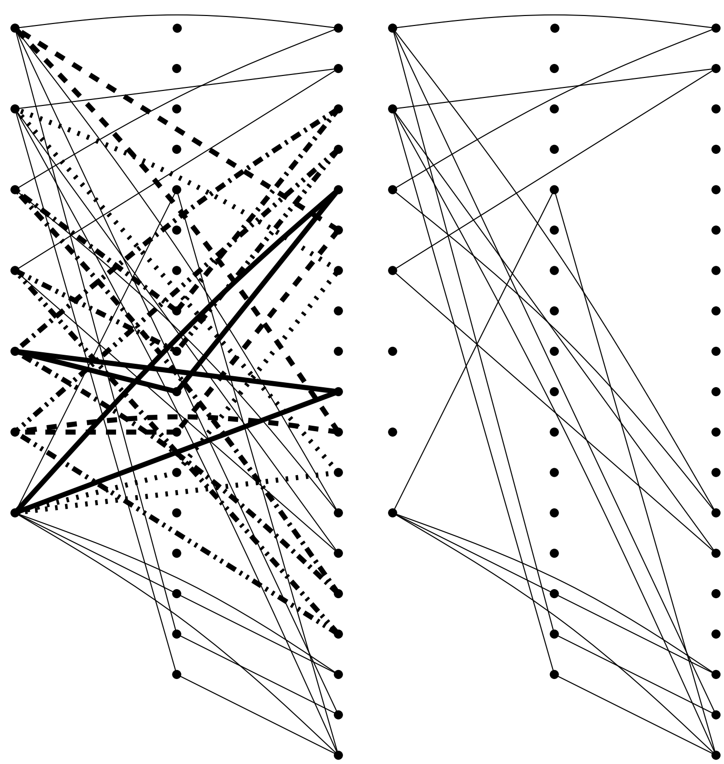


Figure 2.8: Type 2 5-cycles removed and Leave L_4

This new leave has no 5-cycles that we can remove from the graph, so we need to start the process of swapping one 4-path from a 5-cycle we have already defined for a 4-path in the leave of the graph to create a new 5-cycle in order to change the leave and look for an additional 5-cycle. We find the 4-path $((4,1),\ (2,3),\ (2,1),\ (19,3),\ (17,2))$ and see that we need the edge $\{(4,1),\ (17,2)\}$, so we need to find the 5-cycle that contains this edge. The 5-cycle that contains $\{(4,1),(17,2)\}$ is $((4,1),\ (1,3),\ (5,2),\ (8,3),\ (17,2))$, so we now have the 5-cycle $((4,1),\ (2,3),\ (2,1),\ (19,3),\ (17,2))$ (see Figure 2.9). Note that this new 5-cycle is a Type 2 5-cycle, of which we now need 3 more.

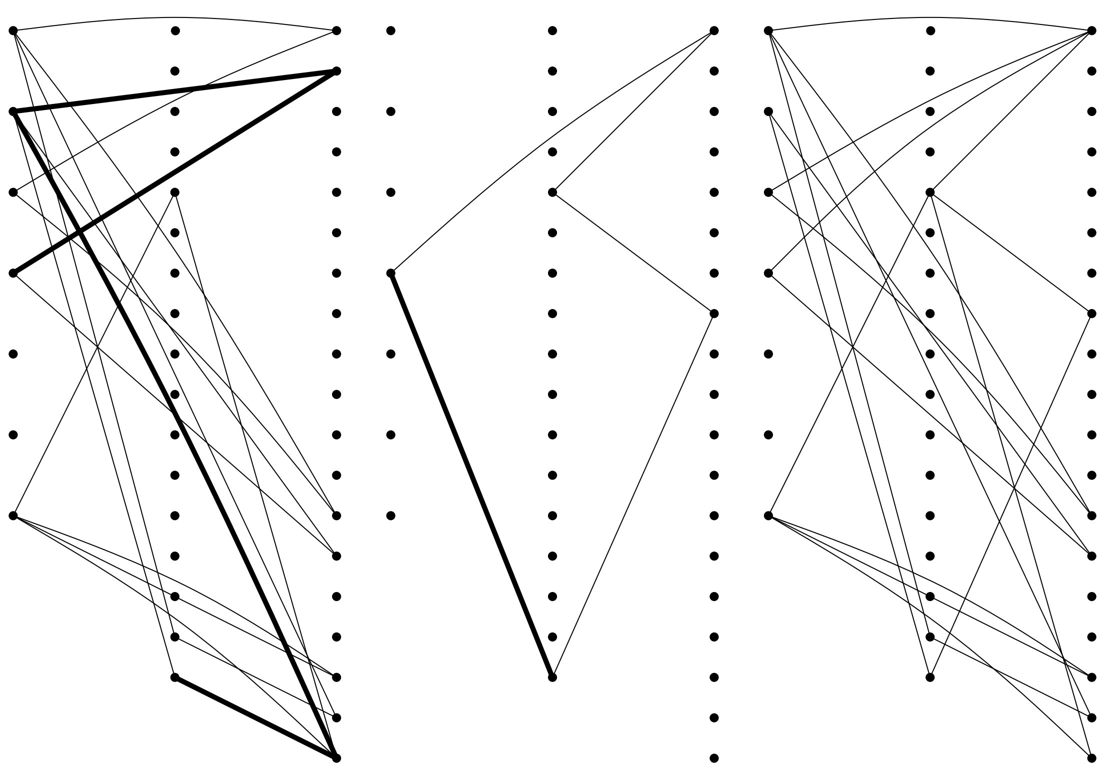


Figure 2.9: First 4-path to be replaced along with the 5-cycle needed and Leave L_5

We see that this leave has no 5-cycles that we can remove from the graph, so we make another swap. We find the 4-path $((4,1),\ (1,3),\ (1,1),\ (18,3),\ (16,2))$ and see that we need the edge $\{(4,1),\ (16,2)\}$, so we need to find the 5-cycle that contains this edge. The 5-cycle that contains $\{(4,1),\ (16,2)\}$ is $((4,1),\ (4,3),\ (4,2),\ (11,3),\ (16,2))$, so we now have the 5-cycle $((4,1),\ (1,3),\ (1,1),\ (18,3),\ (16,2))$ (see Figure 2.10). Note that this new 5-cycle is a Type 2 5-cycle, of which we now need 2 more.

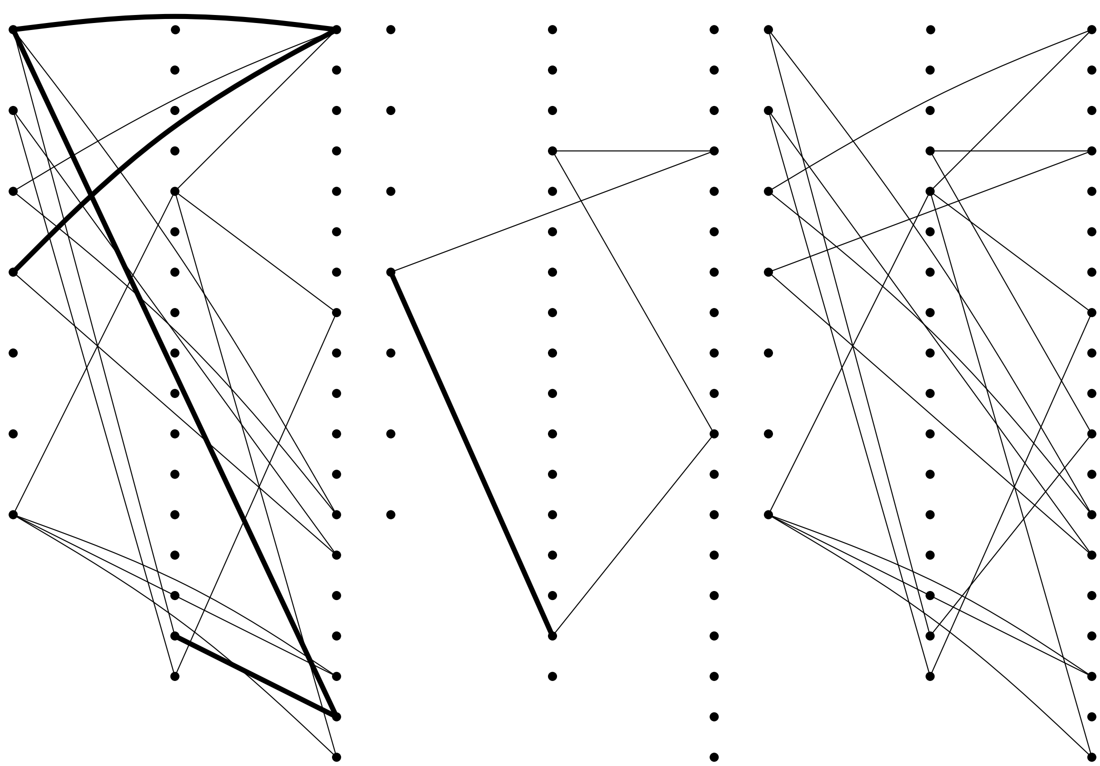


Figure 2.10: Second 4-path to be replaced along with the 5-cycle needed and Leave L_6

We see that this leave has no 5-cycles that we can remove from the graph, so we make another swap. We find the 4-path $((1,1),\ (13,3),\ (3,1),\ (1,3),\ (5,2))$ and see that we need the edge $\{(1,1),\ (5,2)\}$, so we need to find the 5-cycle that contains this edge. The 5-cycle that contains $\{(1,1),\ (5,2)\}$ is $((1,1),\ (2,3),\ (10,2),\ (9,3),\ (5,2))$, so we now have the 5-cycle $((1,1),\ (13,3),\ (3,1),\ (1,3),\ (5,2))$ (see Figure 2.11). Note that this new 5-cycle is a Type 2 5-cycle, of which we now need 1 more.

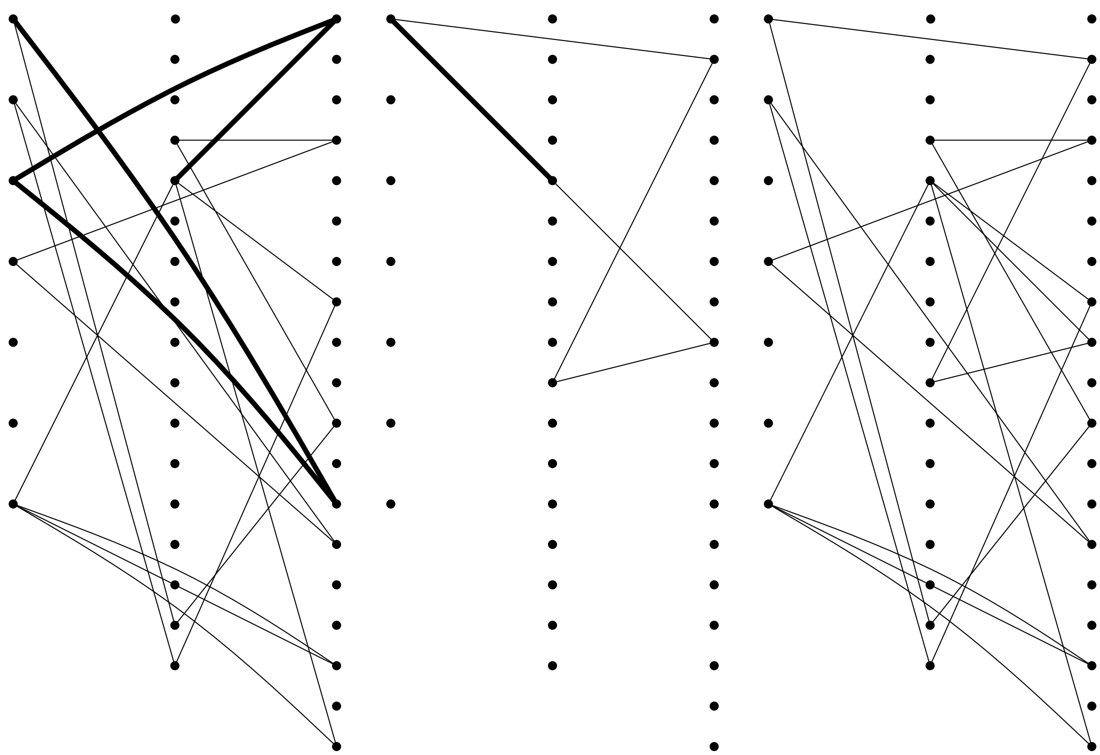


Figure 2.11: Third 4-path to be replaced along with the 5-cycle needed and Leave L_7

We see that this leave has no 5-cycles that we can remove from the graph, so we make another swap. We find the 4-path $((4,1),\ (14,3),\ (2,1),\ (17,2),\ (8,3))$ and see that we need the edge $\{(4,1),\ (8,3)\}$, so we need to find the 5-cycle that contains this edge. The 5-cycle that contains $\{(4,1),\ (8,3)\}$ is $((4,1),\ (8,3),\ (12,2),\ (15,3),\ (7,2))$, so we now have the 5-cycle $((4,1),\ (14,3),\ (2,1),\ (17,2),\ (8,3))$ (see Figure 2.12). Note that this new 5-cycle is a Type 2 5-cycle, of which we now have all 9 needed.

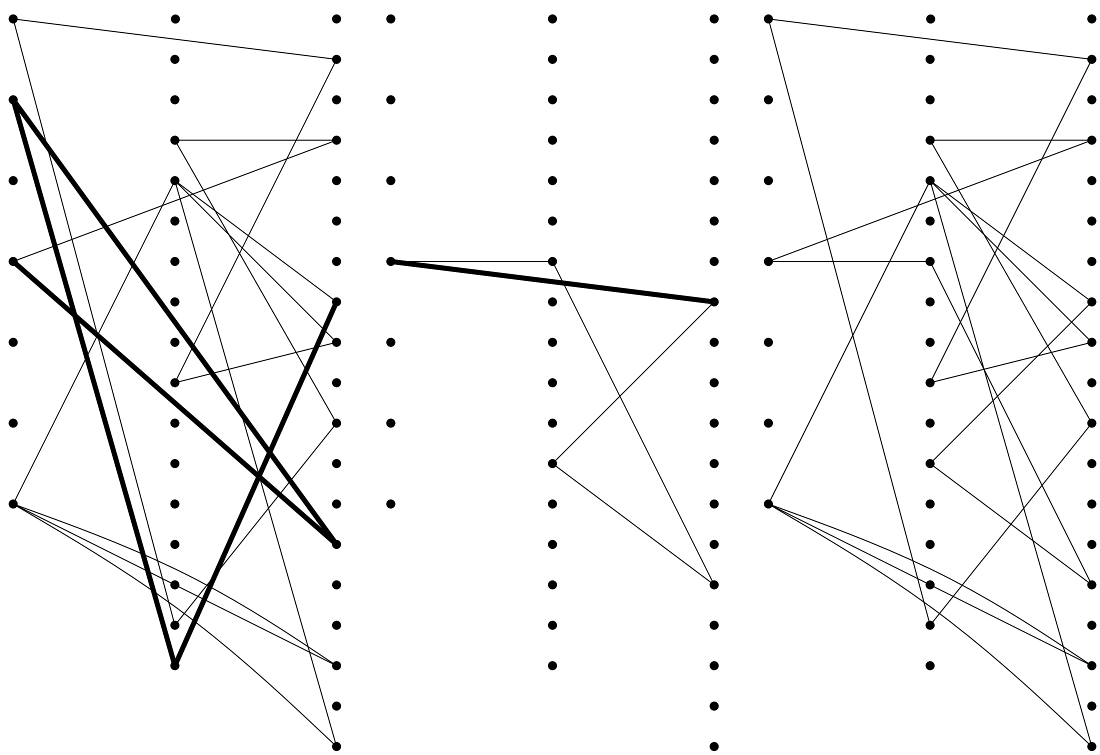


Figure 2.12: Fourth 4-path to be replaced along with the 5-cycle needed the Leave L_8

We see that this leave has no 5-cycles that we can remove from the graph, so we make another swap. We find the 4-path $((7,2),\ (4,1),\ (4,3),\ (4,2),\ (11,3))$ and see that we need the edge $\{(7,2),\ (11,3)\}$, so we need to find the 5-cycle that contains this edge. The 5-cycle that contains $\{(7,2),\ (11,3)\}$ is $((7,2),\ (11,3),\ (12,2),\ (4,3),\ (3,1))$, so we now have the 5-cycle $((7,2),\ (4,1),\ (4,3),\ (4,2),\ (11,3))$ (see Figure 2.13).

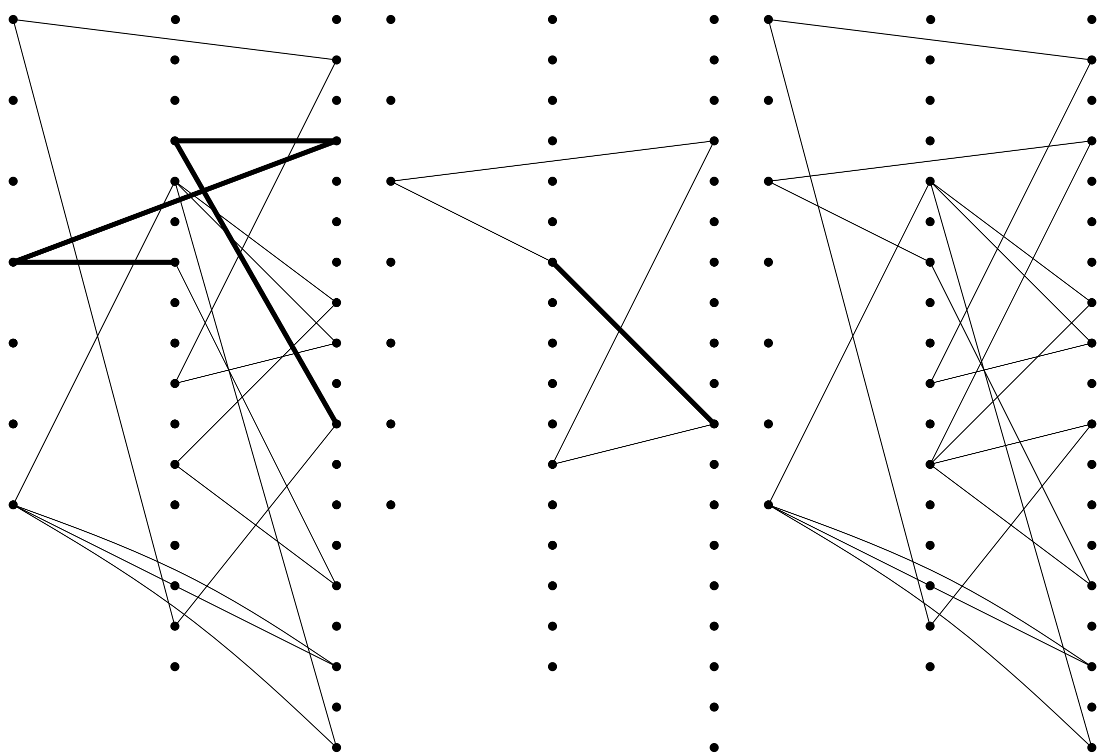


Figure 2.13: Fifth 4-path to be replaced along with the 5-cycle needed and Leave L_9

We see that this leave does have a 5-cycle that we can remove from the graph, this 5-cycle is $((3,1),\ (4,3),\ (12,2),\ (15,3),\ (7,2))$ leaving 15 edges in the leave, L_{10} (see Figure 2.14).

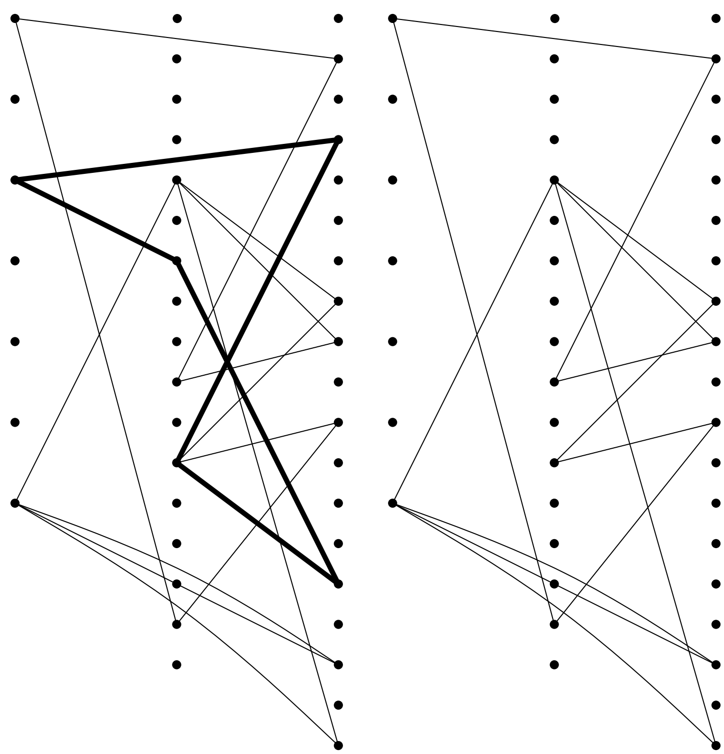


Figure 2.14: 5-cycle removed and Leave L_{10}

We see that this leave has no 5-cycles that we can remove from the graph, so we make another swap. We find the 4-path $((10,2),\, (2,3),\, (1,1),\, (16,2),\, (11,3))$ and see that we need the edge $\{(10,2),\, (11,3)\}$, so we need to find the 5-cycle that contains this edge. The 5-cycle that contains $\{(10,2),\, (11,3)\}$ is $((10,2),\, (11,3),\, (5,2),\, (5,1),\, (4,3))$, so we now have the 5-cycle $((10,2),\, (2,3),\, (1,1),\, (16,2),\, (11,3))$ (see Figure 2.15).

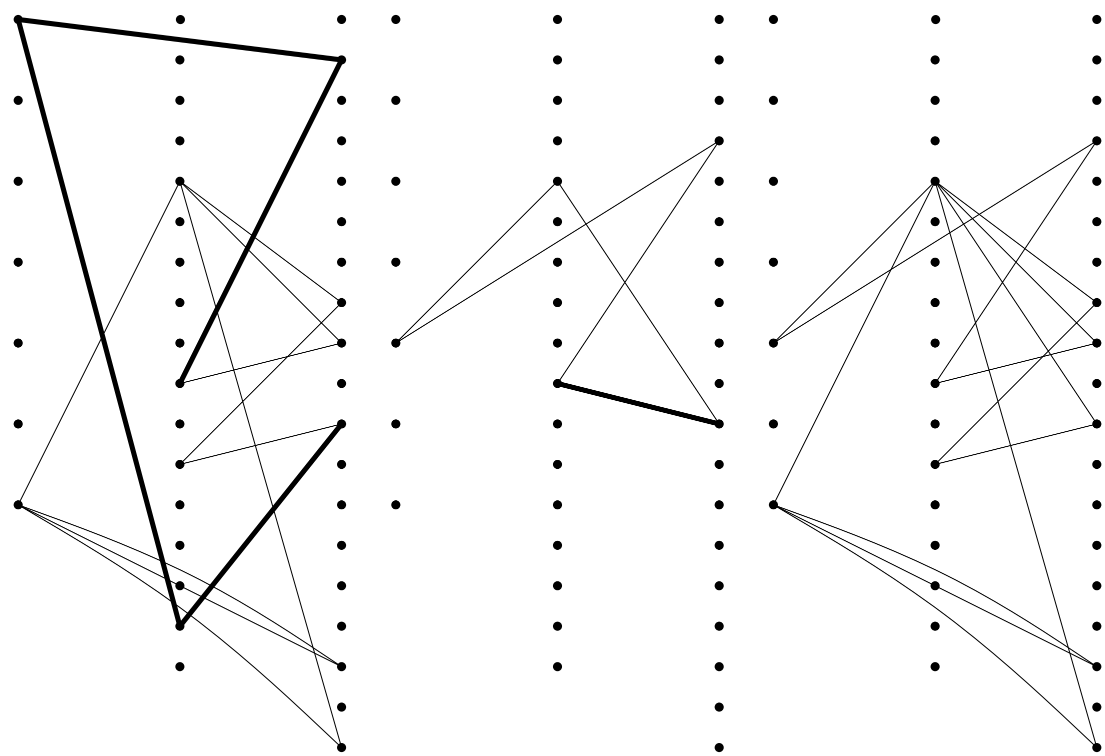


Figure 2.15: Fifth 4-path to be replaced along with the 5-cycle needed and Leave L_{11}

We see that this leave does have a 5-cycle that we can remove from the graph, this 5-cycle is $((5,1),\ (5,2),\ (9,3),\ (10,2),\ (4,3))$ leaving 10 edges in the leave, L_{12} (see Figure 2.16).

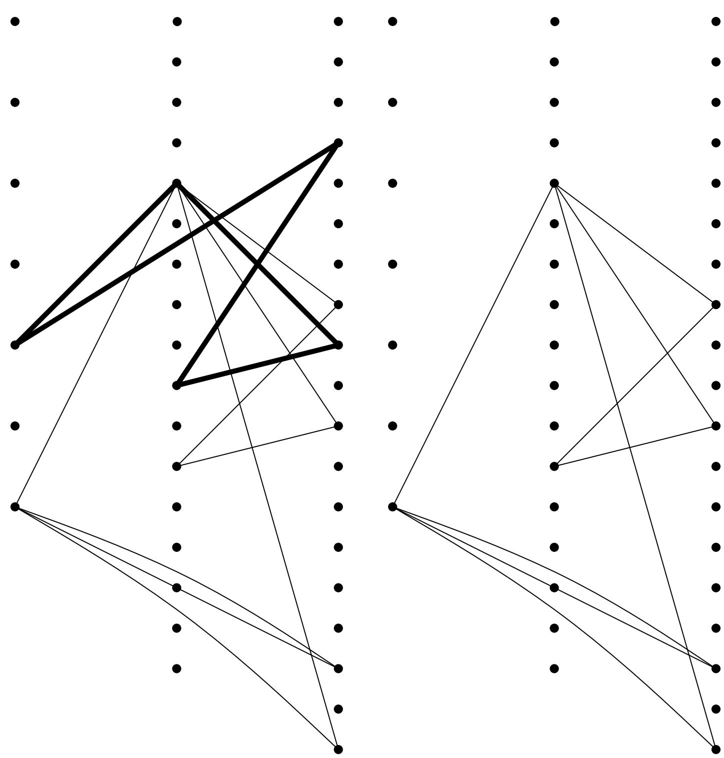


Figure 2.16: 5-cycle removed and Leave L_{12}

We see that this leave has no 5-cycles that we can remove from the graph, so we make another swap. We find the 4-path $((7,1), (5,2), (8,3), (12,2), (11,3))$ and see that we need the edge $\{(7,1), (11,3)\}$, so we need to find the 5-cycle that contains this edge. The 5-cycle that contains $\{(7,1), (11,3)\}$ is $((7,1), (11,3), (15,2), (18,3), (10,2))$, so we now have the 5-cycle $((7,1), (5,2), (8,3), (12,2), (11,3))$ (see Figure 2.17).

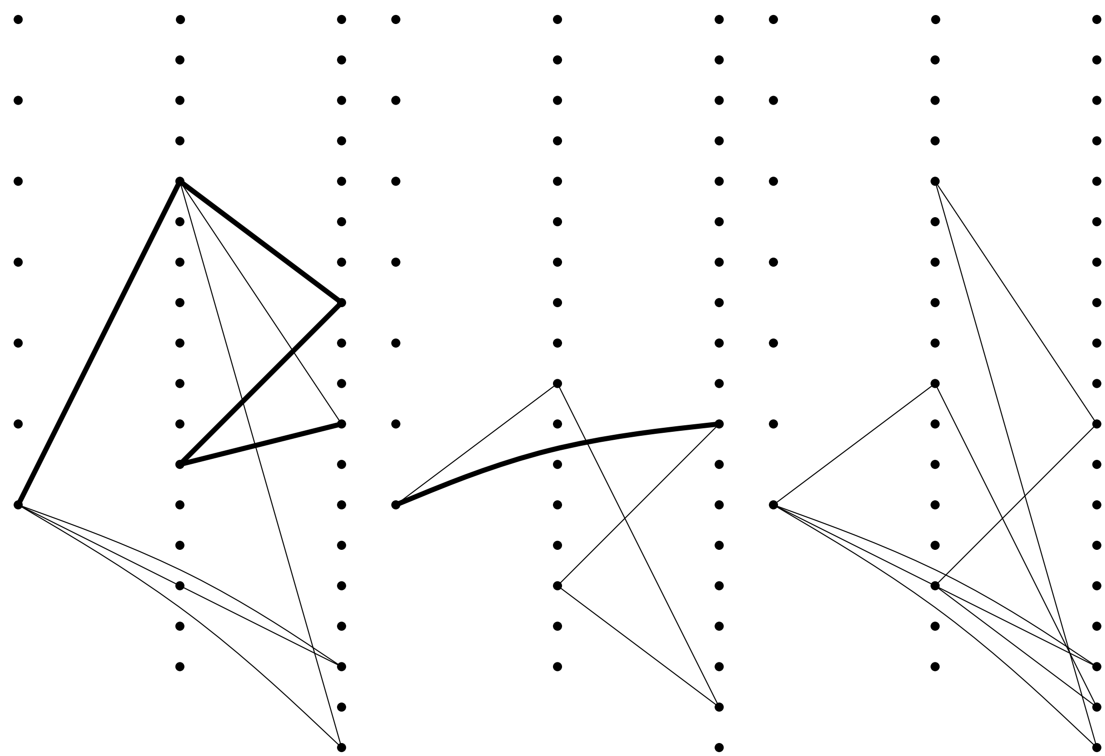


Figure 2.17: Sixth 4-path to be replaced along with the 5-cycle needed and Leave L_{13}

From the 10 edges in this leave we actually have both of the 5-cycles that we need, which is to be expected. If we had only one 5-cycle the leave would only have 5 edges and supposedly no 5-cycles, which is impossible since this would imply that at least one vertex had degree 1 which can not be the case since each vertex in the graph started with even degree and whenever a 5-cycle is removed from the graph the degree of each vertex incident with an edge in the 5-cycle decreased by precisely 2. These 5-cycles are $((7,1), (19,3), (5,2), (11,3), (15,2))$ and $((7,1), (10,2), (18,3), (15,2), (17,3))$ (see Figure 2.18).

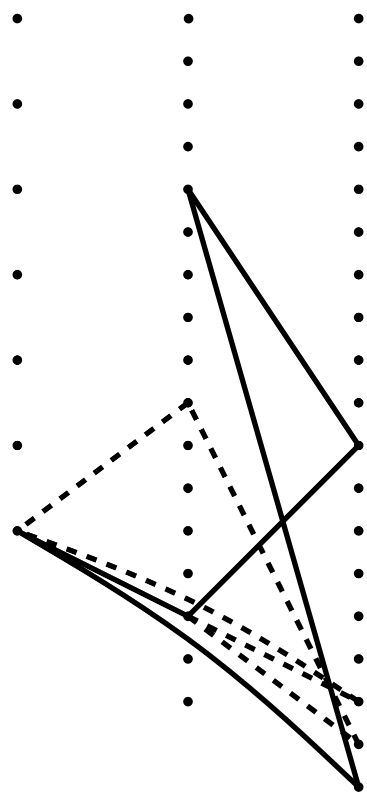


Figure 2.18: Final two 5-cycles

This process of swapping one 5-cycle for another in order to create a new leave represents a trade in which we take seven 5-cycles and twenty edges and decompose the graph induced by these edges into 5-cycles. We take the 5-cycles $((4,1),\ (1,3),\ (5,2),\ (8,3),\ (17,2))$; $((4,1),\ (4,3),\ (4,2),\ (11,3),\ (16,2))$; $((1,1),\ (2,3),\ (10,2),\ (9,3),\ (5,2))$; $((4,1),\ (8,3),\ (12,2),\ (15,3),\ (7,2))$; $((7,2),\ (11,3),\ (12,2),\ (4,3),\ (3,1))$; $((10,2),\ (11,3),\ (5,2),\ (5,1),\ (4,3))$; and $((7,1),\ (11,3),\ (15,2),\ (18,3),\ (10,2))$ along with the edges $\{(1,1),\ (1,3)\}$; $\{(1,1),\ (17,3)\}$; $\{(1,1),\ (18,3)\}$; $\{(1,1),\ (16,2)\}$; $\{(3,1),\ (1,3)\}$; $\{(3,1),\ (17,3)\}$; $\{(16,2),\ (18,3)\}$; $\{(2,1),\ (2,3)\}$; $\{(2,1),\ (14,3)\}$; $\{(2,1),\ (19,3)\}$; $\{(2,1),\ (17,2)\}$; $\{(4,1),\ (2,3)\}$; $\{(4,1),\ (14,3)\}$; $\{(7,1),\ (5,2)\}$; $\{(7,1),\ (15,2)\}$; $\{(7,1),\ (17,3)\}$; $\{(7,1),\ (19,3)\}$; $\{(5,2),\ (19,3)\}$; $\{(15,2),\ (17,3)\}$; and $\{(17,2),\ (19,3)\}$ to get the following eleven 5-cycles $((4,1),\ (2,3),\ (2,1),\ (19,3),\ (17,2))$; $((4,1),\ (1,3),\ (1,1),\ (18,3),\ (16,2))$; $((1,1),\ (13,3),\ (3,1),\ (1,3),\ (5,2))$; $((4,1),\ (14,3),\ (2,1),\ (17,2),\ (8,3))$; $((7,2),\ (4,1),\ (4,3),\ (4,2),\ (11,3))$; $((3,1),\ (4,3),\ (12,2),\ (15,3),\ (7,2))$; $((10,2),\ (2,3),\ (1,1),\ (16,2),\ (11,3))$; $((5,1),\ (5,2),\ (9,3),\ (10,2),\ (4,3))$; $((7,1),\ (5,2),\ (8,3),\ (12,2),\ (11,3))$; $((7,1),\ (19,3),\ (5,2),\ (11,3),\ (15,2))$; and $((7,1),\ (10,2),\ (18,3),\ (15,2),\ (17,3))$.

This uses all of the edges from the leave of the graph of $K_{7,17,19}$, thus we have a decomposition of $K_{7,17,19}$ into 5-cycles.

Chapter 3

Final Comments

Several of the cases that remain to be solved are complete tripartite graphs where the part sizes are relatively prime to each other. It is my belief that the process detailed in the previous chapter will define 5-cycle decompositions for any Complete Tripartite graph $G = K_{r,s,t}$ where r, s, and t are relatively prime to each other. Thus I make the following conjecture, which is made inconsequential if Mahmoodian's original conjecture is proved:

Conjecture 3.1 *Let $K_{r,s,t}$ be a complete tripartite graph with part sizes $r < s < t$. If r, s, and t are pairwise relatively prime and the necessary conditions are satisfied, then $K_{r,s,t}$ admits a 5-cycle decomposition.*

Bibliography

[1] Billington, E.J. and Cavenagh, N.J.: Decomposing complete tripartite graphs into 5-cycles when the partite sets have similar size. Aequationes Math. 82, 277-289 (2011)

[2] Cavenagh, N.J. and Billington, E.J.: On decomposing complete tripartite graphs into 5-cycles. Australas. J. Combin. 22, 41-62 (2000)

[3] Cavenagh, N.J.: Further decompositions of complete tripartite graphs into 5-cycles. Discrete Math. 256, 55-81 (2002)

[4] Mahmoodian, E.S and Mirzakhani, M: Decomposition of complete tripartite graphs into 5-cycles, Combinatorics advances (Tehran, 1994), vol. 329 of Math. Appl., pp. 235-241. KluwerAcad. Publ., Dordrect, 1995

[5] Mahmoodian, E.S., Alipour, S and Mollaahmadi, E.: On decomposing complete tripartite graphs into 5-cycles, Australas. J. Combin. 54, 289-301 (2012)

[6] Sotteau, D.: Decomposition of $K(m, n)$ into cycles (circuits) of length $2k$. J. of Combinatorial Theory B 30, 75-81, 1981

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	18	19	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15		
19	1	2	3	4	5	6	7	10	10	10	10	10	10	10	10	10	10		
	19	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	6		

Figure 1: Covered latin representation of $K_{7,17,19}$ by trades

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
1	59	94	F	45	80	1	31	66	B	17	52	87	F	38	73	63	24
2	25	60	95	G	46	81	3	32	67	C	18	53	88	G	39	74	48
3	2	26	61	96	12	47	M	K	33	68	D	19	54	89	5	40	75
4	76	4	27	62	97	13	82	83	L	34	69	E	20	55	90	6	41
5	42	77	10	28	N	98	14	49	84	H	35	70	F	21	56	91	7
6	8	43	78	11	29	64	99	15	50	85	I	36	71	G	22	57	92
7	93	9	44	79	51	30	65	A	16	P	86	J	37	72	O	23	58

Figure 2: List of 5-cycles to which edges between R and S belong in $K_{r,s,t}$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1	6	60	24	B	45	I	66	1	87	31	I	52	80	73	17	94	38	6	59
2	63	41	81	25	C	46	J	67	3	88	32	J	53	48	74	18	95	39	41
3	80	61	5	M	26	D	47	2	68	12	89	33	80	54	K	75	19	96	40
4	6	41	62	82	83	27	E	48	4	69	13	90	34	48	55	L	76	20	97
5	98	42	K	N	7	84	28	10	49	H	70	14	91	35	K	56	F	77	21
6	22	99	43	L	64	8	85	29	11	50	I	71	15	92	36	L	57	G	78
7	79	23	A	44	H	65	9	86	30	H	51	J	72	16	93	37	P	58	O

Figure 3: List of 5-cycles to which edges between R and T belong in $K_{r,s,t}$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1	1	54	93	37	76	20	59	1	42	88	25	71	8	54	2	37	88	20	71
2	72	3	55	94	38	77	21	60	3	43	89	26	72	9	55	4	38	89	21
3	22	73	5	56	95	39	78	22	61	5	44	90	27	73	10	56	F	39	90
4	91	23	74	82	57	96	40	79	23	62	82	45	91	28	74	11	57	G	40
5	80	92	24	75	7	58	97	51	N	24	O	7	46	92	29	75	12	58	O
6	1	42	93	25	76	8	59	98	42	81	25	64	8	47	93	30	76	13	59
7	60	3	43	94	26	77	9	60	99	43	82	26	65	9	M	94	31	77	14
8	15	61	K	44	95	27	78	10	61	A	44	83	27	66	10	49	95	32	78
9	79	16	62	L	45	96	28	79	11	62	B	45	84	28	67	11	50	96	33
10	34	63	17	N	H	46	97	29	N	12	63	C	46	85	29	68	12	P	97
11	98	35	81	18	64	I	47	98	30	81	13	64	D	47	86	30	69	13	52
12	53	99	36	M	19	65	J	51	99	31	51	14	65	E	M	87	31	70	14
13	15	54	A	37	83	20	66	2	49	A	32	83	15	66	2	49	88	32	71
14	72	16	55	B	38	84	21	67	4	50	B	33	84	16	67	4	50	89	33
15	34	73	17	56	C	39	85	22	68	5	O	C	34	85	17	68	P	P	90
16	91	35	74	18	57	D	40	86	23	69	63	52	D	35	86	18	69	6	52
17	53	92	36	75	19	58	E	48	87	24	70	7	53	E	36	87	19	70	41

Figure 4: List of 5-cycles to which edges between S and T belong in $K_{r,s,t}$