Asymptotic Results in Noncompact Semisimple Lie Groups

by

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Abstract

This dissertation is a collection of results on analysis on real noncompact semisimple Lie groups. Specifically, we examine the convergence patterns of sequences arising from the special group decompositions that exist in this setting.

The dissertation consists of five chapters, the first of which provides a brief introduction to the topics to be studied.

In Chapter 2, we introduce preliminary ideas and definitions regarding Lie groups, their Lie algebras, and the relationships between the two structures.

Chapter 3 specifically examines semisimple Lie algebras and groups; we discuss local (algebra level) and global (group level) decompositions of real and complex semisimple Lie groups, such as the root space decomposition of a complex Lie algebra; the local and global Cartan and Iwasawa decompositions over $\mathbb{R}$; the global Bruhat decomposition; and the restricted root space decomposition of a real Lie algebra. Each of these will play important roles in the remainder.

Chapter 4 presents the iterated Aluthge sequence on $\mathbb{C}_{n\times n}$, and extends the sequence to a real noncompact semisimple Lie group. We use the Cartan decomposition and properties of the group and its adjoint map to show that the iterated Aluthge sequence converges in this setting.

The final chapter discusses the matrix iterated Aluthge sequence and its Lie group generalization using the Bruhat decomposition. We establish convergence of the sequence (under some conditions) in this general setting using the many special properties of the decomposition.

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Chapter 1

Introduction

The topic of this dissertation is the asymptotic behavior of sequences in noncompact connected semisimple Lie groups. The motivation for these problems comes from matrix theory: several well-known matrix decompositions lead to matrix sequences whose asymptotic behavior has been studied extensively. In 2011, Pujals et. al. [7] showed that the iterated Aluthge sequence, which is based on the polar decomposition of a matrix, converges in $\mathbb{C}_{n\times n}$. The behavior of the LR sequence, based on the matrix LU decomposition, was studied by Rutishauser [26]; under certain conditions this sequence also converges. This dissertation generalizes both of these matrix convergence results to the Lie group setting. Several other matrix sequences have led to similar generalizations: Holmes et. al. [18] extended a result on the QR iteration in 2011.

Matrix results provide a natural starting point for the study of general Lie groups. Well-known matrix groups such as $\mathrm{GL}_n(\mathbb{C})$ and $\mathrm{SL}_n(\mathbb{C})$ have a Lie group structure, and the matrix Lie groups are known as the classical groups; their structure often provides clues to the structure of more abstract Lie groups. Using the well-studied classical groups as a model for the abstract groups leads to a much better understanding of the abstract structure than one might otherwise hope to gain.

The main tools in this dissertation are some important decompositions of semisimple Lie algebras, which lead to corresponding decompositions on the group level. For example, the Cartan decomposition of a real semisimple Lie algebra, which is precisely the Hermitian decomposition on the general linear algebra $\mathfrak{gl}_n(\mathbb{C})$, yields the group level Cartan decomposition (which corresponds to the polar decomposition of the general linear group $\mathrm{GL}_n(\mathbb{C})$). In addition, the Iwasawa decomposition (which corresponds to the matrix QR decomposition) and the Bruhat decomposition (which corresponds to the Gelfand-Naimark decomposition in matrix theory) are built from factors derived from algebra level decompositions. The Complete Multiplicative Jordan Decomposition (CMJD) corresponds to the matrix Multiplicative Jordan Decomposition. The Cartan, Iwasawa, and Bruhat decompositions exist for any semisimple Lie group, and indeed for any reductive Lie group.

These decompositions, as well as the tools we use to prove the convergence results, are available to our use because of the remarkable correspondence between a Lie group and its Lie algebra. Indeed, the two structures are diffeomorphic via the exponential map in a neighborhood of the identity, so that results on a Lie algebra often have analogues in the corresponding Lie groups. For example, the Cartan decomposition of a real semisimple Lie algebra as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ implies a decomposition of any Lie group G with Lie algebra $\mathfrak{g}$: G may be decomposed as $G = KP$, where K is the analytic subgroup generated by $\mathfrak{k}$ and $P = \exp \mathfrak{p}$. In particular, properties of the algebra imply that the map $K \times \exp \mathfrak{p} \to G$ is actually a diffeomorphism onto G. Thus the relatively accessible Lie algebra provides a great deal of insight into the structure of G.

The other group decompositions relevant to this dissertation are all derived from the algebra-group correspondence, but these decompositions are not the only beautiful relationships between the two structures. As a particularly relevant example, the Weyl group of a Lie group, which captures the symmetry of the group, and the Weyl group of its Lie algebra, which reflects the algebra's symmetry, are actually isomorphic. Again, we see that the correspondence between the two structures is wonderfully rich.

The organization of this dissertation is as follows: in Chapter 2, we introduce the basic Lie group and Lie algebra definitions and record a few well known results for future reference. In Chapter 3, we discuss the structure of semisimple Lie groups and the relevant Lie group decompositions that lead to questions about sequences. Chapter 4 presents the Aluthge sequence and shows that the sequence converges in a real noncompact semisimple Lie group. Finally, in Chapter 5, we discuss the Bruhat sequence and the conditions under which it converges in a real noncompact semisimple Lie group. We also consider the problem of convergence of the sequence under some relaxed conditions and present some illustrative matrix examples.

Chapter 2

Lie Groups and Lie Algebras

In this chapter, we introduce the background information and notation needed throughout the dissertation. At the most fundamental level, a Lie group is an object that has the structure of a smooth manifold as well as that of a group; in particular, the two different structures (one analytic and one algebraic) are tied together by the requirement that the group operations be smooth mappings on the manifold. Accordingly, we begin by discussing smooth manifolds, following the treatment in [13], [24], and [34].

2.1 Smooth Manifolds

A *topological manifold* is a topological space M that has the local structure of n-dimensional Euclidean space. More precisely, M is a topological manifold of dimension n if

(1) M is a second countable Hausdorff space, and

(2) every point of M has an open neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$.

If M satisfies the properties above, then it is natural to consider the local homeomorphisms, and in particular the relationships between such homeomorphisms. A *chart* on M is a pair (U, φ) consisting of an open subset $U \subset M$ and a homeomorphism $\varphi : U \to \mathbb{R}^n$. We should investigate the behavior of the maps on overlapping charts. To do so, we will need to consider maps on open subsets of $\mathbb{R}^n$. If U and V are open in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and F is a map $F : U \to V$, we say that F is *smooth* or C^∞ if each component function of F has continuous partial derivatives of all orders. If, in addition, F is bijective and F^{-1} is smooth, we say that

F is a *diffeomorphism*. Returning to the manifold M, suppose that charts (U, φ) and (V, ψ) are chosen so that $U \cap V \neq \emptyset$. Then the *transition map* $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is a map between two open subsets of $\mathbb{R}^n$. Two charts (U, φ) and (V, ψ) are called *smoothly compatible* if either $U \cap V = \emptyset$ or $\psi \circ \varphi^{-1}$ is $\mathbb{C}^\infty$.

A collection $\mathcal{A}$ of charts that cover M is called an *atlas*, and if each pair of charts in the atlas $\mathcal{A}$ is smoothly compatible, we say that $\mathcal{A}$ is a *smooth atlas*. If $\mathcal{A}$ is a maximal smooth atlas (in the sense that the addition of any chart (U, φ) to $\mathcal{A}$ would destroy its property of smooth compatibility) we say that $\mathcal{A}$ is a *smooth structure* on M. Concisely, $\mathcal{A} = \{(U_\alpha, \varphi_\alpha) : \alpha \in I\}$ is a smooth structure if

(1) $\displaystyle\bigcup_{\alpha \in I} U_\alpha = M$,

(2) $\varphi_\alpha \circ \varphi_\beta^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is C^∞ for all $\alpha, \beta \in I$, and

(3) the collection is maximal with respect to (2).

The topological manifold M is called a *smooth manifold* (or simply a *manifold*) if it has a smooth structure $\mathcal{A}$, and a chart on M is said to be *smooth* if it is a member of a smooth structure on M.

We would like to extend the idea of smoothness of maps on Euclidean space to maps on manifolds. Accordingly, let M be an m-dimensional manifold and N be an n-dimensional manifold with smooth structures $\mathcal{A}_m$ and $\mathcal{A}_n$, respectively, and $F : M \to N$ a continuous map. The map F is said to be *smooth* if, for every $p \in M$, there are charts $(U, \varphi) \in \mathcal{A}_m$ and $(V, \phi) \in \mathcal{A}_n$ so that $p \in U$, $F(U) \subset V$, and $\phi \circ F \circ \varphi^{-1}$ (which maps an open subset of $\mathbb{R}^m$ to an open subset of $\mathbb{R}^n$) is C^∞ from $\varphi(U)$ to $\phi(V)$. In particular, consider the case where $N = \mathbb{R}$. Then F is called a *smooth function* on M if for each $p \in M$, there is a corresponding smooth chart (U, φ) so that $p \in U$ and $F \circ \varphi^{-1}$ is C^∞.

The manifold analogue for the collection of C^∞ functions on open subsets of Euclidean space is $C^\infty(M)$, which we use to denote the set of all smooth functions $F : M \to \mathbb{R}$. It is clear that $C^\infty(M)$ is a vector space. It becomes a ring if we introduce the product $f \cdot g$, which is calculated by pointwise multiplication. Next, we introduce calculus on M: any linear map $v : C^{\infty}(M) \to \mathbb{R}$ that satisfies the product rule at some $p \in M$, i.e.,

$$v(fg) = f(p)v(g) + v(f)g(p), \qquad \forall f, g \in C^{\infty}(M),$$

is called a *tangent vector* of M at p. The collection of tangent vectors at the point p is a vector space denoted by $T_p(M)$, known as the *tangent space* to M at p. Each $v \in T_p(M)$ (i.e., a vector in the tangent space of M at p) is called a *tangent vector* at p. We may think of $M = \mathbb{R}^n$ as a prototype; the tangent vector v acts on $f \in C^{\infty}(\mathbb{R}^n)$ by taking the directional derivative of f in the direction provided by v, i.e., $v(f) = \langle \nabla f, v \rangle$.

Given a point $p \in M$ and a smooth map $F : M \to N$ we assign to F a corresponding linear map between the tangent spaces $T_p(M)$ and $T_{F(p)}(N)$. This is known as the differential (or *derivative*) of F: the *differential* $dF_p : T_p(M) \to T_{F(p)}(N)$ of F at p is the linear map defined by

$$dF_p(v)(f) = v(f \circ F), \qquad \text{for all } v \in T_p(M), f \in C^{\infty}(N).$$

Notice that dF_p sends a tangent vector $v \in T_p(M)$ to a tangent vector in $T_p(N)$, so that $dF_p(v) : C^{\infty}(N) \to \mathbb{R}$.

Example 2.1. If $M = N = \mathbb{R}$, then under the natural identification $T_p(\mathbb{R}) \cong \mathbb{R}$, the map $dF_p : \mathbb{R} \to \mathbb{R}$ is just multiplication by $F'(p)$, where F' is the usual derivative of elementary calculus.

In a sense, the map dF_p provides a “linear approximation” for F or a “linearization” of F. The rank of the linear map dF_p is well-defined, so that we may define the *rank* of F as $\operatorname{rank} dF_p$. If F is a smooth map so that $\operatorname{rank} F = \dim M$ for each $p \in P$, then F is called an *immersion*. Equivalently, F is an immersion if dF_p is injective for each $p \in M$.

We define the concept of a *submanifold* of M by beginning with a subset $S \subset M$ that is a smooth manifold in its own right; in addition, we want the topological and differential structure of S to align with that of M, so we require the inclusion map $\iota : S \to M$ to be an immersion.

At each point $p \in M$, we have defined a tangent space $T_p(M)$. The disjoint union

$$T(M) = \cup_{p \in M} T_p(M)$$

is called the *tangent bundle* of M. We define the natural projection $\pi : T(M) \to M$ of the tangent bundle onto its manifold via $\pi(X) = p$ for $X \in T_p(M)$. Of course, we may simply view $T(M)$ as a collection of vector spaces, but the definitions actually equip the tangent bundle with quite a bit of structure: $T(M)$ has a natural topology as well as a smooth structure derived from M, so that the tangent bundle of a manifold is again a smooth manifold and in particular, $\pi : T(M) \to M$ is smooth.

With tangent spaces and the tangent bundle in place, we may assign to each $p \in M$ a vector in $T_p(M)$. Such a map $X : M \to T(M)$, so that $X(p) \in T_p(M)$ for all $p \in M$, is called a *vector field* on M. The set of smooth vector fields on M is clearly a vector space over $\mathbb{R}$, and in fact forms a module over the ring $C^\infty(M)$: if X is a vector field on M and $f \in C^\infty(M)$, then fX is the vector field defined by $(fX)(p) = f(p)X_p$.

A *derivation* of an algebra $\mathcal{A}$ over a field $\mathbb{F}$ is a map $D : \mathcal{A} \to \mathcal{A}$ that satisfies the following properties:

(1) $D(af + bg) = aD(f) + bD(g)$ for all $a, b \in \mathbb{F}$, $f, g \in \mathcal{A}$;

(2) $D(fg) = f(Dg) + (Df)g$ for all $f, g \in \mathcal{A}$.

Any smooth vector field X may be thought of as a derivation of $C^\infty(M)$: let $Xf \in C^\infty(M)$ be given by $Xf(p) = X_p f$. Then since $X_p \in T_p(M)$,

$$X(fg)(p) = X_p fg = f(p)(X_p g) + (X_p f)g(p) = f(p)Xg(p) + Xf(p)g(p).$$

In other words, $X(fg) = f(Xg) + (Xf)g$. Each derivation of $C^\infty(M)$ can actually be identified with a smooth vector field: a function $\mathcal{X} : C^\infty(M) \to C^\infty(M)$ is a derivation if and only if it is of the form $\mathcal{X}(f) = Xf$ for some smooth vector field X on M [24, p.86]. If X and Y are smooth vector fields on M, the composition $X \circ Y : C^\infty(M) \to C^\infty(M)$ may not be a smooth vector field; however, the Lie bracket $[X, Y] := X \circ Y - Y \circ X$ always is.

2.2 Lie Groups and Their Lie Algebras

Let $\mathfrak{g}$ be a vector space over a field $\mathbb{F}$, and equip $\mathfrak{g}$ with a product $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, denoted $(X, Y) \mapsto [X, Y]$. The following properties arise naturally for many choices of $[\cdot, \cdot]$:

(1) $[\cdot, \cdot]$ is bilinear.

(2) $[X, X] = 0$ for all $X \in \mathfrak{g}$.

(3) The Jacobi identity $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ holds for all $X, Y, Z \in \mathfrak{g}$.

If $\mathfrak{g}$ and its product $[\cdot, \cdot]$ satisfy the three properties above, we call $[X, Y]$ the *Lie bracket* of X and Y, and call $\mathfrak{g}$ a *Lie algebra* over $\mathbb{F}$. We will focus our attention on $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ in our future discussion; the corresponding Lie algebra is called a real or complex Lie algebra.

Example 2.2. A familiar example of a Lie algebra is the *general linear algebra* $\mathfrak{gl}(V)$, which consists of all linear operators on the vector space V over $\mathbb{F}$ with the Lie bracket defined by

$$[X, Y] = XY - YX, \quad X, Y \in \mathfrak{gl}(V).$$

Example 2.3. The space of smooth vector fields on a manifold M, with

$$[X, Y] = X \circ Y - Y \circ X,$$

has the structure of a Lie algebra over $\mathbb{R}$.

Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras. Both $\mathfrak{g}$ and $\mathfrak{h}$ have the underlying structure of a vector space, so that we have the notion of linear transformation from $\mathfrak{g}$ to $\mathfrak{h}$. Considering the spaces as Lie algebras, we would like to restrict our attention to linear transformations that preserve the algebra structure. Accordingly, we define a *Lie algebra homomorphism* as a linear transformation $\varphi : \mathfrak{g} \to \mathfrak{h}$ such that

$$\varphi([X,Y]) = [\varphi(X), \varphi(Y)], \qquad \text{for all } X, Y \in \mathfrak{g}.$$

Given a vector space V over the field $\mathbb{F}$, a Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ is called a *representation* of $\mathfrak{g}$. Representations are particularly useful, as $\phi(\mathfrak{g})$ is an algebra of matrices which retains inherits the bracket operation from $\mathfrak{g}$.

Because of the bilinearity of the bracket and the Jacobi identity, the linear transformation

$$\mathrm{ad}\ : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$$

defined by

$$\mathrm{ad}\, X(Y) = [X,Y], \quad X, Y \in \mathfrak{g}$$

is a Lie algebra homomorphism and therefore a representation of $\mathfrak{g}$, known as the *adjoint representation* of $\mathfrak{g}$. Clearly the adjoint representation is of prime importance when we study Lie algebras. A vector subspace $\mathfrak{s}$ of $\mathfrak{g}$ that is a Lie algebra in its own right, i.e., $[X,Y] \in \mathfrak{s}$ for all $X, Y \in \mathfrak{s}$, is called a *subalgebra* of $\mathfrak{g}$; it is called an *ideal* if $[X,Y] \in \mathfrak{s}$ for all $X \in \mathfrak{g}$ and $Y \in \mathfrak{s}$.

A *Lie group* G is simultaneously a smooth manifold and a group such that the maps $m : G \times G \to G$ and $i : G \to G$ defined by multiplication and inversion are smooth.

Example 2.4. The set of all $n \times n$ nonsingular complex matrices forms a Lie group, called the *general linear group* and denoted by $\mathrm{GL}_n(\mathbb{C})$. Any closed subgroup of $\mathrm{GL}_n(\mathbb{C})$ is also a Lie group, called a *closed linear group.* Due to their dual topological and algebraic structure, Lie groups have a rich anatomy which provides both analytic and algebraic tools for exploration. Other classical Lie groups are

1. the real general linear group $\mathrm{GL}_n(\mathbb{R}) = \mathrm{GL}_n(\mathbb{C}) \cap \mathbb{R}_{n\times n}$,

2. the special linear group $\mathrm{SL}_n(\mathbb{C}) = \{A \in \mathbb{C}_{n\times n} : \det A = 1\}$,

3. the real special linear group $\mathrm{SL}_n(\mathbb{R}) = \mathrm{SL}_n(\mathbb{C}) \cap \mathbb{R}_{n\times n}$,

4. the unitary group $\mathrm{U}(n) = \{U \in \mathrm{GL}_n(\mathbb{R}) : U^*U = I\} \subset \mathrm{SL}_n(\mathbb{C})$,

5. the orthogonal group $\mathrm{O}(n) = \mathrm{U}(n) \cap \mathbb{R}_{n\times n}$,

6. the special orthogonal group $\mathrm{SO}(n) = \{O \in \mathrm{SL}_n(\mathbb{R}) : O^\top O = I\} \subset \mathrm{O}(n)$,

7. the complex special orthogonal group $\mathrm{SO}_n(\mathbb{C}) = \{O \in \mathrm{SL}_n(\mathbb{C}) : O^\top O = I\}$,

8. the complex symplectic group $\mathrm{Sp}_n(\mathbb{C}) = \{g \in \mathrm{SL}_{2n}(\mathbb{C}) : g^\top J_n g = J_n\}$, where $J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$.

9. the real symplectic group $\mathrm{Sp}_n(\mathbb{R}) = \mathrm{Sp}_n(\mathbb{C}) \cap \mathbb{R}_{n\times n}$,

10. the (compact) symplectic group $\mathrm{Sp}(n) = \mathrm{Sp}(n, \mathbb{C}) \cap \mathrm{U}(2n) = \left\{g = \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix} : g \in \mathrm{U}(2n)\right\}$.

The groups $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{SL}_n(\mathbb{C})$, $\mathrm{SL}_n(\mathbb{R})$, $\mathrm{SO}_n(\mathbb{C})$, $\mathrm{SO}(n)$, $\mathrm{SU}(n)$, $\mathrm{U}(n)$, $\mathrm{Sp}_n(\mathbb{C})$ and $\mathrm{Sp}(n)$ are all connected. The group $\mathrm{GL}_n(\mathbb{R})$ has two components.

Example 2.5. As a more general example, let V be a finite dimensional vector space over $\mathbb{C}$ or $\mathbb{R}$. The group $\mathrm{GL}(V)$ of vector space automorphisms of V, with multiplication given by function composition, is a Lie group.

Let G be a Lie group. Given $g \in G$, we define the left translation $L_g : G \to G$ by $L_g(h) = gh$. Since multiplication is smooth, L_g is a diffeomorphism of G. Because of its manifold structure, we may consider the interaction of a smooth vector field X of G with L_g: X is *left-invariant*, or L_g related to itself, if

$$X \circ L_g = dL_g \circ X, \qquad \text{for all } g \in G.$$

When X is viewed as a derivation on $C^\infty(G)$, left-invariance is equivalent to

$$(Xf) \circ L_g = X(f \circ L_g), \qquad \text{for all } f \in C^\infty(G),\ g \in G.$$

The vector space of left-invariant smooth vector fields on G is closed under the bracket given in Example 2.3, so that it forms a Lie algebra $\mathfrak{g}$. In particular, we refer to $\mathfrak{g}$ as *the Lie algebra of* G. The natural identification $X \mapsto X_e$ of the left-invariant vector field $X \in \mathfrak{g}$ with its image in the tangent space of G at the identity is readily seen to be a vector space isomorphism of $\mathfrak{g}$ with $T_e(G)$. Indeed, we create a bracket operation in $T_e(G)$ by setting $[X_e, Y_e] := [X, Y]_e$, so that $T_e(g)$ is identified with $\mathfrak{g}$ and may itself be viewed as a Lie algebra.

Example 2.6. The Lie algebra of the Lie group $\mathrm{GL}(V)$ is $\mathfrak{gl}(V)$. The Lie algebra of

1. the real general linear group $\mathrm{GL}_n(\mathbb{R})$ is $\mathfrak{gl}_n(\mathbb{R}) = \mathbb{R}_{n\times n}$,
2. the special linear group $\mathrm{SL}_n(\mathbb{C})$ is $\mathfrak{sl}_n(\mathbb{C}) = \{X \in \mathbb{C}_{n\times n} : \operatorname{tr} X = 0\}$.
3. the real special linear group $\mathrm{SL}_n(\mathbb{R})$ is $\mathfrak{sl}_n(\mathbb{C}) \cap \mathbb{R}_{n\times n}$,
4. the unitary group $\mathrm{U}(n)$ is $\mathfrak{u} = \{X \in \mathbb{C}_{n\times n} : X^* = -X\}$,
5. the orthogonal group $\mathrm{O}(n)$ is $\mathfrak{so}(n) = \{X \in \mathbb{R}_{n\times n} : X^\top = -X\}$,
6. the special orthogonal group $\mathrm{SO}(n)$ is also $\mathfrak{so}(n)$,

7. the complex symplectic group $\mathrm{Sp}_n(\mathbb{C})$ is

$$\mathfrak{sp}_n(\mathbb{C}) = \left\{ \begin{pmatrix} X_1 & X_2 \\ X_3 & -X_1^\top \end{pmatrix} : X_i \in \mathbb{C}_{n\times n} \text{ and } X_2 = X_2^\top,\ X_3 = X_3^\top \right\}$$

8. the real symplectic group $\mathrm{Sp}_n(\mathbb{R})$ is

$$\mathfrak{sp}_n(\mathbb{R}) = \left\{ \begin{pmatrix} X_1 & X_2 \\ X_3 & -X_1^\top \end{pmatrix} : X_i \in \mathbb{R}_{n\times n} \text{ and } X_2 = X_2^\top,\ X_3 = X_3^\top \right\}$$

9. and the (compact) symplectic group $\mathrm{Sp}(n)$ is $\mathfrak{sp}(n, \mathbb{R}) = \{X \in \mathfrak{gl}(2n, \mathbb{R}) : X^\top J = -JX\}$ [21, p. 59].

A smooth map $\varphi : G \to H$ between the Lie groups G and H that is also a group homomorphism is called a *smooth homomorphism.* Since a smooth homomorphism φ preserves both the manifold and the group structure of G, the differential map $d\varphi_e : \mathfrak{g} \to \mathfrak{h}$ between the corresponding Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ is actually a Lie algebra homomorphism. We simply write $d\varphi$ for $d\varphi_e$, and $d\varphi$ is called the *derived homomorphism* of φ.

Let G be a Lie group with Lie algebra $\mathfrak{g}$. A *one-parameter subgroup* of G is a smooth homomorphism $\phi : \mathbb{R} \to G$, where we view $\mathbb{R}$ a Lie group under addition. Due to the theorem of existence and uniqueness of solutions of linear ordinary differential equations, the map $\phi \mapsto d\phi(0)$ is a bijection of the set of one-parameter subgroups of G onto $\mathfrak{g}$ [13, p.103]. For each $X \in \mathfrak{g}$, let ϕ_X be the one-parameter subgroup corresponding to X, and define the exponential map $\exp = \exp_{\mathfrak{g}} : \mathfrak{g} \to G$ by $\exp(X) = \phi_X(1)$. Then exp is smooth, and $\phi_X(t) = \exp(tX)$, so that the one-parameter subgroups are the maps $t \mapsto \exp tX$ for $X \in \mathfrak{g}$. The properties of the exponential map lead to the following important theorem.

Theorem 2.7. [13, p.104] There is a neighborhood N_0 of 0 in $\mathfrak{g}$ and an open neighborhood N_e of e in G so that exp is an smooth diffeomorphism of N_0 onto N_e.

In a sense, $\mathfrak{g}$ and G behave alike near their identities. The exponential map interacts particularly well with smooth homomorphisms $\varphi : G \to H$; it is natural in that

$$\varphi \circ \exp_{\mathfrak{g}} = \exp_{\mathfrak{h}} \circ d\varphi. \tag{2.1}$$

If G is a closed linear group, its exponential map is just the well-known matrix exponential function [21, p.76], with $\exp A$ given by

$$\exp A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n.$$

If the submanifold H of G is also a Lie group in its own right, with multiplication induced by the multiplication on G, H is called a *Lie subgroup* of G. If H is also closed in the topology on G, H is called a *closed subgroup*. We would like to have a correlation of some kind between Lie subgroups of G and Lie subalgebras of $\mathfrak{g}$. The following theorem displays an elegant correspondence between the two structures [13, p.112, p.115, p.118]:

Theorem 2.8. Let G be a Lie group with Lie algebra $\mathfrak{g}$. If H is a Lie subgroup of G, then the Lie algebra $\mathfrak{h}$ of H is a subalgebra of $\mathfrak{g}$. Moreover, $\mathfrak{h} = \{X \in \mathfrak{g} : \exp tX \in H \text{ for all } t \in \mathbb{R}\}$. Each subalgebra of $\mathfrak{g}$ is the Lie algebra of exactly one connected Lie subgroup of G.

For $g \in G$, define a map $I_g : G \to G$ by $I_g(x) = gxg^{-1}$. Since multiplication in a Lie group is a smooth operation, I_g is a smooth automorphism; its differential, denoted $\operatorname{Ad} g$, is a Lie algebra automorphism of $\mathfrak{g}$. Then by (2.1), we have

$$\exp(\operatorname{Ad} g(X)) = g(\exp X)g^{-1}, \quad g \in G,\ X \in \mathfrak{g}. \tag{2.2}$$

As a particularly nice example, suppose that G is a closed linear group, i.e., a matrix group. Then $\operatorname{Ad} g$ is given by

$$\operatorname{Ad} g(X) = gXg^{-1}, \quad g \in G.$$

The exponential map is smooth and invertible in a neighborhood of the identity, so that if $X \in \mathfrak{g}$ is close to the identity, $g \mapsto \mathrm{Ad}\, gX$ is smooth as a map from a neighborhood of e in G to $\mathfrak{g}$. In other words, $g \mapsto \mathrm{Ad}\, g$ is a smooth map from a neighborhood of e into $\mathrm{GL}(\mathfrak{g})$. In addition, since $I_{g_1 g_2} = I_{g_1} I_{g_2}$, we see that $\mathrm{Ad}\,(g_1 g_2) = \mathrm{Ad}\,(g_1)\mathrm{Ad}\,(g_2)$. Thus Ad is smooth on all of G, so that $\mathrm{Ad}\, : G \to \mathrm{GL}(\mathfrak{g})$ is a smooth homomorphism; in particular, $\mathrm{Ad}\, G$ is a Lie subgroup of $\mathrm{GL}(\mathfrak{g})$. Ad is actually a representation of G, known as the *adjoint representation*. The smoothness of Ad leads us to consider its differential, which is just $\mathrm{ad}\, : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ [21, p.80]. Another application of (2.1) shows that

$$\mathrm{Ad}\,(\exp X) = \exp(\mathrm{ad}\, X), \quad X \in \mathfrak{g}. \tag{2.3}$$

Note that the exponential map on the left of the equality is $\exp : \mathfrak{g} \to G$, whereas exp on the right hand side is $\exp : \mathfrak{gl}(\mathfrak{g}) \to \mathrm{GL}(\mathfrak{g})$.

The group $\mathrm{Aut}\, \mathfrak{g}$ of all Lie algebra automorphisms of $\mathfrak{g}$ is a closed subgroup of $\mathrm{GL}(\mathfrak{g})$. Since a closed abstract subgroup of a group G is automatically a Lie subgroup [13, p.115], $\mathrm{Aut}\, \mathfrak{g}$ is a Lie subgroup of $\mathrm{GL}(\mathfrak{g})$. Its Lie algebra $\mathrm{Der}\, \mathfrak{g}$ consists of all derivations of $\mathfrak{g}$ [13, p.127]. Due to the Jacobi identity, $\mathrm{ad}\, X : \mathfrak{g} \to \mathfrak{g}$ is a derivation for all $X \in \mathfrak{g}$, so that $\mathrm{ad}\, \mathfrak{g}$ is a subalgebra of $\mathrm{Der}\, \mathfrak{g}$. Indeed, $\mathrm{ad}\, \mathfrak{g}$ generates a connected subgroup $\mathrm{Int}\, \mathfrak{g}$ of $\mathrm{Aut}\, \mathfrak{g}$ [13, p.127], known as the *adjoint group* of $\mathfrak{g}$. Elements of $\mathrm{ad}\, \mathfrak{g}$ are called *inner derivations*, and elements of $\mathrm{Int}\, \mathfrak{g}$ are called *inner automorphisms*. If G is a connected group, then $\mathrm{Int}\, \mathfrak{g} = \mathrm{Ad}\, G$ by (2.3). The adjoint group $\mathrm{Int}\, \mathfrak{g}$ is advantageous over $\mathrm{Ad}\, G$ since it is independent of the choice of the connected Lie group G. The Lie algebra $\mathfrak{g}$ is said to be *compact* if G is compact or, equivalently, the adjoint group $\mathrm{Int}\, \mathfrak{g}$ is compact.

The symmetric bilinear form B on $\mathfrak{g}$ defined by

$$B(X, Y) = \mathrm{tr}\,(\mathrm{ad}\, X\, \mathrm{ad}\, Y), \qquad X, Y \in \mathfrak{g}$$

is called the *Killing form*. Its interaction with the Lie bracket is associative in the sense that

$$B([X,Y],Z) = B(X,[Y,Z]), \quad X,Y,Z \in \mathfrak{g}.$$

If the Killing form has the property that $B(X,Y) = 0$ for all $Y \in \mathfrak{g}$ only when $X = 0$, we say that B is nondegenerate on $\mathfrak{g}$. If σ is an automorphism of $\mathfrak{g}$, then

$$\mathrm{ad}\,(\sigma X) = \sigma \circ \mathrm{ad}\, X \circ \sigma^{-1}$$

so that $B(\sigma X, \sigma Y) = B(X,Y)$. In particular, B is $\mathrm{Ad}\, G$-invariant.

A Lie algebra $\mathfrak{g}$ is abelian if $[\mathfrak{g},\mathfrak{g}] = 0$; it is *simple* if it is not abelian and has no nontrivial ideals; it is *solvable* if $D^k\mathfrak{g} = 0$ for some k, where

$$D^0\mathfrak{g} := \mathfrak{g}, \quad D^{k+1}\mathfrak{g} := [D^k\mathfrak{g}, D^k\mathfrak{g}];$$

it is *nilpotent* if $C_k\mathfrak{g} = 0$ for some k, where

$$C_0\mathfrak{g} = \mathfrak{g}, \quad \text{and } C_{k+1}\mathfrak{g} = [C_k\mathfrak{g}, \mathfrak{g}].$$

If the (unique) maximal solvable ideal of $\mathfrak{g}$, called the *radical* of $\mathfrak{g}$ and denoted by $\mathrm{Rad}\,\mathfrak{g}$, is trivial, $\mathfrak{g}$ is said to be *semisimple*. As semisimple Lie algebras will be a main object of interest in this dissertation, let us record for future reference some equivalent conditions to semisimplicity: $\mathfrak{g}$ is semisimple if and only if its Killing form is nondegenerate. Additionly, $\mathfrak{g}$ is semisimple if and only if it is isomorphic to a direct sum of simple algebras.

The algebra $\mathfrak{g}$ is *reductive* if its center $\mathfrak{z}(\mathfrak{g}) = \mathrm{Rad}\,\mathfrak{g}$ (or, equivalently, $[\mathfrak{g},\mathfrak{g}]$ is semisimple). A Lie group is called *semisimple (simple, reductive, solvable, nilpotent, abelian)* if its Lie algebra is semisimple (simple, reductive, solvable, nilpotent, abelian).

Chapter 3

Structure and Decompositions of Semisimple Lie Groups

In this chapter, we discuss the special structure of semisimple groups which leads to the group decompositions needed in the sequel.

3.1 Real Forms

For any complex Lie algebra $\mathfrak{g}$, restricting the base field $\mathbb{C}$ to $\mathbb{R}$ allows $\mathfrak{g}$ to be viewed as a real Lie algebra $\mathfrak{g}_\mathbb{R}$, called the *realification* of $\mathfrak{g}$. A subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}_\mathbb{R}$ such that $\mathfrak{g}_\mathbb{R} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$ is called a *real form* of $\mathfrak{g}$, and $\mathfrak{g}$ is called the *complexification* of $\mathfrak{g}_0$. If $\mathfrak{g}_0$ is a real form of $\mathfrak{g}$, then each $Z \in \mathfrak{g}$ can be uniquely written as $Z = X + iY$ with $X, Y \in \mathfrak{g}_0$, and the map $\sigma : \mathfrak{g} \to \mathfrak{g}$ given by

$$X + iY \mapsto X - iY \quad \text{for all } X, Y \in \mathfrak{g}_0$$

is called the *conjugation* of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$. It is clear that

(1) $\sigma^2 = 1$,

(2) $\sigma(\alpha X) = \bar{\alpha}\sigma(X)$ for all $X \in \mathfrak{g}$ and $\alpha \in \mathbb{C}$,

(3) $\sigma(X + Y) = \sigma(X) + \sigma(Y)$ for all $X, Y \in \mathfrak{g}$, and

(4) $\sigma[X, Y] = [\sigma X, \sigma Y]$ for all $X, Y \in \mathfrak{g}$.

Because of (2), σ is not an automorphism of $\mathfrak{g}$, but it *is* an automorphism of the real algebra $\mathfrak{g}_\mathbb{R}$.

On the other hand, if $\sigma : \mathfrak{g} \to \mathfrak{g}$ is a bijection such that (1)-(4) above hold, then the set $\mathfrak{g}_0$ of fixed points of σ is a real form of $\mathfrak{g}$ and σ is the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$. Hence there is a one-to-one correspondence between real forms and conjugations of $\mathfrak{g}$.

Let B_0, B, and $B_{\mathbb{R}}$ denote the Killing forms of the Lie algebras $\mathfrak{g}_0$, $\mathfrak{g}$, and $\mathfrak{g}_{\mathbb{R}}$, respectively. Then [13, p.180]

$$\begin{aligned} B_0(X,Y) &= B(X,Y), \quad X,Y \in \mathfrak{g}_0 \\ B_{\mathbb{R}}(X,Y) &= 2\operatorname{Re} B(X,Y), \quad X,Y \in \mathfrak{g}_{\mathbb{R}}. \end{aligned}$$

Since semisimplicity is equivalent to nondegeneracy of the Killing form, $\mathfrak{g}_0$, $\mathfrak{g}$, and $\mathfrak{g}_{\mathbb{R}}$ are all semisimple if any of them is.

Every complex semisimple Lie algebra $\mathfrak{g}$ has a compact real form $\mathfrak{g}_0$, i.e., the group generated by $\mathfrak{g}_0$ is compact [13, p.181]. The compact real forms of complex simple Lie algebras are listed in [13, p.516].

Example 3.1. For $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, the most obvious real form is $\mathfrak{sl}_n(\mathbb{R})$; it is clear that $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sl}_n(\mathbb{R}) \oplus i\mathfrak{sl}_n(\mathbb{R})$. The corresponding conjugation is $X \mapsto \bar{X}$. We note, however, that $\mathfrak{sl}_n(\mathbb{R})$ is not a *compact* real form for $\mathfrak{g}$.

Example 3.2. We construct a compact real form for $\mathfrak{sl}_n(\mathbb{C})$ as follows: if $X \in \mathfrak{sl}_n(\mathbb{C})$, set

$$X_1 = \frac{1}{2}(X - X^*) \in \mathfrak{sl}_n(\mathbb{C}) \text{ and } X_2 = \frac{1}{2}(X + X^*) \in \mathfrak{sl}_n(\mathbb{C}),$$

where $*$ denotes the conjugate transpose. In particular, $X_1^* = -X_1$ and $X_2^* = X_2$ so that X_1 and X_2 are skew-hermitian and hermitian, respectively. Then $X = X_1 + X_2$. The set of all trace zero skew-hermitian matrices is denoted $\mathfrak{su}_n(\mathbb{C}) \subset \mathfrak{sl}_n(\mathbb{C})$; in particular, if $A \in \mathfrak{sl}_n(\mathbb{C})$ is hermitian, then $iA \in \mathfrak{su}_n(\mathbb{C})$, and

$$\mathfrak{g}_{\mathbb{R}} = \mathfrak{su}_n(\mathbb{C}) \oplus i\mathfrak{su}_n(\mathbb{C}).$$

In this case, conjugation is given by $X \mapsto -X^*$. The analytic subgroup of $\mathrm{SL}_n(\mathbb{C})$ generated by $\mathfrak{su}_n(\mathbb{C})$, denoted $\mathrm{SU}_n(\mathbb{C})$, consists of determinant 1 unitary matrices and is compact, so that $\mathfrak{su}_n(\mathbb{C})$ is a compact real form of $\mathfrak{sl}_n(\mathbb{C})$.

3.2 Cartan Decomposition

We are particularly interested in algebras over $\mathbb{R}$. Thus we make the notation more convenient by using $\mathfrak{g}$ to denote a real semisimple Lie algebra, $\mathfrak{g}_{\mathbb{C}}$ its complexification, and σ the conjugation of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{g}$. A (vector space) decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of $\mathfrak{g}$ into a subalgebra $\mathfrak{k}$ and a vector subspace $\mathfrak{p}$ is called a *Cartan decomposition* if there exists a compact real form $\mathfrak{u}$ of $\mathfrak{g}_{\mathbb{C}}$ such that

$$\sigma(\mathfrak{u}) \subset \mathfrak{u}, \quad \mathfrak{k} = \mathfrak{g} \cap \mathfrak{u}, \quad \mathfrak{p} = \mathfrak{g} \cap i\mathfrak{u}.$$

If $\mathfrak{u}$ is any compact real form of $\mathfrak{g}_{\mathbb{C}}$ with a conjugation τ, then there exists an automorphism φ of $\mathfrak{g}_{\mathbb{C}}$ such that the compact real form $\varphi(\mathfrak{u})$ is invariant under σ, which guarantees the existence of a Cartan decomposition of $\mathfrak{g}$. In this case, the involutive automorphism $\theta = \sigma\tau$ is called a *Cartan involution* of $\mathfrak{g}$. The bilinear form B_θ of $\mathfrak{g}$ defined by

$$B_\theta(X, Y) = -B(X, \theta Y), \quad X, Y \in \mathfrak{g}$$

is symmetric and positive definite. The following theorem establishes a one-to-one correspondence between Cartan decompositions of a real semisimple Lie algebra and its Cartan involutions [13, p.184] [25, p.144].

Theorem 3.3. Let $\mathfrak{g}$ be a real semisimple Lie algebra written as the direct sum of subspaces $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. The following statements are equivalent.

(1) $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition.

(2) The map $\theta : X + Y \mapsto X - Y$ $(X \in \mathfrak{k}, Y \in \mathfrak{p})$ is a Cartan involution of $\mathfrak{g}$.

(3) The Killing form is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$, and $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition. It follows that $\mathfrak{k}$ and $\mathfrak{p}$ are the $+1$ and -1 eigenspaces of θ, respectively, and that $\mathfrak{k}$ is a maximal compactly embedded subalgebra of $\mathfrak{g}$. Moreover, $\mathfrak{k}$ and $\mathfrak{p}$ are orthogonal to each other with respect to both the Killing form B and the inner product B_θ.

If $\mathfrak{g}$ is a complex semisimple Lie algebra and $\mathfrak{u}$ is a compact real form of $\mathfrak{g}$, then $\mathfrak{g}_{\mathbb{R}} = \mathfrak{u} \oplus i\mathfrak{u}$ is automatically a Cartan decomposition [13, p.185].

The local (algebra-level) decomposition lifts nicely to a global (group-level) Cartan decomposition. The details are summarized below [13, p.252] [21, p.362].

Theorem 3.4. Let G be a real semisimple Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition corresponding to a Cartan involution θ of $\mathfrak{g}$. Let K be the analytic subgroup of G with Lie algebra $\mathfrak{k}$. Then

(1) K is connected, closed, and contains the center Z of G. Moreover, K is compact if and only if Z is finite. In this case, K is a maximal compact subgroup of G.

(2) There exists an involutive, analytic automorphism Θ of G whose fixed point set is K and whose differential at e is θ.

(3) The map $K \times \mathfrak{p} \to G$ given by $(k, X) \mapsto k \exp X$ is a diffeomorphism onto.

In particular $G = KP$ and every element $g \in G$ can be uniquely written as $g = kp$, where $k \in K$, $p \in P$.

Remark 3.5. If G is compact, $G = K$ and the theorem is trivial.

For any $k \in K$, $\mathrm{Ad}\, k$ leaves $\mathfrak{k}$ invariant because $\mathfrak{k}$ is the Lie algebra of K. Since $\mathrm{Ad}\, k \in \mathrm{Aut}\, \mathfrak{g}$, $\mathrm{Ad}\, k$ leaves invariant the subspace of $\mathfrak{g}$ orthogonal to $\mathfrak{k}$, which is exactly $\mathfrak{p}$. $\mathrm{Ad}\, k$ also leaves B invariant. If $X \in \mathfrak{g}$, write $X = X_\mathfrak{k} + X_\mathfrak{p}$ with $X_\mathfrak{k} \in \mathfrak{k}$ and $X_\mathfrak{p} \in \mathfrak{p}$ and we see that

$$\mathrm{Ad}\, k(\theta(X)) = \mathrm{Ad}\,(k)X_\mathfrak{k} - \mathrm{Ad}\,(k)X_\mathfrak{p} = \theta(\mathrm{Ad}\,(k)X_\mathfrak{k}) + \theta(\mathrm{Ad}\,(k)X_\mathfrak{p}) = \theta(\mathrm{Ad}\,(k)X),$$

i.e., $\operatorname{Ad} k$ commutes with θ. Hence $\operatorname{Ad} k$ leaves B_θ invariant as well.

Example 3.6. Let us work out a Cartan decomposition of $\mathrm{SL}_n(\mathbb{C})$. Since $\mathfrak{su}_n(\mathbb{C})$ is a compact real form of $\mathfrak{sl}_n(\mathbb{C})$, the decomposition

$$\mathfrak{sl}_n(\mathbb{C}) = \mathfrak{su}_n \oplus i\mathfrak{su}_n$$

is a Cartan decomposition with $\mathfrak{k} = \mathfrak{su}_n(\mathbb{C})$, $\mathfrak{p} = i\mathfrak{su}_n(\mathbb{C})$, and Cartan involution the same as the conjugation for the form, i.e., $\theta(X) = -X^*$. As noted above, $\mathfrak{k}$ has analytic subgroup $K = \mathrm{SU}_n(\mathbb{C}) \subset \mathrm{SL}_n(\mathbb{C})$, and $P = \exp \mathfrak{p}$ is the set of determinant 1 matrices g so that $g^* = g$. The map Θ is given by $\Theta(g) = (g^*)^{-1}$.

3.3 Root Space Decomposition

Let $\mathfrak{g}$ be a complex semisimple Lie algebra. An element $X \in \mathfrak{g}$ is called *nilpotent* if $\operatorname{ad} X$ is a nilpotent endomorphism; it is called *semisimple* if $\operatorname{ad} X$ is diagonalizable. Since $\mathfrak{g}$ is semisimple, it possesses nonzero subalgebras consisting of semisimple elements, called *toral subalgebras*. These subalgebras are abelian [20, p.35].

The *normalizer* of a subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ is

$$N_{\mathfrak{g}}(\mathfrak{a}) = \{X \in \mathfrak{g} : \operatorname{ad} X(\mathfrak{a}) \subset \mathfrak{a}\};$$

it is the largest subalgebra of $\mathfrak{g}$ which contains $\mathfrak{a}$ and in which $\mathfrak{a}$ is an ideal. The *centralizer* of $\mathfrak{a}$ in $\mathfrak{g}$ is

$$Z_{\mathfrak{g}}(\mathfrak{a}) = \{X \in \mathfrak{g} : \operatorname{ad} X(\mathfrak{a}) = 0\}.$$

A subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is called a *Cartan subalgebra* of $\mathfrak{g}$ if it is self-normalizing, i.e., $\mathfrak{h} = N_{\mathfrak{g}}(\mathfrak{h})$, and nilpotent. The Cartan subalgebras of $\mathfrak{g}$ are precisely the maximal toral subalgebras of $\mathfrak{g}$ [20, p.80], and all Cartan subalgebras of $\mathfrak{g}$ are conjugate under the adjoint group $\operatorname{Int} \mathfrak{g}$ of inner automorphisms [20, p.82].

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Since $\mathfrak{h}$ is abelian, $\mathrm{ad}_{\mathfrak{g}}\mathfrak{h}$ is a commuting family of semisimple endomorphisms of $\mathfrak{g}$, thus is a simultaneously diagonalizable family. In other words, $\mathfrak{g}$ is the direct sum of the subspaces

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{h}\},$$

where α ranges over the dual $\mathfrak{h}^*$ of $\mathfrak{h}$. Note that $\mathfrak{g}_0 = \mathfrak{h}$ because $\mathfrak{h}$ is self-normalizing. A nonzero $\alpha \in \mathfrak{h}^*$ is called a *root* of $\mathfrak{g}$ relative to $\mathfrak{h}$ if $\mathfrak{g}_\alpha \neq 0$. The set of all roots, denoted by Δ, is call the *root system* of $\mathfrak{g}$ relative to $\mathfrak{h}$. Thus we have the *root space decomposition*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha.$$

The root system Δ characterizes $\mathfrak{g}$ completely.

The restriction of the Killing form on $\mathfrak{h}$ is nondegenerate and is given by

$$B(H, H') = \sum_{\alpha \in \Delta} \alpha(H)\alpha(H'), \quad H, H' \in \mathfrak{h}.$$

Consequently we can explicitly identify $\mathfrak{h}$ with $\mathfrak{h}^*$: each $\alpha \in \mathfrak{h}^*$ corresponds to a unique $H_\alpha \in \mathfrak{h}$ with

$$\alpha(H) = B(H, H_\alpha) \quad \text{for all } H \in \mathfrak{h}.$$

Thus it induces a nondegenerate bilinear form $\langle \cdot, \cdot \rangle$ defined on $\mathfrak{h}^*$ by

$$\langle \alpha, \beta \rangle = B(H_\alpha, H_\beta), \quad \alpha, \beta \in \mathfrak{h}^*.$$

The following is a collection of some properties of the root space decomposition [20, p.36–40]:

(1) Δ is finite and spans $\mathfrak{h}^*$.

(2) If $\alpha, \beta \in \Delta \cup \{0\}$ and $\alpha + \beta \neq 0$, then $B(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$.

(3) If $\alpha \in \Delta$, then $-\alpha \in \Delta$, but no other scalar multiple of α is a root.

(4) If $\alpha \in \Delta$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is one dimensional, with basis H_α.

(5) If $\alpha \in \Delta$, then $\dim \mathfrak{g}_\alpha = 1$.

(6) If $\alpha, \beta \in \Delta$, then $\dfrac{2\langle \beta, \alpha\rangle}{\langle \alpha, \alpha\rangle} \in \mathbb{Z}$ and $\beta - \dfrac{2\langle \beta, \alpha\rangle}{\langle \alpha, \alpha\rangle}\alpha \in \Delta$.

3.4 Restricted Root Space Decomposition

In this section, we follow the treatment in [21]. Let $\mathfrak{g}$ be a real semisimple Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and corresponding Cartan involution θ, and let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. For any element λ of the dual space $\mathfrak{a}^*$ of $\mathfrak{a}$, set

$$\mathfrak{g}_\lambda := \{X \in \mathfrak{g} : \ \mathrm{ad}\,(H)X = \lambda(H)X \ \text{ for all } H \in \mathfrak{a}\}.$$

Analogous to the complex case, if $\lambda \neq 0$ and $\mathfrak{g}_\lambda \neq 0$, we call λ a restricted root of $\mathfrak{g}$, or a root of $\mathfrak{g}$ with respect to $\mathfrak{a}$. We use Σ to denote the set of roots of $(\mathfrak{g}, \mathfrak{a})$.

Set $\mathfrak{m} = Z_k(\mathfrak{a})$, i.e.,

$$\mathfrak{m} = \{X \in \mathfrak{k} : \mathrm{ad}\,(X)H = 0 \text{ for all } H \in \mathfrak{a}\},$$

and

$$\mathfrak{g}_0 = Z_{\mathfrak{g}}(\mathfrak{a}).$$

Since $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$, if $X \in \mathfrak{p}$ so that $\mathrm{ad}\,(H)X = 0$ for all $H \in \mathfrak{a}$, then $X \in \mathfrak{a}$; so $\mathfrak{g}_0 \cap \mathfrak{p} = \mathfrak{a}$.

Proposition 3.7. The restricted root system has the following properties:

1. $\mathfrak{g}$ is the orthogonal direct sum $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda \in \Sigma} \mathfrak{g}_\lambda$.

2. $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$.

3. $\theta(\mathfrak{g}_\lambda) = \mathfrak{g}_{-\lambda}$.

4. $\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m}$.

For each restricted root α, the set

$$P_\alpha = \{X \in \mathfrak{a} : \alpha(X) = 0\}$$

is a subspace of $\mathfrak{a}$ of codimension 1. The subspaces $P_\alpha\,(\alpha \in \Sigma)$ divide $\mathfrak{a}$ into several open convex cones, called *Weyl chambers*. Fix a Weyl chamber $\mathfrak{a}_+$ and refer to it as the *fundamental Weyl chamber*. A root α is *positive* if it is positive on $\mathfrak{a}_+$. If α is not a positive root, then since α is a linear functional and is nonzero on $\mathfrak{a}_+$, it must be negative on $\mathfrak{a}_+$; in this case we call α a negative root. Let Σ^+ denote the set of all positive roots with respect to $\mathfrak{a}_+$, and Σ^- the set of all negative roots. If $\alpha \in \Sigma^+$ and $X \in \mathfrak{g}_\alpha$, write

$$X = X_\mathfrak{k} + X_\mathfrak{p}$$

with $X_\mathfrak{k} \in \mathfrak{k}$ and $X_\mathfrak{p} \in \mathfrak{p}$. Since

$$[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p} \text{ and } [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k},$$

for any $H \in \mathfrak{a}$ we have

$$(\operatorname{ad} H)X_\mathfrak{k} = \alpha(H)X_\mathfrak{p}, \quad (\operatorname{ad} H)X_\mathfrak{p} = \alpha(H)X_\mathfrak{k},$$

which imply

$$(\operatorname{ad} H)^2 X_\mathfrak{k} = \alpha(H)^2 X_\mathfrak{k},$$

$$(\operatorname{ad} H)^2 X_\mathfrak{p} = \alpha(H)^2 X_\mathfrak{p},$$

and

$$\theta(X) = X_{\mathfrak{k}} - X_{\mathfrak{p}} \in \mathfrak{g}_{-\alpha}.$$

For $\alpha \in \Sigma^+$, define

$$\mathfrak{k}_\alpha = \{X \in \mathfrak{k} : (\operatorname{ad} H)^2 X = \alpha(H)^2 X \text{ for all } H \in \mathfrak{a}\},$$
$$\mathfrak{p}_\alpha = \{X \in \mathfrak{p} : (\operatorname{ad} H)^2 X = \alpha(H)^2 X \text{ for all } H \in \mathfrak{a}\}.$$

Example 3.8. Again viewing $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ as a Lie algebra over $\mathbb{R}$, we construct its restricted root space decomposition using the Cartan decomposition

$$\mathfrak{sl}_n(\mathbb{C}) = \mathfrak{su}_n(\mathbb{C}) \oplus i\mathfrak{su}_n(\mathbb{C}),$$

with $\mathfrak{k} = \mathfrak{su}_n(\mathbb{C})$ and $\mathfrak{p} = i\mathfrak{su}_n(\mathbb{C})$. We choose the maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ to be the traceless real diagonal matrices. The restricted root λ_{ij} $(i \neq j)$ corresponds to the two-dimensional root space

$$\mathfrak{g}_{\lambda_{ij}} = \{a_1 E_{ij} + a_2 i E_{ij} : a_1, a_2 \in \mathbb{R}\},$$

where E_{ij} indicates the $n \times n$ matrix with zeros in each entry except for a 1 in the i, j entry. The centralizer $\mathfrak{m}$ of $\mathfrak{a}$ in $\mathfrak{k}$ is the set of traceless diagonal matrices with purely imaginary entries. As $\mathfrak{a}$ commutes only with diagonal matrices, it is clear that $\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m}$. Since the $\mathbf{E_{ij}}$ $(i \neq j)$, together with $\{E_{ii} - E_{i+1,i+1}\}$, where $i < n$ (which generate $\mathfrak{g}_0$), form a basis for $\mathfrak{sl}_n(\mathbb{R})$, it is clear that

$$\mathfrak{sl}_n(\mathbb{C}) = \mathfrak{g}_0 \oplus \bigoplus_{i \neq j} g_{\lambda_{ij}}.$$

Let us examine the roots for $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$. Setting

$$e_1 := E_{11} - E_{22} \text{ and } e_2 := E_{22} - E_{33},$$

we choose $\{e_1, e_2\}$ as a basis for $\mathfrak{a}$; via the Killing form, we see that the angle between e_1 and e_2 is $2\pi/3$.

We may specify the roots by describing their action on the chosen basis. Let λ_1, λ_2, and λ_3 be the roots so that

$$\lambda_1(e_1) = 2,\ \lambda_1(e_2) = -1,$$
$$\lambda_2(e_1) = 1,\ \lambda_2(e_2) = 1,$$
$$\lambda_3(e_1) = -1,\ \lambda_3(e_3) = 2.$$

The remaining (non-zero) roots are $-\lambda_1$, $-\lambda_2$, and $-\lambda_3$. The hyperplanes P_{λ_1}, P_{λ_2}, and P_{λ_3} are given by

$$P_{\lambda_1} = \{ae_1 + 2ae_2 :\ a \in \mathbb{R}\},\ P_{\lambda_2} = \{ae_1 - ae_2 :\ a \in \mathbb{R}\},\ P_{\lambda_3} = \{ae_1 + \frac{a}{2}e_2 :\ a \in \mathbb{R}\}.$$

The hyperplanes subdivide $\mathfrak{a}$ into its Weyl chambers:

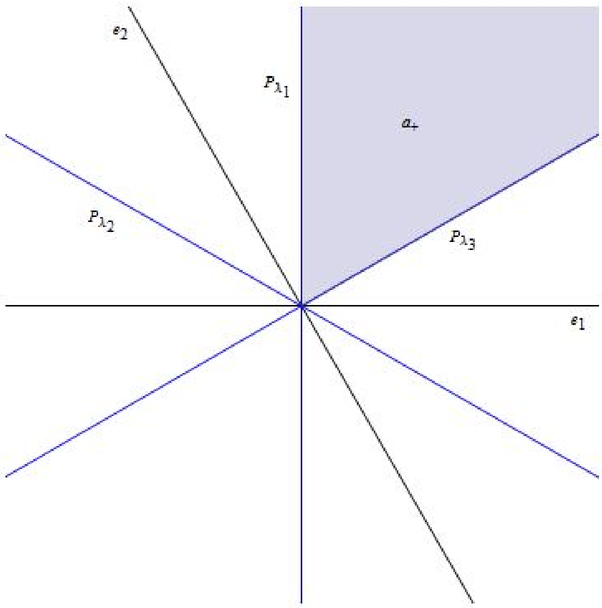

Figure 3.1: Weyl Chambers of $\mathfrak{a}$

If we fix the Weyl chamber $\mathfrak{a}_+ = \{a_1e_1 + a_2e_2 : \ 0 < \frac{1}{2}a_1 < a_2 < a_1\}$, the corresponding positive roots are λ_1, λ_2 and λ_3.

3.5 Iwasawa Decomposition

With notation as in the previous section, we construct another decomposition of a real semisimple Lie algebra, which again lifts to a global decomposition.

Lemma 3.9. [23, p.107]

(1) $\mathfrak{k} = \mathfrak{m} \oplus \sum_{\alpha\in\Sigma^+} \mathfrak{k}_\alpha$ and $\mathfrak{p} = \mathfrak{a} \oplus \sum_{\alpha\in\Sigma^+} \mathfrak{p}_\alpha$ are direct sums whose components are mutually orthogonal under B_θ,

(2) $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} = \mathfrak{k}_\alpha \oplus \mathfrak{p}_\alpha$ for all $\alpha \in \Sigma^+$, and

(3) $\dim \mathfrak{g}_\alpha = \dim \mathfrak{k}_\alpha = \dim \mathfrak{p}_\alpha$ for all $\alpha \in \Sigma^+$.

The spaces $\mathfrak{n} := \bigoplus_{\alpha\in\Sigma^+} \mathfrak{g}_\alpha$ and $\mathfrak{n}^- := \bigoplus_{\alpha\in\Sigma^-} \mathfrak{g}_\alpha$ are subalgebras of $\mathfrak{g}$. Recall that $\theta(g_\lambda) = g_{-\lambda}$ by Proposition 3.7. Thus if $X \in \mathfrak{n}^-$, we see that

$$X = (X + \theta(X)) - \theta(X) \in \mathfrak{k} + \mathfrak{n}.$$

Since $\mathfrak{g}_0 = (\mathfrak{g}_0 \cap \mathfrak{k}) \oplus \mathfrak{a}$, it is clear that $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ as vector spaces. This is actually a direct sum, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$, which is called *Iwasawa decomposition* of $\mathfrak{g}$ [13, p.263] [21, p.373].

The following theorem summarizes the global Iwasawa decomposition [21, p.374].

Theorem 3.10. Let G be a real semisimple Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ be an Iwasawa decomposition. Let K, A, and N be the analytic subgroups of G with Lie algebras $\mathfrak{k}$, $\mathfrak{a}$, and $\mathfrak{n}$, respectively. Then $G = KAN$ and the map

$$(k, a, n) \mapsto kan$$

is a diffeomorphism of $K \times A \times N$ onto G. The groups A and N are simply connected.

As a consequence, if $g \in G$, then g can be written in the form $g = kan$, $k \in K$, $a \in A$, $n \in N$, and the decomposition is unique.

Example 3.11. For $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, we use the Cartan decomposition and choice of $\mathfrak{a}$ from Example 3.8, with $\mathfrak{k} = \mathfrak{su}_n(\mathbb{C})$ and $\mathfrak{a}$ the set of trace zero real diagonal matrices. As indicated in the example, we may choose $\mathfrak{a}_+$ so that $\mathfrak{n}$ consists of strictly upper triangular matrices. The local Cartan decomposition is

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}.$$

We construct the global decomposition by setting $K := \mathrm{SU}_n(\mathbb{C})$, $A := \exp \mathfrak{a}$, and $N := \exp \mathfrak{n}$. Then A is precisely the set of determinant 1 diagonal matrices with positive entries, and N is the set of unit diagonal upper triangular matrices. This choice of factors in the Iwasawa decomposition of $X \in G$ gives the QR decomposition of X; QR factors a matrix as a product of a unitary matrix Q and an upper triangular matrix R. Since $K := \mathrm{SU}_n(\mathbb{C})$ consists of unitary matrices and AN is the group of upper triangular matrices with positive diagonal elements, $Q = k(g)$ and $R = a(g)n(g)$.

Example 3.12. Let G be the real noncompact symplectic group

$$G := \mathrm{Sp}_n(\mathbb{R}) = \{g \in \mathrm{SL}_{2n}(\mathbb{R}) : g^\top J_n g = J_n\}, \qquad J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

By block multiplication, the elements of G are of the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \text{where} \quad A^\top C = C^\top A, \quad B^\top D = D^\top B, \quad A^\top D - C^\top B = I_n. \tag{3.1}$$

The Iwasawa decomposition of $G = KAN$ is given by [31, p.285]

$$\begin{aligned}
K &= \left\{ \begin{pmatrix} C & B \\ -B & C \end{pmatrix} : C + iB \in \mathrm{U}(n) \right\} = \mathrm{O}(2n) \cap \mathrm{Sp}_n(\mathbb{R}), \\
A &= \{\mathrm{diag}\,(a_1, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1}) : a_1, \ldots, a_n > 0\}, \\
N &= \left\{ \begin{pmatrix} C & B \\ 0 & (C^{-1})^\top \end{pmatrix} : C \text{ real unit upper triangular, } CB^\top = BC^\top \right\}.
\end{aligned}$$

3.6 Weyl Groups

Let G be a real semisimple Lie group with Lie algebra $\mathfrak{g}$, with a chosen Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. Let M be the centralizer of $\mathfrak{a}$ in K and M' the normalizer of $\mathfrak{a}$ in K, i.e.,

$$\begin{aligned}
M &= \{k \in K : \mathrm{Ad}\,(k)H = H \text{ for all } H \in \mathfrak{a}\}, \\
M' &= \{k \in K : \mathrm{Ad}\,(k)\mathfrak{a} \subset \mathfrak{a}\}.
\end{aligned}$$

Note that M and M' are also the centralizer and normalizer of A in K, respectively, and that they are closed Lie subgroups of K. More importantly, M is a normal subgroup of M', and the quotient group M'/M is finite because M and M' have the same Lie algebra $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$ [13, p.284]. The finite group $W := W(G, A) = M'/M$ is called the (analytically defined) *Weyl group* of G relative to A. For $w = m_w M \in W$, the linear map

$$\mathrm{Ad}\,(m_w) : \mathfrak{a} \to \mathfrak{a}$$

does not depend on the choice of $m_w \in M'$ representing w. Therefore, $w \mapsto \mathrm{Ad}\,(m_w)$ is well-defined, and we may regard $w \in W$ as the linear map $\mathrm{Ad}\,(m_w) : \mathfrak{a} \to \mathfrak{a}$ and W as a group of linear operators on $\mathfrak{a}$. In particular, it is a faithful representation of W on $\mathfrak{a}$: for if $\mathrm{Ad}\,(m_s)X = \mathrm{Ad}\,(m_t)X$ for all $X \in \mathfrak{a}$, then

$$m_s \exp(X) m_s^{-1} = m_t \exp(X) m_t^{-1} \quad \text{for all } X \in \mathfrak{a}.$$

Since $A = \exp \mathfrak{a}$, we see that $m_s a m_s^{-1} = m_t a m_t^{-1}$ for all $a \in A$, i.e., m_s and m_t are in the same coset. Thus the representation is injective.

For each root α, the reflection s_α about the hyperplane P_α with respect to the Killing form B, is a linear map on $\mathfrak{a}$ given by

$$s_\alpha(H) = H - \frac{2\alpha(H)}{\alpha(H_\alpha)} H_\alpha, \quad \text{for all } H \in \mathfrak{a},$$

where H_α is the element of $\mathfrak{a}$ representing α, i.e., $\alpha(H) = B(H, H_\alpha)$ for all $H \in \mathfrak{a}$. The group $W(\mathfrak{g}, \mathfrak{a})$ generated by $\{s_\alpha : \alpha \in \Sigma\}$ is called the (algebraically defined) *Weyl group* of $\mathfrak{g}$ relative to $\mathfrak{a}$. When viewed as groups of linear operators on $\mathfrak{a}$, the two Weyl groups $W(G, A)$ and $W(\mathfrak{g}, \mathfrak{a})$ coincide [21, p.383].

We record a few of the many remarkable properties of W in the following proposition, which we will have several occasions to use:

Proposition 3.13. [23, p.112] The Weyl group W permutes the Weyl chambers in a simply transitive fashion, i.e., if C_1 and C_2 are Weyl chambers, then there is $s \in W$ so that $s(C_1) = C_2$, and if $s \neq e_W$, then for any chamber C, $S(C) \neq C$.

Example 3.14. Let us compute the Weyl group for $\mathrm{SL}_n(\mathbb{C})$. We use the conventions chosen above for $\mathfrak{a}$ and $\mathfrak{m}$, with $\mathfrak{a}$ the set of traceless (real) diagonal matrices and $\mathfrak{m}$ the set of traceless diagonal matrices with purely imaginary entries. The subgroups generated by $\mathfrak{a}$ and $\mathfrak{m}$ are A, the subgroup of determinant 1 diagonal matrices with real positive entries, and M, the subgroup of determinant 1 diagonal matrices with entries of the form $e^{i\theta}$, $\theta \in \mathbb{R}$.

Let $\mathbf{e_j}$ be the $n \times 1$ column vector with 1 in the jth row and 0s in the remaining entries. The normalizer $M' := N_K(\mathfrak{a})$ is the set of matrices of the form

$$\begin{pmatrix} e^{i\theta_1}\mathbf{e}_{\pi(\mathbf{1})} & e^{i\theta_2}\mathbf{e}_{\pi(\mathbf{2})} & \ldots & e^{i\theta_n}\mathbf{e}_{\pi(\mathbf{n})} \end{pmatrix},$$

where $\theta_i \in \mathbb{R}$ and π is a permutation. Then $M'/M \cong S_n$.

The action of W on $\mathfrak{a}$ is given by permutation of the entries of $X \in \mathfrak{a}$; if

$$X = \operatorname{diag}(a_1, \ldots, a_n),$$

then

$$\omega(X) = \operatorname{diag}(a_{\omega(1)}, \ldots, a_{\omega(n)}), \quad \omega \in S_n.$$

Alternatively, we may identify a permutation $\omega \in S_n$ with the unique permutation matrix (also written as ω) in $\mathrm{SL}_n(\mathbb{C})$, where $\omega \mathbf{e}_i = \mathbf{e}_{\omega(i)}$. The matrix representation of ω under the standard basis is

$$\omega = [\mathbf{e}_{\omega(1)}, \cdots, \mathbf{e}_{\omega(n)}].$$

Thus if $g \in \mathrm{SL}_n(\mathbb{C})$ is written in column form, $g = [\mathbf{g}_1, \cdots, \mathbf{g}_n]$, $g\omega = [\mathbf{g}_{\omega(1)}, \cdots, \mathbf{g}_{\omega(n)}]$. Moreover, if $a = \operatorname{diag}(x_1, \ldots, x_n) \in A$, then the action of ω on a is given by

$$\omega^{-1}\operatorname{diag}(x_1, \ldots, x_n)\omega = \operatorname{diag}(x_{\omega(1)}, \ldots, x_{\omega(n)}). \tag{3.2}$$

3.7 Bruhat Decomposition

Given a real semisimple Lie group G and $s \in W$, we again denote by $m_s \in M'$ a representative such that $s = m_s M$. Moreover, for $s = 1$, we choose the identity of G for m_s.

Because of the global Iwasawa decomposition (3.10), the exponential map is a diffeomorphism from $\mathfrak{a}$, $\mathfrak{n}$, and $\mathfrak{n}^-$ onto A, N, and N^-, respectively. Thus each of A, N, and N^- is a closed subgroup of G [23, p.116]. Since M is also closed and centralizes A, and

MA normalizes N, MAN is a closed subgroup of G [13, p.403]. Thus multiplication from $M \times A \times N$ to MAN is a diffeomorphism onto MAN [21, p.460].

Now by Proposition 3.13 there is $s^* \in W$ so that $s^*(\mathfrak{a}_+) = -\mathfrak{a}_+$. Then $s^* \circ s^*(\mathfrak{a}_+) = \mathfrak{a}_+$, so again by the proposition, $s^* = (s^*)^{-1}$. Then it is clear that $\operatorname{Ad} m_{s^*}(\mathfrak{n}^\pm) = \mathfrak{n}^\mp$, so that

$$m_{s^*} N m_{s^*} = N^-; \tag{3.3}$$

alternatively, we write

$$m_{s^*} N = N^- m_{s^*}. \tag{3.4}$$

We refer to s^* as the *longest element* of W.

The following theorem is one form of the Bruhat decomposition, which relies upon the diffeomorphism outlined above and parameterizes G through W.

Theorem 3.15. [13, p.403] Let G be any noncompact semisimple Lie group. Then G is the disjoint union

$$G = \bigcup_{s \in W} MANm_s MAN. \tag{3.5}$$

If $m_s = m_{s^*}$, the term $MANm_s MAN$ is an open submanifold of G, and each term $MANm_t MAN$ with $t \neq s^*$ is a lower dimensional open submanifold.

Example 3.16. Let $G = \mathrm{SL}_n(\mathbb{R})$, the group of determinant 1 matrices with entries from $\mathbb{R}$. We may choose N (N^-) to consist of upper (respectively lower) triangular matrices with diagonal entries all 1s; MAN is the set of upper triangular matrices, and $W \cong S_n$. Then $s^* \in W$ has matrix representation

$$s^* = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & & \iddots & & \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

Notice that $s^* = (s^*)^{-1}$. It is clear that $s^*Ns^* = N^-$: let $n \in N$, i.e., of the form

$$\begin{pmatrix} 1 & * & \dots & * & * \\ 0 & 1 & \dots & * & * \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & 1 & * \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

Since s^* acts on the left by reversing the order of the columns, s^*n has form

$$\begin{pmatrix} * & * & \dots & * & 1 \\ * & * & \dots & 1 & 0 \\ \vdots & & \iddots & & \vdots \\ * & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix};$$

finally s^* acts on the right by reversing the order of the rows, so we see that s^*ns^* has form

$$\begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ * & 1 & \dots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ * & * & \dots & 1 & 0 \\ * & * & \dots & * & 1 \end{pmatrix}.$$

In particular, $(s^*ns^*)_{ij} = n_{ji}$, and we see that $s^*Ns^* = N^-$, as claimed.

For the purposes of this dissertation, as well as for the classical LU decomposition for $\mathrm{GL}_n(\mathbb{C})$, the decomposition in 3.15 is not the most convenient form. The treatment in [23] yields an alternate (but equivalent, up to the statements on open submanifolds) decomposition.

Recall that M commutes with A, and M and A normalize N and N^-. Since M' normalizes M and A, m_s normalizes M and A for any $s \in W$. Accordingly, we rewrite the terms from Theorem 3.15:

$$MANm_sMAN = NMAm_sMAN = Nm_sMAN.$$

In addition, we note that $m_sG = G$ for all $s \in W$ so that for any $s \in W$,

$$G = \bigcup_{t \in W} MANm_tMAN = \bigcup_{t \in W} m_sNm_tMAN.$$

[23, p.118]. By (3.4), we have

$$m_{s^*}Nm_tMAN = N^-m_{s^*}m_tMAN$$

for any $t \in W$ and

$$\{N^-m_{s^*}m_tMAN : t \in W\} = \{N^-m_tMAN : t \in W\}.$$

Thus the decomposition in Theorem 3.15 is equivalent to the form below.

Theorem 3.17. [23, p. 117] The real semisimple Lie group G has Bruhat decomposition

$$G = \bigcup_{s \in W} N^-m_sMAN, \tag{3.6}$$

which is a disjoint union. Moreover, N^-MAN is a diffeomorphic product and is an open subset of G, and the other cells N^-m_sMAN $(s \neq e_W)$ are lower dimensional submanifolds of G.

An immediate consequence is that, for each $g \in G$, there exists a unique $s \in W$ such that $g \in N^-m_sMAN$.

Given a decomposition of $g \in G$ according to (3.6), say $g = \bar{n} m_s m a n$, we may use the discussion preceding Theorem 3.17 to revert to the form (3.5) in Theorem 3.15; following [18],

$$\begin{aligned} m_{s^*} g &= (m_{s^*} \bar{n} m_{s^*}) m_{s^* s} m a n \\ &\in N m_{s^* s} MAN \\ &\subset MAN m_{s^* s} MAN. \end{aligned}$$

Thus the form (3.5) allows more flexibility on the choice of the MA component. However, the form (3.6) is more convenient for the ensuing calculations, thus we shall use it as the standard Bruhat decomposition in the remainder of this disseration.

Example 3.18. Let us consider the Bruhat decomposition of $\mathrm{SL}_n(\mathbb{C})$. We slightly extend our discussion from the semisimple $\mathrm{SL}_n(\mathbb{C})$ to $\mathrm{GL}_n(\mathbb{C})$, and the arguments from Example 3.14 are still applicable. Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be the standard basis of $\mathbb{C}^n$, i.e., $\mathbf{e}_i$ is a column vector with 1 as the only nonzero entry in the i-th position. Given a matrix $A \in \mathbb{C}_{n\times n}$, let $A(i|j)$ denote the submatrix formed by the first i rows and the first j columns of A, $1 \le i, j \le n$. The Bruhat decomposition of $\mathrm{SL}_n(\mathbb{C})$ is indeed reduced to the well-known Gelfand-Naimark decomposition [13, p.434].

Theorem 3.19. [16] Each $A \in \mathrm{GL}_n(\mathbb{C})$ has $A = L\omega U$, for a permutation matrix ω, a unit lower triangular matrix $L \in \mathrm{GL}_n(\mathbb{C})$, and an upper triangular $U \in \mathrm{GL}_n(\mathbb{C})$. The permutation matrix ω is uniquely determined by A:

$$\operatorname{rank} \omega(i|j) = \operatorname{rank} A(i|j) \qquad \text{for} \qquad 1 \le i, j \le n.$$

Moreover $\operatorname{diag} U$ is uniquely determined by A.

Remark 3.20. Although ω is unique in the Gelfand-Naimark decomposition $A = L\omega U$ of A, the components L and U may be not unique. The following example illustrates this ambiguity:

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

In contrast, the permutation ω' in a Gauss elimination $A = \omega' L' U'$ may be not unique, but L' and U' are uniquely determined by ω'. For example,

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}.$$

Moreover, the ω in a Gelfand-Naimark decomposition $A = L\omega U$ of A can also be a permutation in a Gauss elimination $A = \omega L' U'$ of A. To see this, we notice that $\omega^{-1} A = (\omega^{-1} L \omega) U$ and $\det[(\omega^{-1} L \omega)(k|k)] = 1$ since $(\omega^{-1} L \omega)(k|k)$ is the submatrix formed by choosing the $\omega(1), \cdots, \omega(k)$ rows and columns of L. Therefore, by the LU algorithm [19], $\omega^{-1} L \omega = L_1 U_1$ for some unit lower triangular L_1 and unit upper triangular U_1, and

$$A = L\omega U = \omega(\omega^{-1} L \omega) U = \omega L_1 (U_1 U) = \omega L' U', \tag{3.7}$$

where $L' := L_1$ and $U' := U_1 U$. We also have $\omega^{-1} A = L_1 U_1 U$. Then $u(A)$ can be computed by

$$\mathrm{det}[(\omega^{-1}\mathrm{A})(\mathrm{k}|\mathrm{k})] = \mathrm{det}[(\mathrm{L_1 U_1 U})(\mathrm{k}|\mathrm{k})] = \mathrm{det}[\mathrm{U}(\mathrm{k}|\mathrm{k})] = \prod_{\mathrm{i}=1}^{\mathrm{k}} \mathrm{u_{ii}}. \tag{3.8}$$

Remark 3.21. When ω is the identity, it is well-known [19] that the decomposition $A = LU$ is unique.

3.8 Complete Multiplicative Jordan Decomposition

Let G be a real Lie group with Lie algebra $\mathfrak{g}$. An element $g \in G$ is *elliptic* if $\mathrm{Ad}\,(g) \in \mathrm{Aut}\,\mathfrak{g}$ is diagonalizable over $\mathbb{C}$ with eigenvalues of modulus 1; an element $g \in G$ is *hyperbolic* if $g = \exp X$, where $X \in \mathfrak{g}$ is real semisimple, which is to say $\operatorname{ad} X \in \operatorname{End}\mathfrak{g}$ is diagonalizable over $\mathbb{C}$ with all eigenvalues real; an element $g \in G$ is *unipotent* if $g = \exp X$, where $X \in \mathfrak{g}$ is nilpotent (3.3).

Each $g \in G$ can be uniquely written as $g = ehu$, where e is elliptic, h is hyperbolic, u is unipotent, and the three elements e, h, u commute [22, Proposition 2.1]. This is the complete multiplicative Jordan decomposition (CMJD) of g. We write

$$g = e(g)\, h(g)\, u(g).$$

Let $\overline{\mathfrak{a}}_+$ be the closure of the Weyl chamber $\mathfrak{a}_+$, and let $\overline{A}_+$ be the closure of $A_+ := \exp \mathfrak{a}_+$. Since $\exp : \mathfrak{a} \to A$ is a diffeomorphism onto, we have $\overline{A}_+ = \exp \overline{\mathfrak{a}}_+$ [23]. In addition, diffeomorphisms preserve open sets so that A_+ is open in A. It turns out that $h \in G$ is hyperbolic if and only if it is conjugate to an element of $\overline{A}_+$; in this case, such an element of $\overline{A}_+$ is uniquely determined and we denote it by $b(h)$ [22, Proposition 2.4]. For $g \in G$, we define

$$b(g) := b(h(g)) \in \overline{A}_+.$$

Example 3.22. Follow [13, Lemma 7.1]: viewing $g \in \mathrm{SL}_n(\mathbb{R})$ as an element in $\mathfrak{gl}_n(\mathbb{R})$, the additive Jordan decomposition [19, p.153] for $\mathfrak{gl}_n(\mathbb{R})$ yields

$$g = s + n_1$$

(where $s \in \mathrm{SL}_n(\mathbb{R})$ is semisimple, that is, diagonalizable over $\mathbb{C}$, $n_1 \in \mathfrak{sl}_n(\mathbb{R})$ is nilpotent, and $sn_1 = n_1 s$). Moreover these conditions determine s and n_1 completely [20, Proposition 4.2]. Put $u := 1 + s^{-1}n_1 \in \mathrm{SL}_n(\mathbb{R})$ and we have the multiplicative Jordan decomposition

$$g = su,$$

where s is semisimple, u is unipotent, and $su = us$. By the uniqueness of the additive Jordan decomposition, s and u are also completely determined. Since s is diagonalizable,

$$s = eh,$$

where e is elliptic, h is hyperbolic, $eh = he$, and these conditions completely determine e and h. The decomposition can be obtained by observing that there is $k \in \mathrm{SL}_n(\mathbb{C})$ such that

$$k^{-1}sk = s_1 I_{r_1} \oplus \cdots \oplus s_m I_{r_m},$$

where $s_1 = e^{i\xi_1}|s_1|, \ldots, s_m = e^{i\xi_m}|s_m|$ are the distinct eigenvalues of s with multiplicities $r_1, \ldots, r_m$ respectively. Set

$$e := k(e^{i\xi_1} I_{r_1} \oplus \cdots \oplus e^{i\xi_m} I_{r_m})k^{-1}, \quad h := k(|s_1| I_{r_1} \oplus \cdots \oplus |s_m| I_{r_m})k^{-1}.$$

Since

$$ehu = g = ugu^{-1} = ueu^{-1}uhu^{-1}u,$$

the uniqueness of s, u, e and h implies e, u and h commute. Since g is fixed under complex conjugation, the uniqueness of e, h and u imply $e, h, u \in \mathrm{SL}_n(\mathbb{R})$ [13, p.431]. Thus $g = ehu$ is the CMJD for $\mathrm{SL}_n(\mathbb{R})$. The eigenvalues of h are simply the eigenvalue moduli of s and thus of g.

A matrix in $\mathrm{GL}_n(\mathbb{C})$ is called *elliptic* (respectively *hyperbolic*) if it is diagonalizable with norm 1 (respectively real positive) eigenvalues. It is called *unipotent* if all its eigenvalues are 1. The *complete multiplicative Jordan decomposition* of $g \in \mathrm{GL}_n(\mathbb{C})$ asserts that $g = ehu$ for $e, h, u \in \mathrm{GL}_n(\mathbb{C})$, where e is elliptic, h is hyperbolic, u is unipotent, and these three elements commute. The decomposition is obvious when g is in a Jordan canonical form with diagonal entries (i.e., eigenvalues) $z_1, \cdots, z_n$, in which

$$e = \mathrm{diag}\left(\frac{z_1}{|z_1|}, \cdots, \frac{z_n}{|z_n|}\right), \qquad h = \mathrm{diag}\left(|z_1|, \cdots, |z_n|\right),$$

and $u = h^{-1}e^{-1}g$ is a unit upper triangular matrix.

Chapter 4

Aluthge Iteration

Given $0 < \lambda < 1$, the λ-Aluthge transform of $X \in \mathbb{C}_{n\times n}$ [2]:

$$\Delta_\lambda(X) := P^\lambda U P^{1-\lambda}$$

has been extensively studied, where $X = UP$ is the polar decomposition of X, that is, U is unitary and P is positive semidefinite. If U and P commute, we say that X is *normal.*

The Aluthge transform can be extended to Hilbert space operators [1, 2]; see [9, 10, 29, 33, 36] for some recent research.

Define

$$\Delta_\lambda^m(X) := \Delta_\lambda(\Delta_\lambda^{m-1}(X)), \quad m \in \mathbb{N}$$

with $\Delta_\lambda^1(X) := \Delta_\lambda(X)$ and $\Delta_\lambda^0(X) := X$ so that we have the λ-Aluthge sequence $\{\Delta_\lambda^m(X)\}_{m\in\mathbb{N}}$. It is known that $\{\Delta_\lambda^m(X)\}_{m\in\mathbb{N}}$ converges if $n = 2$ [8], if the eigenvalues of X have distinct moduli [14], and if X is diagonalizable [4, 5, 6]. Very recently Antezana, Pujals and Stojanoff [7] proved the following interesting result using ideas and techniques from dynamical systems and differential geometry.

Theorem 4.1. [7, Theorem 6.1] Let $X \in \mathbb{C}_{n\times n}$ and $0 < \lambda < 1$.

1. The sequence $\{\Delta_\lambda^m(X)\}_{m\in\mathbb{N}}$ converges to a normal matrix $\Delta_\lambda^\infty(X) \in \mathbb{C}_{n\times n}$.

2. The function $\Delta_\lambda^\infty : \mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$ defined by $X \mapsto \Delta_\lambda^\infty(X)$ is continuous.

The convergence problem for $\mathbb{C}_{n\times n}$ is reduced to $\mathrm{GL}_n(\mathbb{C})$ [3] and can be further reduced to $\mathrm{SL}_n(\mathbb{C})$ since $\Delta_\lambda(cX) = c\Delta_\lambda(X)$, $c \in \mathbb{C}$.

Not much is known about the limit $\Delta_\lambda^\infty(X)$. For $X \in \mathrm{SL}_2(\mathbb{C})$ with equal eigenvalue moduli [30],

$$\Delta_\lambda^\infty(X) = \frac{\operatorname{tr} X}{2} I_2 + \frac{\sqrt{4 - \operatorname{tr} X^2}}{2\sqrt{\operatorname{tr}(XX^*) + 2\det X - \operatorname{tr} X^2}}(X - X^*).$$

Our goal in this section is to extend Theorem 4.1 to Lie groups with the right properties.

4.1 The λ-Aluthge Iteration

Let G be a real noncompact connected semisimple Lie group, and let $\mathfrak{g}$ be its Lie algebra, with $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ a fixed Cartan decomposition of $\mathfrak{g}$, and let $G = KP$ be the corresponding global decomposition as in Theorem 3.4. Given $0 < \lambda < 1$, the λ-Aluthge transform of $\Delta_\lambda : G \to G$ is defined as

$$\Delta_\lambda(g) := p^\lambda k p^{1-\lambda},$$

where

$$p^\lambda := \exp(\lambda X) \in P$$

if $p = \exp X$ for some $X \in \mathfrak{p}$. The map $(0,1) \times G \to G$ defined by $(\lambda, g) \mapsto \Delta_\lambda(g)$ is smooth; thus $\Delta_\lambda : G \to G$ is smooth [15]. We define

$$\Delta_\lambda^m(g) := \Delta_\lambda(\Delta_\lambda^{m-1}(g)),$$

with $\Delta_\lambda^1(g) := \Delta_\lambda(g)$ and $\Delta_\lambda^0(g) := g$ so that we have the *generalized λ-Aluthge sequence* $\{\Delta_\lambda^m(g)\}_{m\in\mathbb{N}}$. Clearly $\Delta_\lambda(g) = p^\lambda g (p^\lambda)^{-1}$ so that all members of the Aluthge sequence are in the same conjugacy class.

Lemma 4.2. [15] Let G be a real connected noncompact semisimple Lie group with Lie algebra $\mathfrak{g}$. For any irreducible representation π of G and $0 < \lambda < 1$,

$$\pi \circ \Delta_\lambda = \Delta_\lambda \circ \pi,$$

where Δ_λ on the left is the Aluthge transform of $g \in G$ with respect to the Cartan decomposition of G as $G = KP$, and Δ_λ on the right is the Aluthge transform of $\pi(g) \in \mathrm{GL}(V_\pi)$ with respect to the polar decomposition.

Proof. Let $g = kp$ with $k \in K$, $p \in P$ with $p = \exp(X)$ for some $X \in \mathfrak{p}$. Then

$$\pi(p^\lambda) = \pi(\exp(\lambda X)) = \exp(d\pi(\lambda X)) = \exp(\lambda d\pi(X)) = (\pi(p))^\lambda$$

by 2.1.

Since π is an irreducible representation, the Cartan decomposition of $\pi(g)$ is $\pi(k)\pi(p)$ [15]. So

$$\begin{aligned}
\Delta_\lambda \circ \pi(g) &= \Delta_\lambda(\pi(k)\pi(p)) \\
&= \pi(p)^\lambda \pi(k) \pi(p)^{1-\lambda} \\
&= \pi(p^\lambda)\pi(k)\pi(p^{1-\lambda}) \\
&= \pi(p^\lambda k p^{1-\lambda}) \\
&= \pi \circ \Delta_\lambda(g).
\end{aligned}$$

□

4.2 Normal Elements

An element $g \in G$ is said to be *normal* if $kp = pk$, where $g = kp$ ($k \in K$ and $p \in P$) is the Cartan decomposition of g. Recall that the center Z of G is contained in K 3.4. So $g \in G$ is normal if and only if zg is normal for all $z \in Z$.

Equip $\mathfrak{g}$ once and for all with an inner product [21, p.360] such that the operator $\mathrm{Ad}\, k \in \mathrm{GL}(\mathfrak{g})$ on $\mathfrak{g}$ is orthogonal for all $k \in K$, and $\mathrm{Ad}\, p \in \mathrm{GL}(\mathfrak{g})$ is positive definite for all $p \in P$. Notice that $\mathrm{Ad}\, G = (\mathrm{Ad}\, K)(\mathrm{Ad}\, P)$ is the polar decomposition of $\mathrm{Ad}\, G \subset \mathrm{GL}(\mathfrak{g})$.

Lemma 4.3. 1. The element $g \in G$ is normal if and only if $\mathrm{Ad}\, g \in \mathrm{GL}(\mathfrak{g})$ is normal.

2. Let $0 < \lambda < 1$. An element $g \in G$ is normal if and only if g is invariant under Δ_λ.

Proof. (1) One implication is trivial. For the other implication, consider $g = kp$ such that $\mathrm{Ad}\, g$ is normal, i.e., $\mathrm{Ad}\,(kp) = \mathrm{Ad}\,(pk)$. Since $\ker \mathrm{Ad} = Z \subset K$, $kp = pkz$ where $z \in Z$, i.e., $kpk^{-1} = zp$. Now $kpk^{-1} \in P$ because $\mathfrak{p}$ is invariant under $\mathrm{Ad}\, k$ for all $k \in K$. By the uniqueness of the Cartan decomposition, $z = 1$ and $kpk^{-1} = p$, i.e., $kp = pk$.

(2) Suppose that g is normal, with Cartan decomposition $g = kp$. Then $kpk^{-1} = p$ so that $\exp(\mathrm{Ad}\,(k)X) = \exp X$. Since $\mathrm{Ad}\,(k)\mathfrak{p} = \mathfrak{p}$ and the map $\exp : \mathfrak{p} \to P$ is a diffeomorphism by Theorem 3.4, we have $\mathrm{Ad}\,(k)X = X$. Thus $\mathrm{Ad}\,(k)(tX) = tX$ for all $t \in \mathbb{R}$ so that $kp^\top k^{-1} = p^\top$, i.e., $kp^\top = p^\top k$. As a result $\Delta_\lambda(g) = p^\lambda k p^{1-\lambda} = g$. Conversely if $g = kp$ is invariant under Δ_λ, then $p^\lambda k p^{1-\lambda} = kp$, i.e., $p^\lambda k = kp^\lambda$. So $\exp(\mathrm{Ad}\,(k)\lambda X) = \exp(\lambda X)$ where $\exp X = p$. Again using the diffeomorphism of $\mathfrak{p}$ onto P, $\mathrm{Ad}\,(k)\lambda X = \lambda X$ so that $\mathrm{Ad}\,(k)X = X$ and thus $kp = pk$. □

Lemma 4.4. Let G be a noncompact connected semisimple Lie group and $g \in G$, and let $\varphi : G \to G$ be a diffeomorphism such that $\varphi(cg) = c\varphi(g)$ for each $c \in Z$, where Z is the center of G. If $\{\mathrm{Ad}\, \varphi^m(g)\}_{m\in\mathbb{N}}$ converges to L so that $\mathrm{Ad}^{-1}(L)$ contains some fixed point ℓ of φ, then $\{\varphi^m(g)\}_{m\in\mathbb{N}}$ converges to an element $\varphi^\infty(g) \in G$.

Proof. If $\{\mathrm{Ad}\, \varphi^m(g)\}_{m\in\mathbb{N}}$ converges, then the limit L is of the form $\mathrm{Ad}\, \ell$ for some $\ell \in G$ since $\mathrm{Ad}\, G$ is closed in $\mathrm{GL}(\mathfrak{g})$ [13, p.132]. We may assume that ℓ is a fixed point of φ. Since $(G, \mathrm{Ad}\,)$ is a covering group of $\mathrm{Ad}\, G$ [13, p.272], there is a (local) homeomorphism, induced by Ad, between neighborhoods U of ℓ and $\mathrm{Ad}\, U$ of $\mathrm{Ad}\, \ell$. Thus there is a sequence $\{g_m\}_{m\in\mathbb{N}} \subset U$ converging to ℓ and $\mathrm{Ad}\, g_m = \mathrm{Ad}\, \varphi^m(g)$. Since $\ker \mathrm{Ad} = Z$, there is a sequence $\{z_m\}_{m\in\mathbb{N}} \in Z$ such that $g_m = z_m\varphi^m(g)$, and

$$\lim_{m\to\infty} z_m\varphi^m(g) = \ell. \tag{4.1}$$

Apply φ on (4.1) to have

$$\lim_{m\to\infty} z_m\varphi^{m+1}(g) = \varphi(\ell) = \ell.$$

Hence

$$\lim_{m\to\infty} z_{m+1}z_m^{-1} = 1,$$

where $1 \in G$ denotes the identity element. The converging sequence $\{z_{m+1}z_m^{-1}\}_{m\in\mathbb{N}}$ is contained in the center Z which is discrete [13, p.116]. So $z_{m+1} = z_m = z$ (say) for sufficiently large $m \in \mathbb{N}$. Hence $\{\varphi^m(g)\}_{m\in\mathbb{N}}$ converges to $\varphi^\infty(g) := \ell z^{-1}$. $\square$

4.3 Asymptotic Behavior of the Aluthge Sequence

The main result in this chapter is

Theorem 4.5. [32] Let G be a real connected noncompact semisimple Lie group, and let $g \in G$. Let $0 < \lambda < 1$.

1. The λ-Aluthge sequence $\{\Delta_\lambda^m(g)\}_{m\in\mathbb{N}}$ converges to a normal $\Delta_\lambda^\infty(g) \in G$.
2. The map $\Delta_\lambda^\infty : G \to G$ defined by $g \mapsto \Delta_\lambda^\infty(g)$ is continuous.

Proof. (1) By Lemma 4.2,

$$\mathrm{Ad}\,(\Delta_\lambda^m(g)) = \Delta_\lambda^m(\mathrm{Ad}\,(g)), \qquad m \in \mathbb{N}, \tag{4.2}$$

where Δ_λ on the left is the Aluthge transform of $g \in G$ with respect to the Cartan decomposition $G = KP$ and that on the right is the matrix Aluthge transform of $\mathrm{Ad}\,(g) \in \mathrm{Ad}\,G \subset \mathrm{GL}(\mathfrak{g})$ with respect to the polar decomposition. By Theorem 4.1 $\{\Delta_\lambda^m(\mathrm{Ad}\,(g))\}_{m\in\mathbb{N}}$ converges to a normal $\mathrm{Ad}\,\ell$ for some $\ell \in G$ since $\mathrm{Ad}\,G$ is closed in $\mathrm{GL}(\mathfrak{g})$ [13, p.132]; so does $\{\mathrm{Ad}\,(\Delta_\lambda^m(g))\}_{m\in\mathbb{N}}$. Since ℓ is normal by Lemma 4.3, ℓ is fixed by Δ_λ. Moreover central elements factor out of Δ_λ so that Lemma 4.4 applies immediately, i.e., $\{\Delta_\lambda^m(g)\}_{m\in\mathbb{N}}$ converges to the normal $\Delta_\lambda^\infty(g) := \ell z^{-1} \in G$.

(2) By 4.2,

$$\begin{aligned}
\mathrm{Ad}\,(\Delta_\lambda^\infty(g)) &= \mathrm{Ad}\,(\lim_{m\to\infty}\Delta_\lambda^m(g)) = \lim_{m\to\infty}\mathrm{Ad}\,(\Delta_\lambda^m(g)) \\
&= \lim_{m\to\infty}\Delta_\lambda^m(\mathrm{Ad}\,(g)) = \Delta_\lambda^\infty(\mathrm{Ad}\,(g)).
\end{aligned}$$

So

$$\mathrm{Ad}\circ\Delta_\lambda^\infty = \Delta_\lambda^\infty\circ\mathrm{Ad}\,. \tag{4.3}$$

The $\Delta_\lambda^\infty : \mathrm{GL}(\mathfrak{g}) \to \mathrm{GL}(\mathfrak{g})$ on the right of (4.3) is continuous by Theorem 4.1(b), thus $\mathrm{Ad}\circ\Delta_\lambda^\infty$ is continuous. Since $\mathrm{Ad}\,G \cong G/Z$ [13, p.129], $\mathrm{Ad} : G \to \mathrm{Ad}\,G$ on the left of (4.3) is an open map [13, p.123], [27, p.97]. Hence $\Delta_\lambda^\infty : G \to G$ is continuous.

□

Chapter 5

Bruhat Iteration

5.1 Rutishauser's LR algorithm

Suppose that $A \in \mathrm{GL}_n(\mathbb{C})$ with LU decomposition $A = LU$, where $L \in \mathrm{GL}_n(\mathbb{C})$ is unit lower triangular and $U \in \mathrm{GL}_n(\mathbb{C})$ is upper triangular. Rutishauser's LR algorithm [26] asserts that if $A \in \mathrm{GL}_n(\mathbb{C})$ has distinct eigenvalue moduli, then under certain conditions the following iteration

$$\begin{aligned} A_s &= L_s U_s, \\ A_{s+1} &= U_s L_s = L_{s+1} U_{s+1}, \quad s = 1, 2... \end{aligned}$$

converges to an upper triangular matrix. The result is significant since $A_{s+1} = L_s^{-1} A_s L_s$ so that eigenvalues are preserved during the whole algorithm. Since eigenvalues are continues functions, Rutishauser's algorithm provides a means to approximate eigenvalues of A.

Theorem 5.1. [35] Let $A \in \mathrm{GL}_n(\mathbb{C})$ such that A_s admits LU decomposition for all $s \in \mathbb{N}$ and so that the moduli of the eigenvalues $\lambda_1, \ldots, \lambda_n$ of A are distinct, i.e.,

$$|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n| \ (> 0). \tag{5.1}$$

Let $A = X \mathrm{diag}\,(\lambda_1, \ldots, \lambda_n) X^{-1}$. Assume that X and X^{-1} admit LU decomposition. Then the sequence $\{A_s\}_{s \in \mathbb{N}}$ converges and the limit $\lim_{s \to \infty} A_s$ is an upper triangular matrix R in which $\mathrm{diag}\, R = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_n)$.

J.H. Wilkinson praised the LR iteration as "algorithmic genius" [33, p.vii] and "the most significant advance which has been made in connection with the eigenvalue problem since the advent of automatic computers" with the understanding that "The QR algorithm, which was developed later by Francis, is closely related to the LR algorithm but is based on the use of unitary transformations. In many respects this has proved to be the most effective of known methods for the solution of the general algebraic eigenvalues problem." [33, p.487-488]. See the comments in [12].

Indeed Theorem 5.1 may be generalized due to the argument in [33, p.521-522]; with careful reading (see Remark 5.10), we deduce the theorem below.

Theorem 5.2. Let $A \in \mathrm{GL}_n(\mathbb{C})$ such that A_s admits LU decomposition for all $s \in \mathbb{N}$ and the moduli of the eigenvalues $\lambda_1, \ldots, \lambda_n$ of A are distinct. Let $A = X \operatorname{diag}(\lambda_1, \ldots, \lambda_n) X^{-1}$. Assume that $PY = L_Y U_Y$ ($Y := X^{-1}$) and $XP^\top = L_X U_X$ admit LU decomposition where P is a permutation matrix corresponding to the permutation ω. Then the sequence $\{A_s\}_{s\in\mathbb{N}}$ converges and $\lim_{s\to\infty} A_s$ is an upper triangular matrix R in which $\operatorname{diag} R = \operatorname{diag}(\lambda_{\omega(1)}, \ldots, \lambda_{\omega(n)})$.

The QR algorithm does not imply that its iterates converge to an upper triangular matrix. However, it does assert that, under the assumption of distinct eigenvalue moduli of $A \in \mathrm{GL}_n(\mathbb{C})$, its iterates are "convergent" to an upper triangular form. The LR algorithm, on the other hand, *does* imply convergence, not just form convergence, towards an upper triangular matrix R whose diagonal entries display the eigenvalues of the original matrix. Very recently the QR algorithm has been extended in the context of semisimple Lie groups [18]; see Remark 5.5 for the comparison. We now extend the LR algorithm in the same fashion, and give explicit examples. We remark that the decomposition $PY = L_Y U_Y$ is the matrix version of Gaussian elimination; none of the components P, L_Y, U_Y in the decomposition is unique. However, the Gelfand-Naimark decomposition $Y = L\omega U$ yields a unique ω, where L is unit lower triangular and U is upper triangular. The role of the Gelfand-Naimark decomposition will be played by the Bruhat decomposition in the context of semisimple groups, as we will see in the next section.

5.2 Regular elements

Let G denote a connected real semisimple Lie group with a fixed Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ of the semisimple Lie algebra $\mathfrak{g}$. Let $K \subset G$ be the connected subgroup corresponding to $\mathfrak{k}$. Recall that K is closed. Since $\operatorname{Ad} K \subset \mathrm{O}(\mathfrak{g})$, which is compact, $\operatorname{Ad} K$ is itself a maximal compact subgroup of $\operatorname{Ad} G$ [13, Lemma 1.2, p.253].

Recall that $\overline{\mathfrak{a}}_+$ is the closure of the Weyl chamber $\mathfrak{a}_+$, that $A_+ := \exp \mathfrak{a}_+$, and that $\overline{A}_+ = \exp \overline{\mathfrak{a}}_+$ is the closure of A_+. An element $b \in \overline{A}_+$ is *regular* if $\alpha(\log b) > 0$ for all positive roots α, that is, b is in A_+.

When $G = \mathrm{SL}_n(\mathbb{C})$ or $\mathrm{SL}_n(\mathbb{R})$, the CMJD of $g \in G$ is given in Example 3.22; see [13, p. 430-431]. In this case, $\mathrm{b}(g)$ is regular if and only if g has distinct eigenvalue moduli, which implies that g is diagonalizable, that is, the unipotent part $u(g) = 1$. The following proposition is an extension of this result in the context of a connected real semisimple Lie group G.

Proposition 5.3. [18] Let $g \in G$ such that $b(g) \in A_+$ is regular. Then the unipotent component $u(g)$ in the CMJD of g is the identity and there is $x \in G$ such that $xh(g)x^{-1} = b(g)$ and $xe(g)x^{-1} \in M$.

Let Σ^+ denote the set of positive roots with respect to $\mathfrak{a}_+$, and Σ^- the set of negative roots. For any root α, recall that g_α is the associated root space, and that

$$\mathfrak{n}^- := \sum_{\alpha \in \Sigma^-} g_\alpha.$$

Given $H \in \overline{\mathfrak{a}}_+$, set

$$\mathfrak{n}_H^0 := \sum_{\alpha \in P_H^0} g_\alpha, \text{ where } P_H^0 := \{\alpha \in \Sigma^- : \alpha(H) = 0\},$$

and

$$\mathfrak{n}_H := \sum_{\alpha \in P_H} g_\alpha, \text{ where } P_H := \{\alpha \in \Sigma^- : \alpha(H) < 0\}.$$

Thus $\mathfrak{n}^-$ decomposes as $\mathfrak{n}^- = \mathfrak{n}_H^0 \oplus \mathfrak{n}_H$. Define the map

$$\pi_H^0 : \mathfrak{n}^- \to \mathfrak{n}_H^0$$

by projection onto the first summand. If $H \in \mathfrak{a}_+$, then $P_H^0 = \emptyset$, $\mathfrak{n}_H^0 = 0$, and π_H^0 maps $\mathfrak{n}^-$ to 0.

Lemma 5.4. [18] Let $b \in \overline{A}_+$ and $\ell \in N^-$. Denote $H := \log b \in \overline{\mathfrak{a}}_+$ and $L := \log \ell \in \mathfrak{n}^-$. Then

$$\lim_{i \to \infty} b^i \ell b^{-i} = \exp \pi_H^0(L) \in N^-.$$

In particular, if b is regular, then $\lim\limits_{i \to \infty} b^i \ell b^{-i} = 1$.

Proof. Let $i > 0$, and set $L_i := e^{\mathrm{ad}\,(iH)}(L)$. By (2.2) and (2.3),

$$\begin{aligned} b^i \ell b^{-i} &= \exp(\mathrm{Ad}\,(b^i)(L)) \\ &= \exp(\mathrm{Ad}\,(e^{iH})(L)) \\ &= \exp(e^{\mathrm{ad}\,(iH)}(L)) \\ &= \exp L_i. \end{aligned}$$

For any root α and $L_\alpha \in \mathfrak{g}_\alpha$, we see that

$$\mathrm{ad}\,(iH)(L_\alpha) = i\mathrm{ad}\,(H)(L_\alpha) = i\alpha(H)(L_\alpha).$$

Then

$$e^{\mathrm{ad}\,(iH)}(L_\alpha) = \sum_{j=0}^{\infty} \frac{(\mathrm{ad}\,(iH))^j}{j!}(L_\alpha) = \sum_{j=0}^{\infty} \frac{(i\alpha(H))^j}{j!} L_\alpha = e^{i\alpha(H)} L_\alpha.$$

Now $L \in \mathfrak{n}^-$, so that it may be decomposed as

$$L = \sum_{\alpha \in \Sigma^-} L_\alpha$$

with $L_\alpha \in \mathfrak{g}_\alpha$. Then

$$L_i = e^{\mathrm{ad}\,(iH)}(L) = \sum_{\alpha \in \Sigma^-} e^{i\alpha(H)} L_\alpha.$$

Now for all $\alpha \in \Sigma^-$, either $\alpha(H) = 0$ or $\alpha(H) < 0$ so that

$$\lim_{i \to \infty} b^i \ell b^{-i} = \lim_{i \to \infty} \exp L_i = \exp \lim_{i \to \infty} L_i = \exp \pi_H^0(L).$$

□

Lemma 5.5. [18] Let $\{x_i\}_{i \in \mathbb{N}}$ and $\{y_i\}_{i \in \mathbb{N}}$ be two sequences in G, such that $\lim_{i \to \infty} x_i = 1$ and $\{\mathrm{Ad}\,(y_i)\}_{i \in \mathbb{N}}$ is in a compact subset of $\mathrm{Ad}\,(G)$. Then

$$\lim_{i \to \infty} y_i x_i y_i^{-1} = 1.$$

Proof. Recall that $\exp : \mathfrak{g} \to G$ is a diffeomorphism in a neighborhood of the identity. Since $\lim_{i \to \infty} x_i = 1$, there exist $N \in \mathbb{Z}^+$ and $X_i \in \mathfrak{g}$ (for $i > N$) so that if $i > N$, $x_i = \exp X_i$. Thus $\lim_{i \to \infty} X_i = 0$, and for all $i > N$, $y_i x_i y_i^{-1} = \exp(\mathrm{Ad}\,(y_i) X_i)$.

By way of contradiction, suppose that $\lim_{i \to \infty} y_i x_i y_i^{-1} \neq 1$. Then $\lim_{i \to \infty}(\mathrm{Ad}\,(y_i) X_i) \neq 0$, so that there is an open neighborhood $U \subset \mathfrak{g}$ of 0 and a subsequence $\{t_i\}_{i \in \mathbb{N}}$ of $\mathbb{N}$ so that $t_i > N$ and $\mathrm{Ad}\,(y_{t_i}) X_{t_i} \notin U$ for all $i \in \mathbb{N}$. However, since $\{\mathrm{Ad}\,(y_{t_i})\}$ is in a compact subset of $\mathrm{Ad}\, G$, there is a subsequence $\{s_i\}_{i \in \mathbb{N}}$ of $\{t_i\}_{i \in \mathbb{N}}$ and $y \in G$ so that $\lim_{i \to \infty} y_{s_i} = \mathrm{Ad}\, y$. Thus $\lim_{i \to \infty} \mathrm{Ad}\,(y_{s_i}) X_{s_i} = \mathrm{Ad}\,(y)(0) = 0$, which contradicts the assumption $\mathrm{Ad}\,(y_{t_i}) X_{t_i} \notin U$ for all $i \in \mathbb{N}$. Thus $\lim_{i \to \infty} y_i x_i y_i^{-1} = 1$.

□

Proposition 5.6. Given $\omega \in W$, set $G_\omega := N^- m_\omega MAN$. Then $m_\omega^{-1} G_\omega \subset N^- MAN$.

Proof. Let $g := \bar{n} m_\omega man \in G_\omega$, with $\bar{n} \in N^-$, $m \in M$, $a \in A$, and $n \in N$. By [23, p.117]

$$m_\omega N^- m_\omega^{-1} = N_1 N_2$$

is a diffeomorphic product, where N_1 and N_2 are Lie subgroups of N^- and N, respectively. So

$$m_\omega^{-1} \bar{n}^{-1} m_\omega = \bar{n}' n'$$

with $\bar{n}' \in N^-$ and $n \in N$. Then

$$m_\omega^{-1} \bar{n} m_\omega man = \bar{n}' n' man = \bar{n}' man'' \tag{5.2}$$

with $n'' \in N$ since MA normalizes N. Thus (5.2) is the Bruhat decomposition of $m_\omega^{-1} g \in N^- MAN$.

□

5.3 Bruhat iteration

Let $g = g_1 = \bar{n} man$ be the Bruhat decomposition of $g \in N^- MAN$, where $\bar{n} \in N^-$ and $man \in MAN$. Set $\bar{n}_1 := \bar{n}$ and $u_1 := man$ so that $g_1 = \bar{n}_1 u_1$. Define

$$B(g_1) := u_1 \bar{n}_1 = \bar{n}_1^{-1} g_1 \bar{n}_1$$

and the Bruhat iteration recursively:

$$g_s = B(g_{s-1}) = \bar{n}_{s-1}^{-1} g_{s-1} \bar{n}_{s-1}, \quad s \in \mathbb{N}.$$

Theorem 5.7. Suppose that $g \in G$ with $b(g)$ regular, $\{g_s\}_{s \in \mathbb{N}} \subset N^- MAN$, and $g = xcbx^{-1}$ as in Proposition 5.3. If $x^{-1} = l m_\omega p$, $l \in N^-$, $\omega \in W$, $p \in MAN$, and $x m_\omega \in N^- MAN$, then $g^s \in N^-MAN$. Moreover, there is $p' \in MAN$ so that

$$\lim_{s\to\infty} g_s = p' m_\omega^{-1}(bc) m_\omega (p')^{-1} \in MAN.$$

The MA-component of the limit is $\omega \cdot (bc)$.

Proof. Note that

$$g_s = \bar{n}_{s-1}^{-1} g_{s-1} \bar{n}_{s-1} = \bar{n}_{s-1}^{-1} \cdots \bar{n}_1^{-1} g_1 \bar{n}_1 \cdots \bar{n}_{s-1}. \tag{5.3}$$

Set $t_s := \bar{n}_1 \cdots \bar{n}_s$ and $r_s := u_s \cdots u_1$. Then

$$t_{s-1} g_s = g_1 t_{s-1}. \tag{5.4}$$

Now consider

$$\begin{aligned} t_s r_s &= \bar{n}_1 \cdots \bar{n}_{s-1} (\bar{n}_s u_s) u_{s-1} \cdots u_1 \\ &= \bar{n}_1 \cdots \bar{n}_{s-1} g_s u_{s-1} \cdots u_1 \\ &= g_1 \bar{n}_1 \cdots \bar{n}_{s-1} u_{s-1} \cdots u_1 \\ &= g_1 t_{s-1} u_{s-1}. \end{aligned}$$

Repeated application of this result yields $t_s u_s = g_1^s$, i.e., $g_1^s \in N^-MAN$ for all $s \in N$.

As $t_s \in N^-$ and $r_s \in MAN$ the Bruhat decomposition of g_1^s is given by

$$g_1^s = t_s r_s. \tag{5.5}$$

Let $g_1 = ehu$ be the complete multiplicative Jordan decomposition of g_1. As g_1 is regular, there is an $x \in G$ so that $b := xhx^{-1} \in A_+$. By [18], $u = 1$ and $c := xex^{-1} \in M$, so that $x^{-1} g_1 x = cb \in MA_+$.

Suppose that $x^{-1} = lm_\omega p$, $l \in N^-$, $\omega \in W$, and $p \in MAN$. We want to show that

$$\lim_{s\to\infty} c^s b^s l b^{-s} c^{-s} = 1.$$

Since $c \in M \subset K$, $\{\operatorname{Ad} c^s\}_{s\in\mathbb{N}}$ is contained in the compact set $\operatorname{Ad} K$. By Lemma 5.5 we need only consider the sequence $\{b^s l b^{-s}\}_{s\in\mathbb{N}}$. By Lemma 5.4,

$$\lim_{s\to\infty} b^s l b^{-s} = 1$$

since b is regular. Set

$$l_s := c^s b^s l b^{-s} c^{-s}, \tag{5.6}$$

which is in n^- since MA normalizes N^-. Now

$$g^s = xc^s b^s x^{-1} = xc^s b^s l (c^s b^s)^{-1} c^s b^s m_\omega p = xl_s c^s b^s m_\omega p. \tag{5.7}$$

Since M' normalizes MA, $c^s b^s m_\omega = m_\omega c_s b_s$ for some $c_s \in M$, $b_s \in A$. We have

$$xl_s c^s b^s m_\omega p = xl_s m_\omega c_s b_s p,$$

with $k_s := c_s b_s p \in MAN$. Then (5.7) can be rewritten as

$$g^s = xl_s m_\omega c_s b_s p = xl_s m_\omega k_s. \tag{5.8}$$

Since $l_s \in N^-$ and $k_s \in MAN$, $l_s m_\omega k_s$ is in the cell G_ω. Then by Proposition 5.2 $m_\omega^{-1} l_s m_\omega k_s \in N^- MAN$, i.e.,

$$m_\omega^{-1} l_s m_\omega k_s = l'_s k'_s$$

with $l'_s \in N^-$ and $k'_s \in MAN$. By Proposition 5.6, $m_\omega^{-1} l_s m_\omega = \bar{l}_s p_s$, with $\bar{l}_s \in N^-$ and $p_s \in N$. Since $l_s \to 1$, $m_\omega^{-1} l_s m_\omega = \bar{l}_s p_s \to 1$ as well. By the diffeomorphic product of

N^-MAN, both $\bar{l}_s$ and p_s approach 1. Thus we rewrite (5.8) as

$$g^s = xl_s m_\omega k_s = xm_\omega(m_\omega^{-1} l_s m_\omega)k_s = xm_\omega \bar{l}_s p_s k_s.$$

By the assumption, $xm_\omega \in N^-MAN$, say $xm_\omega = l'p'$. We have $xm_\omega \bar{l}_s p_s k_s = l'p'\bar{l}_s p_s k_s$. Since $l' \in N^-$, $p' \in MAN$, $\bar{l}_s \in N^-$ with $\bar{l}_s \to 1$, and $p_s \to 1$, we have

$$t_s r_s k_s^{-1} \to l'p'.$$

Since the map $N^- \times M \times A \times N \to N^-MAN$ is a diffeomorphism, and $t_s \in N^-$, $r_s k_s^{-1} \in MAN$, $t_s r_s k_s^{-1} \to l'p'$ implies that $t_s \to l'$ and $r_s k_s^{-1} \to p'$ as $s \to \infty$. Since $g_s = t_{s-1}^{-1} g_1 t_{s-1}$ from (5.4), we have

$$g_s \to (l')^{-1} g_1 l' = (xm_\omega(p')^{-1})^{-1} g_1 (xm_\omega(p')^{-1}) = p' m_\omega^{-1}(bc)m_\omega p'.$$

□

We record as a corollary the special case when $\omega = 1$, i.e., x^{-1} is in the large cell.

Corollary 5.8. Suppose that $g \in G$ with $b(g)$ regular, $\{g_s\}_{s\in\mathbb{N}} \subset N^-MAN$, and $g = xcbx^{-1}$ as in Proposition 5.3. If x^{-1} and $x \in N^-MAN$, then $g^s \in N^-MAN$. Moreover, there is $p' \in MAN$ so that

$$\lim_{s\to\infty} g_s = p'(bc)(p')^{-1} \in N.$$

The MA-component of the limit is bc.

Remark 5.9. We remark that the large cell N^-MAN is not closed under inversion. For example, for $G = \mathrm{SL}_2(\mathbb{C})$, note that and

$$A = LU = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

but

$$(LU)^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix},$$

which is not in the large cell.

Remark 5.10. The proofs in the literature of the convergence statements for the LR iteration are not purely group theoretic since they usually make use of the embedding of $\mathrm{GL}_n(\mathbb{C})$ in $\mathbb{C}_{n\times n}$ to accommodate matrix addition at some point (see [33, p.521] for example). Two proofs of the convergence statement are given in [33]. The first is computation-intensive, while the second is much more similar to the proof given here, except that it uses Gaussian elimination instead of Gelfand-Naimark decomposition. However, both proofs are sketched out roughly, and are missing important details, for example, Equation (33.2) in [33, p.521]. Our proof of the generalization only involves purely group theoretic arguments.

Remark 5.11. For $x^{-1} \in N^- m_\omega MAN$, the condition $xm_\omega \in N^-MAN$ is essential to guarantee convergence in Theorem 5.7. Consider Rutishauser's matrix [26, p.52]

$$A_1 := \begin{pmatrix} 1 & -1 & 1 \\ 4 & 6 & -1 \\ 4 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}.$$

Setting

$$x := \begin{pmatrix} 0 & -1 & -1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix},$$

we see that x^{-1} has LU decomposition

$$x^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix},$$

i.e., x^{-1} is in the largest cell, while x is *not* in the largest cell.

We claim that the sequence A_s remains in the largest cell, but diverges. To this end we first establish that

$$A_{s+1} = \begin{pmatrix} 1 & \frac{-1}{5^s} & 1 \\ 4 \cdot 5^s & 6 & -5^s \\ 4 & \frac{4}{5^s} & 1 \end{pmatrix}$$

by induction. A routine calculation shows that

$$A_2 = \begin{pmatrix} 1 & \frac{-1}{5} & 1 \\ 20 & 6 & -5 \\ 4 & \frac{4}{5} & 1 \end{pmatrix}.$$

Suppose that

$$A_s = \begin{pmatrix} 1 & \frac{-1}{5^{s-1}} & 1 \\ 4 \cdot 5^{s-1} & 6 & -5^{s-1} \\ 1 & \frac{-1}{5^{s-1}} & 1 \end{pmatrix}.$$

Then A_s has LU decomposition

$$A_s = \begin{pmatrix} 1 & 0 & 0 \\ 4 \cdot 5^{s-1} & 1 & 0 \\ 4 & \frac{4}{5^s} & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{-1}{5^{s-1}} & 1 \\ 0 & 10 & -5^s \\ 0 & 0 & 1 \end{pmatrix}.$$

So

$$A_{s+1} = \begin{pmatrix} 1 & \frac{-1}{5^{s-1}} & 1 \\ 0 & 10 & -5^s \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 4 \cdot 5^{s-1} & 1 & 0 \\ 4 & \frac{4}{5^s} & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{-1}{5^s} & 1 \\ 4 \cdot 5^s & 6 & -5^s \\ 4 & \frac{4}{5^s} & 1 \end{pmatrix}.$$

The sequence obviously diverges since $4 \cdot 5^s \to \infty$.

Remark 5.12. In Theorem 5.2, the condition is given in terms of Gaussian elimination $PY = L_Y U_Y$ instead of Gelfand-Naimark decomposition $Y = L\omega U$. See [16] for some comparison of the two decompositions. Although ω is unique in the Gelfand-Naimark decomposition $Y = L\omega U$, the components L and U may be not unique. The permutation P may be not unique, but L_Y and U_Y are uniquely determined by the permutation matrix P. Moreover, ω can also be a permutation in a Gauss elimination $Y = \omega L'U'$ [16].

Example 5.13. Consider the real symplectic group ([28, p.129]):

$$G := \mathrm{Sp}_n(\mathbb{R}) = \{g \in \mathrm{SL}_{2n}(\mathbb{R}) : g^\top J_n g = J_n\}, \qquad J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Recall that the elements of G are of the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad A^\top C = C^\top A, \quad B^\top D = D^\top B, \quad A^\top D - C^\top B = I_n \tag{5.9}$$

[28, p.128], and that the Cartan decomposition of G is given by

$$\begin{aligned} K &= \left\{ \begin{pmatrix} C & B \\ -B & C \end{pmatrix} : C + iB \in \mathrm{U}(n) \right\} = O(2n) \cap \mathrm{Sp}_n(\mathbb{R}), \\ A &= \{\mathrm{diag}\,(a_1, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1}) : a_1, \ldots, a_n > 0\}, \\ N &= \left\{ \begin{pmatrix} C & B \\ 0 & (C^{-1})^\top \end{pmatrix} : C \text{ unit upper triangular, } CB^\top = BC^\top \right\} \end{aligned}$$

The centralizer M of A in K is the group of the diagonal matrices in K, i.e., the group of matrices of the form $\mathrm{diag}\,(C, C)$, where $C = \mathrm{diag}\,(\pm 1, \pm 1, \ldots, \pm 1)$ (independent signs here and below). The normalizer M' of A in K is $W'M$ where W' is generated by

$$\begin{gathered} \{E_{k,n+k} - E_{n+k,k} + \textstyle\sum_{i \neq k, n+k} E_{ii} \ : \ k = 1, \ldots, n\} \\ \cup \{\mathrm{diag}\,(C, C) \ : \ C \text{ is a permutation matrix}\}. \end{gathered}$$

Note that $W'M/M \simeq W'/(W' \cap M)$ is isomorphic to the Weyl group. In particular, $W \cong (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$ [20, p.66]. We have $N^- = N^\top = \{n^\top : n \in N\}$.

We may choose $\mathfrak{a}$ to be the set of all real matrices of the form

$$X = \operatorname{diag}(x_1, \ldots x_n, -x_1, \ldots, -x_n) \cong (x_1, \ldots, x_n);$$

the natural basis for $\mathfrak{a}$ is then

$$\{H_i := E_{i,i} - E_{n+i,n+i} : 1 \leq i \leq n\}.$$

The corresponding basis elements L_i of $\mathfrak{a}^*$ are given by $L_i(H_j) = \delta_{ij}$. The Lie algebra $\mathfrak{n}$ of N is

$$\mathfrak{n} = \left\{ \begin{pmatrix} A & B \\ 0 & -A^\top \end{pmatrix} : A, B \in \mathfrak{gl}_n(\mathbb{R}), A \text{ stricly upper triangular}, B = B^\top \right\}.$$

The root system is $\{\pm L_i \pm L_j : i \neq j\} \cup \{2L_i : 1 \leq i \leq n\}$; the positive roots α are $\{L_i - L_j, L_i + L_j, 2L_i : i < j\}$ [11, p.238-240]. The root spaces $\mathfrak{g}_\alpha$ are:

$$\begin{aligned}
\mathfrak{g}_{L_i - L_j} &= E_{ij} - E_{n+j,n+i} \in \mathfrak{n} \\
\mathfrak{g}_{-L_i + L_j} &= E_{ji} - E_{n+i,n+j} \in \mathfrak{n}^- \\
\mathfrak{g}_{L_i + L_j} &= E_{i,n+j} + E_{j,n+i} \in \mathfrak{n} \\
\mathfrak{g}_{-L_i - L_j} &= E_{n+i,j} + E_{n+j,i} \in \mathfrak{n}^- \\
\mathfrak{g}_{2L_i} &= E_{i,n+i} \in \mathfrak{n} \\
\mathfrak{g}_{-2L_i} &= E_{n+i,i} \in \mathfrak{n}^-.
\end{aligned}$$

So $\mathfrak{n}^- = \mathfrak{n}^\top$. Since W contains the negative of the identity $\gamma = -id$, the longest root is $-id$. Now $N = \exp \mathfrak{n}$ and $N^- = \exp \mathfrak{n}^- = \exp \mathfrak{n}^\top = (\exp \mathfrak{n})^\top$. So $\gamma N \gamma^{-1} = N^\top$.

Let $g \in G$ and assume the hypotheses of Theorem 5.7, that is, assume that $ygy^{-1} = cb$ for some $y \in \mathrm{Sp}_n(\mathbb{R})$, $c \in K$ and $b = \mathrm{diag}\,(b_1, \ldots, b_n, b_1^{-1}, \ldots, b_n^{-1}) \in A$ with $bc = cb$, and that y has Bruhat decomposition $y = n^- m_s man \in N^- m_s MAN$. In particular, assume that b is regular, i.e., $b \in A_+^\circ$. Choosing A_+, as usual, to be the set of those matrices in A with first n diagonal entries nonincreasing and ≥ 1, we have $b_1 > b_2 > \cdots > b_n > 1$. Since $bc = cb$, it follows that $c = \mathrm{diag}\,(C, C)$ with $C = \mathrm{diag}\,(\pm 1, \pm 1, \ldots, \pm 1)$.

According to Theorem 5.7,

1. $\lim_{i \to \infty} \mathrm{k}(g_i) = c_s \in M$, where $c_s := (m_s m)^{-1} c (m_s m)$ is of the form $\mathrm{diag}\,(C, C)$ with $C = \mathrm{diag}\,(\pm 1, \pm 1, \ldots, \pm 1)$,

2. $\lim_{i \to \infty} \mathrm{a}(g_i) = b_s \in A$, where $b_s := (m_s m)^{-1} b (m_s m) = m_s^{-1} b m_s$ is of the form $\mathrm{diag}\,(D, D^{-1})$ with D a diagonal matrix having diagonal entries $b_1^{\pm 1}, b_2^{\pm 1}, \ldots, b_n^{\pm 1}$ in some order.

The diagonal entries of $\mathrm{n}(g_i)$ are each 1, so it follows that the diagonal entries of the sequence $\{g_i\}$ converge to the eigenvalues of g.

5.4 Open Problem

One often encounters real matrices $A \in \mathrm{SL}_n(\mathbb{R})$ whose complex eigenvalues occur in complex conjugate pairs, i.e., the hyperbolic component of A is not regular. In order to deal with this case, we would like to relax somewhat the assumption of regularity in Theorem 5.7. In particular, we would like to consider $g \in G$ so that $xgx^{-1} = cb$ for some $y \in G$, $c \in K$, $b \in A_+$, such that $cb = bc$. In $\mathrm{SL}_n(\mathbb{C})$, this corresponds to choosing a matrix X with repeated eigenvalues. In this case, we believe that the convergence pattern for X will be based upon the repeated eigenvalues; if the distinct eigenvalues are $\lambda_1 > \ldots > \lambda_j$, so that λ_i

has multiplicity k_i, then we expect that X (pattern) converges to the block diagonal form

$$\begin{pmatrix} A_{\lambda_1} & * & \dots & * \\ 0 & A_{\lambda_2} & & * \\ & & \ddots & \\ 0 & 0 & \dots & A_{\lambda_j} \end{pmatrix},$$

where A_{λ_i} is a $k_i \times k_i$ matrix with eigenvalues all λ_i. This requires further study.

5.5 Comparison of the Iwasawa and Bruhat Iterations

Given $X \in \mathrm{GL}_n(\mathbb{C})$, the QR algorithm writes X as a product of a unitary matrix Q and upper triangular matrix R,

$$X = QR.$$

We define the QR iteration in a similar fashion to the LR iteration by setting

$$X_1 = X = Q_1 R_1,$$

and defining:

$$X_{s+1} \quad = \quad R_s Q_s = Q_{s+1} R_{s+1}, \quad s = 1, 2...$$

The iteration preserves the eigenvalues of X_1 since $X_{s+1} = R_s X_s R_s^{-1}$. Indeed, if X_1 has distinct eigenvalue moduli, proofs in the literature [17, 35] show that $\{X_i\}_{i\in\mathbb{N}}$ displays pattern convergence to a matrix in upper triangular form. In particular, $\{X_i\}_{i\in\mathbb{N}}$ does not necessarily converge, as the strictly upper triangular entries may behave poorly. However, the diagonal entries will actually converge to the eigenvalues of X_1.

The QR iteration has the advantage of being computationally stable, but as it only guarantees *form* convergence, not actual convergence, it compares poorly with the LR iteration, which does indeed guarantee actual convergence.

The Iwasawa decomposition of $\mathrm{SL}_n(\mathbb{C})$ corresponds to the QR decomposition, thus provides motivation for considering a generalized QR iteration in this context. The Iwasawa iteration of $g \in G$, where G is a real connected semisimple Lie group with Iwasawa decomposition $G = KAN$, is defined by

$$g_1 := g = k_1 a_1 n_1,$$

and

$$g_{s+1} \quad = \quad a_s n_s k_s = k_{s+1} a_{s+1} n_{s+1}, \quad s = 1, 2, ...$$

The asymptotic behavior of this sequence is given in the following theorem:

Theorem 5.14. [18] Let $g \in G$ and assume that $ygy^{-1} = cb$ for some $y \in G$, $c \in K$, $b \in A_+$, so that $cb = bc$. Suppose that y has Bruhat decomposition

$$y = n^- m_s man \in N^- m_s MAN.$$

Let $n_0^- := \exp \pi_H^0(L)$ where $H := \log b \in \mathfrak{a}_+$ and $L := \log n^- \in \mathfrak{n}_-$. Put

$$c_s := (n_0^- m_s m)^{-1} c_s (n_0^- m_s m).$$

Then there exists a sequence $\{d_i\}_{i \in \mathbb{N}}$ in the set $AN\tilde{c}_s AN \cap K$ such that

$$\lim_{i \to \infty} k_i d_i^{-1} = 1.$$

In keeping with the matrix result, the theorem does not suggest actual convergence of the Iwasawa iteration. Instead, it specifies the behavior of the K component of the sequence: it "nearly converges" in the sense that there is a "multiplier sequence" $\{d_i\}$ which can be used to perturb $\{k_i\}$ into a converging sequence. Again, the Bruhat iteration is advantageous over the Iwasawa iteration in that it can guarantee actual convergence, not just form convergence.

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