Path Decompositions of the Kneser Graph

by

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Abstract

Necessary and sufficient conditions are given for the existence of a graph decomposition of the Kneser Graph $KG_{n,2}$ into paths of length three and four, and of the Generalized Kneser Graph $GKG_{n,3,1}$ into paths of length three. A solution is also presented for the problem of embedding maximal H-designs, where H is a path of length three.

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Chapter 1

Introduction

1.1 Definitions

A *graph* G is an ordered pair (V, E) where V is a set of objects known as vertices and E is a set of two element subsets of V called edges. The edge $\{u, v\}$ is said to join the vertices u and v. A *complete graph of order* n, denoted by K_n, is a graph with n vertices in which each pair of vertices is joined by an edge. A *bipartite graph* is a graph G in which the vertices of G can be partitioned into two parts M and N such that the edge set is a subset of $E = \{\{x, y\} \mid x \in M, y \in N\}$; so no edges join two vertices in the same part. If the edge set of G equals E, then G is called a complete bipartite graph with parts M and N and is denoted $K_{m,n}$ or $K(M, N)$, where $|M| = m$ and $|N| = n$. The star of order m is the complete bipartite graph $K_{1,m} = S_m$ (for convenience, we allow the possibility that $m = 0$). The vertex of S_m with degree m when $m \geq 2$ is said to be the center of S_m; if $m = 1$ then either vertex can be designated the center. A *path of length* n, $P_n = (v_0, v_1, \ldots, v_n)$, is a sequence of $n + 1$ distinct vertices such that for $0 \leq i < n$, $\{v_i, v_{i+1}\}$ is an edge in G. A *cycle of length* n is the graph that can be formed from a path of length $n - 1$ by joining the first and last vertices. The join of two graphs G and H, denoted by $G \vee H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{\{a, b\} \mid a \in V(G), b \in V(H)\}$. A graph $G' = (V', E')$ is said to be a *subgraph* of $G = (V, E)$ if $V' \subseteq V$, $E' \subseteq E$.

An H-decomposition of a graph $G = (V, E)$ is a pair (V, B), where B is a collection of edge-disjoint subgraphs of G, each isomorphic to H, whose edges partition $E(G)$. If G is chosen to be the complete graph on n vertices then an H-decomposition of G is also referred to as an H-design of order n. A 3-cycle design of order n is often referred to as a Steiner Triple System. An H-design of a subgraph of K_n is also referred to as a partial H-design of order

n. The *leave* of a partial H-design (V, P) of order n is the graph $L = (V(K_n), E(K_n) \backslash E(P))$ where $E(P) = \{e \mid e \in E(H), H \in P\}$. A partial H-design is said to be *maximal* if its leave contains no subgraphs isomorphic to H.

The set of k element subsets of a set V is denoted $T_k(V)$.

1.2 History and Context

Over the years, many different graph decomposition problems have been studied, using various subgraphs for the decomposition. Perhaps the most common family of decompositions studied are cycle decompositions. One of the earliest graph theoretic problems was posed by Kirkman [16] in 1847 which asks, for a given x, to find the largest subgraph of K_x which has a 3-cycle decomposition.

In the 1960's, a great deal of work was done in solving the problem of cycle designs. In 1965, Kötzig found necessary and sufficient conditions for $4k$-cycle designs of order n where $n \equiv 1 \pmod{8k}$ [18]. The next year, Rosa found p-cycle designs of order n where $p \equiv 2 \pmod 4$ and $n = 2kp + 1$ for any k and p [23]. The solution to the odd cycle decomposition problem waited until 1989 when it was partially solved by Hoffman, Rodger, and Lindner [14] and then until 2001 where it was completely solved by Alspach and Gavlas [1].

G need not be limited to the complete graph. For instance, in 2001, Alspach and Gavlas found necessary and sufficient conditions for the existence even cycle decompositions of $K_{2m} - F$, the complete graph of even order with the edges of a 1-factor F removed [1], and in 2002, Šajna settled the existence problem for odd cycle decompositions of $K_{2m} - F$ [24].

Another graph for which cycle decompositions have been widely studied is the complete multipartite graph. One of the most prominent results for this problem was proved by Sotteau in 1981 [26]. Sotteau showed that there exists a $2k$-cycle decomposition of $K_{m,n}$ if and only if $m, n \geq k$, m and n are even, and $2k|mn$. More recently, stemming from statistical designs, G has been chosen to be the graph formed from a complete multipartite graph with multiplicity λ_2 (i.e., each pair of vertices in different parts are joined by λ_2 edges) by adding a copy of $\lambda_1 K_n$ to each part of size n (i.e., each pair of vertices in the same part are joined by λ_1 edges) and H is a 3-cycle [10], a 4-cycle [11], or a Hamilton cycle [3].

Other types of graph decompositions have also been well studied in the literature. Of particular interest related to this dissertation is the work of Michael Tarsi. In 1979, Tarsi found necessary and sufficient conditions for a decomposition of λK_n into stars of order m [27]. In 1983, he completely solved the problem of decomposing λK_n into paths of length m [28]. In this paper he showed that the obvious necessary conditions for an m-path decomposition of λK_n, namely that $m \mid |E(\lambda K_n)|$ and $n \geq m+1$, are also sufficient. In Chapters 2 and 3 of this dissertation, companion results are obtained by finding necessary and sufficient conditions for decomposing the Kneser and Generalized Kneser graphs into paths of length three (See Theorems 2.1 and 2.2 respectively). In Chapter 4, necessary and sufficient conditions for decomposing the Kneser Graph into paths of length four are found (see Theorem 3.1).

The Kneser Graph $KG_{n,k}$ is defined to be the graph whose vertices are the k-element subsets of some set of n elements, in which two vertices are adjacent if and only if their intersection is empty. The Generalized Kneser Graph, $GKG_{n,k,r}$ is defined to be the graph whose vertices are the k-element subsets of some set of n elements in which two vertices are adjacent if and only if they intersect in precisely r elements. The graph-decomposition problem of finding necessary and sufficient conditions for the existence of P_3-decompositions of $KG_{n,2}$ and $GKG_{n,3,1}$ is completely solved in Theorem 2.1 and 2.2 respectively, and the problem of the existence of P_4-decompositions of $KG_{n,2}$ is completely solved in Theorem 3.1. An explicit construction is provided to find the relevant decompositions.

It is worth noting that Kneser graphs have attracted much interest in the years since Kneser first described them in 1955 [17]. In particular, this interest has centered on solving the conjecture by Kneser that $\chi(KG_{n,k}) = n - 2k + 2$ whenever $n \geq 2k$ [17], where $\chi(G)$ is the chromatic number of G ($KG_{n,k}$ has no edges if $n < 2k$). The first proof of Kneser's conjecture was given by Lovász in 1978. What makes Lovász's proof so interesting is that it wasn't of a combinatorial nature at all, but rather topological. Lovász used the Borsuk-Ulam theorem which states, in essence, that any continuous function from the n-sphere into Euclidean n-space must map some pair of antipodal points to the same point. This application of a theorem in a seemingly unrelated field to what was perceived as a purely combinatorial problem was revelatory, and is considered one of the most influential works in the field of topological combinatorics. In the years after Lovász's proof, several other proofs were published, but all of them were essentially topological in nature [4, 12]. A purely combinatorial proof of Kneser's conjecture wasn't found until 2004, when Matoušek proved the result using Tucker's lemma which deals with vertex labellings in particular triangulations [22].

Another problem regarding Kneser graphs that has received a good deal of attention is to find the values of n and k for which $KG_{n,k}$ contains a Hamilton cycle. In 2000, Chen showed that Kneser graphs are Hamiltonian if $n \geq 3k$ [8]. The current conjecture is that all Kneser Graphs are Hamiltonian if $n \geq 2k + 1$ with the exception of $KG_{5,2}$ which is the Petersen Graph. It has been shown computationally that all connected Kneser graphs with $n \leq 27$ except for the Petersen Graph are indeed Hamiltonian [25]. The veracity of this conjecture in general is still an open problem.

In Chapter 4, the problem of embedding maximal partial 3-path designs is addressed. The embedding problem can be thought of as follows: for each partial H-design (V', P') of order n, find the set of integers M such that $m \in M$ if and only if there exists an H-design (V, P) of K_m such that $V' \subseteq V$ and $P' \subseteq P$. This dissertation also completely solves the problem of embedding maximal partial P_3-designs.

Various embedding problems have been studied extensively in the literature as well. Given the amount of study received by cycle decomposition problems over the years, it is perhaps not surprising that partial cycle system embeddings are also among the most studied embedding problems. In particular, the problem of partial Steiner Triple System embeddings has been a major focus over the years and was only recently solved. In 1977, Lindner conjectured that any partial Steiner Triple System of order u can be embedded in a Steiner Triple System of order v if $v \equiv 1, 3 \pmod 6$ and $v \geq 2u+1$ [20]. The next thirty-two years saw steady progress made towards proving this conjecture. Lindner had already shown in 1975 that any partial Steiner Triple System of order u can be embedded in a Steiner Triple System of order $6u+3$ [19]. In 1980, Anderson, Hilton and Mendelsohn improved the bound to $v \geq 4u+1$ and $v \equiv 1, 3 \pmod 6$ [2]. The bound was improved again in 2004 by Bryant to $3u-2$ [6]. The conjecture was finally proved in 2009 by Bryant and Horsley using a new technique, dubbed "repacking" [7].

1.3 Techniques

In Chapters 2 and 3, where path decompositions of Kneser and Generalized Kneser graphs are considered, the main technique utilized is taking advantage of the highly structured nature of the underlying graph. By cleverly partitioning the element set that generates the vertices, predictable subgraphs can be induced by selecting vertices containing elements in various parts of the partition. These subgraphs can be catalogued in a fairly straightforward manner, and path decompositions of each of them can be found far more simply than trying to decompose the whole graph at once. By finding decompositions of these subgraphs, and by carefully using the partition of the element set to ensure that every edge of the graph appears in exactly one of these subgraphs, all of the edges in the overall graph can be placed into paths.

The construction technique in Chapter 4, where embeddings of partial 3-path systems are considered, uses a similar approach. The key observation here is that maximal partial 3-path systems have easily catalogued leaves. For embedding partial 3-path systems of order n into complete 3-path systems of orders at least $n+2$, by carefully partitioning the components of the leave the embedding problem can be reduced to finding 3-path decompositions of a reasonably small number of fairly basic graphs. In the case of embedding partial 3-path systems or order n into complete 3-path systems of order $n+1$, a different approach was needed. Here, the problem is solved for maximal partial 3-path systems by analyzing what the new 3-paths would look like in terms of how many edges in the leave they would use. From here, the leave is partitioned into the proper number of paths of length two and paths of length one, and the 3-paths required are built up from these smaller paths by taking advantage of the predictable form of the leave.

Chapter 2

P_3-decompositions of $KG_{n,2}$ and $GKG_{n,3,1}$

2.1 Introduction

In this chapter, the problem of finding necessary and sufficient conditions for obtaining 3-path decompositions of $KG_{n,2}$ and $GKG_{n,3,1}$ is completely solved. Recall that $T_k(V)$ is the set of k-element subsets of the set V, and let (a, b, c, d) denote the path, P_3, of length three with edge set $\{\{a, b\}, \{b, c\}, \{c, d\}\}$.

2.2 Building Blocks

The following lemmas will be useful in the constructions to come.

Lemma 2.1. *There exists a P_3-decomposition of each of the following graphs:*

(i) $K_{2,3}$

(ii) $K_{3,3}$

(iii) $K_{n,3k}$ *for any* $n \geq 2$ *and* $k \geq 1$

(iv) $H_4 = K_{3,3} - F$ *with bipartition* $\{\mathbb{Z}_3, \mathbb{Z}_6 \backslash \mathbb{Z}_3\}$ *of* $V(K_{3,3})$*, and where* $E(F) = \{\{i, i+3\} \mid i \in \mathbb{Z}_3\}$

(v) $H_5 = H_4 \cup G'$*, where* $G' = (\mathbb{Z}_9 \backslash \mathbb{Z}_3, \{\{3, 6\}, \{4, 7\}, \{5, 8\}\})$

(vi) H_6*, the bipartite graph with bipartition* $\{T_2(\mathbb{Z}_4), \mathbb{Z}_4\}$ *of* $V(H_6)$ *and* $E(H_6) = \{\{a, b\} \mid b \notin a, a \in T(\mathbb{Z}_4), b \in \mathbb{Z}_4\}$

(vii) $KG_{5,2}$ *(the Petersen Graph)*

(viii) H_8, *the bipartite graph with bipartition* $\{T_2(\mathbb{Z}_5), \mathbb{Z}_5\}$ *of* $V(H_8)$ *and* $E(H_8) = \{\{a, b\} \mid b \notin a, a \in T(\mathbb{Z}_5), b \in \mathbb{Z}_5\}$

Proof. Hoffman and Billington solved cases (i)-(iii) (and much more besides) in [5]. However, in the interest of keeping this discussion self-contained, explicit constructions are given for these cases.

(i) Define $K_{2,3}$ with bipartition $\{\mathbb{Z}_2, \mathbb{Z}_5 \backslash \mathbb{Z}_2\}$ of the vertex set $\mathbb{Z}_5$. Then $(\mathbb{Z}_5, \{(0, 2, 1, 3), (3, 0, 4, 1)\})$ is the required decomposition.

(ii) Define $K_{3,3}$ with bipartition $\{\mathbb{Z}_3, \mathbb{Z}_6 \backslash \mathbb{Z}_3\}$ of the vertex set $\mathbb{Z}_6$. Then $(\mathbb{Z}_6, \{(3, 0, 5, 2), (1, 3, 2, 4), (0, 4, 1, 5)\})$ is the required decomposition.

(iii) Since $n \geq 2$, form a partition, P, of $\mathbb{Z}_n$ into sets of size 2 and 3, and a partition Q of $\mathbb{Z}_{n+3k} \backslash \mathbb{Z}_n$ into sets of size 3. For each $p \in P$ and $q \in Q$, let $(p \cup q, B_{p,q})$ be a P_3-decomposition of $K_{|p|,3}$ with bipartition $\{p, q\}$ of the vertex set. Then $(\mathbb{Z}_{n+3k}, \bigcup_{p \in P, q \in Q} B_{p,q})$ is the required P_3-decomposition of $K_{n,3k}$.

(iv) With bipartition $\{\mathbb{Z}_3, \mathbb{Z}_6 \backslash \mathbb{Z}_3\}$, $(\mathbb{Z}_6, \{(0, 4, 2, 3), (0, 5, 1, 3)\})$ is the required decomposition with $F = \{\{0, 3\}, \{1, 4\}, \{2, 5\}\}$.

(v) $(\mathbb{Z}_9, \{(6, 3, 1, 5), (7, 4, 2, 3), (8, 5, 0, 4)\})$ is the required decomposition.

(vi) $(V(H_6), \{(0, \{2, 3\}, 1, \{0, 2\}), (1, \{0, 3\}, 2, \{1, 3\}), (2, \{0, 1\}, 3, \{0, 2\}), (3, \{1, 2\}, 0, \{1, 3\}))$ is the required decomposition.

(vii) Let $V(KG_{5,2}) = T_2(\mathbb{Z}_5)$. Then $(T_2(\mathbb{Z}_5), \{(\{i, i+1\}, \{i+2, i+3\}, \{i+1, i+4\}, \{i, i+3\}) \mid i \in \mathbb{Z}_5\}$ reducing the sums modulo 5 is the required decomposition.

(viii) $(V(H_8), \{(1, \{0, 2\}, 3, \{0, 1\}), (2, \{0, 3\}, 4, \{0, 2\}), (2, \{0, 4\}, 1, \{0, 3\}), (0, \{1, 2\}, 3, \{0, 4\}), (0, \{1, 3\}, 4, \{1, 2\}), (3, \{1, 4\}, 2, \{1, 3\}), (4, \{2, 3\}, 0, \{1, 4\}), (3, \{2, 4\}, 1, \{2, 3\}), (1, \{3, 4\}, 0, \{2, 4\}), (4, \{0, 1\}, 2, \{3, 4\}))$ is the required decomposition. $\square$

A graph, G, is said to have an Euler tour if there exists a closed walk in G that contains each edge of G exactly once.

The following is well known.

Lemma 2.2. *A connected simple graph, G, has an Euler tour if and only if the degree of every vertex in G is even.*

From this, we can easily obtain the following result.

Lemma 2.3. *If G is a connected bipartite simple graph in which the number of edges is divisible by three and all vertices have even degree, then G has a P_3-decomposition.*

Proof. By Lemma 2.2, let $P = (v_0, v_1, \ldots, v_e)$ be an Euler tour of G. Since G is bipartite, each set of three consecutive edges of P induce a P_3. Therefore, since $e = |E(G)|$ is divisible by three, $(V(G), \{(v_{3i}, v_{3i+1}, v_{3i+2}, v_{3i+3}) \mid i \in \mathbb{Z}_{e/3}\})$ is a P_3-decomposition of G. $\square$

Lemma 2.4. *There exists a P_3-decomposition of each of the following graphs:*

(i) $GKG_{5,3,1}$ *(The Petersen Graph)*

(ii) $GKG_{6,3,1}$

Proof.

(i) $GKG_{5,3,1} = KG_{5,2}$ as can be seen by taking the complement of each vertex. The result follows from Lemma 2.1(vii).

(ii) Partition the vertices of $GKG_{6,3,1}$ into the following two types:

Type 1: $T_3(\mathbb{Z}_5)$, and

Type 2: $T_3(\mathbb{Z}_6) \backslash T_3(\mathbb{Z}_5)$

Let G_1 be the subgraph induced by the Type 1 vertices, G_2 be the subgraph induced by the Type 2 vertices, and G_3 be the bipartite subgraph induced by the edges of the form $\{x, y\}$ where x is a Type 1 vertex and y is a Type 2 vertex. G_1 is clearly a

$GKG_{5,3,1}$ and has a P_3-decomposition by (i). G_2 is isomorphic to $KG_{5,2}$ (all vertices share the element 5, so two are adjacent only if their other two elements are disjoint) and has a P_3-decomposition by Lemma 2.1(vii). G_3 is a bipartite graph that is 6-regular, so $|E(G_3)|$ is a multiple of three. To see that G_3 is connected, for each Type 1 vertex, $\{a, b, c\}$ in G_3 we display a path to each vertex of Type 2 as follows (where a, b, c, d and e are the distinct elements of $\mathbb{Z}_5$): $(\{a, b, c\}, \{a, d, 5\}, \{b, c, d\}, \{a, b, 5\})$, $(\{a, b, c\}, \{a, d, 5\}, \{a, b, e\}, \{d, e, 5\})$, and $(\{a, b, c\}, \{a, d, 5\})$. These account for all pairs of nonadjacent vertices in G_3, so G_3 is connected. Therefore, G_3 is a connected even regular bipartite graph with a multiple of three edges, so by Lemma 2.3, it also has a P_3-decomposition. The union of these three decompositions forms a P_3-decomposition of $GKG_{6,3,1}$. □

2.3 A P_3-Decomposition of $KG_{n,2}$

Theorem 2.1. *$KG_{n,2}$ is P_3-decomposable if and only if $n \neq 4$.*

Proof. If $n \in \{1, 2, 3\}$, then $KG_{n,2}$ has no edges, so the result is vacuously true. Since $KG_{4,2}$ is a 1-factor on six vertices, it is clearly not P_3-decomposable. $KG_{5,2}$ is decomposable by Lemma 2.1(vii).

The remaining cases are proved by induction on n. So now assume that $KG_{w,2}$ is P_3-decomposable for all $w \leq n$ for some $n \geq 5$. It is shown that $G = KG_{n+1,2}$ is P_3-decomposable. Let $\epsilon \in \{0, 1, 2\}$ such that $\epsilon \equiv n \pmod 3$. Let $(T_2(\mathbb{Z}_n), B)$ be a P_3-decomposition of $KG_{n,2}$.

The subgraph of $KG_{n+1,2}$ induced by vertices in $T_2(\mathbb{Z}_{n+1}) \backslash T_2(\mathbb{Z}_n)$ clearly has no edges, since they all share the element n. What remains to be shown is that the subgraph induced by the edges connecting vertices in $T_2(\mathbb{Z}_n)$ to vertices in $T_2(\mathbb{Z}_{n+1}) \backslash T_2(\mathbb{Z}_n)$ has a P_3-decomposition.

Partition $\mathbb{Z}_n$ into $t = (n - \epsilon)/3$ sets: $S_i = \{3i, 3i + 1, 3i + 2\}$ for $i \in \mathbb{Z}_{t-1}$ and $S_{t-1} = \{i \mid n - 3 - \epsilon \leq i \leq n - 1\}$. It is convenient to partition the old vertices, $T(\mathbb{Z}_n)$, into

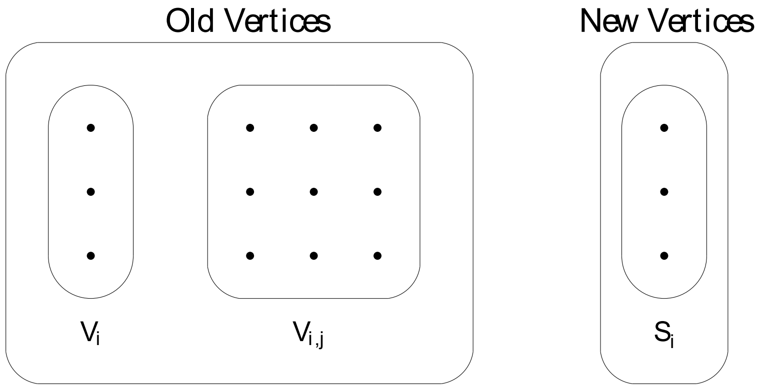

Figure 2.1: Partitioning the Vertices

the following two types (visualized in Figure 2.1):

$V_i = \{\{x, y\} \mid x, y \in S_i, x \neq y\}$ for $i \in \mathbb{Z}_t$, and

$V_{i,j} = \{\{x, y\} \mid x \in S_i, y \in S_j\}$ for $0 \leq i < j < t$.

Further, partition the new vertices into t sets:

$S_i' = \{\{x, n\} \mid x \in S_i\}$ for $i \in \mathbb{Z}_t$.

All of the edges not involving vertices with elements in S_{t-1} are handled first. Of these edges, the edges that require special attention are those joining two vertices in $\{\{v, s\} \mid v \in V_i, s \in S_i'\}$ for some $i \in \mathbb{Z}_{t-1}$. For each $i \in \mathbb{Z}_{t-1}$, these edges induce a matching on six vertices, so they can't be decomposed into three paths in isolation. To decompose these edges, they are combined with edges joining two vertices in $\{\{v, s\} \mid v \in V_{0,i}, s \in S_i'\}$ for some $i \in \mathbb{Z}_{t-1} \setminus \{0\}$ to form 3-paths as described in the next two paragraphs, with $i = 1$ being an even more special case.

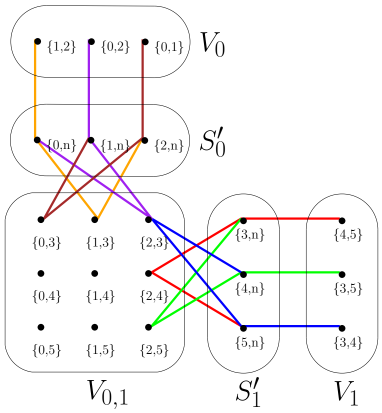

Figure 2.2: The Two H_5's

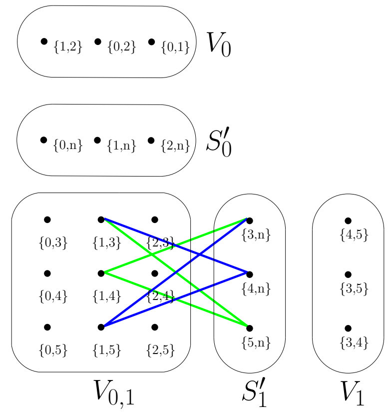

Figure 2.3: Example of the H_4

First, consider the bipartite subgraph G_0 of $KG_{n+1,2}$ induced by the edges joining the vertices in $V_0 \cup V_1 \cup V_{0,1}$ to the vertices in $S'_0 \cup S'_1$. Partition these edges as follows. The edges joining the vertices of $V_1 \cup \{\{0, x\} \mid x \in S_1\}$ to S'_1 induce a subgraph isomorphic to H_5, so by Lemma 2.1(v), there exists a P_3-decomposition of this subgraph. The edges joining the vertices of $V_0 \cup \{\{x, 3\} \mid x \in S_0\}$ to S'_0 also form a subgraph isomorphic to H_5, so by Lemma 2.1(v), there exists a P_3-decomposition of this subgraph as well (See Figure 2.2). Now, for each $k \in \{1, 2\}$ consider the edges joining the vertices $\{\{k, x\} \mid x \in S_1\}$ to the vertices in S'_1. These edges induce a subgraph isomorphic to H_4, so by Lemma 2.1(iv), there exists a P_3-decomposition of this subgraph. Also, for each $k \in \{4, 5\}$ the edges joining the vertices $\{\{x, k\} \mid x \in S_0\}$ to the vertices in S'_0 induce a subgraph isomorphic to H_4, so by Lemma 2.1(iv), there exists a P_3-decomposition of this subgraph (See Figure 2.3 for an example). Figure The union of these sets of 3-paths produce a P_3-decomposition $(V(G_0), B'_0)$ of most of G_0. The edges connecting V_1 to S'_0 and connecting V_0 to S'_1 occur in paths in $B'_{1,0}$ and $B'_{0,1}$, respectively as defined below.

Now, for each $i \in \{\mathbb{Z}_{t-1} \backslash \mathbb{Z}_2\}$, consider the bipartite subgraph G_i of $KG_{n+1,2}$ induced by the edges joining the vertices in $V_i \cup V_{0,i}$ to the vertices in $S'_0 \cup S'_i$. The edges in G_i connecting the vertices of $V_i \cup \{\{0, x\} \mid x \in S_i\}$ to S'_i induce a subgraph isomorphic to H_5, thus it has a P_3-decomposition by Lemma 2.1(v). Now, for each $k \in \{1, 2\}$, the edges joining the vertices in $\{\{k,x\} \mid x \in S_i\}$ to the vertices in S'_i induce a subgraph isomorphic to H_4, so there exists a P_3-decomposition of the subgraph by Lemma 2.1(iv). Further, for each $k \in S_i$, the edges connecting the vertices of $\{\{x,k\} \mid x \in S_0\}$ to the vertices of S'_0 induce a subgraph isomorphic to H_4 which therefore has a decomposition by Lemma 2.1(iv). The union of these decompositions produce a P_3-decomposition, $(V(G_i), B'_i)$, of most of G_i for each $i \in \{\mathbb{Z}_t \backslash \mathbb{Z}_2\}$. The remaining edges in G_i, namely those connecting V_i to S'_0, are in paths in $B'_{i,0}$ as defined below.

For each $i \in \mathbb{Z}_{t-1}$ and for each $j \in \mathbb{Z}_{t-1} \backslash \{i\}$, the bipartite subgraph of G induced by the edges joining vertices in V_i to vertices in S'_j is isomorphic to $K_{3,3}$, so by Lemma 2.1(ii) there exists a P_3-decomposition $(V_i \cup S'_j, B'_{i,j})$ of this subgraph.

For $1 < j < t-1$, the edges connecting vertices of $V_{0,1}$ to vertices of S'_j induce a $K_{9,3}$, and thus this graph has a P_3-decomposition $(V_{0,1} \cup S'_j, B_{0,1,j})$ by Lemma 2.1(iii).

For each $i \in \mathbb{Z}_{t-1} \backslash \mathbb{Z}_2$ and for each $j \in \mathbb{Z}_{t-1} \backslash \{0, i\}$, the edges connecting vertices of $V_{0,i}$ with the vertices of S'_j induce a copy of $K_{9,3}$ so this graph has a P_3-decomposition, $(V_{0,i} \cup S'_j, B_{0,i,j})$, by Lemma 2.1(iii).

For $0 < i < j < t-1$ and $0 \leq k < t-1$, consider the bipartite subgraph of G induced by edges joining vertices in $V_{i,j}$ to vertices in S'_k. This subgraph of G has a P_3-decomposition as follows:

(a) if $k \notin \{i,j\}$, then the subgraph is isomorphic to $K_{9,3}$, so it has a P_3-decomposition $(V_{i,j} \cup S'_k, B_{i,j,k})$ by Lemma 2.1(iii);

(b) if $k = i$, then for each $y \in S_j$ the edges connecting the vertices $\{\{x,y\} \mid x \in S_i\}$ and $S'_{k=i}$ induce a subgraph isomorphic to H_4, which has a P_3-decomposition $(V_{i,j} \cup S'_k, B_{i,j,k})$ by Lemma 2.1(iv);

(c) if $k = j$, then for each $x \in S_i$ the edges connecting the vertices $\{\{x,y\} \mid y \in S_j\}$ and $S'_{k=j}$ form a subgraph isomorphic to H_4, which has a P_3-decomposition $(V_{i,j} \cup S'_k, B_{i,j,k})$ by Lemma 2.1(iv).

The only edges left to consider are all the edges which are incident with a vertex in $S'_{t-1} \cup V_{t-1} \cup V_{i,t-1}$, $i \in \mathbb{Z}_{t-1}$. The handling of these edges depends on the value of ϵ.

First, consider the bipartite subgraph G_{t-1} of $KG_{n+1,2}$ induced by the edges joining the vertices in $V_{t-1} \cup V_{0,t-1}$ to the vertices in $S'_0 \cup S'_{t-1}$. We consider each value of ϵ in turn.

For $\epsilon = 0$, the edges in G_{t-1} connecting the vertices of $V_{t-1} \cup \{\{0, x\} \mid x \in S_{t-1}\}$ to S'_{t-1} induce a subgraph isomorphic to H_5, thus it has a P_3-decomposition by Lemma 2.1(v). Now, for each $k \in \{1, 2\}$, the edges joining the vertices in $\{\{k, x\} \mid x \in S_{t-1}\}$ to the vertices in S'_{t-1} induce a subgraph isomorphic to H_4, so there exists a P_3-decomposition by Lemma 2.1(iv). Further, for each $k \in S_{t-1}$, the edges connecting the vertices of $\{\{x, k\} \mid x \in S_0\}$ to the vertices of S'_0 induce a subgraph isomorphic to H_4 which therefore has a decomposition by Lemma 2.1(iv). Lastly, the edges joining the vertices in v_{t-1} to the vertices in S'_0 induce a subgraph isomorphic to $K_{3,3}$, and so has a P_3-decomposition be Lemma 2.1(ii). The union of these decompositions produce a P_3-decomposition, $(V(G_{t-1}), B'_{t-1})$, of G_{t-1}.

For $\epsilon = 1$ or 2, the edges connecting vertices in V_{t-1} to S'_{t-1} induce a graph isomorphic to H_6 or H_8 respectively, which has a P_3-decomposition by Lemma 2.1(vi)((viii) respectively). Regarding the edges connecting vertices in $V_{0,t-1}$ to S'_{t-1}, for each $y \in S_{t-1}$ the edges connecting the vertices $\{\{x, y\} \mid x \in S_0\}$ and S'_{t-1} induce a subgraph isomorphic to $K_{3,3}$ if $\epsilon = 1$ and $K_{3,4}$ if $\epsilon = 2$, which has a P_3-decomposition by Lemma 2.1(iii) in both cases. For each $k \in S_{t-1}$ the edges connecting the vertices in $\{\{x, k\} \mid x \in S_0\}$ to the vertices in S'_0 induce a subgraph isomorphic to H_4, so there exists a P_3-decomposition by Lemma 2.1(iv). Lastly, the edges joining the vertices in v_{t-1} to the vertices in S'_0 induce a subgraph isomorphic to $K_{3+\epsilon,3}$, and so has a P_3-decomposition be Lemma 2.1(iii). The union of these decompositions produce a P_3-decomposition, $(V(G_{t-1}), B'_{t-1})$, of G_{t-1}.

The rest of the edges are easier to decompose.

For each $i \in \mathbb{Z}_{t-1}$, the bipartite subgraph induced by the edges joining the vertices of V_i to the vertices of S'_{t-1} induce a graph isomorphic $K_{3,3+\epsilon}$, so it has a P_3-decomposition $(V_i \cup S'_{t-1}, B_{i,t-1})$ by Lemma 2.1(iii).

For $0 \leq i < j < t-1$, the bipartite subgraph induced by the edges joining the vertices of $V_{i,j}$ to the vertices of S'_{t-1} induce a graph isomorphic to $K_{9,3+\epsilon}$, so it has a P_3-decomposition $(V_{i,j} \cup S'_{t-1}, B_{i,j,t-1})$ by Lemma 2.1(iii).

Finally, For $0 < i < t-1$ and $0 \leq k < t$, consider the bipartite subgraph of G induced by edges joining vertices in $V_{i,t-1}$ to vertices in S'_k. This subgraph of G has a P_3-decomposition as follows:

(a) if $k \notin \{i, t-1\}$, then the subgraph is isomorphic to $K_{3(3+\epsilon),3}$, so it has a P_3-decomposition $(V_{i,t-1} \cup S'_k, B_{i,t-1,k})$ by Lemma 2.1(iii);

(b) if $k = i$, then for each $y \in S_{t-1}$ the edges connecting the vertices $\{\{x, y\} \mid x \in S_i\}$ and $S'_{k=i}$ induce a subgraph isomorphic to H_4, which has a P_3-decomposition $(V_{i,t-1} \cup S'_k, B_{i,t-1,k})$ by Lemma 2.1(iv);

(c) if $k = t-1$, then

Case $\epsilon = 0$: For each $x \in S_i$ the edges connecting the vertices $\{\{x, y\} \mid y \in S_j\}$ and $S'_{k=j}$ induce a subgraph isomorphic to H_4, which has a P_3-decomposition $(V_{i,j} \cup S'_k, B_{i,j,k})$ by Lemma 2.1(iv).

Case $\epsilon = 1$: For each $y \in S_{t-1}$ the edges connecting the vertices $\{\{x, y\} \mid x \in S_i\}$ and $S'_{k=t-1}$ induce a subgraph isomorphic to $K_{3,3}$, which has a P_3-decomposition $(V_{i,t-1} \cup S'_k, B_{i,t-1,k})$ by Lemma 2.1(ii).

Case $\epsilon = 2$: For each $y \in S_{t-1}$ the edges connecting the vertices $\{\{x, y\} \mid x \in S_i\}$ and $S'_{k=t-1}$ induce a subgraph isomorphic to $K_{4,3}$, which has a P_3-decomposition $(V_{i,t-1} \cup S'_k, B_{i,t-1,k})$ by Lemma 2.1(iii).

This accounts for all new edges.

Let $B_1 = \bigcup_{i \in \mathbb{Z}_t} B'_i$, $B_2 = \bigcup_{0 \le i < j < t} B'_{i,j}$, and $B_3 = \bigcup_{0 \le i < j < t, k \in \mathbb{Z}_t} B'_{i,j,k}$. The required P_3-decomposition of G is given by $(V(G), B \cup B_1 \cup B_2 \cup B_3)$. $\square$

2.4 A P_3-Decomposition of $GKG_{n,3,1}$

Before stating and proving the main result for $GKG_{n,3,1}$, a few definitions and a technical lemma are presented.

A digraph is an ordered quadruple $D = (V, E, t, h)$ where V is a set of vertices, E is a set of ordered pairs of vertices (each element of which is called an arc or directed edge), and $t, h : E \to V$ are functions defined by $t((u, v)) = u$ and $h((u, v)) = v$ for each arc $(u, v) \in E$ ($t(e)$ and $h(e)$ are called the tail and head of arc e, respectively). A complete digraph is a digraph in which $E = V \times V$.

A directed 2-factor of a digraph D is a spanning subdigraph F in which every vertex is the head of exactly one arc and the tail of exactly one arc of F.

Let $D = (V, E, t, h)$ be a digraph, let C be a set of colors, and for each $e \in E$, let $C_e \subseteq C$. A $(C_1, \ldots, C_e)$-coloring of D is a function $c : E \to C$ such that if $e \in E$ then $c(e) \in C_e$ (this is known as a list arc-coloring). A list arc-coloring is said to be proper if no two adjacent arcs receive the same color. In the following lemma, the vertex set is $T_3(\mathbb{Z}_n)$, so we can refer to the intersection of two vertices (it is the intersection of two 3-element sets).

Lemma 2.5. *Let $D = (T_3(\mathbb{Z}_n), E, t, h)$ be a complete digraph. Let $C = \mathbb{Z}_n$ be a set of colors. For each $e \in E$, let $C_e = t(e) \cap h(e)$ (so possibly $C_e = \emptyset$). There exists a proper list arc-colored directed 2-factor of D.*

Proof. Let D, C, and C_e be defined as stated in the lemma. Form a directed 2-factor, F, of D as follows.

First, form a partition, P, of the vertex set $T_3(\mathbb{Z}_n)$ so that two vertices $\{a, b, c\}$ and $\{x, y, z\}$ are in the same element of P if and only if $\{x, y, z\} = \{a + i, b + i, c + i\}$ for some $i \in \mathbb{Z}_n$ with the sums reduced modulo n. If n is not a multiple of three, then P contains

$l = \frac{(n-1)(n-2)}{6}$ sets, each containing n elements. If n is a multiple of three, say $n = 3k$, then P contains $l = \frac{\binom{3k}{3}-k}{3k} = 3\binom{k}{2}$ sets of size n and one set of size k. In either case, let the elements of P of size n be $\{E_0, E_1, \dots E_{l-1}\}$, and if n is a multiple of three then let $E_l = \{\{i, i+k, i+2k\} \mid i \in \mathbb{Z}_k\}$ be the single set of size k.

For $0 \leq i < l$, among the vertices in E_i, let $e_i = \{0, a_i, b_i\}$ with $a_i < b_i$ be one that contains both zero and as small a nonzero element of $\mathbb{Z}_n$ as possible (two such vertices might exist, in which case either can be e_i). Let $d_i = gcd(a_i, n)$. For $0 \leq j < d_i$, and for $0 \leq r < \frac{n}{d_i}$, define the arc $e_{i,r,j} = (\{ra_i+j, (r+1)a_i+j, ra_i+b_i+j\}, \{(r+1)a_i+j, (r+2)a_i+j, (r+1)a_i+b_i+j\})$ in D. Then for each $j \in \mathbb{Z}_{d_i}$, the subgraph $S_{i,j}$ of D induced by $\{e_{i,r,j} \mid 0 \leq r < \frac{n}{d_i}\}$ is a directed cycle. Note that the both the tail and head of each arc $e_{i,r,j}$ contains the element $(r+1)a_i + j$; so let $c(e_{i,r,j}) = (r+1)a_i + j$. Clearly this coloring is proper since consecutive arc colors differ by $a_i \pmod{n}$, where clearly $a_i < n$. If n is not a multiple of three, then $F = \bigcup_{i \in \mathbb{Z}_l, j \in \mathbb{Z}_{d_i}} S_{i,j}$ is a directed 2-factor that is properly list arc-colored as required.

If n is a multiple of three, then F is a properly list arc-colored directed 2-factor that includes all of the vertices in D except for those in E_l. We now insert the k vertices in E_l into an already created directed cycle in F and then give a proper list arc-coloring to the modified cycle. Recall that $E_l = \{i, k+i, 2k+i\} \mid 0 \leq i < k\}$. Consider the colored directed cycle, C', in F containing the vertex $\{0, 1, 1+k\}$. Then $C' = (v'_0, v'_1, \dots, v'_{n-1})$ where for each $j \in \mathbb{Z}_n$, $v'_j = \{0+j, 1+j, 1+k+j\}$ and where the arc (v'_j, v'_{j+1}) is colored $j+1$. For $0 \leq j \leq k$, replace the arc $(\{0+j, 1+j, 1+k+j\}, \{1+j, 2+j, 2+k+j\})$ colored $i+j$ in F with the arcs $(\{0+j, 1+j, 1+k+j\}, \{1+j, 1+k+j, 1+2k+j\})$ colored $1+k+j$ and $(\{1+j, 1+k+j, 1+2k+j\}, \{1+j, 2+j, 2+k+j\})$ colored $1+j$. The resulting cycle is still properly list edge-colored since the only potential conflict is at the vertex $\{0, 1, 1+k\}$ which previously was incident with arcs colored 0 and 1 and now is incident with arcs colored 0 and $1+k$. □

We are now ready to prove our second main result.

Theorem 2.2. *$G = GKG_{n,3,1}$ has a P_3-decomposition for all $n > 0$.*

Proof. For $n \in \{1, 2, 3, 4\}$, G has no edges, so the result is vacuously true. For $n = 5$, G has a P_3-decomposition by Lemma 2.4(i). For $n = 6$, G has a P_3-decomposition by Lemma 2.4(ii).

The remaining cases are proved by induction on n. So now assume that $GKG_{w,3,1}$ is P_3-decomposable for all $w \leq n$ for some $n \geq 6$. It is shown that $H = GKG_{n+2,3,1}$ is P_3-decomposable. Let $(T_3(\mathbb{Z}_n), B_0)$ be a P_3-decomposition of $G = GKG_{n,3,1}$. Partition the vertices of H as follows:

(i) $V_0 = T_3(\mathbb{Z}_n)$ (the vertices of G),

(ii) $V_1 = \{\{a, b, n\} \mid a, b \in \mathbb{Z}_n\}$,

(iii) $V_2 = \{\{a, b, n+1\} \mid a, b \in \mathbb{Z}_n\}$, and

(iv) $V_3 = \{\{a, n, n+1\} \mid a \in \mathbb{Z}_n\}$.

Consider the following subgraphs of H:

(i) H_0 is the subgraph induced by the vertices of V_0,

(ii) H_1 is the subgraph induced by the vertices of $V_1 \cup V_3$,

(iii) H_2 is the subgraph induced by the vertices of $V_2 \cup V_3$,

(iv) H_3 is the bipartite subgraph induced by the edges $\{\{x, y\} \mid x \in V_0, y \in V_1 \cup V_2\}$,

(v) H_4 is the bipartite subgraph induced by the edges $\{\{x, y\} \mid x \in V_1, y \in V_2\}$, and

(vi) H_5 is the bipartite subgraph induced by the edges $\{\{x, y\} \mid x \in V_0, y \in V_3\}$.

These six subgraphs clearly partition the edges of H, so combining P_3-decompositions of each will create a P_3-decomposition of H itself.

Since $H_0 = G$, it has a decomposition $(T_3(\mathbb{Z}_n), B_0)$ by assumption.

Next, notice that in H_1 and H_2, all vertices share the element $x = n$ or $n+1$ respectively; so any two vertices, say $\{a, b, x\}$and $\{c, d, x\}$, are adjacent if and only if $\{a, b\} \cap \{c, d\} = \emptyset$. So H_1 is clearly isomorphic to $KG_{n+1,2}$ with vertex set $\{v \backslash \{n\} \mid v \in V(H_1)\}$ and H_2 is isomorphic to $KG_{n+1,2}$ with vertex set $\{v \backslash \{n+1\} \mid v \in V(H_2)\}$. Therefore, H_1 and H_2 have P_3-decompositions $(V(H_1), B_1)$ and $(V(H_2), B_2)$ respectively by Theorem 2.1.

Next, consider the bipartite subgraph H_3. If $v \in V_0$, then $d_{H_3}(v) = 6\binom{n-3}{2}$, and if $v \in V_1 \cup V_2$, then $d_{H_3}(v) = 2\binom{n-2}{2}$, both of which are even. Also, $|E(H_3)| = 6\binom{n}{2}\binom{n-3}{2}$ which is clearly a multiple of three. Finally, to show H_3 is connected, for each $\{s,t,u\} \in V_0$ we display a path to each vertex in $V_1 \cup V_2$ as follows (where a, b, s, t, and u are distinct elements of $\mathbb{Z}_n$ and $x \in \{n, n+1\}$): $(\{s,t,u\}, \{a,s,x\},$
$\{b,s,u\}, \{a,b,x\})$, $(\{s,t,u\}, \{a,s,x\}, \{a,b,t\}, \{s,t,x\})$, and $(\{s,t,u\}, \{a,s,x\})$.
These account for all pairs of nonadjacent vertices in H_3, so H_3 is easily seen to be connected. Therefore, H_3 has a P_3-decomposition $(V(H_3), B_3)$ by Lemma 2.3.

We also use Lemma 2.3 to find a P_3-decomposition of H_4 as the following shows. H_4 is a $2\binom{n-2}{2}$-regular bipartite graph, so all vertices have even degree. Also, $|E(H_4)| = 2\binom{n}{2}\binom{n-2}{2}$ which is a multiple of three. To see this, note $|E(H_4)|$ is the product of four consecutive integers (one of which must be a multiple of three) divided by two. Finally, to show that H_4 is connected, for each vertex $\{a,b,n\} \in V_1$ we display a path to each vertex in V_2 as follows (where a, b, s, and t are distinct elements of $\mathbb{Z}_n$): $(\{a,b,n\}, \{a,t,n+1\}, \{a,s,n\}, \{a,b,n+1\})$, $(\{a,b,n\}, \{b,s,n+1\}, \{a,s,n\}, \{s,t,n+1\})$, and $(\{a,b,n\}, \{a,c,n+1\})$. These account for all pairs of nonadjacent vertices in H_4, so H_4 is easily seen to be connected. Therefore, H_4 has a P_3-decomposition $(V(H_4), B_4)$ by Lemma 2.3.

Finally, consider H_5. Using Lemma 2.5, let F be a properly list arc-colored 2-factor of the complete digraph with vertex set V_0, with the set of colors $C = \mathbb{Z}_n$, and with lists of colors $(C_0, C_1, \ldots, C_{|E|-1})$ defined by $C_e = t(e) \cap h(e)$ for each $e \in E$. Assume F has components $\{f_0, f_1, \ldots, f_{m-1}\}$. For each $i \in \mathbb{Z}_m$, consider the directed cycle f_i of length l with $E(f_i) = \{e_0, e_1, \ldots, e_{l-1}\}$ where $h(e_k) = t(e_{k+1})$ for $k \in \mathbb{Z}_l$ with additions done modulo l. Form the following 3-paths in H_5: $T_i = \{(t(e_j), \{c(e_j), n, n+1\}, h(e_j), \{h(e_j) \backslash \{c(e_j), c(e_{j+1})\}, n, n+1\}) \mid j \in \mathbb{Z}_l\}$ with subscript additions done modulo l. The edges in T_i exist in H_5 since C_e is a list of the shared elements of $t(e)$ and $h(e)$. H_5 has a P_3-decomposition $(V(H_5), B_5)$ where

$B_5 = \bigcup_{i\in\mathbb{Z}_m} T_i$. To see that each edge in H_5 is in exactly one path in B_5, consider the edge $e = (\{a,b,c\},\{a,n,n+1\})$ in H_5. The vertex $\{a,b,c\}$ is in exactly one component, f_i, of F. Consider the two arcs, e_1 and e_2 in f_i such that $h(e_1) = \{a,b,c\}$ and $t(e_2) = \{a,b,c\}$. There are three possibilities.

1. If $c(e_1) = a$, then e is in $(t(e_1), \{a,n,n+1\}, \{a,b,c\}, \{\{b,c\}\backslash c(e_2), n, n+1)$.

2. If $c(e_2) = a$, then let e_3 be the arc in f_i with $t(e_3) = h(e_2)$. Then e is in $(\{a,b,c\},\{a,n,n+1\}, h(e_2), \{h(e_2)\backslash\{a, c(e_3)\}, n, n+1\})$.

3. If $a \notin \{c(e_1), c(e_2)\}$, the e is in $(t(e_1), \{c(e_1), n, n+1\}, \{a,b,c\}, \{a,n,n+1\})$.

Since F is a properly list arc-colored 2-factor, exactly one of the previous three cases holds. Thus every edge of H_5 is in exactly one path in B_5.

Let $B = \bigcup_{i\in\mathbb{Z}_6} B_i$. Then $(V(H), B)$ is the desired P_3-decomposition. $\square$

Chapter 3

P_4-decompositions of $KG_{n,2}$

3.1 Introduction

In this chapter, the problem of finding necessary and sufficient conditions for obtaining 4-path decompositions of $KG_{n,2}$ is completely solved. Again, recall that $T_k(V)$ is the set of k-element subsets of the set V, and let (a, b, c, d, e) denote the path, P_4, of length four with edge set $\{\{a, b\}, \{b, c\}, \{c, d\}, \{d, e\}\}$.

3.2 Useful Building Blocks

Billington and Hoffman solved a more general problem concerning P_4-decompositions of multipartite graphs [5], but the following will suffice for our purposes.

Lemma 3.1. *The complete bipartite graph K_{a_1,a_2} with $a_1 \leq a_2$ has a P_4-decomposition if and only if $a_1 \geq 2$, $a_2 \geq 3$ and $a_1a_2 \equiv 0 \pmod 4$.*

The next result provides specific ingredients used in the general constructions.

Lemma 3.2. *There exists a P_4-decomposition of:*

(i) the bipartite graph H_1 with partition $\{A = T_2(\mathbb{Z}_4), B = \mathbb{Z}_4\}$ of $V(H_1)$ and $E(H_1) = \{\{a, b\} | a \in A, b \in B, b \notin a\}$,

(ii) the bipartite graph H_2 with partition $\{A = T_2(\mathbb{Z}_6), B = \mathbb{Z}_6\}$ of $V(H_2)$ and $E(H_2) = \{\{a, b\} | a \in A, b \in B, b \notin a\}$,

(iii) $H_3(W, X, Y) = (W \cup X \cup Y, E)$, where W, X, and Y are disjoint sets of size 4, and $E = \{\{(i_1, l_1), (i_2, l_2)\} \mid l_1 \neq l_2, i_1 \in X \cup Y, i_2 \in W\}$,

(iv) $H_4 = (\mathbb{Z}_4 \times \mathbb{Z}_4, \{(i,j),(k,l) \mid i \neq k, j \neq l\})$,

(v) $H_5(W,X,Y,Z) = (W \cup X \cup Y \cup Z, E)$, *where* W, X, Y, *and* Z *are disjoint sets of size* 4, *and* $E = \{\{(i_1,l_1),(i_2,l_2)\} \mid l_1 \neq l_2, i_1 \in X \cup Y \cup Z, i_2 \in W\}$,

(vi) *the bipartite graph* H_6 *with partition* $\{A = T_2(\mathbb{Z}_5) \cup T(\mathbb{Z}_9 \backslash \mathbb{Z}_5), B = \mathbb{Z}_5\}$ *of* $V(H_6)$ *and* $E(H_6) = \{\{a,b\} | a \in A, b \in B, b \notin a\}$, *and*

(vii) K_8.

Proof.

(i) $(V(H_1), \{(0,\{1,2\},3,\{0,1\},2),(3,\{0,2\},1,\{2,3\},0),(0,\{1,3\},2,\{0,3\},1)\})$ is the required decomposition.

(ii) Let $B_1 = \{(1+3i,\{3i,2+3i\},5+3i,\{1+3i,2+3i\},4+3i),(3+3i,\{3i,2+3i\},4+3i,\{3i,1+3i\},5+3i),(2+3i,\{3i,1+3i\},3+3i,\{1+3i,2+3i\},3i) \mid i \in \{0,1\}\}$ with addition done modulo 6 and $B_2 = \{(j+1,\{j,3\},4,\{j,5\},j+2),(\{j,3\},j+2,\{j,4\},3,\{j,5\}),(\{j,3\},5,\{j,4\},j+1,\{j,5\}) \mid j \in \{0,1,2\}\}$ with addition done modulo 3. Then $(V(H_2), B_1 \cup B_2)$ is the required decomposition.

(iii) Let $W = \{w_1,w_2,w_3,w_4\}$, $X = \{x_1,x_2,x_3,x_4\}$, and $Y = \{y_1,y_2,y_3,y_4\}$. Then $(W \cup X \cup Y, \{(x_1,w_3,x_4,w_1,x_3),(x_2,w_4,x_3,w_2,x_4),(y_1,w_3,y_2,w_1,y_4),$ $(y_2,w_4,y_1,w_2,y_3),(w_1,x_2,w_3,y_4,w_2),(w_1,y_3,w_4,x_1,w_2)\})$ is the required decomposition.

(iv) The result follows from (iii), since H_4 is the union of the three graphs $H_3(\mathbb{Z}_4 \times \{i\}, \mathbb{Z}_4 \times \{j\}, \mathbb{Z}_4 \times \{k\})$, where $(i,j,k) \in \{(1,0,2),(3,2,1),(0,3,2)\}$

(v) Let $W = \{w_1,w_2,w_3,w_4\}$, $X = \{x_1,x_2,x_3,x_4\}$, $Y = \{y_1,y_2,y_3,y_4\}$, and $Z = \{z_1,z_2,z_3,z_4\}$. Then $(W \cup X \cup Y \cup Z, \{(x_0,w_2,x_1,w_0,x_2),$
$(x_1,w_3,x_2,w_1,x_3),(y_0,w_1,y_2,w_0,y_1),(y_3,w_2,y_1,w_3,y_2),(z_1,w_0,z_2,w_1,z_3),$

$(z_2, w_3, z_1, w_2, z_3), (z_3, w_0, y_3, w_1, z_0), (w_2, z_0, w_3, x_0, w_1), (w_0, x_3, w_2, y_0, w_3)\})$ is the required decomposition.

(vi) $(V(H_6), \{(\{0,1\}, 3, \{0,2\}, 4, \{5,6\}), (\{0,2\}, 1, \{0,3\}, 4, \{5,7\}),$
$(\{0,3\}, 2, \{0,4\}, 1, \{6,8\}), (\{0,4\}, 3, \{1,2\}, 4, \{5,8\}),$
$(\{1,2\}, 0, \{1,3\}, 4, \{6,7\}), (\{1,3\}, 2, \{1,4\}, 3, \{6,8\}),$
$(\{1,4\}, 0, \{2,3\}, 4, \{6,8\}), (\{2,3\}, 1, \{2,4\}, 3, \{7,8\}),$
$(\{2,4\}, 0, \{3,4\}, 1, \{7,8\}), (\{3,4\}, 2, \{0,1\}, 4, \{7,8\}),$
$(\{5,6\}, 1, \{5,7\}, 3, \{5,8\}), (\{5,6\}, 3, \{6,7\}, 1, \{5,8\}),$
$(\{5,6\}, 0, \{5,7\}, 2, \{5,8\}), (\{5,6\}, 2, \{6,7\}, 0, \{6,8\}),$
$(\{6,8\}, 2, \{7,8\}, 0, \{5,8\})\})$ is the required decomposition.

(vii) Define K_8 on the vertices $\mathbb{Z}_7 \cup \{\infty\}$. Then $(\mathbb{Z}_7 \cup \{\infty\}, \{(\infty, i, i+1, i-1, i+2) \mid i \in \mathbb{Z}_7\})$ with addition done modulo 7 is the required decomposition. □

Since the proof of Theorem 3.1 is based on a recursive construction, the next result gives an important starting point.

Lemma 3.3. *$KG_{16,2}$ is P_4-decomposable.*

Proof. Consider $G = KG_{16,2}$ on vertex set $T(\mathbb{Z}_{16})$. Partition $\mathbb{Z}_{16}$ into four sets $S_i = \{4i, 4i+1, 4i+2, 4i+3\}$ for $i \in \mathbb{Z}_4$. Partition the set of vertices $T(\mathbb{Z}_{16})$ of G into the following two types:

Type 1: $V_i = \{\{x, y\} \mid x, y \in S_i, x \neq y\}$ for each $i \in \mathbb{Z}_4$, and

Type 2: $V_{i,j} = \{\{x, y\} \mid x \in S_i, y \in S_j\}$ for $0 \leq i < j < 4$.

First, the subgraph G' of G induced by the Type 1 vertices is considered. Decompose G' into paths of length four in two steps. First define three pairs of vertices $M_{0,i} = \{\{4i, 4i+$

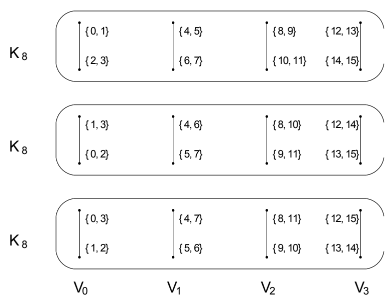

Figure 3.1: The K_8's

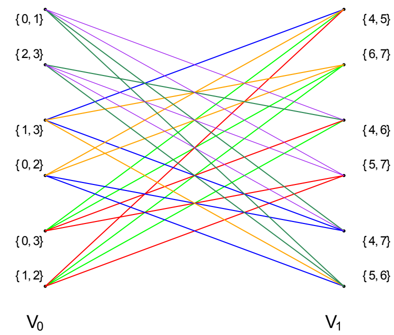

Figure 3.2: $B_{0,1,j}$

$1\}, \{4i+2, 4i+3\}\}$, $M_{1,i} = \{\{4i, 4i+2\}, \{\{4i+1, 4i+3\}\}$, and $M_{2,i} = \{\{4i, 4i+3\}, \{\{4i+1, 4i+2\}\}$. For each $j \in \mathbb{Z}_3$, the subgraph G'_j of G' induced by $\bigcup_{i\in\mathbb{Z}_4} M_{j,i}$ is isomorphic to K_8 (Figure 3.1) which therefore has a P_4-decomposition $(\bigcup_{i\in\mathbb{Z}_4} M_{j,i}, B'_j)$ by Lemma 3.2(vii). Let $B_1 = \bigcup_{j\in\mathbb{Z}_3} B'_j$. Second, for $0 \le i_1 < i_2 \le 3$ and for $j \in \mathbb{Z}_3$, the induced bipartite subgraph $G'_{i_1,i_2,j}$ of G' with bipartition $\{M_{j,i_1}, \bigcup_{k\in\mathbb{Z}_3\setminus\{j\}} M_{k,i_2}\}$ is isomorphic to $K_{2,4}$ so has a P_4-decomposition $(V(G'_{i_1,i_2,j}), B'_{i_1,i_2,j})$ by Lemma 3.1. Let $B_2 = \bigcup_{0\le i_1<i_2\le 3, j\in\mathbb{Z}_3} B'_{i_1,i_2,j}$. (See Figure 3.2 for an example of the decomposition of the other edges beteween V_0 and V_1).

All edges connecting Type 1 vertices have now been placed into 4-paths in $B_1 \cup B_2$. The remaining edges are those connecting Type 2 vertices and those connecting a Type 1 vertex to a Type 2.

The subgraph $G_{i,j}$ of G induced by the vertices in $V_{i,j}$ for $0 \le i < j < 4$ is isomorphic to H_4, so has a P_4-decomposition $(V(G_{i,j}), B_{i,j})$ by Lemma 3.2(iv). Let $B_3 = \bigcup_{0\le i<j<4} B_{i,j}$.

Next, consider the subgraph $G_{i,j,k}$ of G induced by the edges joining the vertices in $V_{i,j}$ to the vertices in V_k where $0 \le i < j < 4$ and $k \in \mathbb{Z}_4$. If $k \notin \{i, j\}$, then $G_{i,j,k}$ is isomorphic to $K_{16,6}$ and has a P_4-decomposition $(V(G_{i,j,k}), B_{i,j,k})$ by Lemma 3.1. If $k \in \{i, j\}$ then without loss of generality assume that $k = i$. Then $G_{i,j,k}$ consists of four edge disjoint copies of $K_{3,4}$ (induced by the edges connecting vertices in $\{\{x, y\} \mid y \in S_j\}$ to vertices in V_k for each $x \in S_{i=k}$). So $G_{i,j,k}$ has a P_4-decomposition $(V(G_{i,j,k}), B_{i,j,k})$ by Lemma 3.1. Let $B_4 = \bigcup_{0\le i<j<4, k\in\mathbb{Z}_4} B_{i,j,k}$.

Finally, consider the subgraph $G_{i,j,k,l}$ of G induced by the edges joining the vertices in $V_{i,j}$ to the vertices in $V_{k,l}$ where $0 \le i < j < 4$, $0 \le k < l < 4$, and $|\{i,j\} \cap \{k,l\}| < 2$. If $\{i,j\} \cap \{k,l\} = \emptyset$, then $G_{i,j,k,l}$ is isomorphic to $KG_{16,16}$ and so has a P_4-decomposition $(V(G_{i,j,k,l}), B_{i,j,k,l})$ by Lemma 3.1. If $|\{i,j\} \cap \{k,l\}| = 1$ then without loss of generality assume that $i = k$. Then $G_{i,j,k,l}$ consists of four edge disjoint copies of $K_{4,12}$ (induced by the edges connecting vertices in $\{\{x,y\} \mid y \in S_j\}$ to vertices in $V_{k,l}$ for each $x \in S_{i=k}$). So $G_{i,j,k,l}$ has a P_4-decomposition $(V(G_{i,j,k,l}), B_{i,j,k,l})$ by Lemma 3.1. Let $B_5 = \bigcup_{0 \le i<j<4, 0\le k<l<4, |\{i,j\}\cap\{k,l\}|<2} B_{i,j,k,l}$

All of the edges have now been placed into 4-paths, thus $(V(G), \bigcup_{1\le i\le 5} B_i)$ is a P_4-decomposition of G. □

3.3 A P_4-Decomposition of $KG_{n,2}$

We are now ready to prove the main theorem.

Theorem 3.1. *$KG_{n,2}$ is P_4-decomposable if and only if $n \equiv 0, 1, 2$ or 3 (mod 16).*

Proof. The necessity follows from the observation that $|E(KG_{n,2})| = n(n-1)(n-2)(n-3)/8$ is a multiple of 4 if and only if $n \equiv 0, 1, 2$ or 3 (mod 16).

If $n \in \{0, 1, 2, 3\}$, then $KG_{n,2}$ has no edges, so the result is vacuously true. $KG_{16,2}$ has a P_4-decomposition by Lemma 3.3. The remaining cases are proved by induction on n. Suppose for some $w \ge 1$ that for all $t \le 16w$ with $t \equiv 0, 1, 2$ or 3 (mod 16) there exists a P_4-decomposition $(V(KG_{t,2}), B)$ of $KG_{t,2}$. It remains to find a P_4-decomposition of $KG_{n,2}$ for each $n = \{16w + 1, 16w + 2, 16w + 3, 16w + 16\}$.

First suppose $n = 16w+16$. By the inductive hypothesis there exists a P_4-decomposition $(V(KG_{16w,2}), B)$ of $KG_{16w,2}$. Let $X = \{x_i \mid i \in \mathbb{Z}_{16}\}$ and $S = \mathbb{Z}_{16w} \cup X$. Consider $G = KG_{16w+16,2}$ on the vertex set $T(S)$. Partition S into $4w + 4$ sets as follows: $S_i = \{4i, 4i + 1, 4i + 2, 4i + 3\}$ for $i \in \mathbb{Z}_{4w}$ and $X_j = \{x_{4j}, x_{4j+1}, x_{4j+2}, x_{4j+3}\}$ for $j \in \mathbb{Z}_4$. Using this partition of S, partition the vertices of G into the following types:

(a) $V_i = \{\{x, y\} \mid x, y \in S_i, x \neq y\}$ for each $i \in \mathbb{Z}_{4w}$,

(b) $V'_i = \{\{x, y\} \mid x, y \in X_i, x \neq y\}$ for each $i \in \mathbb{Z}_4$,

(c) $V_{i,j} = \{\{x, y\} \mid x \in S_i, y \in S_j\}$ for $0 \leq i < j < 4w$,

(d) $V'_{i,j} = \{\{x, y\} \mid x \in X_i, y \in X_j\}$ for $0 \leq i < j < 4$, and

(e) $S_{i,j} = \{\{x, y\} \mid x \in S_i, y \in X_j\}$ for each $i \in \mathbb{Z}_{4w}$ and each $j \in \mathbb{Z}_4$.

It is convenient to refer to the vertices in $(\bigcup_{i \in \mathbb{Z}_{4w}} V_i) \cup (\bigcup_{0 \leq i < j < 4w} V_{i,j})$ as 'old' vertices and the vertices in $(\bigcup_{i \in \mathbb{Z}_4} V'_i) \cup (\bigcup_{0 \leq i < j < 4} V'_{i,j})$ as 'new' vertices. Consider the subgraph G_1 of G induced by the old vertices. G_1 is isomorphic to $KG_{16w,2}$ and therefore has a P_4-decomposition $(V(G_1), B_1)$ by the inductive hypothesis. The subgraph G_2 of G induced by the new vertices is isomorphic to $KG_{16,2}$ and has a P_4-decomposition $(V(G_2), B_2)$ by Lemma 3.3. The bipartite subgraph G_3 of G formed by the edges $\{\{x, y\} \mid x$ is an old vertex, y is a new vertex$\}$ is isomorphic to the complete bipartite graph $K_{\binom{16w}{2},\binom{16}{2}}$ and thus has a P_4-decomposition $(V(G_3), B_3)$ by Lemma 3.1.

The only edges of G left to place into paths are those in the subgraph of G induced by each $S_{i,j}$, those connecting the vertices in each $S_{i,j}$ to the old and new vertices, and those connecting vertices in each $S_{i,j}$ to vertices in each other $S_{k,l}$.

The subgraph $G_{i,j}$ of G induced by the vertices in $S_{i,j}$ for $i \in \mathbb{Z}_{4w}$ and $j \in \mathbb{Z}_4$ is isomorphic to H_4 and thus has a P_4-decomposition $(V(G_{i,j}), B_{i,j})$ by Lemma 3.2(iv). Let $B_4 = \bigcup_{i \in \mathbb{Z}_{4w}, j \in \mathbb{Z}_4} B_{i,j}$.

For each $i, k \in \mathbb{Z}_{4w}$ and $j \in \mathbb{Z}_4$, consider the bipartite subgraph $G_{i,j,k}$ of G induced by the edges joining the vertices in $S_{i,j}$ to the vertices in V_k. If $i \neq k$ then $G_{i,j,k}$ is isomorphic to $K_{16,6}$ and has a P_4-decomposition $(V(G_{i,j,k}), B_{i,j,k})$ by Lemma 3.1. If $i = k$ then $G_{i,j,k}$ consists of four edge disjoint copies of $K_{3,4}$ (induced by the edges connecting the vertices in $\{\{a, y\} \mid y \in X_j\}$ to those in $V_{i=k}$ for each $a \in S_i$). So in all cases $G_{i,j,k}$ has a P_4-decomposition $(V(G_{i,j,k}), B_{i,j,k})$ by Lemma 3.1. Let $B_5 = \bigcup_{i,k\in\mathbb{Z}_{4w}, j\in\mathbb{Z}_4} B_{i,j,k}$.

For each $i \in \mathbb{Z}_{4w}$, $j, k \in \mathbb{Z}_4$, consider the bipartite subgraph $G'_{i,j,k}$ of G induced by the edges joining the vertices in $S_{i,j}$ to the vertices in V'_k. If $j \neq k$, then $G'_{i,j,k}$ is isomorphic to $K_{16,6}$ and has a P_4-decomposition $(V(G'_{i,j,k}), B'_{i,j,k})$ by Lemma 3.1. If $j = k$, then $G'_{i,j,k}$ consists of four edge disjoint copies of $K_{3,4}$ (induced by the edges connecting the vertices in $\{\{a, x\} \mid a \in S_i\}$ to those in V'_k for each $x \in X_k$). So $G'_{i,j,k}$ has a P_4-decomposition $(V(G'_{i,j,k}), B'_{i,j,k})$ by Lemma 3.1. Let $B_6 = \bigcup_{i\in\mathbb{Z}_{4w}, j,k\in\mathbb{Z}_4} B'_{i,j,k}$.

For each $i, k, l \in \mathbb{Z}_{4w}$, $k \neq l$, and $j \in \mathbb{Z}_4$, consider the bipartite subgraph $G_{i,j,k,l}$ of G induced by the edges joining the vertices in $S_{i,j}$ to the vertices in $V_{k,l}$. If $i \notin \{k, l\}$, then $G_{i,j,k,l}$ is isomorphic to $K_{16,16}$ and thus has a P_4-decomposition $(V(G_{i,j,k,l}), B_{i,j,k,l})$ by Lemma 3.1. If $i \in \{k, l\}$ then without loss of generality assume that $i = k$. Then $G_{i,j,k,l}$ consists of four edge disjoint copies of $K_{12,4}$ (induced by the edges connecting the vertices in $\{\{a, y\} \mid y \in X_j\}$ to those in $V_{i=k,l}$ for each $a \in S_{i=k}$). So $G_{i,j,k,l}$ has a P_4-decomposition $(V(G_{i,j,k,l}), B_{i,j,k,l})$ by Lemma 3.1. Let $B_7 = \bigcup_{i,k,l\in\mathbb{Z}_{4w}, j\in\mathbb{Z}_4, k\neq l} B_{i,j,k,l}$.

For each $i \in \mathbb{Z}_{4w}$ and $j, k, l \in \mathbb{Z}_4$ with $k \neq l$, consider the bipartite subgraph $G'_{i,j,k,l}$ of G induced by the edges joining the vertices in $S_{i,j}$ to the vertices in $V'_{k,l}$. If $j \notin \{k, l\}$, then $G'_{i,j,k,l}$ is isomorphic to $K_{16,16}$ and thus has a P_4-decomposition $(V(G'_{i,j,k,l}), B'_{i,j,k,l})$ by Lemma 3.1. If $j \in \{k, l\}$ then without loss of generality assume that $j = k$. Then $G'_{i,j,k,l}$ consists of four edge disjoint copies of $K_{12,4}$ (induced by the edges connecting the vertices in $\{\{a, x\} \mid a \in S_i\}$ to those in $V'_{j=k,l}$ for each $x \in X_{j=k}$). So $G'_{i,j,k,l}$ has a P_4-decomposition $(V(G'_{i,j,k,l}), B'_{i,j,k,l})$ by Lemma 3.1. Let $B_8 = \bigcup_{i\in\mathbb{Z}_{4w}, j,k,l\in\mathbb{Z}_4, k\neq l} B'_{i,j,k,l}$.

Finally, for each $i, k \in \mathbb{Z}_{4w}$ and $j, l \in \mathbb{Z}_4$ with $i \neq k$ and/or $j \neq l$, consider the bipartite subgraph $S'_{i,j,k,l}$ of G induced by the edges joining the vertices in $S_{i,j}$ to the vertices in $S_{k,l}$. If $\{i, j\} \cap \{k, l\} = \emptyset$, then $S'_{i,j,k,l}$ is isomorphic to $K_{16,16}$ and has a P_4-decomposition

$(V(S'_{i,j,k,l}), B'_{i,j,k,l})$ by Lemma 3.1. If $i = k$ and $j \neq l$, then $S'_{i,j,k,l}$ consists of four edge disjoint copies of $K_{12,4}$ (induced by the edges connecting the vertices in $\{\{a, x\} \mid x \in X_j\}$ to those in $S'_{i=k,l}$ for each $a \in S_{i=k}$). If $i \neq k$ and $j = l$, then $S'_{i,j,k,l}$ consists of four edge disjoint copies of $K_{12,4}$ (induced by the edges connecting the vertices in $\{\{a, x\} \mid a \in S_i\}$ to those in $S'_{k,j=l}$ for each $x \in X_{j=l}$). In either case, $S'_{i,j,k,l}$ has a P_4-decomposition $(V(S'_{i,j,k,l}), B'_{i,j,k,l})$ by Lemma 3.1. Let $B_9 = \bigcup_{i,k \in \mathbb{Z}_z, j,l \in \mathbb{Z}_4, i \neq k or j \neq l} B'_{i,j,k,l}$.

$(V(G), \bigcup_{1 \leq i \leq 9} B_i)$ is a P_4-decomposition of $G = KG_{16w+16,2}$.

It now remains to find a P_4-decomposition of $KG_{n,2}$ for each $n \in \{16w+1, 16w+2, 16w+3\}$. To construct these P_4-decompositions, we will extend a P_4-decomposition $(T(\mathbb{Z}_{n-1}), B_0)$ of $KG_{n-1,2}$ to a P_4-decomposition of $KG_{n,2}$. Define $\epsilon \equiv n - 1 (\text{mod } 16)$ with $\epsilon \in \{0, 1, 2\}$.

Partition $\mathbb{Z}_{n-1}$ into $4w$ sets: $S_i = \{4i, 4i + 1, 4i + 2, 4i + 3\}$ for $i \in \mathbb{Z}_{4w-1}$ and $S_{4w-1} = \mathbb{Z}_{n-1} \backslash \mathbb{Z}_{16w-4}$. It is convenient to partition the vertices of the form $T(\mathbb{Z}_{n-1})$, into the following two types:

$V_i = \{\{x, y\} \mid x, y \in S_i, x \neq y\}$ for $i \in \mathbb{Z}_{4w}$, and

$V_{i,j} = \{\{x, y\} \mid x \in S_i, y \in S_j\}$ for $0 \leq i < j < 4w$.

Further, partition the vertices containing the new element $n - 1$ into the $4w$ sets:

$S'_i = \{\{x, n - 1\} \mid x \in S_i\}$ for each $i \in \mathbb{Z}_{4w}$.

The subgraph of $KG_{n,2}$ induced by vertices in $T(\mathbb{Z}_n) \backslash T(\mathbb{Z}_{n-1})$ clearly has no edges, since they all share the element $n - 1$. All that needs to be shown is that the bipartite subgraph

G' induced by the edges connecting verices in $T(\mathbb{Z}_{n-1})$ to vertices in $T(\mathbb{Z}_n)\backslash T(\mathbb{Z}_{n-1})$ has a P_4-decomposition.

It will be helpful in the following discussion to define the following bipartite subgraphs of G':

$G_{i,j}$ is the bipartite subgraph induced by the edges in $\{\{x, y\} \mid x \in V_i, y \in S'_j\}$, and

$G_{i,j,k}$ is the bipartite subgraph induced by the edges in $\{\{x, y\} \mid x \in V_{i,j}, y \in S'_k\}$.

The following parts of the construction are identical for all ϵ.

For each $i, j \in \mathbb{Z}_{4w-1}$, consider $G_{i,j}$. If $i \neq j$, then $G_{i,k}$ is isomorphic to $K_{6,4}$ and therefore has a P_4-decomposition $(V(G_{i,j}), B_{i,j})$ by Lemma 3.1. If $i = j$, then $G_{i,k}$ is isomorphic to H_1 and therefore has a P_4-decomposition $(V(G_{i,j}), B_{i,j})$ by Lemma 3.2(i).

For each $i \in \mathbb{Z}_{4w-1}$, $G_{4w-1,i}$ is isomorphic to $K_{\binom{4+\epsilon}{2},4}$ and therefore has a P_4-decomposition $(V(G_{4w-1,i}), B_{4w-1,i})$ by Lemma 3.1.

For $0 \leq i < j < 4w - 1$ and $k \in \mathbb{Z}_{4w-1}$, consider $G_{i,j,k}$. If $k \notin \{i, j\}$, then $G_{i,j,k}$ is isomorphic to $K_{16,4}$ and therefore has a P_4-decomposition $(V(G_{i,j,k}), B_{i,j,k})$ by Lemma 3.1. If $k \in \{i, j\}$ then without loss of generality assume that $i = k$. Then $G_{i,j,k}$ consists of four edge disjoint copies of $K_{3,4}$ (induced by the edges connecting the vertices in $\{\{a, y\} \mid y \in S_j\}$ to those in $S'_{i=k}$ for each $a \in S_{i=k}$). Thus $G_{i,j,k}$ has a P_4-decomposition $(V(G_{i,j,k}), B_{i,j,k})$ by Lemma 3.1.

For $0 \leq i < j < 4w - 1$ and $k = 4w - 1$, $G_{i,j,k}$ is isomorphic to $K_{16,4+\epsilon}$ and thus has a P_4-decomposition $(V(G_{i,j,4w-1}), B_{i,j,4w-1})$ by Lemma 3.1.

The only edges remaining are those in $G_{i,4w-1}$ where $i \in \mathbb{Z}_{4w}$ and those in $G_{i,4w-1,k}$ where $i \in \mathbb{Z}_{4w-1}$ and $k \in \mathbb{Z}_{4w}$. The way these situations are treated depends on ϵ.

First, suppose that $\epsilon \in \{0, 2\}$.

For each $i \in \mathbb{Z}_{4w}$, consider $G_{i,4w-1}$. If $i \neq 4w-1$ then $G_{i,4w-1}$ is isomorphic to $K_{6,4+\epsilon}$ and therefore has a P_4-decomposition $(V(G_{i,4w-1}), B_{i,4w-1})$ by Lemma 3.1. If $i = 4w-1$ then $G_{i=4w-1,4w-1}$ is isomorphic to H_1 if $\epsilon = 0$ and isomorphic to H_2 if $\epsilon = 2$. In either case, $G_{i=4w-1,4w-1}$ has a P_4-decomposition $(V(G_{4w-1,4w-1}), B_{4w-1,4w-1})$ by Lemma 3.2(i) and (ii) respectively.

Next, for each $i \in \mathbb{Z}_{4w-1}$ and each $k \in \mathbb{Z}_{4w}$, consider $G_{i,4w-1,k}$. If $k \notin \{i, 4w-1\}$ then $G_{i,4w-1,k}$ is isomorphic to $K_{4(4+\epsilon),4}$ and therefore has a P_4-decomposition $(V(G_{i,4w-1,k}), B_{i,4w-1,k})$ by Lemma 3.1. If $k = i$, then $G_{i,4w-1,k}$ is the disjoint union of the graphs $H_3(\{\{a, n-1\} \mid a \in S_i\}, \{\{a, n-2j\} \mid a \in S_i\}, \{\{a, n-2j-1\} \mid a \in S_i\})$ for each $j \in \{1, 2\}$ if $\epsilon = 0$ and for each $j \in \{1, 2, 3\}$ if $\epsilon = 2$. Each of these subgraphs has a P_4-decomposition by Lemma 3.2(iii), so their union forms a P_4-decomposition $(V(G_{i,4w-1,k=i}), B_{i,4w-1,k=i})$ of $G_{i,4w-1,k=i}$. If $k = 4w-1$ then $G_{i,4w-1,k=4w-1}$ consists of $4+\epsilon$ edge disjoint copies of $K_{4,3+\epsilon}$ (induced by the edges connecting the vertices in $\{\{x, a\} \mid x \in S_i\}$ to those in $S'_{k=4w-1}$ for each $a \in S_{k=4w-1}$). So $G_{i,4w-1,k=4w-1}$ has a P_4-decomposition $(V(G_{i,4w-1,k}), B_{i,4w-1,k})$ by Lemma 3.1.

Let $B_1 = \bigcup_{i,j \in \mathbb{Z}_t} B_{i,j}$ and $B_2 = \bigcup_{0 \leq i < j \leq t-1, k \in \mathbb{Z}_t} B_{i,j,k}$. The required P_4-decomposition of G is given by $(V(G), \bigcup_{i \in \mathbb{Z}_3} B_i)$.

Finally, suppose that $\epsilon = 1$.

For each $j \in \mathbb{Z}_{2w}$, consider $G'_j = G_{2j,4w-1} \cup G_{2j+1,4w-1}$. If $j \neq 2w-1$, then G'_j is isomorphic to $K_{12,5}$ and therefore has a P_4-decomposition $(V(G'_j), B'_{j,4w-1})$ by Lemma 3.1. If $j = 2w-1$, then G'_j is isomorphic to H_6 and therefore has a P_4-decomposition $(V(G'_j), B_{j,4w-1})$ by Lemma 3.2(vi).

For each $i \in \mathbb{Z}_{4w-1}$ and each $k \in \mathbb{Z}_{4w}$, consider $G_{i,4w-1,k}$. If $k \notin \{i, 4w-1\}$, then $G_{i,4w-1,k}$ is isomorphic to $K_{20,4}$ and therefore has a P_4-decomposition $(V(G_{i,4w-1,k}), B_{i,4w-1,k})$ by Lemma 3.1. If $k = i$, then $G_{i,4w-1,k}$ is the disjoint union of the graphs $H_5(\{\{a, n-1\} \mid a \in S_i\}, \{\{a, n-2\} \mid a \in S_i\}, \{\{a, n-3\} \mid a \in S_i\}, \{\{a, n-4\} \mid a \in S_i\})$ and $H_3(\{\{a, n-1\} \mid a \in S_i\}, \{\{a, n-5\} \mid a \in S_i\}, \{\{a, n-6\} \mid a \in S_i\})$ and so has a P_4-decomposition $(V(G_{i,4w-1,k=i}), B_{i,4w-1,k=i})$ by Lemma 3.2 (v) and (iii). If $k = 4w-1$, then

$G_{i,4w-1,k=4w-1}$ consists of five edge disjoint copies of $K_{4,4}$ (induced by the edges connecting the vertices in $\{\{x,a\} \mid x \in S_i\}$ to those in $S'_{k=4w-1}$ for each $a \in S_{k=4w-1}$). So $G_{i,4w-1,k=4w-1}$ has a P_4-decomposition $(V(G_{i,4w-1,k}), B_{i,4w-1,k})$ by Lemma 3.1.

Let $B_1 = \bigcup_{i\in\mathbb{Z}_{4w}, j\in\mathbb{Z}_{4w-1}} B_{i,j}$, $B_2 = \bigcup_{j\in\mathbb{Z}_{2w}} B'_{j,4w-1}$ and $B_3 = \bigcup_{0\le i<j\le t-1, k\in\mathbb{Z}_{4w}} B_{i,j,k}$. The required P_4-decomposition of G is given by $(V(G), \bigcup_{0\le i\le 3} B_i)$.

□

Chapter 4

Embedding Partial P_3-systems

4.1 Introduction

In this chapter, the problem of embedding maximal partial 3-path designs is completely solved. Recall that the embedding problem is that for each partial H-design (V', P') of order n, find the set of integers M such that $m \in M$ if and only if there exists an H-design (V, P) of K_m such that $V' \subseteq V$ and $P' \subseteq P$. Recall also that the existence of P_3-designs was settled by Tarsi [28].

Theorem 4.1. *There exists a P_m-design of order n if and only if $\frac{n(n+1)}{2} \equiv 0$ (mod m) and $n \geq m+1$.*

4.2 Building Blocks

In the discussion that follows we will use the following definitions. A star of order k, S_k, is the complete bipartite graph $K_{1,k}$ (possibly $k = 0$). The vertex $c(S_k)$ of S_k with degree k when $k \geq 2$ is said to be the center of S_k; if $k = 1$ then either vertex can be designated to be the center. The *leave* of a partial H-design (V, P) of order n is the graph $L = (V(K_n), E(K_n) \backslash E(P))$. A partial H-design is said to be *maximal* if its leave has no proper subgraphs isomorphic to H. Let (a, b, c, d) denote the path, P_3, of length three with edge set $\{\{a, b\}, \{b, c\}, \{c, d\}\}$. The disjoint union of two graphs G and H, denoted $G + H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$.

The Billington and Hoffman result [5] will be very useful in the constructions to come:

Lemma 4.1. *The complete bipartite graph K_{a_1,a_2} with $a_1 \leq a_2$ has a P_3-decomposition if and only if $a_1 \geq 2$, $a_2 \geq 3$ and $a_1a_2 \equiv 0$ (mod 3).*

The following lemmas will be useful in the constructions to come.

Lemma 4.2. *There exists a P_3-decomposition of the following graphs:*

(i) $H_1 = (S_3 \vee K_2^C) - F$ *where* $F = \{\{x, c(S_3)\} \mid x \in V(K_2^C)\}$,

(ii) $H_2 = (S_3 \vee K_3^C) - F$ *where* $F = \{\{x, c(S_3)\} \mid x \in V(K_3^C)\}$,

(iii) $H_3 = (S_3 \vee K_4^C) - F$ *where* $F = \{\{x, c(S_3)\} \mid x \in V(K_4^C)\}$,

(iv) $H_4 = ((S_2 + S_1) \vee K_2^C) - F$ *where* $F = \{\{x, c\} \mid x \in V(K_2^C), c \in \{c(S_2), c(S_1)\}\}$,

(v) $H_5 = ((S_2 + S_1) \vee K_3^C) - F$ *where* $F = \{\{x, c\} \mid x \in V(K_3^C), c \in \{c(S_2), c(S_1)\}\}$,

(vi) $H_6 = ((S_2 + S_1) \vee K_4^C) - F$ *where* $F = \{\{x, c\} \mid x \in V(K_4^C), c \in \{c(S_2), c(S_1)\}\}$, *and*

(vii) $H_7 = S_1 \vee K_4^C$.

Proof. For each of the following, let $V(K_n^C) = \{v_1, v_2, \ldots, v_n\}$, $V(S_3) = \{c, a_1, a_2, a_3\}$ where c is the center, $V(S_2) = \{c_2, a_1, a_2\}$ and $V(S_1) = \{c_1, b_1\}$ where c_2 and c_1 are the centers of S_2 and S_1 respectively.

(i) $(V(H_1), \{(c, a_3, v_1, a_2), (a_1, c, a_2, v_2), (a_3, v_2, a_1, v_1)\})$ is the desired decomposition.

(ii-iii) Let $n \in \{2, 3\}$. The subgraph of H_n obtained by removing the edges in the two three paths (c, a_1, v_1, a_2) and (a_2, c, a_3, v_1) is isomorphic to $K_{n,3}$, so has a P_3-decomposition $(V(H_n), B_n)$ by Lemma 4.1. Then $(V(H_n), B_n \cup \{(c, a_1, v_1, a_2), (a_2, c, a_3, v_1)\})$ gives the desired decomposition.

(iv) $(V(H_4), \{(c_1, b_1, v_1, a_2), (a_1, c_2, a_2, v_2), (b_1, v_2, a_1, v_1)\})$ is the desired decomposition.

(v-vi) Let $n \in \{5, 6\}$. The subgraph of H_n obtained by removing the edges in the two three paths (c_1, b_1, v_1, a_1) and (a_1, c_2, a_2, v_1) is isomorphic to $K_{n-3,3}$, so has a P_3-decomposition $(V(H_n), B_n)$ by Lemma 4.1. Then $(V(H_n), B_n \cup \{(c_1, b_1, v_1, a_1), (a_1, c_2, a_2, v_1)\})$ gives the desired decomposition.

(vii) $(V(H_7), \{(v_1, c_1, b_1, v_4), (v_1, b_1, v_2, c_1), (v_4, c_1, v_3, b_1)\}$ is the desired decomposition. $\square$

Lemma 4.3. *There exists a P_3-decomposition of the following graphs:*

(i) $G_1 = ((S_1 + S_1 + S_1) \vee K_2^C)$,

(ii) $G_2 = ((S_1 + S_1 + S_1) \vee K_3^C)$, *and*

(iii) $G_3 = ((S_1 + S_1 + S_1) \vee K_4^C)$.

Proof. For each of the following, let $V(K_n^C) = \{v_1, v_2, \ldots, v_n\}$ and let the vertex sets for the three one stars be $\{a_1, a_2\}$, $\{b_1, b_2\}$, and $\{c_1, c_2\}$.

(i) $(V(G_1), \{(a_1, a_2, v_1, b_1), (b_1, b_2, v_2, a_2), (b_1, v_2, c_1, c_2), (b_2, v_1, a_1, v_2),$ $(v_2, c_2, v_1, c_1)\})$ is the required decomposition.

(ii-iii) Let $n \in \{2, 3\}$. The subgraph of G_n obtained by removing the edges in the three three paths $(a_1, a_2, v_1, b_1), (b_1, b_2, v_1, c_1)$, and (c_1, c_2, v_1, a_1) is isomorphic to $K_{n,6}$, so has a P_3-decomposition $(V(G_n), B_n)$ by Lemma 4.1. Then $(V(G_n), B_n \cup \{(a_1, a_2, v_1, b_1),$ $(b_1, b_2, v_1, c_1), (c_1, c_2, v_1, a_1)\})$ gives the desired decomposition. $\square$

Lemma 4.4. *There exists a P_3-decomposition of K_4.*

Proof. Let $V(K_4) = \mathbb{Z}_4$. Then $(V(K_4), \{(0, 1, 2, 3), (0, 2, 1, 3)\}$ is the required decomposition. $\square$

Lemma 4.5. *There exists a P_3-decomposition of $K_3 \vee K_m^C$ for all $m \geq 1$.*

Proof. Let $V(K_3) = \{a, b, c\}$ and $V(K_m^C) = \mathbb{Z}_m$. For $m = 1$, the graph is isomorphic to K_4 and has a P_3-decomposition by Lemma 4.4. For $m = 2$, $(V(K_2^C), \{(1, a, c, 2),$ $(a, b, 1, c), (c, b, 2, a)\})$ is the required decomposition. For $m \geq 3$, consider the subgraph G of $K_3 \vee K_m^C$ induced by the vertices in $\{a, b, c, 1\}$. G is isomorphic to K_4, and has a P_3-decomposition by Lemma 4.4. The remaining edges in $(K_3 \vee K_m^C) - G$ are isomorphic to $K_{3,m-1}$ and have a P_3-decomposition by Lemma 4.1. The union of these decompositions give the required decomposition. $\square$

4.3 Embedding Partial P_3-designs.

We now prove the main results.

Theorem 4.2. *Let $k \geq n+2$. A partial P_3-design of order $n \geq 2$ can be embedded in a P_3-design of order k if and only if $k \equiv 0$ or 1 (mod 3) and $k \geq 4$.*

Proof. Suppose there exists a P_3-design of order k. Then it must be the case that the number of edges in K_k is a multiple of three. This occurs when $k \equiv 0$ or 1 (mod 3). Further, since K_3 has no P_3-design, it must be the case that $k \geq 4$. So the necessity is proved.

To prove the sufficiency, first note that if $n \leq 2$ then the result follows from Theorem 4.1, so assume that $n \geq 3$. Begin by adding copies of P_3 to the given partial P_3-design until a maximal partial P_3-design (V, B) of order n is obtained. The result will follow once it is shown that (V, B) can be embedded in P_3-designs of orders $n+3$ and $n+4$ when $n \equiv 0$ (mod 3), orders $n+2$ and $n+3$ when $n \equiv 1$ (mod 3), and order $n+2$ when $n \equiv 2$ (mod 3), since the embedding for all values larger than these can be obtained through repeated application of these small embeddings.

Since (V, B) is a maximal partial P_3-design, each component of its leave, L, must be a K_3 or a star (possibly with no edges), for otherwise it would contain a path of length three. Let $T = \{T_1, T_2, \ldots, T_t\}$ be the set of components of L which are isomorphic to K_3, let P be the subgraph of L consisting of the components isomorphic to S_i with $i \geq 2$, let R be the subgraph of L consisting of components that are isomorphic to $S_1 = K_2$, and I be the set of isolated vertices in L (copies of S_0).

Let $m \in \{2, 3, 4\}$ such that $n+m \equiv 0$ or 1 (mod 3). Construct a P_3-design $(V \cup V', B')$ of order $n+m$ with $B \subseteq B'$ as follows.

For each $1 \leq i \leq t$, the subgraph T_i' of K_{n+m} induced by the edges in $E(T_i) \cup \{\{x, y\} \mid x \in V(T_i), y \in V'\}$ is isomorphic to $K_3 \vee K_m^C$ and has a P_3-decomposition $(V(T_i'), B_{T_i})$ by Lemma 4.5. Let $B_T = \bigcup_{1 \leq i \leq t} B_{T_i}$.

Let $\{R_i \mid 1 \leq i \leq \lceil |E(R)|/3 \rceil\}$ be a partition of $E(R)$ with $|R_i| = 3$ for each $i \leq |E(R)|/3$. For each $i \leq |E(R)|/3$, the subgraph of K_{n+m} induced by the edges in $E(R_i) \cup \{\{x, y\} \mid x \in V(R_i), y \in V'\}$ is isomorphic to $(S_1 + S_1 + S_1) \vee K_m^C$, so has a P_3-decomposition $(V(R_i), B_{R_i})$ by Lemma 4.3. Let $B_R = \bigcup_{1 \leq i \leq |E(R)|/3} B_{R_i}$. If $|E(R)| \equiv 1$ or $2 \pmod 3$, then the subgraph induced by $R_{\lceil |E(R)|/3 \rceil}$ is isomorphic to S_1 or $(S_1 + S_1)$ respectively; the edges in these subgraphs have not yet been placed into 3-paths.

Let $C_1, C_2, \ldots, C_l$ be the components of P. Let $\{P_i \mid 1 \leq i \leq \lceil |E(P)|/3 \rceil\}$ be a partition of $E(P)$ with $|P_i| = 3$ for each $i \leq |E(P)|/3$ such that for all $i < j$, if $e_1 \in (P_i \cap E(C_k))$ and $e_2 \in (P_j \cap E(C_l))$ then $k \leq l$. This partion ensures that for each $i \leq |E(P)|/3$, P_i induces a subgraph of L isomorphic to either S_3 or $(S_2 + S_1)$, and that $P_{\lceil |E(P)|/3 \rceil}$ induces a subgraph of L isomorphic to S_1 if $|E(P)| \equiv 1 \pmod 3$ and to S_2 if $|E(P)| \equiv 2 \pmod 3$. For each $i \leq |E(P)|/3$, let C'_i be the collection of centers of any stars of P with edges in P_i and consider the subgraph of K_{n+m} induced by the edges in $E(P_i) \cup \{\{x, y\} \mid x \in V(P_i) \backslash C'_i, y \in V'\}$. This subgraph is isomorphic to either $(S_3 \vee K_m^C) - F$ or $((S_2 + S_1) \vee K_m^C) - F$ as defined in Lemma 4.2 and thus has a P_3-decomposition $(V(P_i), B_{P_i})$ by Lemma 4.2. Let $B_P = \bigcup_{1 \leq i \leq |E(P)|/3} B_{P_i}$.

Note that we have now placed all edges in L into paths of length three except for those in $R_{\lceil |E(R)|/3 \rceil}$ and $P_{\lceil |E(P)|/3 \rceil}$ whenever $|E(R)|$ and $|E(P)|$ respectively is not a multiple of three. At this point, the edges between $3(\lfloor |E(P)|/3 \rfloor + \lfloor |E(R)|/3 \rfloor + |T|)$ vertices in L and each of the vertices in V' now occur in 3-paths. We now consider two cases in turn.

First, suppose that $n \equiv 0$ or $1 \pmod 3$, then $|E(L)| \equiv 0 \pmod 3$. Therefore, if the number of edges in either R or P is not divisible by 3, then the subgraph induced by $R_{\lceil |E(R)|/3 \rceil} \cup P_{\lceil |E(P)|/3 \rceil}$ is isomorphic either to $(S_2 + S_1)$ or $(S_1 + S_1 + S_1)$. If $R_{\lceil |E(R)|/3 \rceil} \cup P_{\lceil |E(P)|/3 \rceil}$ is isomorphic to $(S_2 + S_1)$ then the subgraph of K_{n+m} induced by the edges in $E(R_{\lceil |E(R)|/3 \rceil} \cup P_{\lceil |E(P)|/3 \rceil}) \cup \{\{x, y\} \mid x \in V(R_{\lceil |E(R)|/3 \rceil} \cup P_{\lceil |E(P)|/3 \rceil}) \backslash C', y \in V'\}$ (where C' is the set of centers of the two stars in question) is isomorphic to $((S_2 + S_1) \vee K_m^C) - F$ as defined in Lemma 4.2 and thus has a P_3-decomposition $(V(R_{\lceil |E(R)|/3 \rceil} \cup P_{\lceil |E(P)|/3 \rceil}), B^*)$ by Lemma 4.2. If $R_{\lceil |E(R)|/3 \rceil} \cup P_{\lceil |E(P)|/3 \rceil}$ is isomorphic to $(S_1 + S_1 + S_1)$, then the subgraph of K_{n+m} induced by the edges in $E(R_{\lceil |E(R)|/3\rceil} \cup P_{\lceil |E(P)|/3\rceil}) \cup \{\{x,y\} \mid x \in V(R_{\lceil |E(R)|/3\rceil} \cup P_{\lceil |E(P)|/3\rceil}), y \in V'\}$ is isomorphic to $(S_1 + S_1 + S_1) \vee K_m^C$ and has a P_3-decomposition $(V(R_{\lceil |E(R)|/3\rceil} \cup P_{\lceil |E(P)|/3\rceil}), B^*)$ by Lemma 4.3 (let $C' = \emptyset$ in this case). Note that all edges in L have now been placed into paths of length three.

All that remains when $n \equiv 0$ or $1 \pmod 3$ is to place the edges connecting the vertices in $I' = I \cup C' \cup \{\bigcup_{i \leq |E(P)|/3} C_i'\}$ to the vertices in V' and the edges in the subgraph of K_{n+m} induced by the vertices in V' into paths of length three. Note that: exactly three vertices in $V(P_i)$ have all of the edges connecting them to the vertices in V' placed into paths of length three in B_{P_i}; all six vertices in $V(R_i)$ have all of the edges connecting them to the vertices in V' placed into paths of length three in B_{R_i}; and all three vertices in $V(T_i)$ have all of the edges connecting them to the vertices in V' placed into paths of length three B_{T_i}. This immediately implies that $|I'| \equiv n \pmod 3$.

If $n \equiv 0 \pmod 3$, then the bipartite subgraph induced by the edges $\{\{x,y\} \mid x \in I', y \in V'\}$ is isomorphic to $K_{|I'|,m}$ and has a P_3-decomposition (I', B_I) by Lemma 4.1. Note that all the subconstructions used so far have at least one path either of the form $p_1 = (a,u,b,v)$ or $p_2 = (a,b,u,c)$ where $\{a,b,c\} \in V(L)$ and $\{u,v\} \in V'$. If $m = 3$, either replace p_1 with the paths (a,u,v,w) and (w,u,b,v) or replace p_2 with the paths (c,u,v,w) and (w,u,b,a), thereby placing edges joining vertices in V' into 3-paths. If $m = 4$, then by Lemma 4.4, let (V', B_N) be a P_3-design of order 4. The union of all 3-paths thus defined completes the case where $n \equiv 0 \pmod 3$.

If $n \equiv 1 \pmod 3$, then $m \in \{2,3\}$ and $|I'| \equiv 1 \pmod 3$. Again, note that all the subconstructions used so far have at least one path either of the form $p_1 = (a,u,b,v)$ or $p_2 = (a,b,u,c)$ where $\{a,b,c\} \in V(L)$ and $\{u,v\} \in V'$. If $V(I') = \{w\}$ and $m = 2$, then replace p_1 with the paths (a,u,v,w) and (w,u,b,v), or replace p_2 with the paths (c,u,v,w) and (w,u,b,a). If $V(I') = \{w\}$ and $m = 3$, then by Lemma 4.4, let $(V' \cup \{w\}, B_N)$ be a P_3-design of order 4. If $|I'| \geq 4$ and $m = 2$, then partition I' into two sets I_1' and I_2' with $|I_1'| = 4$ and $|I_2'| = |I'| - 4 \equiv 0 \pmod 3$. The subgraph of K_{n+m} induced by the vertices in

$V' \cup I'_1$ is isomorphic to H_7 and has a P_3-decomposition $(V' \cup I'_1, B_N)$ by Lemma 4.2. The bipartite subgraph induced by the edges $\{\{x, y\} \mid x \in I'_2, y \in V'\}$ is isomorphic to $K_{|I'|-4,2}$ and has a P_3-decomposition $(V' \cup I'_2, B_I)$ by Lemma 4.1. If $|I'| \geq 4$ and $m = 3$, then let $a \in I'$. The subgraph of K_{n+m} induced by the vertices in $V' \cup \{a\}$ is isomorphic to K_4 and has a P_3-decomposition $(V' \cup \{a\}, B_N)$ by Lemma 4.4. The remaining edges, namely those in $\{\{x, y\} \mid x \in I' \backslash \{a\}, y \in V'\}$, induce a bipartite subgraph isomorphic to $K_{|I'|-1,3}$ and has a P_3-decomposition $(V' \cup I' \backslash \{a\}, B_I)$ by Lemma 4.1. The union of all 3-paths thus defined completes the case where $n \equiv 1 \pmod 3$.

Finally consider the case where $n \equiv 2 \pmod 3$, so $m = 2$. In this case $|E(L)| \equiv 1 \pmod 3$, so there must be exactly 1 or 4 edges in $R_{\lceil |E(R)|/3 \rceil} \cup P_{\lceil |E(P)|/3 \rceil}$. Therefore, $R_{\lceil |E(R)|/3 \rceil} \cup P_{\lceil |E(P)|/3 \rceil}$ must induce a graph isomorphic to S_1 or to $(S_2 + S_1 + S_1)$ (note that $R_{\lceil |E(R)|/3 \rceil} \cup P_{\lceil |E(P)|/3 \rceil}$ can't induce a subgraph isomorphic to $S_2 + S_2$, since $P_{\lceil |E(P)|/3 \rceil}$ induces a subgraph isomorphic to either S_1 or S_2 and $R_{\lceil |E(R)|/3 \rceil}$ induces a subgraph isomorphic to either S_1 or $S_1 + S_1$).

If $R_{\lceil |E(R)|/3 \rceil} \cup P_{\lceil |E(P)|/3 \rceil}$ induces a graph isomorphic to S_1, then the subgraph of K_{n+2} induced by the vertices in $V' \cup R_{\lceil |E(R)|/3 \rceil} \cup P_{\lceil |E(P)|/3 \rceil}$ is isomorphic to K_4 and has a P_3-decomposition $(V' \cup R_{\lceil |E(R)|/3 \rceil} \cup P_{\lceil |E(P)|/3 \rceil}, B_N)$ by Lemma 4.4. This leaves the edges connecting the vertices in V' to those in $I'' = I \cup \{\bigcup_{i \leq |E(P)|/3} C'_i\}$. Note that since $R_{\lceil |E(R)|/3 \rceil} \cup P_{\lceil |E(P)|/3 \rceil}$ induces a graph on exactly two vertices, $|I''| \equiv 0 \pmod 3$. Thus the bipartite subgraph induced by the edges $\{\{x, y\} \mid x \in I'', y \in V'\}$ is isomorphic to $K_{|I'|,2}$, so has a P_3-decomposition $(V' \cup I'', B_I)$ by Lemma 4.1.

If $R_{\lceil |E(R)|/3 \rceil} \cup P_{\lceil |E(P)|/3 \rceil}$ induces a graph isomorphic to $(S_2 + S_1 + S_1)$, then partition the edges in $R_{\lceil |E(R)|/3 \rceil} \cup P_{\lceil |E(P)|/3 \rceil}$ into P'_1 and P'_2 such that P'_1 induces a subgraph isomorphic to $(S_2 + S_1)$ and P'_2 induces a subgraph isomorphic to S_1. Consider the the subgraph of K_{n+m} induced by the edges in $E(P'_1) \cup \{\{x, y\} \mid x \in V(P'_1) \backslash C', y \in V'\}$ where C' is the set of centers of the two stars in P'_1. This subgraph is isomorphic to $((S_2 + S_1) \vee K_m^C) - F$ as defined in Lemma 4.2 and thus has a P_3-decomposition $(V(P'_1), B^*)$ by Lemma 4.2. The subgraph of

K_{n+2} induced by the vertices in $V' \cup P_2'$ is isomorphic to K_4 and has a P_3-decomposition by Lemma 4.4. This leaves the bipartite subgraph induced by the edges $\{\{x, y\} \mid x \in I', y \in V'\}$ with I' as defined above. This subgraph is isomorphic to $K_{|I'|,2}$, so has a P_3-decomposition $(V' \cup I', B_I)$ by Lemma 4.1.

Let $B' = B \cup B_T \cup B_R \cup B_P \cup B_N \cup B_I \cup B^*$. Then $(V(K_{n+m}, B')$ is an embedding of $(V(K_n, B)$ as required. □

Theorem 4.3. *A maximal partial P_3-design (V, B) of order n can be embedded in a P_3-design of order $n + 1$ if and only if*

(i) $n \equiv 0$ *or* 2 *(mod* 3*),*

(ii) $|E(L)| \geq n/2$, *and*

(iii) $n \neq 2$

Proof. To prove the necessity, assume that (V, B) is embedded in a P_3-design $(V \cup \{v\}, B')$ of order $n + 1$. By Theorem 4.1, $n + 1 \equiv 0$ or $1 \pmod 3$, so $n \equiv 0$ or $2 \pmod 3$. Clearly $n \neq 2$ since by Theorem 4.1 there is no P_3-design of order 3. To show the necessity of the condition $|E(L)| \geq n/2$, note that since (V, B) is maximal, each new 3-path in $B' \backslash B$ must either be of the form (a, b, c, v) using two edges of L or (a, b, v, c) using one edge in L, where $\{a, b, c\} \in V(L)$. Define paths of the first form as Type 1 and of the second as Type 2. Let x be the number of paths of Type 1 in any embedding and let y be the number of paths of Type 2. By considering the edges in L in paths in $B' \backslash B$ it follows that $2x + y = |E(L)|$. Also, the n edges joining vertices in L to v are all in paths in $B \backslash B'$, so $x + 2y = n$. Therefore $x = (2|E(L)| - n)/3$ and $y = (2n - |E(L)|)/3$. Since x must be a nonnegative integer, $|E(L)| \geq n/2$. (Note that $y \geq 0$ implies that $|E(L)| \leq 2n$. But since each component in each maximal P_3-design is K_3 or a star, in fact $|E(L)| \leq n$, so this apparent necessary condition is always satisfied.)

To show the sufficiency, we will embed a maximal partial P_3-design (V, B) of order n satisfying the necessary conditions into a P_3-design $(V \cup \{v\}, B')$ of order $n+1$. If $n = 3$ then

$L = K_3$ so the result follows from Theorem 4.1, so we can assume that $n \geq 5$. Recall that in the leave L of any maximal partial P_3-designs, each component must be a K_3 or a star (possibly with no edges), otherwise it would contain a path of length three. In particular, this means that $|E(L)| \leq n$.

As is shown above, we must construct $x = (2|E(L)| - n)/3$ 3-paths of Type 1 and $y = (2n - |E(L)|)/3$ 3-paths of Type 2. Note that since $|E(L)| \leq n$, $y \geq (2n - n)/3 \geq 5/3$ so $y \geq 2$. To be able to construct x 3-paths of Type 1, clearly there must be at least x edge-disjoint 2-paths in L. To show that there are sufficiently many 2-paths in L to use as building blocks for the Type 1 paths, consider the value $2|E(L)| - n$. Let L' be the subgraph of L induced by the components of L not isomorphic to S_0 or S_1. Since each component of L isomorphic to S_0 has one vertex and zero edges and each component of L isomorphic to S_1 has two vertices and one edges, it follows that $n - |V(L')| \geq 2(|E(L)| - |E(L')|)$, so $2|E(L')| - |V(L')| \geq 2|E(L)| - n$. Now consider a maximal P_2-decomposition $(V(L'), D)$ of the subgraph induced by L'. Since D contains exactly two edges in each component isomorphic to K_3 and contains all of the edges of each component that is a star except possibly one, it follows that the number of edges in D is at least two thirds of the total number of edges in L'; so $|D| \geq |E(L')|/3$. Also note that $|E(L')| \leq |V(L')|$ since L' consists entirely of disjoint stars and $K_3's$, so $|E(L')| \geq 2|E(L')| - |V(L')|$. So $|D| \geq |E(L')|/3 \geq (2|E(L')| - |V(L')|)/3 \geq (2|E(L)| - n)/3 = x$, ensuring that there are indeed at least x 2-paths in L.

Construct the Type 1 3-paths as follows. Let $\{X_i' \mid i \in \mathbb{Z}_x\}$ be any set of x edge disjoint paths of length two in L. For each $i \in \mathbb{Z}_x$, extend $X_i' = (a, b, d)$ to the 3-path $X_i = (a, b, d, v)$. Let $X = \{X_i \mid i \in \mathbb{Z}_x\}$. Note that because each component of L is a K_3 or a star,

(1) for each edge $\{a, b\}$ in L that occurs in no 3-path in X, at least one of the edges $\{a, v\}$ and $\{b, v\}$ does not occur in a 3-path in X.

Also note that if c is the center of a star then the edge of the form $\{c, v\}$ has not been placed into a 3-path in X, since no center can be an end of a 2-path in L.

Let $Y'' = \{Y_i'' \mid i \in \mathbb{Z}_y\}$ be the subset of $E(L)$ consisting of the $|E(L)| - 2x = y$ edges not occurring in paths in X. Finally we show how to place into 3-paths the edges in Y'' together with the edges in the set W consisting of the $n - x = 2y$ edges incident with v that occur in no 3-path in X. It is easy to direct the edges in Y'' so that in the resulting directed graph D:

(i) if (a, b) is the arc in D corresponding to $Y_i'' = \{a, b\}$ then $\{b, v\}$ occurs in no path in X (this is possible by (1)),

(ii) each vertex in D has in-degree at most 1, and

(iii) each center c of each star incident with an edge in Y'' has in-degree exactly 1.

Note that since each component of L is a K_3 or a star, by (ii) it follows that:

(iv) the only vertices with out-degree greater than 1 are centers of stars.

For each $i \in \mathbb{Z}_y$, if $Y_i'' = \{a, b\}$ corresponds to (a, b) in D then define the 2-path $Y_i' = (a, b, v)$; name these so that if b is the center of a star, then i is as small as possible. More formally, if (a_1, b_1) and (a_2, b_2) are arcs in D corresponding to Y_i'' and Y_j'' respectively and if b_1 is the center of a star and b_2 is not the center of a star, then $i < j$. Let $Y' = \bigcup_{i \in \mathbb{Z}_y} Y_i'$.

Let Z be the set consisting of the y edges in W occurring in no path in Y'. Then Z can be partitioned into sets Z_1 and Z_2 where $Z_1 = \{\{u, v\} \mid (u, c) \in D, c \text{ is the center of a star in } L\}$. By (iii) at most one vertex (namely v) is incident with more than one edge in Z_1, so we can name the vertices in D so that $Z_1 = \{\{u_i, v\} \mid i \in \mathbb{Z}_{|Z_1|}\}$ where for each $i \in \mathbb{Z}_{|Z_1|}$, (u_i, c_i) is the arc in D corresponding to Y_i'', and c_i is the center of a star in L. Then we can let $Z_2 = \{\{u_i, v\} \mid i \in \mathbb{Z}_y \backslash \mathbb{Z}_{|Z_1|}\}$. It turns out that the vertices in $\{u_i \mid i \in \mathbb{Z}_y \backslash \mathbb{Z}_{|Z_1|}\}$ are isolated vertices in D. To see this, notice that if $u \in \{u_i \mid i \in \mathbb{Z}_y \backslash \mathbb{Z}_{|Z_1|}\}$, then:

(i) u is an isolated vertex in L, or

(ii) u appears in a 3-path in X of the form (a, u, b, v) where $\{a, u\}$ and $\{u, b\}$ are edges in a component of L isomorphic to K_3, or

(iii) u appears in a 3-path in X of the form (u, c, a, v) where $\{c, u\}$ and $\{c, a\}$ are edges in a component of L isomorphic to a star with center c, or

(iv) u is the center or a star in L all of whose edges appear in 3-paths in X.

For each $i \in \mathbb{Z}_y$, extend $Y_i' = (a, b, v) = (u_i, b, v)$ to the 3-path $Y_i = (u_i, b, v, u_{i+1})$ reducing the sum modulo y. Let $Y = \{Y_i \mid i \in \mathbb{Z}_y\}$. Note that each element in Y is a 3-path, since $u_i \neq u_{i+1}$ since $y \geq 2$.

All edges have now been placed into 3-paths, so $(V(L) \cup \{v\}, B \cup X \cup Y)$ embeds (V, B) into a complete P_3-design of order $n + 1$ as desired. $\square$

4.4 Further Comments

It is natural to consider embeddings of all partial P_3-designs, not just maximal ones. While Theorem 4.2 does find embeddings for all partial P_3-designs, Theorem 4.3 does not.

The difficulty when embedding partial P_3-designs of order n into P_3-designs of order $n + 1$ arises because of condition (ii) in Theorem 4.3. Indeed, it is easy to find partial P_3-designs of order n that can be embedded in a P_3-design of order $n + 1$, but can also be extended to a maximal partial P_3-design which cannot be embedded in a P_3-design of order $n + 1$. As an example, consider the partial P_3-design (V, B) of order 6 in which B is empty (the leave is K_6 itself). This can easily be embedded in a P_3-designs of order 7 by Tarsi's result in [28], since any P_3-design of order 7 embeds (V, B). (V, B) can be extended to a maximal partial P_3-design (V, B') in which the leave is empty ((V, B') is a complete design). By Theorem 4.3, (V, B') cannot be embedded in a P_3-designs of order 7.

It is clear that new necessary conditions must be introduced to completely solve this problem.

Chapter 5

Future Directions

Identifying and classifying underlying structures and decompositions of Kneser and Generalized Kneser Graphs seems to a be a fertile area for future results. In particular, the (Generalzed) Kneser Graphs have very strong underlying recursive structures that lend themselves readily to inductive design constructions. As an example of this, consider the Kneser Graph $G = KG_{n,k}$ on element set $\mathbb{Z}_n$. For any element i in the element set, the subgraph G_i of G induced by the vertices not containing the element i is clearly isomorphic to $KG_{n-1,k}$ on the element set $\mathbb{Z}_n \backslash \{i\}$. Note that the collection of vertices in G containing the element i is an independent set (no two are adjacent). This set is denoted V_i. G can therefore be viewed as the edge-disjoint union of G_i and the bipartite subgraph B_i of G induced by the edges $\{\{a, b\} \mid a \in V(G_i), b \in V_i\}$. So in this case, if we have some H-decomposition of G_i, and we can find an H-decomposition of B_i, we immediately have an H-decomposition of G. Note that B_i is not only bipartite, but any two vertices in a given partition have the same degree in B_i. Graph decompositions of bipartite graphs have been well studied in the literature and give a firm grounding for these types of constructions (see for example [5]).

This approach not only allows one to show the existence of a given decomposition, but actually creates a structure for building explicit constructions inductively. It can also be generalized fairly readily to Generalized Kneser Graphs and illustrates the interrelationship between Kneser Graphs and Generalized Kneser Graphs. To illustrate this, consider the Generalized Kneser Graph $G = GKG_{n,k,r}$ on element set $\mathbb{Z}_n$. The subgraph G'_i of G induced by the vertices containing the element i is isomorphic to $GKG_{n-1,k-1,r-1}$ on the element set $\mathbb{Z}_n \backslash \{i\}$. The subgraph G_i of G induced by the vertices not containing the element i is isomorphic to $GKG_{n-1,k,r}$ on the element set $\mathbb{Z}_n \backslash \{i\}$. G can therefore be viewed as the edge-disjoint union of G_i, G'_i and the bipartite subgraph B_i of G induced by the edges $\{\{a, b\} \mid a \in V(G_i), b \in V(G'_i)\}$, with similar possibilities for analysis as above. Note that the Kneser Graph *is* a Generalized Kneser Graph with $r = 0$, so repeated applications of this procedure can partially reduce the task of decomposition of Generalized Kneser Graphs to the decompositions of the Kneser Graph.

In each of the results discussed in Chapter 2, the obvious necessary condition that the number of edges in the graph be a multiple of the path length turned out to also be sufficient for the existence of the relevant path decomposition. This, plus another small observation, leads to the following conjecture.

Conjecture 1. *The Kneser Graph $KG_{n,k}$ has a P_l-decomposition if and only if:*

(i) $n > 2k$*,*

(ii) $l \mid \binom{n}{k}\binom{n-k}{k}/2 = |E(KG_{n,k})|$*,*

(iii) and $2o(KG_{n,k}) \leq |E(KG_{n,k})|/l$ *where* $o(G)$ *denotes the number of vertices in* G *with odd degree.*

The first two conditions simply ensure that the graph is connected and has a number of edges that is a multiple of the path length. The connectivity is important, since if $n = 2k$, then the graph is a 1-factor and if $n < 2k$, the graph has no edges The third condition comes from the observation that removing a path from a graph will change the parity of exactly two vertices in the graph (the endpoints of the path). If the number of paths in a proposed path decomposition of a graph is less than half the number of odd vertices, then the proposed path decomposition is clearly impossible, since a decomposition can be thought of as a way to remove all edges from a graph, leaving each vertex with degree zero. There would be an insufficient number of paths to turn all of the odd vertices to the even even zero. If this conjecture can be established, it should be possible to extend the result to a similar result for Generalized Kneser Graphs using the observations of the recursive nature of these graphs discussed above.

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