Analysis and finite element approximation for nonlinear problems in poroelasticity and bioconvection

by

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Abstract

This dissertation is concerned with nonlinear systems of partial differential equation with solution dependent physical coefficients satisfying the Nemytskii assumption. Such equations arise from two important application fields: poroelasticity and bioconvection.

First, we consider a quasi-static poroelasticity model with dilatation dependent hydraulic conductivity and an implicit time derivative. We derive the existence and uniqueness of solutions using the modified Rothe's method, Brouwer's fixed point Theorem and the Sobolev embedding Theorem. Next we construct a finite element approximation with linear elements and establish the optimal error estimate. We then conduct numerical examples to verify the convergence and simulate the diffusion in a fluid saturated sponge.

Second, we study the bioconvection model, a coupled Navier-Stokes type equation, with concentration dependent viscosity. We combine the theory of the Navier-Stokes equation and the modified Rothe's method to establish existence and uniqueness of solutions of both steady and time dependent bioconvection. After that we perform finite element analysis with Taylor-Hood elements and prove the convergence theorem. Finally numerical examples are constructed using lab data to verify the convergence of the numerical scheme and simulate the convection pattern formed by micro-organisms inside a culture fluid.

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Chapter 1

Introduction

This dissertation is devoted to the study of the well posedness and numerical approximations of systems of partial differential equation (PDE) with solution dependent physical parameters arising from two application fields: poroelasticity and bioconvection.

In mathematical modeling, we usually consider a linear system under certain ideal homogeneous assumptions about the physical parameters. In practice, however, the physical parameters are generally inhomogeneous and related to the solution of the system of PDE. A generic PDE with solution dependent physical coefficient can be written as

$$L\Big(x, t, u(t,x), \lambda(t,x,u)\Big) = f\,, \quad x \in \Omega\,, \quad t > 0\,, \tag{1.1}$$

where Ω is the physical domain, t refers to time, L denotes a general differential operator and $\lambda(t,x,u)$ is the physical parameter depending on the solution u. When L is a linear differential operator, (1.1) is often reduced to a quasi-linear PDE. Theoretical analysis of this problem concerning the well posedness and regularity of solutions can be found in [1], [2], and [3]. Finite element approximations of some simple quasi-linear elliptic PDEs and parabolic PDEs (1.1) are presented in [4], [5] and [6].

In this dissertation, we consider two mathematical models taking the form of coupled PDEs with solution dependent coefficients. The first model arises in poroelasitcity with dilatation dependent hydraulic conductivity and the second model is about bioconvections with concentration dependent viscosity. Though the physical backgrounds of the two problems are quite different, we will use similar approaches to study the well posedness and numerical approximations of the two underlying PDE systems. For instance we will use the modified

Rothe's method to study the existence of weak solutions of both of the two evolution PDE systems. We will use the fact that the physical parameters in both of the two models give rise to Nemiskii operators to derive the well-posedness results under minimum regularity assumptions on the input data.

1.1 Poroelasticity with dilatation dependent hydraulic conductivity

Motivation. We are surrounded by porous elastic solid materials: natural (e.g., rocks, soils, shale, living tissue, the brain, the heart) and man-made (e.g., cement, concrete, filters, foams, ceramics, gels, clays). Porous material has a solid matrix structure with small pores inside which contain air or fluid. Because of their ubiquity and unique properties, porous materials are of great interest to natural scientists and engineers ([7, 8, 9, 10]). Porous media finds applications in diverse areas include reservoir engineering [11], biomechanics [12, 13, 14] and environmental engineering [15, 16, 11].

Mathematical model. Poromechanics is a branch of physics and specifically continuum mechanics and acoustics that studies the behaviour of fluid-saturated porous media. The particular mathematical model we are interested in describes the swelling and shrinking of an elastic deforming porous medium coupled with the fluid.

Due to the elastic nature of the porous medium, the study of fluid saturated porous media is called poroelasticity. Terzaghi [17] first derived a one dimensional model in soil mechanics to involve the influence of the fluid inside a solid body. The model was later extended to three dimension by Rendulik [18]. A mathematical formulation is derived in Biot's work [19] and studies in his other work between 1955 and 1962 [20, 21, 22, 23, 24], which are later considered to be the foundation of modern poroelasticity theory.

The following equations are the essence of Biot's poroelasticity theory [19] (details of the theory can be found in [25]) in which the coupled constitutive equations takes the form

$$\begin{cases} \dfrac{\partial}{\partial t}(\text{momentum}) + \text{stress} = \text{external force}\,, \\ \dfrac{\partial}{\partial t}(\text{fluid content}) + \text{flux} = \text{external fluid source}\,. \end{cases} \tag{1.2}$$

Next we convert (1.2) into a coupled system of partial differential equations for the fluid pressure and the displacement of the poroelastic medium. To this end, we denote the pressure by p and the displacement by $\mathbf{u}$. According to Hooke's law in 3-D the effective stress caused by the deformation of the solid matrix is given by

$$\tau_e = 2\mu\varepsilon + \lambda tr(\varepsilon) I\,. \tag{1.3}$$

Here $\varepsilon = \dfrac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$ is the strain, $tr(\varepsilon)$ is the trace of ε, I is the identity matrix, and λ, μ are the Lamé constants, corresponding to the dilatation and shear modulus respectively, given by

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}\,, \quad \mu = \frac{E}{2(1+\nu)}\,,$$

where the Young's modulus E captures the elastic stiffness in the direction of a load and the Poisson's ratio ν describes the stretch (compression) in the direction perpendicular to the load. By introducing λ and μ, the effective stress is written in a symmetric form. To include the effect of the fluid pressure inside the porous body, we introduce the addition stress due to the pore pressure p_p:

$$p_p = \alpha p\,, \tag{1.4}$$

where the Biot-Willis constant α [23] satisfies $\alpha = 1 - K/K_s$ with K being the bulk modulus of the porous matrix and K_s is the bulk modulus of the solid material. In most situations $\alpha \approx 1$, corresponding to an incompressible solid matrix. In a certain models of secondary consolidation in clays, we also add an additional term

$$\tau_s = \lambda_* \frac{\partial}{\partial t} \nabla \cdot \mathbf{u} \,, \tag{1.5}$$

where λ_* is the coefficient of the secondary consolidation [26]. Together, the total stress is given by

$$\tau = \tau_e + p_p + \tau_s \,. \tag{1.6}$$

In the second equation of (1.2), the fluid content is given by

$$\eta := \eta_f + \eta_d = c_0 p + \alpha \nabla \cdot \mathbf{u} \,. \tag{1.7}$$

Here $\eta_f = c_0 p$ measures the fluid content that can be forced into the medium by pressure increments within constant volume with the constant $c_0 \geq 0$ combining the porosity and the compressibility, and $\eta_d = \alpha \nabla \cdot \mathbf{u}$ denotes the fluid content due to the change of the pores size, i.e., the dilatation of the void volume, which is proportional to $\nabla \cdot \mathbf{u}$, the total dilatation of the body. If $\alpha = 1$, the solid matrix is incompressible. Thus the total dilatation is the same as the dilatation of the pores. According to Darcy's law,

$$q = -\kappa \nabla p \tag{1.8}$$

represents the linear relationship between fluid flux and the pressure drop, where the hydraulic conductivity $\kappa > 0$ measures the permeability and the viscosity of the fluid. Now letting the function $\mathbf{f}$ denote the volume-distributed external force, g the fluid source density, and ρ the density of the medium, we substitute (1.3), (1.4), (1.5), (1.6), (1.7) and (1.8) into (1.2) and apply divergence theorem to obtain the fully dynamic system

$$\begin{cases} \rho \dfrac{\partial^2}{\partial t^2} \mathbf{u} - \lambda_* \nabla (\dfrac{\partial}{\partial t} \nabla \cdot \mathbf{u}) - (\lambda + \mu) \nabla (\nabla \cdot \mathbf{u}) - \mu \Delta \mathbf{u} + \alpha \nabla p = \mathbf{f} \,, \\ \hfill \dfrac{\partial}{\partial t} (c_0 p + \alpha \nabla \cdot \mathbf{u}) - \nabla \cdot (\kappa \nabla p) = g \,, \end{cases} \tag{1.9}$$

subject to initial and boundary conditions. The first equation is a hyperbolic linear elasticity equation for the displacement of the poroelastic medium while the second equation is an implicit parabolic equation for the fluid's pressure.

In general, complicated boundary conditions must be considered. In particular, the boundary may have to be partitioned into disjoint, regular open set, on which various boundary conditions are imposed. For example, we may partition the boundary Γ as $\Gamma = \bar{\Gamma}_c \cup \bar{\Gamma}_t$ with $\Gamma_c \cap \Gamma_t = \emptyset$. We say Γ_c is the clamped boundary on which the Dirichlet condition $\mathbf{u}|_{\Gamma_c} = \mathbf{u}_b$ is given, and Γ_t is the traction boundary on which the normal stress $\sigma|_{\Gamma_t} \cdot \mathbf{n} = s$ is prescribed. For the pressure, we may partition Γ as $\Gamma = \bar{\Gamma}_d \cup \bar{\Gamma}_f$ with $\Gamma_d \cap \Gamma_f = \emptyset$. We say Γ_d is the drained boundary on which the Dirichlet condition $p|_{\Gamma_d} = p_b$ is given, and Γ_f is the flux boundary on which the fluid flux $(\kappa(\nabla \cdot \mathbf{u})\nabla p)|_{\Gamma_f} \cdot \mathbf{n} = r$ is prescribed. On $\Gamma_t \cap \Gamma_f$, we have the balance of force and flux at the same time. To handle this, we introduce β to be the surface fraction of the sealed portion of $\Gamma_t \cap \Gamma_f$. Then $1 - \beta$ corresponds to the exposed portion. Next we define the characteristic function

$$\chi_{tf} = \begin{cases} 1\,, & x \in \Gamma_t \cap \Gamma_f\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

On the sealed part, the pore pressure contributes to the total force. Thus

$$\Big[(\lambda + \mu)\nabla \cdot \mathbf{u} I + \mu \nabla \mathbf{u}\Big] \cdot \mathbf{n} - \beta \alpha p \cdot \mathbf{n} \chi_{tf} = s \quad \text{on} \quad \Gamma_t\,,$$

On the exposed part, the changing rate of the fluid content caused by the dilatation must be included in the balance of flux. As a result we have

$$-\frac{\partial}{\partial t}\Big((1-\beta)\alpha \mathbf{u} \cdot \mathbf{n}\Big)\chi_{tf} + \kappa \nabla p \cdot \mathbf{n} = r \quad \text{on} \quad \Gamma_f\,.$$

Quasi-static poroelasticity. In this dissertation we restrict our considerations to the linear quasi-static flow in a saturated deformable poroelastic medium, i.e., we neglect the effects of $\rho \dfrac{\partial^2}{\partial t^2}\mathbf{u}$ and the secondary consolidation (1.5). Then the system takes the form

$$\begin{cases} -(\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}) - \mu\Delta\mathbf{u} + \alpha\nabla p = \mathbf{f}\,, \\ \quad \dfrac{\partial}{\partial t}(c_0 p + \alpha\nabla\cdot\mathbf{u}) - \nabla\cdot(\kappa\nabla p) = g\,. \end{cases} \tag{1.10}$$

In this case, we focus on a quasi-static problem with a coupling of an elliptic equation and a parabolic equation.

In a homogeneous and isotropic medium, where the permeability and the viscosity are constants, κ is also a constant. In this case, the well posedness and regularity of system (1.10) was studied in [25] as an application of the semi-group theory to linear degenerate evolution equations in Hilbert spaces. Galerkin and discontinuous Galerkin approximations for this linear system can be found in [27] and [28].

In this dissertation, we consider the case when the hydraulic conductivity κ depends on the dilatation $\nabla\cdot\mathbf{u}$, i.e., $\kappa = \kappa(\nabla\cdot\mathbf{u})$. A typical example for dilatation dependent hydraulic conductivity can be found in [29], which is used in calculating the pressure drop of a fluid flowing through a packed bed of solids. Difficulties in studying (1.10) arise from

1. Nonlinearity of the system due to the dependence of κ on $\nabla\cdot\mathbf{u}$.

2. Implicit evolution in the second equation of (1.10) due to the term $\frac{\partial}{\partial t}(\alpha\nabla\cdot\mathbf{u})$.

A proof of the existence and uniqueness of a solution of similar type of equations using Rothe's method can be found in Chapter 5 of [1] under the assumption that $\kappa:\mathbb{R}\to\mathbb{R}$ is continuous and satisfies

$$0 < \kappa_* \le \kappa(x) \le \kappa^* \quad \forall x\in\mathbb{R}\,, \tag{1.11}$$

for some constants κ_*, κ^*. In general, the uniqueness of the solution usually requires Lipschitz continuity condition of κ and additional coefficient assumptions. The finite element approximations of similar problems can be found in Chapter 8.7 of [4] and in Chapter 13 of [5]. However, none of these studies involves implicit evolution equations.

Linear implicit evolution equations have been studies in [30], [31], and [2] using the semi-group theory. The quasi-static poroelasticity model with constant hydraulic conductivity was studies in [25], also using the semi-group theory on Hilbert spaces. Formally, note that the first equation in (1.10) is of second order for displacement $\mathbf{u}$ and first order for pressure p. Therefore $c_0 p$ and $\alpha\nabla\cdot\mathbf{u}$ should have the same regularity.

1.2 Bioconvection

Motivation. Bio-convection occurs due to on average upwardly swimming micro-organisms which are slightly denser than water. The micro-organisms swim upward to meet the sunlight. When the micro-organisms on the surface become too dense, they sink under the effect of gravity. Repeating this process, a convection pattern is formed.

Mathematical model. A fluid dynamical model treating the micro-organisms as collections of particles was derived by M. Levandowsky, W. S. Hunter and E. A. Spiegel [32] and independently by Y. Moribe [33]. We describe the model as follows. Let $\Omega\subset\mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega$. At point $x\in\Omega$, let $\mathbf{u}(x)=\{\mathbf{u}_j(x)\}_{j=1}^3$ and $p(x)$ denote the velocity and pressure of the culture fluid while $c(x)$ refers to the concentration of the micro-organism. Note that we use the volume concentration

$$c = nv_0\,,$$

where n is the number of organisms per unit volume and v_0 is the volume of an individual organism. Let ρ_0 be the density of the micro-organism and ρ_m be the density of the culture fluid. Then the density of the suspension is the sum

$$\rho=\rho_0 c+\rho_m(1-c)=\rho_m(1+\gamma c)\,, \tag{1.12}$$

where $\gamma=\rho_0/\rho_m-1$. Assume that the organisms affect the fluid dynamics only through their influence on its density and that the suspension is nearly incompressible. Then the fluid satisfies the Navier-Stokes equation

$$\begin{cases} \rho_m\Big(\dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\Big) - \nabla\cdot(\nu\nabla\mathbf{u}) + \nabla p = -g\rho i_3 + \mathbf{f}\,, \\ \nabla\cdot\mathbf{u} = 0\,. \end{cases} \tag{1.13}$$

Here ν is the kinematic viscosity of the culture fluid, g is the acceleration of gravity, $\mathbf{f}$ is the volume-distributed external force, and $i_3 = (0, 0, 1)$ is the vertical unit vector. For the concentration c, mass conservation gives

$$\frac{D}{Dt}\,c + \nabla\cdot q = 0\,. \tag{1.14}$$

Here $\dfrac{D}{Dt} = \dfrac{\partial}{\partial t} + \mathbf{u}\cdot\nabla$ is the material derivative and q is the flux of micro-organisms which is given by

$$q = -\theta\nabla c + Uci_3\,, \tag{1.15}$$

where θ and U are the diffusion rate and mean upward swimming velocity of the micro-organism, respectively.

Combining (1.12), (1.13), (1.14), and (1.15), we derive the fully dynamic system, in Ω

$$\begin{cases} \rho_m\Big(\dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\Big) - \nabla\cdot(\nu(c)\nabla\mathbf{u}) + \nabla p = -g\rho_m(1+\gamma c)i_3 + \mathbf{f}\,, \\ \nabla\cdot\mathbf{u} = 0\,, \\ \dfrac{\partial c}{\partial t} - \theta\Delta c + \mathbf{u}\cdot\nabla c + U\dfrac{\partial c}{\partial x_3} = 0\,. \end{cases} \tag{1.16}$$

We assume the following boundary conditions for $\mathbf{u}$ and c, on Ω

$$\begin{cases} \mathbf{u} = 0\,, \\ \theta\dfrac{\partial c}{\partial \mathbf{n}} - Ucn_3 = 0\,. \end{cases} \tag{1.17}$$

The second equation of (1.17) refers to a zero flux condition on the boundary and $\mathbf{n} = (n_1, n_2, n_3)$ is the outward pointing unit normal vector on $\partial\Omega$. We further assume the fixed total mass for the micro-organisms, i.e.,

$$\frac{1}{|\Omega|}\int_\Omega c(x)dx = \alpha\,, \tag{1.18}$$

for some constant α. This means that no micro-organisms are allowed to leave or enter the container. Finally the complete system describing the motion of micro-organisms takes the form, in Ω

$$\begin{cases} \rho_m\Big(\dfrac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}\Big)-\nabla\cdot(\nu(c)\nabla\mathbf{u})+\nabla p=-g\rho_m(1+\gamma c)i_3+\mathbf{f}\,, \\ \nabla\cdot\mathbf{u}=0\,, \\ \dfrac{\partial c}{\partial t}-\theta\Delta c+\mathbf{u}\cdot\nabla c+U\dfrac{\partial c}{\partial x_3}=0\,, \\ \mathbf{u}=0\,,\quad \theta\dfrac{\partial c}{\partial \mathbf{n}}-Ucn_3=0\,,\quad \text{on}\quad \partial\Omega \\ \dfrac{1}{|\Omega|}\displaystyle\int_\Omega c(x)dx=\alpha\,. \end{cases} \tag{1.19}$$

Note that the bioconvection model (1.19) is a special case of a more general equation describing the diffusion of an admixture in a region [34].

In an ideal Newtonian fluid, the viscosity ν is a constant. In this case, the existence of the solution as well as the positivity of the concentration are proved in [35] where the author considered both the stationary and evolutionary cases. The evolutionary case of system (1.16) with constant viscosity ν is studied numerically in [36]. The numerical study of slightly different bioconvection models can be found in [37], [38], [39], [40] and [41].

In practice, the viscosity is related to the concentration of the solute. Albert Einstein first showed in his Ph.D thesis [42] that

$$\frac{\nu}{\nu_0}=1+\xi c \tag{1.20}$$

where ν is the viscosity of the suspension, ν_0 is the viscosity of the pure solution, c is the volume fraction of the particle spheres and ξ is a proportionality coefficient chosen (experimentally) to be 2.5. Equation (1.20) is valid only for low concentration cases $0 < c <$ 10%. Therefore the result was later extended to add the c^2 term by Batchelor [43] for larger concentration ($c \geq 10\%$). When the concentration is much higher, the relative viscosity $\frac{\nu}{\nu_0}$ varies as an exponential function of the concentration c ([44], [45] and [46]).

A recent work [47] showed the existence and uniqueness of a periodic solution of (1.19) under the assumption that $\nu(\cdot)$ is a C^1 function and that for some positive constants ν_* and ν^*

$$\nu_* < \nu(x) < \nu^* \quad \forall x \in R \quad \text{and} \quad \sup_{x \in R} \nu'(x) < \infty \,.$$

In this dissertation, we relax the above condition to assume that $\nu : \mathbb{R} \to \mathbb{R}$ is continuous and

$$0 < \nu_* \leq \nu(x) \leq \nu^* \quad \forall x \in \mathbb{R} \,, \tag{1.21}$$

for some constants ν_* and ν^*.

1.3 Plan of dissertation

In the rest of Chapter 1 we introduce notations and assumptions that will be used throughout the thesis. In chapter 2 the existence of a weak solution of (1.10) with homogeneous boundary conditions will be proved using the modified Rothe's approach ([48], [49] and [1]) and analysis of implicit evolutionary equations [25] based on existing results for the steady poroelasticity [50]. The uniqueness will also be proved under certain regularity assumptions on the exact solution. A fully discrete finite element approximation using linear elements and backward Euler scheme will be studied and an optimal priori error estimate will be proved using approximation theory and functional analysis ([5], Chapter 13). Numerical experiment will be constructed to demonstrate the efficiency and the accuracy of the numerical method. Then a numerical experiment with dilatation dependent hydraulic conductivity $\kappa(\nabla\cdot\mathbf{u})$ will be used to simulate a fluid saturated sponge. In chapter 3 and 4, the existence and uniqueness of a solution of (1.19) will be proved for both steady and the fully dynamic case using the modified Rothe's method combined with the theory of the Navier-Stokes equations [51, 52]. A complete finite element method will be constructed using Taylor-Hood elements [53] and a convergence theorem will be proved for both steady and evolutionary cases. The error estimate will be established for the steady bioconvection using theories in [52, 54, 55]. Numerical examples will be constructed to illustrate the convergence rate and several practical simulations will be studied for the steady bioconvection flow. We conclude this dissertation in Chapter 5 with remarks and a plan for the future work.

1.4 Notations and Assumptions

Throughout the paper, we consider system (1.10) and (1.19) on an open bounded region Ω in $\mathbb{R}^2$ or $\mathbb{R}^3$ with a smooth boundary Γ and on a time interval $I=(0,T]$. We denote by $C_0^\infty(\Omega)$ the space of infinitely differentiable functions with compact support in Ω and by $L^2(\Omega)$ the space of square integrable functions on Ω. Let $W^{k,p}(\Omega)$ be the Sobolev space consisting of functions in $L^p(\Omega)$ with each of their partial derivatives through order k also in $L^p(\Omega)$. Specifically, $H^k(\Omega)$ denotes the Hilbert space $W^{k,2}(\Omega)$. The space $H_0^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the $H^1(\Omega)$ norm. As usual, $H^{-1}(\Omega)$ and $(H^1(\Omega))^*$ denote the dual of $H_0^1(\Omega)$ and $H^1(\Omega)$, respectively. Let $\mathbf{H}^k(\Omega)=\Big(H^k(\Omega)\Big)^d$, $\mathbf{H}_0^1(\Omega)=\Big(H_0^1(\Omega)\Big)^d$, and $\mathbf{L}^2(\Omega)=\Big(L^2(\Omega)\Big)^d$, $d=2,3$, with $\|\cdot\|_k$ and $\|\cdot\|$ denoting the respective norms of the two spaces. Let $H^{1/2}(\Gamma)$ and $\mathbf{H}^{1/2}(\Gamma)$ denote the trace space of $H^1(\Omega)$ and $\mathbf{H}^1(\Omega)$ while $H^{-1/2}(\Gamma)$ is the dual space of $H^{1/2}(\Gamma)$. We shall use $(\cdot,\cdot)$ to denote both the L^2 and $\mathbf{L}^2$ inner product and denote by $\langle\cdot,\cdot\rangle$ the duality pairing. The Poincaré inequality implies that there exists a constant C_p such that

$$\|w\|_1\le C_p\|\nabla w\|\,,\quad \forall w\in\mathbf{H}_0^1(\Omega)\text{ or }H_0^1(\Omega)\,. \tag{1.22}$$

Throughout the paper, we will use C as a generic constant whose value may vary from one occurence to the next.

Chapter 2

Poroelasticity

2.1 Steady poroelasticity

We first introduce some existing results for steady poroelasticty which is described by the system (1.10) without the time derivative $\dfrac{\partial}{\partial t}(c_0 p + \alpha \nabla \cdot \mathbf{u})$, that is

$$\begin{cases} -(\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}) - \mu\Delta\mathbf{u} + \alpha\nabla p = \mathbf{f}\,, \\ \qquad\qquad -\nabla\cdot\Big(\kappa(\nabla\cdot\mathbf{u})\nabla p\Big) = g\,, \end{cases} \tag{2.1}$$

with boundary conditions

$$\mathbf{u}|_{\Gamma} = \mathbf{u}_b\,,$$

and

$$-\nabla\cdot\Big(\kappa(\nabla\cdot\mathbf{u})\nabla p\Big)|_{\Gamma}\cdot\mathbf{n} = r\,.$$

We assume that the body force $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$, the fluid source $g \in (H^1(\Omega))^*$, $\mathbf{u}_b \in \mathbf{H}^{1/2}(\Gamma)$, and that $r \in H^{-1/2}(\Omega)$. Furthermore we assume that the data satisfies the following compatibility condition

$$\langle r, 1\rangle_{\Gamma} = \langle g, 1\rangle_{\Omega}\,.$$

We denote by Q the quotient space $H^1(\Omega)/\mathbb{R}$ and define the bilinear forms

$$\begin{cases} e(\mathbf{u},\mathbf{v}) := \Big((\lambda+\mu)(\nabla\cdot\mathbf{u}), \nabla\cdot\mathbf{v}\Big) + (\mu\nabla\mathbf{u}, \nabla\mathbf{v}) \quad \forall\mathbf{u},\mathbf{v}\in\mathbf{H}^1(\Omega)\,, \\ b(p,\mathbf{v}) := \displaystyle\int_{\Omega} p\nabla\cdot\mathbf{v} \quad \forall p\in Q\,, \quad \mathbf{v}\in\mathbf{H}^1(\Omega)\,. \end{cases} \tag{2.2}$$

Multiply the first equation and second equation in (2.1) by test functions $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$ and $q \in Q$ respectively, the weak formulation of (2.1) takes the form

Definition 2.1. Given $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$, $g \in (H^1(\Omega))^*$, $\mathbf{u}_b \in \mathbf{H}^{1/2}(\Gamma)$, and $r \in H^{-1/2}(\Omega)$, find $(\mathbf{u}, p) \in \mathbf{H}^1(\Omega) \times Q$ with $\mathbf{u}|_\Gamma = \mathbf{u}_b$ such that

$$\begin{cases} e(\mathbf{u}, \mathbf{v}) - b(p, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle_\Omega \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega)\,, \\ \Big(\kappa(\nabla \cdot \mathbf{u}) \nabla p, \nabla q\Big) = \langle g, q \rangle_\Omega + \langle l, q \rangle_\Gamma \quad \forall q \in Q\,. \end{cases} \tag{2.3}$$

Assumption (1.11) assures that $\kappa(\nabla \cdot \mathbf{u}(x))$ is a so called Nemytskii operator whose definition and properties are stated in the following Lemma.

Lemma 2.2 ([56])**.** *Assume that a function $f : \Omega \times \mathbb{R}^m \to \mathbb{R}$ satisfies the Carathéodory conditions:*

(i) $f(x, u)$ is a continuous function of u for almost all $x \in \Omega$;

(ii) $f(x, u)$ is a measurable function of x for all $u \in \mathbb{R}^m$. Furthermore for some constant C and $g \in L^q(\Omega)$

$$|f(x, u)| \le C|u|^{p-1} + g(x) \quad x \in \Omega,\ u \in \mathbb{R}^m$$

where $1 < q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then the Nemytskii operator $F(u) : \Omega \to \mathbb{R}$ defined by

$$F(u)(x) = f(x, u(x))$$

is a bounded and continuous map from $L^p(\Omega; \mathbb{R}^m)$ into $L^q(\Omega; \mathbb{R})$.

The existence of a weak solution of (2.3) is given in the following Theorem.

Theorem 2.3 (Y. Cao, S. Chen and A. J. Meir)**.** *Given $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$, $g \in (H^1(\Omega))^*$, $\mathbf{u}_b \in \mathbf{H}^{1/2}(\Gamma)$, and $r \in H^{-1/2}(\Omega)$, System (2.3) admits at least one weak solution satisfying $\mathbf{u}|_\Gamma = \mathbf{u}_b$.*

Remark 2.4. The uniqueness of (2.3) can be obtained under additional coefficient assumptions and regularity assumptions if we assume that κ is Lipschitz continuous (see [50]).

Finite element approximation. We construct the Galerkin finite element approximation fort he weak solution $(\mathbf{u}, p)$ of (2.3) on a convex polygonal, or polyhedral domain Ω. Let τ_h be a family of quasi-uniform triangulations τ_h satisfying $\max_{\tau \in \tau_h} diam\ \tau \leq h$. Then we define the finite dimensional subspace $Q_h \subset Q$ and $\mathbf{V}_h \subset \mathbf{H}_0^1(\Omega)$ with the following approximation properties

$$\lim_{h \to 0} \inf_{\mathbf{v}_h \in \mathbf{V}_h} \|\mathbf{v} - \mathbf{v}_h\|_1 = 0 \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega)\,,$$

and

$$\lim_{h \to 0} \inf_{q_h \in Q_h} \|q - q_h\|_1 = 0 \quad \forall q \in Q\,.$$

The numerical scheme is to seek $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ such that

$$\begin{cases} e(\mathbf{u}_h, \mathbf{v}_h) - b(p_h, \mathbf{v}_h) = \langle \mathbf{f}, \mathbf{v}_h \rangle_\Omega \quad \forall \mathbf{v}_h \in \mathbf{V}_h\,, \\ \Big(\kappa(\nabla \cdot \mathbf{u}_h)\nabla p_h, \nabla q_h\Big) = \langle g, q_h \rangle_\Omega + \langle l, q_h \rangle_\Gamma \quad \forall q_h \in Q_h\,. \end{cases} \tag{2.4}$$

Following an argument similar to the proof of Theorem 2.3, we can show that the solution $(\mathbf{u}_h, p_h)$ of (2.4) exists and the convergence $(\mathbf{u}_h, p_h)$ to the exact solution $(\mathbf{u}, p)$ of (2.3) is established in what follows.

Theorem 2.5 (Y. Cao, S. Chen and A. J. Meir)**.** *Assume that the weak solution* $(\mathbf{u}, p) \in \mathbf{H}^1(\Omega) \times Q$ *of* (2.3) *is unique. In addition, assume that* $p \in W^{1,\infty}(\Omega)$*. Then*

$$\lim_{h \to 0}(\|\mathbf{u} - \mathbf{u}_h\|_1 + \|p - p_h\|_1) = 0\,.$$

2.2 Existence and uniqueness of a weak solution of quasi-static poroelasticity

We consider the quasi-static poroelasticity (1.10). For brevity, we assume that both $\mathbf{u}$ and p satisfy homogeneous boundary conditions. Also for brevity, we set $\alpha = 1$ (corresponding to an incompressible solid matrix). For $\alpha \neq 1$, one may convert it to the $\alpha = 1$ case by rescaling the problem. We also assume that the external force $\mathbf{f} = \mathbf{0}$. The nonzero case can be handled through a simple transformation (see [25]).

The weak formulation. By definition (2.2), the weak formulation of system (1.10) is given in the following definition

Definition 2.6. Given g in $L^2(I; L^2(\Omega))$ and l in $L^2(\Omega)$, a pair $(\mathbf{u}, p)$ in $\mathbf{H}_0^1(\Omega) \times H_0^1(\Omega)$ is said to be a weak solution of system (1.10) if it satisfies for all $t \in I$

$$\begin{cases} e(\mathbf{u}, \mathbf{v}) = -(\nabla p, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega)\,, \\ \langle \dfrac{\partial}{\partial t}(c_0 p + \nabla \cdot \mathbf{u}), q\rangle + \Big(\kappa(\nabla \cdot \mathbf{u})\nabla p, \nabla q\Big) = (g, q) \quad \forall q \in H_0^1(\Omega)\,, \\ c_0 p(\cdot, 0) + \nabla \cdot \mathbf{u}(\cdot, 0) = l\,. \end{cases} \tag{2.5}$$

It is easy to verify that the bilinear form $e(\cdot,\cdot)$ satisfied the hypothesis of Lax-Milgram Theorem. Thus for a fixed $t \in I$ and $p(\cdot, t)$ in $L^2(\Omega)$, the first equation of (2.5) can be solved for $\mathbf{u}$. Define the operator $B : L^2(\Omega) \to L^2(\Omega)$ such that for $p \in L^2(\Omega)$, $Bp = \nabla \cdot \mathbf{u}$ where $\mathbf{u}$ satisfies

$$\begin{cases} e(\mathbf{u}, \mathbf{v}) = -(\nabla p, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega)\,, \\ \qquad \mathbf{u} = 0 \quad \text{on } \partial\Omega\,. \end{cases}$$

The following Lemma can be found in [25].

Lemma 2.7. *The operator $B : L^2(\Omega) \to L^2(\Omega)$ defined above is linear, continuous, monotone and self-adjoint with $Ker(B) = Ker(\nabla)$ and $Rg(B) = Ker(\nabla)^{\perp}$.*

Remark 2.8. Due to the assumption that the boundary is smooth, B is also continuous from $H_0^1(\Omega)$ into itself. Therefore there exist constants $C_{B_0} and C_{B_1} > 0$ such that

$$\begin{aligned} \|Bq\| &\leq C_{B_0}\|q\| \quad \forall q \in L^2(\Omega)\,, \\ \|Bq\|_1 &\leq C_{B_1}\|q\|_1 \quad \forall q \in H_0^1(\Omega)\,. \end{aligned} \tag{2.6}$$

Moreover, from Lemma 2.7 and the homogeneous boundary condition, $Ker(B) = Ker(\nabla) = \{0\}$. As a result, the mapping $B : L^2(\Omega) \to L^2(\Omega)$ is one to one thus B is a continuous bijection from $L^2(\Omega)$ into itself. According to the bounded inverse Theorem, B and $c_0 + B$ have bounded inverses.

Substituting $\nabla \cdot \mathbf{u} = Bp$ in the second equation of (2.5), we obtain the decoupled initial value problem for p.

$$\begin{cases} \langle \dfrac{\partial}{\partial t}(c_0 + B)p, q\rangle + \Big(\kappa(Bp)\nabla p, \nabla q\Big) = (g, q) \quad \forall q \in H_0^1(\Omega)\,, \\ (c_0 + B)p(\cdot, 0) = l\,. \end{cases} \tag{2.7}$$

To prove the existence of a weak solution, we use the modified Rothe's method to construct a convergent sequence of approximate solutions of (2.7) using the backward Euler approximation of the time derivative in (2.7). Let $k = T/n$ for some positive integer n. Partition I uniformly with time step k and denote nodal points by $t_i = t_i^n = ik$, for $i = 1, 2, \ldots, n$. Let $p_n^0 \in L^2(\Omega)$ be such that $(c_0 + B)p_n^0 = l$ and define

$$\begin{cases} g_n^i := 1/k \displaystyle\int_{t_{i-1}}^{t_i} g(t)dt\,, \\ \delta(c_0 + B)p_n^i := (c_0 + B)(p_n^i - p_n^{i-1})/k\,, \quad i = 1, \ldots, n\,. \end{cases} \tag{2.8}$$

We apply the following scheme inductively to obtain a sequence p_n^i, $i = 1, \cdots, n$.

$$\Big(\delta(c_0 + B)p_n^i, q\Big) + \Big(\kappa(Bp_n^i)\nabla p_n^i, \nabla q\Big) = (g_n^i, q) \quad \forall q \in H_0^1(\Omega)\,. \tag{2.9}$$

Multiplying the above equation by k, we have

$$\Big((c_0+B)p_n^i, q\Big) + k\Big(\kappa(Bp_n^i)\nabla p_n^i, \nabla q\Big) = k(g_n^i, q) + \Big((c_0+B)p_n^{i-1}, q\Big) \quad \forall q \in H_0^1(\Omega)\,. \tag{2.10}$$

To prove the existence of a solution of (2.10), we need the following direct corollary of Brouwer's fixed point Theorem.

Lemma 2.9. *Let H be a finite-dimensional Hilbert space with scalar product $(\cdot,\cdot)$ and the corresponding norm $|\cdot|$. Let Φ be a continuous mapping from H to H and assume that there exists $\mu > 0$ such that:*

$$(\Phi(u), u) \geq 0\,, \quad \forall u \in H \quad \textit{with} \quad |u| = \mu\,.$$

Then there exists an elment $u \in H$ such that:

$$\Phi(u) = 0\,, \quad \textit{with} \quad |u| \leq \mu\,.$$

The following Lemma shows that (2.10) is well posed.

Lemma 2.10. *Given p_n^{i-1} in $L^2(\Omega)$, and $g_n^i \in L^2(\Omega)$, equation (2.10) has a weak solution p_n^i in $H_0^1(\Omega)$, $i = 1, \cdots, n$.*

Proof. For notational simplicity, we write $\bar{g} = kg_n^i + (c_0+B)p_n^{i-1}$ and $\bar{p} = p_n^i$. Given $\bar{g}$ in $L^2(\Omega)$, we show that there exists a $\bar{p}$ in $H_0^1(\Omega)$ such that

$$\Big((c_0+B)\bar{p}, q\Big) + k\Big(\kappa(B\bar{p})\nabla\bar{p}, \nabla q\Big) = (\bar{g}, q) \quad \forall q \in H_0^1(\Omega)\,. \tag{2.11}$$

For this purpose we let $\{q_i\}_{i=1}^{\infty}$ be an orthonormal basis of $H_0^1(\Omega)$. Denote by V_m the finite dimensional space spanned by $\{q_1, q_2, \ldots, q_m\}$ and define the mapping $\Phi_m : V_m \to V_m$ by

$$(\Phi_m q, w) = \Big((c_0+B)q, w\Big) + k\Big(\kappa(Bq)\nabla q, \nabla w\Big) - (\bar{g}, w) \quad \forall w \in V_m\,.$$

From (1.11), (1.22), and the monotonicity of B,

$$\begin{aligned}(\Phi_m q, q) &= \Big((c_0 + B)q, q\Big) + k\Big(\kappa(Bq)\nabla q, \nabla q\Big) - (\bar{g}, q)\\ &\geq c_0\|q\|^2 + (Bq, q) + kk_*\|\nabla q\|^2 - \|\bar{g}\|\|q\|\\ &\geq (kk_*\|q\|_1/C_p^2 - \|\bar{g}\|)\|q\|_1\,.\end{aligned}$$

Thus $(\Phi_m q, q) \geq 0$, for all q with $\|q\|_1 = C_p^2\|\bar{g}\|/(kk_*)$. Because V_m is finite dimensional and Φ defined above is continuous, from Lemma 2.9, there exists $\bar{p}_m$ in V_m, such that $\|\bar{p}_m\|_1 \leq C_p^2\|\bar{g}\|/(kk_*)$ and $\bar{p}_m$ satisfies $\Phi_m\bar{p}_m = 0$, i.e.

$$\Big((c_0 + B)\bar{p}_m, q\Big) + k\Big(\kappa(B\bar{p}_m)\nabla\bar{p}_m, \nabla q\Big) = (\bar{g}, q)\,, \quad \forall q \in V_j\,, \quad j \leq m\,. \tag{2.12}$$

Since

$$\|\bar{p}_m\|_1 \leq C_p^2\|\bar{g}\|/(kk_*)\,, \tag{2.13}$$

i.e., $\{\bar{p}_m\}_{m=1}^\infty$ is a uniformly bounded sequence in $H_0^1(\Omega)$, there exsits a subsequence of $\{\bar{p}_m\}_{m=1}^\infty$, still denoted by $\{\bar{p}_m\}_{m=1}^\infty$, and a function $\bar{p} \in H_0^1(\Omega)$ such that

$$\bar{p}_m \rightharpoonup \bar{p} \quad \text{in} \quad H_0^1(\Omega)\,. \tag{2.14}$$

Due to the compact embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$,

$$\bar{p}_m \to \bar{p} \quad \text{in} \quad L^2(\Omega)\,. \tag{2.15}$$

We now show that the weak limit $\bar{p}$ is a solution of (3.9). Choose a test function

$$q \in W^{1,\infty}(\Omega) \cap H_0^1(\Omega)\,. \tag{2.16}$$

Using (2.6), (2.15), (2.16), and applying the Cauchy-Schwarz inequality we have

$$\begin{aligned}\Big|\Big((c_0+B)\bar{p}_m,q\Big)-\Big((c_0+B)\bar{p},q\Big)\Big| \\ &\le \|(c_0+B)(\bar{p}_m-\bar{p})\|\|q\| \\ &\le (c_0+C_{B_0})\|q\|\|\bar{p}_m-\bar{p}\|\to 0 \quad \text{as} \quad m\to\infty\,.\end{aligned}$$

The boundedness of B, the Nemytskii property of κ and the fact $\bar{p}_m\to\bar{p}$ imply that

$$\kappa(B\bar{p}_m)\to\kappa(B\bar{p}) \quad \text{in} \quad L^2(\Omega) \quad \text{as} \quad m\to\infty\,. \tag{2.17}$$

From (1.11), (2.13), (2.14), (2.15), (2.16) and (2.17)

$$\begin{aligned}\Big|\Big(\kappa(B\bar{p})\nabla\bar{p},\nabla q\Big)-\Big(\kappa(B\bar{p}_m)\nabla\bar{p}_m,\nabla q\Big)\Big| \\ &\le \Big|\Big(\kappa(B\bar{p})\nabla(\bar{p}-\bar{p}_m),\nabla q\Big)\Big|+\Big|\Big(\big(\kappa(B\bar{p})-\kappa(B\bar{p}_m)\big)\nabla\bar{p}_m,\nabla q\Big)\Big| \\ &\le k^*\Big|\Big(\nabla(\bar{p}-\bar{p}_m),\nabla q\Big)\Big|+\|\kappa(B\bar{p})-\kappa(B\bar{p}_m)\|\|\nabla\bar{p}_m\| \\ &\to 0 \quad m\to\infty\,.\end{aligned}$$

Because $W^{1,\infty}(\Omega)$ is dense in $H_0^1(\Omega)$, for all q in $H_0^1(\Omega)$ we have that

$$\lim_{m\to\infty}\Big((c_0+B)\bar{p}_m,q\Big)+\Big(\kappa(B\bar{p}_m)\nabla\bar{p}_m,\nabla q\Big)=\Big((c_0+B)\bar{p},q\Big)+\Big(\kappa(B\bar{p})\nabla\bar{p},\nabla q\Big)\,.$$

Combining with (2.12), we obtain

$$\Big((c_0+B)\bar{p},q\Big)+\Big(\kappa(B\bar{p})\nabla\bar{p},\nabla q\Big)=(\bar{g},q) \quad \forall q\in V_j\,. \tag{2.18}$$

Since finite linear combinations of $\{q_j\}_{j=1}^{\infty}$ are dense in $H_0^1(\Omega)$, we conclude that $\bar{p}$ is a solutin of (3.9). $\square$

Lemma 2.11 (Discrete energy estimate)**.** *There exists a constant C independent of n such that*

$$k\sum_{i=1}^{n}\|p_n^i\|_1^2 \le C\,.$$

Proof. Writing (2.9) with the test function $q = p_n^i$, multiplying by k and summing from $i = 2$ to any $n_0 \le n$, we obtain

$$\sum_{i=2}^{n_0}\Big((c_0+B)(p_n^i - p_n^{i-1}), p_n^i\Big) + k\sum_{i=2}^{n_0}\Big(\kappa(Bp_n^i)\nabla p_n^i, \nabla p_n^i\Big) = k\sum_{i=2}^{n_0}(g_n^i, p_n^i)\,. \tag{2.19}$$

Using summation by parts and the fact that B is self-adjoint, we have that

$$\begin{aligned}
&\sum_{i=2}^{n_0}\Big((c_0+B)(p_n^i - p_n^{i-1}), p_n^i\Big)\\
&\quad= \frac{1}{2}\sum_{i=2}^{n_0}\Big[\Big((c_0+B)(p_n^i - p_n^{i-1}), p_n^i + p_n^{i-1}\Big) + \Big((c_0+B)(p_n^i - p_n^{i-1}), p_n^i - p_n^{i-1}\Big)\Big]\\
&\quad= \frac{1}{2}\sum_{i=2}^{n_0}\Big[\Big((c_0+B)p_n^i, p_n^i\Big) - \Big((c_0+B)p_n^{i-1}, p_n^{i-1}\Big) + \Big((c_0+B)(p_n^i - p_n^{i-1}), p_n^i - p_n^{i-1}\Big)\Big]\\
&\quad= \frac{1}{2}\sum_{i=2}^{n_0}\Big((c_0+B)(p_n^i - p_n^{i-1}), p_n^i - p_n^{i-1}\Big) + \frac{1}{2}\Big((c_0+B)p_n^{n_0}, p_n^{n_0}\Big) - \frac{1}{2}\Big((c_0+B)p_n^1, p_n^1\Big)\,.
\end{aligned}$$

It follows from (1.11) and (2.19) that

$$\begin{aligned}
&\frac{1}{2}\sum_{i=2}^{n_0}\Big((c_0+B)(p_n^i - p_n^{i-1}), p_n^i - p_n^{i-1}\Big) + \frac{1}{2}\Big((c_0+B)p_n^{n_0}, p_n^{n_0}\Big) + k\sum_{i=2}^{n_0}k_*\|\nabla p_n^i\|^2\\
&\qquad\le k\sum_{i=2}^{n_0}(g_n^i, p_n^i) + \frac{1}{2}\Big((c_0+B)p_n^1, p_n^1\Big)\,,
\end{aligned} \tag{2.20}$$

and the monotonicity of B leads to

$$k\sum_{i=2}^{n_0}k_*\|\nabla p_n^i\|^2 \le k\sum_{i=2}^{n_0}(g_n^i, p_n^i) + \frac{1}{2}\Big((c_0+B)p_n^1, p_n^1\Big)\,. \tag{2.21}$$

To estimate the second term on the right hand side of the above inequality, we write (2.9) with $i = 1$, multiply by k and set the test function $q = p_n^1$ to obtain

$$\begin{aligned}
&\Big((c_0 + B)p_n^1, p_n^1\Big) + kk_*\|\nabla p_n^1\|^2 \\
&\qquad \le \Big((c_0 + B)p_n^1, p_n^1\Big) + k\Big(\kappa(Bp_n^1)\nabla p_n^1, \nabla p_n^1\Big) \\
&\qquad = k(g_n^1, p_n^1) + (l, p_n^1)\,.
\end{aligned}$$

The monotonicity of B, Young's inequality, and inequality (2.21) yield

$$\begin{aligned}
k\sum_{i=1}^{n_0}\|\nabla p_n^i\|^2 &\le Ck\sum_{i=1}^{n_0}(g_n^i, p_n^i) + (l, p_n^1) \\
&\le k\sum_{i=1}^{n_0}(C(\varepsilon)\|g_n^i\|^2 + \varepsilon\|p_n^i\|^2) + C(\varepsilon)\|l\|^2\,,
\end{aligned} \tag{2.22}$$

where $C(\varepsilon)$ is a constant depending on ε. Choosing a sufficiently small $\varepsilon > 0$ such that $\varepsilon C_p^2 < 1$ where C_p is given by (1.22) and noticing that for some constant C independent of n

$$k\sum_{i=1}^{n_0}\|g_n^i\|^2 \le C\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\|g(t)\|^2dt = C\|g\|^2_{L^2(I;L^2(\Omega))}\,, \tag{2.23}$$

we obtain that for any $n_0 \le n$

$$k\sum_{i=1}^{n_0}\|p_n^i\|_1^2 \le \frac{C}{1 - \varepsilon C_p^2}\|g\|_{L^2(I;L^2(\Omega))}\,.$$

□

Lemma 2.12 (see [1], p. 327)**.** *For φ in $C^\infty(I)$, define two piecewise constant functions φ_n and $\tilde{\varphi}_n$ such that*

$$\begin{cases}
\varphi_n(t) = \varphi(t_i)\,, & t \in (t_{i-1}, t_i]\,, \\
\tilde{\varphi}_n(t) = \Big(\varphi(t_{i+1}) - \varphi(t_i)\Big)/k\,, & t \in (t_{i-1}, t_i]\,,
\end{cases}$$

for $i = 1, 2, \ldots, n$, with

$$\begin{cases} \varphi_n(0) = \varphi(t_1)\,, \\ \tilde{\varphi}_n(0) = \tilde{\varphi}(t_1)\,, \quad \tilde{\varphi}_n(t_{n+1}) = \tilde{\varphi}(t_n)\,. \end{cases}$$

Then

$$\|\varphi_n - \varphi\|_{L^2(I)} \leq C(\varphi)k\,, \quad \|\tilde{\varphi}_n - \varphi'\|_{L^2(I)} \leq C(\varphi)k^{1/2}$$

where the constant $C(\varphi)$ depends on φ only.

Lemma 2.13 (Aubin-Lions, see [57])**.** *Let X_0, X, X_1 be three Banach spaces with $X_0 \subset X \subset X_1$. Suppose that X_0 is compactly embedded in X and X is continuously embedded in X_1. Furthermore assume that X_0 andd X_2 are reflexive spaces. Let $1 < p < \infty$ and $1 < q < \infty$. Define*

$$W = \{u \in L^p([0,T]; X_0);\ u' \in L^q([0,T]; X_1\}\,.$$

Then the embedding of W into $L^p([0,T]; X)$ is compact.

Remark 2.14. Specifically, let $X_0 = H_0^1(\Omega)$, $X = L^2(\Omega)$, $X_1 = H^{-1}(\Omega)$ the dual space of $H_0^1(\Omega)$ and $p = q = 2$. Then

$$\begin{aligned} W = \{u \in L^2(I; H_0^1(\Omega));\ u' \in L^q(I; H^{-1}(\Omega))\} \\ \hookrightarrow L^2(I; L^2(\Omega)) \quad \text{compactly}\,. \end{aligned} \tag{2.24}$$

Armed with the above Lemmas, we are now ready to show the existence of a solution of equation (2.7).

Theorem 2.15. *Given l in $L^2(\Omega)$ and g in $L^2(I; L^2(\Omega))$, equation (2.7) has at least one solution p in $L^2(I; H_0^1(\Omega))$.*

Proof. For each positive integer n, define the piecewise constant function $P_n \in L^2(I; H_0^1(\Omega))$ as

$$P_n(t) = p_n^i\,, \quad t \in (t_{i-1}, t_i]\,, \tag{2.25}$$

for $i = 1, 2, \ldots, n$ with $P_n(0) = p_n^1$. From Lemma 2.11

$$\|P_n\|^2_{L^2(I;H_0^1(\Omega))} = k \sum_{i=1}^{n} \|p_n^i\|_1^2 \le C\,. \tag{2.26}$$

Hence there exists a weak limit of the sequence $\{P_n\}_{n=1}^{\infty}$ which is the candidate solution of (2.7). However, the derivative of P_n is zero for a.e. t. To approximate the weak time derivative we define $\tilde{P}_n \in L^2(I; H_0^1(\Omega))$ as

$$\tilde{P}_n(t) = p_n^{i-1} + (t - t_{i-1})(p_n^i - p_n^{i-1})/k\,, \quad t \in (t_{i-1}, t_i]\,, \quad i = 1, 2, \ldots, n\,. \tag{2.27}$$

Obviously $\tilde{P}_n$ is piecewise linear in t. Moreover, the time derivative of $\tilde{P}_n$ is a piecewise constant in t satisfying

$$\tilde{P}_n' = \delta p_n^i = (p_n^i - p_n^{i-1})/k\,, \quad t \in (t_{i-1}, t_i]\,, \quad i = 1, 2, \ldots, n\,. \tag{2.28}$$

It is easy to verify that both $\{P_n\}_{n=1}^{\infty}$ and $\{\tilde{P}_n\}_{n=1}^{\infty}$ satisfy the same energy estimate and have the same weak and strong limit. Therefore $\|\tilde{P}_n(t)\|_{L^2(I;H_0^1(\Omega))}$ is also uniformly bounded. Due to the boundedness of B on $H_0^1(\Omega)$, there exists a constant C such that

$$\|(c_0 + B)\tilde{P}_n(t)\|_{L^2(I;H_0^1(\Omega))} \le C\,. \tag{2.29}$$

Meanwhile, it follows from (1.11), (2.9) and the Cauchy-Schwarz inequality that

$$\left|\Big((c_0 + B)\delta p_n^i, q\Big)\right| = |(g_n^i, q) - (\kappa(Bp_n^i)\nabla p_n^i, \nabla q)| \le (\|g_n^i\| + k^*\|p_n^i\|_1)\|q\|_1 \quad \forall q \in H_0^1(\Omega)\,.$$

Hence

$$\|(c_0 + B)\delta p_n^i\|_{-1} \le \|g_n^i\| + k^*\|p_n^i\|_1\,.$$

It follows from Lemma 2.11 and (2.23) that

$$\begin{aligned}\|(c_0+B)\tilde{P}_n'\|^2_{L^2(I;H^{-1}(\Omega))} &= k\sum_{i=1}^{n}\|(c_0+B)\delta p_n^i\|^2_{-1}\\ &\le k\sum_{i=1}^{n}\|g_n^i\|^2 + k\sum_{i=1}^{n}k^*\|p_n^i\|_1^2\\ &\le C\,.\end{aligned} \tag{2.30}$$

Thus we can find a subsequence of $\{(c_0+B)\tilde{P}_n\}_{n=1}^{\infty}$, still denoted as $\{(c_0+B)\tilde{P}_n\}_{n=1}^{\infty}$, and a function r in $L^2(I;H_0^1(\Omega))$ with r' in $L^2(I;H^{-1}(\Omega))$ ([58], p. 356) such that

$$\begin{cases}(c_0+B)\tilde{P}_n \rightharpoonup r \quad \text{in} \quad L^2(I;H_0^1(\Omega))\,,\\ (c_0+B)\tilde{P}_n' \rightharpoonup r' \quad \text{in} \quad L^2(I;H^{-1}(\Omega))\,.\end{cases}$$

From the embedding (2.24)

$$(c_0+B)\tilde{P}_n \to r \quad \text{in} \quad L^2(I;L^2(\Omega))\,.$$

Due to the relationship between the piecewise constant P_n and the piecewise linear functions $\tilde{P}_n$ we have that

$$(c_0+B)P_n \to r \quad \text{in} \quad L^2(I;L^2(\Omega))\,. \tag{2.31}$$

Since c_0+B is invertible on $L^2(\Omega)$ (see Remark 2.8), there exists a p in $L^2(I;L^2(\Omega))$ satisfying $(c_0+B)p=q$, such that

$$P_n \to p = (c_0+B)^{-1}r \quad \text{in} \quad L^2(I;L^2(\Omega))\,. \tag{2.32}$$

Recall that $\{P_n\}_{n=1}^{\infty}$ is bounded in $L^2(I;H_0^1(\Omega))$ (see (2.26)). Hence there exists $\bar{p}$ in $L^2(I;H_0^1(\Omega))$ such that

$$P_n \rightharpoonup \bar{p} \quad \text{in} \quad L^2(I;H_0^1(\Omega)). \tag{2.33}$$

Then $p = \bar{p}$ because the weak limit is unique.

Next we prove that p satisfies (2.7). Define

$$g_n(t) = g_n^i\,, \quad t \in (t_{i-1}, t_i]\,, \quad i = 1, 2, \ldots, n\,.$$

Proceeding as in Lemma 2.12, we obtain that

$$\|g_n - g\|_{L^2(I;L^2(\Omega))} \to 0\,, \quad \text{as} \quad n \to \infty\,. \tag{2.34}$$

Let $\bar{q}$ be a test function of the form $\bar{q} = q\varphi$ satisfying

$$q \in W^{1,\infty}(\Omega) \cap H_0^1(\Omega) \quad \textit{and} \quad \varphi(t) \in C_0^\infty(I)\,. \tag{2.35}$$

It follows from summation by parts that

$$\begin{aligned}
&\sum_{i=1}^{n} \Big((c_0{+}B)(p_n^i - p_n^{i-1}), q\Big)\varphi(t_i) \\
&\qquad = \Big((c_0 + B)p_n^n, q\Big)\varphi(t_n) - \Big((c_0 + B)p_n^0, q\Big)\varphi(t_1) \\
&\qquad\quad + k\sum_{i=1}^{n-1} \Big((c_0 + B)p_n^i, q\Big)\Big(\varphi(t_{i+1}) - \varphi(t_i)\Big)/k\,.
\end{aligned}$$

Multiplying (2.9) with $k\varphi(t_i)$, summing up from $i = 1$ to n, and using the summation above, we obtain

$$\begin{aligned}\Big((c_0 + B)p_n^n, q\Big)\varphi(T) - \Big((c_0 + B)p_n^0, q\Big)\varphi(t_1) \\ + \sum_{i=1}^{n-1}\Big((c_0 + B)p_n^i, q\Big)\Big(\varphi(t_{i+1}) - \varphi(t_i)\Big) \\ + k\sum_{i=1}^{n}\Big(\kappa(Bp_n^i)\nabla p_n^i, \nabla q\Big)\varphi(t_i) \\ =k\sum_{i=1}^{n}(g_n^i, q)\varphi(t_i)\,.\end{aligned}$$

Recall that $\varphi(T) = 0$. The above equation takes the form

$$\begin{aligned}-(l,q)\varphi(k) - \int_0^T\Big((c_0 + B)P_n, q\Big)\tilde{\varphi}_n\, dt \\ + \int_0^T\Big(\kappa(BP_n)\nabla P_n, \nabla q\Big)\varphi_n\, dt \\ = \int_0^T(g_n, q)\varphi_n\,.\end{aligned} \tag{2.36}$$

Letting $n \to \infty$ in (2.36) and using the continuity of $\varphi(t)$, we have that

$$-(l, q)\varphi(k) \to -(l, q)\varphi(0) = 0\,. \tag{2.37}$$

Notice that

$$\begin{aligned}\int_0^T\Big((c_0 + B)P_n, q\Big)\tilde{\varphi}_n\, dt = \int_0^T\Big((c_0 + B)P_n, q\Big)(\tilde{\varphi}_n - \varphi')\, dt \\ + \int_0^T\Big((c_0 + B)P_n, q\Big)\varphi'\, dt\,.\end{aligned}$$

It follows from Lemma 2.12, (2.6), and (2.26) that

$$\begin{aligned}
\Big|\int_0^T \Big((c_0+B)P_n, q\Big)(\tilde{\varphi}_n - \varphi')\ dt\Big| \\
&\le \|q\| \int_0^T \|(c_0+B)P_n\| |\tilde{\varphi}_n - \varphi'|\ dt \\
&\le C\|q\| \|P_n\|_{L^2(I;L^2(\Omega))} \|\tilde{\varphi}_n - \varphi'\|_{L^2(I)} \to 0 \quad as \quad n\to\infty\,.
\end{aligned}$$

Since $q\varphi' \in L^2(I; H_0^1(\Omega))$, the above and (2.33) yields that

$$\begin{aligned}
\int_0^T \Big((c_0+B)P_n, q\Big)\varphi'\ dt \\
&\to \int_0^T \Big((c_0+B)p, q\Big)\varphi'\ dt \\
&= -\int_0^T \Big((c_0+B)p', q\Big)\varphi\ dt\,.
\end{aligned}$$

Hence

$$\int_0^T \Big((c_0+B)P_n, q\Big)\tilde{\varphi}_n\ dt \to -\int_0^T \Big((c_0+B)p', q\Big)\varphi\ dt\,, \quad \text{as} \quad n\to\infty\,. \tag{2.38}$$

Next we prove the convergence of the third term $\int_0^T \Big(\kappa(BP_n)\nabla P_n, \nabla q\Big)\varphi_n\ dt$ on the left hand side of (2.36) to $\int_0^T \Big(\kappa(Bp)\nabla p, \nabla q\Big)\varphi\ dt$. Write

$$\begin{aligned}
\int_0^T \Big(\kappa(BP_n)\nabla P_n, \nabla q\Big)\varphi_n\ dt - \int_0^T \Big(\kappa(Bp)\nabla p, \nabla q\Big)\varphi\ dt \\
&= \int_0^T \Big(\kappa(BP_n)\nabla P_n, \nabla\Big)(\varphi_n - \varphi)\ dt \\
&+ \int_0^T \Big(\kappa(Bp)\nabla(P_n - p), \nabla q\Big)\varphi\ dt \\
&+ \int_0^T \Big(\big(\kappa(BP_n) - \kappa(Bp)\big)\nabla P_n, \nabla q\Big)\varphi\ dt\,. \\
&:= \mathrm{I} + \mathrm{II} + \mathrm{III}
\end{aligned}$$

Using (1.11), Lemma 2.12, (2.26) and (4.11), we have that

$$
\begin{aligned}
\mathrm{I} &= \Big| \int_0^T \Big(\kappa(BP_n)\nabla P_n, \nabla q\Big)\Big(\varphi_n(t) - \varphi(t)\Big)\ dt\Big| \\
&\qquad \leq k^* \|\nabla q\|_1 \|P_n\|_{L^2(I;H_0^1(\Omega))} \|\varphi_n(t) - \varphi(t)\|_{L^2(I)} \to 0, \quad \text{as} \quad n \to \infty.
\end{aligned}
$$

The test function $\nabla q \varphi$ belongs to $L^2(I; L^2(\Omega))$. Therefore the weak convergence of P_n to p in $L^2(I; L^2(\Omega)$ and (1.11) imply that

$$
\mathrm{II} = \int_0^T \Big(\kappa(Bp)\nabla(P_n - p), \nabla q\Big)\varphi(t)\ dt \to 0, \quad \text{as} \quad n \to \infty.
$$

For III, we first use (1.11) and (2.32) to deduce that

$$
\kappa(BP_n) \to \kappa(Bp) \quad \text{in} \quad L^2(I; L^2(\Omega))\,.
$$

Thus

$$
\begin{aligned}
\mathrm{III} &= \Big| \int_0^T \Big(\big(\kappa(BP_n) - \kappa(Bp)\big)\nabla P_n, \nabla q\Big)\varphi(t)\ dt\Big| \\
&\leq C \int_0^T \|\kappa(BP_n) - \kappa(Bp)\| \|\nabla P_n\| dt \\
&\leq C \|\kappa(BP_n) - \kappa(Bp)\|_{L^2(I;L^2(\Omega))} \|P_n\|_{L^2(I;H_0^1(\Omega))} \\
&\to 0 \quad \text{as} \quad n \to \infty\,.
\end{aligned}
$$

Combining the above estimates we obtain

$$
\int_0^T \Big(\kappa(BP_n)\nabla P_n, \nabla q\Big)\varphi_n\ dt \to \int_0^T \Big(\kappa(Bp)\nabla p, \nabla q\Big)\varphi\ dt\,, \quad \text{as} \quad n \to \infty. \tag{2.39}
$$

From a similar argument, it is straightforward to show that

$$
\int_0^T (g_n, q)\varphi_n\ dt = \int_0^T (g, q)\varphi_n(t)\ dt \to \int_0^T (g, q)\varphi(t)\ dt\,. \tag{2.40}
$$

Combing (2.37), (2.38), (2.39) and (2.40), we conclude that for any test function $\bar{q}$ of the form $q\varphi(t)$ satisfying (2.35),

$$-\int_0^T \Big((c_0+B)p', \bar{q}\Big)\, dt + \int_0^T \Big(\kappa(Bp)\nabla p, \nabla \bar{q}\Big)\, dt = \int_0^T (g, \bar{q})\, dt\,. \tag{2.41}$$

The test functions $\bar{q}$ defined in (2.35) is dense in $L^2(I; H_0^1(\Omega))$. Therefore (2.41) holds for all the test function q in $L^2(I; H_0^1(\Omega))$, i.e., p satisfies (2.7) for a.e. $t \in I$.

Finally we consider the initial condition. The facts

$$(c_0+B)p \in L^2(I; H_0^1(\Omega)) \quad \textit{and} \quad (c_0+B)p' \in L^2(I; H^{-1}(\Omega))$$

imply that $(c_0+B)p$ belongs to $C(I; L^2(\Omega))$, the space of all continuous functions that value in $L^2(\Omega)$ (see [58], p. 288). Hence the initial condition $(c_0+B)p(\cdot,0) = l$ should be given in $L^2(\Omega)$. □

In the next Theorem we give some conditions that guarantee the uniqueness of solutions of equation (2.7).

Theorem 2.16. *Assume*

(H1) κ is Lipschitz continuous with Lipschitz constant k_l, i.e.,

$$|\kappa(x) - \kappa(y)| < k_l|x-y|\,, \quad \forall x\,, y \in \mathbb{R}\,;$$

(H2) $\nabla p \in L^\infty(\Omega)$ and there exists a constant C such that $\|\nabla p\|_\infty \le C$;

(H3) $c_0 k_* / (C_p^2 k^* C_{B_1}) > 1$.

Then the solution p of equation (2.7) is unique.

Proof. Suppose p_1, p_2 both satisfy (2.7), i.e.,

$$\langle \frac{\partial}{\partial t}(c_0+B)p_1, q\rangle + \Big(\kappa(Bp_1)\nabla p_1, \nabla q\Big) = (g,q)\,, \quad \forall q \in H_0^1(\Omega)\,,$$
$$\langle \frac{\partial}{\partial t}(c_0+B)p_2, q\rangle + \Big(\kappa(Bp_2)\nabla p_2, \nabla q\Big) = (g,q)\,, \quad \forall q \in H_0^1(\Omega)\,.$$

Taking the substraction and set $q = (c_0+B)(p_1-p_2)$ we have

$$\begin{aligned}
&\langle (c_0+B)(p_1-p_2)', (c_0+B)(p_1-p_2)\rangle \\
&\qquad + \Big(\kappa(Bp_1)\nabla p_1 - \kappa(Bp_2)\nabla p_2, \nabla(c_0+B)(p_1-p_2)\Big) \\
&= \langle (c_0+B)(p_1-p_2)', (c_0+B)(p_1-p_2)\rangle \\
&\qquad + \Big(\big(\kappa(Bp_1)-\kappa(Bp_2)\big)\nabla p_1, \nabla(c_0+B)(p_1-p_2)\Big) \\
&\qquad + \Big(\kappa(Bp_2)\nabla(p_1-p_2), \nabla c_0(p_1-p_2)\Big) \\
&\qquad + \Big(\kappa(Bp_2)\nabla(p_1-p_2), \nabla B(p_1-p_2)\Big) \\
&= 0\,.
\end{aligned} \tag{2.42}$$

Recall

$$\langle (c_0+B)(p_1-p_2)', (c_0+B)(p_1-p_2)\rangle = \frac{1}{2}\frac{d}{dt}\|(c_0+B)(p_1-p_2)\|^2\,. \tag{2.43}$$

Then (H1), (H2), (2.6), and Young's inequality yields that

$$\begin{aligned}
&|((\kappa(Bp_1)-\kappa(Bp_2))\nabla p_1, \nabla(c_0+B)(p_1-p_2))| \\
&\quad \le \varepsilon\|(\kappa(Bp_1)-\kappa(Bp_2)\|^2 + C(\varepsilon)\|(c_0+B)(p_1-p_2)\|_1^2 \\
&\quad \le \varepsilon\|p_1-p_2\|^2 + C(\varepsilon)\|p_1-p_2\|_1^2\,.
\end{aligned} \tag{2.44}$$

It follows from (1.11), (1.22) and (2.6) that

$$\Big(\kappa(Bp_2)\nabla(p_1-p_2), \nabla c_0(p_1-p_2)\Big) \ge (c_0 k_*/C_p^2)\|p_1-p_2\|_1^2 \tag{2.45}$$

and

$$|(\kappa(Bp_2)\nabla(p_1-p_2),\nabla B(p_1-p_2))| \le k^* C_{B_1}\|p_1-p_2\|_1^2\,. \tag{2.46}$$

(2.43), (2.44), (2.45), (2.46), and (2.43) yield

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(c_0+B)(p_1-p_2)\|^2 - C(\varepsilon)\|p_1-p_2\|^2 \\
&\qquad + (c_0 k_*/C_p^2 - k^* C_{B_1} - \varepsilon)\|p_1-p_2\|_1^2 \\
&\quad \le 0\,.
\end{aligned} \tag{2.47}$$

Hypothesis (H3) allows us to choose small $\varepsilon > 0$ such that $c_0 k_*/C_p^2 - k^* C_{B_1} - \varepsilon > 0$. The boundedness of the inverse of B and (2.47) give that

$$\frac{1}{2}\frac{d}{dt}\|(c_0+B)(p_1-p_2)\|^2 \le C\|p_1-p_2\|^2 \le C\|(c_0+B)(p_1-p_2)\|^2\,.$$

It follows from the Gronwall's inequality that $(c_0+B)(p_1-p_2)=0$. Therefore $p_1=p_2$ because c_0+B has a bounded inverse. $\square$

Remark 2.17. After obtaining a solution p of equation (2.7), we can solve the first equation of (2.5) for $\mathbf{u}$ with p substituted to the right hand side. According to the inverse estimate of the elliptic equation

$$\|\mathbf{u}(t)\|_1 \le C\|p(t)\|\,, \quad \forall t \in I\,,$$

$\mathbf{u}$ belongs to $L^2(I;\mathbf{H}_0^1(\Omega))$ and the pair $(\mathbf{u},p)$ is the solution of system (2.5). Furthermore, the linear dependence of $\mathbf{u}$ on p guarantees the uniqueness of $\mathbf{u}$ as long as p is unique.

2.3 Numerical approximation with the finite element method

In this section, we consider the numerical approximation to the solutions of (2.7). In particular we will derive error estimates for a fully discretized numerical scheme using backward Euler method for the temporal discretization and finite element method on the spatial dimension. Throughout this section, we assume that the weak solution (u, p) of (1.10) is unique, i.e., we assume that hypothesis (H1), (H2) and (H3) are satisfied.

We start by constructing the finite element spaces as follows. Let τ_h be a family of quasi-uniform triangulations (see [5], p. 2-3) of a convex polygonal, or polyhedral, domain Ω satisfying $\max\limits_{\tau \in \tau_h} \operatorname{diam} \tau \leq h$ (where τ is a geometrical element, e.g., a triangle, or a tetrahedron). Let Q_h be the space consisting of continuous functions on Ω which are linear on each triangle, or tetrahedron, and vanish on $\partial\Omega$. Let $\{P_j\}_{j=1}^{N_h}$ be the set of all the interior vertices of the triangulation. Assume that Φ_j^p is the pyramid function which value 1 at P_j and vanishes at all the other vertices. It is easy to see that $\{\Phi_j^p\}_{j=1}^{N_h}$ forms a basis of Q_h. Let $\mathbf{V}_h = (Q_h)^d$, $d = 2, 3$. Then $\{(\Phi_j^p, 0), (0, \Phi_j^p)\}_{j=1}^{N_h}$ ($\{(\Phi_j^p, 0, 0), (0, \Phi_j^p, 0), (0, 0, \Phi_j^p)\}_{j=1}^{N_h}$) forms a basis of $\mathbf{V}_h$ for d=2 or d=3, respectively.

As in the previous section, we denote the time stepsize by k, that is, $k = \frac{T}{N}$, for some positive integer N, and $t_n = nk$, $n = 0 \cdots, N$. Let $\mathbf{u}_h \in \mathbf{V}_h$ and $p_h \in Q_h$ be the approximation solution of (2.5). We write $P^n = p_h(t_n)$ and $\mathbf{U}^n = \mathbf{u}_h(t_n)$ to be the approximation of $\mathbf{u}$, p at t_n. Define $\bar{\partial} P^n := (P^n - P^{n-1})/k$, $n = 1 \cdots, N$. The fully discretized finite element approximation with backward Euler method is to find $\mathbf{U}^n \in \mathbf{V}_h$, $P^n \in Q_h$, for $n = 1, \cdots, N$, such that

$$\begin{cases} e(\mathbf{U}^n, \mathbf{v}) = -(\nabla P^n, \mathbf{v})\,, \quad \forall \mathbf{v} \in \mathbf{V}_h\,, \\ (c_0 \bar{\partial} P^n, q) + \bar{\partial}(\nabla \cdot \mathbf{U}^n, q) + \Big(\kappa(\nabla \cdot \mathbf{U}^n) \nabla P^n, \nabla q\Big) = \Big(g(t_n), q\Big)\,, \quad \forall q \in Q_h\,. \end{cases} \tag{2.48}$$

Here we set $c_0 P^0 + \nabla \cdot U^0 = l_0$, where l_0 is the approximation to l in Q_h.

Remark 2.18. We can prove the existence of a weak solution of (2.48) following an argument similar to the one used for the continuous problem.

Next we consider the error estimates of the approximate solutions given by (2.48). We need the Ritz-Galerkin type projections of the steady state problem corresponding to (2.5). Given $r \in H_0^1(\Omega)$, define the projeciton $R_h r$ of r onto Q_h by

$$\Big(\kappa(\nabla \cdot \mathbf{u})\nabla(r - R_h r), \nabla q\Big) = \Big(\kappa(Bp)\nabla(r - R_h r), \nabla q\Big) = 0\,, \quad \forall q \in Q_h\,. \tag{2.49}$$

We first introduce the following Lemmas which can be found in [27] and [5].

Lemma 2.19 ([27], Lemma 3.4.2)**.** *Let B be a continuous linear operator on a Banach space X and let $f : [0,T] \to X$ be continuously differentiable with respect to t. Then*

$$B\frac{\partial f}{\partial t} = \frac{\partial}{\partial t} B f\,.$$

Lemma 2.20 ([5], p. 3)**.** *Let Q_h be given as above. Define the interpolation operator $I_h : H^2(\Omega) \cap H_0^1(\Omega) \to Q_h$ such that for any $q \in H^2(\Omega) \cap H_0^1(\Omega)$. Then we have the estimate*

$$\|q - I_h q\| + h\|\nabla(q - I_h q)\| \le C h^2 \|q\|_2\,.$$

Lemma 2.21 ([5], Lemma 13.1)**.** *Assume that $q \in H^2(\Omega) \cap H_0^1(\Omega)$. Then*

$$\|\nabla(q - R_h q)\| \le C_1 h \|q\|_2\,,$$

and

$$\|q - R_h q\| \le C_2 h^2 \|q\|_2\,,$$

where C_1 *and* C_2 *depend on* $\mathbf{u}$ *and* p.

To estimate the error between between p^n and P^n, we define

$$\rho^n := p^n - R_h p^n\,, \quad \theta^n := R_h p^n - P^n\,. \tag{2.50}$$

With these we can write the difference between p^n and P^n as

$$p^n - P^n = \rho^n + \theta^n\,. \tag{2.51}$$

Lemma 2.22. *Assume that* κ *is differentiable,* $p \in C^1(I; H^2(\Omega) \cap H_0^1(\Omega))$ *and* $\mathbf{u} \in C^1(I; W^{1,\infty}(\Omega))$. *Then*

$$\|\rho(t)\| + h\|\nabla\rho(t)\| \leq C(\mathbf{u}, p)h^2\,, \quad t \in (0, T]\,,$$

and

$$\|\rho_t(t)\| + h\|\nabla\rho_t(t)\| \leq C(\mathbf{u}, p, p_t)h^2\,, \quad t \in (0, T]\,.$$

Proof. The first estimate follows directly from Lemma 2.21. Differentiating (2.49) with respect to t and setting $r = p$ we obtain

$$\Big(\kappa(Bp)\nabla\rho_t, \nabla q\Big) + \Big(\big(\kappa(Bp)\big)_t \nabla\rho, \nabla q\Big) = 0\,, \quad \forall q \in Q_h\,. \tag{2.52}$$

Hypothesis (H1) guarantees that κ' is uniformly bounded. Thus

$$\big(\kappa(Bp)\big)_t = \big(\kappa(\nabla\cdot\mathbf{u})\big)_t = \kappa'(\nabla\cdot\mathbf{u})(\nabla\cdot\mathbf{u})_t$$

is also uniformly bounded due to the assumptions on the regularity of $\mathbf{u}$. It follows from (1.11), (2.49), (H1), and (2.52) that

$$
\begin{aligned}
k_* \|\nabla \rho_t\|^2 &\leq \Big(\kappa(Bp)\nabla\rho_t, \nabla\rho_t\Big) \\
&= \Big(\kappa(Bp)\nabla\rho_t, \nabla(p_t - q)\Big) + \Big(\kappa(Bp)\nabla\rho_t, \nabla(q - R_h p_t)\Big) \\
&= \Big(\kappa(Bp)\nabla\rho_t, \nabla(p_t - q)\Big) - \Big(\big(\kappa(Bp)\big)_t \nabla\rho, \nabla(q - R_h p_t)\Big) \\
&\leq C(\|\nabla\rho_t\|\|\nabla(p_t - q)\| + C\|\nabla\rho\|\|\nabla(q - R_h p_t)\|)\,.
\end{aligned}
$$

Taking $w = I_h p_t$, applying Lemma 2.20 and Young's inequality twice we obtain

$$
\begin{aligned}
k_* \|\nabla\rho_t\|^2 &\leq Ch\|p_t\|_2\|\nabla\rho_t\| + C\|\nabla\rho\|\Big(\|\nabla(I_h p_t - p_t)\| + \|\nabla(p_t - R_h p_t)\|\Big) \\
&\leq Ch\|p_t\|_2\|\nabla\rho_t\| + Ch\|\nabla\rho\|\|p_t\|_2 + \|\nabla\rho\|\|\nabla\rho_t\| \\
&\leq \varepsilon\|\nabla\rho_t\|^2 + h^2 C(\varepsilon)\|p_t\|_2^2 + Ch^2\|p_t\|_2^2 + \varepsilon\|\nabla\rho_t\|^2 + C(\varepsilon)\|\nabla\rho\|^2\,.
\end{aligned}
$$

Recall the first estimate that $\|\nabla\rho\| \leq Ch^2$. Choose $\varepsilon > 0$ such that $\varepsilon < k^*$. Then

$$
\|\nabla\rho_t\|^2 \leq C(h^2\|p_t\|_2^2 + \|\nabla\rho\|^2) \leq Ch^2\,.
$$

Next we estimate $\|\rho_t\|$ using a standard duality argument. For $\varphi \in L^2(\Omega)$, let $\psi \in H_0^1(\Omega)$ be the solution of

$$
-\nabla\cdot\Big(\kappa(Bp)\nabla\psi\Big) = -\kappa(Bp)\Delta\psi - \nabla\kappa(Bp)\cdot\nabla\psi = \varphi\,.
$$

The existence of φ is guaranteed by the Lax-Milgram Lemma. Furthermore there exists a constant C such that

$$
\|\psi\|_2 \leq C\|\varphi\|.
$$

Using integration by parts and (2.52) we have that for any $q \in Q_h$

$$
\begin{aligned}
(\rho_t, \varphi) &= \Big(\kappa(Bp)\nabla\rho_t, \nabla\psi\Big) \\
&= \Big(\kappa(Bp)\nabla\rho_t, \nabla(\psi - q)\Big) + \Big(\kappa(Bp)\nabla\rho_t, \nabla q\Big) \\
&= \Big(\kappa(Bp)\nabla\rho_t, \nabla(\psi - q)\Big) - \Big((\kappa(Bp))_t\nabla\rho, \nabla q\Big) \\
&= \Big(\kappa(Bp)\nabla\rho_t, \nabla(\psi - q)\Big) + \Big(\big(\kappa(Bp)\big)_t\nabla\rho, \nabla(\psi - q)\Big) - \Big(\nabla\rho, \big(\kappa(Bp)\big)_t\nabla\psi\Big).
\end{aligned}
$$

Integration by parts yields

$$
-\Big(\nabla\rho, \big(\kappa(Bp)\big)_t\nabla\psi\Big) = \Big(\rho, \big(\kappa(Bp)\big)_t\Delta\psi\Big) + \Big(\rho, \nabla\cdot\big((\kappa(Bp))_t\big)\nabla\psi\Big).
$$

Recall that both $\kappa(Bp)$ and $\big(\kappa(Bp)\big)_t$ are uniformly bounded. Choosing $w = I_h\psi$ and using the previous estimates, we obtain

$$
\begin{aligned}
|(\rho_t, \varphi)| &\le C\Big(\|\nabla\rho_t\|\|\nabla(\psi - I_h\psi\| + \|\nabla\rho\|\|\nabla(\psi - I_h\psi)\| + \|\rho\|\|\Delta\psi\|\Big) \\
&\le C\Big(\|\nabla\rho_t\|h\|\psi\|_2 + \|\nabla\rho\|h\|\psi\|_2 + \|\rho\|\|\Delta\psi\|\Big) \\
&\le C(p)h^2\|\psi\|_2 \le C(p)h^2\|\varphi\|.
\end{aligned}
$$

The statement above holds for any $\varphi \in L^2(\Omega)$. Therefore $\|\rho_t(t)\| \le Ch^2$. $\square$

Lemma 2.23. $\|\nabla R_h p(t)\|_{L^\infty(\Omega)} \le C(p)$ *for any* $p(t) \in H^2(\Omega) \cap H_0^1(\Omega)$*,* $\forall t \in [0, T]$*.*

Proof. For $q \in H^2(\Omega) \cap H_0^1(\Omega)$, we have the inverse estimate(since $\nabla R_h p(t)$ is constant on teach triangle)

$$
\|\nabla q\|_{L^\infty(\Omega)} \le Ch^{-1}\|\nabla q\|, \quad \text{for } \; q \in Q_h,
$$

which implies that

$$
\begin{aligned}
\|\nabla(R_h p - I_h p)\|_{L^\infty(\Omega)} &\le Ch^{-1}\|\nabla(R_h p - I_h p)\| \\
&\le Ch^{-1}\Big(\|\nabla(R_h p - p)\| + \|\nabla(p - I_h p)\|\Big) \\
&\le Ch^{-1}\Big(C(p)h + Ch\|p\|_2\Big) \le C(p)\,.
\end{aligned}
$$

Using the fact that $\|\nabla I_h p\|_{L^\infty(\Omega)} \le C\|\nabla p\|_{L^\infty(\Omega)}$ we obtain

$$
\|\nabla R_h p\|_{L^\infty(\Omega)} \le \|\nabla(R_h p - I_h p)\|_{L^\infty(\Omega)} + \|\nabla I_h p\|_{L^\infty(\Omega)} \le C(p)\,.
$$

□

Writing $\mathbf{u}^n = \mathbf{u}(t_n)$, $p^n = p(t_n)$ and $p_t^n = p_t(t_n)$, we are now ready to derive the error estimates for the finite element approximation (2.48).

Theorem 2.24. *Assume*

1. *The hypothesis of Theorem 2.16 holds;*
2. *κ is differentiable;*
3. *$p \in C^2(I; H^2(\Omega) \cap H_0^1(\Omega))$ and $\mathbf{u} \in C^1(I, W^{1,\infty}(\Omega))$.*

Then there exists $k_0 > 0$ such that for $k \le k_0$, there exists constants C_1 and C_2 depending on $\mathbf{u}$, p, p_t, p_{tt}, and l, such that

$$
\|\mathbf{u}(t_n) - \mathbf{U}^n\| + \|p^n - P^n\| \le C_1\|l^0 - l\| + C_2(h^2 + k)\,.
$$

Proof. We first estimate $\|p(t_n) - P^n\|$. In light of (2.51) and Lemma 2.22, it suffices to estimate θ^n defined in (2.50).

From (2.49)

$$
\begin{aligned}
&\Big(\kappa(Bp^n)\nabla p^n - \kappa(BP^n)\nabla P^n, \nabla q\Big) \\
&\qquad = \Big(\kappa(Bp^n)\nabla R_h p^n, \nabla q\Big) - \Big(\kappa(BP^n)\nabla Rhp^n, \nabla q\Big) \\
&\qquad + \Big(\kappa(BP^n)\nabla R_h p^n, \nabla q\Big) - \Big(\kappa(BP^n)\nabla P^n, \nabla q\Big), \quad \forall q \in Q_h.
\end{aligned}
$$

Substracting (2.48) from the continuous equation (2.9) and using the above equation we have that

$$
\begin{aligned}
0 =&\Big((c_0 + B)(p_t^n - \bar{\partial} P^n), q\Big) + \Big(\kappa(Bp^n)\nabla p^n - \kappa(BP^n)\nabla P^n, q\Big) \\
=&\Big((c_0 + B)(p_t^n - \bar{\partial} p^n), q\Big) + \Big((c_0 + B)\bar{\partial}\rho^n, q\Big) + \Big((c_0 + B)\bar{\partial}\theta^n, q\Big) \\
&+ \Big(\big(\kappa(Bp^n) - \kappa(BP^n)\big)\nabla R_h p^n, \nabla q\Big) + \Big(\kappa(BP^n)\nabla\theta^n, \nabla q\Big).
\end{aligned}
$$

Letting $q = \theta^n$ in the above equation we obtain

$$
\begin{aligned}
&\Big((c_0 + B)\bar{\partial}\theta^n, \theta^n\Big) + \Big(\kappa(BP^n)\nabla\theta^n, \nabla\theta^n\Big) \\
&\quad = \Big((c_0 + B)(p_t^n - \bar{\partial} p^n), \theta^n\Big) + \Big((c_0 + B)\bar{\partial}\rho^n, \theta^n\Big) \\
&\quad + \Big(\big(\kappa(Bp^n) - \kappa(BP^n)\big)\nabla R_h p^n, \nabla\theta^n\Big).
\end{aligned}
\tag{2.53}
$$

The fact that B is self-adjoint and monotone leads to

$$
\begin{aligned}
&\Big((c_0 + B)\bar{\partial}\theta^n, \theta^n\Big) \\
&= \frac{1}{2}\Big[\Big((c_0 + B)\bar{\partial}\theta^n, \theta^n - \theta^{n-1}\Big) + \Big((c_0 + B)\bar{\partial}\theta^n, \theta^n + \theta^{n-1}\Big)\Big] \\
&= \frac{k}{2}\Big((c_0 + B)\bar{\partial}\theta^n, \bar{\partial}\theta^n\Big) + \frac{1}{2}\Big[\Big((c_0 + B)\theta^n, \theta^n\Big) - \Big((c_0 + B)\theta^{n-1}, \theta^{n-1}\Big)\Big] \\
&\geq \frac{1}{2}\bar{\partial}\Big((c_0 + B)\theta^n, \theta^n\Big).
\end{aligned}
$$

It follows from (1.11), (2.53), (H1), Remark 2.8, Lemma 2.23, and Young's inequality that

$$
\begin{aligned}
&\frac{1}{2}\bar{\partial}\Big((c_0+B)\theta^n,\theta^n\Big)+k_*\|\nabla\theta^n\|^2\\
&\le C(\varepsilon)\|(c_0+B)(p_t^n-\bar{\partial}p^n)\|^2+\varepsilon\|\theta^n\|^2\\
&\quad+C(\varepsilon)\|(c_0+B)\bar{\partial}\rho^n\|^2+\varepsilon\|\theta^n\|^2\\
&\quad+C(\varepsilon)\|\kappa(BP^n)-\kappa(Bp^n)\|^2+\varepsilon\|\nabla\theta^n\|^2\\
&\le C(\varepsilon)\|p_t^n-\bar{\partial}p^n\|^2+C(\varepsilon)\|\bar{\partial}\rho^n\|^2\\
&\quad+C(\varepsilon)\|P^n-p^n\|^2+(2C_p^2+1)\varepsilon\|\nabla\theta\|^2\,,
\end{aligned}
\tag{2.54}
$$

where C_p is given by (1.22). Choose $\varepsilon>0$ such that $(2C_p+1)\varepsilon\le k^*$. Then

$$
\frac{1}{2}\bar{\partial}\Big((c_0+B)\theta^n,\theta^n\Big)\le C\Big(\|p_t^n-\bar{\partial}p^n\|^2+\|\bar{\partial}\rho^n\|^2+\|P^n-p^n\|^2\Big)\,.
$$

Multiplying the above inequality by k, we have that

$$
\begin{aligned}
&\Big((c_0+B)\theta^n,\theta^n\Big)-\Big((c_0+B)\theta^{n-1},\theta^{n-1}\Big)\\
&\le Ck\Big(\|p_t^n-\bar{\partial}p^n\|^2+\|\bar{\partial}\rho^n\|^2+\|P^n-p^n\|^2\Big)\\
&\le Ck\Big(\|p_t^n-\bar{\partial}p^n\|^2+\|\bar{\partial}\rho^n\|^2+\|\rho^n\|^2+\|\theta_n\|^2+(B\theta^n,\theta^n)\Big)\\
&\le Ck\Big(\|p_t^n-\bar{\partial}p^n\|^2+\|\bar{\partial}\rho^n\|^2+\|\rho^n\|^2\Big)+Ck\Big((c_0+B)\theta^n,\theta^n\Big)\,.
\end{aligned}
$$

Equivalently ([5], p. 238)

$$
(1-Ck)\Big((c_0+B)\theta^n,\theta^n\Big)\le\Big((c_0+B)\theta^{n-1},\theta^{n-1}\Big)+CkR_n\,,
$$

where the remainder R_n is given by

$$
R_n=\|p_t^n-\bar{\partial}p^n\|^2+\|\bar{\partial}\rho^n\|^2+\|\rho^n\|^2\,.
$$

For sufficiently small k, the above equation is equivalent to

$$\Big((c_0+B)\theta^n,\theta^n\Big) \le (1+Ck)\Big((c_0+B)\theta^{n-1},\theta^{n-1}\Big) + CkR_n.$$

Inductively for $n \ge 2$ we have that

$$\begin{aligned}\Big((c_0+B)\theta^n,\theta^n\Big) &\le (1+Ck)^{n-1}\Big((c_0+B)\theta^1,\theta^1\Big) + Ck\sum_{j=2}^{n}(1+Ck)^{n-j}R_j \\ &\le C\Big((c_0+B)\theta^1,\theta^1\Big) + Ck\sum_{j=2}^{n}R_j\,.\end{aligned} \tag{2.55}$$

It follows from Lemma 2.22 that

$$\|\rho^j\| \le Ch^2 \tag{2.56}$$

and

$$\|\bar{\partial}\rho^j\| = \frac{1}{k}\|\int_{t_{j-1}}^{t_j}\rho_t(s)\;ds\| \le Ch^2\,. \tag{2.57}$$

From (H3)

$$\begin{aligned}\|p_t^j - \bar{\partial}p^j\| &= \frac{1}{k}\|kp_t^j - p(t_j) + p(t_{j-1})\| \\ &= \frac{1}{k}\|\int_{t_{j-1}}^{t_j}(s-t_{j-1})p_{tt}\;ds\| \\ &\le k\int_0^{t_n}\|p_{tt}\|\;ds \le Ck\,.\end{aligned} \tag{2.58}$$

Combining (2.56), (2.57), and (2.58), we obatin

$$R_j \le C(h^2+k)^2\,,\quad 2\le j\le n. \tag{2.59}$$

For $n=1$, denote $\rho(l) = l - R_h l$, $(c_0+B)\theta^0 = \theta(l) = R_h l - l_0$ Then (2.54) yields

$$\begin{aligned}
&\frac{1}{k}\Big((c_0{+}B)\theta^1 - \theta(l), \theta^1\Big) + C_1\|\nabla\theta^1\|^2 \\
&\qquad \le C_2\Big(\|(c_0+B)(p_t^1 - \bar{\partial}p^1)\|^2 + \|(c_0+B)\bar{\partial}\rho^1\|^2 + \|\kappa(BP^1) - \kappa(Bp^1)\|^2\Big)\,.
\end{aligned}$$

Applying the Young's inequality we have

$$\begin{aligned}
&\Big((c_0{+}B)\theta^1, \theta^1\Big) + C_1\|\nabla\theta^1\|^2 \\
&\quad \le C_2 k\Big(\|(c_0+B)(p_t^1 - \bar{\partial}p^1)\|^2 + \|(c_0+B)\bar{\partial}\rho^1\|^2 \\
&\qquad + \|P^1 - p^1\|^2\Big) + (\theta(l), \theta^1)\,. \\
&\quad \le C_2\Big(\|(c_0+B)(p_t^1 - \bar{\partial}p^1)\|^2 + \|(c_0+B)\bar{\partial}\rho^1\|^2 \\
&\qquad + \|\theta^1\|^2 + \|\rho^1\|^2\Big) + C(\varepsilon)\|\theta(l)\|^2 + \varepsilon\|\theta^1\|^2\,.
\end{aligned}$$

Choose sufficiently small $\varepsilon > 0$ such that $\varepsilon < C_1$. Then the above inequality leads to

$$\begin{aligned}
&(1 - Ck)\Big((c_0+B)\theta^1, \theta^1\Big) \\
&\qquad \le\ C\Big(\|\theta(l)\|^2 + \|(c_0+B)(p_t^1 - \bar{\partial}p^1)\|^2 + \|(c_0+B)\bar{\partial}\rho^1\|^2 + \|\rho^1\|^2\Big)
\end{aligned} \tag{2.60}$$

Proceeding as in (2.56), (2.57), (2.58), using Lemma 2.22, we have

$$\|\rho^1\| \le Ch^2\,.$$

As a consequence of the boundedness of B and Lemma 2.22

$$\|\bar{\partial}(c_0+B)\rho^1\| = \frac{1}{k}\|\int_0^{t_1}(c_0+B)\rho_t(s)\,ds\| \le Ch^2$$

and

$$
\begin{aligned}
&\|(c_0{+}B)(p_t^1 - \bar{\partial} p^1)\| \\
&\quad = \frac{1}{k}\|k(c_0+B)p_t^1 - (c_0+B)p(k) + (c_0+B)p(0)\| \\
&\quad = \frac{1}{k}\| \int_0^{t_1} s(c_0+B)p_{tt}\, ds\| \\
&\quad \le k \int_0^{t_1} \|(c_0+B)p_{tt}\|\, ds = Ck\,.
\end{aligned}
$$

Thus

$$((c_0+B)\theta^1, \theta^1) \le C\|\theta(l)\|^2 + C(h^2+k)\,. \tag{2.61}$$

It follows from Lemma 2.22 that

$$\|\theta(l)\| \le \|l_0 - l\| + \|\rho(l)\| \le \|l - l_0\| + Ch^2\,.$$

Hence (2.59) and (2.61) yield

$$\|\theta^n\| \le C\|l - l_0\| + C(h^2+k)\,.$$

Hence

$$\|p^n - P^n\| \le C\|l - l_0\| + C(h^2+k)\,.$$

To obtain the estimate for $\mathbf{u}(t_n) - \mathbf{U}^n$, we define the projection $\mathbf{R}_h : \mathbf{H}_0^1(\Omega) \to \mathbf{V}_h$ by

$$e(\mathbf{u} - \mathbf{R}_h\mathbf{u}, \mathbf{v}) = 0\,, \quad \forall \mathbf{v} \in \mathbf{V}_h\,, \quad \mathbf{u} \in \mathbf{H}_0^1(\Omega). \tag{2.62}$$

We split the error $\mathbf{u}^n - \mathbf{U}^n$ as

$$\mathbf{u}^n - \mathbf{U}^n = \mathbf{u}^n - \mathbf{R}_h\mathbf{u}^n + \mathbf{R}_h\mathbf{u}^n - \mathbf{U}^n = \rho_{\mathbf{u}}^n + \theta_{\mathbf{u}}^n\,,$$

where $\rho_{\mathbf{u}}^n = \mathbf{u}^n - \mathbf{R}_h \mathbf{u}^n$ and $\theta_{\mathbf{u}}^n = \mathbf{R}_h \mathbf{u}^n - \mathbf{U}^n$. Similarly to Lemma 2.22, it is easy to verify that the projection error $\rho_{\mathbf{u}}$ satisfies

$$\|\rho_{\mathbf{u}}\| \le C(\mathbf{u}) h^2 .$$

Substracting the first equation in (2.48) from (2.5), using (2.62), we have that

$$e(\theta_{\mathbf{u}}^n, \mathbf{v}) = a\Big(\big(\mathbf{U}^n - \mathbf{u}^n\big), \mathbf{v}\Big) = -\Big(\nabla(P^n - p^n), \mathbf{v}\Big)\,, \quad \forall \mathbf{v} \in \mathbf{V}_h\,.$$

With $\mathbf{v} = \theta_{\mathbf{u}}(t_n)$ in the above equation, Young's inequality and the inverse estimate of elliptic equation yield

$$\|\theta_{\mathbf{u}}^n\|_1 \le \|p^n - P^n\| \le C\|l - l_0\| + C(h^2 + k)$$

from which the assertion in Theorem follows. □

2.4 Numerical example

We first test the convergence rate in two dimension for brevity. Let $\Omega = [-1,1] \times [-1,1]$, $I = [0,1]$. We choose Kozeny-Carmen-type ([59], [60], [61]) hydraulic conductivity κ, which is defined by

$$\kappa(s) = \begin{cases} \frac{k_0}{\eta} \frac{\phi^3(s)}{(1-\phi(s))^2}\,, & \frac{\phi_0}{\phi_0 - 1} < s_* < s < s^* < 1\,, \\ k_*\,, & s \le s_*, \\ k^*\,, & s^* \le s\,. \end{cases} \tag{2.63}$$

Here

$$\phi(s) = \phi_0 + (1 - \phi_0)s$$

and

$$k_* = \frac{k_0}{\mu} \frac{(\phi_0 + (1-\phi_0)s_*)^3}{(1-s_*)^2(1-\phi_0)^2}\,, \qquad k^* = \frac{k_0}{\mu} \frac{(\phi_0 + (1-\phi_0)s^*)^3}{(1-s^*)^2(1-\phi_0)^2}\,,$$

where k_0, η, ϕ_0, s_*, and s^* are given constants.

Table 2.1: The converge rate for a fixed time step in the $L^2(I;L^2(\Omega))$ norm

h	$\|p-p_h\|$	$\|u-u_h\|$	conv. rate p	conv. rate u
1/4	0.1023	0.0295		
1/8	0.0554	0.0072	0.88	2.03
1/16	0.0145	0.0018	1.93	2.00
1/32	0.0037	4.61e-4	1.97	1.97
1/64	9.20e-4	1.16e-4	2.01	1.99
1/128	2.30e-4	2.89e-5	2.00	2.00

It is easy to see that the continuous function κ given by (2.63) satisfies (1.11) and (H1). We set $\lambda = \mu = c_0 = \alpha = 1$ in system (1.10) and $\kappa_0 = \eta = 1$, $\phi_0 = 0.5$, $s_* = -0.75$, $s^* = 0.75$ in our numerical experiment. Also we choose the forcing term $\mathbf{f}$, g in (1.10) and the initial value l in (2.5) such that $(\mathbf{u}, p)$ defined by

$$\mathbf{u} = e^{-t}(\sin(\pi x)\sin(\pi y), \sin(\pi x)\sin(\pi y))^T$$

and

$$p = t\sin(\pi x)\sin(\pi y)$$

is the exact solution of (1.10). Notice that the solution $(\mathbf{u}, p)$ defined above is analytic. Therefore (H2) is satisfied.

For each time step, we solve the nonlinear problem using the Picard iteration. We set the tolerance of this iteration to be 10^{-10} .

To verify the convergence rate in spatial dimension, we let the time step k be a fixed value $k = 10^{-4}$. Table 2.1 and Table 2.2 list the errors and rates of convergence in spatial dimension. Figure 2.1 plots the errors in both $L^2(I;L^2(\Omega))$ norm and $L^2(I;H^1(\Omega))$ norm. Both the tables and the figure verify the second order convergence rate for the finite element approximation. Moreover, the H^1 norms of $\mathbf{u}$ and p both have first order convergence, which is natural for linear finite elements with pyramid basis.

Table 2.2: The converge rate for a fixed time step in the $L^2(I; H_0^1(\Omega))$ norm

h	$\|p - p_h\|_1$	$\|u - u_h\|_1$	conv. rate p	conv. rate u
1/4	0.7685	0.3061		
1/8	0.5556	0.1646	0.47	0.90
1/16	0.2767	0.0838	1.01	0.97
1/32	0.1385	0.0421	1.00	0.99
1/64	0.0693	0.0211	1.00	1.00
1/128	0.0346	0.0105	1.00	1.00

Table 2.3: The converge rate for time step $k = h^2$ in the $L^2(I; L^2(\Omega))$ norm

h	k	$\|p - p_h\|$	$\|u - u_h\|$	conv. rate p	conv. rate u
1/4	1/16	0.0980	0.0290		
1/8	1/64	0.0519	0.0070	0.92	2.05
1/16	1/256	0.0135	0.0018	1.94	1.96
1/32	1/1024	0.0034	4.52e-4	1.99	1.99

Table 2.4: The converge rate for time step $k = h^2$ in the $L^2(I; H^1(\Omega))$ norm

h	k	$\|p - p_h\|_1$	$\|u - u_h\|_1$	conv. rate p	conv. rate u
1/4	1/16	0.7530	0.2041		
1/8	1/64	0.5347	0.1629	0.49	0.33
1/16	1/256	0.2662	0.0829	1.07	0.97
1/32	1/1024	0.1332	0.0416	1.00	0.99

Figure 2.1: The plot of errors for a fixed time step

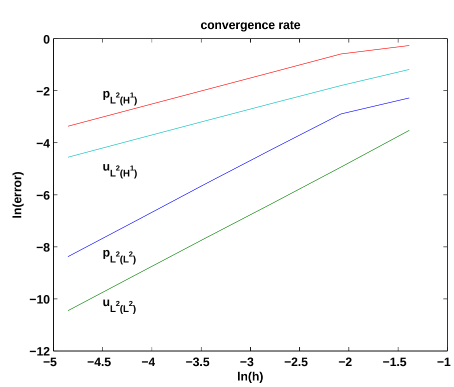
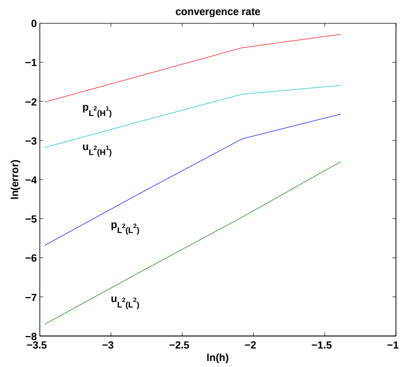


Figure 2.2: The plot of errors for time step $k = h^2$

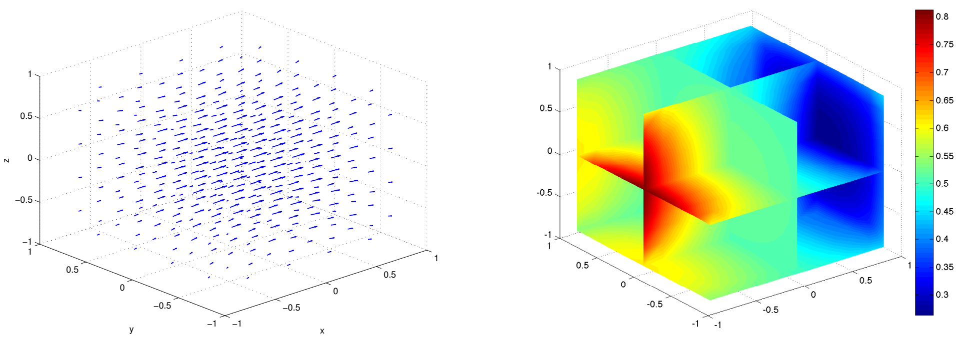


To see the convergence rate with respect to the time step k, we set $k = h^2$. The errors are presented in table 2.3, table 2.4 and figure 2.2, from which we know that when k is small, the convergence rate with respect to k is the same as h^2.

We next show some simulation of a real life example. Our intention is to show the effect of a nonlinear hydraulic conductivity κ thus we choose a model in three dimensions and we only consider the numerical approximation of the steady system to save computations. Suppose we have a $2 \times 2 \times 2$ inches cube sponge on $\Omega = [-1, 1] \times [-1, 1] \times [-1, 1]$. We assume that there is no external force of fluid resources, i.e., $\mathbf{f} = 0$ and $g = 0$. We assume zero boundary conditions of both $\mathbf{u}$ and p except that $p = 1$ on $x = -1$ (the left surface) and $p = 0$ on $x = 1$ (the right surface). In this case, there is a pressure drop from the left surface to the right surface. If κ is a constant, we expect a linear pressure with constant pressure drop. As a result, we have uniform flux pointing to the right, parallel to the x axis.

Case 1. We want to show the improvements of simulation when we introduce the nonlinear hydraulic conductivity κ in (2.63). The flux and hydraulic conductivity are then given in figure 2.3. From the graph we can see that the flux is not uniform due to the nonlinear κ and the value of κ varies from 0.3 to 0.8 at different part of the elastic body. The highest κ occurs close to the left surface since the pore structure has been pushed to the right by the high pressure. As a result, the size of the pores become bigger which leads

Figure 2.3: Case 1



(a) Flux (b) Hydraulic conductivity

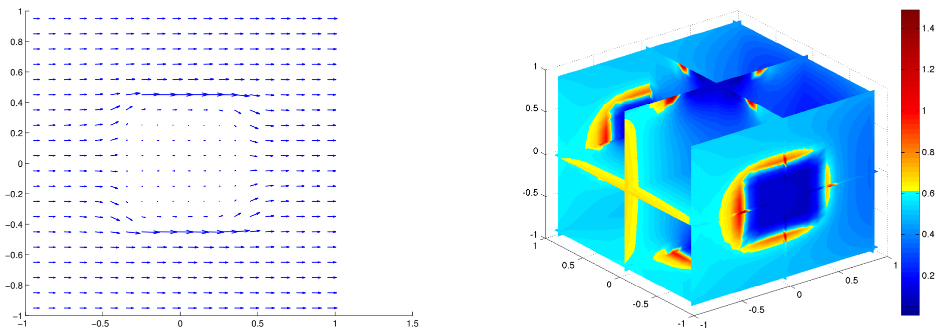
to a higher κ. The lower κ near the right surface where most of solid matrix stays is due to the small size of the pores.

Case 2. We apply a displacement of compression in the middle of the front and back surface, that is

$$\mathbf{u} = \begin{cases} (0, 0.1, 0)\,, & y = -1\,, -0.25 \leq x, z \leq 0.25\,, \\ (0, -0.1, 0)\,, & y = 1\,, -0.25 \leq x, z \leq 0.25\,, \\ 0\,, & \text{otherwise}\,, \end{cases}$$

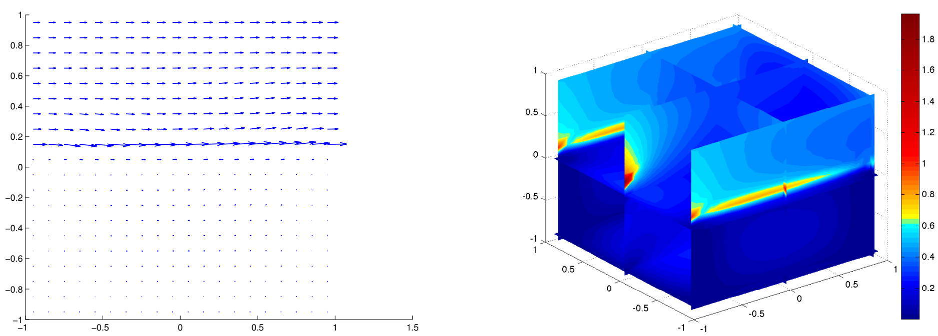
and we assume the same boundary condition of p as in case 1. The flux and κ are shown in figure 2.4. We observe two facts. First, we have a very low hydraulic conductivity in the middle and the flux are avoiding where we pressed the body. Only very small flux can pass through the middle and this effect becomes weaker as the flux are away from the front surface into the heart of the object. In real life case, this is because the pores close to the front and back surface become small when we press the sponge. Another fact is that the highest flow occurs right around where we pressed the sponge. This is because of the elastic nature of the sponge that the water originally kept in the pores near the middle of the front and back surface was forced to come out due to pressing.

Figure 2.4: Case 2



(a) Flux on $y = 1$ (b) Hydraulic conductivity

Figure 2.5: Case 3



(a) Flux on $y = 1$ (b) Hydraulic conductivity

Case 3 We observe another case similar to case 2. We squeeze the lower half of the sponge, that is

$$\mathbf{u} = \begin{cases} (0, 0.1, 0)\,, & y = -1\,, -1 \le z \le 0\,, \\ (0, -0.1, 0)\,, & y = 1\,, -1 \le z \le 0\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

The result is shown in figure 2.5. Again, we observe a low hydraulic conductivity in the lower half of the sponge and a high hydraulic conductivity around where we squeeze it.

2.5 Conclusion

We studied the mathematical model of flows through elastic porous media. The PDE system under consideration is comprised of the linear elasticity equation for the deformation of the matrix and an implicit parabolic equation for the fluid pressure with hydraulic conductivity dependent on the displacement. We showed the existence of a weak solution under minimum regularity assumption on the input data and its uniqueness under stronger conditions. We then focused our attention on the finite element approximations of the weak solution and derived rigorous error estimates. We used our model to simulate the distribution of flux and hydraulic conductivity inside a sponge. In the simulation we showed that the nonlinear hydraulic conductivity κ defined in (2.63) captures the distribution of the hydraulic conductivity at different part of the elastic body, which cannot be obtained by the linear system with a constant hydraulic conductivity assumption. The difference is significant and has a huge effect on the flow through the porous media. We established a better approximation to simulate the flow through a porous media, which matches the common sense.

Chapter 3

Steady bioconvection

3.1 Existence and uniqueness of a weak solution

The weak formulation. we consider a steady case of system (1.19). Choose the pressure space $L^2_0(\Omega) = L^2(\Omega)/\mathbb{R}$ to be the quotient subspace of $L^2(\Omega)$, the subspace of functions which are orthogonal to constants. It is easy to see that ([62], p. 23)

$$L^2_0(\Omega) = \{p \in L^2(\Omega);\ \int_\Omega p\ dx = 0\}\,.$$

Define the following bilinear and trilinear forms

$$\begin{cases} a(c,r) = (\nabla c, \nabla r) \quad \forall c, r \in H^1(\Omega)\,, \\ B_2(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \displaystyle\int_\Omega \mathbf{u} \cdot \nabla \mathbf{v}\ \mathbf{w}\ dx \quad \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{H}^1_0(\Omega)\,, \\ B_3(\mathbf{u}, c, r) = \displaystyle\int_\Omega \mathbf{u} \cdot \nabla c\ r dx, \quad \forall \mathbf{u} \in \mathbf{H}^1_0(\Omega), \quad c, r \in H^1(\Omega)\,, \\ b(q, \mathbf{v}) = -(q, \nabla \cdot \mathbf{v}) \quad \forall q \in L^2_0(\Omega)\,, \quad \mathbf{v} \in \mathbf{H}^1_0(\Omega) \end{cases} \tag{3.1}$$

and set

$$\tilde{H} = H^1(\Omega) \cap L^2_0(\Omega) = \{c \in H^1(\Omega)\ :\ \int_\Omega c\ dx = 0\}\,.$$

Condition (1.18) is equivalent to requiring $c - \frac{\alpha}{|\Omega|} \in \tilde{H}$. For brevity, we write $\alpha = \frac{\alpha}{|\Omega|}$ and we assume $\rho_m = 1$, otherwise we rescale the problem. Also for brevity, define an auxiliary concentration $c_\alpha = c - \alpha$ and $\mathbf{f}_\alpha = \mathbf{f} - g\rho_m\gamma\alpha i_3$. Multiplying the equations of (1.19) by test functions and integrating by parts we obtain the following weak formulation for system (1.19) (without confussion, we write $c = c_\alpha$ and $\mathbf{f} = \mathbf{f}_\alpha$).

Definition 3.1. Given $\mathbf{f}$ in $\mathbf{L}^2(\Omega)$, a triple $(\mathbf{u}, p, c) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times \tilde{H}$ is said to be a weak solution of the steady bioconvection if it satisfies

$$
\begin{cases}
\Big(\nu(c+\alpha)\nabla\mathbf{u}, \nabla\mathbf{v}\Big) + B_2(\mathbf{u}, \mathbf{u}, \mathbf{v}) + b(p, \mathbf{v}) \\
\qquad = -\Big(g(1+\gamma c)i_3, \mathbf{v}\Big) + (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega)\,, \\
b(q, \mathbf{u}) = 0 \quad \forall q \in L_0^2(\Omega)\,, \\
\theta a(c, r) + B_3(\mathbf{u}, c, r) - U(c, \dfrac{\partial r}{\partial x_3}) = U\alpha(\dfrac{\partial r}{\partial x_3}, 1) \quad \forall r \in \tilde{H}\,.
\end{cases}
\tag{3.2}
$$

We adopt the theory for solving Navier-Stokes type equations (see Chapter 4 of [62] and Chapter 2-3 of [52]). Define

$$
\mathbf{V} = \{\mathbf{u} \in \mathbf{H}_0^1(\Omega) \ : \ \nabla \cdot \mathbf{u} = 0 \text{ in } \Omega\}\,.
$$

Then to solve system (3.2), it suffices to solve the following auxiliary problem ([62]). Find a pair $(\mathbf{u}, c) \in \mathbf{V} \times \tilde{H}$ such that

$$
\begin{cases}
\Big(\nu(c+\alpha)\nabla\mathbf{u}, \nabla\mathbf{v}\Big) + B_2(\mathbf{u}, \mathbf{u}, \mathbf{v}) = -\Big(g(1+\gamma c)i_3 + \mathbf{f}, \mathbf{v}\Big) \quad \forall \mathbf{v} \in \mathbf{V}\,, \\
\theta a(c, r) + B_3(\mathbf{u}, c, r) - U(c, \dfrac{\partial r}{\partial x_3}) = U\alpha(\dfrac{\partial r}{\partial x_3}, 1) \quad \forall r \in \tilde{H}\,.
\end{cases}
\tag{3.3}
$$

Remark 3.2. Obviously, if $(\mathbf{u}, c, p)$ is a solution of system (3.2), then $(\mathbf{u}, c)$ must be a solution of (3.3). The converse is also true since the bilinear form $b(\cdot, \cdot)$ defined above satiesfies the inf-sup condition (see [62], p. 113), i.e., for some $\beta > 0$

$$
\sup_{\mathbf{v} \in \mathbf{H}_0^1(\Omega)} \frac{(q, \nabla \cdot \mathbf{v})}{\|\mathbf{v}\|_1} \geq \beta \|q\| \quad \forall q \in L_0^2(\Omega)\,.
$$

Existence. To prove the existence of a weak solution of (3.3), we construct a sequence of approximate weak solutions using the Galerkin method. This will also be helpful in our later discussion and analysis of the finite element method. The following inequalities can be found in [32]:

$$\begin{cases} \|\mathbf{v}\|_1 \leq C_\Omega \|\nabla \mathbf{v}\| & \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega)\,, \\ \|r\|_1 \leq C_\Omega \|\nabla r\| & \forall r \in \tilde{H}\,, \end{cases} \tag{3.4}$$

for some constant C_Ω independent of $\mathbf{v}$ and r. The first inequality is Poincaré's inequality while the second is due to the fact that $\int_\Omega r dx = 0$. We can obviously use the same constant C_Ω in both inequalities. From Lemma 2.2, it is straightforward that $\nu(x)$ satisfying (1.21) is a Nemytskii operator.

Next we study the properties of the trilinear forms B_2 and B_3.

Lemma 3.3. *The trilinear form $B_2(\cdot,\cdot,\cdot)$ and $B_3(\cdot,\cdot,\cdot)$ are continuous on $\mathbf{H}^1(\Omega)$.*

Proof. From Holder's inequality and the Sobolev imbedding theorem in three dimensions we have

$$B_2(\mathbf{u},\mathbf{v},\mathbf{w}) \leq C\|\mathbf{u}\|_{\mathbf{L}^4(\Omega)}\|\nabla \mathbf{v}\|_{\mathbf{L}^2(\Omega)}\|\mathbf{w}\|_{\mathbf{L}^4(\Omega)} \leq C_{B_2}\|\mathbf{u}\|_1\|\mathbf{v}\|_1\|\mathbf{w}\|_1\,. \tag{3.5}$$

Similarly

$$B(\mathbf{u},c,r) \leq C_{B_3}\|\mathbf{u}\|_1\|c\|_1\|r\|_1\,. \tag{3.6}$$

□

Lemma 3.4. *Assume that $\mathbf{v},\mathbf{w} \in \mathbf{H}_0^1(\Omega)$, $c, r \in \tilde{H}$, and $\mathbf{u} \in \mathbf{V}$, then*

$$\begin{cases} B_2(\mathbf{u},\mathbf{v},\mathbf{w}) + B_2(\mathbf{u},\mathbf{w},\mathbf{v}) = 0\,, \\ B_3(\mathbf{u},c,r) + B_3(\mathbf{u},r,c) = 0\,, \end{cases} \tag{3.7}$$

and

$$B_2(\mathbf{u},\mathbf{v},\mathbf{v}) = 0\,, \quad B_3(\mathbf{u},r,r) = 0\,. \tag{3.8}$$

Proof. Notice that properties (3.7) and (3.8) are equivalent. Thus it suffices to prove (3.8). According to Green's formula, if $\mathbf{u} \in \mathbf{V}$

$$B_2(\mathbf{u},\mathbf{v},\mathbf{v}) = \frac{1}{2}\sum_{i,j=1}^{3}\int_\Omega \mathbf{u}_j \frac{\partial(\mathbf{v}_i^2)}{\partial x_j} = -\frac{1}{2}\sum_{i=1}^{3}\int_\Omega \nabla\cdot\mathbf{u}\ \mathbf{v}_i^2\ dx = 0$$

and

$$B_3(\mathbf{u},r,r) = \frac{1}{2}\sum_{j=1}^{3}\int_\Omega \mathbf{u}_j \frac{\partial(r^2)}{\partial x_j} = -\frac{1}{2}\int_\Omega \nabla\cdot\mathbf{u}\ r^2\ dx = 0\,.$$

□

Since $\mathbf{V}$ and $\tilde{H}$ are both separable Hilbert spaces, there exist sequences $\{\mathbf{v}_j\}_{j=1}^\infty$ and $\{r_j\}_{j=1}^\infty$ such that $\{\mathbf{v}_j\}_{j=1}^\infty$ and $\{r_j\}_{j=1}^\infty$ are orthonormal bases of $\mathbf{V}$ and $\tilde{H}$, respectively. Let $\mathbf{V}_m$ and H_m be the spaces spanned by $\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_m\}$ and $\{r_1,r_2,\ldots,r_m\}$ respectively. We seek $(\mathbf{u}^m,c^m) \in \mathbf{V}_m \times H_m$ such that

$$\begin{cases} \Big(\nu(c^m+\alpha)\nabla\mathbf{u}^m,\nabla\mathbf{v}\Big) + B_2(\mathbf{u}^m,\mathbf{u}^m,\mathbf{v}) = -\Big((g+\gamma c^m)i_3,\mathbf{v}\Big) + (\mathbf{f},\mathbf{v}) \quad \forall\mathbf{v}\in\mathbf{V}_m\,, \\ \theta a(c^m,r) + B_3(\mathbf{u}^m,c^m,r) - U(c^m,\dfrac{\partial r}{\partial x_3}) = U\alpha(\dfrac{\partial r}{\partial x_3},1) \quad \forall r\in H_m\,. \end{cases} \tag{3.9}$$

Proceeding as in Lemma 2.10, using properties (3.5) and (3.6), we can prove the existence of a weak solution of (3.9) for any integar $m > 0$ either by a direct corollary of Brouwer fixed point theorem or using Riesz' theorem.

Next we show that $\{\mathbf{u}^m\}_{m=1}^\infty$ and $\{c^m\}_{m=1}^\infty$ are uniformly bounded in $\mathbf{V}$ and $\tilde{H}$ respectively.

Lemma 3.5. *Assume that*

$$\frac{\theta}{C_\Omega^2} > U\,. \tag{3.10}$$

Then there exists a constant C independent of m such that

$$\|c^m\|_1 + \|\mathbf{u}^m\|_1 < C.$$

Proof. Let $\mathbf{v} = \mathbf{u}^m$, $r = c^m$ in (3.9). Equation (3.8) implies that

$$\begin{cases} \Big(\nu(c^m + \alpha)\nabla\mathbf{u}^m, \nabla\mathbf{u}^m\Big) = -\Big((g + \gamma c^m)i_3, \mathbf{u}^m\Big) + (\mathbf{f}, \mathbf{u}^m)\,, \\ \theta a(c^m, c^m) - U(c^m, \dfrac{\partial c^m}{\partial x_3}) = U\alpha(\dfrac{\partial c^m}{\partial x_3}, 1)\,. \end{cases}$$

Thus it follows from (1.21) and Young's inequality that

$$\begin{aligned} \nu_*\|\nabla\mathbf{u}^m\|^2 &\le \Big(\nu(c^m + \alpha)\nabla\mathbf{u}^m, \nabla\mathbf{u}^m\Big) \\ &\le |-\Big((g + \gamma c^m)i_3, \mathbf{u}^m\Big) + (\mathbf{f}, \mathbf{u}^m)| \\ &\le \Big(\gamma\|c^m\| + \|\mathbf{f} - gi_3\|\Big)\|\mathbf{u}^m\| \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} \theta\|\nabla c^m\|^2 &\le \theta a(c^m, c^m) \\ &\le \Big|U(c^m, \frac{\partial c^m}{\partial x_3}) + U\alpha(\frac{\partial c^m}{\partial x_3}, 1)\Big| \\ &\le U\|c^m\|_1^2 + U\alpha|\Omega|^{\frac{1}{2}}\|c^m\|_1\,. \end{aligned} \tag{3.12}$$

Using the above inequality, (3.4), and assumption (3.10) we obtain

$$\|c^m\|_1 \le \left(\frac{\theta}{C_\Omega^2} - U\right)^{-1} U\alpha|\Omega|^{\frac{1}{2}}\,. \tag{3.13}$$

Substituting (3.13) into (3.11) gives

$$\|\mathbf{u}_m\|_1 \le \frac{C_\Omega^2}{\nu_*}\Big((\frac{\theta}{C_\Omega^2} - U)^{-1}\gamma U\alpha|\Omega|^{\frac{1}{2}}) + \|\mathbf{f} - gi_3\|\Big)\,.$$

□

We are now ready to show the existence of a solution of (3.3).

Theorem 3.6. *Assume that* (1.21) *and* (3.10) *hold, and* $\mathbf{f} \in \mathbf{L}^2(\Omega)$, *then system* (3.3) *has a weak solution.*

Proof. Consider sequences of solutions $\{\mathbf{u}^m\}_{m=1}^{\infty}$ and $\{c^m\}_{m=1}^{\infty}$ defined by (3.9). According to Lemma 3.5, both sequences are bounded therefore there exist $\mathbf{u} \in \mathbf{V}$ and $c \in \tilde{H}$ (via subsequences if necessary) such that

$$\mathbf{u}^m \rightharpoonup \mathbf{u} \quad \text{in} \quad \mathbf{V} \quad \text{and} \quad c^m \rightharpoonup c \quad \text{in} \quad \tilde{H} \quad as \quad m \to \infty \,. \tag{3.14}$$

Then the Sobolev embedding theorem implies that

$$\mathbf{u}^m \to \mathbf{u} \quad \text{in} \quad \mathbf{L}^2(\Omega) \quad \text{and} \quad c^m \to c \quad \text{in} \quad L^2(\Omega) \quad as\ m \to \infty \,. \tag{3.15}$$

We now show that the weak limit $(\mathbf{u}, c)$ is a solution of (3.3). Let $\mathbf{v}$ and r be test functions such that

$$\mathbf{v} \in \mathbf{V} \cap (C_0^\infty(\Omega))^3 \,, \quad r \in C^\infty(\Omega) \cap \tilde{H} \,. \tag{3.16}$$

First notice that

$$\begin{aligned}
&\Big(\nu(c+\alpha)\nabla \mathbf{u}, \nabla \mathbf{v}\Big) - \Big(\nu(c^m+\alpha)\nabla \mathbf{u}^m, \nabla \mathbf{v}\Big) \\
&\quad = \Big(\nu(c+\alpha)\nabla(\mathbf{u}-\mathbf{u}^m), \nabla \mathbf{v}\Big) + \Big(\big(\nu(c+\alpha) - \nu(c^m+\alpha)\big)\nabla \mathbf{u}^m, \nabla \mathbf{v}\Big) \\
&\quad := \mathrm{I} + \mathrm{II} \,.
\end{aligned}$$

From (1.21) and (3.14) it follows that

$$|\mathrm{I}| \le \nu^* |(\nabla(\mathbf{u}^m - \mathbf{u}), \nabla \mathbf{v})| \to 0 \quad \text{as } m \to \infty \,.$$

The limits (3.15) and the fact that ν is a Nemytskii operator imply that

$$\nu(c^m+\alpha) \to \nu(c+\alpha) \quad \text{in} \quad L^2(\Omega) \quad \text{as } m \to \infty \,. \tag{3.17}$$

Hence from (3.16), Lemma 3.5, and Holder's inequality

$$|\mathrm{II}| \le C\|\nu(c^m+\alpha)-\nu(c+\alpha)\|\|\mathbf{u}^m\|_1 \to 0 \quad as\ m\to\infty\,.$$

Combing the above estimates we obtain

$$\Big(\nu(c^m+\alpha)\nabla\mathbf{u}^m,\nabla\mathbf{v}\Big)\to\Big(\nu(c+\alpha)\nabla\mathbf{u},\nabla\mathbf{v}\Big) \quad \text{as} \quad m\to\infty\,. \tag{3.18}$$

Next using Green's formula

$$B_2(\mathbf{u}^m,\mathbf{u}^m,\mathbf{v}) = \sum_{i,j=1}^{3}\int_\Omega \mathbf{u}_j^m(\frac{\partial \mathbf{u}_i^m}{\partial x_j}\mathbf{v}_i)\ dx = -\sum_{i,j=1}^{3}\int_\Omega \mathbf{u}_i^m\mathbf{u}_j^m(\frac{\partial \mathbf{v}_i}{\partial x_j})\ dx\,.$$

By assumption (3.16), $\frac{\partial \mathbf{v}}{\partial x_j}$ is uniformly bounded. The fact $\mathbf{u}^m\to\mathbf{u}$ in $\mathbf{L}^2(\Omega)$ implies that $\mathbf{u}_i^m\mathbf{u}_j^m\to\mathbf{u}_i\mathbf{u}_j$ in $\mathbf{L}^1(\Omega)$ as $m\to\infty$. Therefore

$$\lim_{m\to\infty} B_2(\mathbf{u}^m,\mathbf{u}^m,\mathbf{v}) = -\sum_{i,j=1}^{3}\int_\Omega \mathbf{u}_i\mathbf{u}_j(\frac{\partial \mathbf{v}_i}{\partial x_j})\ dx = -B_2(\mathbf{u},\mathbf{v},\mathbf{u}) = B_2(\mathbf{u},\mathbf{u},\mathbf{v})\,.$$

Following the same argument, we have

$$B_2(\mathbf{u}^m,\mathbf{u}^m,\mathbf{v})\to B_2(\mathbf{u},\mathbf{u},\mathbf{v})\,,\quad B_3(\mathbf{u}^m,c^m,r)\to B_3(\mathbf{u},c,r) \quad \text{as} \quad m\to\infty\,. \tag{3.19}$$

It follows from the weak convergence of $\{c^m\}_{n=1}^{\infty}$ to c that

$$\begin{cases} \Big(g(1+\gamma c^m)i_3,\mathbf{v}\Big)\to\Big(g(1+\gamma c)i_3,\mathbf{v}\Big) \quad \text{as } m\to\infty\,, \\ \theta a(c^m,r)\to\theta a(c,r) \quad \text{as } m\to\infty\,, \\ U(c^m,\dfrac{\partial r}{\partial x_3})\to U(c,\dfrac{\partial r}{\partial x_3}) \quad \text{as } m\to\infty\,. \end{cases} \tag{3.20}$$

Because the test functions $\mathbf{v}$, r defined in (3.16) are dense in $\mathbf{V}$ and $\tilde{H}$ respectively, conclusions (3.18), (3.19), and (3.20) hold for $\forall\mathbf{v}\in\mathbf{V}$ and $\forall r\in\tilde{H}$. Letting $m\to\infty$ in (3.9) and using the above results we obtain

$$\begin{cases} \Big(\nu(c+\alpha)\nabla\mathbf{u},\nabla\mathbf{v}\Big)+B_2(\mathbf{u},\mathbf{u},\mathbf{v})=-\Big((g+\gamma c)i_3,\mathbf{v}\Big)+(\mathbf{f},\mathbf{v}) \quad \forall\mathbf{v}\in\mathbf{V}_m\,, \\ \theta a(c,r)+B_3(\mathbf{u},c,r)-U(c,\dfrac{\partial r}{\partial x_3})=U\alpha(\dfrac{\partial r}{\partial x_3},1) \quad \forall r\in H_m\,. \end{cases}$$

Since the basis $\{\mathbf{v}_j\}_{j=1}^{\infty}$ and $\{r_j\}_{j=1}^{\infty}$ are dense in $\mathbf{V}$ and $\tilde{H}$, we conclude that $(\mathbf{u},c)$ is a solution of (3.3). □

Uniqueness. First notice that the bilinear form $b(\cdot,\cdot)$ satisfies the inf-sup condition (see Remark 3.2). Therefore for each solution $(\mathbf{u},c)\in\mathbf{V}\times\tilde{H}$ of system (3.3), there exists a unique $p\in L_0^2(\Omega)$ satisfying system (3.2) (see [62], p. 113). To prove the uniqueness of the solution of (3.2), it suffices to prove that system (3.3) has a unique solution.

Following the proof of Lemma 3.5 we can obtain a priori estimates for $\mathbf{u}$ and c.

$$\|\mathbf{u}\|_1\le C_3 \quad \textit{and} \quad \|c\|_1\le C_4\,. \tag{3.21}$$

where

$$C_3=\frac{C_\Omega^2}{\nu_*}(\gamma C_4+\|\mathbf{f}-gi_3\|)\,,\quad C_4=\frac{U\alpha|\Omega|^{\frac{1}{2}}}{\frac{\theta}{C_\Omega^2}-U}\,.$$

Theorem 3.7. *Assume that*

(H4) The hypothesis of Theorem 3.6 holds;

(H5) The viscosity $\nu(\cdot)$ is Lipschitz continuous, i.e., there exists a constant $\nu_L>0$ such that

$$|\nu(x_1)-\nu(x_2)|\le\nu_L|x_1-x_2| \quad \forall x_1,x_2\in\mathbb{R}\,;$$

(H6) There exists a constant C_0 such that $\|\nabla\mathbf{u}\|_\infty\le C_0$;

(H7) The following inequality

$$\frac{\nu_*}{C_\Omega^2} - \left(\frac{C_{B_3} C_4}{\frac{\theta}{C_\Omega^2} - U} (\nu_L C_0 + g\gamma) + C_{B_2} C_3 \right) > 0$$

holds.

Then the solution $(\mathbf{u}, c)$ *of system* (3.3) *is unique.*

Proof. Let $(\mathbf{u}, c)$ and $(\bar{\mathbf{u}}, \bar{c})$ be two different solutions of (3.3). Substituting both solutions into (3.3) with $\mathbf{v} = \mathbf{u} - \bar{\mathbf{u}}$ and $r = c - \bar{c}$, taking the the difference of the two equations, we have

$$\begin{aligned} &\Big(\nu(c+\alpha)\nabla\mathbf{u}, \nabla(\mathbf{u}-\bar{\mathbf{u}})\Big) - \Big(\nu(\bar{c}+\alpha)\nabla\bar{\mathbf{u}}, \nabla(\mathbf{u}-\bar{\mathbf{u}})\Big) \\ &\qquad + B_2(\mathbf{u}, \mathbf{u}, \mathbf{u}-\bar{\mathbf{u}}) - B_2(\bar{\mathbf{u}}, \bar{\mathbf{u}}, \mathbf{u}-\bar{\mathbf{u}}) \\ &= - g\gamma\Big((c-\bar{c})i_3, \mathbf{u}-\bar{\mathbf{u}}\Big) \end{aligned} \tag{3.22}$$

and

$$\theta a(c-\bar{c}, c-\bar{c}) + B_3(\mathbf{u}, c, c-\bar{c}) - B_3(\bar{\mathbf{u}}, \bar{c}, c-\bar{c}) - U(c-\bar{c}, \frac{\partial(c-\bar{c})}{\partial x_3}) = 0\,. \tag{3.23}$$

The skew symmetry (3.8) leads to the identity

$$\begin{cases} B_2(\mathbf{u}, \mathbf{u}, \mathbf{u}-\bar{\mathbf{u}}) - B_2(\bar{\mathbf{u}}, \bar{\mathbf{u}}, \mathbf{u}-\bar{\mathbf{u}}) = B_2(\mathbf{u}-\bar{\mathbf{u}}, \mathbf{u}, \mathbf{u}-\bar{\mathbf{u}})\,, \\ B_3(\mathbf{u}, c, c-\bar{c}) - B_3(\bar{\mathbf{u}}, \bar{c}, c-\bar{c}) = B_3(\mathbf{u}-\bar{\mathbf{u}}, c, c-\bar{c})\,. \end{cases} \tag{3.24}$$

Thus it follows from (3.6), (3.4), (3.21), (3.23), and (3.24) that

$$\begin{aligned} \frac{\theta}{C_\Omega^2}\|c-\bar{c}\|_1^2 &\le |B_3(\mathbf{u}-\bar{\mathbf{u}}, c, c-\bar{c})| + U(c-\bar{c}, \frac{\partial(c-\bar{c})}{\partial x_3}) \\ &\le C_{B_3}C_4|\|c-\bar{c}\|_1\|\mathbf{u}-\bar{\mathbf{u}}\|_1 + U\|c-\bar{c}\|_1^2\,. \end{aligned}$$

Then from (3.10)

$$\|c-\bar{c}\|_1 \leq \frac{C_{B_3}C_4}{\frac{\theta}{C_\Omega^2}-U}\|\mathbf{u}-\bar{\mathbf{u}}\|_1\,. \tag{3.25}$$

Substituting the above estimate into (3.22) and combining (1.21), (3.4), (3.5), (3.21), (H6) and (3.24), we obatin

$$\begin{aligned}
&\frac{\nu_*}{C_\Omega^2}\|\mathbf{u}-\bar{\mathbf{u}}\|_1^2 \leq \Big(\nu(c+\alpha)\nabla(\mathbf{u}-\bar{\mathbf{u}}),\nabla(\mathbf{u}-\bar{\mathbf{u}})\Big)\\
&\leq \Big|\Big(\big(\nu(c+\alpha)-\nu(\bar{c}+\alpha)\big)\nabla\bar{\mathbf{u}}),\nabla(\mathbf{u}-\bar{\mathbf{u}})\Big)\Big| + |B_2(\mathbf{u}-\bar{\mathbf{u}},\mathbf{u},\mathbf{u}-\bar{\mathbf{u}})| + g\gamma\big|\big((c-\bar{c})i_3,\mathbf{u}-\bar{\mathbf{u}}\big)\big|\\
&\leq \nu_L C_0\|c-\bar{c}\|_1\|\mathbf{u}-\bar{\mathbf{u}}\|_1 + C_{B_2}\|\mathbf{u}-\bar{\mathbf{u}}\|_1^2\|\mathbf{u}\|_1 + g\gamma\|c-\bar{c}\|_1\|\mathbf{u}-\bar{\mathbf{u}}\|_1\\
&\leq \left(\frac{C_{B_3}C_4}{\frac{\theta}{C_\Omega^2}-U}(\nu_L C_0+g\gamma)+C_{B_2}C_3\right)\|\mathbf{u}-\bar{\mathbf{u}}\|_1^2\,.
\end{aligned}$$

From assumption (H7) we conclude that

$$\|\mathbf{u}-\bar{\mathbf{u}}\|_1 = \|c-\bar{c}\|_1 = 0\,.$$

□

Remark 3.8. In practice, it is necessary to verify condition (3.10) and (H6). Since the micro-organism is slightly denser than water, $\gamma = \rho_0/\rho_m - 1$ is small. Therefore to fulfill (3.10) and (H6), ν_* and θ must be sufficiently large while U and C_Ω are sufficiently small. In other words, the model is valid for a suspension containing culture fluid with large viscosity, large diffusion rate, slowly upswimming micro-organisms in a relatively small container.

3.2 Numerical approximation using the finite element method

In this section, we construct and analyze a finite element method for approximating weak solutions of (3.2). Throughout this section, we assume that the hypotheses of Theorem 3.7 hold.

Let τ_h be a regular triangulation of Ω and $\mathbf{X}_h$, M_h, and S_h be finite element subspaces of $\mathbf{H}_0^1(\Omega)$, $L_0^2(\Omega)$, and $\tilde{H}$, respectively. Assume that the following discrete inf-sup condition holds.

$$\sup_{\mathbf{v}\in\mathbf{X}_h} \frac{b(\mathbf{v},q)}{\|\mathbf{v}\|_{\mathbf{X}_h}} \geq \beta\|q\|_{M_h} \quad \forall q \in M_h\,, \tag{3.26}$$

where $\beta > 0$ is a fixed constant. Furthermore we assume that $\mathbf{X}_h$, M_h, and S_h satisfy the following approximation properties.

$$\inf_{\mathbf{v_h}\in\mathbf{X}_h} \|\mathbf{v}-\mathbf{v}_h\|_1 \leq Ch^s\|\mathbf{v}\|_{s+1} \qquad \forall \mathbf{v}\in\mathbf{H}^{s+1}(\Omega)\,, \quad 0 < s \leq k\,, \tag{3.27}$$

$$\inf_{q_h\in M^h} \|q-q_h\| \leq Ch^s\|q\|_{s} \qquad \forall q\in H^{s}(\Omega)\,, \quad 0 < s \leq k\,, \tag{3.28}$$

$$\inf_{t_h\in S_h} \|t-t_h\|_1 \leq Ch^s\|t\|_{s+1} \qquad \forall t\in H^{s+1}(\Omega)\,, \quad 0 < s \leq k\,, \tag{3.29}$$

for $k = 2, 3$. For the construction of these spaces, see [63, 62, 52]. Next we define the discrete divergence free space

$$\mathbf{V}_h = \{\mathbf{v}\in\mathbf{X}_h : \quad (\nabla\cdot\mathbf{v}, q_h) = 0 \quad \forall q_h \in M_h\}\,.$$

Notice that in general, $\mathbf{V}_h$ is not a subspace of $\mathbf{V}$. Thus in general the identity (3.8) does not hold. To obtain a skew symmetry similar to (3.8) on $\mathbf{V}_h$, we define auxiliary forms $\hat{B}_2$ and $\hat{B}_3$ by

$$\begin{cases} \hat{B}_2(\mathbf{u},\mathbf{v},\mathbf{w}) = \dfrac{1}{2}B_2(\mathbf{u},\mathbf{v},\mathbf{w}) - \dfrac{1}{2}B_2(\mathbf{u},\mathbf{w},\mathbf{v})\,, \\ \hat{B}_3(\mathbf{u},c,r) = \dfrac{1}{2}B_3(\mathbf{u},c,r) - \dfrac{1}{2}B_3(\mathbf{u},r,c)\,. \end{cases}$$

It is easy to verify that

$$\begin{aligned} \hat{B}_2(\mathbf{u},\mathbf{v},\mathbf{w}) = B_2(\mathbf{u},\mathbf{w},\mathbf{v}) \quad \forall \mathbf{u}\in\mathbf{V}\,, \quad \forall \mathbf{v},\mathbf{w}\in\mathbf{H}_0^1(\Omega)\,, \\ \hat{B}_3(\mathbf{u},c,r) = B_3(\mathbf{u},c,r) \quad \forall \mathbf{u}\in\mathbf{V}\,, \quad \forall c,r\in\tilde{H}\,. \end{aligned} \tag{3.30}$$

In addition, we have the identity

$$\hat{B}_2(\mathbf{u},\mathbf{v},\mathbf{v}) = 0\,, \quad \hat{B}_3(\mathbf{u},c,c) = 0 \quad \forall \mathbf{u},\mathbf{v} \in \mathbf{H}_0^1(\Omega)\,, \quad \forall c \in \tilde{H}\,, \tag{3.31}$$

and the tricontinuous property

$$\begin{cases} \hat{B}_2(\mathbf{u},\mathbf{v},\mathbf{w}) \le C_{B_2}\|\mathbf{u}\|_1\|\mathbf{v}\|_1\|\mathbf{w}\|_1 \quad \forall \mathbf{u},\mathbf{v},\mathbf{w} \in \mathbf{H}_0^1(\Omega)\,, \\ \hat{B}_3(\mathbf{u},c,r) \le C_{B_3}\|\mathbf{u}\|_1\|c\|_1\|r\|_1 \quad \forall \mathbf{u} \in \mathbf{H}_0^1(\Omega)\,, \quad \forall c,r \in \tilde{H}\,, \end{cases} \tag{3.32}$$

where C_{B_2} and C_{B_3} are the same as in (3.5) and (3.6).

We define the finite element approximation of (3.3) as follows.

Definition 3.9. Find $(\mathbf{u}_h, p_h, c_h) \in \mathbf{X}_h \times M_h \times S_h$, such that

$$\begin{cases} \Big(\nu(c_h+\alpha)\nabla\mathbf{u}_h, \nabla\mathbf{v}\Big) + \hat{B}_2(\mathbf{u}_h,\mathbf{u}_h,\mathbf{v}) - (p_h, \nabla\cdot\mathbf{v}) \\ \qquad = -\Big(g(1+\gamma c_h\Big)i_3,\mathbf{v}) + (\mathbf{f},\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{X}_h\,, \\ (\nabla\cdot\mathbf{u}_h, q_h) = 0 \quad \forall q_h \in M_h\,, \\ \theta a(c_h,r) + \hat{B}_3(\mathbf{u}_h,c_h,r) - U(c_h, \dfrac{\partial r}{\partial x_3}) = U\alpha(\dfrac{\partial r}{\partial x_3},1) \quad \forall r \in S_h\,. \end{cases} \tag{3.33}$$

Analogous to the continuous case, we first solve an auxiliary discrete system.

Find $(\mathbf{u}_h, c_h) \in \mathbf{V}_h \times S_h$ such that

$$\begin{cases} \Big(\nu(c_h+\alpha)\nabla\mathbf{u}_h, \nabla\mathbf{v}\Big) + \hat{B}_2(\mathbf{u}_h,\mathbf{u}_h,\mathbf{v}) \\ \qquad = -\Big(g(1+\gamma c_h)i_3,\mathbf{v}\Big) + (\mathbf{f},\mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}_h\,, \\ \theta a(c_h,r) + \hat{B}_3(\mathbf{u}_h,c_h,r) - U(c_h, \dfrac{\partial r}{\partial x_3}) = U\alpha(\dfrac{\partial r}{\partial x_3},1)\,, \quad \forall r \in S_h\,. \end{cases} \tag{3.34}$$

The skew symmetry (3.31) and inequality (3.32) guarantee the existence of a weak solution of (3.34) by using anargument similar to the one used in the continuous case. A solution

$(\mathbf{u}_h, p_h, c_h)$ of (3.33) can be completed by solving (see [62], p. 59, Theorem I.4.1)

$$
\begin{aligned}
(p_h, \nabla\cdot\mathbf{v}) = \Big(\nu(c_h+\alpha)\nabla\mathbf{u}_h, \nabla\mathbf{v}\Big) + \hat{B}_2(\mathbf{u}_h, \mathbf{u}_h^n, \mathbf{v}) \\
+\Big(g(1+\gamma c_h)i_2, \mathbf{v}) - (\mathbf{f}^n, \mathbf{v}) \quad \forall \mathbf{v}\in\mathbf{X}_h\,.
\end{aligned}
\tag{3.35}
$$

The right hand side defines a functional on $\mathbf{X}_h$ which vanishes on $\mathbf{V}_h$. Due to (4.55) and the property of Lagrange multipliers, equation (4.66) is always solvable and the solution $p_h \in M_h$ is unique in the quotient space M_h/N_h where

$$
N_h = \{q_h \in S_h \,:\, (q_h, \nabla\cdot\mathbf{v}) = 0\,, \quad \forall \mathbf{v}\in\mathbf{X}_h\}\,.
$$

Following the same approach as in Lemma 3.5, we can show that $\|\mathbf{u}_h\|_1$ and $\|c_h\|_1$ are uniformly bounded, i.e., there exist constants C_3 and C_4 independent of h such that

$$
\|\mathbf{u}_h\|_1 \le C_3\,, \quad \|c_h\|_1 \le C_4. \tag{3.36}
$$

To carry out the error estimate, we introduce the Ritz Galerkin projections $r_h : \mathbf{H}_0^1(\Omega) \to \mathbf{V}_h$, $s_h : \tilde{H} \to S_h$, and the L^2 projection $\pi_h : L_0^2(\Omega) \to M_h$. In this way we split the errors into two parts

$$
\begin{cases}
\mathbf{u} - \mathbf{u}_h = \mathbf{u} - r_h\mathbf{u} + r_h\mathbf{u} - \mathbf{u}_h := \rho_{\mathbf{u}}^h + \theta_{\mathbf{u}}^h\,, \\
p - p_h = p - \pi_h p + \pi_h p - p_h := \rho_p^h + \theta_p^h\,, \\
c - c_h = c - s_h c + s_h c - c_h := \rho_c^h + \theta_c^h\,.
\end{cases}
\tag{3.37}
$$

From the approximation properties (3.27) – (3.29) we known that

$$
\begin{cases}
\|r_h\mathbf{u}\|_1 \le C(\mathbf{u})\,, \quad \|\rho_{\mathbf{u}}^h\|_1 \le Ch^s\|\mathbf{v}\|_{s+1}\,, \quad \mathbf{u}\in\mathbf{H}^{s+1}(\Omega)\,, \quad 0 < s \le k\,, \\
\|s_h c\|_1 \le C(c)\,, \quad \|\rho_c^h\|_1 \le Ch^s\|c\|_{s+1}\,, \quad c\in\mathbf{H}^{s+1}(\Omega)\,, \quad 0 < s \le k\,, \\
\|\pi_h p\| \le C(p)\,, \quad \|\rho_p^h\| \le Ch^s\|p\|_s\,, \quad p\in H^s(\Omega)\,, \quad 0 < s \le k\,.
\end{cases}
\tag{3.38}
$$

Theorem 3.10. *Assume that the hypotheses of Theorem 3.6 and Theorem 3.7 hold. Then for* $\mathbf{u} \in H^{s+1}(\Omega)$, $p \in H^s(\Omega)$, *and* $c \in H^{s+1}(\Omega)$, *there exists a constant* C *independent of* h *such that*

$$\|\mathbf{u} - \mathbf{u}_h\|_1 + \|c - c_h\|_1 + \|p - p_h\| \leq Ch^s , \quad 0 < s \leq k . \tag{3.39}$$

Proof. Due to (3.38), it suffices to estimate $\theta_{\mathbf{u}}^h$, θ_p^h, and θ_c^h. Subtracting (3.33) from (3.2) with $\mathbf{v} = \theta_{\mathbf{u}}^h$, $r = \theta_c^h$ and using (3.30) we have that

$$\begin{aligned}
&\Big(\nu(c+\alpha)\nabla\mathbf{u},\nabla\theta_{\mathbf{u}}^h\Big) - \Big(\nu(c_h+\alpha)\nabla\mathbf{u}_h, \nabla\theta_{\mathbf{u}}^h\Big) \\
&\qquad + \hat{B}_2(\mathbf{u},\mathbf{u},\theta_{\mathbf{u}}^h) - \hat{B}_2(\mathbf{u}_h,\mathbf{u}_h,\theta_{\mathbf{u}}^h) \\
&\qquad + b(p-p_h,\theta_{\mathbf{u}}^h) \\
&\quad = - g\gamma\Big((c-c_h)i_3,\theta_{\mathbf{u}}^h\Big)
\end{aligned} \tag{3.40}$$

and

$$\theta a(c-c_h,\theta_c^h) + \hat{B}_3(\mathbf{u},c,\theta_c^h) - \hat{B}_3(\mathbf{u}_h,c_h,\theta_c^h) - U(c-c_h,\frac{\partial\theta_c^h}{\partial x_3}) = 0 . \tag{3.41}$$

It follows from (1.21), (3.31), and (3.32) that

$$\begin{aligned}
\frac{\theta}{C_\Omega^2}\|\theta_c^h\|_1^2 &\leq \theta a(\theta_c^h,\theta_c^h) \\
&= - \theta a(\rho_c^h,\theta_c^h) - \hat{B}_3(\theta_{\mathbf{u}}^h,c_h,\theta_c^h) - \hat{B}_3(r_h\mathbf{u},\rho_c^h,\theta_c^h) \\
&\quad - \hat{B}_3(\rho_{\mathbf{u}}^h,c,\theta_c^h) + U(\theta_c^h,\frac{\partial\theta_c^h}{\partial x_3}) + U(\rho_c^h,\frac{\partial\theta_c^h}{\partial x_3}) \\
&\leq \theta\|\theta_c^h\|_1\|\rho_c^h\|_1 + U\|\theta_c^h\|_1^2 + U\|\rho_c^h\|_1\|\theta_c^h\|_1 \\
&\quad + C_{B_3}\|\theta_c^h\|_1\Big(\|\theta_{\mathbf{u}}^h\|_1\|c_h\|_1 + \|r_h\mathbf{u}\|_1\|\rho_c^h\|_1 + \|\rho_{\mathbf{u}}^h\|_1\|c\|_1\Big) .
\end{aligned}$$

Due to assumption (3.10), moving the term $U\|\theta_c^h\|_1^2$ to the left and dividing by $\|\theta_c^h\|_1$, we have from (3.21) and (3.36) that

$$\|\theta_c^h\|_1 \leq \frac{1}{\frac{\theta}{C_\Omega^2} - U}\Big((\theta + U + C_{B_3}\|r_h\mathbf{u}\|_1)\|\rho_c^h\|_1 + C_{B_3}C_4(\|\theta_{\mathbf{u}}^h\|_1 + \|\rho_{\mathbf{u}}^h\|_1)\Big)\,. \tag{3.42}$$

Notice that $\theta_{\mathbf{u}}^h \in \mathbf{V}_h$ and $\theta_p^h \in M_h$. The definition of $\mathbf{V}_h$ implies that $b(\theta_p^h, \theta_{\mathbf{u}}^h) = 0$. Therefore

$$b(p - p_h, \theta_{\mathbf{u}}^h) = b(\rho_p^h, \theta_{\mathbf{u}}^h)\,.$$

Then according to (1.21), (H5), (H6), (3.21), (3.32), and (3.36), equation (3.40) yields

$$\begin{aligned}
\frac{\nu_*}{C_\Omega^2}\|\theta_{\mathbf{u}}^h\|_1^2 &\leq \Big(\nu(c_h+\alpha)\nabla\theta_{\mathbf{u}}^h, \nabla\theta_{\mathbf{u}}^h\Big) \\
&= \Big(\nu(c_h+\alpha)\nabla\rho_{\mathbf{u}}^h, \nabla\theta_{\mathbf{u}}^h\Big) + \Big(\big(\nu(c_h+\alpha) - \nu(c+\alpha)\big)\nabla\mathbf{u}, \nabla\theta_{\mathbf{u}}^h\Big) \\
&\quad - \hat{B}_2(\theta_{\mathbf{u}}^h, \mathbf{u}_h, \theta_{\mathbf{u}}^h) - B_2(r_h\mathbf{u}, \rho_{\mathbf{u}}^h, \theta_{\mathbf{u}}^h) - \hat{B}_2(\rho_{\mathbf{u}}^h, \mathbf{u}, \theta_{\mathbf{u}}^h) \\
&\quad + b(\rho_p^h, \theta_{\mathbf{u}}^h) + g\gamma\Big((c - c_h)i_3, \theta_{\mathbf{u}}^h\Big) \\
&\leq \nu^*\|\rho_{\mathbf{u}}^h\|_1\|\theta_{\mathbf{u}}^h\|_1 + (\nu_L C_0 + g\gamma)\|c_h - c\|_1\|\theta_{\mathbf{u}}^h\|_1 \\
&\quad + C_{B_2}\|\theta_{\mathbf{u}}^h\|_1\Big(C_3\|\theta_{\mathbf{u}}^h\|_1 + \|r_h\mathbf{u}\|_1\|\rho_{\mathbf{u}}^h\|_1 + C_3\|\rho_{\mathbf{u}}^h\|_1\Big) + \|\rho_p^h\|\|\theta_{\mathbf{u}}^h\|_1\,.
\end{aligned} \tag{3.43}$$

From (3.42)

$$\begin{aligned}
\|c_h - c\|_1 &\leq \|\rho_c^h\|_1 + \|\theta_c^h\|_1 \\
&\leq \|\rho_c^h\|_1 + \frac{1}{\frac{\theta}{C_\Omega^2} - U}\Big((\theta + U + C_{B_3}\|r_h\mathbf{u}\|_1)\|\rho_c^h\|_1 + C_{B_3}C_4(\|\theta_{\mathbf{u}}^h\|_1 + \|\rho_{\mathbf{u}}^h\|_1)\Big)\,.
\end{aligned}$$

Dividing (3.43) by $\|\theta_{\mathbf{u}}^h\|_1$ we obtain

$$
\begin{aligned}
&\left(\frac{\nu_*}{C_\Omega^2} - C_{B_2}C_3 - (\nu_L C_0 + g\gamma)\frac{C_{B_3}C_4}{\frac{\theta}{C_\Omega^2} - U}\right)\|\theta_{\mathbf{u}}^h\|_1 \\
&\leq \Big(\nu^* + C_{B_2}\|r_h\mathbf{u}\|_1 + C_{B_2}C_3 + \frac{C_{B_3}C_4(\nu_L C_0 + g\gamma)}{\frac{\theta}{C_\Omega^2} - U}\Big)\|\rho_{\mathbf{u}}^h\|_1 \\
&\quad + (\nu_L C_0 + g\gamma)\left(1 + \frac{\theta + U + C_{B_3}\|r_h\mathbf{u}\|_1}{\frac{\theta}{C_\Omega^2} - U}\right)\|\rho_c^h\|_1 + \|\rho_p^h\|\,.
\end{aligned}
$$

From assumption (H7) and (3.38)

$$
\|\theta_{\mathbf{u}}^h\|_1 \leq C\Big(\|\rho_{\mathbf{u}}^h\|_1 + \|\rho_c^h\|_1 + \|\rho_p^h\|\Big)\,, \tag{3.44}
$$

and from (3.42) and (3.38)

$$
\|\theta_c^h\|_1 \leq C\Big(\|\rho_{\mathbf{u}}^h\|_1 + \|\rho_c^h\|_1 + \|\rho_p^h\|\Big)\,. \tag{3.45}
$$

Combining (4.67), (3.44) and (3.45) we have that

$$
\|\mathbf{u} - \mathbf{u}_h\|_1 + \|c - c_h\|_1 \leq C\Big(\|\rho_{\mathbf{u}}^h\|_1 + \|\rho_c^h\|_1 + \|\rho_p^h\|\Big)\,. \tag{3.46}
$$

It remains to estimate $\|p - p_h\|$. Subtracting (3.33) from (3.2) gives

$$
\begin{aligned}
-b(\mathbf{v}, \theta_p^h) =& \Big(\nu(c+\alpha)\nabla\mathbf{u}, \nabla\mathbf{v}\Big) - \Big(\nu(c_h+\alpha)\nabla\mathbf{u}_h, \nabla\mathbf{v}\Big) \\
&+ \hat{B}_2(\mathbf{u}, \mathbf{u}, \mathbf{v}) - \hat{B}_2(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}) + b(\rho_p^h, \mathbf{v}) + g\gamma\Big((c - c_h)i_3, \mathbf{v}\Big)\,.
\end{aligned}
$$

From (3.26), (3.32), (3.21), (3.36), and (3.46) it follows that

$$\begin{aligned}
\|\theta_p^h\| &\leq \frac{1}{\beta} \sup_{\mathbf{v} \in \mathbf{X}_h} \frac{1}{\|\mathbf{v}\|_1} \Big((\nu(c_h + \alpha)\nabla(\mathbf{u} - \mathbf{u}_h), \nabla \mathbf{v}\Big) \\
&\quad + \Big(\big(\nu(c+\alpha) - \nu(c_h + \alpha)\big)\nabla \mathbf{u}, \nabla \mathbf{v}\Big) \\
&\quad + \hat{B}_2(\mathbf{u} - \mathbf{u}_h, \mathbf{u}, \mathbf{v}) + \hat{B}_2(\mathbf{u}_h, \mathbf{u} - \mathbf{u}_h, \mathbf{v}) \\
&\quad + b(\rho_p^h, \mathbf{v}) + g\gamma\big((c - c_h)i_3, \mathbf{v}\big)\Big) \\
&\leq \frac{1}{\beta} \sup_{\mathbf{v} \in \mathbf{X}_h} \frac{1}{\|\mathbf{v}\|_1} \Big((\nu^* + 2C_{B_2}C_3)\|\mathbf{u} - \mathbf{u}_h\|_1 \|\mathbf{v}\|_1 \\
&\quad + (\nu_L C_0 + \gamma)\|c - c_h\|_1 \|\mathbf{v}\|_1 + \|\rho_p^h\| \|\mathbf{v}\|_1 \Big) \\
&\leq C\Big(\|\mathbf{u} - \mathbf{u}_h\|_1 + \|c - c_h\|_1 + \|\rho_p^h\| \Big) .
\end{aligned}$$

Combing the above estimate with (3.44) and (3.45), we obtain

$$\|p - p_h\|_1 \leq C\Big(\|\rho_{\mathbf{u}}^h\|_1 + \|\rho_c^h\|_1 + \|\rho_p^h\| \Big) . \tag{3.47}$$

Then the estimate of the pressure error follows from (3.38). □

3.3 Numerical experiments

In this section we describe five numerical experiments that were conducted. The first one uses artificial data to verify the error estimates while the other four use data obtained from lab experiments. We used Taylor-Hood finite element spaces ([53]) for $\mathbf{V}_h$ and M_h, and continuous piecewise quadratic function spaces for S_h. In particular, the theoretical convergence rate is given by

$$\|\rho_{\mathbf{u}}^h\| = O(h^m) , \quad \|\rho_{\mathbf{u}}^h\|_1 = O(h^{m-1}) , \quad \|\rho_p^h\| = O(h^{m-1}) , \quad \|\rho_c^h\| = O(h^m) ,$$

Table 3.1: Convergence rate

h	$\|p-p_h\|$	$\|u-u_h\|$	$\|c-c_h\|$	$\|p-p_h\|_1$	$\|u-u_h\|_1$	$\|c-c_h\|_1$
1/2	0.2520	0.0078	0.0049	0.9846	0.0854	0.0460
1/4	0.0323	0.0010	6.8E-04	0.3847	0.0207	0.0118
1/8	0.0055	1.31E-04	8.88E-05	0.1786	0.0050	0.030
1/16	0.0011	1.65E-05	1.13E-05	0.0877	0.0012	7.45E-04
1/32	2.28E-04	2.07E-06	1.42E-06	0.0436	3.09E-04	1.86E-04
1/64	4.90E-05	2.60E-07	1.80E-07	0.0216	7.68E-05	4.7E-05
conv. rate	2.22	2.99	2.97	1.02	2.01	2.00

where

$$\mathbf{u} \in \mathbf{H}^m(\Omega) \cap \mathbf{H}_0^1(\Omega)\,, \quad p \in H^{m-1}(\Omega) \cap L_0^2(\Omega)\,, \quad c \in H^m(\Omega) \cap L_0^2(\Omega)\,, \quad m = 2, 3\,.$$

In the numerical experiment we used the following parameters

$$\gamma = 0.1\,, \quad U = 0.1\,, \quad \theta = 1\,,$$

and

$$\nu(x) = \sin^2 x + 1, \quad x \in \mathbb{R}\,.$$

The forcing term $\mathbf{f}$ was chosen so that the exact solution is

$$\begin{cases} \mathbf{u} = (\sin \pi x \sin \pi y, \sin \pi x \sin \pi y)^T\,, \\ p = \sin \pi x \sin \pi y\,, \\ c = \sin \pi x \sin \pi y\,. \end{cases}$$

The numerical errors for different mesh sizes are listed in Table 3.1. The convergence rates listed in the table are consistent with our theoretical result.

Example 2. In this example we consider a 10 cm $\times$ 10 cm container filled with microorganisms in a suspension under zero external force, i.e., $\mathbf{f} \equiv 0$. For computational simplicity, we study the domain in two dimensional horizontal – vertical plan . The parameters of the model, obtained from lab experiments (see [36]) are given in Table 3.2.

Table 3.2: Parameter values

ν_0	g	γ	θ	U
cm^2/sec	m/sec^2		cm^2/sec	cm/sec
0.01	9.81	0.1	0.0025	0.01

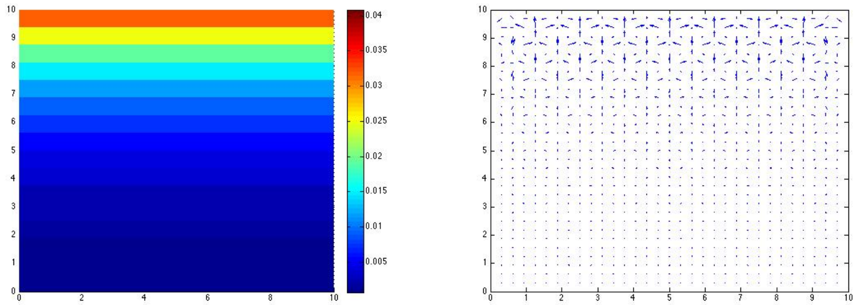
Define

$$\nu(c) = \begin{cases} \nu_0\,, & c < 0\,, \\ \nu_0(1 + 2.5\ c + 5.3\ c^2)\,, & 0 < c < 10\%\,, \\ \nu_0 \exp(\dfrac{2.5\ c}{1 - 1.4\ c})\,, & 10\% < c < 60\%\,, \\ \nu_0 \exp(9.375)\,, & c > 60\%\,. \end{cases} \tag{3.48}$$

Here ν_0 is the viscosity of the culture fluid. The viscosity (3.48) combines the work of Batchelor [43] for low concentration and Mooney [44] for high concentration. Note that $\exp(\frac{2.5\ c}{1-1.4\ c})$ is bounded below by $\nu_* = \nu_0$ but tends to infinity when the maximum concentration $\varphi_m = \frac{1}{1.4}$ is reached since the suspension behaves as a solid. In this case no movement of neighboring particles are allowed. Therefore we set the upper bound $\nu^* = \nu_0 \exp(9.375)$ so that the viscosity defined in (3.48) satisfies property (1.21).

We first chose $\alpha = 1\%$. The velocity and concentration are given in Figure 3.1. We can see that a bioconvection pattern can not be formed and the concentration has a homogeneous horizontal distribution. This is because the right hand side of the first equation in (1.16) is almost equal to $-g$. As a result, $\mathbf{u} \approx 0$ while p is almost linear with $\nabla p \approx -g$ and $\frac{\partial c}{\partial x} \approx 0$ because of the nearly zero velocity $\mathbf{u}$. In this case, the micro-organisms do not move and the concentration stays linear in the vertical direction with zero horizontal gradient. From observed experiments, for a shallow container with low concentration of micro-organisms, the micro-organisms will stay at the surface of the suspension due to the upswimming. Actually, the effect of gravity is due to high density of the micro-organisms but the micro-organisms are almost isolated thus gravity can be neglected. In fact, bioconvection only occurs for

Figure 3.1: Concentration and velocity field for $\alpha = 1\%$



sufficiently deep container. The higher the concentration is, the shallower the container can be. We conclude that a %1 concentration is not enough to form a bioconvection pattern in a 10 cm deep container.

Example 3. In this example, we assign the same parameter as in Example 2 except that $\alpha = 20\%$. Figure 3.2 shows the concentration distribution and the velocity filed using streamlines. Here the color denotes the magnitude of the velocity. The figure shows that a bioconvection pattern can be formed for sufficiently large concentration. Our simulation result is consistent with the results obtained in [36]. From the figure we can see that two convections, separated from the center, flow steadily in opposite directions. The highest velocity occurs in the center, where the concentration is low, due to the upswimming under small effect of gravity. Another high speed motion is observed on the left and right hand sides of the container, which is caused mostly by gravity due to high concentration in the upper left and right corners. Only a few micro-organisms remains at the bottom while most of the micro-organisms stay close to the surface.

Example 4. In this example, $\alpha = 20\%$ and constant viscosity $\nu(c) \equiv 0.01$. The result is shown in Figure 3.3. From the graph, we can see that both example 3 and example 4 capture the motion of the bioconvection but the concentration distribution and velocity field are slightly different. The velocity in the nonlinear case are slower and smoother due to a relatively higher viscosity. The difference is more notable in regions where the concentration is high. Example 3 reflects higher concentrated micro-organisms at the top corners since more

Figure 3.2: Concentration and velocity field for $\alpha = 20\%$

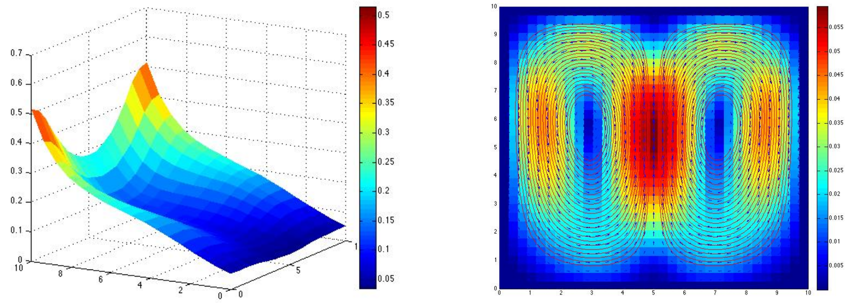
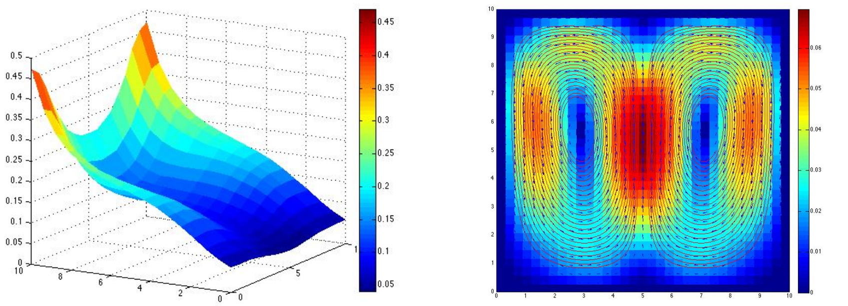


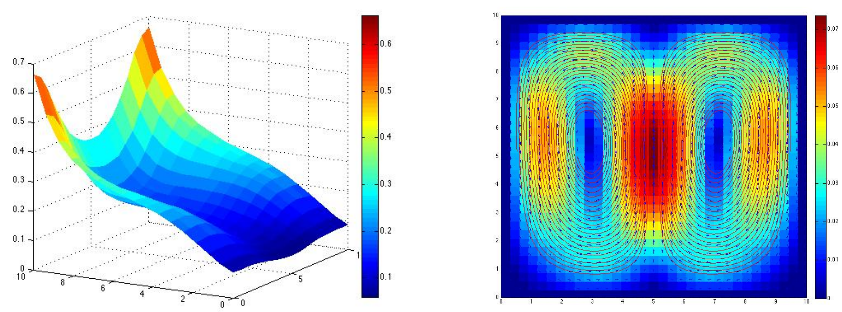
Figure 3.3: Concentration and velocity field for $\alpha = 20\%$ with constant viscosity



micro-organisms are washed up by the drag force and stay there due to the high viscosity. One can see that the involvement of a nonhomogeneous viscosity improves the accuracy of the simulation.

Example 5. In the last numerical experiment, all data are the same as in Example 3 except that $\alpha = 30\%$. The velocity field and concentration distribution are given in figure 3.4. As the concentration increases, the effect of the gravity becomes more significant. However, once the pattern is formed, the distribution of the concentration stays the same.

Figure 3.4: Concentration and velocity field for $\alpha = 30\%$



Chapter 4

Time dependent bioconvection

4.1 Existence and uniqueness of a solution

The weak formulation. In this chapter, we consider system (1.19) in a two dimensional domain Ω. Similarly to the steady bioconvection case we choose $L_0^2(\Omega)$ as the functional space for the pressure. Using the same auxiliary concentration $c = c_\alpha$, force $\mathbf{f} = \mathbf{f}_\alpha$ and the same bilinear and trilinear forms defined in (3.1), we define the weak solution of system (1.19) as follows.

Definition 4.1. Given $\mathbf{f}$ in $L^2(I;\mathbf{L}^2(\Omega))$, $\mathbf{u}_0 \in \mathbf{L}^2(\Omega)$, $c_0 \in L^2(\Omega)$, a triple $(\mathbf{u}, p, c) \in L^2(I;\mathbf{H}_0^1(\Omega)) \times L^2(I;L_0^2(\Omega)) \times L^2(I;\tilde{H})$ is said to be a weak solution of system (1.19) if for any $t \in (0,T]$

$$\begin{cases} \langle \mathbf{u}', \mathbf{v}\rangle + \Big(\nu(c+\alpha)\nabla \mathbf{u}, \nabla \mathbf{v}\Big) + B_2(\mathbf{u},\mathbf{u},\mathbf{v}) + b(p,\mathbf{v}) \\ \quad = -\Big(g(1+\gamma c\Big)i_2, \mathbf{v}) + (\mathbf{f},\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega)\,, \\ b(q,\mathbf{u}) = 0 \quad \forall q \in L_0^2(\Omega)\,, \\ \langle c', r\rangle + \theta a(c,r) + B_3(\mathbf{u},c,r) - U(c, \dfrac{\partial r}{\partial x_2}) = 0 \quad \forall r \in \tilde{H}\,, \\ \mathbf{u}(0) = \mathbf{u}_0\,, \quad c(0) = c_0\,. \end{cases} \tag{4.1}$$

Analogous to the steady state case, to solve system (4.1), it suffices to solve the following associated problem.

Given $\mathbf{f}$ in $L^2(I;\mathbf{L}^2(\Omega))$, $\mathbf{u}_0 \in \mathbf{L}^2(\Omega)$, and $c_0 \in L^2(\Omega)$, find a pair $(\mathbf{u}, c) \in L^2(I;\mathbf{V}) \times L^2(I;\tilde{H})$ such that for any $t \in (0, T]$,

$$\begin{cases} \langle \mathbf{u}', \mathbf{v}\rangle + \Big(\nu(c+\alpha)\nabla\mathbf{u}, \nabla\mathbf{v}\Big) + B_2(\mathbf{u},\mathbf{u},\mathbf{v}) = -\Big(g(1+\gamma c)i_2, \mathbf{v}\Big) + (\mathbf{f},\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}\,, \\ (c', r) + \theta a(c, r) + B_3(\mathbf{u}, c, r) - U(c, \dfrac{\partial r}{\partial x_2}) = U\alpha(\dfrac{\partial r}{\partial x_2}, 1) \quad \forall r \in \tilde{H}\,, \\ \mathbf{u}(0) = \mathbf{u}_0\,, \quad c(0) = c_0 - \alpha\,. \end{cases} \tag{4.2}$$

Remark 4.2. Similarly to the steady case, if $(\mathbf{u}, c)$ is a solution of system (4.2), then $(\mathbf{u}, p, c)$, the solution of (4.1), can be recovered because the bilinear form $b(\cdot,\cdot)$ defined above satiesfies the inf-sup condition (3.2), i.e., for some $\beta > 0$

$$\sup_{\mathbf{v}\in\mathbf{H}_0^1(\Omega)} \frac{(q, \nabla\cdot\mathbf{v})}{\|\mathbf{v}\|_1} \geq \beta\|q\|\,, \quad \forall q \in L_0^2(\Omega)\,.$$

For more details see [62], p. 59, Theorem I.4.1.

Existence. To establish the existence of a weak solution, we use the same modified Rothe's method developed in chapter 2 to construct a convergent sequence of approximate solutions of (4.2) using the backward Euler approximation of the time derivative $\mathbf{u}'$ and c'. To this end we let $k = T/n$ for some positive integer n, partition I uniformly with time step k, and denote nodal points $t_i = t_i^n = ik$, for $i = 1, 2, \ldots, n$. Let $\mathbf{u}_n^0 = \mathbf{u}_0$ and $c_n^0 = c_0$ and define

$$\begin{cases} \mathbf{f}_n^i := 1/k \displaystyle\int_{t_{i-1}}^{t_i} \mathbf{f}(t)\, dt\,, \\ \delta\mathbf{u}_n^i := (\mathbf{u}_n^i - \mathbf{u}_n^{i-1})/k\,, \\ \delta c_n^i := (c_n^i - c_n^{i-1})/k\,. \end{cases}$$

For each integer $n > 0$, we apply the following scheme inductively to find $\mathbf{u}_n^i \in \mathbf{V}$ and $c_n^i \in \tilde{H}$, the approximations of the solution $\mathbf{u}$ and c at t_i, $i = 1, 2, \ldots, n$ respectively, with

$\mathbf{u}_n^0 = \mathbf{u}_0$ and $c_n^0 = c_0$.

$$\begin{cases} (\delta \mathbf{u}_n^i, \mathbf{v}) + \Big(\nu(c_n^i + \alpha)\nabla \mathbf{u}_n^i, \nabla \mathbf{v}\Big) + B_2(\mathbf{u}_n^i, \mathbf{u}_n^i, \mathbf{v}) \\ \quad = -\Big(g(1+\gamma c_n^i)i_2, \mathbf{v}\Big) + (\mathbf{f}_n^i, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}\,, \\ (\delta c_n^i, r) + \theta a(c_n^i, r) + B_3(\mathbf{u}_n^i, c_n^i, r) - U(c_n^i, \dfrac{\partial r}{\partial x_2}) = U\alpha(\dfrac{\partial r}{\partial x_2}, 1) \quad \forall r \in \tilde{H}\,. \end{cases} \tag{4.3}$$

Multiplying both sides of (4.3) by k, we obtain the following scheme.

Given $\mathbf{u}_n^{i-1} \in \mathbf{V}$, $c_n^{i-1} \in \tilde{H}$, seek $(\mathbf{u}_n^i, c_n^i) \in \mathbf{V} \times \tilde{H}$ such that

$$\begin{cases} (\mathbf{u}_n^i, \mathbf{v}) + k\Big(\nu(c_n^i + \alpha)\nabla \mathbf{u}_n^i, \nabla \mathbf{v}\Big) + kB_2(\mathbf{u}_n^i, \mathbf{u}_n^i, \mathbf{v}) \\ \quad = -k\Big(g(1+\gamma c_n^i)i_2, \mathbf{v}\Big) + (k\mathbf{f}_n^i + \mathbf{u}_n^{i-1}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}\,, \\ (c_n^i, r) + k\theta a(c_n^i, r) + kB_3(\mathbf{u}_n^i, c_n^i, r) - kU(c_n^i, \dfrac{\partial r}{\partial x_2}) \\ \quad = kU\alpha(\dfrac{\partial r}{\partial x_2}, 1) + (c_n^{i-1}, r) \quad \forall r \in \tilde{H}\,. \end{cases} \tag{4.4}$$

Using (3.5), (3.6), (3.7), and (3.8), we establish the existence of a weak solution of (4.4) as follows.

Lemma 4.3. *Given $\mathbf{u}_n^{i-1} \in \mathbf{V}$, $c_n^{i-1} \in \tilde{H}$, assume that*

$$\frac{\theta}{C_\Omega^2} > U\,. \tag{4.5}$$

Then system (4.4) has a weak solution $(\mathbf{u}_n^i, c_n^i) \in \mathbf{V} \times \tilde{H}$.

Proof. Analogous to the proof of Theorem 3.6, since $\mathbf{V}$ and $\tilde{H}$ are both separable, there exist $\{\mathbf{v}_j\}_{j=1}^\infty$ and $\{r_j\}_{j=1}^\infty$ orthonormal bases of $\mathbf{V}$ and $\tilde{H}$ respectively. Let $\mathbf{V}_m$, H_m be finite subspaces of $\mathbf{V}$, $\tilde{H}$ spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ and $\{r_1, r_2, \ldots, r_m\}$. We consider the finite dimensional approximations of $\mathbf{u}$ and c in $\mathbf{V}_m$ and H_m, i.e., we seek $\mathbf{u}^m \in \mathbf{V}_m$, $c^m \in H_m$

such that

$$\begin{cases} (\mathbf{u}^m, \mathbf{v}) + k\Big(\nu(c^m+\alpha)\nabla\mathbf{u}^m, \nabla\mathbf{v}\Big) + kB_2(\mathbf{u}^m, \mathbf{u}^m, \mathbf{v}) \\ \qquad = -k\Big(g(1+\gamma c^m)i_2, \mathbf{v}\Big) + (k\mathbf{f}_n^i + \mathbf{u}_n^{i-1}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}_m\,, \\ (c^m, r) + k\theta a(c^m, r) + kB_3(\mathbf{u}^m, c^m, r) - kU(c^m, \dfrac{\partial r}{\partial x_2}) \\ \qquad = kU\alpha(\dfrac{\partial r}{\partial x_2}, 1) + (c_n^{i-1}, r) \quad \forall r \in H_m\,. \end{cases} \tag{4.6}$$

For any integer $m > 0$, the existence of a solution $(\mathbf{u}^m, c^m)$ of (4.6) is guaranteed by Lemma 2.9. To show that $\{\mathbf{u}^m\}_{m=1}^\infty$ and $\{c^m\}_{m=1}^\infty$ are bounded sequence in $\mathbf{V}_m$ and H_m respectively, we take $\mathbf{v} = \mathbf{u}^m$ and $r = c^m$ in (4.6), and use (1.21) and (3.8) to find

$$\begin{aligned} \|\mathbf{u}^m\|^2 + \frac{k\nu_*}{C_\Omega^2}\|\mathbf{u}^m\|_1^2 \le &(\mathbf{u}^m, \mathbf{u}^m) + k\Big(\nu(c^m+\alpha)\nabla\mathbf{u}^m, \nabla\mathbf{u}^m\Big) \\ \le &\Big| -k\Big(g(1+\gamma c^m)i_2, \mathbf{u}^m\Big) + (k\mathbf{f}_n^i + \mathbf{u}_n^{i-1}, \mathbf{u}^m)\Big| \\ \le &\varepsilon\|\mathbf{u}^m\|^2 + C(\varepsilon)\|c^m\|^2 + C(\varepsilon)\|kf_n^i + \mathbf{u}_n^{i-1} - kgi_2\|^2 \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} \|c^m\|^2 + \frac{k\theta}{C_\Omega^2}\|c^m\|_1^2 \le &(c^m, c^m) + k\theta a(c^m, c^m) \\ = &\Big|kU(c^m, \frac{\partial c^m}{\partial x_2}) + kU\alpha(\frac{\partial c^m}{\partial x_2}, 1) + (c_n^{i-1}, c^m)\Big| \\ \le &kU\|c^m\|_1^2 + \varepsilon\|c^m\|_1^2 + C(\varepsilon)(1 + \|c_n^{i-1}\|^2)\,. \end{aligned}$$

By assumption (4.5), we may choose a sufficiently small $\varepsilon > 0$ such that $\varepsilon < \dfrac{k\theta}{c_\Omega^2} - kU$. Then the above inequality gives

$$\|c^m\|_1 \le C\,. \tag{4.8}$$

Furthermore substituting (4.8) into (4.7) and choosing $0 < \varepsilon < 1$ we have that

$$\|\mathbf{u}^m\|_1 \le C\,.$$

This means that the sequences $\{\mathbf{u}^m\}_{m=1}^{\infty}$ and $\{c^m\}_{m=1}^{\infty}$ are bounded in $\mathbf{V}$ and $\tilde{H}$ respectively. Therefore there exist $\mathbf{u}_n^i \in \mathbf{V}$ and $c_n^i \in \tilde{H}$ such that

$$\mathbf{u}^m \rightharpoonup \mathbf{u}_n^i \quad \text{in} \quad \mathbf{V} \quad \text{and} \quad c^m \rightharpoonup c_n^i \quad \text{in} \quad \tilde{H} \quad \text{as} \quad m \to \infty\,. \tag{4.9}$$

Due to Sobolev embedding theorem, this yields

$$\mathbf{u}^m \to \mathbf{u}_n^i \quad \text{in} \quad \mathbf{L}^2(\Omega) \quad \text{and} \quad c^m \to c_n^i \quad \text{in} \quad L^2(\Omega) \quad as\ m \to \infty\,. \tag{4.10}$$

Next we show that the weak limit $(\mathbf{u}, c)$ is a solution of (4.4). Choose test functions

$$\mathbf{v} \in \mathbf{V} \cap (C_0^{\infty}(\Omega))^3\,, \quad r \in C^{\infty}(\Omega) \cap \tilde{H}\,. \tag{4.11}$$

Similarly to the strategy used in the proof of Theorem 3.6, we let m tend zero and use the weak and strong convergence of $\{\mathbf{u}^m\}_{n=1}^{\infty}$ and $\{c^m\}_{n=1}^{\infty}$ to conclude that

$$\begin{cases} \Big(\nu(c^m+\alpha)\nabla\mathbf{u}^m, \nabla\mathbf{v}\Big) \to \Big(\nu(c+\alpha)\nabla\mathbf{u}_n^i, \nabla\mathbf{v}\Big) \quad \text{as} \quad m \to \infty\,, \\ B_2(\mathbf{u}^m, \mathbf{u}^m, \mathbf{v}) \to B_2(\mathbf{u}_n^i, \mathbf{u}_n^i, \mathbf{v})\,, \quad B_3(\mathbf{u}^m, c^m, r) \to B_3(\mathbf{u}_n^i, c_n^i, r) \quad \text{as} \quad m \to \infty\,, \\ (\mathbf{u}^m, \mathbf{v}) \to (\mathbf{u}_n^i, \mathbf{v})\,, \quad (c^m, r) \to (c_n^i, r) \quad \text{as} \quad m \to \infty\,, \\ (g(1+\gamma c^m)i_2, \mathbf{v}) \to (g(1+\gamma c_n^i)i_2, \mathbf{v})\,, \quad \theta a(c^m, r) \to \theta a(c_n^i, r) \quad \text{as} \quad m \to \infty\,, \\ U(c^m, \dfrac{\partial r}{\partial x_2}) \to U(c_n^i, \dfrac{\partial r}{\partial x_2}) \quad \text{as} \quad m \to \infty\,. \end{cases}$$

Since $\mathbf{v}$ and r defined in (4.11) are dense in $\mathbf{V}$ and $\tilde{H}$, the limits $\mathbf{u}_n^i$ and c_n^i satisfy (4.6), i.e.,

$$\begin{cases} (\mathbf{u}_n^i, \mathbf{v}) + k\Big(\nu(c_n^i+\alpha)\nabla\mathbf{u}_n^i, \nabla\mathbf{v}\Big) + kB_2(\mathbf{u}_n^i, \mathbf{u}_n^i, \mathbf{v}) \\ \qquad = -k\Big(g(1+\gamma c_n^i)i_2, \mathbf{v}\Big) + (k\mathbf{f}_n^i + \mathbf{u}_n^{i-1}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}_m\,, \\ (c_n^i, r) + k\theta a(c_n^i, r) + kB_3(\mathbf{u}_n^i, c_n^i, r) - kU(c_n^i, \dfrac{\partial r}{\partial x_2}) \\ \qquad = kU\alpha(\dfrac{\partial r}{\partial x_2}, 1) + (c_n^{i-1}, r) \quad \forall r \in H_m\,. \end{cases} \tag{4.12}$$

Because such test functions $\mathbf{v}$ and r are dense in $\mathbf{V}$ and $\tilde{H}$, the above system holds for all $\mathbf{v} \in \mathbf{V}$ and $r \in \tilde{H}$. This means that $(\mathbf{u}_n^i, c_n^i)$, is a solution of (4.4). $\square$

By the modified Rothe's method, we now construct a sequence to approximate the solution of (4.2). To this end, for each positive integer n, we define two piecewise constant functions

$$\mathbf{U}_n(t) = \mathbf{u}_n^i \,, \quad C_n(t) = c_n^i \,, \quad t \in (t_{i-1}, t_i) \,, \quad i = 1, 2, \dots, n \,, \tag{4.13}$$

with $\mathbf{U}_n(0) = \mathbf{u}_0$ and $C_n(0) = c_0$. Notice that piecewise constant function does not have derivative in $L^2(I; L^2(\Omega))$. Therefore to approximate the time derivative in (3.3), we define the following two piecewise linear functions

$$\begin{aligned} \tilde{\mathbf{U}}_n(t) &= \mathbf{u}_n^i + (t - t_{i-1})(\mathbf{u}_n^i - \mathbf{u}_n^{i-1})/k \,, \\ \tilde{C}_n(t) &= c_n^i + (t - t_{i-1})(c_n^i - c_n^{i-1})/k \,, \\ t &\in (t_{i-1}, t_i] \,, \quad i = 1, 2, \dots, n \,, \end{aligned} \tag{4.14}$$

with $\tilde{\mathbf{U}}_n(0) = \mathbf{u}_0$ and $\tilde{C}_n(0) = c_0$. The following estimate is true for any integer $n > 0$.

Lemma 4.4. *Assume (4.5) is satisfied, then for any positive integer $n_0 \leq n$, we have the estimate*

$$\|\mathbf{u}_n^{n_0}\| + \|c_n^{n_0}\| + k \sum_{i=1}^{n_0} (\|\mathbf{u}_n^i\|_1^2 + \|c_n^i\|_1^2 + \|\delta \mathbf{u}_n^i\|_{\mathbf{V}'}^2 + \|c_n^i\|_{\tilde{H}'}^2) \leq C \,, \tag{4.15}$$

where C is a constant independent of n and n_0.

Proof. Taking $\mathbf{v} = \mathbf{u}_n^i$ and $r = c_n^i$ in (4.4), we deduce from (3.8) that

$$\begin{cases} (\delta \mathbf{u}_n^i, \mathbf{u}_n^i) + \Big(\nu(c_n^i + \alpha)\nabla \mathbf{u}_n^i, \nabla \mathbf{u}_n^i\Big) = -\Big(g\big(1 + \gamma c_n^i\big)i_2, \mathbf{u}_n^i\Big) + (\mathbf{f}_n^i, \mathbf{u}_n^i) \,, \\ (\delta c_n^i, c_n^i) + \theta a(c_n^i, c_n^i) - U(c_n^i, \dfrac{\partial c_n^i}{\partial x_2}) = \dfrac{U\alpha}{|\Omega|}(\dfrac{\partial c_n^i}{\partial x_2}, 1) \,. \end{cases} \tag{4.16}$$

Notice the identities

$$
\begin{cases}
(\mathbf{u}_n^i - \mathbf{u}_n^{i-1}, \mathbf{u}_n^i) = 1/2(\|\mathbf{u}_n^i - \mathbf{u}_n^{i-1}\|^2 + \|\mathbf{u}_n^i\|^2 - \|\mathbf{u}_n^{i-1}\|^2)\,, \\
(c_n^i - c_n^{i-1}, c_n^i) = 1/2(\|c_n^i - c_n^{i-1}\|^2 + \|c_n^i\|^2 - \|c_n^{i-1}\|^2)\,.
\end{cases}
\tag{4.17}
$$

Similarly to the way of obtaining (2.20), we multiply (4.16) and sum from $i = 1$ to n_0. Applying (1.21) and Young's inequality we find that

$$
\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{n_0} \|\mathbf{u}_n^i - \mathbf{u}_n^{i-1}\|^2 + \frac{1}{2}\|\mathbf{u}_n^{n_0}\|^2 + \frac{k\nu_*}{C_\Omega^2}\sum_{i=1}^{n_0}\|\mathbf{u}_n^i\|_1^2 \\
&\quad \leq k\sum_{i=1}^{n_0}\Big| -\Big(g(1+\gamma c_n^i)i_2, \mathbf{u}_n^i\Big) + (\mathbf{f}_n^i, \mathbf{u}_n^i)\Big| + \frac{1}{2}\|\mathbf{u}_n^0\|^2 \\
&\quad \leq k\sum_{i=1}^{n_0}\Big(\varepsilon\|\mathbf{u}_n^i\|^2 + C(\varepsilon)\big(\|\mathbf{f}_n^i\|^2 + \|c_n^i\|^2 + 1\big)\Big) + \frac{1}{2}\|\mathbf{u}_n^0\|^2
\end{aligned}
\tag{4.18}
$$

and

$$
\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{n_0} \|c_n^i - c_n^{i-1}\|^2 + \frac{1}{2}\|c_n^{n_0}\|^2 + \frac{k\theta}{C_\Omega^2}\sum_{i=1}^{n_0}\|c_n^i\|_1^2 \\
&\quad \leq k\sum_{i=1}^{n_0}|U(c_n^i, \frac{\partial c_n^i}{\partial x_2}) + U\alpha(\frac{\partial c_n^i}{\partial x_2}, 1)| + \frac{1}{2}\|c_n^0\|^2 \\
&\quad \leq k\sum_{i=1}^{n_0}\Big(U\|c_n^i\|_1^2 + \varepsilon\|c_n^i\|^2 + C(\varepsilon)\Big) + \frac{1}{2}\|c_n^0\|^2\,.
\end{aligned}
\tag{4.19}
$$

By (4.5), we may choose $\varepsilon > 0$ to be sufficiently small such that $\varepsilon < \dfrac{k\theta}{C_\Omega^2} - kU$. Then (4.19) gives

$$
\|c_n^{n_0}\|^2 + k\sum_{i=1}^{n_0}\|c_n^i\|_1^2 \leq C\,. \tag{4.20}
$$

In addition we have

$$
\|\mathbf{f}_n^i\|^2 \leq 1/k^2\left(\int_{t_{i-1}}^{t^i}\mathbf{f}(t)\;dt\right) \leq 1/k\int_{t_{i-1}}^{t^i}\|\mathbf{f}(t)\|^2\;dt\,.
$$

Hence

$$k\sum_{i=1}^{n_0}\|\mathbf{f}_n^i\|^2 \le \sum_{i=1}^{n_0}\int_{t_{i-1}}^{t^i}\|\mathbf{f}(t)\|^2\,dt \le \|\mathbf{f}\|_{L^2(I;\mathbf{L}^2(\Omega))}\,. \tag{4.21}$$

Substituting (4.20) and (4.21) into (4.18) we conclude that

$$\|\mathbf{u}_n^{n_0}\|^2 + k\sum_{i=1}^{n_0}\|\mathbf{u}_n^i\|_1^2 \le Ck\sum_{i=1}^{n_0}\left(\|c_n^i\|_1^2 + \|\mathbf{f}_n^i\|^2 + 1\right) + \frac{1}{2}\|\mathbf{u}_n^0\|^2 \le C\,. \tag{4.22}$$

To obtain an estimate for $\delta\mathbf{u}_n^i$ and δc_n^i, we first recall the following estimate ([52], p. 292, Lemma 3.3 and Lemma 3.4)

$$\|r\|_{L^4(\Omega)} \le 2^{\frac{1}{4}}\|r\|^{\frac{1}{2}}\|\nabla r\|^{\frac{1}{2}}\,,\quad \forall r\in H^1(\Omega)\,. \tag{4.23}$$

Applying Holder's inequality to (3.7) we find

$$\begin{aligned}
|B_3(\mathbf{u},c,r)| = |B_3(\mathbf{u},r,c)| &\le \|\mathbf{u}\|_{\mathbf{L}^4(\Omega)}\|\nabla r\|\|c\|_{L^4(\Omega)}\\
&\le \left(\sum_{i=1}^{2}\|\mathbf{u}_i\|_{L^4(\Omega)}^2\right)^{\frac{1}{2}}\|c\|_{L^4(\Omega)}\|r\|_1\\
&\le C\left(\sum_{i=1}^{2}\|\mathbf{u}_i\|\|\nabla\mathbf{u}_i\|\right)^{\frac{1}{2}}\|c\|^{\frac{1}{2}}\|\nabla c\|^{\frac{1}{2}}\|r\|_1\\
&\le C(\|\mathbf{u}\|\|\mathbf{u}\|_1\|c\|\|c\|_1)^{\frac{1}{2}}\|r\|_1\quad \forall\mathbf{u}\in\mathbf{V}\,,\quad \forall c,r\in H^1(\Omega)\,.
\end{aligned} \tag{4.24}$$

Similarly

$$|B_2(\mathbf{u},\mathbf{v},\mathbf{w})| \le C(\|\mathbf{u}\|\|\mathbf{u}\|_1\|\mathbf{v}\|\|\mathbf{v}\|_1)^{\frac{1}{2}}\|\mathbf{w}\|_1\quad \forall\mathbf{u}\in\mathbf{V}\,,\quad \forall\mathbf{v},\mathbf{w}\in\mathbf{H}_0^1(\Omega)\,. \tag{4.25}$$

Recall, from (4.22), that $\|\mathbf{u}_n^i\|$ and $\|c_n^i\|$ are uniformly bounded with respect to i. Combining (4.20), (4.22), (1.21), and (4.25), we apply Cauchy-Schwarz inequality to deduce from (4.3) that

$$
\begin{aligned}
(\delta \mathbf{u}_n^i,\mathbf{v}) &= -\Big(\nu(c_n^i+\alpha)\nabla \mathbf{u}_n^i,\nabla \mathbf{v}\Big) - B_2(\mathbf{u}_n^i,\mathbf{u}_n^i,\mathbf{v}) - \Big(g(1+\gamma c_n^i)i_2,\mathbf{v}\Big) + (\mathbf{f}_n^i,\mathbf{v}) \\
&\le \nu^*\|\mathbf{u}_n^i\|_1\|\mathbf{v}\|_1 + C\|\mathbf{u}_n^i\|\|\mathbf{u}_n^i\|_1\|\mathbf{v}\|_1 + C\|c_n^i\|_1\|\mathbf{v}\|_1 + \|\mathbf{f}_n^i - g i_2\|\|\mathbf{v}\|_1\,, \qquad (4.26)\\
&\le C\Big(\|\mathbf{u}_n^i\|_1 + \|c_n^i\|_1 + \|\mathbf{f}_n^i - g i_2\|\Big)\|\mathbf{v}\|_1 \quad \forall \mathbf{v}\in \mathbf{H}_0^1(\Omega)
\end{aligned}
$$

and

$$
\begin{aligned}
(\delta c_n^i,r) &= -\theta a(c_n^i,r) - B(\mathbf{u}_n^i,c_n^i,r) + U(c_n^i,\frac{\partial r}{\partial x_2}) + \frac{U\alpha}{|\Omega|}(\frac{\partial r}{\partial x_2},1) \\
&\le C\|c_n^i\|_1\|r\|_1 + C(\|\mathbf{u}_n^i\|\|\mathbf{u}_n^i\|_1\|c_n^i\|\|c_n^i\|_1)^{\frac12}\|r\|_1 + C\|c_n^i\|_1\|r\|_1 + C\|r\|_1 \qquad (4.27)\\
&\le C\Big(\|c_n^i\|_1 + \|\mathbf{u}_n^i\|_1 + 1\Big)\|r\|_1 \quad \forall r\in \tilde{H}\,.
\end{aligned}
$$

Consequently

$$
\begin{cases}
\|\delta \mathbf{u}_n^i\|_{\mathbf{V}'} \le C\Big(\|c_n^i\|_1 + \|\mathbf{u}_n^i\|_1 + \|\mathbf{f}_n^i\|\Big)\,,\\
\|\delta c_n^i\|_{\tilde{H}'} \le C\Big(\|c_n^i\|_1 + \|\mathbf{u}_n^i\|_1 + 1\Big)\,.
\end{cases}
$$

Using (4.20), (4.22), and (4.21)

$$
k\sum_{i=1}^{n_0}\Big(\|\delta \mathbf{u}_n^i\|_{\mathbf{V}'}^2 + \|\delta c_n^i\|_{\tilde{H}'}^2\Big) \le C\,.
$$

□

In view of Lemma 4.4 there exists a constant C such that

$$
\|\mathbf{U}_n\|_{L^2(I;\mathbf{V})}^2 + \|C_n\|_{L^2(I;\tilde{H})}^2 = k\sum_{i=1}^{n}(\|\mathbf{u}_n^i\|_1^2 + \|c_n^i\|_1^2) \le C\,. \qquad (4.28)
$$

Thus there exist $\mathbf{u}\in\mathbf{V}$, $c\in\tilde{H}$ such that

$$
\mathbf{U}_n \rightharpoonup \mathbf{u} \quad \text{in} \quad L^2(I;\mathbf{V}) \quad \text{and} \quad C_n \rightharpoonup c \quad \text{in} \quad L^2(I;\tilde{H})\,. \qquad (4.29)
$$

It is straight forward to verify that the piecewise constant function defined in (4.13) share the same weak and strong limits with the piecewise linear function in (4.14). As a result

$$\tilde{\mathbf{U}}_n \rightharpoonup \mathbf{u} \quad \text{in} \quad L^2(I;\mathbf{H}_0^1(\Omega)) \quad \text{and} \quad \tilde{C}_n \rightharpoonup c \quad \text{in} \quad L^2(I;\tilde{H})\,.$$

Moreover Lemma 4.4 implies that

$$\|\tilde{\mathbf{U}}_n'\|^2_{L^2(I;\mathbf{V}')} + \|\tilde{C}_n'\|^2_{L^2(I;\tilde{H}'} = k\sum_{i=1}^{n}(\|\delta\mathbf{u}_n^i\|^2_{\mathbf{V}} + \|\delta c_n^i\|^2_{\tilde{H}'}) \leq C\,.$$

Hence there exist $\bar{\mathbf{u}} \in L^2(I;\mathbf{V}')$ and $\bar{c} \in L^2(I;\tilde{H}')$ such that

$$\tilde{\mathbf{U}}_n' \rightharpoonup \bar{\mathbf{u}} \quad \text{in} \quad L^2(I;\mathbf{V}') \quad \text{and} \quad \tilde{C}_n' \rightharpoonup c \quad \text{in} \quad L^2(I;\tilde{H}')\,. \tag{4.30}$$

It is easy to verify that ([58], p. 356)

$$\bar{\mathbf{u}} = \mathbf{u}' \quad \text{and} \quad \bar{c} = c'\,. \tag{4.31}$$

Then in view of Lemma 2.24, we set $X_0 = \mathbf{V}$, $X = \mathbf{L}^2(\Omega) \cap \mathbf{V}$, and $X_1 = \mathbf{V}'$ to use the compact embedding

$$\{\mathbf{u} \in L^2(I;\mathbf{V});\ \mathbf{u}' \in L^q(I;\mathbf{V}')\} \hookrightarrow L^2(I;\mathbf{L}^2(\Omega) \cap \mathbf{V})\,,$$

and we set $X_0 = \tilde{H}$, $X = L_0^2(\Omega)$, and $X_1 = \tilde{H}'$ to obtain that

$$\{c \in L^2(I;\tilde{H});\ u' \in L^q(I;\tilde{H}')\} \hookrightarrow L^2(I;L_0^2(\Omega))\,.$$

Consequently

$$\tilde{\mathbf{U}}_n \to \mathbf{u} \quad \text{in} \quad L^2(I;\mathbf{L}^2(\Omega)) \quad \text{and} \quad \tilde{C}_n \to c \quad \text{in} \quad L^2(I;L^2(\Omega))\,.$$

Thus

$$\mathbf{U}_n \to \mathbf{u} \quad \text{in} \quad L^2(I;\mathbf{L}^2(\Omega)) \quad \text{and} \quad C_n \to c \quad \text{in} \quad L^2(I;L^2(\Omega))\,. \tag{4.32}$$

Armed with the above Lemmas and Lemma 2.12, we are now ready to show that $(\mathbf{u}, c)$ is a solution of (4.2) with weak derivative $\mathbf{u}'$, c' as defined in (4.30) and (4.31) .

Theorem 4.5 (Existence)**.** *Suppose* (4.5) *is satisfied. Given* $\mathbf{f} \in L^2(I;\mathbf{L}^2(\Omega))$*,* $\mathbf{u}_0 \in \mathbf{L}^2(\Omega)$*, and* $c_0 \in L^2(\Omega)$*, system* (4.2) *has a weak solution* $(\mathbf{u}, c) \in L^2(I;\mathbf{V}) \times L^2(I;\tilde{H})$ *with* $\mathbf{u}' \in L^2(I;\mathbf{V}')$ *and* $c' \in L^2(I;\tilde{H}')$*.*

Proof. Define

$$\mathbf{f}_n(t) = \mathbf{f}_n^i\,, \quad t \in (t_{i-1}, t_i]\,, \quad i = 1, 2, \ldots, n\,.$$

Proceeding as in Lemma 2.12, we have

$$\|\mathbf{f}_n - \mathbf{f}\|_{L^2(I;\mathbf{L}^2(\Omega))} \to 0 \quad \text{as } n \to \infty\,. \tag{4.33}$$

Construct the test functions of the form $\tilde{\mathbf{v}} = \mathbf{v}\varphi$ and $\tilde{r} = r\varphi$ with

$$\mathbf{v} \in (C_0^\infty(\Omega))^3 \cap \mathbf{V}\,, \quad r \in C_0^\infty(\Omega) \cap \tilde{H}\,, \quad \text{and} \quad \varphi(t) \in C_0^\infty(I)\,. \tag{4.34}$$

We have the following identities

$$\begin{aligned} &\sum_{i=1}^{n}(\mathbf{u}_n^i - \mathbf{u}_n^{i-1}, \mathbf{v})\varphi(t_i) \\ &\qquad = (\mathbf{u}_n^n, \mathbf{v})\varphi(t_n) - (\mathbf{u}_n^0, \mathbf{v})\varphi(t_1) - k\sum_{i=1}^{n-1}(\mathbf{u}_n^i, \mathbf{v})\Big(\varphi(t_{i+1}) - \varphi(t_i)\Big)/k \end{aligned}$$

and

$$\sum_{i=1}^{n}(c_n^i - c_n^{i-1}, r)\varphi(t_i)$$
$$= (c_n^n, r)\varphi(t_n) - (c_n^0, r)\varphi(t_1) - k\sum_{i=1}^{n-1}(c_n^i, r)\Big(\varphi(t_{i+1}) - \varphi(t_i)\Big)/k\,.$$

Multiplying (4.3) with $k\varphi(t_i)$ and summing from $i = 1$ to n, we obtain

$$\begin{aligned}
&(\mathbf{u}_n^n,\mathbf{v})\varphi(T) - (\mathbf{u}_n^0, \mathbf{v})\varphi(t_1) - k\sum_{i=1}^{n-1}(\mathbf{u}_n^i, \mathbf{v})\Big(\varphi(t_{i+1}) - \varphi(t_i)\Big)/k \\
&\quad + k\sum_{i=1}^{n}\Big(\nu(c_n^i + \alpha)\nabla\mathbf{u}_n^i), \nabla\mathbf{v}\Big)\varphi(t_i) + B_2(\mathbf{u}_n^i, \mathbf{u}_n^i, \mathbf{v})\varphi(t_i) \\
&= k\sum_{i=1}^{n}\Big(- \big(g(1 + c_n^i)i_2, \mathbf{v}\big) + (\mathbf{f}_n^i, \mathbf{v})\Big)\varphi(t_i)
\end{aligned}$$

and

$$\begin{aligned}
&(c_n^n,r)\varphi(T) - (c_n^0, r)\varphi(t_1) - k\sum_{i=1}^{n-1}(c_n^i, r)\Big(\varphi(t_{i+1}) - \varphi(t_i)\Big)/k \\
&\quad + k\sum_{i=1}^{n}\left(\theta a(c_n^i, r) + kB_3(\mathbf{u}_n^i, c_n^i, r) - U(c_n^i, \frac{\partial r}{\partial x_2})\right)\varphi(t_i) \\
&= k\sum_{i=1}^{n}\frac{U\alpha}{|\Omega|}(\frac{\partial r}{\partial x_2}, 1)\varphi(t_i)\,.
\end{aligned}$$

From the definition of $\mathbf{U}_n$, C_n, $\tilde{\mathbf{U}}_n$, and $\tilde{C}_n$ the above equations yield

$$\begin{cases} -(\mathbf{u}_0,\mathbf{v})\varphi(k) - \displaystyle\int_0^T (\mathbf{U}_n, \mathbf{v})\tilde{\varphi}_n \ dt \\ \qquad + \displaystyle\int_0^T \Big(\nu(C_n+\alpha)\nabla\mathbf{U}_n, \nabla\mathbf{v}\Big)\varphi_n \ dt + \int_0^T B_2(\mathbf{U}_n, \mathbf{U}_n, \mathbf{v})\varphi_n \ dt \\ \qquad = -\displaystyle\int_0^T (g(1+C_n)i_2, \mathbf{v})\varphi_n \ dt + \int_0^T (\mathbf{f}_n, \mathbf{v})\varphi_n \ dt\,, \\ -(c_0,r)\varphi(k) + \displaystyle\int_0^T (C_n, r)\tilde{\varphi}_n \ dt \\ \qquad + \displaystyle\int_0^T \theta a(C_n, r)\varphi_n \ dt + \int_0^T B_3(\mathbf{U}_n, C_n, r)\varphi_n \ dt \\ \qquad - \displaystyle\int_0^T U(C_n, \frac{\partial r}{\partial x_2})\varphi_n \ dt = \int_0^T \frac{U\alpha}{|\Omega|}(\frac{\partial r}{\partial x_2}, 1)\varphi_n \ dt\,. \end{cases} \tag{4.35}$$

The continuity of φ implies (notice that $\varphi(0) = 0$) that

$$(\mathbf{u}_0, \mathbf{v})\varphi(k) \to 0\,, \quad \text{and} \quad (c_0, \mathbf{v})\varphi(k) \to 0 \quad \text{as} \quad n \to \infty\,. \tag{4.36}$$

From Lemma 2.12, (4.28), and Holder's inequality it follows that

$$\Big|\int_0^T (\mathbf{U}_n, \mathbf{v})(\tilde{\varphi}_n - \varphi') \ dt\Big| \le \|\mathbf{v}\|\|\mathbf{U}_n\|_{L^2(I;\mathbf{L}^2(\Omega))}\|\tilde{\varphi}_n - \varphi'\|_{L^2(I)} \to 0 \quad \text{as} \quad n \to \infty$$

and

$$\Big|\int_0^T (C_n, r)(\tilde{\varphi}_n - \varphi') \ dt\Big| \le \|r\|\|C_n\|_{L^2(I;L^2(\Omega))}\|\tilde{\varphi}_n - \varphi'\|_{L^2(I)} \to 0 \quad \text{as} \quad n \to \infty\,.$$

Together with (4.29) we have

$$\begin{cases} \displaystyle\int_0^T (\mathbf{U}_n, \mathbf{v})\tilde{\varphi}_n \; dt = \int_0^T (\mathbf{U}_n, \mathbf{v})(\tilde{\varphi}_n - \varphi') \; dt + \int_0^T (\mathbf{U}_n, \mathbf{v})\varphi' \; dt \\ \qquad\qquad\qquad\qquad \displaystyle\to \int_0^T (\mathbf{u}, \mathbf{v})\varphi' \; dt \,, \\ \displaystyle\int_0^T (C_n, r)\tilde{\varphi}_n \; dt = \int_0^T (C_n, r)(\tilde{\varphi}_n - \varphi') \; dt + \int_0^T (C_n, r)\varphi' \; dt \\ \qquad\qquad\qquad\qquad \displaystyle\to \int_0^T (c, r)\varphi' \; dt \end{cases} \tag{4.37}$$

as n $\to \infty$. Next the Nemytskii property (1.21) leads to

$$\nu(C_n + \alpha) \to \nu(c + \alpha) \quad \text{in} \quad L^2(I; L^2(\Omega)) \quad \text{as} \quad n \to \infty \,.$$

Thus (1.21), (4.29), (4.28), Lemma 2.12, and Holder's inequality yield

$$\begin{aligned} &\Big| \int_0^T \Big(\nu(C_n + \alpha)\nabla \mathbf{U}_n, \nabla \mathbf{v}\Big)\varphi_n \; dt - \int_0^T \Big(\nu(C + \alpha)\nabla \mathbf{u}, \nabla \mathbf{v}\Big)\varphi_n \; dt\Big| \\ &\quad = \Big| \int_0^T \Big(\nu(C_n + \alpha)\nabla \mathbf{U}_n, \nabla \mathbf{v}\Big)(\varphi_n - \varphi) \; dt \\ &\qquad + \int_0^T \Big(\nu(C + \alpha)(\nabla \mathbf{U}_n - \nabla \mathbf{u}), \nabla \mathbf{v})\Big)\varphi \; dt \\ &\qquad + \int_0^T \Big(\big(\nu(C_n + \alpha) - \nu(C + \alpha)\big)\nabla \mathbf{U}_n, \nabla \mathbf{v}\Big)\varphi \; dt\Big| \\ &\quad \le \nu^* \|\mathbf{v}\|_1 \|\mathbf{U}_n\|_{L^2(I;\mathbf{V})} \|\varphi_n - \varphi\|_{L^2(I)} \\ &\qquad + \nu^* \Big| \int_0^T (\nabla \mathbf{U}_n - \nabla \mathbf{u}, \nabla \mathbf{v})\varphi \; dt\Big| \\ &\qquad + \|\mathbf{v}\|_1 \|\nu(C_n + \alpha) - \nu(C + \alpha)\|_{L^2(I;L^2(\Omega))} \|\mathbf{U}_n\|_{L^2(I;\mathbf{V})} \\ &\qquad \to 0 \,, \quad \text{as} \quad n \to \infty \,. \end{aligned} \tag{4.38}$$

Combing (4.28), Lemma 2.12, and Holder's inequality we let n tend ∞ to find

$$\begin{cases} \Big| \displaystyle\int_0^T \Big(g(1+C_n)i_2, \mathbf{v}\Big)(\varphi_n - \varphi)\ dt\Big| \le \Big(\|\mathbf{v}\|_1 \|C_n\|_{L^2(I;L^2(\Omega))} + C\Big)\|\varphi_n - \varphi\|_{L^2(I)} \to 0\,, \\ \Big| \displaystyle\int_0^T \theta a(C_n, r)(\varphi_n - \varphi)\ dt\Big| \le \theta \|r\|_1 \|C_n\|_{L^2(I;\mathbf{V})} \|\varphi_n - \varphi\|_{L^2(I)} \to 0\,, \\ \Big| \displaystyle\int_0^T U(C_n, \frac{\partial r}{\partial x_2})(\varphi_n - \varphi)\ dt\Big| \le U\|r\|_1 \|C_n\|_{L^2(I;L^2(\Omega))} \|\varphi_n - \varphi\|_{L^2(I)} \to 0\,, \\ \Big| \displaystyle\int_0^T \frac{U\alpha}{|\Omega|}(\frac{\partial r}{\partial x_2}, 1)(\varphi_n - \varphi)\ dt\Big| \le C\|r\|_1 \|\varphi_n - \varphi\|_{L^2(I)} \to 0\,. \end{cases}$$

Hence by (4.29) we conclude that as $n \to \infty$

$$\begin{cases} \displaystyle\int_0^T \Big(g(1+C_n)i_2, \mathbf{v}\Big)\varphi_n\ dt = \int_0^T \Big(g(1+C_n)i_2, \mathbf{v}\Big)(\varphi_n - \varphi)\ dt \\ \qquad\qquad + \displaystyle\int_0^T \Big(g(1+C_n)i_2, \mathbf{v}\Big)\varphi\ dt \\ \qquad\qquad \to \displaystyle\int_0^T \Big(g(1+C)i_2, \mathbf{v}\Big)\varphi\ dt\,, \\ \displaystyle\int_0^T \theta a(C_n, r)\varphi_n\ dt = \int_0^T \theta a(C_n, r)(\varphi_n - \varphi)\ dt + \int_0^T \theta a(C_n, r)\varphi\ dt \\ \qquad\qquad \to \displaystyle\int_0^T \theta a(C, r)\varphi\ dt\,, \\ \displaystyle\int_0^T U(C_n, \frac{\partial r}{\partial x_2})\varphi_n\ dt = \int_0^T U(C_n, \frac{\partial r}{\partial x_2})(\varphi_n - \varphi)\ dt + \int_0^T U(C_n, \frac{\partial r}{\partial x_2})\varphi\ dt \\ \qquad\qquad \to \displaystyle\int_0^T U(C, \frac{\partial r}{\partial x_2})\varphi\ dt\,, \\ \displaystyle\int_0^T \frac{U\alpha}{|\Omega|}(\frac{\partial r}{\partial x_2}, 1)\varphi_n\ dt \to \int_0^T \frac{U\alpha}{|\Omega|}(\frac{\partial r}{\partial x_2}, 1)\varphi\ dt\,. \end{cases} \tag{4.39}$$

Note that

$$\begin{aligned}
\int_0^T (\mathbf{f}_n, \mathbf{v})\varphi_n \, dt &= \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \Big(1/k \int_{t_{i-1}}^{t_i} \mathbf{f}(\tau) d\tau, \mathbf{v}\Big) \varphi(t_i) \, dt \\
&= \sum_{i=1}^{n} \left(\int_{t_{i-1}}^{t_i} \mathbf{f}(\tau) d\tau, \mathbf{v} \right) \varphi(t_i) \\
&= \sum_{i=1}^{n} \left(\int_{t_{i-1}}^{t_i} \mathbf{f}(\tau), \mathbf{v} \right) \varphi_n(\tau) \, d\tau \\
&= \int_0^T \Big(\mathbf{f}(t), \mathbf{v}\Big) \varphi_n(t) \, dt \, .
\end{aligned}$$

Together with (4.33) and Lemma 2.12

$$\begin{aligned}
&\Big| \int_0^T (\mathbf{f}_n, \mathbf{v})\varphi_n \, dt - \int_0^T (\mathbf{f}, \mathbf{v})\varphi \, dt \Big| \\
&\qquad \leq \Big| \int_0^T (\mathbf{f}, \mathbf{v})(\varphi_n - \varphi) \, dt \Big| \\
&\qquad \leq \|\mathbf{v}\|_1 \|\mathbf{f}\|_{L^2(I;\mathbf{L}^2(\Omega))} \|\varphi_n - \varphi\|_{L^2(I)} \to 0 \quad \text{as} \quad n \to \infty \, .
\end{aligned} \tag{4.40}$$

Next we show that

$$\int_0^T (B_2(\mathbf{U}_n, \mathbf{U}_n, \mathbf{v})\varphi_n \, dt \to \int_0^T B_2(\mathbf{u}, \mathbf{u}, \mathbf{v})\varphi \, dt \quad \text{as} \quad n \to \infty \, . \tag{4.41}$$

Write

$$\begin{aligned}
&\left(\int_0^T (B_2(\mathbf{U}_n, \mathbf{U}_n, \mathbf{v})\varphi_n \, dt - \int_0^T B_2(\mathbf{u}, \mathbf{u}, \mathbf{v})\varphi \, dt \right) \\
&= \int_0^T B_2(\mathbf{U}_n, \mathbf{U}_n, \mathbf{v})(\varphi_n - \varphi) \, dt + \int_0^T \Big(B_2(\mathbf{U}_n, \mathbf{U}_n, \mathbf{v}) - B_2(\mathbf{u}, \mathbf{u}, \mathbf{v}) \Big) \varphi \, dt \, .
\end{aligned}$$

By (4.25), Lemma 2.12, Lemma 4.4, and the uniform boundedness of $\|\mathbf{u}_n^i\|$ we have that

$$
\begin{aligned}
&|\int_0^T (B_2(\mathbf{U}_n, \mathbf{U}_n, \mathbf{v})(\varphi_n - \varphi)\ dt| \\
&\qquad\qquad \leq C|\int_0^T \|\mathbf{U}_n\|\|\mathbf{U}_n\|_1\|\mathbf{v}\|_1(\varphi_n - \varphi)\ dt| \\
&\qquad\qquad \leq C\|\mathbf{v}\|_1\|\mathbf{U}_n\|_{L^2(I;\mathbf{V})}\|\varphi_n - \varphi\|_{L^2(I)} \to 0 \quad \text{as} \quad n \to \infty\,.
\end{aligned}
$$

By (3.7),(4.32), and (4.34), we use integration by parts to find that as $n \to \infty$

$$
\begin{aligned}
\int_0^T B_2(\mathbf{U}_n, \mathbf{U}_n, \mathbf{v})\varphi\ dt &= -\int_0^T B_2(\mathbf{U}_n, \mathbf{v}, \mathbf{U}_n)\varphi\ dt \\
&= \sum_{i,j=1}^{2} \int_0^T \int_\Omega (\mathbf{U}_n)_i(\frac{\partial v_j}{\partial x_i})(\mathbf{U}_n)_j\ dx\ \varphi\ dt \\
&\to \sum_{i,j=1}^{2} \int_0^T \int_\Omega \mathbf{u}_i(\frac{\partial v_j}{\partial x_i})\mathbf{u}_j\ dx\ \varphi\ dt \\
&= -\int_0^T B_2(\mathbf{u}, \mathbf{v}, \mathbf{u})\varphi\ dt \\
&= \int_0^T B_2(\mathbf{u}, \mathbf{u}, \mathbf{v})\varphi\ dt\,,
\end{aligned}
$$

from which (4.41) follows. Analogously, (4.24), Lemma (4.4), and Holder's inequality yield

$$
\begin{aligned}
&|\int_0^T B_3(\mathbf{U}_n, C_n, r)(\varphi_n - \varphi)\ dt| \\
&\quad \leq C\int_0^T \|\mathbf{U}_n\|\|\mathbf{U}_n\|_1\|C_n\|\|C_n\|_1)^{\frac{1}{2}})\|\mathbf{v}\|_1|\varphi_n - \varphi|\ dt \\
&\quad \leq C\int_0^T (\|\mathbf{U}_n\|_1 + \|C_n\|_1)\|\mathbf{v}\|_1|\varphi_n - \varphi|\ dt \\
&\quad \leq C\|\mathbf{v}\|_1\Big(\|\mathbf{U}_n\|_{L^2(I;V)} + \|C_n\|_{L^2(I;\tilde{H})}\Big)\|\varphi_n - \varphi\|_{L^2(I)} \to 0 \quad \text{as} \quad n \to \infty
\end{aligned}
$$

and

$$
\int_0^T B_3(\mathbf{U}_n, \mathbf{U}_n, \mathbf{v})\varphi\ dt \to \int_0^T B_3(\mathbf{u}, \mathbf{u}, \mathbf{v})\varphi\ dt\,.
$$

Together the above estimates give

$$\int_0^T B(\mathbf{U}_n, C_n, r)\varphi_n \, dt \to \int_0^T B(\mathbf{u}, c, r)\varphi \, dt \,, \quad \text{as} \quad n \to \infty \,. \tag{4.42}$$

Combing (4.36), (4.37), (4.38), (4.39), (4.40), (4.41), and (4.42), we deduce from system (4.35) that

$$\begin{cases} -\displaystyle\int_0^T (\mathbf{u}, \tilde{\mathbf{v}}')dt + \int_0^T \Big(\nu(c+\alpha)\nabla\mathbf{u}, \nabla\tilde{\mathbf{v}}\Big) \, dt + \int_0^T B_2(\mathbf{u}, \mathbf{u}, \tilde{\mathbf{v}}) \, dt \\ \quad = -\displaystyle\int_0^T \Big(g(1+\gamma c)i_2, \tilde{\mathbf{v}}\Big) \, dt + \int_0^T (\mathbf{f}, \tilde{\mathbf{v}}) \, dt \,, \\ -\displaystyle\int_0^T (c, \tilde{r}') \, dt + \int_0^T \theta a(c, \tilde{r}) \, dt + \int_0^T B_3(\mathbf{u}, c, \tilde{r}) \, dt - \int_0^T U(c, \frac{\partial \tilde{r}}{\partial x_2}) \, dt \\ \quad = \displaystyle\int_0^T \frac{U\alpha}{|\Omega|}(\frac{\partial \tilde{r}}{\partial x_2}, 1) \, dt \,, \end{cases} \tag{4.43}$$

for any $\tilde{\mathbf{v}}$ and $\tilde{r}$ of the form in (4.11). Because such $\tilde{\mathbf{v}}$ and $\tilde{r}$ are dense in $L^2(I;\mathbf{V})$ and $L^2(I;\tilde{H})$ respectively, system (4.43) holds for all $\tilde{\mathbf{v}} \in L^2(I;\mathbf{V})$ and $\tilde{r} \in L^2(I;\tilde{H})$. Hence $(\mathbf{u}, c) \in \mathbf{V}\times\tilde{H}$ satisfies (4.2) in sense of distribution. Since the weak derivative $\mathbf{u}' \in L^2(I;\mathbf{V}')$ and $c' \in L^2(I;\tilde{H}')$ both exist, the pointwise version (4.2) is true for a.e.$t \in I$, i.e., $(\mathbf{u}, c)$ is the solution of (4.2). □

Uniqueness. Analogous to the steady case, the bilinear form $b(\cdot,\cdot)$ satisfies the inf-sup condition (3.2). Thus for each solution $(\mathbf{u}, c)$ of system (4.2), there exists a unique $p \in L^2(I; L^2_0(\Omega))$ satisfying system (4.1) (see [62], p. 59, Theorem I.4.1). As a result, to prove the uniqueness of solution to system (4.1), it suffices to prove that system (4.2) has a unique solution $(\mathbf{u}, c)$.

Theorem 4.6. *Suppose*

(H8) The hypotheses of Theorem 4.5 hold;

(H9) The viscosity $\nu(\cdot)$ is Lipschitz continuous, i.e., there exists $\nu_L > 0$ such that

$$|\nu(x_1) - \nu(x_2)| \leq \nu_L |x_1 - x_2| \quad \forall x_1, x_2 \in \mathbb{R}\,;$$

(H10) There exists a constant C_0 such that

$$\|\nabla \mathbf{u})\|_\infty \leq C_0 \quad \forall t \in I\,.$$

Then the solution $(\mathbf{u}, c)$ of system (4.2) is unique.

Proof. Let $(\mathbf{u}_1, c_2)$ and $(\mathbf{u}_2, c_2)$ be two different solutions of (4.2). Substituting $(\mathbf{u}, c)$ with $(\mathbf{u}_1, c_1)$ and $(\mathbf{u}_2, c_2)$ in (4.2) we have

$$\begin{cases} \langle \mathbf{u}_1', \mathbf{v}\rangle + \Big(\nu(c_1+\alpha)\nabla\mathbf{u}_1, \nabla\mathbf{v}\Big) + B_2(\mathbf{u}_1, \mathbf{u}_1, \mathbf{v}) \\ \qquad = -\Big(g(1+\gamma c_1)i_2, \mathbf{v}\Big) + (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}\,, \\ (c_1', r) + \theta a(c_1, r) + B_3(\mathbf{u}_1, c_1, r) - U(c_1, \dfrac{\partial r}{\partial x_2}) = U\alpha(\dfrac{\partial r}{\partial x_2}, 1) \quad \forall r \in \tilde{H}\,, \\ \mathbf{u}_1(0) = \mathbf{u}_0\,, \quad c_1(0) = c_0 - \alpha \end{cases} \tag{4.44}$$

and

$$\begin{cases} \langle \mathbf{u}_2', \mathbf{v}\rangle + \Big(\nu(c_2+\alpha)\nabla\mathbf{u}_2, \nabla\mathbf{v}\Big) + B_2(\mathbf{u}_2, \mathbf{u}_2, \mathbf{v}) \\ \qquad = -\Big(g(1+\gamma c_2)i_2, \mathbf{v}\Big) + (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}\,, \\ (c_2', r) + \theta a(c_2, r) + B_3(\mathbf{u}_2, c_2, r) - U(c_2, \dfrac{\partial r}{\partial x_2}) = U\alpha(\dfrac{\partial r}{\partial x_2}, 1) \quad \forall r \in \tilde{H}\,, \\ \mathbf{u}_2(0) = \mathbf{u}_0\,, \quad c_2(0) = c_0 - \alpha\,. \end{cases} \tag{4.45}$$

Subtracting (4.45) from (4.44) with $\mathbf{v} = \mathbf{u}_1 - \mathbf{u}_2$ and $r = c_1 - c_2$, we find that

$$\begin{aligned} &\langle (\mathbf{u}_1 - \mathbf{u}_2)', \mathbf{u}_1 - \mathbf{u}_2 \rangle + \Big(\nu(c_1 + \alpha)\nabla \mathbf{u}_1 - \nu(c_2 + \alpha)\nabla \mathbf{u}_2, \nabla(\mathbf{u}_1 - \mathbf{u}_2) \Big) \\ &\qquad + B_2(\mathbf{u}_1, \mathbf{u}_1, \mathbf{u}_1 - \mathbf{u}_2) - B_2(\mathbf{u}_2, \mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2) \\ &\quad = - g\gamma \Big((c_1 - c_2) i_2, \mathbf{u}_1 - \mathbf{u}_2 \Big) \end{aligned} \tag{4.46}$$

and

$$\begin{aligned} \langle (c_1 - c_2)', c_1 - c_2 \rangle &+ \theta a(c_1 - c_2, c_1 - c_2) \\ &+ B_3(\mathbf{u}_1, c_1, c_1 - c_2) - B_3(\mathbf{u}_2, c_2, c_1 - c_2) \\ &- U\Big(c_1 - c_2, \frac{\partial (c_1 - c_2)}{\partial x_2} \Big) \\ = 0\,. \end{aligned} \tag{4.47}$$

From (3.8)

$$\begin{cases} B_2(\mathbf{u}_1, \mathbf{u}_1, \mathbf{u}_1 - \mathbf{u}_2) - B_2(\mathbf{u}_2, \mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2) = B_2(\mathbf{u}_1 - \mathbf{u}_2, \mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2)\,, \\ B_3(\mathbf{u}_1, c_1, c_1 - c_2) - B_3(\mathbf{u}_2, c_2, c_1 - c_2) = B_3(\mathbf{u}_1 - \mathbf{u}_2, c_2, c_1 - c_2)\,. \end{cases}$$

Using (3.4) and(4.24), we apply Young's inequality to deduce from (4.47) that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|c_1 - c_2\|^2 + \frac{\theta}{C_\Omega^2}\|c_1 - c_2\|_1^2 \\ &\quad \le \langle (c_1 - c_2)', c_1 - c_2 \rangle + \theta a(c_1 - c_2, c_1 - c_2) \\ &\quad \le |B_3(\mathbf{u}_1 - \mathbf{u}_2, c_2, c_1 - c_2)| + U\|c_1 - c_2\|\|c_1 - c_2\|_1 \\ &\quad \le C\Big(\|\mathbf{u}_1 - \mathbf{u}_2\|\|\mathbf{u}_1 - \mathbf{u}_2\|_1 \|c_2\|\|c_2\|_1 \Big)^{\frac{1}{2}} \|c_1 - c_2\|_1 + U\|c_1 - c_2\|_1^2 \\ &\quad \le (\varepsilon + U)\|c_1 - c_2\|_1^2 + C(\varepsilon)\Big(\|\mathbf{u}_1 - \mathbf{u}_2\|\|\mathbf{u}_1 - \mathbf{u}_2\|_1 \|c_2\|\|c_2\|_1 \Big)\,. \end{aligned} \tag{4.48}$$

We rewrite (4.46) in the following form

$$
\begin{aligned}
&\langle(\mathbf{u}_1-\mathbf{u}_2)',\mathbf{u}_1-\mathbf{u}_2\rangle+\Big(\big(\nu(c_1+\alpha)-\nu(c_2+\alpha)\big)\nabla\mathbf{u}_1,\nabla(\mathbf{u}_1-\mathbf{u}_2)\Big)\\
&\qquad\qquad+\Big(\nu(c_2+\alpha)\nabla(\mathbf{u}_1-\mathbf{u}_2),\nabla(\mathbf{u}_1-\mathbf{u}_2)\Big)+B_2(\mathbf{u}_1-\mathbf{u}_2,\mathbf{u}_2,\mathbf{u}_1-\mathbf{u}_2)\\
&\qquad=-\,g\gamma\Big((c_1-c_2)i_3,\mathbf{u}_1-\mathbf{u}_2\Big)\,.
\end{aligned}
$$

Then using (1.21), (3.4), (4.25), (H9), and Young's inequality yield that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\mathbf{u}_1-\mathbf{u}_2\|^2+\frac{\nu_*}{C_\Omega^2}\|\mathbf{u}_1-\mathbf{u}_2\|_1^2\\
&\quad\leq\langle(\mathbf{u}_1-\mathbf{u}_2)',\mathbf{u}_1-\mathbf{u}_2\rangle+\Big(\nu(c_2+\alpha)\nabla(\mathbf{u}_1-\mathbf{u}_2),\nabla(\mathbf{u}_1-\mathbf{u}_2)\Big)\\
&\quad\leq\bigg|\Big(\big(\nu(c_1+\alpha)-\nu(c_2+\alpha)\big)\nabla\mathbf{u}_1,\nabla(\mathbf{u}_1-\mathbf{u}_2)\Big)\\
&\qquad+B_2(\mathbf{u}_1-\mathbf{u}_2,\mathbf{u}_2,\mathbf{u}_1-\mathbf{u}_2)+g\gamma\Big((c_1-c_2)i_3,\mathbf{u}_1-\mathbf{u}_2\Big)\bigg|\\
&\quad\leq C_0\nu_L\|c_1-c_2\|\|\mathbf{u}_1-\mathbf{u}_2\|_1+\|\mathbf{u}_1-\mathbf{u}_2\|\|c_1-c_2\|\\
&\qquad+C\Big(\|\mathbf{u}_1-\mathbf{u}_2\|\|\mathbf{u}_1-\mathbf{u}_2\|_1\|\mathbf{u}_2\|\|\mathbf{u}_2\|_1\Big)^{\frac{1}{2}}\|\mathbf{u}_1-\mathbf{u}_2\|_1\\
&\quad\leq\varepsilon\|\mathbf{u}_1-\mathbf{u}_2\|_1^2+C(\varepsilon)\Big(\|\mathbf{u}_1-\mathbf{u}_2\|\|\mathbf{u}_1-\mathbf{u}_2\|_1\|\mathbf{u}_2\|\|\mathbf{u}_2\|_1+\|c_1-c_2\|^2\Big)\,.
\end{aligned}
\tag{4.49}
$$

Hypothesis (4.5) implies that we may choose a sufficiently small $\varepsilon>0$ such that

$$
\varepsilon<\min\{\frac{\nu_*}{C_\Omega^2},\frac{\theta}{C_\Omega^2}-U\}\,.
$$

Taking the sum of (4.48) and (4.49) and applying Young's inequality we find that

$$\begin{aligned}
&\frac{d}{dt}\|\mathbf{u}_1-\mathbf{u}_2\|^2+\|\mathbf{u}_1-\mathbf{u}_2\|_1^2+\frac{d}{dt}\|c_1-c_2\|^2+\|c_1-c_2\|_1^2\\
&\leq C\Big(\|\mathbf{u}_1-\mathbf{u}_2\|\|\mathbf{u}_1-\mathbf{u}_2\|_1\|c_2\|\|c_2\|_1\\
&\quad+\|\mathbf{u}_1-\mathbf{u}_2\|\|\mathbf{u}_1-\mathbf{u}_2\|_1\|\mathbf{u}_2\|\|\mathbf{u}_2\|_1\\
&\quad+\|c_1-c_2\|^2\Big)\\
&\leq \varepsilon\|\mathbf{u}_1-\mathbf{u}_2\|_1^2\Big(\|c_2\|^2+\|\mathbf{u}_2\|^2\Big)\\
&\quad+C(\varepsilon)\|\mathbf{u}_1-\mathbf{u}_2\|^2\Big(\|c_2\|_1^2+\|\mathbf{u}_2\|_1^2\Big)\\
&\quad+C\|c_1-c_2\|^2\,.
\end{aligned}$$

Proceeding as in Lemma 4.4, we can show that $\|\mathbf{u}_2\|$ and $\|c_2\|$ are uniformly bounded with respect to t. Hence choosing $\varepsilon>0$ such that

$$\varepsilon<\frac{1}{\max\{\|c_2\|^2\,,\|\mathbf{u}_2\|^2\}}\,,$$

we obtain that

$$\begin{aligned}
&\frac{d}{dt}\|\mathbf{u}_1-\mathbf{u}_2\|^2+\frac{d}{dt}\|c_1-c_2\|^2\\
&\qquad\leq C\|\mathbf{u}_1-\mathbf{u}_2\|^2\Big(\|c_2\|_1^2+\|\mathbf{u}_2\|_1^2\Big)+C\|c_1-c_2\|^2\,.
\end{aligned}$$

Using Gronwall's inequality we conclude that for any $t\in I$ (recall that both $\|\mathbf{u}_2\|_1^2$ and $\|c_2\|_1^2$ are integrable on I)

$$\begin{aligned}
\|\mathbf{u}_1-\mathbf{u}_2\|^2(t)+&\|c_1-c_2\|^2(t)\\
&\leq\|\mathbf{u}_1-\mathbf{u}_2\|^2(0)\exp(\int_0^t\Big(\|\mathbf{u}_2\|_1^2+\|c_2\|_1^2\Big)\,ds\\
&\quad+C\|c_1-c_2\|^2(0)\,.
\end{aligned}$$

Thus

$$\|\mathbf{u}_1 - \mathbf{u}_2\| = \|c_1 - c_2\| = 0\,, \quad a.e.t \in I\,. \tag{4.50}$$

$\square$

Remark 4.7. Comparing Theorem 4.6 with Theorem 3.7, we can see condition (H7) is not needed to prove the uniqueness of the solution, i.e., the uniqueness of solution of time dependent bioconvection can be obtained under weaker conditions than in the steady case. This is general for partial differential equations with solution dependent coefficient. We also note that we restrict the equation to a two dimensional model.

4.2 Numerical approximation

In this section, we consider the semi-discrete finite element approximation to solutions of (4.1). Throughout this section, we assume that the weak solution $(\mathbf{u}, p, c)$ of (3.2) exists and is unique.

We use the same finite element spaces as for the steady bioconvection. Let τ_h be a family of quasi-uniform triangulations of the convex polygonal domain Ω satisfying

$$\max_{\tau\in\tau_h} \operatorname{diam} \tau \le h\,,$$

where h refers to the mesh size. Let $\mathbf{X}_h$, M_h, and S_h be the corresponding finite dimensional subspaces of $\mathbf{H}_0^1(\Omega)$, $L_0^2(\Omega)$, and $\tilde{H}$, respectively, that satisfy the following approximation properties

$$\inf_{\mathbf{v_h}\in\mathbf{X}_h} \|\mathbf{v} - \mathbf{v}_h\|_1 \le Ch^s\|\mathbf{v}\|_{s+1} \qquad \forall \mathbf{v} \in \mathbf{H}^{s+1}(\Omega)\,, \quad 0 < s \le k\,, \tag{4.51}$$

$$\inf_{q_h\in M^h} \|q - q_h\| \le Ch^s\|q\|_s \qquad \forall q \in H^s(\Omega)\,, \quad 0 < s \le k\,, \tag{4.52}$$

$$\inf_{t_h\in S_h} \|t - t_h\|_1 \le Ch^s\|t\|_{s+1} \qquad \forall t \in H^{s+1}(\Omega)\,, \quad 0 < s \le k\,, \tag{4.53}$$

for $k = 2, 3$. For the construction of these spaces, see [63, 62, 52]. Let $\mathbf{U}_h$ and C_h be the approximation of the initial conditions $\mathbf{u}_0$ and c_0 in $\mathbf{X}_h$ and S_h respectively. The semi-discrete finite dimensional approximation of (4.1) is defined as follows.

Given $\mathbf{U}_h \in \mathbf{X}_h$, $C_h \in S_h$, find $(\mathbf{u}_h, p_h, c_h) \in \mathbf{X}_h \times M_h \times S_h$ such that for any $t \in I$

$$\begin{cases} \langle \mathbf{u}_h', \mathbf{v}\rangle + \Big(\nu(c_h + \alpha)\nabla \mathbf{u}_h, \nabla \mathbf{v}\Big) + B_2(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}) + b(p_h, \mathbf{v}) \\ \quad = -\Big(g(1+\gamma c_h)\Big) i_2, \mathbf{v}) + (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{X}_h\,, \\ b(q, \mathbf{u}_h) = 0 \quad \forall q \in M_h\,, \\ \langle c_h', r\rangle + \theta a(c_h, r) + B_3(\mathbf{u}_h, c_h, r) - U(c_h, \dfrac{\partial r}{\partial x_2}) = 0 \quad \forall r \in S_h\,, \\ \mathbf{u}_h(0) = \mathbf{U}_h\,, \quad c_h(0) = C_h\,. \end{cases} \tag{4.54}$$

As for the steady biocovnection, we assume that these finite spaces have explicit bases and that a discrete version of the inf-sup condition (3.26) is satisfied, i.e., for some $\beta > 0$,

$$\sup_{\mathbf{v} \in \mathbf{X}_h} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_{\mathbf{X}_h}} \geq \beta \|q\|_{M_h} \quad \forall q \in M_h\,. \tag{4.55}$$

For the construction of these spaces, see [63, 62, 52]. Define the discrete divergence free space

$$\mathbf{V}_h = \{\mathbf{v} \in \mathbf{X}_h \;:\quad (\nabla \cdot \mathbf{v}, q_h) = 0\,, \quad \forall q_h \in M_h\}$$

and the auxiliary forms $\hat{B}_2$ and $\hat{B}_3$

$$\begin{cases} \hat{B}_2(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \dfrac{1}{2} B_2(\mathbf{u}, \mathbf{v}, \mathbf{w}) - \dfrac{1}{2} B_2(\mathbf{u}, \mathbf{w}, \mathbf{v})\,, \\ \hat{B}_3(\mathbf{u}, c, r) = \dfrac{1}{2} B_3(\mathbf{u}, c, r) - \dfrac{1}{2} B_3(\mathbf{u}, r, c)\,. \end{cases}$$

We recall the following properties

$$\begin{aligned} \hat{B}_2(\mathbf{u}, \mathbf{v}, \mathbf{w}) &= B_2(\mathbf{u}, \mathbf{w}, \mathbf{v}) \quad \forall \mathbf{v}, \mathbf{w} \in \mathbf{H}_0^1(\Omega)\,, \\ \hat{B}_3(\mathbf{u}, c, r) &= B_3(\mathbf{u}, c, r) \quad \forall \mathbf{u} \in \mathbf{V}\,, \quad \forall c, r \in \tilde{H}\,, \end{aligned} \tag{4.56}$$

and the identities

$$\hat{B}_2(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0\,, \quad \hat{B}_3(\mathbf{u}, c, c) = 0 \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{H}_0^1(\Omega)\,, \quad \forall c \in \tilde{H}\,, \tag{4.57}$$

and the tricontinuous properties

$$\begin{cases} \hat{B}_2(\mathbf{u}, \mathbf{v}, \mathbf{w}) \le C_{B_2} \|\mathbf{u}\|_1 \|\mathbf{v}\|_1 \|\mathbf{w}\|_1 \quad \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{H}_0^1(\Omega)\,, \\ \hat{B}_3(\mathbf{u}, c, r) \le C_{B_3} \|\mathbf{u}\|_1 \|c\|_1 \|r\|_1 \quad \forall \mathbf{u} \in \mathbf{H}_0^1(\Omega)\,, \quad \forall c, r \in H^1(\Omega)\,. \end{cases} \tag{4.58}$$

It is straight forward to verify that the discrete versions of estimates (4.25) and (4.24) hold, that is,

$$|B_3(\mathbf{u}, c, r)| \le C(\|\mathbf{u}\| \|\mathbf{u}\|_1 \|c\| \|c\|_1)^{\frac{1}{2}} \|r\|_1 \quad \forall \mathbf{u} \in \mathbf{V}\,, \quad \forall c, r \in H^1(\Omega)\,, \tag{4.59}$$

and

$$|B_2(\mathbf{u}, \mathbf{v}, \mathbf{w})| \le C(\|\mathbf{u}\| \|\mathbf{u}\|_1 \|\mathbf{v}\| \|\mathbf{v}\|_1)^{\frac{1}{2}} \|\mathbf{w}\|_1 \quad \forall \mathbf{u} \in \mathbf{V}\,, \quad \mathbf{v}\,, \mathbf{w} \in \mathbf{H}_0^1(\Omega)\,. \tag{4.60}$$

We first consider the discrete version of (4.2).

Find a pair $(\mathbf{u}_h, c_h)$ such that for $\forall t \in I$,

$$\begin{cases} (\mathbf{u}_h', \mathbf{v}) + \Big(\nu(c_h + \alpha)\nabla \mathbf{u}_h, \nabla \mathbf{v}\Big) + \hat{B}_2(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}) \\ \quad = -(g(1 + \gamma c_h) i_2, \mathbf{v}) + (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}_h\,, \\ (c_h', r) + \theta a(c_h, r) + \hat{B}_3(\mathbf{u}_h, c_h, r) - U(c_h, \dfrac{\partial r}{\partial x_2}) = U\alpha(\dfrac{\partial r}{\partial x_2}, 1) \quad \forall r \in S_h\,. \end{cases} \tag{4.61}$$

The existence of the above scheme is guaranteed by general ordinary differential equation theory. Taking $\mathbf{v} = \mathbf{u}_h$ and $r = c_h$ in (4.61), using (1.21), (1.22), and (3.31), and applying

Young's inequality we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\mathbf{u}_h\|^2 + \frac{\nu_*}{C_\Omega^2}\|\mathbf{u}_h\|_1^2 \\
&\qquad \le (\mathbf{u}_h', \mathbf{u}_h) + \Big(\nu(c_h+\alpha)\nabla\mathbf{u}_h, \nabla\mathbf{u}_h\Big) \\
&\qquad \le |-\Big(g(1+\gamma c_h)i_2, \mathbf{u}_h\Big) + (\mathbf{f}, \mathbf{u}_h)| \le \varepsilon\|\mathbf{u}_h\|_1^2 + C\Big(\|\mathbf{f}-gi_2\|^2 + \|c_h\|^2\Big)
\end{aligned} \tag{4.62}$$

and

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|c_h\|^2 + \frac{\theta}{C_\Omega^2}\|c_h\|_1^2 &\le (c_h', c_h) + \theta a(c_h, c_h) \\
&\le |U(c_h, \frac{\partial c_h}{\partial x_2}) + U\alpha(\frac{\partial c_h}{\partial x_2}, 1)| \\
&\le (\varepsilon + U)\|c_h\|_1^2 + C\,.
\end{aligned} \tag{4.63}$$

By (4.5) we can choose small $\varepsilon > 0$ such that $\varepsilon < \dfrac{\theta}{C_\Omega^2} - U$. Then (4.63) gives

$$\frac{d}{dt}\|c_h\|^2 + \|c_h\|_1^2 \le C\,.$$

Thus (4.62) yields

$$\frac{d}{dt}\|\mathbf{u}_h\|^2 + \|\mathbf{u}_h\|_1^2 \le \|\mathbf{f}-gi_2\|^2 + \|c_h\|^2 \le C\,.$$

Integrating (4.62) and (4.63) with respect to t and taking the sum, we obtain the estimate

$$\|\mathbf{u}_h\|^2 + \|c_h\|^2 + \int_0^t \Big(\|\mathbf{u}_h\|_1^2 + \|c_h\|_1^2\Big)\,d\tau \le C \quad \forall t \in I\,. \tag{4.64}$$

Following the same argument, we establish a similar estimate for the exact solution $(\mathbf{u}, c)$, that is

$$\|\mathbf{u}\|^2 + \|c\|^2 + \int_0^t \Big(\|\mathbf{u}\|_1^2 + \|c\|_1^2\Big)\,ds \le C \quad \forall t \in I\,. \tag{4.65}$$

After $(\mathbf{u}_h, c_h)$ are computed, we compute $p_h \in M_h$ by solving

$$\begin{aligned}(p_h, \nabla \cdot \mathbf{v}) =& (\mathbf{u}_h', \mathbf{v}) + \Big(\nu(c_h+\alpha)\nabla \mathbf{u}_h, \nabla \mathbf{v}\Big) + B_2(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}) \\ &+ \Big(g(1+\gamma c_h) i_2, \mathbf{v}\Big) - (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{X}_h\,.\end{aligned} \tag{4.66}$$

As in the steady case, by the property of Lagrange multiplier and (4.55), the above equation is always solvable and the solution $p_h \in M_h$ is unique in the quotient space M_h/N_h where

$$N_h = \{q_h \in M_h\, : (q_h, \nabla \cdot \mathbf{v}) = 0 \quad \forall \mathbf{v} \in \mathbf{X}_h\}\,.$$

In this way, the pressure p_h depends continuously on the discrete solution $\mathbf{u}_h$.

To obtain the error estimates, we use the Ritz Galerkin projections ([62], p. 132-139) $r_h : \mathbf{H}_0^1(\Omega) \to \mathbf{V}_h$, $s_h : \tilde{H} \to S_h$, and the L^2 projection $\pi_h : L_0^2(\Omega) \to M_h$ to split the errors into two parts:

$$\begin{cases} \mathbf{u} - \mathbf{u}_h = \mathbf{u} - r_h\mathbf{u} + r_h\mathbf{u} - \mathbf{u}_h := \rho_{\mathbf{u}}^h + \theta_{\mathbf{u}}^h\,, \\ p - p_h = p - \pi_h p + \pi_h p - p_h := \rho_p^h + \theta_p^h\,, \\ c - c_h = c - s_h c + s_h c - c_h := \rho_c^h + \theta_c^h\,. \end{cases} \tag{4.67}$$

The convergence of the projection error is guaranteed by (4.51), that is

$$\|\rho_{\mathbf{u}}^h\|_1 \to 0\,, \quad \|\rho_p^h\|_1 \to 0\,, \quad \|\rho_c^h\|_1 \to 0 \quad \text{as} \quad h \to 0\,. \tag{4.68}$$

We also have the following estimate for the projection

$$\|r_h\mathbf{u}\|_1 \le C(\|\mathbf{u}\|_1)\,, \quad \|s_h c\|_1 \le C(\|c\|_1)\,, \quad \|\pi_h p\| \le C(\|p\|_1)\,. \tag{4.69}$$

Then the main convergence theorem of the numerical approximation is stated as follows.

Theorem 4.8. *Assume that*

(H11) The assumptions of Theorem 4.6 hold;

(H12) $\mathbf{u} \in C^1(I; \mathbf{H}_0^1(\Omega)) \cap \mathbf{V}$ *and* $c \in C^1(I; H_0^1(\Omega)) \cap \tilde{H}$ *(see [52], p. 211).*

Then the solution $(\mathbf{u}_h, c_h)$ *converges to the exact solution* $(\mathbf{u}, c)$*, i.e.,*

$$\|\mathbf{u} - \mathbf{u}_h\| + \|c - c_h\| \to 0 \quad \text{as} \quad h \to 0\,.$$

Proof. In light of (4.68), it suffices to estimate $\theta_{\mathbf{u}}^h$, θ_p^h and θ_c^h. Subtracting (4.1) from (4.54) with $\mathbf{v} = \theta_{\mathbf{u}}^h$, $r = \theta_c$ we have that

$$\begin{aligned}(\mathbf{u}_h' - \mathbf{u}', \theta_{\mathbf{u}}^h) + &\Big(\nu(c_h + \alpha)\nabla \mathbf{u}_h, \nabla \theta_{\mathbf{u}}^h\Big) - \Big(\nu(c + \alpha)\nabla \mathbf{u}, \nabla \theta_{\mathbf{u}}^h\Big) \\ &+ \hat{B}_2(\mathbf{u}_h, \mathbf{u}_h, \theta_{\mathbf{u}}^h) - \hat{B}_2(\mathbf{u}, \mathbf{u}, \theta_{\mathbf{u}}^h) \\ &+ b(p - p_h, \theta_{\mathbf{u}}^h) \\ &= - g\gamma\Big((c_h - c))i_2, \theta_{\mathbf{u}}^h\Big)\end{aligned} \tag{4.70}$$

and

$$(c_h' - c', \theta_c^h) + \theta a(c_h - c, \theta_c^h) + \hat{B}_3(\mathbf{u}_h, c_h, \theta_c^h) - \hat{B}_3(\mathbf{u}, c, \theta_c^h) - U(c_h - c, \frac{\partial \theta_c^h}{\partial x_2}) = 0\,. \tag{4.71}$$

Notice that $\theta_{\mathbf{u}}^h \in \mathbf{V}_h$ and $\theta_p^h \in M_h$. Therefore the definition of $\mathbf{V}_h$ implies that $b(\theta_p^h, \theta_{\mathbf{u}}^h) = 0$, i.e.,

$$b(p - p_h, \theta_{\mathbf{u}}^h) = b(\rho_p^h, \theta_{\mathbf{u}}^h)\,.$$

The following identities are guaranteed by (4.57) .

$$\hat{B}_2(\mathbf{u}_h, \mathbf{u}_h, \theta_{\mathbf{u}}^h) - \hat{B}_2(\mathbf{u}, \mathbf{u}, \theta_{\mathbf{u}}^h) = \hat{B}_2(\theta_{\mathbf{u}}^h, \mathbf{u}_h, \theta_{\mathbf{u}}^h) + \hat{B}_2(r_h\mathbf{u}, \rho_{\mathbf{u}}^h, \theta_{\mathbf{u}}^h) + \hat{B}_2(\rho_{\mathbf{u}}^h, \mathbf{u}, \theta_{\mathbf{u}}^h)$$

and

$$\hat{B}_3(\mathbf{u}_h, c_h, \theta_c^h) - \hat{B}_3(\mathbf{u}, c, \theta_c^h) = \hat{B}_3(\theta_{\mathbf{u}}^h, c_h, \theta_c^h) + \hat{B}_3(r_h\mathbf{u}, \rho_c^h, \theta_c^h) + \hat{B}_3(\rho_{\mathbf{u}}^h, c, \theta_c^h)\,.$$

Then combing (1.21), (3.4), (4.60), (4.59), (4.64), (4.65), and (H12), we apply Young's inequality to deduce from (4.70) and (4.71) that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\theta_u^h\|^2 + \frac{\nu_*}{C_\Omega^2}\|\theta_{\mathbf{u}}^h\|_1^2 \\
&\quad \leq ((\theta_{\mathbf{u}}^h)', \theta_{\mathbf{u}}^h) + \Big(\nu(c_h+\alpha)\nabla\theta_{\mathbf{u}}^h, \nabla\theta_{\mathbf{u}}^h\Big) \\
&\quad \leq |((\rho_{\mathbf{u}}^h)', \theta_u^h) + \Big(\nu(c+\alpha)\nabla\rho_{\mathbf{u}}^h, \nabla\theta_{\mathbf{u}}^h\Big) + \Big(\big(\nu(c_h+\alpha) - \nu(c+\alpha)\big)\nabla\mathbf{u}, \nabla\theta_u^h\Big) \\
&\qquad + \hat{B}_2(\theta_{\mathbf{u}}^h, \mathbf{u}_h, \theta_{\mathbf{u}}^h) + \hat{B}_2(r_h\mathbf{u}, \rho_{\mathbf{u}}^h, \theta_{\mathbf{u}}^h) + \hat{B}_2(\rho_{\mathbf{u}}^h, \mathbf{u}, \theta_{\mathbf{u}}^h) \\
&\qquad + b(\rho_p^h, \theta_{\mathbf{u}}^h) + g\gamma\Big((c_h - c)i_2, \theta_{\mathbf{u}}^h\Big)| \\
&\quad \leq \|(\rho_{\mathbf{u}}^h)'\|\|\theta_{\mathbf{u}}^h\| + \nu^*\|\theta_{\mathbf{u}}^h\|_1\|\rho_{\mathbf{u}}^h\|_1 + C_0\nu_L\|c - c_h\|\|\theta_{\mathbf{u}}^h\|_1 \\
&\qquad + \|\rho_p^h\|\|\theta_{\mathbf{u}}^h\|_1 + g\gamma\|c_h - c\|\|\theta_{\mathbf{u}}^h\| \\
&\qquad + C\|\theta_{\mathbf{u}}\|_1\Big((\|\theta_{\mathbf{u}}^h\|\|\theta_{\mathbf{u}}^h\|_1\|\mathbf{u}_h\|\|\mathbf{u}_h\|_1)^{\frac{1}{2}} + \|r_h\mathbf{u}\|_1\|\rho_{\mathbf{u}}^h\|_1 + \|\rho_{\mathbf{u}}^h\|_1\|\mathbf{u}\|_1\Big) \\
&\quad \leq \varepsilon\|\theta_{\mathbf{u}}\|_1^2 + C(\|\mathbf{u}\|_1, \varepsilon)\Big(\|(\rho_{\mathbf{u}}^h)'\|^2 + \|\rho_{\mathbf{u}}^h\|_1^2 + \|\theta_{\mathbf{u}}\|\|\theta_{\mathbf{u}}\|_1\|\mathbf{u}_h\|\|\mathbf{u}_h\|_1 \\
&\qquad + \|\rho_p^h\|^2 + \|\theta_c^h\|^2 + \|\rho_c^h\|^2\Big)
\end{aligned} \tag{4.72}
$$

and

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\theta_c^h\|^2 + \frac{\theta}{C_\Omega^2}\|\theta_c^h\|_1^2 \\
&\quad \leq ((\theta_c^h)', \theta_c^h) + \theta a(\theta_c^h, \theta_c^h) \\
&\quad \leq \Big|((\rho_c^h)', \theta_c^h) + \theta a(\rho_c^h, \theta_c^h) + U(c_h - c, \frac{\partial\theta_c^h}{\partial x_2}) \\
&\qquad + \hat{B}_3(\theta_{\mathbf{u}}^h, c_h, \theta_c^h) + \hat{B}_3(r_h\mathbf{u}, \rho_c^h, \theta_c^h) + \hat{B}_3(\rho_{\mathbf{u}}^h, c, \theta_c^h)\Big| \\
&\quad \leq \|(\rho_c^h)'\|\|\theta_c^h\| + \theta\|\theta_c^h\|_1\|\rho_c^h\|_1 + U\|c_h - c\|\|\theta_c^h\|_1 \\
&\qquad + C\|\theta_c^h\|_1\Big((\|\theta_{\mathbf{u}}^h\|\|\theta_{\mathbf{u}}^h\|_1\|c_h\|\|c_h\|_1)^{\frac{1}{2}} + \|r_h\mathbf{u}\|_1\|\rho_c^h\|_1 + \|\rho_{\mathbf{u}}^h\|_1\|c\|_1\Big) \\
&\quad \leq \varepsilon\|\theta_c^h\|_1^2 + C(\mathbf{u}, c)\Big(\|(\rho_c^h)'\|^2 + \|\rho_c^h\|_1^2 + \|\rho_{\mathbf{u}}^h\|_1^2 \\
&\qquad + \|\theta_{\mathbf{u}}^h\|\|\theta_{\mathbf{u}}^h\|_1\|c_h\|\|c_h\|_1 + \|\rho_c^h\|^2 + \|\theta_c^h\|^2\Big).
\end{aligned} \tag{4.73}
$$

Taking the sum of (4.72) and (4.73), applying Young's inequality again we obtain

$$
\begin{aligned}
&\frac{d}{dt}\|\theta_{\mathbf{u}}^h\|^2 + \|\theta_{\mathbf{u}}^h\|_1^2 + \frac{d}{dt}\|\theta_c^h\|^2 + \|\theta_c^h\|_1^2 \\
&\qquad \le \varepsilon(\|\theta_{\mathbf{u}}^h\|_1^2 + \|\theta_c^h\|_1^2) \\
&\qquad\quad + C\Big(\|(\rho_{\mathbf{u}}^h)'\|^2 + \|(\rho_c^h)'\|^2 + \|\rho_{\mathbf{u}}^h\|_1^2 + \|\rho_c^h\|_1^2 + \|\rho_c^h\|^2 + \|\theta_c^h\|^2 + \|\rho_p^h\|^2 \\
&\qquad\quad + \|\theta_{\mathbf{u}}^h\|\|\theta_{\mathbf{u}}^h\|_1\|\mathbf{u}_h\|\|\mathbf{u}_h\|_1 + \|\theta_{\mathbf{u}}^h\|\|\theta_{\mathbf{u}}^h\|_1\|c_h\|\|c_h\|_1\Big) \\
&\qquad \le \varepsilon\Big(1 + \|c_h\|^2 + \|\mathbf{u}_h\|^2\Big)\|\theta_{\mathbf{u}}^h\|_1^2 + \varepsilon\|\theta_c^h\|_1^2 \\
&\qquad\quad + C\Big(\|(\rho_{\mathbf{u}}^h)'\|^2 + \|(\rho_c^h)'\|^2 + \|\rho_{\mathbf{u}}^h\|_1^2 + \|\rho_c^h\|_1^2 \\
&\qquad\quad + \|\rho_p^h\|^2 + \|\rho_c^h\|^2 + \|\theta_c\|^2 + \big(\|\mathbf{u}_h\|_1^2 + \|c_h\|_1^2\big)\|\theta_{\mathbf{u}}^h\|^2\Big)\,.
\end{aligned}
\tag{4.74}
$$

Proceeding as in Lemma 4.4, we can prove the uniform boundedness of $\|\mathbf{u}_h\|$ and $\|c_h\|$ with respect to h. Choose $\varepsilon > 0$ such that

$$
\varepsilon < \min\{\frac{1}{1 + \|\mathbf{u}_h\|^2 + \|c_h\|^2}\,, 1\}\,.
$$

Then (4.74) leads to

$$
\begin{aligned}
&\frac{d}{dt}\Big(\|\theta_u\|^2 + \|\theta_c\|^2\Big) \\
&\qquad \le C\Big(\|(\rho_{\mathbf{u}}^h)'\|^2 + \|(\rho_c^h)'\|^2 + \|\rho_{\mathbf{u}}^h\|_1^2 + \|\rho_c^h\|_1^2 + \|\rho_c^h\|^2 + \|\rho_p^h\|^2\Big) \\
&\qquad\quad + C\Big(1 + \|\mathbf{u}_h\|_1^2 + \|c_h\|_1^2\Big)\Big(\|\theta_{\mathbf{u}}\|^2 + \|\theta_c\|^2\Big)\,.
\end{aligned}
$$

Integrating with respect to s from 0 to t gives

$$
\begin{aligned}
\|\theta_u^h\|^2 + \|\theta_c^h\|^2 \le &\|\theta_{\mathbf{u}}^h\|^2(0) + \|\theta_c^h\|^2(0) \\
&+ C\int_0^t \Big(\|(\rho_{\mathbf{u}}^h)'\|^2 + \|(\rho_c^h)'\|^2 + \|\rho_{\mathbf{u}}^h\|_1^2 + \|\rho_c^h\|_1^2 + \|\rho_c^h\|^2 + \|\rho_p^h\|^2\Big)\,ds \\
&+ C\int_0^t \Big(1 + \|\mathbf{u}_h\|_1^2 + \|c_h\|_1^2\Big)\Big(\|\theta_{\mathbf{u}}^h\|^2 + \|\theta_c^h\|^2\Big)\,ds\,.
\end{aligned}
$$

Applying Gronwall's inequality we have

$$
\begin{aligned}
\|\theta_u^h\|^2 + \|\theta_c^h\|^2 \leq & \Big(\|\theta_{\mathbf{u}}^h\|^2(0) + \|\theta_c^h\|^2(0)\Big) \exp\left(C\int_0^t (1 + \|\mathbf{u}_h\|_1^2 + \|c_h\|_1^2)\ ds\right) \\
& + C\int_0^t \Big(\|(\rho_{\mathbf{u}}^h)'\|^2 + \|(\rho_c^h)'\|^2 + \|\rho_{\mathbf{u}}^h\|_1^2 + \|\rho_c^h\|_1^2 + \|\rho_c^h\|^2 + \|\rho_p^h\|^2\Big)\ ds \\
\leq & \ C\Big(\|\mathbf{U}_h - \mathbf{u}_0\|^2 + \|C_h - c_0\|^2\Big) \\
& + \int_0^t \Big(\|\rho_{\mathbf{u}}'\|^2 + \|\rho_c'\|^2 + \|\rho_{\mathbf{u}}\|_1^2 + \|\rho_c\|_1^2 + \|\rho_c\|^2 + \|\rho_p^h\|^2\Big) ds)\,,
\end{aligned}
\tag{4.75}
$$

since $\|\mathbf{u}_h\|_1^2$ and $\|c_h\|_1^2$ are integrable on I according to a similar argument as in Lemma 4.4. □

Remark 4.9. The proof of the convergence of pressure p_h remains open.

4.3 Numerical experiments

In this section we describe a numerical experiment in two dimensions to verify the convergence of numerical scheme (4.54). In this experiment, the parameters used were as in Section 3.3. Construct the Taylor-Hood element and assign the parameters as

$$\gamma = 0.1\,, \quad U = 0.1\,, \quad \theta = 1\,, \quad \alpha = 10\%\,,$$

and

$$\nu(x) = \sin^2 x + 1\,, \quad x \in \mathbb{R}\,.$$

The forcing term $\mathbf{f}$ was chosen so that the exact solution is

$$
\begin{cases}
\mathbf{u} = \sqrt{t}(\sin\pi x \sin\pi y, \sin\pi x \sin\pi y)^T\,, \\
p = \sqrt{t}\sin\pi x \sin\pi y\,, \\
c = \sqrt{t}\sin\pi x \sin\pi y\,.
\end{cases}
$$

Table 4.1: Convergence rate in $L^2(I; L^2(\Omega))$

h	$\|\mathbf{u} - \mathbf{u}_h\|$	$\|p - p_h\|$	$\|c - c_h\|$
1/2	0.0067	0.2386	0.0128
1/4	7.72e-04	0.0531	0.0014
1/8	8.84e-05	0.0128	1.75e-04
1/16	1.16e-05	0.0030	2.20e-05
1/32	1.51e-06	7.23e-04	3.06e-06
conv. rate	2.94	2.05	2.85

Table 4.2: Convergence rate in $L^2(I; H^1(\Omega))$

h	$\|\mathbf{u} - \mathbf{u}_h\|_1$	$\|p - p_h\|_1$	$\|c - c_h\|_1$
1/2	0.1065	0.5044	0.0415
1/4	0.0264	0.2298	0.0103
1/8	0.0066	0.1070	0.0027
1/16	0.0016	0.0474	6.85e-04
1/32	0.0004	0.0233	1.71e-04
conv. rate	2.00	1.02	2.00

Since our focus is on the convergence of the numerical solution with respect to the mesh size h, we used the time step $k = 10^{-5}$. The numerical errors for different mesh sizes are shown in table 4.1 and 4.2, from which we can see that the error tends zero as h become smaller just as stated in Theorem 4.8. Furthermore the numerical method achieves the optimal convergence order although we did not prove it.

4.4 Conclusion

In chapter 3 and 4, we studied the mathematical model of bioconvection caused by average upswimming micro-organisms. The PDE system consists of a Navier-Stokes type equation for the velocity and pressure coupled with a parabolic equation for the concentration. The viscosity is assumed to be dependent on the concentration. We established the existence and uniqueness of a weak solution of both steady and evolutionary bioconvection. We then constructed finite element approximation of the weak solutions and proved the convergence of the numerical approximation to the exact solution. We used our numerical method to simulate the velocity and concentration distribution inside a small container. The uniqueness result and the convergence theorem for the evolutionary case are only valid in the two dimensional case. The convergence of the discrete pressure remains open for the time dependent bioconvection.

Chapter 5

Conclusion and future work

Conclusion. We studied two systems of partial differential equations with coefficients that depend on the solution. The first, a quasi-static poroelasticity is a system comprised of the equation of linear elasticity and and a nonlinear diffusion equation, and the second, a Navier-Stokes type system, both systems were studied using the modified Rothe's method to handle the solution dependent coefficients. We established the existence and uniqueness of solutions, and conducted numerical experiments approximating weak solutions of both systems using the finite element method . Numerical simulations were constructed to show the accuracy of the models.

Future work. Many equations remain unresolved and various extensions to the present work are possible.

1. More efficient numerical methods may be considered, including finite volume method and discontinuous Galerkin methods.

2. The equation of quasi-static poroelasticty can be considered subject to general boundary conditions. Introducing general boundary condition may change the null space of operator B defined in chapter 2. The energy estimate may need to be modified.

3. We may consider the fully dynamic model (1.9) with secondary consolidation, that is, with the two terms $\rho\frac{\partial^2}{\partial t^2}\mathbf{u}$ and $\lambda_*\nabla\frac{d}{dt}(\nabla\cdot\mathbf{u})$. Both cases result in coupled systems of hyperbolic and parabolic equations.

4. Many poroelasticity problems have multi-scale features. For instance, we consider a rock with small pores. On the macro-scale we consider a deformation equation of the rock while on the micro-scale we solve a diffusion equation inside the pores. In Biot's model the two equations are coupled through the Biot-Willis constant α. However, in view of

multi-scale finite element method, the two equations can be coupled through mathematical homogenization by constructing multi-scale finite elements. More work and details can be found in [64].

5. System (1.19) can be extended to equations that model the convection caused by the admixture in the atmosphere and ocean which is important in the study of earth ecology and which involves large scale computation and multi-species creatures [34].

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