Spatial Spread Dynamics of Monostable Equations in Spatially Locally Inhomogeneous Media with Temporal Periodicity

by

Liang Kong

A dissertation submitted to the Graduate Faculty of
Auburn University
in partial fulfillment of the
requirements for the Degree of
Doctor of Philosophy

Auburn, Alabama
August 03, 2013

Keywords: Monostable equation, random dispersal, nonlocal dispersal, discrete dispersal, spreading speed, principal eigenvalue/eigenfunction.

Approved by

Wenxian Shen, Chair, Professor of Mathematics and Statistics
Georg Hetzer, Professor of Mathematics and Statistics
Xiaoying Han, Associate Professor of Mathematics and Statistics
Bertram Zinner, Associate Professor of Mathematics and Statistics

Abstract

This dissertation is devoted to the study of semilinear dispersal evolution equations of the form

$$u_t(t,x) = (\mathcal{A}u)(t,x) + u(t,x)f(t,x,u(t,x)), \quad x \in \mathcal{H},$$

where $\mathcal{H} = \mathbb{R}^N$ or $\mathbb{Z}^N$, $\mathcal{A}$ is a random dispersal operator or nonlocal dispersal operator in the case $\mathcal{H} = \mathbb{R}^N$ and is a discrete dispersal operator in the case $\mathcal{H} = \mathbb{Z}^N$, and f is periodic in t, asymptotically periodic in x (i.e. $f(t,x,u) - f_0(t,x,u)$ converges to 0 as $\|x\| \to \infty$ for some time and space periodic function $f_0(t,x,u)$), and is of KPP type in u.

These type of equations are called as Monostable or KPP type equations, which arise in modeling the population dynamics of many species which exhibit local, nonlocal and discrete internal interactions and live in locally spatially inhomogeneous media with temporal periodicity. The following main results are proved in the dissertation.

Firstly, it is proved that Liouville type property holds for such equations, that is, time periodic strictly positive solutions are unique. It is proved that if time periodic strictly positive solutions (if exists) are globally stable with respect to strictly positive perturbations. Moreover, it is proved that if the trivial solution $u = 0$ of the limit equation of such an equation is linearly unstable, then the equation has a time periodic strictly positive solution.

Secondly, spatial spreading speeds of such equations is investigated. It is also proved that if $u \equiv 0$ is a linearly unstable solution to the time and space periodic limit equation of such an equation, then the original equation has a spatial spreading speed in every direction. Moreover, it is proved that the localized spatial inhomogeneity neither slows down nor speeds up the spatial spreading speeds. In addition, in the time dependent case, various spreading features of the spreading speeds are obtained.

Finally, the effects of temporal and spatial variations on the uniform persistence and spatial spreading speeds of such equations are considered. As in the periodic media case, it is shown that temporal and spatial variations favor the population's persistence and do not reduce the spatial spreading speeds.

Acknowledgments

First and foremost I want to thank my advisor Dr. Wenxian Shen. It has been an honor to be her Ph.D. student. I will forever be thankful to all her contributions of time, ideas, and funding to make my Ph.D. experience stimulating and productive. The positivity and enthusiasm she has for her research was contagious and motivational for me. The patience and caring she gave inspire me not only to be a better mathematician, but also to be a better person.

I would like to show my gratitude to my committee members, Drs. Georg Hetzer, Xiaoying Han and Bertram Zinner, for their time and attention during busy semesters.

I would also like to thank my parents for their unconditional love and support during the long years of my education. I am truly indebted and thankful to my wife, Li Cheng, for her sacrifice and support to ensure me stay focus on my graduate study.

In addition, I would like thank the NSF for the financial support (NSF-DMS-0907752).

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Chapter 1

Introduction

In this dissertation, we investigate the Liouville type property and spatial spreading dynamics of monostable evolution equations in locally spatially inhomogeneous periodic media, in particular, we consider the existence, uniqueness, and stability of time periodic positive solutions and spatial spreading speeds of monostable type dispersal evolution equations in periodic media with localized spatial inhomogeneity. We also study the influence of the inhomogeneity of the underlying media on the spatial spreading speeds of the monostable equations.

Our model equations are of the form,

$$u_t(t,x) = \mathcal{A}u + uf_0(t,x,u), \quad x \in \mathcal{H}, \tag{1.1}$$

where $\mathcal{H} = \mathbb{R}^N$ or $\mathbb{Z}^N$; in the case $\mathcal{H} = \mathbb{R}^N$,

$$\mathcal{A}u = \Delta u$$

or

$$(\mathcal{A}u)(t,x) = \int_{\mathbb{R}^N} \kappa(y-x)u(t,y)dy - u(t,x)$$

($\kappa(\cdot)$ is a smooth non-negative convolution kernel supported on a ball centered at the origin and $\int_{\mathbb{R}^N} \kappa(z)dz = 1$), and in the case $\mathcal{H} = \mathbb{Z}^N$,

$$(\mathcal{A}u)(t,j) = \sum_{k\in K} a_k(u(t,j+k) - u(t,j))$$

($a_k > 0$ and $K = \{k \in \mathcal{H} \,|\, \|k\| = 1\}$); and $f_0(t+T,x,u) = f_0(t,x+p_i\mathbf{e_i},u) = f_0(t,x,u)$ ($T \in \mathbb{R}$ and $p_i \in \mathcal{H}$ are given constants), and $\partial_u f_0(t,x,u) < 0$ for $u \geq 0$, $f_0(t,x,u) < 0$ for $u \gg 1$.

Among others, equation (1.1) is used to model the evolution of population density of a species. The case that $\mathcal{H} = \mathbb{R}^N$ and $\mathcal{A}u = \Delta u$ indicates that the environment of the underlying model problem is not patchy and the internal interaction of the organisms is random and local (i.e. the organisms move randomly between the adjacent spatial locations, such $\mathcal{A}$ is referred to as a *random dispersal operator*) (see [1], [2], [9], [21], [23], [24], [45], [56], [77], [79], [81], [82], [86], etc., for the application in this case). If the environment of the underlying model problem is not patchy and the internal interaction of the organisms is nonlocal, $(\mathcal{A}u)(t,x) = \int_{\mathbb{R}^N} \kappa(y-x)u(t,y)dy - u(t,x)$ is often adopted (such $\mathcal{A}$ is referred to as a *nonlocal dispersal operator*) (see [3], [10], [16], [22], [26], [41], etc.). The case that $\mathcal{H} = \mathbb{Z}^N$ and $(\mathcal{A}u)(t,j) = \sum_{k\in K} a_k(u(t,j+k) - u(t,j))$ (which is referred to as a *discrete dispersal operator*) arises when modeling the population dynamics of species living in patchy environments (see [21], [55], [56], [77], [78], [81], [82], [83], etc.). The periodicity of $f_0(t,x,u)$ in t and x reflects the periodicity of the environment. In literature, equation (1.1) is called of Fisher or KPP type due to the pioneering works of Fisher [24] and Kolmogorov, Petrowsky, Piscunov [45] on the following special case of (1.1),

$$u_t = u_{xx} + u(1-u), \quad x \in \mathbb{R}. \tag{1.2}$$

Central problems about (1.1) include the existence, uniqueness, and stability of time and space periodic positive solutions and spatial spreading speeds. Such problems have been extensively studied (see [1]-[8], [11]-[13], [18], [20], [19], [25], [27]-[30], [35], [39], [40], [44], [49]-[52], [54], [57]-[62], [65]-[76], [80]-[82], [84], etc.). It is known that time and space periodic positive solutions of (1.1) (if exist) are unique, which is referred to as the *Liouville type property* for (1.1). If $u \equiv 0$ is linearly unstable with respect to spatially periodic perturbations, then (1.1) has a unique stable time and space periodic positive solution $u_0^*(t,x)$ and for any $\xi \in \mathbb{R}^N$ with $\|\xi\| = 1$, (1.1) has a spreading speed $c_0^*(\xi)$ in the direction of ξ (see section 2.4 for detail).

In reality, the underlying media of many biological problems is non-periodically inhomogeneous. It is therefore of great importance to investigate the dynamics of monostable evolution equations in various types of non-periodically inhomogeneous media, for example, in almost periodic media, in periodic media with locally spatial perturbations, etc.. There are many works on various extensions of the spatial spreading dynamics of monostable evolution equations in periodic media, see, for example, [4], [31], [37], [59], [68]-[72], etc..

The aim of the current dissertation is to investigate the dynamics of KPP type equations in periodic media with spatially localized inhomogeneity, in particular, to deal with the extensions of the above results for (1.1) to KPP type equations in periodic media with spatially localized inhomogeneity. We hence consider

$$u_t = \mathcal{A}u + uf(t,x,u), \quad x \in \mathcal{H}, \tag{1.3}$$

where $\mathcal{A}$ and $\mathcal{H}$ are as in (1.1), $\partial_u f(t,x,u) < 0$ for $u \ge 0$, $f(t,x,u) < 0$ for $u \gg 1$, $f(t+T,x,u) = f(t,x,u)$, and $|f(t,x,u) - f_0(t,x,u)| \to 0$ as $\|x\| \to \infty$ uniformly in (t,u) on bounded sets ($f_0(t,x,u)$ is as in (1.1)) (See (H1) in Chapter 2 for detail). We show that localized inhomogeneity does not destroy the existence and uniqueness of time periodic positive solutions and it neither slow down nor speed up the spatial spreading speeds. We also show that temporal and spatial inhomogeneity does not slow down the spatial spreading speeds. More precisely, we prove

- (Liouville type property or uniqueness of time periodic strictly positive solutions) *Time periodic strictly positive solutions of* (1.3) *(if exist) are unique (see Theorem 2.1(1)).*

- (Stability of time periodic strictly positive solutions) *If* (1.3) *has a time periodic strictly positive solution $u^*(t,x)$, then it is asymptotically stable (see Theorem 2.1(2)).*

• (Existence of time periodic strictly positive solutions) *If* $u = 0$ *is a linearly unstable solution of* (1.1) *with respect to periodic perturbations, then* (1.3) *has a time periodic strictly positive solution* $u^*(t,x)$ *(see Theorem 2.1(3)).*

• (Tail property of time periodic strictly positive solutions) *If* $u = 0$ *is a linearly unstable solution of* (1.1) *with respect to periodic perturbations, then* $u^*(t,x) - u_0^*(t,x) \to 0$ *as* $\|x\| \to \infty$ *uniformly in* t *(see Theorem 2.1(4)).*

• (Spatial spreading speeds) *If* $u = 0$ *is a linearly unstable solution of* (1.1) *with respect to periodic perturbations, then for each* $\xi \in \mathbb{R}^N$ *with* $\|\xi\| = 1$, $c_0^*(\xi)$ *is the spreading speed of* (1.3) *in the direction of* ξ *(see Theorem 2.2).*

• (Effect of temporal variation) *If* $u = 0$ *is a linearly unstable solution of*

$$u_t(t,x) = \mathcal{A}u + u\hat{f}_0(x,u), \quad x \in \mathcal{H}, \tag{1.4}$$

where $\mathcal{H}$ *and* $\mathcal{A}$ *is as in* (1.1) *and* $\hat{f}_0$ *is the time average of* $f_0(t,x,u)$ *(see* (2.19)*), then* (1.3) *has a time periodic strictly positive solution* $u^*(t,x)$ *and for each* $\xi \in \mathbb{R}^N$ *with* $\|\xi\| = 1$, $c_0^*(\xi)$ *is the spreading speed of* (1.3) *in the direction of* ξ*. Moreover,*

$$c_0^*(\xi) \ge \hat{c}_0^*(\xi),$$

where $\hat{c}_0^*(\xi)$ *is the spatial spreading speed of* (1.4) *in the direction of* ξ *(see Theorem 2.4 (1)).*

• (Effect of spatial variation) *If* $u = 0$ *is a linearly unstable solution of*

$$u_t(t,x) = \mathcal{A}u + u\hat{\hat{f}}_0(u), \quad x \in \mathcal{H}, \tag{1.5}$$

where $\mathcal{H}$ *and* $\mathcal{A}$ *is as in* (1.1) *and* $\hat{\hat{f}}_0(u)$ *is the spatial average of* $\hat{f}_0(x,u)$ *(see* (2.21)*), then* (1.3) *has a time periodic strictly positive solution* $u^*(t,x)$ *and for each* $\xi \in \mathbb{R}^N$ *with* $\|\xi\| = 1$,

$c_0^*(\xi)$ *is the spreading speed of* (1.3) *in the direction of* ξ. *Moreover,*

$$c_0^*(\xi) \geq \hat{c}_0^*(\xi) \geq \hat{\hat{c}}_0^*(\xi),$$

where $\hat{\hat{c}}_0^*(\xi)$ *is the spatial spreading speed of* (1.5) *in the direction of* ξ *(see Theorem 2.4 (2)).*

We remark that, in the case that $\mathcal{H} = \mathbb{R}^N$ and $\mathcal{A} = \Delta$, Liouvile type property of (1.3) is discussed in [8] and the methods used in [8] quite rely on the special properties of parabolic equations. The current thesis recovers the results obtained in [8] by different methods, which apply to all three different type dispersal operators.

The rest of the dissertation is organized as follows. In chapter 2, we introduce the standing notions, hypotheses, and definitions, and state the main results of the dissertation. In chapter 3, we present some preliminary materials to be used in the proofs of the main results. We study the existence, uniqueness, and stability of time periodic positive solutions of (1.3) in chapter 4. In chapter 5, we explore the spreading speeds of (1.3). We give another elegant method working on time independent case in chapter 6. In chapter 7, we consider the temporal and spatial variations on the spatial spreading dynamics of monostable stable equations. In chapter 8, We will address some remarks and open problems.

Chapter 2

Notations, Hypotheses, Definitions, and Main Results

In this chapter, we first introduce some standing notations, hypotheses, and definitions. We then state the main results of the paper.

2.1 Notions, hypotheses and definitions

In this section, we introduce standing notions, hypotheses, and definitions. Throughout this subsection,

$$\mathcal{H} = \mathbb{R}^N \text{ or } \mathbb{Z}^N \text{ and } p_i \in \mathcal{H} \text{ with } p_i > 0 \text{ (} i = 1, 2, \cdots, N \text{)} \tag{2.1}$$

Let

$$X = C^b_{\text{unif}}(\mathcal{H}, \mathbb{R}) := \{u \in C(\mathcal{H}, \mathbb{R}) \,|\, u \text{ is uniformly continuous and bounded on } \mathcal{H}\} \tag{2.2}$$

with norm $\|u\| = \sup_{x \in \mathcal{H}} |u(x)|$,

$$X^+ = \{u \in X \,|\, u(x) \geq 0 \;\; \forall x \in \mathcal{H}\}, \tag{2.3}$$

and

$$X^{++} = \{u \in X^+ \,|\, \inf_{x \in \mathcal{H}} u(x) > 0\}. \tag{2.4}$$

Let

$$X_p = \{u \in X \,|\, u(\cdot + p_i \mathbf{e_i}) = u(\cdot)\}, \tag{2.5}$$

$$X_p^+ = X^+ \cap X_p, \tag{2.6}$$

and

$$X_p^{++} = X^{++} \cap X_p. \tag{2.7}$$

For given $u, v \in X$, we define

$$u \leq v \ (u \geq v) \quad \text{if } v - u \in X^+ \ (u - v \in X^+) \tag{2.8}$$

and

$$u \ll v \ (u \gg v) \quad \text{if } v - u \in X^{++} \ (u - v \in X^{++}). \tag{2.9}$$

Let $\mathcal{H}_i$ and $\mathcal{A}_i : \mathcal{D}(\mathcal{A}_i) \subset X \to X$ $(i = 1, 2, 3)$ be defined by

$$\mathcal{H}_1 = \mathbb{R}^N, \ (\mathcal{A}_1 u)(x) = \Delta u(x) \quad \forall \ u \in \mathcal{D}(\mathcal{A}_1), \tag{2.10}$$

where $\mathcal{D}(\mathcal{A}_1) = \{u \in X \,|\, \partial_{x_j} u(\cdot), \partial^2_{x_j x_k} u(\cdot) \in X, \ 1 \leq j, k \leq N\}$,

$$\mathcal{H}_2 = \mathbb{R}^N, \ (\mathcal{A}_2 u)(x) = \int_{\mathbb{R}^N} \kappa(y - x) u(y) dy - u(x) \quad \forall \ u \in \mathcal{D}(\mathcal{A}_2) = X, \tag{2.11}$$

and

$$\mathcal{H}_3 = \mathbb{Z}^N, \ (\mathcal{A}_3 u)(j) = \sum_{k \in K} a_k (u(j + k) - u(j)) \quad \forall u \in \mathcal{D}(\mathcal{A}_3) = X. \tag{2.12}$$

Let

$$\mathcal{X}_p = \{u \in C(\mathbb{R} \times \mathcal{H}, \mathbb{R}) \,|\, u(t + T, x + p_i \mathbf{e_i}) = u(t, x)\} \tag{2.13}$$

with norm $\|u\| = \max_{t\in\mathbb{R},x\in\mathcal{H}} |u(t,x)|$. For given $\xi \in S^{N-1}$ and $\mu \in \mathbb{R}$, let $\mathcal{A}_{\xi,\mu} : \mathcal{D}(\mathcal{A}_{\xi,\mu}) \subset \mathcal{X}_p \to \mathcal{X}_p$ be defined by

$$(\mathcal{A}_{\xi,\mu}u)(t,x) = \begin{cases} \Delta u(t,x) - 2\mu\xi\cdot\nabla u(t,x) + \mu^2 u(t,x) \text{ if } \mathcal{H}=\mathcal{H}_1,\ \mathcal{A}=\mathcal{A}_1 \\ \\ \int_{\mathbb{R}^N} e^{-\mu(y-x)\cdot\xi}\kappa(y-x)u(t,y)dy - u(t,x) \text{ if } \mathcal{H}=\mathcal{H}_2,\ \mathcal{A}=\mathcal{A}_2 \\ \\ \sum_{k\in K} a_k(e^{-\mu k\cdot\xi}u(t,j+k) - u(t,j)) \text{ if } \mathcal{H}=\mathcal{H}_3,\ \mathcal{A}=\mathcal{A}_3 \end{cases} \tag{2.14}$$

for $u \in \mathcal{D}(\mathcal{A}_{\xi,\mu})$. Observe that

$$\mathcal{A}_{\xi,0} = \mathcal{A} \quad \forall\, \xi \in S^{N-1}.$$

For any given $a \in \mathcal{X}_p$, $\xi \in S^{N-1}$, and $\mu \in \mathbb{R}$, let $\sigma(-\partial_t + \mathcal{A}_{\xi,\mu} + a(\cdot,\cdot)\mathcal{I})$ be the spectrum of the operator $-\partial_t + \mathcal{A}_{\xi,\mu} + a(\cdot,\cdot)\mathcal{I} : \mathcal{D}(-\partial_t + \mathcal{A}_{\xi,\mu} + a(\cdot,\cdot)\mathcal{I}) \subset \mathcal{X}_p \to \mathcal{X}_p$,

$$\big((-\partial_t + \mathcal{A}_{\xi,\mu} + a(\cdot,\cdot)\mathcal{I})u\big)(t,x) = -u_t(t,x) + (\mathcal{A}_{\xi,\mu}u)(t,x) + a(t,x)u(t,x).$$

Let $\lambda_{\xi,\mu}(a)$ be defined by

$$\lambda_{\xi,\mu}(a) = \sup\{\mathrm{Re}\lambda \mid \lambda \in \sigma(-\partial_t + \mathcal{A}_{\xi,\mu} + a(\cdot,\cdot)\mathcal{I})\}. \tag{2.15}$$

We call $\lambda_{\xi,\mu}(a)$ the *principal spectrum point* of $-\partial_t + \mathcal{A}_{\xi,\mu} + a(\cdot,\cdot)\mathcal{I}$. It equals the principal eigenvalue of $-\partial_t + \mathcal{A}_{\xi,\mu} + a(\cdot,\cdot)\mathcal{I}$ if it exists (see Definition 3.1 for the definition of principal eigenvalue).

It is clear that $\lambda_{\xi,0}(a)$ is independent of $\xi \in S^{N-1}$ and we may put

$$\lambda(a) = \lambda_{\xi,0}(a).$$

Observe that $\sigma(-\partial_t+\mathcal{A}_{\xi,\mu}+a(\cdot,\cdot)\mathcal{I})$ is the spectrum of the following eigenvalue problem,

$$-\partial_t u + \mathcal{A}_{\xi,\mu} u + a(\cdot,\cdot)\mathcal{I}u = \lambda u, \quad u \in \mathcal{X}_p. \tag{2.16}$$

When $a(t,x) \equiv a(x)$ is independent of t, (2.16) reduces to

$$\mathcal{A}_{\xi,\mu} u + a(\cdot)\mathcal{I}u = \lambda u, \quad u \in X_p. \tag{2.17}$$

In the following, to indicate the dependence of X, X_p, $\mathcal{X}_p$ on the media, we may put

$$X_i = X = C^b_{\rm unif}(\mathcal{H}_i,\mathbb{R}) \quad \text{in the case} \quad \mathcal{H} = \mathcal{H}_i,$$

$$X_{i,p} = X_p = \{u \in X_i \,|\, u(\cdot + p_i \mathbf{e_i}) = u(\cdot)\} \quad \text{in the case} \quad \mathcal{H} = \mathcal{H}_i,$$

etc..

Consider (1.3). We introduce the following standing hypotheses.

(H0) *$f(t+T,x,u) = f(t,x,u)$ for $(t,x,u) \in \mathbb{R}\times\mathcal{H}\times\mathbb{R}$ ($T>0$ is a given positive number); $f(t,x,u)$ is C^1 in t,u and $f(t,x,u)$, $f_t(t,x,u)$, $f_u(t,x,u)$ are uniformly continuous in $(t,x,u)\in\mathbb{R}\times\mathcal{H}\times E$ (E is any bounded subset of $\mathbb{R}$); $f(t,x,u)<0$ for all $t\in\mathbb{R}$, $x\in\mathcal{H}$, and $u\ge M_0$ ($M_0>0$ is some given constant); and $\inf_{t\in\mathbb{R},x\in\mathcal{H}} f_u(t,x,u)<0$ for all $u\ge 0$.*

(H1) *$f_0(t,x,u)$ satisfies (H0), $f_0(t,x+p_i\mathbf{e_i},u) = f_0(t,x,u)$ for $(t,x,u)\in\mathbb{R}\times\mathcal{H}\times\mathbb{R}$, and $|f(t,x,u)-f_0(t,x,u)|\to 0$ as $\|x\|\to\infty$ uniformly in (t,u) on bounded sets.*

(H1)$'$ *$f_0(u)$ satisfies (H0), $f(t,x,u)\equiv f(x,u)$, and $f(x,u)-f_0(u)\to 0$ as $\|x\|\to\infty$.*

(H2) $\lambda(f_0(\cdot,\cdot,0))>0$.

Observe that $f(t,x,u) = 1-u$ is a typical example which satisfies (H0) and (H0) is referred to as Fisher or KPP type condition. Throughout this section, we assume (H0). By general semigroup theory (see [32], [64]), for any $u_0\in X$, (1.3) has a unique (local) solution

$u(t,\cdot;u_0)$ with $u(0,\cdot;u_0) = u_0(\cdot)$. Furthermore, if $f(t, x+p_i\mathbf{e_i}, u) = f(t,x,u)$ and $u_0 \in X_p$, then $u(t,\cdot;u_0) \in X_p$. To indicate the dependence of $u(t,x;u_0)$ on f, we may write $u(t,x;u_0)$ as $u(t,x;u_0,f)$.

Observe also that assumption (H1) reflects the localized spatial inhomogeneity of the media. (H1)$'$ is a special case of (H1). Assumption (H2) is the linear instability condition of the trivial solution of (1.3) in the case $f(t,x,u) = f_0(t,x,u)$. If $f_0(t,x,u) \equiv f_0(u)$, then (H2) becomes $f_0(0) > 0$.

Let $\hat{f}(x,u)$, $\hat{f}_0(x,u)$, $\hat{\hat{f}}(u)$, and $\hat{\hat{f}}_0(u)$ be defined as follows.

$$\hat{f}(x,u) = \frac{1}{T}\int_0^T f(t,x,u)dt, \tag{2.18}$$

$$\hat{f}_0(x,u) = \frac{1}{T}\int_0^T f_0(t,x,u)dt, \tag{2.19}$$

and

$$\hat{\hat{f}}(u) = \lim_{R\to\infty} \frac{1}{|B(0,R)|}\int_{B(0,R)} \hat{f}(x,u)dx, \tag{2.20}$$

$$\hat{\hat{f}}_0(u) = \lim_{R\to\infty} \frac{1}{|B(0,R)|}\int_{B(0,R)} \hat{f}_0(x,u)dx, \tag{2.21}$$

where

$$B(0,R) = \{x \in \mathcal{H} \,|\, |x_i| \le R,\ i = 1,2,\cdots,N\}$$

and $|B(0,R)|$ is the Lebesgue measure of $B(0,R)$ in the case $\mathcal{H} = \mathbb{R}^N$ and $|B(0,R)$ is the cardinality of $B(0,R)$ in the case $\mathcal{H} = \mathbb{Z}^N$.

Observe that if $f(t,x,u)$ satisfies (H0), then so are $\hat{f}(x,u)$ and $\hat{\hat{f}}(u)$. If $f(t,x,u)$ and $f_0(t,x,u)$ satisfy (H0) and (H1), then so are $\hat{f}(x,u)$ and $\hat{\hat{f}}(u)$. If $\hat{f}(x,u)$ and $\hat{f}_0(x,u)$ satisfy (H0) and (H1), then so are $\hat{\hat{f}}(u)$ and $\hat{\hat{f}}_0(u)$.

Let

$$S^{N-1} = \{\xi \in \mathbb{R}^N \,|\, \|\xi\| = 1\}. \tag{2.22}$$

For given $\xi \in S^{N-1}$ and $u \in X^+$, we define

$$\liminf_{x\cdot\xi\to-\infty} u(x) = \liminf_{r\to-\infty} \inf_{x\in\mathcal{H}, x\cdot\xi\le r} u(x).$$

For given $u : [0,\infty)\times\mathcal{H}\to\mathbb{R}$ and $c > 0$, we define

$$\liminf_{x\cdot\xi\le ct, t\to\infty} u(t,x) = \liminf_{t\to\infty} \inf_{x\in\mathcal{H}, x\cdot\xi\le ct} u(t,x),$$

$$\limsup_{x\cdot\xi\ge ct, t\to\infty} u(t,x) = \limsup_{t\to\infty} \sup_{x\in\mathcal{H}, x\cdot\xi\ge ct} u(t,x).$$

The notions $\limsup_{|x\cdot\xi|\le ct, t\to\infty} u(t,x)$, $\limsup_{|x\cdot\xi|\ge ct, t\to\infty} u(t,x)$, $\limsup_{\|x\|\le ct, t\to\infty} u(t,x)$, and $\limsup_{\|x\|\ge ct, t\to\infty} u(t,x)$ are defined similarly. We define $X^+(\xi)$ by

$$X^+(\xi) = \{u \in X^+ \mid \liminf_{x\cdot\xi\to-\infty} u(x) > 0, \quad u(x) = 0 \text{ for } x\cdot\xi \gg 1\}. \tag{2.23}$$

Definition 2.1 (Time periodic strictly positive solution)**.** *A solution $u(t,x)$ of* (1.3) *on $t \in \mathbb{R}$ is called a* time periodic strictly positive solution *if $u(t+T,x) = u(t,x)$ for $(t,x) \in \mathbb{R}\times\mathcal{H}$ and $\inf_{(t,x)\in\mathbb{R}\times\mathcal{H}} u(t,x) > 0$.*

Definition 2.2 (Spatial spreading speed)**.** *For given $\xi \in S^{N-1}$, a real number $c^*(\xi)$ is called the* spatial spreading speed *of* (1.3) *in the direction of ξ if for any $u_0 \in X^+(\xi)$,*

$$\liminf_{x\cdot\xi\le ct, t\to\infty} u(t,x;u_0) > 0 \quad \forall c < c^*(\xi)$$

and

$$\limsup_{x\cdot\xi\ge ct, t\to\infty} u(t,x;u_0) = 0 \quad \forall c > c^*(\xi).$$

2.2 Main Results

In this section, we state the main results of this dissertation.

The first theorem is about time periodic strictly positive solutions.

Theorem 2.1 (Time periodic strictly positive solutions)**.** *Consider* (1.3) *and assume (H0).*

(1) (*Liouville type property or uniqueness*) *If* (1.3) *has a time periodic strictly positive solution, then it is unique.*

(2) (*Stability*) *Assume that* $u^*(t,x)$ *is a time periodic strictly positive solution of* (1.3). *Then it is stable and for any* $u_0 \in X^{++}$, $\lim_{t\to\infty} \|u(t,\cdot;u_0,f(\cdot+\tau,\cdot,\cdot)) - u^*(t+\tau,\cdot)\|_{X_i} = 0$ *uniformly in* $\tau \in \mathbb{R}$.

(3) (*Existence*) *Assume also (H1) and (H2). Then* (1.3) *has a unique time periodic strictly positive solution* $u^*(t,x)$.

(4) (*Tail property*) *Assume also (H1) and (H2). Then* $u^*(t,x) - u_0^*(t,x) \to 0$ *as* $\|x\| \to \infty$ *uniformly in* $t \in \mathbb{R}$, *where* $u^*(t,x)$ *is as in (3) and* $u_0^*(t,x)$ *is the unique time and space periodic positive solution of* (1.1) *(see Proposition 3.8 for the existence and uniqueness of* $u_0^*(t,x)$*).*

Remark 2.1. *(1) Theorem 2.1 indicates that localized spatial inhomogeneity does not destroy the Liouville type property of* (1.3)*, in particular, it does not destroy the existence of time periodic positive solution. Moreover, it shows that localized spatial inhomogeneity does not affect the behavior of the time periodic positive solution as the space variable goes to* ∞.

(2) Assume (H0) *and* (H1)$'$. *Then* $u_0^*(t,x)$ *is a positive constant, denoted by* u^0, *such that* $f_0(u^0) = 0$.

(3) Biologically, Theorem 2.1 implies that if $u = 0$ *is a linearly unstable solution of the limit equation of* (1.3)*, then the population will persist and is eventually time periodic.*

The second theorem is about spatial spreading speeds.

Theorem 2.2 (Existence of spreading speeds)**.** *Consider* (1.3) *and assume (H0)-(H2). Then for any given* $\xi \in S^{N-1}$, (1.3) *has a spreading speed* $c^*(\xi)$ *in the direction of* ξ*. Moreover, for any* $u_0 \in X^+(\xi)$,

$$\limsup_{x\cdot\xi \le ct, t\to\infty} |u(t,x;u_0) - u^*(t,x)| = 0 \quad \forall c < c^*(\xi). \tag{2.24}$$

and

$$c^*(\xi) = c_0^*(\xi)$$

where $c_0^*(\xi)$ *is the spatial spreading speeds of* (1.1) *in the direction of* ξ *(see Proposition 3.9 for the existence and characterization of* $c_0^*(\xi)$*).*

Remark 2.2. *(1) Theorem 2.2 shows that localized spatial inhomogeneity does not affect the existence of spatial spreading speeds of monostable equations. Moreover, it shows that localized spatial inhomogeneity neither slows down nor speeds up the spatial spreading speeds.*

(2) Assume (H0), (H1)$'$, *and (H2). Then we have the following explicit expression for* $c_0^*(\xi)$,

$$c_0^*(\xi) = \inf_{\mu>0} \frac{f_0(0)+\mu^2}{\mu} = 2\sqrt{f_0(0)} \quad \textit{in the case} \quad \mathcal{H}=\mathcal{H}_1,\ \mathcal{A}=\mathcal{A}_1,$$

$$c_0^*(\xi) = \inf_{\mu>0} \frac{\int_{\mathbb{R}^N} e^{-\mu z\cdot\xi}\kappa(z)dz - 1 + f_0(0)}{\mu} \quad \textit{in the case} \quad \mathcal{H}=\mathcal{H}_2,\ \mathcal{A}=\mathcal{A}_2$$

and

$$c_0^*(\xi) = \inf_{\mu>0} \frac{\sum_{k\in K} a_k(e^{-\mu k\cdot\xi} - 1) + f_0(0)}{\mu} \quad \textit{in the case} \quad \mathcal{H}=\mathcal{H}_3,\ \mathcal{A}=\mathcal{A}_3.$$

For time independent case, we have the following additional result regarding the spreading speeds.

Theorem 2.3 (Spreading features of spreading speeds). *Assume* (H2) *and* $(\mathrm{H1})'$ *and* $f_0(0) > 0$. *Then for any given* $\xi \in S^{N-1}$, *the following hold.*

(1) For each $u_0 \in X^+$ *satisfying that* $u_0(x) = 0$ *for* $x \in \mathcal{H}$ *with* $|x \cdot \xi| \gg 1$,

$$\limsup_{|x\cdot\xi| \geq ct, t\to\infty} u(t,x;u_0) = 0 \quad \forall c > \max\{c_0^*(\xi), c_0^*(-\xi)\}.$$

(2) For each $\sigma > 0$, $r > 0$, *and* $u_0 \in X^+$ *satisfying that* $u_0(x) \geq \sigma$ *for* $x \in \mathcal{H}$ *with* $|x\cdot\xi| \leq r$,

$$\limsup_{|x\cdot\xi| \leq ct, t\to\infty} |u(t,x;u_0) - u^*(x)| = 0 \quad \forall 0 < c < \min\{c_0^*(\xi), c_0^*(-\xi)\}.$$

(3) For each $u_0 \in X^+$ *satisfying that* $u_0(x) = 0$ *for* $x \in \mathcal{H}_i$ *with* $\|x\| \gg 1$,

$$\limsup_{\|x\| \geq ct, t\to\infty} u(t,x;u_0) = 0 \quad \forall c > \sup_{\xi \in S^{N-1}} c_0^*(\xi).$$

(4) For each $\sigma > 0$, $r > 0$, *and* $u_0 \in X^+$ *satisfying that* $u_0(x) \geq \sigma$ *for* $\|x\| \leq r$,

$$\limsup_{\|x\| \leq ct, t\to\infty} |u_0(t,x;u_0) - u^*(x)| = 0 \quad \forall 0 < c < \inf_{\xi \in S^{N-1}} c_0^*(\xi).$$

The last theorem is about the effect of temporal and spatial variations on the time periodic positive solutions and spatial spreading speeds.

Theorem 2.4. *Assume that* (H0) *and* H(1) *are satisfied.*

(1) If $\lambda(\hat{f}_0(\cdot,0)) > 0$, *then* (1.3) *has a unique time periodic strictly positive solution* $u^*(t,x)$ *and has a spreading speed* $c^*(\xi)$ *in the direction of* ξ. *Moreover, for any* $u_0 \in X^+(\xi)$,

$$\limsup_{x\cdot\xi \leq ct, t\to\infty} |u(t,x;u_0) - u^*(t,x)| = 0 \quad \forall c < c^*(\xi). \tag{2.25}$$

and

$$c^*(\xi) = c_0^*(\xi) \geq \hat{c}_0^*(\xi),$$

where $c_0^*(\xi)$ *is as in Theorem 2.2 and* $\hat{c}_0^*(\xi)$ *is the spatial spreading speeds of* (1.1) *with* $f(t,x,u)=\hat{f}_0(x,u)$ *in the direction of* ξ.

(2) If $\lambda(\hat{\hat{f}}_0(0))>0$, *then* (1.3) *has a unique time periodic strictly positive solution* $u^*(t,x)$ *and has a spreading speed* $c^*(\xi)$ *in the direction of* ξ. *Moreover, for any* $u_0\in X^+(\xi)$,

$$\limsup_{x\cdot\xi\le ct, t\to\infty}|u(t,x;u_0)-u^*(t,x)|=0 \quad \forall c<c^*(\xi). \tag{2.26}$$

and

$$c^*(\xi)=c_0^*(\xi)\ge\hat{c}_0^*(\xi)\ge\hat{\hat{c}}_0^*(\xi),$$

where $c_0^*(\xi)$ *and* $\hat{c}_0^*(\xi)$ *are as in (1) and* $\hat{\hat{c}}_0^*(\xi)$ *is is the spatial spreading speeds of* (1.1) *with* $f(t,x,u)=\hat{\hat{f}}_0(u)$ *in the direction of* ξ.

Remark 2.3. *Theorem 2.4 shows that temporal and spatial variations favor the persistence of population and do not reduce the spatial spreading speeds of monostable evolution equations in periodic media with localized spatial inhomogeneity.*

Chapter 3

Preliminary

In this chapter, we present some preliminary materials to be used in later chapters, including a comparison principle for solutions of (1.3); convergence of solutions of (1.3) on compact sets and strip type sets; monotonicity of part metric between two positive solutions of (1.3); the principal eigenvalues theory for time periodic dispersal operators; and the existence, uniqueness, and stability of time and space periodic positive solutions of (1.1) and spatial spreading speeds of (1.1).

3.1 Comparison principle and global existence

In this section, we consider comparison principle and global existence of solutions of (1.3). Throughout this subsection, we assume (H0).

Let $\Omega \subset \mathcal{H}$ be a convex region of $\mathcal{H}$. A given continuous and bounded function $u : [0,\tau) \times \mathbb{R}^N \to \mathbb{R}$, is called a *super-solution* (*sub-solution*) of (1.3) on $[0,\tau) \times \Omega$ if

$$u_t(t,x) \geq (\leq)(\mathcal{A}u)(t,x) + u(t,x)f(t,x,u(t,x)) \quad \forall (t,x) \in (0,\tau) \times \bar{\Omega}. \tag{3.1}$$

Proposition 3.1 (Comparison principle). *(1) Suppose that $u^1(t,x)$ and $u^2(t,x)$ are sub- and super-solutions of* (1.3) *on $[0,\tau)\times\Omega$ with $u^1(t,x) \leq u^2(t,x)$ for $x \in \mathcal{H}\backslash\Omega$, $t \in [0,\tau)$ and $u^1(0,x) \leq u^2(0,x)$ for $x \in \bar{\Omega}$. Then $u^1(t,x) \leq u^2(t,x)$ for $x \in \Omega$ and $t \in [0,\tau)$. Moreover, if $u^1(0,x) \not\equiv u^2(0,x)$ for $x \in \Omega$, then $u^1(t,x) < u^2(t,x)$ for $t \in (0,\tau)$ and $x \in \Omega$.*

(2) If $u_{01}, u_{02} \in X$ and $u_{01} \le u_{02}$, then $u(t,\cdot;u_{01}) \le u(t,\cdot;u_{02})$ for $t > 0$ at which both $u(t,\cdot;u_{01})$ and $u(t,\cdot;u_{02})$ exist. Moreover, if $u_{01} \ne u_{02}$, then $u(t,x;u_{01}) < u(t,x;u_{02})$ for all $x \in \mathcal{H}$ and $t > 0$ at which both $u(t,\cdot;u_{01})$ and $u(t,\cdot;u_{02})$ exist.

(3) If $u_{01}, u_{02} \in X$ and $u_{01} \ll u_{02}$, then $u(t,\cdot;u_{01}) \ll u(t,\cdot;u_{02})$ for $t > 0$ at which both $u(t,\cdot;u_{01})$ and $u(t,\cdot;u_{02})$ exist.

Proof. (1) The case that $\mathcal{H} = \mathcal{H}_1(= \mathbb{R}^N)$ and $\mathcal{A} = \mathcal{H}_1(= \Delta)$ follows from comparison principle for parabolic equations. We prove the case that $\mathcal{H} = \mathcal{H}_2(= \mathbb{R}^N)$ and $\mathcal{A} = \mathcal{A}_2$. The case that $\mathcal{H} = \mathcal{H}_3(= \mathbb{Z}^N)$ and $\mathcal{A} = \mathcal{A}_3$ can be proved similarly.

Observe that for any $t \in [0,\tau)$,

$$\begin{aligned}\int_{\mathbb{R}^N} \kappa(y-x)u^1(t,y)dy &= \int_{\mathbb{R}^N\setminus\Omega} \kappa(y-x)u^1(t,y)dy + \int_{\Omega} \kappa(y-x)u^1(t,y)dy\\ &\le \int_{\mathbb{R}^N\setminus\Omega} \kappa(y-x)u^2(t,y)dy + \int_{\Omega} \kappa(y-x)u^1(t,y)dy. \end{aligned} \tag{3.2}$$

Let $v(t,x) = u^2(t,x) - u^1(t,x)$. By (3.2),

$$\begin{aligned} v_t(t,x) &\ge \int_{\Omega} \kappa(y-x)v(t,y)dy - v(t,x) + u^2(t,x)f(t,x,u^2(t,x)) - u^1(t,x)f(t,x,u^1(t,x))\\ &= \int_{\Omega} \kappa(y-x)v(t,y)dy - v(t,x) + a(t,x)v(t,x), \quad x \in \bar{\Omega},\ t \in (0,\tau), \end{aligned}$$

where

$$a(t,x) = f(t,x,u^2(t,x)) + u^1(t,x)\int_0^1 \partial_u f(t,x,su^2(t,x) + (1-s)u^1(t,x))ds.$$

The rest of the proof follows from the arguments of [43, Proposition 2.4].

(2) It follows from (1) with $u^1(t,x) = u(t,x;;u_{01})$, $u^2(t,) = u(t,x;u_{02})$, and $\Omega = \mathcal{H}$.

(3) We provide a proof for the case that $\mathcal{H} = \mathcal{H}_2$ and $\mathcal{A} = \mathcal{A}_2$. Other cases can be proved similarly. Take any $\tau > 0$ such that both $u(t,\cdot;u_{01})$ and $u(t,\cdot;u_{02})$ exist on $[0,\tau]$. It suffices to prove that $u(t,\cdot;u_{02}) \gg u(t,\cdot;u_{01})$ for $t \in [0,\tau]$. To this end, let

$w(t,x) = u(t,x;u_{02}) - u(t,x;u_{01})$. Then $w(t,x)$ satisfies the following equation,

$$w_t(t,x) = \int_{\mathbb{R}^N} \kappa(y-x)w(t,y)dy - w(t,x) + a(t,x)w(t,x),$$

where

$$a(t,x) = f(x,u(t,x;u_{02})) + u(t,x;u_{01})\int_0^1 \partial_u f(x, su(t,x;u_{02}) + (1-s)u(t,x;u_{01}))ds.$$

Let $M > 0$ be such that $M \geq \sup_{x\in\mathbb{R}^N, t\in[0,\tau]}(1-a(t,x))$ and $\tilde{w}(t,x) = e^{Mt}w(t,x)$. Then $\tilde{w}(t,x)$ satisfies

$$\tilde{w}_t(t,x) = \int_{\mathbb{R}^N} \kappa(y-x)\tilde{w}(t,y)dy + [M-1+a(t,x)]\tilde{w}(t,x).$$

Let $\mathcal{K}: X \to X$ be defined by

$$(\mathcal{K}u)(x) = \int_{\mathbb{R}^N} \kappa(y-x)u(y)dy \quad \text{for} \quad u \in X. \tag{3.3}$$

Then $\mathcal{K}$ generates an analytic semigroup on X and

$$\tilde{w}(t,\cdot) = e^{\mathcal{K}t}(u_{02}-u_{01}) + \int_0^t e^{\mathcal{K}(t-\tau)}(M-1+a(\tau,\cdot))\tilde{w}(\tau,\cdot)d\tau.$$

Observe that $e^{\mathcal{K}t}u_0 \geq 0$ for any $u_0 \in X^+$ and $t \geq 0$ and $e^{\mathcal{K}t}u_0 \gg 0$ for any $u_0 \in X^{++}$ and $t \geq 0$. Observe also that $u_{02} - u_{01} \in X_2^{++}$. By (2), $\tilde{w}(\tau,\cdot) \geq 0$ and hence $(M-1+a(\tau,\cdot))\tilde{w}(\tau,\cdot) \geq 0$ for $\tau \in [0,T]$. It then follows that $\tilde{w}(t,\cdot) \gg 0$ and then $w(t,\cdot) \gg 0$ (i.e. $u(t,\cdot;u_{02}) \gg u(t,\cdot;u_{01})$) for $t \in [0,\tau]$. □

Proposition 3.2 (Global existence). *For any given $u_0 \in X^+$, $u(t,\cdot;u_0)$ exists for all $t \geq 0$.*

Proof. Let $u_0 \in X^+$ be given. There is $M \gg 1$ such that $0 \le u_0(x) \le M$ and $f(t,x,M) < 0$ for all $x \in \mathcal{H}$. Then by Proposition 3.1,

$$0 \le u(t,\cdot;u_0) \le M$$

for any $t > 0$ at which $u(t,\cdot;u_0)$ exists. It is then not difficult to prove that for any $\tau > 0$ such that $u(t,\cdot;u_0)$ exists on $(0,\tau)$, $\lim_{t\to\tau} u(t,\cdot;u_0)$ exists in X. This implies that $u(t,\cdot;u_0)$ exists and $u(t,\cdot;u_0) \ge 0$ for all $t \ge 0$. $\square$

3.2 Convergence on compact subsets and strip type subsets

In this section, we explore the convergence property of solutions of (1.3) on compact subsets and strip type subsets. As mentioned before, to indicate the dependence of solutions of (1.3) on the nonlinearity, we may write $u(t,\cdot;u_0)$ as $u(t,\cdot;u_0,f)$.

Proposition 3.3. *Suppose that $u_{0n}, u_0 \in X^+$ $(n = 1,2,\cdots)$ with $\{\|u_{0n}\|\}$ being bounded, and f_n, g_n $(n = 1,2\cdots)$ satisfy (H0) with $f_n(t,x,u)$, $g_n(t,x,u)$, and $\partial_u f_n(t,x,u)$ being bounded uniformly in $x \in \mathcal{H}$ and (t,u) on bounded subsets.*

(1) (Convergence on compact subsets) *If $u_{0n}(x) \to u_0(x)$ as $n \to \infty$ uniformly in x on bounded sets and $f_n(t,x,u) - g_n(t,x,u)$ as $n \to \infty$ uniformly in (t,x,u) on bounded sets, then for each $t > 0$, $u(t,x;u_{0n},f_n) - u(t,x;u_0,g_n) \to 0$ as $n \to \infty$ uniformly in x on bounded sets.*

(2) (Convergence on strip type subsets) *If $u_{0n}(x) \to u_0(x)$ as $n \to \infty$ uniformly in x on any set E with $\{x\cdot\xi \,|\, x \in E\}$ being a bounded set of $\mathbb{R}$ and $f_n(t,x,u) - g_n(t,x,u) \to 0$ as $n \to \infty$ uniformly in (t,x,u) on any set E with $\{(t,x\cdot\xi,u)\,|\,(t,x,u) \in E\}$ being a bounded set of $\mathbb{R}^3$, then for each $t > 0$, $u(t,x;u_{0n},f_n) - u(t,x;u_0,g_n) \to 0$ as $n \to \infty$ uniformly in x on any set E with $\{x\cdot\xi \,|\, x \in E\}$ being a bounded set of $\mathbb{R}$.*

Proof. (1) We prove the case that $\mathcal{H} = \mathcal{H}_2$ and $\mathcal{A}_2$. Other cases can be proved similarly.

Let $v^n(t,x) = u(t,x;u_{0n},f_n) - u(t,x;u_0,g_n)$. Then $v^n(t,x)$ satisfies

$$v_t^n(t,x) = \int_{\mathbb{R}^N} \kappa(y-x)v^n(t,y)dy - v^n(t,x) + a_n(t,x)v^n(t,x) + b_n(t,x),$$

where

$$\begin{aligned} a_n(t,x) =& f_n(t,x,u(t,x;u_{0n},f_n)) \\ &+ u(t,x;u_0,f_n)\cdot \int_0^1 \partial_u f_n(t,x,su(t,x;u_{0n},f_n)+(1-s)u(t,x;u_0,g_n))ds \end{aligned}$$

and

$$b_n(t,x) = u(t,x;u_0,g_n)\cdot \big(f_n(t,x,u(t,x;u_0,g_n)) - g_n(t,x,u(t,x;u_0,g_n))\big).$$

Observe that $\{a_n(t,x)\}$ is uniformly bounded and continuous in t and x and $b_n(t,x)\to 0$ as $n\to\infty$ uniformly in (t,x) on bounded sets of $[0,\infty)\times\mathbb{R}^N$.

Take a $\rho>0$. Let

$$X(\rho) = \{u\in C(\mathbb{R}^N,\mathbb{R}) \,|\, u(\cdot)e^{-\rho\|\cdot\|}\in X\}$$

with norm $\|u\|_\rho = \|u(\cdot)e^{-\rho\|\cdot\|}\|$. Note that $\mathcal{K}: X(\rho)\to X(\rho)$ also generates an analytic semigroup, where $\mathcal{K}$ is as in (3.3), and there are $M>0$ and $\omega>0$ such that

$$\|e^{(\mathcal{K}-\mathcal{I})t}\|_{X(\rho)} \le Me^{\omega t} \quad \forall t\ge 0,$$

where $\mathcal{I}$ is the identity map on $X(\rho)$. Hence

$$\begin{aligned} v^n(t,\cdot) =& e^{(\mathcal{K}-\mathcal{I})t}v^n(0,\cdot) + \int_0^t e^{(\mathcal{K}-\mathcal{I})(t-\tau)}a_n(\tau,\cdot)v^n(\tau,\cdot)d\tau \\ &+ \int_0^t e^{(\mathcal{K}-\mathcal{I})(t-\tau)}b_n(\tau,\cdot)d\tau \end{aligned}$$

and then

$$
\begin{aligned}
\|v^n(t,\cdot)\|_{X(\rho)} &\le Me^{\omega t}\|v^n(0,\cdot)\|_{X(\rho)} + M \sup_{\tau\in[0,t],x\in\mathbb{R}^N}|a_n(\tau,x)| \int_0^t e^{\omega(t-\tau)}\|v^n(\tau,\cdot)\|_{X(\rho)}d\tau \\
&\quad + M\int_0^t e^{\omega(t-\tau)}\|b_n(\tau,\cdot)\|_{X(\rho)}d\tau \\
&\le Me^{\omega t}\|v^n(0,\cdot)\|_{X(\rho)} + M \sup_{\tau\in[0,t],x\in\mathbb{R}^N}|a_n(\tau,x)| \int_0^t e^{\omega(t-\tau)}\|v^n(\tau,\cdot)\|_{X(\rho)}d\tau \\
&\quad + \frac{M}{\omega}\sup_{\tau\in[0,t]}\|b_n(\tau,\cdot)\|_{X(\rho)}e^{\omega t}.
\end{aligned}
$$

By Gronwall's inequality,

$$
\|v^n(t,\cdot)\|_{X(\rho)} \le e^{(\omega+M\sup_{\tau\in[0,t],x\in\mathbb{R}^N}|a_n(\tau,x)|)t}\Big(M\|v^n(0,\cdot)\|_{X(\rho)} + \frac{M}{\omega}\sup_{\tau\in[0,t]}\|b_n(\tau,\cdot)\|_{X(\rho)}\Big).
$$

Note that

$$
\|v^n(0,\cdot)\|_{X(\rho)} \to 0
$$

and

$$
\sup_{\tau\in[0,t]}\|b_n(\tau,\cdot)\|_{X(\rho)} \to 0 \quad \text{as} \quad n\to\infty.
$$

It then follows that

$$
\|v^n(t,\cdot)\|_{X(\rho)} \to 0 \quad \text{as} \quad n\to\infty
$$

and then

$$
u(t,x;u_{0n},f_n) - u(t,x;u_0,g_n) \quad \text{as} \quad n\to\infty
$$

uniformly in x on bounded sets.

(2) It can be proved by similar arguments as in (1) with $X(\rho)$ being replaced by $X_\xi(\rho)$, where

$$
X_\xi(\rho) = \{u\in C(\mathcal{H},\mathbb{R})\,|\,u_{\xi,\rho}\in X\},
$$

with norm $\|u\|_{X_\xi(\rho)} = \|u_{\xi,\rho}\|_X$, where $u_{\xi,\rho}(x) = e^{-\rho|x\cdot\xi|}u(x)$. $\square$

3.3 Part metric

In this section, we investigate the decreasing property of the so called part metric between two positive solutions of (1.3). Throughout this subsection, we also assume (H0).

First, we introduce the notion of part metric. For given $u, v \in X^{++}$, define

$$\rho(u, v) = \inf\{\ln \alpha \,|\, \frac{1}{\alpha} u \le v \le \alpha u,\ \alpha \ge 1\}.$$

Observe that $\rho(u, v)$ is well defined and there is $\alpha \ge 1$ such that $\rho(u, v) = \ln \alpha$. Moreover,

$$\rho(u, v) = \rho(v, u)$$

and

$$\rho(u, v) = 0 \quad \text{iff} u \equiv v.$$

In literature, $\rho(u, v)$ is called the *part metric* between u and v.

Proposition 3.4 (Strict decreasing of part metric)**.** *For any $\epsilon > 0$, $\sigma > 0$, $M > 0$, and $\tau > 0$ with $\epsilon < M$ and $\sigma \le \ln \frac{M}{\epsilon}$, there is $\delta > 0$ such that for any $u_0, v_0 \in X^{++}$ with $\epsilon \le u_0(x) \le M$, $\epsilon \le v_0(x) \le M$ for $x \in \mathcal{H}$ and $\rho(u_0, v_0) \ge \sigma$, there holds*

$$\rho(u(\tau, \cdot; u_0), u(\tau, \cdot; v_0)) \le \rho(u_0, v_0) - \delta.$$

Proof. We give a proof for the case that $\mathcal{H} = \mathcal{H}_1$ and $\mathcal{A} = \mathcal{A}_1$. Other cases can be proved similarly.

Let $\epsilon > 0$, $\sigma > 0$, $M > 0$, and $\tau > 0$ be given and $\epsilon < M$, $\sigma < \ln \frac{M}{\epsilon}$. First, note that by Proposition 3.1, there are $\epsilon_1 > 0$ and $M_1 > 0$ such that for any $u_0 \in X^{++}$ with $\epsilon \le u_0(x) \le M$ for $x \in \mathbb{R}^N$, there holds

$$\epsilon_1 \le u(t, x; u_0) \le M_1 \quad \forall\ t \in [0, \tau],\ x \in \mathbb{R}^N. \tag{3.4}$$

Let

$$\delta_1 = \epsilon_1^2 e^{\sigma}(1-e^{\sigma}) \sup_{t\in[0,\tau],x\in\mathbb{R}^N,u\in[\epsilon_1,M_1M/\epsilon]} f_u(t,x,u). \tag{3.5}$$

Then $\delta_1 > 0$ and there is $0 < \tau_1 \le \tau$ such that

$$\frac{\delta_1}{2}\tau_1 < e^{\sigma}\epsilon_1 \tag{3.6}$$

and

$$\Big|\frac{\delta_1}{2} tvf_u(t,x,w)\Big| + \Big|\frac{\delta_1}{2}tf(t,x,v-\frac{\delta_1}{2}t)\Big| \le \frac{\delta_1}{2} \tag{3.7}$$

for any $t\in[0,\tau_1],\ x\in\mathbb{R}^N,\ v,w\in[0,M_1M/\epsilon]$.

Let

$$\delta_2 = \frac{\delta_1\tau_1}{2M_1}. \tag{3.8}$$

Then $\delta_2 < e^{\sigma}$ and $0 < \frac{\delta_2\epsilon}{M} < 1$. Let

$$\delta = -\ln\big(1-\frac{\delta_2\epsilon}{M}\big). \tag{3.9}$$

Then $\delta > 0$. We prove that δ defined in (3.9) satisfies the property in the proposition.

For any $u_0, v_0 \in X^{++}$ with $\epsilon \le u_0(x) \le M$ and $\epsilon \le v_0(x) \le M$ for $x\in\mathbb{R}$ and $\rho(u_0,v_0)\ge\sigma$, there is $\alpha^*\ge 1$ such that

$$\rho_1(u_0,v_0) = \ln\alpha^*$$

and

$$\frac{1}{\alpha^*}u_0 \le v_0 \le \alpha^* u_0.$$

Note that $e^{\sigma}\le\alpha^*\le\frac{M}{\epsilon}$. We first show that $\rho(u(t,\cdot;u_0),u(t,\cdot;v_0))$ is non-increasing in $t>0$.

By Proposition 3.1,

$$u(t,\cdot;v_0)\le u(t,\cdot;\alpha^*u_0) \quad \text{for} \quad t>0.$$

Let

$$v(t,x) = \alpha^* u(t,x;u_0).$$

We then have

$$\begin{aligned} v_t(t,x) &= \Delta v(t,x) + v(t,x) f(t,x,u(t,x;u_0)) \\ &= \Delta v(t,x) + v(t,x) f(t,x,v(t,x)) + v(t,x) f(t,x,u(t,x;u_0)) \\ &\quad - v(t,x) f(t,x,v(t,x)) \\ &\geq \Delta v(t,x) + v(t,x) f(t,x,v(t,x)) \quad \forall t > 0. \end{aligned}$$

By Proposition 3.1 again,

$$u(t,\cdot;\alpha^* u_0) \leq \alpha^* u(t,\cdot;u_0)$$

and hence

$$u(t,\cdot;v_0) \leq \alpha^* u(t,\cdot;u_0)$$

for $t > 0$. Similarly, we can prove that

$$\frac{1}{\alpha^*} u(t,\cdot;u_0) \leq u(t,\cdot;v_0)$$

for $t > 0$. It then follows that

$$\rho(u(t,\cdot;u_0), u(t,\cdot;v_0)) \leq \rho(u_0,v_0) \quad \forall\ t \geq 0$$

and then

$$\rho(u(t_2,\cdot;u_0), u(t_2,\cdot;v_0)) \leq \rho(u(t_1,\cdot;u_0), u(t_1,\cdot;v_0)) \quad \forall\ 0 \leq t_1 \leq t_2.$$

Next, we prove that

$$\rho(u(\tau,\cdot;u_0),u(\tau,\cdot;v_0)) \le \rho(u_0,v_0)-\delta.$$

Note that $e^{\sigma} \le \alpha^* \le \frac{M}{\epsilon}$ and

$$\begin{aligned}
v_t(t,x) &= \Delta v(t,x) + v(t,x) f(t,x,u(t,x;u_0)) \\
&= \Delta v(t,x) + v(t,x) f(t,x,v(t,x)) + v(t,x) f(t,x,u(t,x;u_0)) \\
&\quad - v(t,x) f(t,x,v(t,x)) \\
&\ge \Delta v(t,x) + v(t,x) f(t,x,v(t,x)) + \delta_1 \quad \forall 0 < t \le \tau_1.
\end{aligned}$$

This together with (3.6), (3.7) implies that

$$(v(t,x)-\frac{\delta_1}{2}t)_t \ge \Delta\big(v(t,x)-\frac{\delta_1}{2}t\big) + \big(v(t,x)-\frac{\delta_1}{2}t\big) f\big(t,x,v(t,x)-\frac{\delta_1}{2}t\big)$$

for $0 < t \le \tau_1$. Then by Proposition 3.1 again,

$$u(t,\cdot;\alpha^* u_0) \le \alpha^* u(t,\cdot;u_0) - \frac{\delta_1}{2}t \quad \text{for} \quad 0 < t < \tau_1.$$

By (3.8),

$$u(\tau_1,\cdot;v_0) \le (\alpha^*-\delta_2) u(\tau_1,\cdot;u_0).$$

Similarly, it can be proved that

$$\frac{1}{\alpha^*-\delta_2} u(\tau_1,\cdot;u_0) \le u(\tau_1,\cdot;v_0).$$

It then follows that

$$\rho(u(\tau_1,\cdot;u_0),u(\tau_1,\cdot;v_0)) \le \ln(\alpha^*-\delta_2) = \ln\alpha^* + \ln(1-\frac{\delta_2}{\alpha^*}) \le \rho(u_0,v_0)-\delta.$$

and hence

$$\rho(u(\tau,\cdot;u_0),u(\tau,\cdot;v_0)) \le \rho(u(\tau_1,\cdot;u_0),u(\tau_1,\cdot;v_0)) \le \rho(u_0,v_0)-\delta.$$

$\square$

3.4 Principal eigenvalue theory

In this section, we recall some principal eigenvalue theory for time periodic dispersal operators to be used in this thesis. We first recall some properties for general time and space periodic dispersal operators and then recall some special properties for time independent and space periodic dispersal operators.

3.4.1 Principal eigenvalues of time periodic dispersal operators

In this subsection, we recall some principal eigenvalue theory for both time and space periodic dispersal operators.

Let $\mathcal{X}_p$ and $\mathcal{A}_{\xi,\mu}$ be as in (2.13) and (2.14), respectively ($\xi \in S^{N-1}$ and $\mu \in \mathbb{R}$). For given $a \in \mathcal{X}_p$ and $\xi \in S^{N-1}$, $\mu \in \mathbb{R}$, let $\lambda_{\xi,\mu}(a)$ be as in (2.15).

Definition 3.1. *A real number λ_0 is said to be the principal eigenvalue of $-\partial_t+\mathcal{A}_{\xi,\mu}+a(\cdot,\cdot)\mathcal{I}$ if λ_0 is an isolated eigenvalue of $-\partial_t+\mathcal{A}_{\xi,\mu}+a(\cdot,\cdot)\mathcal{I}$ with a positive eigenfunction $\phi(\cdot,\cdot)$ (i.e. $\phi(t,x)>0$ for $(t,x)\in\mathbb{R}\times\mathcal{H}$ and $\phi(\cdot,\cdot)\in\mathcal{X}_p$) and for any $\lambda\in\sigma(-\partial_t+\mathcal{A}_{\xi,\mu}+a(\cdot,\cdot)\mathcal{I})$,* $\mathrm{Re}\mu \le \lambda_0$.

We remark that $\lambda_{\xi,\mu}(a)\in\sigma(-\partial_t+\mathcal{A}_{\xi,\mu}+a(\cdot,\cdot)\mathcal{I})$ and if $-\partial_t+\mathcal{A}_{\xi,\mu}+a(\cdot,\cdot)\mathcal{I}$ admits a principal eigenvalue λ_0, then $\lambda_0=\lambda_{\xi,\mu}(a)$. We also remark that in the case that $\mathcal{H}=\mathcal{H}_i$ and $\mathcal{A}=\mathcal{A}_i$ with $i=1$ or 3, principal eigenvalue of $-\partial_t+\mathcal{A}_{\xi,\mu}+a(\cdot,\cdot)\mathcal{I}$ always exists. But in the case that $\mathcal{H}=\mathcal{H}_2$ and $\mathcal{A}=\mathcal{A}_2$, $-\partial_t+\mathcal{A}_{\xi,\mu}+a(\cdot,\cdot)\mathcal{I}$ may not have a principal eigenvalue (see [17] and [74] for examples).

For given $a \in \mathcal{X}_p$, let

$$\hat{a}(x) = \frac{1}{T}\int_0^T a(t,x)dt.$$

The following proposition is established in [65] regarding principal eigenvalues of time periodic nonlocal dispersal operators.

Proposition 3.5. *(1) If $\hat{a}(\cdot)$ is C^N and there is $x_0 \in \mathbb{R}^N$ such that $\hat{a}(x_0) = \max_{x\in\mathbb{R}^N} \hat{a}(x_0)$ and the partial derivatives of $\hat{a}(x)$ up to order $N-1$ at x_0 are zero, then for any $\xi \in S^{N-1}$ and $\mu \in \mathbb{R}$, $\lambda_{\xi,\mu}(a)$ is the principal eigenvalue of $-\partial_t + \mathcal{A}_{\xi,\mu} + a(\cdot,\cdot)\mathcal{I}$.*

(2) Let $a(\cdot,\cdot) \in \mathcal{X}_p$ be given. For any $\epsilon > 0$, there is $a^{\pm}(\cdot,\cdot) \in \mathcal{X}_p$ such that $\lambda_{\xi,\mu}(a^{\pm})$ are principal eigenvalues of $-\partial_t + \mathcal{A}_{\xi,\mu} + a^{\pm}(\cdot,\cdot)\mathcal{I}$,

$$a^-(t,x) \le a(t,x) \le a^+(x,) \quad \forall (t,x) \in \mathbb{R}\times\mathcal{H},$$

and

$$\lambda_{\xi,\mu}(a^+) - \epsilon \le \lambda_{\xi,\mu}(a) \le \lambda_{\xi,\mu}(a^-) + \epsilon.$$

Proof. We only need to prove the case that $\mathcal{H} = \mathcal{H}_2$ and $\mathcal{A} = \mathcal{A}_2$.

(1) It follows from [65, Theorem B(1)].

(2) It follows from [65, Proposition 3.10, Lemma 4.1]. □

The following proposition shows that the temporal variation does not reduce the principal spectrum point of dispersal operators.

Proposition 3.6. *For any given $\xi \in S^{N-1}$, $\mu \in \mathbb{R}$, and $a \in \mathcal{X}_p$,*

$$\lambda_{\xi,\mu}(a) \ge \lambda_{\xi,\mu}(\hat{a}).$$

Proof. It follows from Theorem 6.5 in [73] (see also [42] for the case that $\mathcal{H} = \mathcal{H}_1$ and $\mathcal{A} = \mathcal{A}_1$ and see [43] for the case that $\mathcal{H} = \mathcal{H}_2$ and $\mathcal{A} = \mathcal{A}_2$, □

3.4.2 Principal eigenvalues of spatially periodic dispersal operators

In this subsection, we present some special principal eigenvalue theories for time independent but spatially periodic dispersal operators with random, nonlocal, and discrete dispersals.

Recall that, if $a(t,x) \equiv a(x)$, the eigenvalue problem (2.16) reduces to the eigenvalue problem (2.17). To be more precise, when $\mathcal{H} = \mathcal{H}_1$ and $\mathcal{A} = \mathcal{A}_1$, (2.16) reduces to

$$\begin{cases} \Delta u(x) - 2\mu\xi \cdot \nabla u(x) + (a(x) + \mu^2)u(x) = \lambda u(x), \quad x \in \mathbb{R}^N \\ u(x + p_i \mathbf{e_i}) = u(x), \quad x \in \mathbb{R}^N. \end{cases} \tag{3.10}$$

When $\mathcal{H} = \mathcal{H}_2$ and $\mathcal{A} = \mathcal{A}_2$, (2.16) reduces to

$$\begin{cases} \int_{\mathbb{R}^N} e^{-\mu(y-x)\cdot\xi} \kappa(y-x)u(y)dy - u(x) + a(x)u(x) = \lambda u(x), \quad x \in \mathbb{R}^N \\ u(x + p_i \mathbf{e_i}) = u(x), \quad x \in \mathbb{R}^N. \end{cases} \tag{3.11}$$

When $\mathcal{H} = \mathcal{H}_3$ and $\mathcal{A} = \mathcal{A}_3$, (2.16) reduces to

$$\begin{cases} \sum_{k\in K} a_k(e^{-\mu k\cdot\xi} u(j+k) - u(j)) + a(j)u(j) = \lambda u(j), \quad j \in \mathbb{Z}^N \\ u(j + p_i \mathbf{e_i}) = u(j), \quad j \in \mathbb{Z}^N. \end{cases} \tag{3.12}$$

Observe that when $\mu = 0$, (3.10), (3.11), and (3.12) are independent of ξ. Observe also that if $u(t,x) = e^{-\mu(x\cdot\xi - \frac{\lambda}{\mu}t)}\phi(x)$ is a solution of

$$u_t(t,x) = \Delta u(t,x) + a(x)u(t,x), \quad x \in \mathbb{R}^N \tag{3.13}$$

with $\phi(\cdot) \in X_{1,p} \setminus \{0\}$, or a solution of

$$u_t(t,x) = \int_{\mathbb{R}^N} k(y-x)u(t,y)dy - u(t,x) + a(x)u(t,x), \quad x \in \mathbb{R}^N \tag{3.14}$$

with $\phi(\cdot) \in X_{2,p} \setminus \{0\}$, or a solution of

$$u_t(t,j) = \sum_{k \in K} a_k(u(t,x+j) - u(t,j)) + a(j)u(t,j), \quad j \in \mathbb{Z}^N \tag{3.15}$$

with $\phi(\cdot) \in X_{3,p} \setminus \{0\}$, then λ is an eigenvalue of (3.10) or (3.11) or (3.12) with $\phi(\cdot)$ being a corresponding eigenfunction. If $a(x) = f(x,0)$, then (3.13) (resp. (3.14), (3.15)) is the linearized equation of (1.3) with $f(t,x,u) = f(x,u)$ and $\mathcal{H} = \mathcal{H}_1$ and $\mathcal{A}_1$ (resp. $\mathcal{H} = \mathcal{H}_2$ and $\mathcal{A} = \mathcal{A}_2$, $\mathcal{H} = \mathcal{H}_3$ and $\mathcal{A} = \mathcal{A}_3$) at $u = 0$.

For given $a_i(\cdot) \in X_{i,p}$, let $\hat{\hat{a}}_i$ be the space average of $a_i(\cdot)$ ($i = 1,2,3$), that is,

$$\begin{cases} \hat{\hat{a}}_i = \frac{1}{|D_i|} \int_{D_i} a_i(x)dx \quad \text{for} \quad i = 1,2 \\ \\ \hat{\hat{a}}_3 = \frac{1}{\#D_3} \sum_{j \in D_3} a_3(j), \end{cases} \tag{3.16}$$

where

$$D_i = [0,p_1] \times [0,p_2] \times \cdots \times [0,p_N] \cap \mathcal{H}_i, \;\; i = 1,2,3 \tag{3.17}$$

and

$$\begin{cases} |D_i| = p_1 \times p_2 \times \cdots \times p_N \text{ for } i = 1,2 \\ \\ \#D_3 = \text{ the cardinality of } D_3. \end{cases} \tag{3.18}$$

The following proposition shows a relation between $\lambda_{\mu,\xi}(a_i)$ and $\lambda_{\mu,\xi}(\hat{\hat{a}}_i)$ for $a_i \in X_{i,p}$.

Proposition 3.7 (Influence of spatial variation)**.** *For given* $1 \le i \le 3$, $\mu \in \mathbb{R}$, $\xi \in S^{N-1}$, *and* $a_i \in X_{i,p}$, *there holds*

$$\lambda_{\mu,\xi}(a_i) \ge \lambda_{\mu,\xi}(\hat{\hat{a}}_i).$$

Proof. The case $i = 1$ is well known. The cases $i = 2$ and 3 follow from [35, Theorem 2.1]. □

We remark that $\lambda_{\mu,\xi}(\hat{\hat{a}}_i)$ ($a_i \in X_{i,p}$, $i = 1, 2, 3$) have the following explicit expressions,

$$\begin{cases} \lambda_{\mu,\xi}(\hat{\hat{a}}_1) = \hat{\hat{a}}_1 + \mu^2 \\ \\ \lambda_{\mu,\xi}(\hat{\hat{a}}_2) = \int_{\mathbb{R}^N} e^{-\mu z\cdot\xi}\kappa(z)dz - 1 + \hat{\hat{a}}_2 \\ \\ \lambda_{\mu,\xi}(\hat{\hat{a}}_3) = \sum_{k\in K} a_k(e^{-\mu k\cdot\xi} - 1) + \hat{\hat{a}}_3. \end{cases} \tag{3.19}$$

3.5 Positive solutions and spreading speeds of KPP equations

In this section, we recall some existing results on the existence, uniqueness, and stability of time and space periodic positive solutions and spatial spreading speeds of (1.1).

3.5.1 Time periodic positive solutions and spreading speeds of KPP equations in periodic media

A solution $u(t, x)$ of (1.1) is called *time and space periodic solution* if it is a solution on $t \in \mathbb{R}$ and $u(t + T, x) = u(t, x + p_i\mathbf{e_i}) = u(t, x)$ for $t \in \mathbb{R}$, $x \in \mathcal{H}$, and $i = 1, 2, \cdots, N$. It is called a *positive solution* if $u(t, x) > 0$ for all t in the existence interval and $x \in \mathcal{H}$.

Proposition 3.8. *Consider* (1.1) *and assume that f_0 satisfies (H0) and $f_0(t, x + p_i\mathbf{e_i}, u) = f_0(t, x, u)$ ($i = 1, 2, \cdots, N$).*

(1) (Uniqueness of periodic positive solutions) If (1.1) *has a time and space periodic positive solution, then it is unique.*

(2) (Stability of periodic positive solutions) If (1.1) *has a time and space periodic positive solution $u^*(t, x)$, then it is globally asymptotically stable with respect to perturbations in $X_p^+ \setminus \{0\}$.*

(3) (Existence of periodic positive solutions) If $\lambda(f_0(\cdot,\cdot,0)) > 0$, then (1.1) *has a time and space periodic positive solution.*

Proof. The case that $\mathcal{H} = \mathcal{H}_1$ and $\mathcal{A} = \mathcal{A}_1$ follows from the results in [57]. The case that $\mathcal{H} = \mathcal{H}_2$ and $\mathcal{A} = \mathcal{A}_2$ follows from the results in [65]. The case that $\mathcal{H} = \mathcal{H}_3$ and $\mathcal{A} = \mathcal{A}_3$ can be proved by the similar arguments as in [65]. □

Corollary 3.1. *(1) If $\lambda(\hat{f}_0(\cdot,0)) > 0$, then* (1.1) *has a time and space periodic positive solution.*

(2) If $\lambda(\hat{\hat{f}}_0(0)) > 0$, then (1.1) *has a time and space periodic positive solution.*

Proof. (1) follows from Propositions 3.6 and 3.8.

(2) follows from Propositions 3.6, 3.7 and 3.8. □

Proposition 3.9. *Consider* (1.1). *Assume that f_0 satisfies (H0), $f_0(t,x+p_i\mathbf{e_i},u) = f_0(t,x,u)$ ($i = 1,2,\cdots,N$), and $\lambda(f_0(\cdot,\cdot,0)) > 0$. Then for any given $\xi \in S^{N-1}$,* (1.1) *has a spatial spreading speed $c_0^*(\xi)$ in the direction of ξ. Moreover,*

$$c_0^*(\xi) = \inf_{\mu>0} \frac{\lambda_{\xi,\mu}(f_0(\cdot,\cdot,0))}{\mu} \tag{3.20}$$

and for any $c < c_0^*(\xi)$ and $u_0 \in X^+(\xi)$,

$$\limsup_{x\cdot\xi\le ct,t\to\infty} |u(t,x;u_0,f_0(\cdot,\cdot+y,\cdot)) - u_0^*(t,x+y)| = 0$$

uniformly in $y \in \mathcal{H}$.

Proof. The cases that $\mathcal{H} = \mathcal{H}_i$ and $\mathcal{A} = \mathcal{A}_i$ for $i = 1,3$ follow from the results in [82] (see also [51], [61], [62]). The case that $\mathcal{H} = \mathcal{H}_2$ and $\mathcal{A} = \mathcal{A}_2$ follows from the results in [66]. □

To indicate the dependence of $c_0^*(\xi)$ on f_0, we may denote $c_0^*(\xi)$ by $c_0^*(\xi, f_0)$.

Corollary 3.2. *Consider* (1.1). *Assume that* f_0 *satisfies (H0),* $f_0(t, x+p_i\mathbf{e_i}, u) = f_0(t, x, u)$ *(*$i = 1, 2, \cdots, N$*), and* $\lambda(\hat{\hat{f}}_0(0)) > 0$*. Then for any given* $\xi \in S^{N-1}$, (1.1) *has a spatial spreading speed* $c_0^*(\xi)$ *in the direction of* ξ *and*

$$c_0^*(\xi, f_0) \geq c_0^*(\hat{f}_0) \geq c_0^*(\hat{\hat{f}}_0).$$

Proof. First, by Propositions 3.6, 3.7, and 3.9, $c_0^*(\xi, f_0$, $c_0^*(\xi, \hat{f}_0)$, and $c_0^*(\xi, \hat{\hat{f}}_0)$ exist. Moreover, by Propositions 3.6 and 3.7 again, and by (3.20),

$$c_0^*(\xi, f_0) \geq c_0^*(\hat{f}_0) \geq c_0^*(\hat{\hat{f}}_0).$$

□

3.5.2 KPP equations in spatially periodic media

In this subsection, we recall some additional spatial spreading dynamics of KPP equations in spatially periodic media.

Consider (1.1). Throughout this subsection, we assume that $f_0(t, x, u) \equiv f_0(x, u)$.

Proposition 3.10 (Spreading speeds)**.** *Consider* (1.1). *Assume that* $f_0(t, x, u) = f_0(x, u)$ *satisfies (H0),* $f_0(x + p_i\mathbf{e_i}, u) = f_0(x, u)$ *(*$i = 1, 2, \cdots, N$*), and* $\lambda(f_0(\cdot, 0)) > 0$*. Then for any given* $\xi \in S^{N-1}$, (1.1) *has a spatial spreading speed* $c_0^*(\xi)$ *in the direction of* ξ*. Moreover,* $c_0^*(\xi)$ *is of the following spreading features.*

(1) For each $u_0 \in X^+$ *satisfying that* $u_0(x) = 0$ *for* $x \in \mathcal{H}$ *with* $|x \cdot \xi| \gg 1$,

$$\limsup_{|x\cdot\xi| \geq ct, t\to\infty} u(t, x; u_0, f_0(\cdot,\cdot)) = 0 \quad \forall c > \max\{c_0^*(\xi), c_0^*(-\xi)\}.$$

(2) For each $\sigma > 0$, $r > 0$, *and* $u_0 \in X^+$ *satisfying that* $u_0(x) \geq \sigma$ *for* $x \in \mathcal{H}$ *with* $|x\cdot\xi| \leq r$,

$$\limsup_{|x\cdot\xi| \leq ct, t\to\infty} |u(t, x; u_0, f_0(\cdot,\cdot)) - u_0^*(x; f_0(\cdot,\cdot))| = 0$$

for all $0 < c < \min\{c_0^*(\xi), c_0^*(-\xi)\}$*.*

(3) For each $u_0 \in X^+$ *satisfying that* $u_0(x) = 0$ *for* $x \in \mathcal{H}$ *with* $\|x\| \gg 1$*,*

$$\limsup_{\|x\| \ge ct, t\to\infty} u(t, x; u_0, f_0(\cdot, \cdot)) = 0 \quad \forall c > \sup_{\xi \in S^{N-1}} c_0^*(\xi).$$

(4) For each $\sigma > 0$*,* $r > 0$*, and* $u_0 \in X^+$ *satisfying that* $u_0(x) \ge \sigma$ *for* $x \in \mathcal{H}$ *with* $\|x\| \le r$*,*

$$\limsup_{\|x\| \le ct, t\to\infty} |u(t, x; u_0, f_0(\cdot, \cdot)) - u_0^*(x)| = 0 \quad \forall 0 < c < \inf_{\xi \in S^{N-1}} c_0^*(\xi).$$

Proof. The cases $\mathcal{H} = \mathcal{H}_i$ and $\mathcal{A} = \mathcal{A}_i$ with $i = 1, 3$ follow from [51, Theorems 3.1-3.4 and Corollary 3.1] (see also [82, Theorems 1.2-2.3]) and the case $\mathcal{H} = \mathcal{H}_2$ and $\mathcal{A} = \mathcal{A}_2$ follows from [75, Theorems D and E]. □

Chapter 4

Existence, Uniqueness, and Stability of Time Periodic Strictly Positive Solutions

In this chapter, we explore the existence, uniqueness, and stability of time periodic strictly positive solutions of (1.3) and prove Theorem 2.1.

4.1 Uniqueness and stability

In this subsection, we prove the uniqueness and stability of time periodic strictly positive solutions of (1.3) (if exist), i.e. prove Theorem 2.1(1) and (2).

Proof of Theorem 2.1 (1). Suppose that there are two time periodic strictly positive solutions $u^1(t,x)$ and $u^2(t,x)$. Then

$$\rho(u^1(t+T,\cdot),u^2(t+T,\cdot)) = \rho(u^1(t,\cdot),u^2(t,\cdot))$$

for any $t \in \mathbb{R}$. By Proposition 3.4, we must have

$$u^1(t,x) \equiv u^2(t,x).$$

Thus time periodic positive solutions are unique. $\square$

Proof of Theorem 2.1 (2). First of all, for any $u_0 \in X^{++}$, by Proposition 3.4,

$$\rho(u(t,\cdot;u_0),u^*(t,\cdot)) \le \rho(u_0,u^*(0,\cdot)) \quad \forall\ t \ge 0. \tag{4.1}$$

This implies that $u^*(t,x)$ is stable with respect to perturbations in X^{++}.

Next, for any given $u_0 \in X^{++}$, we show

$$\|u(t, \cdot; u_0, f(\cdot + \tau, \cdot, \cdot)) - u^*(t + \tau, \cdot)\| \to 0 \tag{4.2}$$

as $t \to \infty$ uniformly in $\tau \in \mathbb{R}$. Thanks to the periodicity of $f(t, x, u)$ and $u^*(t, x)$ in t, we only need to show that the limit in (4.2) exists and is uniformly in $\tau \in [0, T]$. Moreover, note that

$$u(t, x; u_0, f(\cdot + \tau, \cdot, \cdot)) = u(t - T + \tau, x; u(T - \tau, \cdot; u_0, f(\cdot + \tau, \cdot, \cdot)), f)$$

for any $t \geq T - \tau$. By Proposition 3.1,

$$\inf_{\tau \in [0,T], x \in \mathcal{H}} u(T - \tau, x; u_0, f(\cdot + \tau, \cdot, \cdot)) > 0.$$

It then suffices to prove that the limit in(4.2) exists for $\tau = 0$.

Let $\alpha_0 \geq 1$ be such that $\rho(u_0, u^*(0, \cdot)) = \ln \alpha_0$ and

$$\frac{1}{\alpha_0} u^*(0, x) \leq u_0(x) \leq \alpha_0 u^*(0, x) \quad \forall\, x \in \mathcal{H}.$$

By Proposition 3.4, there is $\alpha_\infty \geq 1$ such that

$$\lim_{t \to \infty} \rho(u(t, \cdot; u_0), u^*(t, \cdot)) = \ln \alpha_\infty.$$

Moreover, by (4.1),

$$\rho(u(t, \cdot; u_0), u^*(t, \cdot)) \leq \rho(u_0, u^*(0, \cdot)) = \ln \alpha_0$$

and hence

$$\frac{1}{\alpha_0} u^*(t, x) \leq u_0(t, x; u_0) \leq \alpha_0 u^*(t, x) \quad \forall\, t > 0,\ x \in \mathcal{H}. \tag{4.3}$$

If $\alpha_\infty = 1$, then for any $\epsilon > 0$, there is $\tau > 0$ such that for $t \geq \tau$,

$$\rho(u(t, \cdot; u_0), u^*(t, \cdot)) \leq \ln(1 + \epsilon).$$

This implies that

$$\frac{1}{1+\epsilon} u^*(t, x) \leq u(t, x; u_0) \leq (1 + \epsilon) u^*(t, x) \quad \forall\ t \geq \tau,\ x \in \mathcal{H}.$$

Hence

$$|u(t, x; u_0) - u^*(t, x)| \leq \epsilon u^*(t, x) \quad \forall\ t \geq \tau,\ x \in \mathcal{H}.$$

It then follows that

$$\lim_{t \to \infty} \|u(t, \cdot; u_0) - u^*(t, \cdot)\| = 0.$$

Assume $\alpha_\infty > 1$. By (4.3), there are $\epsilon > 0,\ M > 0$, and $\sigma > 0$ such that

$$\epsilon \leq u(t, x; u_0) \leq M,\ \ \epsilon \leq u^*(t, x) \leq M \quad \forall\ t \geq 0,\ x \in \mathcal{H}$$

and

$$\rho(u(t, \cdot; u_0), u^*(t, \cdot)) \geq \sigma \quad \forall t \geq 0.$$

By Proposition 3.4 again, there is $\delta > 0$ such that for any $n \geq 1$,

$$\rho(u(nT, \cdot; u_0), u^*(nT, \cdot)) \leq \rho(u((n-1)T, \cdot; u_0), u^*((n-1)T, \cdot)) - \delta$$

and hence

$$\rho(u(nT, \cdot; u_0), u^*(nT, \cdot)) \leq \rho(u_0, u^*(0, \cdot)) - n\delta \quad \forall\ n \geq 1.$$

Let $n \to \infty$, we have

$$\lim_{n \to \infty} \rho(u(nT, \cdot; u_0), u^*(nT, \cdot)) = -\infty.$$

This is a contradiction.

Therefore, we must have $\alpha_\infty = 1$ and

$$\lim_{t\to\infty} \|u(t,\cdot;u_0) - u^*(t,\cdot)\| = 0.$$

$\square$

4.2 Existence

In this subsection, we show the existence of time periodic strictly positive solutions of (1.3), i.e., show Theorem 2.1(3). To this end, we first prove some lemmas.

Throughout this subsection, we assume the conditions in Theorem 2.1(3). Then by Proposition 3.8, (1.1) has a unique time and space periodic positive solution $u_0^*(t,x)$. Let $\delta_0 > 0$ be such that

$$0 < \delta_0 < \inf_{(t,x)\in\mathbb{R}\times\mathcal{H}} u_0^*(t,x).$$

Let $\psi_0 : \mathbb{R} \to \mathbb{R}^+$ be a non-increasing smooth function such that

$$0 \le \psi_0(\cdot) \le \delta_0, \quad \liminf_{r\to-\infty} \psi_0(r) > 0, \quad \psi_0(r) = 0 \quad \forall\ r \gg 1. \tag{4.4}$$

Lemma 4.1. *For give $\xi \in S^{N-1}$, let $u_0(x) = \psi_0(x\cdot\xi)$ and $u_{n,\xi}(x) = u_0(x+n\xi)$ $(n \in \mathbb{N})$. There are $K \ge 0$ and $n^* \ge 0$ such that*

$$u(KT,\cdot;u_{n^*,\xi},f) \ge u_{n^*,\xi}(\cdot).$$

Proof. Let $\epsilon > 0$ be such that

$$\delta_0 < \inf_{(t,x)\in\mathbb{R}\times\mathcal{H}} u_0^*(t,x) - \epsilon.$$

Fix $0 < c^{'} < c_0^*(\xi)$. By the spreading properties of (1.1), there is $K \in \mathbb{N}$ such that,

$$u(KT, x; u_0, f_0(\cdot, \cdot + y, \cdot) \geq u_0^*(t, x + y) - \epsilon/2 \tag{4.5}$$

for $x \cdot \xi \leq c^{'} KT$ and $y \in \mathcal{H}$. Without loss of generality, we may assume that

$$u_0(x) = 0 \quad \text{for} \quad x \cdot \xi \geq c^{'} KT.$$

Observe that

$$f(t, x - n\xi, u) - f_0(t, x - n\xi, u) \to 0$$

as $n \to \infty$ uniformly in (t, x, u) for (t, u) in bounded sets and x in sets with $x \cdot \xi$ being bounded. Then by Proposition 3.3, there exists $n^* \in \mathbb{N}$ such that

$$u(KT, x; u_0, f(\cdot, \cdot - n\xi, \cdot)) \geq u(KT, x; u_0, f_0(\cdot, \cdot - n \cdot \xi, \cdot) - \epsilon/2, \quad n \geq n^* \tag{4.6}$$

for $c^{'} KT - 1 \leq x \cdot \xi \leq c^{'} KT$. This together with (4.5) implies that

$$u(KT, x; u_0, f(\cdot, \cdot - n\xi, \cdot)) \geq u_0^*(KT, x - n\xi) - \epsilon, \quad n \geq n^* \tag{4.7}$$

for $x \cdot \xi \in [c^{'} KT - 1, c^{'} KT]$.

Note that

$$u(KT, x + n\xi; u_0, f(\cdot, \cdot - n\xi, \cdot)) = u(KT, x; u_0(\cdot + n\xi), f).$$

We then have

$$u(KT, x; u_0(\cdot + n^*\xi), f) \geq u_0^*(KT, x) - \epsilon \tag{4.8}$$

for $x \cdot \xi \in [c^{'} KT - 1 - n^*, c^{'} KT - n^*]$.

By comparison principle and (4.7),

$$\begin{aligned}
u(KT,x;u_0(\cdot+n^*\xi),f) &= u(KT,x+(n^*+1)\xi;u_0(\cdot-\xi),f(\cdot,\cdot-(n^*+1)\xi,\cdot)\\
&\geq u(KT,x+(n^*+1)\xi;u_0,f(\cdot,\cdot-(n^*+1)\xi,\cdot)\\
&\geq u_0^*(KT,x)-\epsilon
\end{aligned}$$

for $x\cdot\xi\in[c^{'}KT-(n^*+2),c^{'}KT-(n^*+1)]$. This together with (4.8) implies that

$$u(KT,x;u_0(\cdot+n^*\xi),f)\geq u_0^*(KT,x)-\epsilon$$

for $x\cdot\xi\in[c^{'}KT-(n^*+2),c^{'}KT-n^*]$.

By induction, we have

$$u(KT,x;u_0(\cdot+n^*\xi),f)\geq u_0^*(KT,x)-\epsilon$$

for $x\cdot\xi\in(-\infty,c^{'}KT-n^*]$. The lemma then follows from the fact that

$$u_0(x+n^*\xi)\leq\delta_0<\inf_{(t,x)\in\mathbb{R}\times\mathcal{H}}u_0^*(t,x)-\epsilon$$

for all $x\in\mathcal{H}$ and $u_0(x+n^*\xi)=0$ for $x\cdot\xi\geq c^{'}KT-n^*$. □

Lemma 4.2. *Let $u_0(\cdot)$, K, and n^* be as in Lemma 4.1. For any $M\gg 1$, there exists $\tilde{\delta}_0>0$ such that $u(t,x;u_{n^*,\xi})\geq\tilde{\delta}_0$ for $t\geq KT$ and $x\cdot\xi\leq M$.*

Proof. By Proposition 3.1 and Lemma 4.1,

$$u(mKT,\cdot;u_{n^*,\xi},f)\geq u_{n^*,\xi}(\cdot)\quad\forall\ m\geq 1. \tag{4.9}$$

Moreover, by the arguments of Lemma 4.1,

$$u(mKT, x; u_{n^*,\xi}, f) \geq \delta_0 \quad \forall\, m \geq 1,\ x \cdot \xi \leq -n^*.$$

It then suffices to prove that for any $M \gg 1$,

$$\inf_{t \in [KT,(K+1)T], x \in \mathcal{H}, x\cdot\xi \in [-n^*,M]} u(t, x; u_{n^*,\xi}, f) > 0. \tag{4.10}$$

Suppose that (4.10) does not hold. Then there are $t_n \in [KT, (K+1)T]$ and $x_n \in \mathcal{H}$ with $x_n \cdot \xi \in [-n^*, M]$, $\|x_n\| \to \infty$ such that

$$u(t_n, x_n; u_{n^*,\xi}, f) \to 0 \quad \text{as} \quad n \to \infty. \tag{4.11}$$

Without loss of generality, we may assume that $t_n \to t^*$ as $n \to \infty$ for some $t^* \in [KT, (K+1)T]$. We then have

$$\|u(t_n, \cdot; u_{n^*,\xi}, f) - u(t^*, \cdot; u_{n^*,\xi}, f)\| \to 0 \quad \text{as} \quad n \to \infty$$

and hence

$$u(t_n, x_n; u_{n^*,\xi}, f) - u(t^*, x_n; u_{n^*,\xi}, f) \to 0 \quad \text{as} \quad n \to \infty.$$

Observe that

$$u(t^*, x_n; u_{n^*,\xi}, f) = u(t^*, 0; u_{n^*,\xi}(\cdot + x_n), f(\cdot, \cdot + x_n, \cdot)) \quad \forall\, n \geq 1$$

and

$$f(t, x + x_n, u) - f_0(t, x + x_n, u) \to 0$$

as $n \to \infty$ uniformly in (t,x,u) on bounded sets. Observe also that there is $n^{**} \geq n^*$ such that

$$u_{n^*,\xi}(\cdot + x_n) \geq u_{n^{**},\xi}(\cdot) \quad \forall\, n \geq 1.$$

By Propositions 3.1 and 3.3, we have

$$u(t^*, 0; u_{n^*,\xi}(\cdot + x_n), f(\cdot, \cdot + x_n, \cdot)) \geq u(t^*, 0; u_{n^{**},\xi}(\cdot), f(\cdot, \cdot + x_n, \cdot))$$

and

$$u(t^*, 0; u_{n^{**},\xi}(\cdot), f(\cdot, \cdot + x_n, \cdot)) - u(t^*, 0; u_{n^{**},\xi}(\cdot), f_0(\cdot, \cdot + x_n, \cdot)) \to 0 \tag{4.12}$$

as $n \to \infty$. Without loss of generality, we may also assume that there is $\tilde{x} \in \mathcal{H}$ such that

$$f_0(t, x + x_n, u) \to f_0(t, x + \tilde{x}, u) \quad \text{as} \quad n \to \infty$$

uniformly in (t,x,u) on bounded sets. Then by Proposition 3.3 again,

$$u(t^*, 0; u_{n^{**},\xi}(\cdot), f_0(\cdot, \cdot + x_n, \cdot)) \to u(t^*, 0; u_{n^{**},\xi}(\cdot), f_0(\cdot, \cdot + \tilde{x}, \cdot)) \quad \text{as} \quad n \to \infty. \tag{4.13}$$

By Proposition 3.1,

$$u(t^*, 0; u_{n^{**},\xi}(\cdot), f_0(\cdot, \cdot + \tilde{x}, \cdot)) > 0. \tag{4.14}$$

It then follows from (4.12), (4.13), and (4.14) that

$$\liminf_{n \to \infty} u(t_n, x_n; u_{n^*,\xi}, f) > 0,$$

which contradicts to (4.11). Therefore, (4.10) holds and the lemma thus follows. □

Observe that for any $M \geq M_0$, $u(t,x) \equiv M$ is a supersolution of (1.3) on $\mathcal{H}$. Hence

$$u(T, x; M, f) \leq M$$

and then by Proposition 3.1, $u(nT, x; M, f)$ decreases at n increases. Define

$$u^+(x) := \lim_{n\to\infty} u(nT, x; M, f). \tag{4.15}$$

Then $u^+(x)$ is a Lebesgue measurable and upper semi-continuous function. In the following, we fix an $M \geq \max\{M_0, \delta_0\}$.

Lemma 4.3. *There exists $\bar{\delta} > 0$ such that $u^+(x) \geq \bar{\delta}$ for $x \in \mathbb{R}^{\mathbb{N}}$.*

Proof. Let $\psi_0(\cdot)$ be as in (4.4) and

$$u_{\pm i}(x) = \psi_0(\pm x \cdot \mathbf{e_i}), \quad i = 1, 2, \cdots, N.$$

By Proposition 3.1,

$$u(t, \cdot; M, f) \geq u(t, \cdot; u_{\pm i}, f)$$

for $t \geq 0$ and $i = 1, 2, \cdots, N$. By Lemma 4.2, there is $\bar{\delta}$ and $\bar{T} > 0$ such that

$$u(t, x; u_{\pm i}, f) \geq \bar{\delta} \quad \forall\ t \geq \bar{T},\ x \cdot \mathbf{e_i} \leq 0,\ i = 1, 2, \cdots, N.$$

It then follows that

$$u(t, x; M, f) \geq \bar{\delta} \quad \forall\ t \geq \bar{T},\ x \in \mathcal{H}.$$

This implies that

$$u(mT, x; M, f) \geq \bar{\delta} \quad \forall\ m \gg 1,\ x \in \mathcal{H}$$

and then

$$u^+(x) \geq \bar{\delta} \quad \forall\ x \in \mathcal{H}.$$

The lemma thus follows. $\square$

Now we prove the existence of time periodic positive solutions

Proof of Theorem 2.1(3). We first claim that

$$\rho(u(nT,\cdot;\bar{\delta}/2),u(nT,\cdot;M)) \to 0 \tag{4.16}$$

as $n \to \infty$. Assume this is not true. Let

$$\rho_n = \rho(u(nT,\cdot;\bar{\delta}/2),u(nT,\cdot;M))$$

and

$$\rho_\infty = \lim_{n\to\infty} \rho_n$$

(the existence of this limit follows from Proposition 3.4). Then $\rho_\infty > 0$,

$$\frac{\bar{\delta}}{e^{\rho_0}} \le \frac{1}{e^{\rho_0}} u(nT,\cdot;M) \le u(nT,\cdot;\bar{\delta}/2) \le e^{\rho_0} u(nT,\cdot;M) \le e^{\rho_0} M$$

for $n = 0,1,2,\cdots$ and

$$\rho(u(nT,\cdot;\bar{\delta}/2),u(nT,\cdot;M)) \ge \rho_\infty$$

for $n = 1,2,\cdots$. By Proposition 3.4, there is $\delta > 0$ such that

$$\rho(u(nT,\cdot;\bar{\delta}/2),u(nT,\cdot;M)) \le \rho_0 - n\delta \quad \forall\ n = 1,2,\cdots.$$

This implies that $\rho_\infty = -\infty$, a contradiction. Therefore, (4.16) holds.

By (4.16), there is $K_1 \ge 1$ such that

$$\bar{\delta}/2 \le u(K_1 T,\cdot,\bar{\delta}/2)$$

and then

$$\bar{\delta}/2 \le u(nK_1T,\cdot;\bar{\delta}/2) \le u(nK_1T,\cdot;M) \le M \quad \forall\ n = 1,2,\cdots.$$

It then follows that

$$u(nK_1T, x; M) \geq u^+(x) \geq u(nK_1T, x; \bar{\delta}/2) \quad \forall\, x \in \mathcal{H},\ n = 1, 2, \cdots.$$

Therefore

$$\begin{aligned} 0 \leq u(nK_1T, x; M) - u^+(x) &\leq u(nK_1T, x; M) - u(nK_1T, x; \bar{\delta}/2) \\ &\leq u(nK_1T, x; M)(1 - \frac{1}{e^{\rho_n}}) \\ &\leq M(1 - \frac{1}{e^{\rho_n}}). \end{aligned}$$

This implies that

$$\lim_{n\to\infty} u(nK_1T, x; M) = u^+(x)$$

uniformly in $x \in \mathcal{H}$ and $u^+(\cdot) \in X^{++}$. Moreover, by

$$u(nK_1T, \cdot; M) \geq u(kT, \cdot; M) \geq u((n+1)K_1, \cdot; M) \quad \forall\, nK_1 \leq k \leq (n+1)K_1,$$

we have

$$\lim_{k\to\infty} u(kT, x; M) = u^+(x)$$

uniformly in $x \in \mathcal{H}$ and then

$$u(T, \cdot; u^+) = u^+(\cdot).$$

This implies that $u^*(t, x) = u(t, x; u^+)$ is a time periodic strictly positive solutions of (1.3).

$\square$

4.3 Tail property

In this section, we prove the tail property of time periodic strictly positive solutions of (1.3). Throughout this subsection, we assume the conditions in Theorem 2.1 (4).

Proof of Theorem 2.1 (4). Suppose that $u^*(t,x)$ is a time periodic strictly positive solution of (1.3). Observe that $u^*(t,x) = u(t,x;u^+)$, where u^+ is as in the proof of Theorem 2.1(3). We claim

$$\lim_{r\to\infty} \sup_{x\in\mathcal{H},\|x\|\geq r} |u^*(t,x) - u_0^*(t,x)| = 0. \tag{4.17}$$

To prove (4.17), we first show that

$$\lim_{r\to\infty} \sup_{x\in\mathcal{H},\|x\|\geq r} |u^+(x) - u_0^+(x)| = 0. \tag{4.18}$$

Recall

$$u^+(x) := \lim_{n\to\infty} u(nT,x;M,f)$$

and

$$u_0^+(x) := \lim_{n\to\infty} u(nT,x;M,f_0).$$

Assume (4.18) is not true. Then there exists $\epsilon_0 > 0$ and $\{x_n\} \in \mathbb{R}$ with $\|x_k\| \to \infty$ such that

$$|u^+(x_k) - u_0^+(x_k)| > \epsilon_0$$

for $k \geq 1$.

Since both

$$u(nT,x;M,f) \to u^+(x)$$

and

$$u(nT,x;M,f_0) \to u_0^+(x)$$

uniformly on $x \in \mathcal{H}$, there is N such that for $n \geq N$,

$$|u^+(nT,x_k;M,f) - u^+(nT,x_k;M,f_0)| > \epsilon_0 \quad \forall\ k \geq 1. \tag{4.19}$$

Note that there is $\tilde{x}_0 \in \mathcal{H}$ such that

$$f_0(t, x + x_k, u) \to f_0(t, x + \tilde{x}_0, u)$$

as $k \to \infty$ uniformly in (t, x, u) on bounded sets. Note also that

$$f(t, x + x_k, u) - f_0(t, x + x_k, u) \to 0$$

as $k \to \infty$ uniformly in (t, x, u) on bounded sets. Hence

$$f(t, x + x_k, u) \to f_0(t, x + \tilde{x}_0, u)$$

as $k \to \infty$ uniformly in (t, x, u) on bounded sets. Then by Proposition 3.3,

$$\begin{aligned}
&|u(NT, x_k; M, f) - u(NT, x_k; M, f_0)| \\
&= |u(NT, 0; M, f(\cdot, \cdot + x_k; \cdot)) - u(NT, 0; M, f_0(\cdot, \cdot + x_k; \cdot))| \\
&\le |u(NT, 0; M, f(\cdot, \cdot + x_k; \cdot)) - u(NT, 0; M, f_0(\cdot, \cdot + \tilde{x}_0; \cdot))| \\
&+ |u(NT, 0; M, f_0(\cdot, \cdot + \tilde{x}_0; \cdot)) - u(NT, 0; M, f_0(\cdot, \cdot + x_k; \cdot))| \\
&\to 0
\end{aligned}$$

as $k \to \infty$. This contradicts to (4.19). Therefore, (4.18) holds.

Now we prove (4.17). Recall that

$$u^*(t, x) = u(t, x; u^+, f)$$

and

$$u_0^*(t, x) = u(t, x; u_0^+, f_0).$$

Suppose that (4.17) does not hold for some $t > 0$. Then there are $x_k \in \mathcal{H}$ with $\|x_k\| \to \infty$ and $\epsilon_0 > 0$ such that

$$|u(t, x_k; u^+, f) - u(t, x_k; u_0^+, f_0)| \geq \epsilon_0 \quad \forall\, k \geq 1.$$

Hence

$$\begin{aligned}|u(t, x_k; u^+, f) - u(t, x; u_0^+, f)| &= |u(t, 0; u^+(\cdot + x_k), f(\cdot + x_k)) - u(t, 0; u_0^+(\cdot + x_k), f_0(\cdot + x_k))| \\ &\geq \epsilon_0\end{aligned}$$

for all $k \geq 1$. By (H1), (4.18), Proposition 3.3, and the arguments in the proof of (4.18),

$$\lim_{k \to \infty} [u(t, 0; u^+(\cdot + x_k), f(\cdot + x_k)) - u(t, 0; u_0^+(\cdot + x_k), f_0(\cdot + x_k))] = 0.$$

This is a contradicts again. Therefore, (4.17) holds. $\square$

Chapter 5

Spatial Spreading Speeds

In this chapter, we investigate the spatial spreading speeds of (1.3) and prove Theorem 2.2. We first prove a Lemma.

Throughout this section, we assume the conditions in Theorem 2.2. Let $u_0^*(t,x)$ be the unique time and space periodic positive solution of (1.1). Let $\delta_0 > 0$ be such that

$$0 < \delta_0 < \inf_{(t,x)\in\mathbb{R}\times\mathcal{H}} u_0^*(t,x).$$

Lemma 5.1. *Let $\xi \in S^{N-1}$, $c > 0$ and $u_0 \in X^+$ be given. If*

$$\liminf_{x\cdot\xi\le ct,t\to\infty} u(t,x;u_0,f) > 0,$$

then for any $0 < c^{'} < c$,

$$\limsup_{x\cdot\xi\le c^{'}t,t\to\infty} |u(t,x;u_0,f) - u^*(t,x)| = 0.$$

Proof. It can be proved by the similar arguments as in [46, Lemma 5.1 (1)]. For completeness, we provide a proof in the following.

Suppose that $\liminf_{x\cdot\xi\le ct,t\to\infty} u(t,x;u_0,f) > 0$. Then there are δ and $T^* > 0$ such that

$$u(t,x;u_0,f) \ge \delta \quad \forall (t,x) \in \mathbb{R}^+ \times \mathcal{H},\ x\cdot\xi \le ct,\ t \ge T^*.$$

Assume that the conclusion of (1) is not true. Then there are $0 < c^{'} < c$, $\epsilon_0 > 0$, $x_n \in \mathcal{H}$, and $t_n \in \mathbb{R}^+$ with $x_n \cdot \xi \le c^{'}t_n$ and $t_n \to \infty$ such that

$$|u(t_n, x_n; u_0, f) - u^*(t_n, x_n)| \geq \epsilon_0 \quad \forall n \geq 1. \tag{5.1}$$

Note that there are $k_n \in \mathbb{Z}^+$ and $\tau_n \in [0, T]$ such that $t_n = k_n T + \tau_n$. Without loss of generality, we may assume that

$$\tau_n \to \tau^* \quad \text{and} \quad x_n \to x^*$$

as $n \to \infty$ in the case that $\{\|x_n\|\}$ is bounded (this implies that $f(t + t_n, x + x_n, u) \to f(t + \tau^*, x + x^*, u)$ uniformly in (t, x, u) in bounded sets) and

$$f(t + t_n, x + x_n, u) - f_0(t + \tau^*, x + x_n, u) \to 0$$

as $n \to \infty$ uniformly in (t, x, u) on bounded sets in the case that $\{\|x_n\|\}$ is unbounded.

Let $\tilde{u}_0 \in X^+$,

$$\tilde{u}_0(x) = \delta \quad \forall x \in \mathcal{H}.$$

Let

$$M = \sup_{x \in \mathcal{H}} u_0(x).$$

By Theorem 2.1, there is $\tilde{T} > 0$ such that

$$|u(t, x; M, f) - u^*(t, x)| < \frac{\epsilon_0}{2} \quad \forall\, t \geq \tilde{T},\ x \in \mathcal{H}, \tag{5.2}$$

$$|u(\tilde{T}, x; \tilde{u}_0, f(\cdot + \tau, \cdot + x^*, \cdot)) - u^*(\tilde{T} + \tau, x + x^*)| < \frac{\epsilon_0}{2}, \quad \forall\, x \in \mathcal{H},\ \tau \in \mathbb{R} \tag{5.3}$$

and

$$|u(\tilde{T}, x; \tilde{u}_0, f_0(\cdot + \tau, \cdot, \cdot)) - u_0^*(\tilde{T} + \tau, x)| < \frac{\epsilon_0}{2} \quad \forall\, x \in \mathcal{H},\ \tau \in \mathbb{R}. \tag{5.4}$$

Without loss of generality, we may assume that $t_n - \tilde{T} \geq T^*$ for $n \geq 1$. Let $\tilde{u}_{0n} \in X^+$ be such that

$$\tilde{u}_{0n}(x) = \delta \quad \text{for} \quad x \cdot \xi \leq \frac{c' + c}{2}(t_n - \tilde{T}),$$

$$0 \le \tilde{u}_{0n}(x) \le \delta \quad \text{for} \quad \frac{c^{'} + c}{2}(t_n - \tilde{T}) \le x \cdot \xi \le c(t_n - \tilde{T}),$$

and

$$\tilde{u}_{0n}(x) = 0 \quad \text{for} \quad x \cdot \xi \ge c(t_n - \tilde{T}).$$

Then

$$u(t_n - \tilde{T}, \cdot; u_0, f) \ge \tilde{u}_{0n}(\cdot)$$

and hence

$$\begin{aligned} u(t_n, x_n; u_0, f) &= u(\tilde{T}, x_n; u(t_n - \tilde{T}, \cdot; u_0, f), f(\cdot + t_n - \tilde{T}, \cdot, \cdot)) \\ &= u(\tilde{T}, 0; u(t_n - \tilde{T}, \cdot + x_n; u_0, f), f(\cdot + t_n - \tilde{T}, \cdot + x_n, \cdot)) \\ &\ge u(\tilde{T}, 0; \tilde{u}_{0n}(\cdot + x_n), f(\cdot + t_n - \tilde{T}, \cdot + x_n, \cdot)). \end{aligned} \tag{5.5}$$

Observe that

$$\tilde{u}_{0n}(x + x_n) \to \tilde{u}_0(x)$$

as $n \to \infty$ uniformly in x on bounded sets. In the case that

$$f(t_n + t, x + x_n, u) - f_0(t + \tau^*, x + x_n, u) \to 0$$

as $n \to \infty$, by Proposition 3.3,

$$u(\tilde{T}, 0; \tilde{u}_{0n}(\cdot + x_n), f(\cdot + t_n - \tilde{T}, \cdot + x_n, \cdot)) - u(\tilde{T}, 0; \tilde{u}_0, f_0(\cdot + \tau^* - \tilde{T}, \cdot + x_n, \cdot)) \to 0$$

as $n \to \infty$. Then by (5.4) and (5.5),

$$u(t_n, x_n; u_0, f) > u_0^*(\tau^*, x_n) - \epsilon_0/2 \quad \text{for} \quad n \gg 1. \tag{5.6}$$

By Theorem 2.1(4),

$$u_0^*(\tau^*, x_n) > u^*(\tau^*, x_n) - \epsilon_0/2 \quad \text{for} \quad n \gg 1. \tag{5.7}$$

By Proposition 3.1 and (5.2),

$$u(t_n, x_n; u_0, f) \le u(t_n, x_n; M, f) \le u^*(t_n, x_n) + \epsilon_0 \quad \forall\ n \gg 1.$$

(5.6), (5.7), and the continuity of $u^*(t, x)$ imply,

$$|u(t_n, x_n; u_0, f) - u^*(t_n, x_n)| < \epsilon_0 \quad \text{for} \quad n \gg 1.$$

This contradicts to (5.1).

In the case that $x_n \to x^*$, by Proposition 3.3 again,

$$u(\tilde{T}, 0; \tilde{u}_{0n}(\cdot + x_n), f(\cdot + t_n - \tilde{T}, \cdot + x_n, \cdot)) \to u(\tilde{T}, 0; \tilde{u}_0, f(\cdot + \tau^* - \tilde{T}, \cdot + x^*, \cdot))$$

as $n \to \infty$. By (5.3) and (5.5),

$$u(t_n, x_n; u_0, f) > u^*(\tau^*, x^*) - \epsilon_0/2 \quad \text{for} \quad n \gg 1. \tag{5.8}$$

By the continuity of $u^*(\cdot, \cdot)$,

$$u^*(\tau^*, x^*) > u^*(\tau_n, x_n) - \epsilon_0/2 \quad \text{for} \quad n \gg 1. \tag{5.9}$$

By Proposition 3.1 and (5.2),

$$u(t_n, x_n; u_0, f) \le u(t_n, x_n; M, f) \le u^*(t_n, x_n) + \epsilon_0 \quad \forall\ n \gg 1.$$

This together with (5.8), and (5.9) implies that

$$|u(t_n, x_n; u_0, f) - u^*(t_n, x_n)| < \epsilon_0 \quad \text{for} \quad n \gg 1.$$

This contradicts to (5.1) again.

Hence

$$\limsup_{x\cdot\xi\le c' t, t\to\infty} |u(t,x;u_0,f)-u^*(t,x)|=0$$

for all $0<c^{'}<c$. $\square$

Observe that for any given $\xi\in S^{N-1}$, there is $\mu^*(\xi)>0$ such that

$$c_0^*(\xi)=\frac{\lambda_{\xi,\mu^*(\xi)}(f_0(\cdot,\cdot,0))}{\mu^*(\xi)}$$

and

$$\frac{\lambda_{\xi,\mu}(f_0(\cdot,\cdot,0))}{\mu}>c_0^*(\xi)\quad \forall\ 0<\mu<\mu^*(\xi).$$

We now prove Theorem 2.2.

Proof of Theorem 2.2. We first show that for any $c>c_0^*(\xi)$,

$$\limsup_{x\cdot\xi\ge ct, t\to\infty} u(t,x;u_0,f)=0. \tag{5.10}$$

Let $0<\mu<\mu^*(\xi)$ be such that

$$c=\frac{\lambda_{\xi,\mu}(f_0(\cdot,\cdot,0))}{\mu}.$$

By Proposition 3.5, for any $\epsilon>0$, there are $\delta>0$ and $a(\cdot,\cdot)\in\mathcal{X}_p$ such that

$$a(t,x)\ge f_0(t,x,0)+\delta,$$

$$c_0^*(\xi)<\frac{\lambda_{\xi,\mu}(a)}{\mu}<\frac{\lambda_{\xi,\mu}(f_0(\cdot,\cdot,0))}{\mu}<c,$$

and $\lambda_{\xi,\mu}(a)$ is the principal eigenvalue of $-\partial_t+\mathcal{A}_{\xi,\mu}+a(\cdot,\cdot)\mathcal{I}$ with a positive principal eigenfunction $\phi(\cdot,\cdot)\in\mathcal{X}_p$. Let

$$u_M(t,x)=Me^{-\mu(x\cdot\xi-\frac{\lambda_{\xi,\mu}(a)}{\mu}t)}\phi(t,x).$$

It is not difficult to verify that $u_M(t,x)$ is a solution of

$$u_t = \mathcal{A}u + a(t,x)u, \quad x \in \mathcal{H}. \tag{5.11}$$

Observe that

$$f_0(t,x,u) \le f_0(t,x,0) \le a(t,x) - \delta \quad \forall\, t \in \mathbb{R},\ x \in \mathcal{H},\ u \ge 0.$$

This together with (H1) implies that there is $\tilde{M} > 0$ such that

$$f(t,x,u) \le f(t,x,0) \le a(t,x) \quad \forall\, t \in \mathbb{R},\ x \cdot \xi \ge \tilde{M},\ u \ge 0. \tag{5.12}$$

Let $M \ge \|u_0\|$ be such that

$$f(t,x,u_M(t,x)) \le a(t,x) \quad \forall\, t \ge 0,\ x \cdot \xi \le \tilde{M}. \tag{5.13}$$

It then follows from (5.11), (5.12), and (5.13) that $u_M(t,x)$ is a super-solution of (1.3). By Proposition 3.1,

$$u(t,x;u_0,f) \le u_M(t,x) = Me^{-\mu(x\cdot\xi - \frac{\lambda_{\xi,\mu}(a)}{\mu}t)}\phi(t,x) \quad \forall t \ge 0,\ x \in \mathcal{H}.$$

This implies that (5.10) holds.

Next, we prove that for any $c < c_0^*(\xi)$,

$$\limsup_{x\cdot\xi \le ct, t\to\infty} |u(t,x;u_0,f) - u^*(t,x)| = 0. \tag{5.14}$$

First of all, by Proposition 3.5, there is $\epsilon > 0$ such that

$$c < \inf_{\mu>0} \frac{\lambda_{\xi,\mu}(f_0(\cdot,\cdot,0) - \epsilon)}{\mu} < c_0^*(\xi).$$

By (H1), there is $M > 0$ such that

$$f(t, x, u) \geq f_0(t, x, u) - \epsilon \quad \forall\, t \in \mathbb{R},\ x \cdot \xi \geq M,\ 0 \leq u \leq 1. \tag{5.15}$$

By Lemma 4.2, there are $\tilde{\delta} > 0$ and $\tilde{T} > 0$ such that

$$u(t, x; u_0, f) \geq \tilde{\delta} \quad \forall\ t \geq \tilde{T},\ x \cdot \xi \leq M. \tag{5.16}$$

Consider equation

$$u_t = (\mathcal{A}u)(t, x) + [f_0(t, x, u) - \epsilon - K^* u]u(x), \quad x \in \mathcal{H} \tag{5.17}$$

By Proposition 3.8, (5.17) has a unique time and space periodic solution $u^*_{0,K^*}(t, x,)$. Let $K^* \gg 1$ be such that

$$u_{0,K^*}(\tilde{T}, x) \leq \tilde{\delta}.$$

Let $\tilde{u}_0 \in X^+(\xi)$ be such that

$$\tilde{u}_0(x) \leq \min\{u^*_{0,K^*}(\tilde{T}, x), u(\tilde{T}, x; u_0, f)\}.$$

By Proposition 3.1,

$$u(t + \tilde{T}, x; u_0, f) \geq u(t, x; \tilde{u}_0, \tilde{f}_0(\cdot + \tilde{T}, \cdot, \cdot)) \quad \forall\ t \geq 0,\ x \cdot \xi \geq M,$$

where $\tilde{f}_0(t, x, u) = f_0(t, x, u) - \epsilon - K^* u$. By Proposition 3.9, for any $c < c' < \inf_{\mu > 0} \frac{\lambda_{\xi,\mu}(f_0 - \epsilon)}{\mu}$,

$$\limsup_{x \cdot \xi \leq c' t, t \to \infty} |u(t, x; \tilde{u}_0, \tilde{f}_0(\cdot + \tilde{;} \cdot, \cdot)) - u^*_{0,K^*}(t + \tilde{T}, x)| = 0. \tag{5.18}$$

By (5.16) and (5.18),

$$\liminf_{x\cdot\xi\le c' t, t\to\infty} u(t,x;u_0,f) > 0.$$

This together with Lemma 5.1 implies that (5.14) holds. $\square$

Chapter 6

Another Method to Show the Time Independent Case

In this chapter, we consider time independent monostable equations and provide some other method to prove Theorems 2.1 and 2.2. We also show Theorem 2.3.

6.1 Another method to prove Theorem 2.1

In this section, we investigate the existence of positive stationary solutions of (1.3) in the special case that $f(t,x,u) \equiv f(x,u)$ and $f_0(x,u) \equiv f_0(u)$ and give another proof of Theorem 2.1.

Throughout this section, we assume $f(t,x,u) \equiv f(x,u)$, $f_0(x,u) \equiv f_0(u)$ and (H0), (H1)$'$ and (H2). We first prove some lemmas.

For convenience, we denote $f(x,u)$ and $f_0(u)$ by $f_i(x,u)$ and $f_i^0(u)$, respectively, in the case $\mathcal{H} = \mathcal{H}_i$ and $\mathcal{A} = \mathcal{A}_i$ for $i = 1,2,3$. We may write $u(t,x;u_0,f_i)$ as $u_i(t,x,u_0)$.

Lemma 6.1. *For any $1 \le i \le 3$ and $\epsilon > 0$, there are $p = (p_1, p_2, \cdots, p_N) \in \mathbb{N}^N$ and $h_i \in X_{i,p} \cap C^N(\mathcal{H}_i, \mathbb{R})$ such that*

$$f_i(x,0) \ge h_i(x) \quad \text{for} \quad x \in \mathcal{H}_i,$$

$$\hat{\hat{h}}_i \ge f_i^0(0) - \epsilon \quad (\text{hence} \quad \lambda(h_i(\cdot)) \ge f_i^0(0) - \epsilon),$$

and for the cases that $i = 1$ and 2, the partial derivatives of $h_i(x)$ up to order $N-1$ are zero at some $x_0 \in \mathcal{H}_i$ with $h_i(x_0) = \max_{x \in \mathcal{H}_i}(x)$, where $\hat{\hat{h}}_i$ is the average of $h_i(\cdot)$ (see (3.16) for the definition).

Proof. Fix $1 \le i \le 3$. By $(\mathrm{H1})'$, there are $0 < \epsilon_0 \ll 1$ and $L_0 > 0$ such that

$$f_i(x,0) \ge f_i^0(0) - \epsilon_0 \quad \text{for} \quad x \in \mathcal{H}_i,\ \|x\| \ge L_0.$$

Let

$$M_0 = \inf_{x \in \mathcal{H}_i, 1 \le i \le 3} f_i(x,0).$$

Let $h_0 : \mathbb{R} \to [0,1]$ be a smooth function such that

$$h_0(s) = \begin{cases} 1 \quad \text{for} \quad |s| \le 1 \\ \\ 0 \quad \text{for} \quad |s| \ge 2. \end{cases}$$

For any $p = (p_1, p_2, \cdots, p_N) \in \mathbb{N}^N$ with $p_j > 4L_0$, let $h_i \in X_{i,p} \cap C^N(\mathcal{H}_i, \mathbb{R})$ $(i = 1,2,3)$ be such that

$$h_i(x) = f_i^0(0) - \epsilon_0 - h_0\Big(\frac{\|x\|^2}{L_0^2}\Big)(f_i^0(0) - \epsilon_0 - M_0)$$

$$\text{for} \quad x \in \Big([-\frac{p_1}{2}, \frac{p_1}{2}] \times [-\frac{p_2}{2}, \frac{p_2}{2}] \times \cdots \times [-\frac{p_N}{2}, \frac{p_N}{2}]\Big) \cap \mathcal{H}_i.$$

Then

$$f_i(x,0) \ge h_i(x) \quad \forall x \in \mathcal{H}_i,\ 1 \le i \le 3.$$

It is clear that for $i = 1$ or 2, the partial derivatives of $h_i(x)$ up to order $N-1$ are zero at some $x_0 \in \mathcal{H}_i$ with $h_i(x_0) = \max_{x \in \mathcal{H}_i} h_i(x) (= f_i^0(0) - \epsilon_0)$. For given $\epsilon > 0$, choosing $p_j \gg 1$, we have

$$\hat{\hat{h}}_i > f_i^0(0) - \epsilon.$$

By Proposition 3.7, $\lambda(h_i(\cdot)) \ge \lambda(\hat{\hat{h}}_i) = \hat{\hat{h}}_i$ and hence

$$\lambda(h_i(\cdot)) \ge f_i^0(0) - \epsilon.$$

The lemma is thus proved. $\square$

Lemma 6.2. *Suppose that $\tilde{u}_2^*:\mathbb{R}^N\to[\sigma_0,M_0]$ is Lebesgue measurable, where σ_0 and M_0 are two positive constants. If*

$$\int_{\mathbb{R}^N}\kappa(y-x)\tilde{u}_2^*(y)dy-\tilde{u}_2^*(x)+\tilde{u}_2^*(x)\tilde{f}_2(x,\tilde{u}_2^*(x))=0\quad\forall x\in\mathbb{R}^N,$$

where $\tilde{f}_2(x,u)=f_2(x,u)$ or $f_2^0(u)$ for all $x\in\mathbb{R}^N$ and $u\in\mathbb{R}$, then $\tilde{u}_2^*(\cdot)\in X_2^{++}$.

Proof. We prove the case that $\tilde{f}_2(x,u)=f_2(x,u)$. The case that $\tilde{f}_2(x,u)=f_2^0(u)$ can be proved similarly.

Let

$$h^*(x)=\int_{\mathbb{R}^N}\kappa(y-x)\tilde{u}_2^*(y)dy\quad\text{for}\quad x\in\mathbb{R}^N.$$

Then $h^*(\cdot)$ is C^1 and has bounded first order partial derivatives. Let

$$F(x,\alpha)=h^*(x)-\alpha+\alpha f_2(x,\alpha)\quad\forall x\in\mathbb{R}^N,\ \alpha\in\mathbb{R}.$$

Then $F:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ is C^1 and $F(x,\tilde{u}_2^*(x))=0$ for each $x\in\mathbb{R}^N$. If $\alpha^*>0$ is such that $F(x,\alpha^*)=0$, then

$$-1+f_2(x,\alpha^*)=-\frac{h^*(x)}{\alpha^*}<0$$

and hence

$$\partial_\alpha F(x,\alpha^*)=-1+f_2(x,\alpha^*)+\alpha^*\partial_u f_2(x,\alpha^*)<0.$$

By Implicit Function Theorem, $\tilde{u}_2^*(x)$ is C^1 in x. Moreover,

$$\frac{\partial\tilde{u}_2^*(x)}{\partial x_j}=\frac{\frac{\partial h^*(x)}{\partial x_j}}{-1+f(x,\tilde{u}_2^*(x))+\partial_u f_2(x,\tilde{u}_2^*(x))\tilde{u}_2^*(x)}\quad\forall x\in\mathbb{R}^N,\ 1\le j\le N.$$

Therefore, $\tilde{u}_2^*$ has bounded first order partial derivatives. It then follows that $\tilde{u}_2^*(x)$ is uniformly continuous in $x\in\mathbb{R}^N$ and then $\tilde{u}_2^*\in X_2^{++}$. $\square$

Lemma 6.3. *Suppose that* $u_i^*(\cdot) \in X_i^{++}$ *and* $u = u_i^*(\cdot)$ *is a stationary solution of* (1.3) *with* $\mathcal{H} = \mathcal{H}_i$, $\mathcal{A} = \mathcal{A}_i$, *and* $f(t, x, u) = f_i(x, u)$. *Then*

$$u_i^*(x) \to u_i^0 \quad \text{as} \quad \|x\| \to \infty,$$

where u_i^0 *is a positive constant such that* $f_i^0(u_i^0) = 0$.

Proof. We first prove that

$$u_1^*(x) \to u_1^0 \quad \text{as} \quad \|x\| \to \infty.$$

Assume that $u_1^*(x) \not\to u_1^0$ as $\|x\| \to \infty$. Then there are $\epsilon_0 > 0$ and $x_n \in \mathbb{R}^N$ such that

$$\|x_n\| \to \infty$$

and

$$|u_1^*(x_n) - u_1^0| \geq \epsilon_0 \quad \text{for} \quad n = 1, 2, \cdots.$$

By the uniform continuity of $u_1^*(x)$ in $x \in \mathbb{R}^N$, without loss of generality, we may assume that there is a continuous function $\tilde{u}_1^* : \mathbb{R}^N \to [\sigma_0, M_0]$ for some $\sigma_0, M_0 > 0$ such that

$$u_1(x + x_n) \to \tilde{u}_1^*(x)$$

as $n \to \infty$ uniformly in x on bounded sets. Moreover, by a priori estimates for parabolic equations, $\tilde{u}_1^*$ is $C^{2+\alpha}$ for some $\alpha > 0$ and we may also assume that

$$\Delta u_1(x + x_n) \to \Delta \tilde{u}_1^*(x)$$

as $n \to \infty$ uniformly in x on bounded sets. This together with $f_1(x + x_n, u) \to f_1^0(u)$ as $n \to \infty$ uniformly in x on bounded sets and in $u \in \mathbb{R}$ implies that

$$\Delta \tilde{u}_1^* + \tilde{u}_1^* f_1^0(\tilde{u}_1^*) = 0, \quad x \in \mathbb{R}^N.$$

By Proposition 3.8, we must have

$$\tilde{u}_1^*(x) \equiv u_1^*(x; f_1^0(\cdot)) \equiv u_1^0$$

and hence

$$u_1^*(x_n) \to u_1^0 \quad \text{as} \quad n \to \infty.$$

This is a contradiction. Therefore

$$u_1^*(x) \to u_1^0 \quad \text{as} \quad \|x\| \to \infty.$$

Next, we prove that

$$u_2^*(x) \to u_2^0 \quad \text{as} \quad \|x\| \to \infty.$$

Similarly, assume that $u_2^*(x) \not\to u_2^0$ as $\|x\| \to \infty$. Then there are $\epsilon_0 > 0$ and $x_n \in \mathbb{R}^N$ such that $\|x_n\| \to \infty$ and

$$|u_2^*(x_n) - u_2^0| \geq \epsilon_0 \quad \text{for} \quad n = 1, 2, \cdots.$$

By the uniform continuity of $u_2^*(x)$ in $x \in \mathbb{R}^N$, without loss of generality, we may assume that there is a continuous function $\tilde{u}_2^* : \mathbb{R}^N \to [\sigma_0, M_0]$ for some $\sigma_0, M_0 > 0$ such that

$$u_2^*(x + x_n) \to \tilde{u}_2^*(x)$$

as $n \to \infty$ uniformly in x on bounded sets. By the Lebesgue Dominated Convergence Theorem, we have

$$\int_{\mathbb{R}^N} \kappa(y - x)\tilde{u}_2^*(y)dy - \tilde{u}_2^*(x) + \tilde{u}_2^*(x) f_2^0(\tilde{u}_2^*(x)) = 0 \quad \forall x \in \mathbb{R}^N.$$

By Lemma 6.2, $\tilde{u}_2^* \in X_2^{++}$. By Proposition 3.8 again, we have $\tilde{u}_2^*(x) \equiv u_2^0$ and then $u_2^*(x_n) \to u_2^0$ as $n \to \infty$. This is a contradiction. Therefore $u_2^*(x) \to u_2^0$ as $\|x\| \to \infty$.

Finally, it can be proved by the similar arguments as in the case $i = 2$ that

$$u_3^*(j) \to u_3^0 \quad \text{as} \quad \|j\| \to \infty.$$

$\square$

Lemma 6.4. *There is $u_i^- \in X_i^{++}$ such that for any $\delta > 0$ sufficiently small, $u(t, x; \delta u_i^-)$ is increasing in $t > 0$ and $u^{-,*,\delta} \in X_i^{++}$, where $u^{-,*,\delta}(x) = \lim_{t\to\infty} u(t, x; \delta u_i^-)$, and hence $u = u^{-,*,\delta}(\cdot)$ is a stationary solution of (1.3) in X_i^{++} in the case $\mathcal{H} = \mathcal{H}_1$, $\mathcal{A} = \mathcal{A}_i$, and $f(t, x, u) = f_i(x, u)$ $(i = 1, 2, 3)$.*

Proof. Fix $1 \le i \le 3$. Let $M^* > 0$ be such that $f_i(x, M^*) < 0$. Let $\epsilon > 0$ be such that

$$f_i^0(0) - \epsilon > 0.$$

By Lemma 6.1, there are $p \in \mathbb{N}^N$ and $h_i(\cdot) \in X_{i,p} \cap C^N(\mathcal{H}_i, \mathbb{R})$ such that

$$f_i(x, 0) \ge h_i(x), \text{ and } \hat{h}_i \ge f_i^0(0) - \epsilon (> 0).$$

Moreover, for $i = 1$ or 2, the partial derivatives of $h_i(x)$ up to order $N - 1$ are zero at some $x_0 \in \mathcal{H}_i$ with $h_i(x_0) = \max_{x\in\mathcal{H}_i} h_i(x)$. Let u_i^- be the positive principal eigenfunction of $\mathcal{A}_i + h_i(\cdot)\mathcal{I}$ with $\|u_i^-\| = 1$ (the existence of u_i^- is well known in the case that $i = 1$ or 3 and follows from Proposition 3.5 in the case that $i = 2$). It is not difficult to verify that $u = \delta u_i^-$ is a sub-solution of (1.3) for any $\delta > 0$ sufficiently small. It then follows that for any $\delta > 0$ sufficiently small,

$$\delta u_i^-(\cdot) \le u(t_1, \cdot; \delta u_i^-) \le u_i(t_2, \cdot; \delta u_i^-) \quad \forall 0 < t_1 < t_2.$$

This implies that there is a Lebesgue measurable function $u_i^{-,*,\delta} : \mathcal{H}_i \to [\sigma_0, M_0]$ for some $\sigma_0, M_0 > 0$ such that

$$\lim_{t\to\infty} u(t,x;\delta u_i^-) = u_i^{-,*,\delta}(x) \quad \forall x \in \mathcal{H}_i.$$

Moreover, by regularity and a priori estimates for parabolic equations, $u_1^{-,*,\delta} \in X_1^{++}$. It is clear that $u_3^{-,*,\delta} \in X_3^{++}$. By Lemma 6.2, $u_2^{-,*,\delta} \in X_2^{++}$. Therefore for $1 \le i \le 3$, $u_i^{-,*,\delta} \in X_i^{++}$ and $u = u_i^{-,*,\delta}(\cdot)$ is a stationary solution of (1.3) in X_i^{++} $(i = 1,2,3)$. □

Lemma 6.5. *Let $M \gg 1$ be such that $f_i(x, M) < 0$ for $x \in \mathcal{H}_i$ $(i = 1,2,3)$. Then $\lim_{t\to\infty} u(t,x;u_0)$ exists for every $x \in \mathcal{H}_i$, where $u_0(x) \equiv M$. Moreover, $u_i^{+,*,M}(\cdot) \in X_i^{++}$, where $u_i^{+,*,M}(x) := \lim_{t\to\infty} u_i(t,x;u_0)$, and hence $u = u_i^{+,*,M}(\cdot)$ is a stationary solution of (1.3) in X_i^{++} in the case $\mathcal{H} = \mathcal{H}_i$ and $\mathcal{A} = \mathcal{A}_i$ $(i = 1,2,3)$.*

Proof. Fix $1 \le i \le 3$. For any $M > 1$ with $f_i(x,M) < 0$ for all $x \in \mathcal{H}_i$, $u = M$ is a super-solution of (1.3). Hence

$$u(t_2,\cdot;M) \le u(t_1,\cdot;M) \le M \quad \forall 0 \le t_1 < t_2.$$

It then follows that $\lim_{t\to\infty} u(t,x;M)$ exists for all $x \in \mathbb{R}^N$. Let

$$u_i^{+,*,M}(x) = \lim_{t\to\infty} u(t,x;M).$$

We have

$$u_i^{+,*,M}(x) \ge u_i^{-,*,\delta}(x) \quad \text{for} \quad 0 < \delta \ll 1.$$

By the similar arguments as in Lemma 6.4, $u_i^{+,*,M} \in X_i^{++}$ and $u = u_i^{+,*,M}(\cdot)$ is a stationary solution of (1.3) in X_i^{++} $(i = 1,2,3)$. □

Proof of Theorem 2.1. We first prove (3). Let $1 \le i \le 3$ be given. By Lemmas 6.4 and 6.5, (1.3) has stationary solutions in X_i^{++}. We denote it by $u_i^*(\cdot)$.

We next prove (1). We claim that stationary solution of (1.3) in X_i^{++} is unique. In fact, suppose that $u_i^{1,*}$ and $u_i^{2,*}$ are two stationary solutions of (1.3) in X_i^{++}. Assume that $u_i^{1,*} \neq u_i^{2,*}$. Then there is $\alpha^* > 1$ such that

$$\rho_i(u_i^{1,*}, u_i^{2,*}) = \ln \alpha^* > 0.$$

Note that

$$\frac{1}{\alpha^*} u_i^{1,*} \leq u_i^{2,*} \leq \alpha^* u_i^{1,*}.$$

By Lemma 6.3,

$$\lim_{\|x\| \to \infty} u_i^{1,*}(x) = u_i^0$$

and

$$\lim_{\|x\| \to \infty} u_i^{2,*}(x) = u_i^0.$$

This implies that there is $\epsilon > 0$ such that

$$\frac{1}{\alpha^* - \epsilon} u_i^{1,*}(x) \leq u_i^{2,*}(x) \leq (\alpha^* - \epsilon) u_i^{1,*}(x) \quad \text{for} \quad \|x\| \gg 1.$$

By Proposition 3.1 and the arguments in Proposition 3.4,

$$\frac{1}{\alpha^*} u_i^{1,*}(x) < u_i^{2,*}(x) < \alpha^* u_i^{1,*}(x) \quad \forall x \in \mathbb{R}^N.$$

It then follows that for $0 < \epsilon \ll 1$,

$$\frac{1}{\alpha^* - \epsilon} u_i^{1,*}(x) \leq u_i^{2,*}(x) \leq (\alpha^* - \epsilon) u_i^{1,*}(x) \quad \forall x \in \mathbb{R}^N$$

and then $\rho_i(u_i^{1,*}, u_i^{2,*}) \leq \ln(\alpha^* - \epsilon)$, this is a contradiction. Therefore $u_i^{1,*} = u_i^{2,*}$ and (1.3) has a unique stationary solution u_i^* in X_i^{++}.

We now prove (2). Fix $1 \le i \le 3$. For any $u_0 \in X_i^{++}$, there is $\delta > 0$ sufficiently small and $M > 0$ sufficiently large such that $\delta u_i^- \le u_0 \le M$ and $u = \delta u_i^-$ is a sub-solution of (1.3) (u_i^- is as in Lemma 6.4) and $u = M$ is a super-solution of (1.3). Then

$$\delta u_i^- \le u_i(t, \cdot; \delta u_i^-) \le u_i(t, \cdot; u_0) \le u_i(t, \cdot; M) \le M \quad \forall t \ge 0.$$

By (1), Lemmas 6.4 and 6.5, and Dini's Theorem,

$$u_i(t, x; \delta u_i^-) < u_i^*(x) < u_i(t, x; M) \quad \forall t > 0,\ x \in \mathcal{H}_i$$

and

$$\lim_{t\to\infty} u_i(t, x; \delta u_i^-) = \lim_{t\to\infty} u_i(t, x; M) = u_i^*(x)$$

uniformly in x on bounded sets. It then follows that

$$\lim_{t\to\infty} u_i(t, x; u_0) = u_i^*(x)$$

uniformly in x on bounded sets.

We claim that $\|u_i(t, \cdot; u_0) - u_i^*(\cdot)\| \to 0$ as $t \to \infty$. Assume the claim is not true. Then there are $\epsilon_0 > 0$, $t_n \to \infty$, and x_n with $\|x_n\| \to \infty$ such that

$$|u_i(t_n, x_n; u_0) - u_i^*(x_n)| \ge \epsilon_0 \quad \forall n \in \mathbb{N}.$$

Then by Lemma 6.3,

$$|u_i(t_n, x_n; u_0) - u_i^0| \ge \frac{\epsilon_0}{2} \quad \forall n \gg 1.$$

Let $\tilde{\delta} > 0$ and $\tilde{M} > 0$ be such that

$$\tilde{\delta} \le u_i(t, \cdot; u_0) \le \tilde{M} \quad \forall t \ge 0.$$

For any $\epsilon > 0$, let $T > 0$ be such that

$$|u_i(T, \cdot; \tilde{\delta}, f_i^0(\cdot)) - u_i^0| < \epsilon, \quad |u_i(T, \cdot; \tilde{M}, f_i^0(\cdot)) - u_i^0| < \epsilon. \tag{6.1}$$

Observe that

$$\tilde{\delta} \le u_i(t_n - T, x_n + x; u_0) \le \tilde{M}$$

and

$$u_i(t_n, x_n + \cdot; u_0) = u_i(T, x_n + \cdot; u_i(t_n - T, \cdot; u_0)) = u_i(T, \cdot; u_i(t_n - T, \cdot + x_n; u_0), f_i(\cdot + x_n, \cdot))$$

for $n \gg 1$. Then

$$u_i(T, \cdot; \tilde{\delta}, f_i(\cdot + x_n)) \le u_i(t_n, x_n + \cdot; u_0) \le u_i(T, \cdot; \tilde{M}, f_i(\cdot + x_n, \cdot)). \tag{6.2}$$

Observe also that

$$f_i(x + x_n, u) \to f_i^0(u)$$

as $n \to \infty$ uniformly in (x, u) on bounded sets. Then by Proposition 3.3,

$$u_i(T, x; \tilde{\delta}, f_i(\cdot + x_n, \cdot)) \to u_i(T, x; \tilde{\delta}, f_i^0(\cdot))$$

and

$$u_i(T, x; \tilde{M}, f_i(\cdot + x_n, \cdot)) \to u_i(T, x; \tilde{M}, f_i^0(\cdot))$$

as $n \to \infty$ uniformly in x on bounded sets. This together with (6.1) implies that

$$|u_i(T, 0; \tilde{\delta}, f_i(\cdot + x_n, \cdot)) - u_i^0| < 2\epsilon, \quad |u_i(T, 0; \tilde{M}, f_i(\cdot + x_n, \cdot)) - u_i^0| < 2\epsilon \quad \text{for} \quad n \gg 1$$

and then by (6.2),

$$|u_i(t_n, x_n; u_0) - u_i^0| < 2\epsilon \quad \text{for} \quad n \gg 1.$$

Hence

$$\lim_{n\to\infty} u_i(t_n, x_n; u_0) = u_i^0,$$

which is a contradiction. Therefore

$$\|u_i(t, \cdot; u_0) - u_i^*(\cdot)\| \to 0$$

as $t \to \infty$.

Finally, note that (4) follows from Lemma 6.3. $\square$

6.2 Spatial Spreading Speeds and Proofs of Theorems 2.2 and 2.3

In this section, we explore the spreading speeds of (1.3) in the special case that $f(t, x, u) = f(x, u)$ and $f_0(x, u) = f_0(u)$. We provide another proof of Theorem 2.2 and give a proof of Theorem 2.3.

Throughout this section, we assume $f(t, x, u) \equiv f(x, u)$ and $f_0(x, u) \equiv f_0(u)$, and assume (H0), $(\mathrm{H1})'$ and (H2).

For convenience again, we denote $f(x, u)$ and $f_0(u)$ by $f_i(x, u)$ and $f_i^0(u)$, respectively, in the case $\mathcal{H} = \mathcal{H}_i$ and $\mathcal{A} = \mathcal{A}_i$ for $i = 1, 2, 3$. We may write $u(t, x; u_0, f_i)$ as $u_i(t, x, u_0)$ or $u_i(t, x; u_0, f_i)$, write $c_0^*(\xi, f_i^0)$ as $c_i^0(\xi)$, and $\lambda_{\xi,\mu}(f_i^0(0))$ as $\lambda_i(\xi, \mu, f_i^0(0))$.

We first prove two lemmas.

Lemma 6.6. *Let $\xi \in S^{N-1}$, $c > 0$, $1 \le i \le 3$, and $u_0 \in X_i^+$ be given.*

(1) If $\liminf_{x\cdot\xi\le ct, t\to\infty} u_i(t, x; u_0) > 0$, then for any $0 < c' < c$,

$$\limsup_{x\cdot\xi\le c't, t\to\infty} |u_i(t, x; u_0) - u_i^*(x)| = 0.$$

(2) If $\liminf_{|x\cdot\xi|\le ct, t\to\infty} u_i(t,x;u_0) > 0$*, then for any* $0 < c' < c$*,*

$$\limsup_{|x\cdot\xi|\le c' t, t\to\infty} |u_i(t,x;u_0) - u_i^*(x)| = 0.$$

(3) If $\liminf_{\|x\|\le ct, t\to\infty} u_i(t,x;u_0) > 0$*, then for any* $0 < c' < c$*,*

$$\limsup_{\|x\|\le c' t, t\to\infty} |u_i(t,x;u_0) - u_i^*(x)| = 0.$$

Proof. (1) Suppose that $\liminf_{x\cdot\xi\le ct, t\to\infty} u_i(t,x;u_0) > 0$. Then there are δ and $T > 0$ such that

$$u_i(t,x;u_0) \ge \delta \quad \forall (t,x) \in \mathbb{R}^+ \times \mathcal{H}_i,\ x\cdot\xi \le ct,\ t \ge T.$$

Assume that the conclusion of (1) is not true. Then there are $0 < c' < c$, $\epsilon_0 > 0$, $x_n \in \mathcal{H}_i$, and $t_n \in \mathbb{R}^+$ with $x_n \cdot \xi \le c' t_n$ and $t_n \to \infty$ such that

$$|u_i(t_n, x_n; u_0) - u_i^*(x_n)| \ge \epsilon_0 \quad \forall n \ge 1. \tag{6.3}$$

Without loss of generality, we may assume that

$$x_n \to x^*$$

as $n \to \infty$ in the case that $\{\|x_n\|\}$ is bounded (this implies that $f_i(x + x_n, u) \to f_i(x + x^*, u)$ uniformly in (x,u) in bounded sets) and

$$f_i(x + x_n, u) \to f_i^0(u)$$

as $n \to \infty$ uniformly in (x,u) on bounded sets in the case that $\{\|x_n\|\}$ is unbounded.

Let $\tilde{u}_0 \in X_i^+$,

$$\tilde{u}_0(x) = \delta \quad \forall x \in \mathcal{H}_i.$$

By Theorem 2.1, there is $\tilde{T} > 0$ such that

$$u_i(\tilde{T}, x; u_0) - u_i^*(x) < \epsilon_0 \quad \forall x \in \mathcal{H}_i, \tag{6.4}$$

$$|u_i(\tilde{T}, x; \tilde{u}_0, f_i(\cdot + x^*, \cdot)) - u_i^*(x + x^*)| < \frac{\epsilon_0}{2}, \tag{6.5}$$

and

$$|u_i(\tilde{T}, x; \tilde{u}_0, f_i^0) - u_i^0| < \frac{\epsilon_0}{2}. \tag{6.6}$$

Without loss of generality, we may assume that $t_n - \tilde{T} \geq T$ for $n \geq 1$. Let $\tilde{u}_{0n} \in X_i^+$ be such that

$$\begin{cases} \tilde{u}_{0n}(x) = \delta \quad \text{for} \quad x \cdot \xi \leq \frac{c' + c}{2}(t_n - \tilde{T}) \\ \\ 0 \leq \tilde{u}_{0n}(x) \leq \delta \quad \text{for} \quad \frac{c' + c}{2}(t_n - \tilde{T}) \leq x \cdot \xi \leq c(t_n - \tilde{T}) \\ \\ \tilde{u}_{0n}(x) = 0 \quad \text{for} \quad x \cdot \xi \geq c(t_n - \tilde{T}). \end{cases}$$

Then

$$u_i(t_n - \tilde{T}, \cdot; u_0) \geq \tilde{u}_{0n}(\cdot)$$

and hence

$$\begin{aligned} u_i(t_n, x_n; u_0) &= u_i(\tilde{T}, x_n; u_i(t_n - \tilde{T}, \cdot; u_0)) \\ &= u_i(\tilde{T}, 0; u_i(t_n - \tilde{T}, \cdot + x_n; u_0), f_i(\cdot + x_n, \cdot)) \\ &\geq u_i(\tilde{T}, 0; \tilde{u}_{0n}(\cdot + x_n), f_i(\cdot + x_n, \cdot)). \end{aligned} \tag{6.7}$$

Observe that

$$\tilde{u}_{0n}(x + x_n) \to \tilde{u}_0$$

as $n \to \infty$ uniformly in x on bounded sets. In the case that

$$f_i(x + x_n, u) \to f_i^0(u),$$

by Proposition 3.3,

$$u_i(\tilde{T}, 0; \tilde{u}_{0n}(\cdot + x_n), f_i(\cdot + x_n, \cdot)) \to u_i(\tilde{T}, 0; \tilde{u}_0, f_i^0(\cdot))$$

as $n \to \infty$. By (6.6) and (6.7),

$$u_i(t_n, x_n; u_0) > u_i^0 - \epsilon_0/2 \quad \text{for} \quad n \gg 1. \tag{6.8}$$

By Lemma 6.3,

$$u_i^0 > u_i^*(x_n) - \epsilon_0/2 \quad \text{for} \quad n \gg 1. \tag{6.9}$$

By (6.4), (6.8), and (6.9),

$$|u_i(t_n, x_n; u_0) - u_i^*(x_n)| < \epsilon_0 \quad \text{for} \quad n \gg 1.$$

This contradicts to (6.3).

In the case that

$$x_n \to x^*,$$

by Proposition 3.3 again,

$$u_i(\tilde{T}, 0; \tilde{u}_{0n}(\cdot + x_n), f_i(\cdot + x_n, \cdot)) \to u_i(\tilde{T}, 0; \tilde{u}_0, f_i(\cdot + x^*, \cdot))$$

as $n \to \infty$. By (6.5) and (6.7),

$$u_i(t_n, x_n; u_0) > u_i^*(x^*) - \epsilon_0/2 \quad \text{for} \quad n \gg 1. \tag{6.10}$$

By the continuity of $u_i^*(\cdot)$,

$$u_i^*(x^*) > u_i^*(x_n) - \epsilon_0/2 \quad \text{for} \quad n \gg 1. \tag{6.11}$$

By (6.4), (6.10), and (6.11),

$$|u_i(t_n, x_n; u_0) - u_i^*(x_n)| < \epsilon_0 \quad \text{for} \quad n \gg 1.$$

This contradicts to (6.3) again.

Hence

$$\lim_{x\cdot\xi \le c' t, t\to\infty} |u_i(t, x; u_0) - u_i^*(x)| = 0$$

for all $0 < c' < c$.

(2) It can be proved by the similar arguments as in (1).

(3) It can also be proved by the similar arguments as in (1). □

Lemma 6.7. *Let $M > 0$ be such that $f_i(x, u) < 0$ for $x \in \mathcal{H}_i$, $u \in [0, M]$, and $i = 1, 2, 3$. Then for any $\epsilon > 0$, there are $p \in \mathbb{N}^N$ and $g_i : \mathcal{H}_i \times \mathbb{R} \to \mathbb{R}$ such that for any $u \in \mathbb{R}$, $g_i(\cdot, u) \in X_{i,p}$, $g_i(\cdot, \cdot)$ satisfies (H0) and (H2), and*

$$f_i(x, u) \ge g_i(x, u) \quad \forall x \in \mathcal{H}_i,\ u \in [0, M],$$

$$\hat{g}_i(0) \ge f_i^0(0) - \epsilon,$$

where $\hat{g}_i(\cdot)$ is as in (2.20) $(i = 1, 2, 3)$.

Proof. By Lemma 6.1, for any $\epsilon > 0$, there are $p \in \mathbb{N}^N$ and $h_i(\cdot) \in X_{i,p} \cap C^N(\mathcal{H}_i, \mathbb{R})$ such that

$$f_i(x, 0) \ge h_i(x)\ \forall x \in \mathcal{H}_i$$

and

$$\hat{\hat{h}}_i \geq f_i^0(0) - \epsilon$$

for $i = 1, 2, 3$. Fix $1 \leq i \leq 3$ and choose $M_i > 0$ such that

$$f_i(x, u) \geq g_i(x, u) := h_i(x) - M_i u \quad \text{for} \quad x \in \mathcal{H}_i,\ 0 \leq u \leq M.$$

It is not difficult to see that $g_i(\cdot, \cdot)$ $(1 \leq i \leq 3)$ satisfy the lemma. $\square$

Recall that for given $\xi \in S^{N-1}$,

$$c_1^0(\xi) = \inf_{\mu>0} \frac{f_1^0(0) + \mu^2}{\mu} = 2\sqrt{f_1^0(0)},$$

$$c_2^0(\xi) = \inf_{\mu>0} \frac{\int_{\mathbb{R}^N} e^{-\mu z\cdot\xi}\kappa(z)dz - 1 + f_2^0(0)}{\mu},$$

and

$$c_3^0(\xi) = \inf_{\mu>0} \frac{\sum_{k\in K} a_k(e^{-\mu k\cdot\xi} - 1) + f_3^0(0)}{\mu}.$$

Observe that $\lambda_i(\mu, \xi, f_i^0(0))$ $(i = 1, 2, 3)$ exist,

$$\lambda_1(\mu, \xi, f_1^0(0)) = f_1^0(0) + \mu^2,$$

$$\lambda_2(\mu, \xi, f_2^0(0)) = \int_{\mathbb{R}^N} e^{-\mu z\cdot\xi}\kappa(z)dz - 1 + f_2^0(0),$$

and

$$\lambda_3(\mu, \xi, f_3^0(0)) = \sum_{k\in K} a_k(e^{-\mu k\cdot\xi} - 1) + f_3^0(0).$$

If no confusion occurs, we may denote $\lambda_i(\mu, \xi, f_i^0(0))$ by $\lambda_i(\mu, \xi)$ $(i = 1, 2, 3)$. Observe also that

$$v_1(t, x) = e^{-\mu(x\cdot\xi - \frac{\lambda_1(\mu,\xi)}{\mu}t)},$$

$$v_2(t, x) = e^{-\mu(x\cdot\xi - \frac{\lambda_2(\mu,\xi)}{\mu}t)},$$

and

$$v_3(t,j) = e^{-\mu(j\cdot\xi - \frac{\lambda_3(\mu,\xi)}{\mu}t)}$$

are solutions of

$$v_t(t,x) = \Delta v(t,x) + f_1^0(0)v(t,x), \quad x \in \mathbb{R}^N, \tag{6.12}$$

$$v_t(t,x) = \int_{\mathbb{R}^N} \kappa(y-x)v(t,y)dy - v(t,x) + f_2^0(0)v(t,x), \quad x \in \mathbb{R}^N, \tag{6.13}$$

and

$$v_t(t,j) = \sum_{k\in K} a_k(v(t,j+k) - v(t,j)) + f_3^0(0)v(t,j), \quad j \in \mathbb{Z}^N, \tag{6.14}$$

respectively.

We now give another proof of Theorem 2.2.

Proof of Theorem 2.2. Fix $\xi \in S^{N-1}$ and $1 \le i \le 3$. We first prove that for any $c^{'} > c_i^0(\xi)$ and $u_0 \in X_i^+(\xi)$,

$$\limsup_{x\cdot\xi \ge c' t, t\to\infty} u_i(t,x;u_0) = 0. \tag{6.15}$$

To this end, take a c such that $c^{'} > c > c_i^*(\xi)$. Note that there is $\mu_i^* > 0$ such that

$$c_i^0(\xi) = \frac{\lambda_i(\xi,\mu_i^*)}{\mu_i^*}$$

and there is $\mu \in (0,\mu_i^*)$ such that

$$c = \frac{\lambda_i(\mu,\xi)}{\mu}.$$

Take $d > M > 0$ such that

$$u_0(x) \le M \quad \text{and} \quad u_0(x) \le de^{-\mu x\cdot\xi} \quad \forall x \in \mathcal{H}_i,$$

$$f_i(x,M) < 0 \quad \forall x \in \mathcal{H}_i, \tag{6.16}$$

and

$$f_i(x,u) = f_i^0(u) \quad \text{for} \quad x\cdot\xi \geq -\frac{1}{\mu}\ln\frac{M}{d}(>0). \tag{6.17}$$

Observe that by (6.16) and (H2), for $(t,x)\in(0,\infty)\times\mathcal{H}_i$ with $de^{-\mu(x\cdot\xi-ct)}\geq M$, i.e., $x\cdot\xi\leq -\frac{1}{\mu}\ln\frac{M}{d}+ct$,

$$f_i(x,de^{-\mu(x\cdot\xi-ct)})<0<f_i^0(0).$$

By (6.17), for $(t,x)\in(0,\infty)\times\mathcal{H}_i$ with $de^{-\mu(x\cdot\xi-ct)}\leq M$, i.e, $x\cdot\xi\geq -\frac{1}{\mu}\ln\frac{M}{d}+ct$,

$$f_i(x,de^{-\mu(x\cdot\xi-ct)}) = f_i^0(de^{-\mu(x\cdot\xi-ct)})\leq f_i^0(0).$$

It then follows that $u=de^{-\mu(x\cdot\xi-ct)}$, which is a solution of (6.12) or (6.13) or (6.14) if $i=1$ or 2 or 3, is a super-solution of (1.3) and hence by Proposition 3.1,

$$u_i(t,x;u_0)\leq de^{-\mu(x\cdot\xi-ct)} \quad \forall t>0\ x\in\mathcal{H}_i. \tag{6.18}$$

This implies that (6.15) holds.

Next, we prove that for any $c^{'}<c_i^0(\xi)$ and any $u_0\in X_i^+(\xi)$,

$$\limsup_{x\cdot\xi\leq c^{'}t,t\to\infty}|u_i(t,x;u_0)-u_i^*(x)|=0. \tag{6.19}$$

To this end, take a $c\in\mathbb{R}$ such that

$$c^{'}<c<c_i^0(\xi).$$

Let $M>0$ be such that

$$u_0(x)\leq M \quad \text{and} \quad f_i(x,M)<0 \quad \text{for all} \quad x\in\mathcal{H}_i.$$

Then $u \equiv M$ is a super-solution of (1.3) and

$$u_i(t, x; u_0) \le M \quad \forall t \ge 0, \; x \in \mathcal{H}_i.$$

For any $\epsilon > 0$, let $g_i(\cdot, \cdot)$ be as in Lemma 6.7. By Corollary 3.2, for $\epsilon > 0$ sufficiently small,

$$c_i^*(\xi, g_i(\cdot, \cdot)) \ge c_i^*(\xi, \hat{\hat{g}}_i(\cdot)) > c.$$

By Propositions 3.1 and 3.10,

$$\liminf_{x\cdot\xi \le ct, t\to\infty} u_i(t, x; u_0) \ge \liminf_{x\cdot\xi \le ct, t\to\infty} u_i(t, x; u_0, g_i) > 0.$$

(6.19) then follows from Lemma 6.6.

By (6.15) and (6.19), $c_i^*(\xi)$ exists and $c_i^*(\xi) = c_i^0(\xi)$ for $i = 1, 2, 3$. Moreover, (2.26) holds □

Finally, prove Theorem 2.3.

Proof of Theorem 2.3. (1) Fix $\xi \in S^{N-1}$ and $1 \le i \le 3$. Let $u_0 \in X_i^+$ satisfy that $u_0(x) = 0$ for $x \in \mathcal{H}_i$ with $|x \cdot \xi| \gg 1$. Then there are $u_0^+ \in X_i^+(\xi)$ and $u_0^- \in X_i^+(-\xi)$ such that

$$u_0(x) \le u_0^{\pm}(x) \quad \forall x \in \mathcal{H}_i.$$

By Proposition 3.1 and Theorem 2.2,

$$\limsup_{x\cdot\xi \ge c' t, t\to\infty} u_i(t, x; u_0) \le \limsup_{x\cdot\xi \ge c' t, t\to\infty} u_i(t, x; u_i^+) = 0 \quad \forall c' > c_i^*(\xi)$$

and

$$\limsup_{x\cdot(-\xi) \ge c' t, t\to\infty} u_i(t, x; u_0) \le \limsup_{x\cdot(-\xi) \ge c' t, t\to\infty} u_i(t, x; u_i^-) = 0 \quad \forall c' > c_i^*(-\xi)$$

It then follows that

$$\limsup_{|x\cdot\xi|\geq c^{'}t, t\to\infty} u_i(t,x;u_0)=0 \quad \forall c^{'}>\max\{c_i^*(\xi),c_i^*(-\xi)\}.$$

(2) Fix $\xi\in S^{N-1}$ and $1\leq i\leq 3$. For given $0<c^{'}<\min\{c_i^*(\xi),c_i^*(-\xi)\}$, take a $c>0$ such that

$$c^{'}<c<\min\{c_i^*(\xi),c_i^*(-\xi)\}.$$

For given $u_0\in X_i^+$ satisfying the condition in Theorem 2.1 (2), let $M>0$ be such that

$$u_0(x)\leq M \quad \text{and} \quad f_i(x,M)<0 \quad \text{for all} \quad x\in\mathcal{H}_i.$$

Then $u\equiv M$ is a super-solution of (1.3) and

$$u_i(t,x;u_0)\leq M \quad \forall t\geq 0,\ x\in\mathcal{H}_i.$$

For any $\epsilon>0$, let $g_i(\cdot,\cdot)$ be as in Lemma 6.7. By Corollary 3.2, for $\epsilon>0$ sufficiently small,

$$c_i^*(\xi,g_i(\cdot,\cdot))\geq c_i^*(\xi,\hat{g}_i(\cdot))>c.$$

By Propositions 3.1 and 3.10,

$$\liminf_{|x\cdot\xi|\leq ct, t\to\infty} u_i(t,x;u_0)\geq \liminf_{|x\cdot\xi|\leq ct, t\to\infty} u_i(t,x;u_0,g_i)>0.$$

It then follows from Lemma 6.6 that

$$\limsup_{|x\cdot\xi|\leq c^{'}t, t\to\infty} |u_i(t,x;u_0)-u_i^*(x)|=0.$$

(3) Fix $\xi \in S^{N-1}$ and $1 \le i \le 3$. Let

$$c > \sup_{\xi \in S^{N-1}} c_i^*(\xi).$$

Let $u_0 \in X_i^+$ be such that

$$u_0(x) = 0 \quad \text{for} \quad \|x\| \gg 1.$$

Note that for every given $\xi \in S^{N-1}$, there is $\tilde{u}_0(\cdot;\xi) \in X_i^+(\xi)$ such that

$$u_0(\cdot) \le \tilde{u}_0(\cdot;\xi).$$

By Proposition 3.1,

$$0 \le u_i(t,x;u_0) \le u_i(t,x;\tilde{u}_0(\cdot;\xi))$$

for $t > 0$ and $x \in \mathcal{H}_i$. It then follows from Theorem 2.2 that

$$0 \le \limsup_{x\cdot\xi \ge ct, t\to\infty} u_i(t,x;u_0) \le \limsup_{x\cdot\xi \ge ct, t\to\infty} u_i(t,x;\tilde{u}_0(\cdot;\xi)) = 0.$$

Take any $c^{'} > c$. Consider all $x \in \mathcal{H}_i$ with $\|x\| = c^{'}$. By the compactness of $\partial B(0,c^{'}) = \{x \in \mathcal{H}_i | \, \|x\| = c^{'}\}$, there are $\xi_1, \xi_2, \cdots, \xi_L \in S^{N-1}$ such that for every $x \in \partial B(0,c^{'})$, there is l $(1 \le l \le L)$ such that

$$x \cdot \xi_l \ge c.$$

Hence for every $x \in \mathcal{H}_i$ with $\|x\| \ge c^{'}t$, there is $1 \le l \le L$ such that

$$x \cdot \xi_l = \frac{\|x\|}{c^{'}}\Big(\frac{c^{'}}{\|x\|}x\Big) \cdot \xi_l \ge \frac{\|x\|}{c^{'}} c \ge ct.$$

By the above arguments,

$$0 \le \limsup_{x\cdot\xi_l \ge ct, t\to\infty} u_i(t,x;u_0) \le \limsup_{x\cdot\xi_l \ge ct, t\to\infty} u_i(t,x;\tilde{u}_0(\cdot;\xi_l)) = 0$$

for $l = 1, 2, \cdots L$. This implies that

$$\limsup_{\|x\| \geq c' t, t \to \infty} u_i(t, x; u_0) = 0.$$

Since $c' > c$ and $c > \sup_{\xi \in S^{N-1}} c_i^*(\xi)$ are arbitrary, we have that for $c > \sup_{\xi \in S^{N-1}} c_i^*(\xi)$,

$$\limsup_{\|x\| \geq ct, t \to \infty} u_i(t, x; u_0) = 0.$$

(4) It can be proved by similar arguments as in (2). To be more precise, for given $0 < c' < \min\{c_i^*(\xi) \,|\, \xi \in S^{N-1}\}$, take a $c > 0$ such that

$$c' < c < \min\{c_i^*(\xi) \,|\, \xi \in S^{N-1}\}.$$

For given $u_0 \in$ satisfying the condition in Theorem 2.3 (4), let $M > 0$ be such that

$$u_0(x) \leq M \quad \text{and} \quad f_i(x, M) < 0 \quad \text{for all} \quad x \in \mathcal{H}_i.$$

Then $u \equiv M$ is a super-solution of (1.3) and

$$u_i(t, x; u_0) \leq M \quad \forall t \geq 0,\ x \in \mathcal{H}_i.$$

For any $\epsilon > 0$, let $g_i(\cdot, \cdot)$ be as in Lemma 6.7. By Corollary 3.2, for $\epsilon > 0$ sufficiently small,

$$c_i^*(\xi, g_i(\cdot, \cdot)) \geq c_i^*(\xi, \hat{g}_i(\cdot)) > c.$$

By Propositions 3.1 and 3.10,

$$\liminf_{\|x\| \leq ct, t \to \infty} u_i(t, x; u_0) \geq \liminf_{\|x\| \leq ct, t \to \infty} u_i(t, x; u_0, g_i) > 0.$$

It then follows from Lemma 6.6 that

$$\limsup_{\|x\|\le c' t, t\to\infty} |u_i(t,x;u_0) - u_i^*(x)| = 0.$$

$\square$

Chapter 7

Effects of Temporal and Spatial Variations

In this chapter, we prove Theorem 2.4 on the effects of temporal and spatial variations on the spatial spreading dynamics of monostable equations.

Proof of Theorem 2.4. (1) Suppose that $\lambda(\hat{f}_0(\cdot,0)) > 0$. By Proposition 3.6,

$$\lambda(f_0(\cdot,\cdot,0)) \geq \lambda(\hat{f}_0(\cdot,0)) > 0.$$

Then by Theorem 2.1, (1.3) has a time periodic strictly positive solution $u^*(t,x)$. By Theorem 2.2, (1.3) has a spatial spreading speed $c^*(\xi)$ in the direction of ξ for each $\xi \in S^{N-1}$, and

$$c^*(\xi) = c_0^*(\xi).$$

Recall that

$$c_0^*(\xi) = \inf_{\mu>0} \frac{\lambda_{\xi,\mu}(f_0(\cdot,\cdot,0))}{\mu}.$$

By Proposition 3.6 again,

$$\lambda_{\xi,\mu}(f_0(\cdot,\cdot,0)) \geq \lambda_{\xi,\mu}(\hat{f}_0(\cdot,0)).$$

It then follows that

$$c^*(\xi) = c_0^*(\xi) \geq \inf_{\mu>0} \frac{\lambda_{\xi,\mu}(\hat{f}_0(\cdot,0))}{\mu} = \hat{c}_0^*(\xi).$$

(2) Suppose that $\lambda(\hat{\hat{f}}_0(0)) > 0$. By Proposition 3.7, we have

$$\lambda(\hat{f}_0(\cdot,0)) \geq \lambda(\hat{\hat{f}}_0(0)) > 0.$$

It then follows from (1) that (1.3) has a time periodic strictly positive solution $u^*(t,x)$ and has a spatial spreading speed $c^*(\xi)$ in the direction of ξ for each $\xi \in S^{N-1}$.

Recall again that

$$c_0^*(\xi) = \inf_{\mu>0} \frac{\lambda_{\xi,\mu}(f_0(\cdot,\cdot,0))}{\mu}.$$

By Proposition 3.7 again,

$$\lambda_{\xi,\mu}(\hat{f}_0(\cdot,0)) \geq \lambda_{\xi,\mu}(\hat{\hat{f}}_0(0)).$$

This together with (1) implies that

$$c^*(\xi) = c_0^*(\xi) \geq \hat{c}_0^*(\xi) \geq \hat{\hat{c}}_0^*(\xi)$$

for each $\xi \in S^{N-1}$. $\square$

Chapter 8

Concluding Remarks and Future Plan

8.1 Concluding Remarks

In this dissertation, we studied the semilinear dispersal evolution equations of the form

$$u_t(t,x) = (\mathcal{A}u)(t,x) + u(t,x)f(t,x,u(t,x)), \quad x \in \mathcal{H},$$

where $\mathcal{H} = \mathbb{R}^N$ or $\mathbb{Z}^N$, $\mathcal{A}$ is a random dispersal operator or nonlocal dispersal operator in the case $\mathcal{H} = \mathbb{R}^N$ and is a discrete dispersal operator in the case $\mathcal{H} = \mathbb{Z}^N$, and f is periodic in t, asymptotically periodic in x (i.e. $f(t,x,u) - f_0(t,x,u)$ converges to 0 as $\|x\| \to \infty$ for some time and space periodic function $f_0(t,x,u)$), and is of KPP type in u. It is proved that Liouville type property for such equations holds, that is, time periodic strictly positive solutions are unique. It is also proved that if $u \equiv 0$ is a linearly unstable solution to the time and space periodic limit equation of such an equation, then it has a unique stable time periodic strictly positive solution and has a spatial spreading speed in every direction. Moreover, we developed multiple methods to achieve the two results, and considered the effects of temporal and spatial variations of the media on the uniform persistence and spatial spreading speeds of monostable equations.

It should be pointed out that the Liouville type property for (1.3) in the case that $\mathcal{H} = \mathbb{R}^N$, $\mathcal{A} = \Delta u$, and $f(t,x,u) = f(x,u)$ has been proved in [8]. However, many techniques developed in [8] are difficult to apply to (1.3). Several important new techniques are developed in the current dissertation. Both the results and techniques established in the current paper can be extended to more general cases (say, cases that $\mathcal{A}$ is some linear combination of random and nonlocal dispersal operators and/or $f(t,x,u)$ is almost periodic in t and asymptotically periodic in x).

It should also be pointed out that, if $u=0$ is a linearly unstable solution of (1.1) with respect to periodic perturbations, then (1.1) has traveling wave solutions connecting 0 and $u_0^*(\cdot,\cdot)$ and propagating in the direction of ξ with speed $c>c_0^*(\xi)$ for any $\xi\in S^{N-1}$. But (1.3) may have no traveling wave solutions connecting 0 and $u^*(\cdot,\cdot)$ (see [63]) (hence, localized spatial inhomogeneity may prevent the existence of traveling wave solutions).

8.2 Future plans

8.2.1 Single-species population model

For the single-species model, we established the existence of strictly positive solution u^* and spreading speeds. Next topic I am interested in is to generalize traveling wave solution, which is a global-in-time solution, connecting the positive solution u^* and trivial solution $u=0$, in the locally inhomogeneous media. Due to the spatial heterogeneity, the global-in-time wave-like solutions lose some classical features, compared to homogeneous equations, such as, constant in shape (up to a spatial shift) and the unchanged speed for each profile. Recently, James Nolen together his collaborates considered (1.3) in temporal independent setting, and showed that a sufficiently strong localized inhomogeneity may prevent existence of global-in-time wave-like solutions. They created a time-global bump-like solution, which blocks any wave-like solution. I am curious to know whether such interesting scenario happen in more general heterogeneous media (say, a spatially locally perturbed spatial periodic media). How about the existence of wave-like solutions in nonlocal and discrete dispersal framework?

As a short term plan, the following problem is listed in my agenda.

- Explore the existence, uniqueness and stability of of global-in-time wave-like solutions in spatially locally perturbed media.

8.2.2 Multi-species population model

Comparing with single species model, much less is known about the multi-species system, especially in the heterogeneous (say, spatially locally inhomogeneous) environments.

For instance, for two species competitive system, main results established in the literature are related to spatial-temporal independent nonlinearities,

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + uf(u,v), \quad x \in \mathbb{R}^N, \\ \\ \frac{\partial v}{\partial t} = \Delta v + vg(u,v), \quad x \in \mathbb{R}^N, \end{cases}$$

where $f(u,v) < 0$ and $g(u,v) < 0$ for $u, v \geq 0$ with $u^2 + v^2 \gg 1$, $f_u(u,v) < 0$, $f_v(u,v) < 0$, $g_u(u,v) < 0$, and $g_v(u,v) < 0$ for $u, v \geq 0$ (see [28], [36], [38], [47], [48], [53], etc.)

Here are three problems being considered.

- Explore the existence of traveling wave solutions to the systems in periodic media with nonlocal dispersal.
- Extend the concept of spreading speeds for multi-species system in homogeneous and periodic media to general inhomogeneous media.
- Investigate the existence of spreading speeds to the systems in the locally inhomogeneous media.

To attack the above problems, we will first extend the existing results on uniform persistence, coexistence, and extinction of two species competition systems in homogeneous and periodic media (see [15], [33], [34], etc.) to locally spatially inhomogeneous media.

Regarding the long term research plan, I have following problems in mind.

- Study the monostable equations with more general nonlinearity, that is almost periodic or random stationary ergodic in time.

- Investigate the dynamics of other types of equations, such as ignition type equations and bistable type equations which arise in Ising and combustion models and phase transition models.

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