Structure Theory and a Generalization of the Isomorphism Theorems

by

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Abstract

A general format in which the mathematical structure of topological spaces, algebraic structures, and graphs can be expressed is described. A generalization of the fundamental homomorphism theorem and the isomorphism theorems of algebra is proved.

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Chapter 1

Definitions

Definition The statement that r *is a relation* means r is a set of ordered pairs. If S is a set then $r(S)$ denotes the set to which an element y belongs if and only if there is an element $x \in S$ such that $(x, y) \in r$.

Theorem 1.1. *Suppose each of r and s is a relation, $r \subseteq s$, each of U and V is a set, and $U \subseteq V$. Then $r(U) \subseteq s(V)$.*

Proof:

$$
\begin{aligned}
& y \in r(U) \\
\Longrightarrow\ & \exists x \in U \text{ such that } (x, y) \in r \\
\Longrightarrow\ & x \in V \text{ and } (x, y) \in s \\
\Longrightarrow\ & y \in s(V)
\end{aligned}
$$

So $r(U) \subseteq s(V)$. □

Definition Suppose each of r and s is a relation. The *composition of r and s* is the relation to which a pair (x, z) belongs if and only if there is an element y such that $(x, y) \in s$ and $(y, z) \in r$. Denote the composition of r and s by rs.

Theorem 1.2. *Suppose each of r, s, and t is a relation. Then $(rs)t = r(st)$.*

Proof:

$$\begin{aligned}
& (a,d) \in (rs)t \\
\iff & \exists c \text{ such that } (a,c) \in rs \text{ and } (c,d) \in t \\
\iff & \exists b \text{ and a } c \text{ such that } (a,b) \in r, (b,c) \in s, \text{ and } (c,d) \in t \\
\iff & \exists b \text{ such that } (a,b) \in r \text{ and } (b,d) \in st \\
\iff & (a,d) \in r(st)
\end{aligned}$$

So $(rs)t = r(st)$. □

Theorem 1.3. *Suppose each of f, g, r, and s is a relation, $f \subseteq g$, and $r \subseteq s$. Then $rf \subseteq sg$.*

Proof:

$$\begin{aligned}
& (x,z) \in rf \\
\implies & \exists y \text{ such that } (x,y) \in f \text{ and } (y,z) \in r \\
\implies & (x,y) \in g \text{ and } (y,z) \in s \\
\implies & (x,z) \in sg
\end{aligned}$$

So $rf \subseteq sg$. □

Definition Suppose r is a relation. The *inverse of* r is the relation to which a pair (x,y) belongs if and only if (y,x) is in r. Denote the inverse of r by r^{-1}.

Theorem 1.4. *Suppose r is a relation. Then $(r^{-1})^{-1} = r$.*

Proof:

$$
\begin{aligned}
& (x,y) \in (r^{-1})^{-1} \\
\iff & (y,x) \in r^{-1} \\
\iff & (x,y) \in r
\end{aligned}
$$

So $(r^{-1})^{-1} = r$. □

Theorem 1.5. *Suppose each of r and s is a relation. Then $(rs)^{-1} = s^{-1}r^{-1}$.*

Proof:

$$
\begin{aligned}
& (x,z) \in (rs)^{-1} \\
\iff & (z,x) \in rs \\
\iff & \exists y \text{ such that } (y,x) \in r \text{ and } (z,y) \in s \\
\iff & \exists y \text{ such that } (x,y) \in r^{-1} \text{ and } (y,z) \in s^{-1} \\
\iff & (x,z) \in s^{-1}r^{-1}
\end{aligned}
$$

So $(rs)^{-1} = s^{-1}r^{-1}$. □

Definition Suppose r is a relation. The statement that *D is the domain of r* means D is the set to which an element x belongs if and only if x is the first element of a pair in r. Denote the domain of r by $\text{dom}(r)$.

Definition Suppose r is a relation. The statement that *R is the image of r* means R is the set to which an element x belongs if and only if x is the second element of a pair in r. Denote the image of r by $\text{im}(r)$.

Theorem 1.6. *Suppose r is a relation. Then* $\text{im}(r) = r(\text{dom}(r))$.

Proof:

$$
\begin{aligned}
& y \in \operatorname{im}(r) \\
\iff & \exists (x, y) \in r \\
\iff & \exists x \in \operatorname{dom}(r) \text{ such that } (x, y) \in r \\
\iff & y \in r(\operatorname{dom}(r))
\end{aligned}
$$

So $\operatorname{im}(r) = r(\operatorname{dom}(r))$. □

Theorem 1.7. *Suppose each of r and s is a relation. Then $\operatorname{im}(rs) = r(\operatorname{im}(s))$.*

Proof:

$$
\begin{aligned}
& z \in \operatorname{im}(rs) \\
\iff & \exists (x, z) \in rs \\
\iff & \exists y \text{ such that } (y, z) \in r \text{ and } \exists (x, y) \in s \\
\iff & \exists y \in \operatorname{im}(s) \text{ such that } (y, z) \in r \\
\iff & z \in r(\operatorname{im}(s))
\end{aligned}
$$

So $\operatorname{im}(rs) = r(\operatorname{im}(s))$. □

Definition The statement that f *is a function* means f is a relation such that no two pairs in f share the same first element. If $(x, y) \in f$, then denote y by $f(x)$.

Theorem 1.8. *Suppose each of f and g is a function. Then fg is a function.*

Proof:

$$
\begin{aligned}
&(x, z_1) \in fg \text{ and } (x, z_2) \in fg \\
\Longrightarrow\ &\exists y_1 \text{ such that } (x, y_1) \in g \text{ and } (y_1, z_1) \in f \\
\text{and } &\exists y_2 \text{ such that } (x, y_2) \in g \text{ and } (y_2, z_2) \in f \\
\Longrightarrow\ &y_1 = y_2 \text{ and } (y_1, z_1) \in f \text{ and } (y_1, z_2) = (y_2, z_2) \in f \text{ (since } g \text{ is a function)} \\
\Longrightarrow\ &z_1 = z_2 \text{ (since } f \text{ is a function)}
\end{aligned}
$$

So no two pairs of fg contain the first element. So fg is a function. □

Definition The statement that f *is an injection* means f is a function and f^{-1} is a function.

Theorem 1.9. *Suppose each of f and g is an injection. Then fg is a injection.*

Proof:

$$
\begin{aligned}
&f \text{ is an injection and } g \text{ is an injection} \\
\Longrightarrow\ &f \text{ is a function, } g \text{ is a function, } f^{-1} \text{ is a function, and } g^{-1} \text{ is a function} \\
\Longrightarrow\ &fg \text{ is a function and } (fg)^{-1} = g^{-1}f^{-1} \text{ is a function} \\
\Longrightarrow\ &fg \text{ is an injection} \quad \Box
\end{aligned}
$$

Definition The statement that f *is a surjection with respect to Y* means f is a function with image Y.

Theorem 1.10. *Suppose S is a set, and f is a surjection with respect to S, and g is a surjection with respect to* $\text{dom}(f)$. *Then fg is a surjection with respect to S.*

Proof:

$$
\begin{aligned}
& z \in S \\
\iff & z \in \operatorname{im}(f) \\
\iff & \exists y \in \operatorname{dom}(f) = \operatorname{im}(g) \text{ such that } (y, z) \in f \\
\iff & \exists x \in \operatorname{dom}(g) \text{ and } \exists y \in \operatorname{dom}(f) \text{ such that } (x, y) \in g \text{ and } (y, z) \in f \\
\iff & \exists x \in \operatorname{dom}(g) \text{ such that } (x, z) \in fg \\
\iff & z \in \operatorname{im}(fg)
\end{aligned}
$$

So $S = \operatorname{im}(fg)$ and thus fg is a surjection with respect to S. □

Definition The statement that f *is a bijection with respect to* Y means f is an injection and a surjection with respect to Y.

Definition The notation $r : X \to Y$ means r is a relation and X is the domain of r and the image of r is a subset of Y, and henceforth if the terms *surjection* or *bijection* are used to describe r they will be understood to be with respect to Y.

Definition Suppose A is a set. Denote by 1_A the relation $\{(a, a) \mid a \in A\}$.

Theorem 1.11. *Suppose A is a set. Then $1_A = 1_A^{-1}$.*

Proof:

$$
\begin{aligned}
& (a, a) \in 1_A \\
\iff & (a, a) \in 1_A^{-1}
\end{aligned}
$$

So $1_A = 1_A^{-1}$. □

Definition Suppose A is a set and r is a relation. Denote by $r|_A$ the relation to which a pair (x, y) belongs if and only if $(x, y) \in r$ and $x \in A$.

Theorem 1.12. *Suppose A is a set and r is a relation. Then $r = r|_A$ if and only if* $\mathrm{dom}(r) \subseteq A$.

Proof: Suppose $r = r|_A$.

$$\begin{aligned} & a \in \mathrm{dom}(r) \\ \Longrightarrow\ & \exists b \text{ such that } (a,b) \in r \\ \Longrightarrow\ & \exists b \text{ such that } (a,b) \in r|_A \\ \Longrightarrow\ & a \in A \end{aligned}$$

So $\mathrm{dom}(r) \subseteq A$.

Suppose $\mathrm{dom}(r) \subseteq A$.

$$\begin{aligned} & (a,b) \in r \\ \Longleftrightarrow\ & a \in \mathrm{dom}(r) \subseteq A \text{ and } (a,b) \in r \\ \Longleftrightarrow\ & (a,b) \in r|_A \end{aligned}$$

So $r = r|_A$. □

Theorem 1.13. *Suppose A is a set, and r is a relation. Then $r1_A = r|_A$.*

Proof:

$$\begin{aligned} & (x,z) \in r1_A \\ \Longleftrightarrow\ & \exists y \text{ such that } (x,y) \in 1_A \text{ and } (y,z) \in r \\ \Longleftrightarrow\ & \exists y \text{ such that } x \in A,\ x = y, \text{ and } (y,z) \in r \\ \Longleftrightarrow\ & x \in A \text{ and } (x,z) \in r \\ \Longleftrightarrow\ & (x,z) \in r|_A \end{aligned}$$

So $r1_A = r|_A$. □

Definition Suppose A is a set and r is a relation. Denote by $r|^A$ the relation to which a pair (x, y) belongs if and only if $(x, y) \in r$ and $y \in A$.

Theorem 1.14. *Suppose A is a set and r is a relation. Then $r = r|^A$ if and only if $\mathrm{im}(r) \subseteq A$.*

Proof: Suppose $r = r|^A$.

$$\begin{aligned} & a \in \mathrm{im}(r) \\ \Longrightarrow\ & \exists b \text{ such that } (b, a) \in r \\ \Longrightarrow\ & \exists b \text{ such that } (b, a) \in r|^A \\ \Longrightarrow\ & a \in A \end{aligned}$$

So $\mathrm{im}(r) \subseteq A$.

Suppose $\mathrm{im}(r) \subseteq A$.

$$\begin{aligned} & (b, a) \in r \\ \Longleftrightarrow\ & a \in \mathrm{im}(r) \subseteq A \text{ and } (b, a) \in r \\ \Longleftrightarrow\ & (b, a) \in r|^A \end{aligned}$$

So $r = r|^A$. □

Theorem 1.15. *Suppose A is a set, and r is a relation. Then $1_A r = r|^A$.*

Proof:

$$
\begin{aligned}
&(x,z) \in 1_A r \\
\iff &\exists y \text{ such that } (x,y) \in r \text{ and } (y,z) \in 1_A \\
\iff &\exists y \text{ such that } (x,y) \in r,\ y \in A, \text{ and } y = z \\
\iff &(x,z) \in r \text{ and } z \in A \\
\iff &(x,z) \in r|^A
\end{aligned}
$$

So $1_A r = r|^A$. □

Theorem 1.16. *Suppose each of A and B is a set, and $f : A \to B$ is a function. Then $1_A \subseteq f^{-1}f$, and $1_A = f^{-1}f$ if and only if f is an injection.*

Proof:

$$
\begin{aligned}
&(a,a) \in 1_A \\
\implies &a \in A \\
\implies &(a, f(a)) \in f \text{ and } (f(a), a) \in f^{-1} \\
\implies &(a,a) \in f^{-1}f
\end{aligned}
$$

So $1_A \subseteq f^{-1}f$.

Suppose $1_A = f^{-1}f$.

$$
\begin{aligned}
&(b, a_1) \in f^{-1} \text{ and } (b, a_2) \in f^{-1} \\
\implies &(a_1, b) \in f \text{ and } (b, a_2) \in f^{-1} \\
\implies &(a_1, a_2) \in f^{-1}f = 1_A \\
\implies &a_1 = a_2
\end{aligned}
$$

So f^{-1} is a function and thus f is an injection.

Suppose f is an injection.

$$
\begin{aligned}
& (a_1, a_2) \in f^{-1}f \\
\Longrightarrow\ & \exists b \text{ such that } (a_1, b) \in f \text{ and } (b, a_2) \in f^{-1} \\
\Longrightarrow\ & (b, a_1) \in f^{-1} \text{ and } (b, a_2) \in f^{-1} \\
\Longrightarrow\ & a_1 = a_2 \text{ (since } f^{-1} \text{ is a function)} \\
\Longrightarrow\ & (a_1, a_2) \in 1_A
\end{aligned}
$$

So $f^{-1}f \subseteq 1_A$, and thus $1_A = f^{-1}f$. □

Theorem 1.17. *Suppose each of A and B is a set, and $f : A \to B$ is a function. Then $ff^{-1} \subseteq 1_B$, and $ff^{-1} = 1_B$ if and only if f is a surjection.*

Proof: Suppose each of b_1 and b_2 is in B.

$$
\begin{aligned}
& (b_1, b_2) \in ff^{-1} \\
\Longrightarrow\ & \exists a \in A \text{ such that } (b_1, a) \in f^{-1} \text{ and } (a, b_2) \in f \\
\Longrightarrow\ & (a, b_1) \in f \text{ and } (a, b_2) \in f \\
\Longrightarrow\ & b_1 = b_2 \text{ (since } f \text{ is a function)} \\
\Longrightarrow\ & (b_1, b_2) \in 1_B
\end{aligned}
$$

So $ff^{-1} \subseteq 1_B$.

Suppose $ff^{-1} = 1_B$.

Suppose $b \in B$.

$$
\begin{aligned}
&(b, b) \in 1_B = ff^{-1} \\
\implies &\exists a \in A \text{ such that } (b, a) \in f^{-1} \text{ and } (a, b) \in f \\
\implies &a \in A \text{ such that } b = f(a) \subseteq \text{im}(f)
\end{aligned}
$$

So f is a surjection.

Suppose f is a surjection.

Suppose $(b, b) \in 1_B$ (so $b \in B$). There is an $a \in A$ such that $f(a) = b$.

$$
\begin{aligned}
&(a, b) \in f \text{ and } (b, a) \in f^{-1} \\
\implies &(b, b) \in ff^{-1}
\end{aligned}
$$

So $1_B \subseteq ff^{-1}$, and thus $ff^{-1} = 1_B$. □

Theorem 1.18. *Suppose A is a set. Then 1_A is a bijection with respect to A.*

Proof:

1. 1_A is an injection:

$$1_A = 1_A 1_A = 1_A^{-1} 1_A$$

   So 1_A is an injection.

2. 1_A is a surjection with respect to A:

$$1_A = 1_A 1_A = 1_A 1_A^{-1}$$

So $A = \mathrm{im}(1_A)$ and thus 1_A is a surjection with respect to A.

So 1_A is a bijection with respect to A. □

Definition Suppose each of A and B is a set. Denote by $A \times B$ the relation to which an ordered pair (a, b) belongs if and only if $a \in A$ and $b \in B$.

Definition Let I be a set. The statement that $(A, \mathcal{R})$ is an *I-structure* means A is a set, and $\mathcal{R}$ is a set of relations each of which is a subset of $I \times A$. A is called the *base set of* $(A, \mathcal{R})$ and I is called the *index set of* $(A, \mathcal{R})$.

Definition Suppose each of $(A, \mathcal{R})$ and $(B, \mathcal{S})$ is an I-structure. The statement that a function $\alpha : A \to B$ is *preservative* means for each $r \in \mathcal{R}$, $\alpha r \in \mathcal{S}$.

Definition Suppose each of $(A, \mathcal{R})$ and $(B, \mathcal{S})$ is an I-structure. The statement that a function $\alpha : A \to B$ is *saturating* means for each $s \in \mathcal{S}$ such that $\mathrm{im}(s) \subseteq \mathrm{im}(\alpha)$, there is an $r \in \mathcal{R}$ such that $\alpha r = s$.

Definition Suppose each of $(A, \mathcal{R})$ and $(B, \mathcal{S})$ is an I-structure. The statement that a function $\alpha : A \to B$ is *continuous* means for each $s \in \mathcal{S}$, $\alpha^{-1}s \in \mathcal{R}$.

Definition Suppose each of $(A, \mathcal{R})$ and $(B, \mathcal{S})$ is an I-structure. The statement that a function $\alpha : A \to B$ is *conservative* means for each $r \in \mathcal{R}$, there is an $s \in \mathcal{S}$ such that $\mathrm{im}(s) \subseteq \mathrm{im}(\alpha)$ and $\alpha^{-1}s = r$.

Definition Suppose $\varphi : A \to B$ is a function and each of $(A, \mathcal{R})$ and $(B, \mathcal{S})$ is an I-structure. The statement that *φ is an I-structure homomorphism from $(A, \mathcal{R})$ to $(B, \mathcal{S})$* means φ is preservative and saturating.

Definition Suppose $\varphi : A \to B$ is a function and each of $(A, \mathcal{R})$ and $(B, \mathcal{S})$ is an I-structure. The statement that *φ is an I-structure cohomomorphism from $(A, \mathcal{R})$ to $(B, \mathcal{S})$* means φ is continuous and conservative.

Definition Suppose $\varphi : A \to B$ is a function and each of $(A, \mathcal{R})$ and $(B, \mathcal{S})$ is an I-structure. The statement that φ *is an I-structure isomorphism from* $(A, \mathcal{R})$ *to* $(B, \mathcal{S})$ means φ is a continuous, preservative bijection.

Definition The statement that an I-structure $(A, \mathcal{R})$ and an I-structure $(B, \mathcal{S})$ are *isomorphic* means there is an isomorphism $\varphi : A \to B$ from $(A, \mathcal{R})$ to $(B, \mathcal{S})$. In this case $(A, \mathcal{R}) \cong (B, \mathcal{S})$ denotes "$(A, \mathcal{R})$ and $(B, \mathcal{S})$ are isomorphic".

Lemma 1.19.1. *Suppose each of $(A, \mathcal{R})$ and $(B, \mathcal{S})$ is an I-structure and $\varphi : A \to B$ is a continuous function. Then φ is saturating.*

Proof: Suppose $s \in \mathcal{S}$ such that $\text{im}(s) \subseteq \text{im}(\varphi)$.

φ is continuous, so $\varphi^{-1}s \in \mathcal{R}$.

So $\varphi^{-1}s \in \mathcal{R}$ and $s = 1_{\text{im}(\varphi)}s = \varphi\varphi^{-1}s$. So φ is saturating. $\square$

Theorem 1.19. *Suppose each of $(A, \mathcal{R})$ and $(B, \mathcal{S})$ is an I-structure and $\alpha : A \to B$ is a function. Then α is a bijective I-structure homomorphism if and only if α is an I-structure isomorphism.*

Proof: Suppose α is a bijective I-structure homomorphism.

α is bijective and preservative, so it remains only to show that α is continuous.

Suppose $s \in \mathcal{S}$. α is surjective, so $\text{im}(s) \subseteq B = \text{im}(\alpha)$. α is saturating, so there is an $r \in \mathcal{R}$ such that $\alpha r = s$.

$$\alpha^{-1}s = \alpha^{-1}\alpha r = 1_A r = r \in \mathcal{R}$$

So α is continuous and is thus an isomorphism.

Suppose α is an I-structure isomorphism.

α is bijective and preservative. α is continuous, so by Lemma 1.19.1, α is saturating.

So α is a bijective homomorphism. □

Lemma 1.20.1. *Suppose each of $(A, \mathcal{R})$ and $(B, \mathcal{S})$ is an I-structure and $\varphi : A \to B$ is a conservative function. Then φ is preservative.*

Proof: Suppose $r \in \mathcal{R}$.

φ is conservative, so there is an $s \in \mathcal{S}$ such that $\mathrm{im}(s) \subseteq \mathrm{im}(\varphi)$ and $\varphi^{-1}s = r$.

So $\varphi r = \varphi\varphi^{-1}s = 1_{\mathrm{im}(\varphi)}s = s \in \mathcal{S}$.

So φ is preservative. □

Theorem 1.20. *Suppose each of $(A, \mathcal{R})$ and $(B, \mathcal{S})$ is an I-structure and $\alpha : A \to B$ is a cohomomorphism. Then α is a homomorphism.*

Proof: α is conservative, so by Lemma 1.20.1, α is preservative.

α is continuous, so by Lemma 1.19.1, α is saturating.

α is both preservative and saturating, so α is a homomorphism. □

Theorem 1.21. *Suppose each of $(A, \mathcal{R})$ and $(B, \mathcal{S})$ is an I-structure and $\alpha : A \to B$ is a function. Then α is a bijective cohomomorphism if and only if α is an isomorphism.*

Proof: Suppose α is a bijective cohomomorphism.

α is bijective and continuous. α is conservative, so by Lemma 1.20.1, α is preservative.

So α is an isomorphism.

Suppose α is an isomorphism.

α is bijective and continuous, so it remains only to show that α is conservative.

Suppose $r \in \mathcal{R}$. α is preservative, so $\alpha r \in \mathcal{S}$. $\mathrm{im}(\alpha r) \subseteq \mathrm{im}(\alpha)$.

$$\alpha^{-1}\alpha r = 1_A r = r$$

So α is conservative and is thus a bijective cohomomorphism. □

Definition Suppose each of $(A, \mathcal{R})$ and $(B, \mathcal{S})$ is an I-structure. The statement that a function $\varphi : A \to B$ is a *structure monomorphism* means φ is an injective I-structure homomorphism.

Definition Suppose each of $(A, \mathcal{R})$ and $(B, \mathcal{S})$ is an I-structure. The statement that a function $\varphi : A \to B$ is a *structure epimorphism* means φ is a surjective I-structure homomorphism.

Theorem 1.22. *Suppose each of $M = (A, \mathcal{R})$, $N = (B, \mathcal{S})$, $L = (C, \mathcal{T})$ is an I-structure, $\alpha : A \to B$ is an epimorphism from M to N, and $\beta : B \to C$ is a homomorphism from N to L. Then $\beta\alpha$ is a homomorphism from M to L.*

Proof: Suppose $r \in \mathcal{R}$. α is preservative, so $\alpha r \in \mathcal{S}$. β is preservative, so $\beta\alpha r \in \mathcal{T}$. So $\beta\alpha$ is preservative.

Suppose $t \in \mathcal{T}$ such that $\mathrm{im}(t) \subseteq \mathrm{im}(\beta\alpha)$. β is saturating, and $\mathrm{im}(t) \subseteq \mathrm{im}(\beta\alpha) \subseteq \mathrm{im}(\beta)$, so there is an $s \in \mathcal{S}$ such that $\beta s = t$.

$\mathrm{im}(s) \subseteq B = \mathrm{im}(\alpha)$, so there is an $r \in \mathcal{R}$ such that $\alpha r = s$.

So $r \in \mathcal{R}$ such that $\beta\alpha r = \beta s = t$. So $\beta\alpha$ is saturating.

$\beta\alpha$ is both preservative and saturating, and is thus a homomorphism. □

Definition Suppose each of $(A, \mathcal{R})$ and $(B, \mathcal{S})$ is an I-structure. The statement that a function $\varphi : A \to B$ is a *structure comonomorphism* means φ is an injective I-structure cohomomorphism.

Definition Suppose each of $(A, \mathcal{R})$ and $(B, \mathcal{S})$ is an I-structure. The statement that a function $\varphi : A \to B$ is a *structure coepimorphism* means φ is a surjective I-structure cohomomorphism.

Theorem 1.23. *Suppose each of $M = (A, \mathcal{R})$, $N = (B, \mathcal{S})$, $L = (C, \mathcal{T})$ is an I-structure, $\alpha : A \to B$ is an coepimorphism from M to N, and $\beta : B \to C$ is a cohomomorphism from N to L. Then $\beta\alpha$ is a cohomomorphism from M to L.*

Proof: Suppose $t \in \mathcal{T}$. β is continuous, so $\beta^{-1}t \in \mathcal{S}$. α is continuous, so $(\beta\alpha)^{-1}t = \alpha^{-1}\beta^{-1}t \in \mathcal{R}$. So $\beta\alpha$ is continuous.

Suppose $r \in \mathcal{R}$. α is conservative, so there is an $s \in \mathcal{S}$ such that $\mathrm{im}(s) \subseteq \mathrm{im}(\alpha)$ and $\alpha^{-1}s = r$.

$s \in \mathcal{S}$, so there is an $t \in \mathcal{T}$ such that $\mathrm{im}(t) \subseteq \mathrm{im}(\beta)$ and $\beta^{-1}t = s$.

So $t \in \mathcal{T}$ such that $(\beta\alpha)^{-1}t = \alpha^{-1}\beta^{-1}t = \alpha^{-1}s = r$. So $\beta\alpha$ is conservative.

$\beta\alpha$ is both continuous and conservative, and is thus a cohomomorphism. □

Chapter 2

Equivalence Relations

Definition Suppose r is a relation. The statement that r is *symmetric* means $r^{-1} \subseteq r$.

Lemma 2.1.1. *Let r be a symmetric relation. Then $r^{-1} = r$.*

Proof:

$$\begin{aligned} &(x, y) \in r \\ \implies &(y, x) \in r^{-1} \subseteq r \\ \implies &(x, y) \in r^{-1} \end{aligned}$$

So $r \subseteq r^{-1}$ and since $r^{-1} \subseteq r$, $r^{-1} = r$. □

Definition Suppose r is a relation. The statement that r is *transitive* means $rr \subseteq r$.

Definition Suppose r is a relation. The statement that r is an *equivalence relation* means r is symmetric and transitive.

Definition Suppose r is a relation. The statement that r *is* reflexive *with respect to A* means $1_A \subseteq r$.

Lemma 2.1.2. *Suppose r is an equivalence relation. Then r is reflexive with respect to* $\mathrm{dom}(r)$.

Proof:

$$
\begin{aligned}
& (a,a) \in 1_{\mathrm{dom}(r)} \\
\iff & a \in \mathrm{dom}(r) \\
\iff & \exists b \text{ such that } (a,b) \in r \\
\iff & \exists b \text{ such that } (a,b) \in r \text{ and } (b,a) \in r^{-1} = r \\
\iff & (a,a) \in rr
\end{aligned}
$$

So $1_{\mathrm{dom}(r)} \subseteq rr \subseteq r$. □

Remark Suppose A is a set. Then 1_A is an equivalence relation.

Lemma 2.1.3. *Suppose r is a reflexive relation. Then for each $a \in \mathrm{dom}(r)$, $a \in r(\{a\})$.*

Proof: Suppose $a \in \mathrm{dom}(r)$.

$1_{\mathrm{dom}(r)} \subseteq r$, so $a \in \{a\} = 1_{\mathrm{dom}(r)}(\{a\}) \subseteq r(\{a\})$. □

Lemma 2.1.4. *Suppose r is an equivalence relation. Then $rr = r$.*

Proof:

$$r = 1_{\mathrm{im}(r)}r = 1_{\mathrm{dom}(r^{-1})}r = 1_{\mathrm{dom}(r)}r \subseteq rr$$

So $r \subseteq rr$. Then since $rr \subseteq r$, $rr = r$. □

Definition Suppose A is a set. The statement that $\mathcal{P}$ is a *partition* of A means if $P \in \mathcal{P}$ then $P \subseteq A$, and if $a \in A$ then a belongs to exactly one element of $\mathcal{P}$.

Theorem 2.1. *Suppose r is an equivalence relation. Then r induces a partition $\mathcal{P}$ of $\mathrm{dom}(r)$ by $\mathcal{P} = \{r(\{a\}) \mid a \in \mathrm{dom}(r)\}$, each member of which is nonempty.*

Proof:

$$\begin{aligned}
& P \in \mathcal{P} \\
\Longrightarrow\ & P = r(\{a\}) \text{ for some } a \in \operatorname{dom}(r) \\
\Longrightarrow\ & r(\{a\}) = r^{-1}(\{a\}) \subseteq \operatorname{dom}(r)
\end{aligned}$$

So each member of $\mathcal{P}$ is a subset of $\operatorname{dom}(r)$.

Suppose $a \in \operatorname{dom}(r)$.

$$a \in \{a\} = 1_{\operatorname{dom}(r)}(\{a\}) \subseteq r(\{a\})$$

So a belongs to one member of $\mathcal{P}$.

Suppose $b \in \operatorname{dom}(r)$ and $a \in r(\{b\})$. Then $(b, a) \in r$ and $r = r^{-1}$ so $(a, b) \in r$.

$$\begin{aligned}
& p \in r(\{a\}) \\
\Longrightarrow\ & (a, p) \in r \\
\Longrightarrow\ & (b, p) \in rr \\
\Longrightarrow\ & (b, p) \in r \\
\Longrightarrow\ & p \in r(\{b\})
\end{aligned}
\qquad\qquad
\begin{aligned}
& p \in r(\{b\}) \\
\Longrightarrow\ & (b, p) \in r \\
\Longrightarrow\ & (a, p) \in rr \\
\Longrightarrow\ & (a, p) \in r \\
\Longrightarrow\ & p \in r(\{a\})
\end{aligned}$$

So $r(\{a\}) = r(\{b\})$.

So a belongs to no more than one member of $\mathcal{P}$.

So $\mathcal{P}$ is a partition.

If $P \in \mathcal{P}$, then $P = r(\{a\})$ for some $a \in \mathrm{dom}(r)$ so by Lemma 2.1.3, $a \in r(\{a\}) = P$, so P is nonempty.

So each member of $\mathcal{P}$ is nonempty. □

Theorem 2.2. *Suppose A is a set and f is a function with domain A. Then $f^{-1}f$ is an equivalence relation on A.*

Proof:

1. $f^{-1}f$ is symmetric: $(f^{-1}f)^{-1} = f^{-1}(f^{-1})^{-1} = f^{-1}f$
2. $f^{-1}f$ is transitive: $f^{-1}ff^{-1}f = f^{-1}1_{\mathrm{im}(f)}f = f^{-1}f$

So $f^{-1}f$ is an equivalence relation. □

Definition Suppose A is a set and f is a function with domain A. Then denote by A/f the partition of A $\{f^{-1}f(\{a\}) \mid a \in A\}$.

Theorem 2.3. *Suppose A is a set and r is an equivalence relation with domain A. Suppose $\mathcal{P}$ is the partition of A induced by r, and $\pi : A \to \mathcal{P}$ is the function which assigns each member of A to its part in $\mathcal{P}$. Then $r = \pi^{-1}\pi$.*

Proof:

$$
\begin{aligned}
& (a_1, a_2) \in r \\
\iff & \exists P \in \mathcal{P} \text{ such that } a_1 \in P \text{ and } a_2 \in P \\
\iff & \exists P \in \mathcal{P} \text{ such that } (a_1, P) \in \pi \text{ and } (a_2, P) \in \pi \\
\iff & \exists P \in \mathcal{P} \text{ such that } (a_1, P) \in \pi \text{ and } (P, a_2) \in \pi^{-1} \\
\iff & (a_1, a_2) \in \pi^{-1}\pi
\end{aligned}
$$

So $r = \pi^{-1}\pi$. □

Definition Suppose f is a function with domain A. Denote by $\pi_f : A \to A/f$ the function such that for each $a \in A$, $\pi_f(a)$ is the part in A/f to which a belongs.

Theorem 2.4. *Suppose A is a set, and f is a function with domain A. Then π_f is a surjection with respect to A/f.*

Proof: Suppose $P \in A/f$. P is nonempty, so there is an $a \in P$, and by the definition of π_f, $\pi_f(a) = P$. So π_f is a surjection with respect to A/f. □

Theorem 2.5. *Suppose A is a set, and f is a function with domain A. Then for each $a \in A$, $\pi_f(a) = f^{-1}f(\{a\})$.*

Proof: For each $a \in A$, $\pi_f(a)$ is the part in A/f to which a belongs.

By Lemma 2.1.3, a belongs to $f^{-1}f(\{a\}) \in A/f$, so $\pi_f(a) = f^{-1}f(\{a\})$. □

Theorem 2.6. *Suppose A is a set, and f is a function with domain A. Then $\pi_f^{-1}\pi_f = f^{-1}f$.*

Proof:

$$
\begin{aligned}
& (a_1, a_2) \in \pi_f^{-1}\pi_f \\
\implies & \exists P \in A/f \text{ such that } (a_1, P) \in \pi_f \text{ and } (P, a_2) \in \pi_f^{-1} \\
\implies & (a_1, P) \in \pi_f \text{ and } (a_2, P) \in \pi_f \\
\implies & f^{-1}(\{f(a_1)\}) = f^{-1}f(\{a_1\}) = \pi_f(a_1) = P = \pi_f(a_2) = f^{-1}f(\{a_2\}) = f^{-1}(\{f(a_2)\}) \\
\implies & f(a_1) = 1_{\mathrm{im}(f)}f(a_1) = ff^{-1}f(a_1) = ff^{-1}f(a_2) = 1_{\mathrm{im}(f)}f(a_2) = f(a_2) \\
\implies & (a_1, f(a_1)) \in f \text{ and } (f(a_1), a_2) \in f^{-1} \\
\implies & (a_1, a_2) \in f^{-1}f
\end{aligned}
$$

So $\pi_f^{-1}\pi_f \subseteq f^{-1}f$.

$$\begin{aligned}
&(a_1, a_2) \in f^{-1}f \\
\implies &\pi_f(a_1) = f^{-1}(f(\{a_1\})) = f^{-1}(f(\{a_2\})) = \pi_f(a_2) \\
\implies &(a_1, \pi_f(a_1)) \in \pi_f \text{ and } (a_2, \pi_f(a_1)) \in \pi_f \\
\implies &(a_1, \pi_f(a_1)) \in \pi_f \text{ and } (\pi_f(a_1), a_2) \in \pi_f^{-1} \\
\implies &(a_1, a_2) \in \pi_f^{-1}\pi_f
\end{aligned}$$

So $f^{-1}f \subseteq \pi_f^{-1}\pi_f$.

So $\pi_f^{-1}\pi_f = f^{-1}f$. □

Theorem 2.7. *Suppose A is a set, and f is a function with domain A. Then $A/\pi_f = A/f$.*

Proof:

$$A/f = \{f^{-1}f(\{a\}) \mid a \in A\} = \{\pi_f^{-1}\pi_f(\{a\}) \mid a \in A\} = A/\pi_f \quad \square$$

Theorem 2.8. *Suppose A is a set, and f is a function with domain A. Then $\pi_{\pi_f} = \pi_f$.*

Proof: Suppose $a \in A$.

$$\pi_f(a) = f^{-1}f(\{a\}) = \pi_f^{-1}\pi_f(\{a\}) = \pi_{\pi_f}(a)$$

Since this is true for each $a \in A$, $\pi_{\pi_f} = \pi_f$. □

Theorem 2.9. *Suppose A is a set, and f is a function with domain A. Then if $P \in A/f$ then $P = \pi_f^{-1}(\{P\})$ and $\pi_f(P) = \{P\}$.*

Proof: Suppose $P \in A/f$ and $a \in P$.

$$P = \pi_f(a) = f^{-1}f(\{a\}) = \pi_f^{-1}\pi_f(\{a\}) = \pi_f^{-1}(\pi_f(\{a\})) = \pi_f^{-1}(\{\pi_f(a)\}) = \pi_f^{-1}(\{P\})$$

So $P = \pi_f^{-1}(\{P\})$.

$$\begin{aligned} P =& \pi_f^{-1}(\{P\}) \\ \implies \pi_f(P) =& \pi_f(\pi_f^{-1}(\{P\})) = \pi_f\pi_f^{-1}(\{P\}) = 1_{A/f}(\{P\}) = \{P\} \end{aligned}$$

So $\pi_f(P) = \{P\}$. □

Theorem 2.10. *Suppose A is a set, f is a function with domain A, and each of a_1 and a_2 is in A. Then the following are equivalent:*

1. *There is a $P \in A/f$ such that a_1 and a_2 belong to P.*
2. $\pi_f(a_1) = \pi_f(a_2)$
3. $a_2 \in \pi_f^{-1}(\pi_f(\{a_1\}))$
4. $(a_1, a_2) \in \pi_f^{-1}\pi_f$
5. $(a_1, a_2) \in f^{-1}f$
6. $a_2 \in f^{-1}(f(\{a_1\}))$
7. $f(a_1) = f(a_2)$

Proof: $1 \implies 2$:

Suppose there is a $P \in A/f$ such that a_1 and a_2 belong to P.

$a_1 \in P$ so $\pi_f(a_1) = P$ and $a_2 \in P$ so $\pi_f(a_2) = P$.

So $\pi_f(a_1) = P = \pi_f(a_2)$. $\square$

$2 \implies 3$:

Suppose $\pi_f(a_1) = \pi_f(a_2)$.

$$
\begin{aligned}
& 1_A \subseteq \pi_f^{-1}\pi_f \\
\implies & \{a_2\} = 1_A(\{a_2\}) \subseteq \pi_f^{-1}\pi_f(\{a_2\}) = \pi_f^{-1}(\{\pi_f(a_2)\}) = \pi_f^{-1}(\{\pi_f(a_1)\}) = \pi_f^{-1}(\pi_f(\{a_1\})) \\
\implies & a_2 \in \pi_f^{-1}(\pi_f(\{a_1\})) \quad \square
\end{aligned}
$$

$3 \implies 4$:

Suppose $a_2 \in \pi_f^{-1}(\pi_f(\{a_1\})) = \pi_f^{-1}\pi_f(\{a_1\})$.

Then there is a pair $(x, a_2) \in \pi_f^{-1}\pi_f$ such that $x \in \{a_1\}$. So $x = a_1$ and $(a_1, a_2) \in \pi_f^{-1}\pi_f$. $\square$

$4 \implies 5$:

Suppose $(a_1, a_2) \in \pi_f^{-1}\pi_f$.

$\pi_f^{-1}\pi_f = f^{-1}f$, so $(a_1, a_2) \in \pi_f^{-1}\pi_f = f^{-1}f$. $\square$

$5 \implies 6$:

Suppose $(a_1, a_2) \in f^{-1}f$.

Then $a_1 \in \{a_1\}$ such that $(a_1, a_2) \in f^{-1}f$, so $a_2 \in f^{-1}f(\{a_1\}) = f^{-1}(f(\{a_1\}))$. $\square$

$6 \implies 7$:

Suppose $a_2 \in f^{-1}(f(\{a_1\}))$.

$$\begin{aligned} & a_2 \in f^{-1}(f(\{a_1\})) \\ \implies & \{a_2\} \subseteq f^{-1}f(\{a_1\}) \\ \implies & \{f(a_2)\} = f(\{a_2\}) \subseteq f(f^{-1}f(\{a_1\})) = ff^{-1}f(\{a_1\}) = 1_{\text{im}(f)}f(\{a_1\}) = f(\{a_1\}) = \{f(a_1)\} \\ \implies & f(a_1) = f(a_2) \quad \square \end{aligned}$$

$7 \implies 1$:

Suppose $f(a_1) = f(a_2)$.

Consider $f^{-1}f(\{a_1\}) \in A/f$.

By Lemma 2.1.3:

$$\begin{aligned} & a_1 \in f^{-1}f(\{a_1\}) \\ \text{and } & a_2 \in f^{-1}f(\{a_2\}) = f^{-1}(\{f(a_2)\}) = f^{-1}(\{f(a_1)\}) = f^{-1}(f(\{a_1\})) = f^{-1}f(\{a_1\}) \end{aligned}$$

So $f^{-1}f(\{a_1\}) \in A/f$ such that a_1 and a_2 belong to $f^{-1}f(\{a_1\})$. $\square$

Chapter 3

Structurizations

Remark In this chapter 1 is used set theoretically, i.e., $1 = \{0\}$.

Definition Suppose (X, τ) is a topological space[2]. Then the *structurization of* (X, τ) is the 1-structure $(X, \dot{\tau})$, where $\dot{\tau}$ is $\{1 \times S \mid S \in \tau\}$.

Example Consider the set $\mathbb{R}$ with the standard topology $\tau_{\mathbb{R}}$. Then the structurization of $(\mathbb{R}, \tau_{\mathbb{R}})$ is $(\mathbb{R}, \mathcal{R})$, where $\mathcal{R} = \{1 \times S \mid S \in \tau_{\mathbb{R}}\}$. E.g., $1 \times (-3, \infty) \in \mathcal{R}$.

Theorem 3.1. *Suppose each of* (X, τ_X) *and* (Y, τ_Y) *is a topological space, and* $(X, \dot{\tau}_X)$ *is the structurization of* (X, τ_X)*, and* $(Y, \dot{\tau}_Y)$ *is the structurization of* (Y, τ_Y)*. Then* $\alpha : X \to Y$ *is preservative if and only if* α *is an open function with respect to* (X, τ_X) *and* (Y, τ_Y)*.*

Proof: Suppose $\alpha : X \to Y$ is preservative.

Suppose $S \in \tau_X$. $1 \times S \in \dot{\tau}_X$. Define $r = 1 \times S$. $\alpha r \in \dot{\tau}_Y$ so $\alpha r = 1 \times T$ for some T in τ_Y. $\alpha[S] = \alpha[r[1]] = \alpha r[1] = T \in \tau_Y$.

So α is an open function with respect to (X, τ_X) and (Y, τ_Y).

Suppose $\alpha : X \to Y$ is an open function.

Suppose $r \in \dot{\tau}_X$. Then $r = 1 \times S$ for some S in τ_X. Since α is open, $\alpha[S] \in \tau_Y$, so $\alpha r = \alpha(1 \times S) = 1 \times \alpha[S] \in \dot{\tau}_Y$.

So α is preservative with respect to the 1-structures $(X, \dot{\tau}_X)$ and $(Y, \dot{\tau}_Y)$. □

Theorem 3.2. *Suppose each of* (X, τ_X) *and* (Y, τ_Y) *is a topological space, and* $(X, \dot{\tau}_X)$ *is the structurization of* (X, τ_X)*, and* $(Y, \dot{\tau}_Y)$ *is the structurization of* (Y, τ_Y)*. Then* $\alpha : X \to Y$ *is continuous if and only if* α *is a continuous function with respect to* (X, τ_X) *and* (Y, τ_Y)*.*

Proof: Suppose $\alpha : X \to Y$ is (structurally) continuous.

Suppose $T \in \tau_Y$. Then $1 \times T \in \dot{\tau}_Y$. Define $s = 1 \times T$. $\alpha^{-1}s \in \dot{\tau}_X$ so $\alpha^{-1}(T) = \alpha^{-1}[s[1]] = \alpha^{-1}s[1] \in \tau_X$.

So α is a (topologically) continuous function with respect to (X, τ_X) and (Y, τ_Y).

Suppose $\alpha : X \to Y$ is a (topologically) continuous function.

Suppose $s \in \dot{\tau}_Y$. Then $s = 1 \times T$ for some $T \in \tau_Y$. Since α is continuous, $\alpha^{-1}[T] \in \tau_X$, so $\alpha^{-1}s = \alpha^{-1}(1 \times T) = 1 \times \alpha^{-1}[T] \in \dot{\tau}_X$

So α is (structurally) continuous with respect to the 1-structures $(X, \dot{\tau}_X)$ and $(Y, \dot{\tau}_Y)$. □

Theorem 3.3. *Suppose each of* (X, τ_X) *and* (Y, τ_Y) *is a topological space, and the 1-structure* $(X, \dot{\tau}_X)$ *is the structurization of* (X, τ_X)*, and the 1-structure* $(Y, \dot{\tau}_Y)$ *is the structurization of* (Y, τ_Y)*. Then* (X, τ_X) *and* (Y, τ_Y) *are homeomorphic if and only if* $(X, \dot{\tau}_X)$ *and* $(Y, \dot{\tau}_Y)$ *are isomorphic.*

Proof: Let $\varphi : A \to B$ be a function.

φ is a homeomorphism

$\iff \varphi$ is bijective, open, and (topologically) continuous

$\iff \varphi$ is bijective, preservative, and (structurally) continuous (Thm 3.1, Thm 3.2)

$\iff \varphi$ is a 1-structure isomorphism □

Lemma 3.4.1. *Suppose J is a nonempty set, and for each $j \in J$, r_j is a relation, and φ is a function. Then $\varphi^{-1}(\bigcap_{j\in J} r_j) = \bigcap_{j\in J}(\varphi^{-1} r_j)$.*

Proof:

$$\begin{aligned}
& (x, y) \in \varphi^{-1}(\bigcap_{j\in J} r_j) \\
\iff & \exists z \text{ such that } (z, y) \in \varphi^{-1} \text{ and } (x, z) \in \bigcap_{j\in J} r_j \\
\iff & \exists z \text{ such that } (z, y) \in \varphi^{-1} \text{ and } (x, z) \in r_j \text{ for each } j \in J \\
\iff & (x, y) \in \varphi^{-1} r_j \text{ for each } j \in J \\
\iff & (x, y) \in \bigcap_{j\in J}(\varphi^{-1} r_j)
\end{aligned}$$

So $\varphi^{-1}(\bigcap_{j\in J} r_j) = \bigcap_{j\in J}(\varphi^{-1} r_j)$. □

Lemma 3.4.2. *Suppose J is a set, and for each $j \in J$ r_j is a relation, and φ is a function. Then $\varphi(\bigcup_{j\in J} r_j) = \bigcup_{j\in J}(\varphi r_j)$.*

Proof:

$$\begin{aligned}
& (x, y) \in \varphi(\bigcup_{j\in J} r_j) \\
\iff & \exists z \text{ such that } (z, y) \in \varphi \text{ and } (x, z) \in \bigcup_{j\in J} r_j \\
\iff & \exists z \text{ such that } (z, y) \in \varphi \text{ and } (x, z) \in r_j \text{ for some } j \in J \\
\iff & (x, y) \in \varphi r_j \text{ for some } j \in J \\
\iff & (x, y) \in \bigcup_{j\in J}(\varphi r_j)
\end{aligned}$$

So $\varphi(\bigcup_{j\in J} r_j) = \bigcup_{j\in J}(\varphi r_j)$. □

Theorem 3.4. *Suppose each of $(A, \mathcal{R})$ and $(B, \mathcal{S})$ is a 1-structure, $\varphi : A \to B$ is a co-epimorphism, and $(A, \mathcal{R})$ is the structurization of a topological space. Then $(B, \mathcal{S})$ is the structurization of a topological space.*

Proof: φ is a cohomomorphism, so by Theorem 1.20, φ is a homomorphism.

1. $\varnothing \in \mathcal{R}$, and φ is preservative, so $\varnothing = \varphi\varnothing \in \mathcal{S}$.

2. $1 \times A \in \mathcal{R}$, so suppose $r = 1 \times A$. φ is surjective, and φ is preservative, so $1 \times B = 1 \times \varphi[A] = \varphi(1 \times A) = \varphi r \in \mathcal{S}$.

3. Suppose J is a set, and for each $j \in J$, $s_j \in \mathcal{S}$.

   φ is continuous, so for each $j \in J$, $\varphi^{-1}s_j \in \mathcal{R}$. $(A, \mathcal{R})$ is the structurization of a topological space, so $\bigcup_{j \in J} \varphi^{-1}s_j \in \mathcal{R}$.

   φ is preservative and surjective, so by Lemma 3.4.2,

$$\bigcup_{j \in J} s_j = \bigcup_{j \in J} \varphi\varphi^{-1}s_j = \varphi \bigcup_{j \in J} \varphi^{-1}s_j \in \mathcal{S}$$

4. Suppose each of s_0 and s_1 is in $\mathcal{S}$. φ is continuous, so each of $\varphi^{-1}s_0$ and $\varphi^{-1}s_1$ is in $\mathcal{R}$. $(A, \mathcal{R})$ is the structurization of a topological space, so by Lemma 3.4.1, $\varphi^{-1}(s_0 \cap s_1) = (\varphi^{-1}s_0) \cap (\varphi^{-1}s_1) \in \mathcal{R}$.

   φ is preservative and surjective, so $s_0 \cap s_1 = \varphi\varphi^{-1}(s_0 \cap s_1) \in \mathcal{S}$.

So by the above, $(B, \mathcal{S})$ is the structurization of a topological space.

Theorem 3.5. *Suppose each of $(A, \mathcal{R})$ and $(B, \mathcal{S})$ is a 1-structure, $\varphi : A \to B$ is a comonomorphism, and $(B, \mathcal{S})$ is the structurization of a topological space. Then $(A, \mathcal{R})$ is the structurization of a topological space.*

Proof: Suppose $(B, \mathcal{S})$ is the structurization of a topological space.

1. $\varnothing \in \mathcal{S}$, and φ is continuous, so $\varnothing = \varphi^{-1}\varnothing \in \mathcal{R}$.

2. $1 \times B \in \mathcal{S}$, so suppose $s = 1 \times B$. φ is continuous, so $1 \times A = 1 \times \varphi^{-1}[B] = \varphi^{-1}(1 \times B) = \varphi^{-1}s \in \mathcal{R}$.

3. Suppose J is a set, and for each $j \in J$, $r_j \in \mathcal{R}$.

   φ is preservative, so for each $j \in J$, $\varphi r_j \in \mathcal{S}$. $(B, \mathcal{S})$ is the structurization of a topological space, so $\bigcup_{j \in J} \varphi r_j \in \mathcal{S}$.

   φ is continuous and injective, so by Lemma 3.4.2,

$$\bigcup_{j \in J} r_j = \varphi^{-1}\varphi(\bigcup_{j \in J} r_j) = \varphi^{-1}(\bigcup_{j \in J} \varphi r_j) \in \mathcal{R}$$

4. Suppose each of r_0 and r_1 is in $\mathcal{R}$. φ is preservative, so each of φr_0 and φr_1 is in $\mathcal{S}$. $(B, \mathcal{S})$ is the structurization of a topological space, so $(\varphi r_0) \cap (\varphi r_1) \in \mathcal{S}$.

   φ is continuous and injective, so by Lemma 3.4.1, $r_0 \cap r_1 = (\varphi^{-1}\varphi r_0) \cap (\varphi^{-1}\varphi r_1) = \varphi^{-1}((\varphi r_0) \cap (\varphi r_1)) \in \mathcal{R}$.

So by the above, $(A, \mathcal{R})$ is the structurization of a topological space. $\square$

Definition The statement that $\mathcal{F}$ is a type means $\mathcal{F}$ is a function with domain a set of symbols and image a subset of the cardinal numbers[3].

Definition Let $\mathcal{F}$ be a type. The statement that $\mathbf{A} = (A, F)$ is an algebra of type $\mathcal{F}$ means A is a set, F is a set of functions each having image a subset of A, and there is a bijection $g : \mathrm{dom}(\mathcal{F}) \to F$ such that for each $f \in \mathrm{dom}(\mathcal{F})$, $\mathrm{dom}(g(f)) = A^{\mathcal{F}(f)}$. For each $f \in \mathrm{dom}(\mathcal{F})$, denote $g(f)$ by $f^{\mathbf{A}}$.

Definition Let $\mathbf{A} = (A, F)$ be an algebra of type $\mathcal{F}$. Define I to be the set of symbols $\bigcup_{f \in \mathrm{dom}(\mathcal{F})} \left(\{p_f\} \cup \bigcup_{q \in \mathcal{F}(f)} \{q_f\} \right)$. The *structurization of* (A, F) is the I-structure $(A, \mathcal{R})$ where $\mathcal{R}$ is the set of functions to which a function r belongs if and only if there is an f in $\mathrm{dom}(\mathcal{F})$ and an element a in $A^{\mathcal{F}(f)}$ such that the domain of r is $\{p_f\} \cup \bigcup_{q \in \mathcal{F}(f)} \{q_f\}$ and for each element q in $\mathcal{F}(f)$, $r(q_f) = a(q)$, and $r(p_f) = f^{\mathbf{A}}(a)$.

Example Suppose $\mathcal{F} = \{(e, 0), ({}^{-1}, 1), (\cdot, 2)\}$ is the type associated with groups. e is the symbol corresponding with the 0-ary function that for each group, picks out the identity element of the group, ${}^{-1}$ is the symbol corresponding with the unary function that associates each element of the group with its inverse, and $\cdot$ is the symbol corresponding with the binary function of the group.

Consider the dihedral group $D_6 = \{\epsilon, \sigma, \sigma^2, \tau, \tau\sigma, \tau\sigma^2\}$.

Then the structurization of D_6 is the I-structure $(D_6, \mathcal{R})$, with $I = \{p_e, p_{-1}, 0_{-1}, p_., 0_., 1_.\}$

and

$$\mathcal{R} = \{$$

$$\{(p_e, \epsilon)\},$$

$$\{(p_{-1}, \epsilon), (0_{-1}, \epsilon)\}, \{(p_{-1}, \sigma), (0_{-1}, \sigma^2)\}, \{(p_{-1}, \sigma^2), (0_{-1}, \sigma)\},$$
$$\{(p_{-1}, \tau), (0_{-1}, \tau)\}, \{(p_{-1}, \tau\sigma), (0_{-1}, \tau\sigma)\}, \{(p_{-1}, \tau\sigma^2), (0_{-1}, \tau\sigma^2)\},$$

$$\{(p., \epsilon), (0., \epsilon), (1., \epsilon)\}, \{(p., \sigma), (0., \sigma), (1., \epsilon)\}, \{(p., \sigma^2), (0., \sigma^2), (1., \epsilon)\},$$
$$\{(p., \tau), (0., \tau), (1., \epsilon)\}, \{(p., \tau\sigma), (0., \tau\sigma), (1., \epsilon)\}, \{(p., \tau\sigma^2), (0., \tau\sigma^2), (1., \epsilon)\},$$

$$\{(p., \sigma), (0., \epsilon), (1., \sigma)\}, \{(p., \sigma^2), (0., \sigma), (1., \sigma)\}, \{(p., \epsilon), (0., \sigma^2), (1., \sigma)\},$$
$$\{(p., \tau\sigma), (0., \tau), (1., \sigma)\}, \{(p., \tau\sigma^2), (0., \tau\sigma), (1., \sigma)\}, \{(p., \tau), (0., \tau\sigma^2), (1., \sigma)\},$$

$$\{(p., \sigma^2), (0., \epsilon), (1., \sigma^2)\}, \{(p., \epsilon), (0., \sigma), (1., \sigma^2)\}, \{(p., \sigma), (0., \sigma^2), (1., \sigma^2)\},$$
$$\{(p., \tau\sigma^2), (0., \tau), (1., \sigma^2)\}, \{(p., \tau), (0., \tau\sigma), (1., \sigma^2)\}, \{(p., \tau\sigma), (0., \tau\sigma^2), (1., \sigma^2)\},$$

$$\{(p., \tau), (0., \epsilon), (1., \tau)\}, \{(p., \tau\sigma^2), (0., \sigma), (1., \tau)\}, \{(p., \tau\sigma), (0., \sigma^2), (1., \tau)\},$$
$$\{(p., \epsilon), (0., \tau), (1., \tau)\}, \{(p., \sigma^2), (0., \tau\sigma), (1., \tau)\}, \{(p., \sigma), (0., \tau\sigma^2), (1., \tau)\},$$

$$\{(p., \tau\sigma), (0., \epsilon), (1., \tau\sigma)\}, \{(p., \tau), (0., \sigma), (1., \tau\sigma)\}, \{(p., \tau\sigma^2), (0., \sigma^2), (1., \tau\sigma)\},$$
$$\{(p., \sigma), (0., \tau), (1., \tau\sigma)\}, \{(p., \epsilon), (0., \tau\sigma), (1., \tau\sigma)\}, \{(p., \sigma^2), (0., \tau\sigma^2), (1., \tau\sigma)\},$$

$$\{(p., \tau\sigma^2), (0., \epsilon), (1., \tau\sigma^2)\}, \{(p., \tau\sigma), (0., \sigma), (1., \tau\sigma^2)\}, \{(p., \tau), (0., \sigma^2), (1., \tau\sigma^2)\},$$
$$\{(p., \sigma^2), (0., \tau), (1., \tau\sigma^2)\}, \{(p., \sigma), (0., \tau\sigma), (1., \tau\sigma^2)\}, \{(p., \epsilon), (0., \tau\sigma^2), (1., \tau\sigma^2)\}$$

$$\}$$

Theorem 3.6. *Suppose each of $\mathbf{A} = (A, F)$ and $\mathbf{B} = (B, G)$ is an algebra of type $\mathcal{F}$, and $(A, \mathcal{R})$ is the structurization of (A, F) and $(B, \mathcal{S})$ is the structurization of (B, G). Then a function $\varphi : A \to B$ is an algebraic homomorphism if and only if it is a $\bigcup_{f \in \mathrm{dom}(\mathcal{F})} \left(\{p_f\} \cup \bigcup_{q \in \mathcal{F}(f)} \{q_f\} \right)$-structure homomorphism.*

Proof: Suppose φ is an algebraic homomorphism.

Suppose $r \in \mathcal{R}$. Then there is an f in $\mathrm{dom}(\mathcal{F})$ and an element a in $A^{\mathcal{F}(f)}$ such that the domain of r is $\{p_f\} \cup \bigcup_{q \in \mathcal{F}(f)} \{q_f\}$ and for each element q in $\mathcal{F}(f)$, $r(q_f) = a(q)$, and $r(p_f) = f^{\mathbf{A}}(a)$.

φa is an element in $B^{\mathcal{F}(f)}$, so there is an $s \in \mathcal{S}$ such that the domain of s is $\{p_f\} \cup \bigcup_{q \in \mathcal{F}(f)} \{q_f\}$ and for each element q in $\mathcal{F}(f)$, $s(q_f) = \varphi a(q) = \varphi(a(q)) = \varphi(r(q_f)) = \varphi r(q_f)$, and $s(p_f) = f^{\mathbf{B}}(\varphi a) = \varphi(f^{\mathbf{A}}(a)) = \varphi(r(p_f)) = \varphi r(p_f)$.

So $\varphi r = s \in \mathcal{S}$ and φ is preservative.

Suppose $s \in \mathcal{S}$ such that $\mathrm{im}(s) \subseteq \mathrm{im}(\varphi)$.

There is an f in $\mathrm{dom}(\mathcal{F})$ and an element b in $B^{\mathcal{F}(f)}$ such that the domain of s is $\{p_f\} \cup \bigcup_{q \in \mathcal{F}(f)} \{q_f\}$, and for each element q in $\mathcal{F}(f)$, $s(q_f) = b(q)$, and $s(p_f) = f^{\mathbf{B}}(b)$.

Moreover, for each $q \in \mathcal{F}(f)$, since $b(q) = s(q_f) \in \mathrm{im}(s) \subseteq \mathrm{im}(\varphi)$, there is an $a_q \in A$ such that $b(q) = \varphi(a_q)$.

Define $a : \mathcal{F}(f) \to A$ such that for each $q \in \mathcal{F}(f)$, $a(q) = a_q$. a is an element in $A^{\mathcal{F}(f)}$, so there is an $r \in \mathcal{R}$ such that the domain of r is $\{p_f\} \cup \bigcup_{q \in \mathcal{F}(f)} \{q_f\}$ and for each element q in $\mathcal{F}(f)$, $r(q_f) = a(q)$, and $r(p_f) = f^{\mathbf{A}}(a)$. Note for each $q \in \mathcal{F}(f)$, $b(q) = \varphi(a_q) = \varphi(a(q)) = \varphi a(q)$, so $b = \varphi a$.

For each q in $\mathcal{F}(f)$, $s(q_f) = b(q) = \varphi a(q) = \varphi(a(q)) = \varphi(r(q_f)) = \varphi r(q_f)$.

$s(p_f) = f^{\mathbf{B}}(b) = f^{\mathbf{B}}(\varphi a) = \varphi(f^{\mathbf{A}}(a)) = \varphi(r(p_f)) = \varphi r(p_f)$.

So r is a relation in $\mathcal{R}$ such that $\varphi r = s$, and φ is saturating.

Thus φ is an $\bigcup_{f \in \mathrm{dom}(\mathcal{F})} \left(\{p_f\} \cup \bigcup_{q \in \mathcal{F}(f)} \{q_f\}\right)$-structure homomorphism.

Suppose φ is an $\bigcup_{f \in \mathrm{dom}(\mathcal{F})} \left(\{p_f\} \cup \bigcup_{q \in \mathcal{F}(f)} \{q_f\}\right)$-structure homomorphism.

Suppose f is in $\mathrm{dom}(\mathcal{F})$ and a is an element in $A^{\mathcal{F}(f)}$.

Then there is an $r \in \mathcal{R}$ such that such that the domain of r is $\{p_f\} \cup \bigcup_{q \in \mathcal{F}(f)} \{q_f\}$ and for each element q in $\mathcal{F}(f)$, $r(q_f) = a(q)$, and $r(p_f) = f^{\mathbf{A}}(a)$.

φ is an I-structure homomorphism, so $\varphi r \in \mathcal{S}$.

Since φr is in $\mathcal{S}$, and has domain $\{p_f\} \cup \bigcup_{q \in \mathcal{F}(f)} \{q_f\}$, and for each $q \in \mathcal{F}(f)$, $\varphi r(q_f) = \varphi(r(q_f)) = \varphi(a(q)) = \varphi a(q)$, f is the member of $\mathrm{dom}(\mathcal{F})$ and φa is the element in $B^{\mathcal{F}(f)}$ such that $\varphi r(p_f) = f^{\mathbf{B}}(\varphi a)$.

$\varphi(f^{\mathbf{A}}(a)) = \varphi(r(p_f)) = \varphi r(p_f) = f^{\mathbf{B}}(\varphi a)$.

So φ is an algebraic homomorphism. $\square$

Theorem 3.7. *Suppose each of* $\mathbf{A} = (A, F)$ *and* $\mathbf{B} = (B, G)$ *is an algebra of type* $\mathcal{F}$*, and* $(A, \mathcal{R})$ *is the structurization of* (A, F) *and* $(B, \mathcal{S})$ *is the structurization of* (B, G)*. Then a function* $\varphi : A \to B$ *is an algebraic isomorphism if and only if it is an* $\bigcup_{f\in \mathrm{dom}(\mathcal{F})} \left(\{p_f\} \cup \bigcup_{q\in\mathcal{F}(f)} \{q_f\} \right)$*-structure isomorphism.*

Proof: Suppose $\varphi : A \to B$ is a function.

$$\begin{aligned} &\varphi \text{ is an algebraic isomorphism} \\ \iff &\varphi \text{ is a bijective algebraic homomorphism} \\ \iff &\varphi \text{ is a bijective structure homomorphism} \\ \iff &\varphi \text{ is an } I\text{-structure isomorphism} \quad \square \end{aligned}$$

Remark Graph will hereafter be used to refer to graphs which may contain loops and multiple edges[4].

Definition Let G be a graph. Define $V(G)$ to be the vertex set of G.

Definition Let G be a graph. Define $E(G)$ to be the edge set of G.

Definition Let G be a graph. The *structurization of* G is the $\mathbb{N}\backslash\{0\}$-structure $(A, \mathcal{R})$ where $\mathcal{R}$ is the set of relations to which a relation $f : \mathbb{N}\backslash\{0\} \to V(G)$ belongs if and only if f contains either one or two pairs each having the same first element n, and there are at least n edges joining the vertices in $f[\{n\}]$. (If there is exactly one pair (n, v) in f, then the n edges correspond to n loops at v).

Theorem 3.8. *Suppose each of* G *and* H *is a graph. Suppose the* $\mathbb{N}\backslash\{0\}$*-structure* $(V(G), \mathcal{R})$ *is the structurization of* G *and the* $\mathbb{N}\backslash\{0\}$*-structure* $(V(H), \mathcal{S})$ *is the structurization of* H*. Then the graphs* G *and* H *are isomorphic if and only if* $(V(G), \mathcal{R})$ *and* $(V(H), \mathcal{S})$ *are isomorphic.*

Proof: Suppose G and H are isomorphic, and $\varphi : V(G) \to V(H)$ is a graph isomorphism. φ is a bijection.

Suppose $r \in \mathcal{R}$. Suppose r contains exactly one element (n, v). Then there are at least n loops at vertex v in G, so there are at least n loops at vertex $\varphi(v)$ in H. Thus there is a $s \in \mathcal{S}$ such that s contains exactly one element $(n, \varphi(v))$, so $s[\{n\}] = \varphi[r[\{n\}]] = \varphi r[\{n\}]$. So $s = \varphi r$.

Suppose r contains exactly two elements (n, v) and (n, w). Then there are at least n edges connecting vertices v and w in G, so there are at least n edges connecting vertices $\varphi(v)$ and $\varphi(w)$ in H. Thus there is a $s \in \mathcal{S}$ such that s contains exactly two elements $(n, \varphi(v))$ and $(n, \varphi(w))$, so $s[\{n\}] = \varphi[r[\{n\}]] = \varphi r[\{n\}]$. So $s = \varphi r$.

So in both cases there is a s in $\mathcal{S}$ such that $s = \varphi r$, so φ is preservative.

Suppose $s \in \mathcal{S}$. Suppose s contains exactly one element (n, v). Then there are at least n loops at vertex w in H, so there are at least n loops at vertex $\varphi^{-1}(v)$ in G. Thus there is a $r \in \mathcal{R}$ such that r contains exactly one element $(n, \varphi^{-1}(v))$, so $r[\{n\}] = \varphi^{-1}[s[\{n\}]] = \varphi^{-1}s[\{n\}]$. So $r = \varphi^{-1}s$.

Suppose s contains exactly two elements (n, v) and (n, w). Then there are at least n edges connecting vertices v and w in H, so there are at least n edges connecting vertices $\varphi^{-1}(v)$ and $\varphi^{-1}(w)$ in G. Thus there is a $r \in \mathcal{R}$ such that r contains exactly two elements $(n, \varphi^{-1}(v))$ and $(n, \varphi^{-1}(w))$, so $r[\{n\}] = \varphi^{-1}[s[\{n\}]] = \varphi^{-1}s[\{n\}]$. So $r = \varphi^{-1}s$.

So φ is continuous and thus φ is a $\mathbb{N}\backslash\{0\}$-structure isomorphism. $\square$

Suppose $(V(G), \mathcal{R})$ and $(V(H), \mathcal{S})$ are isomorphic and $\alpha : V(G) \to V(H)$ is a $\mathbb{N}\backslash\{0\}$-structure isomorphism and hence a bijection.

Suppose $n \in \mathbb{N}$, and there are exactly n loops at vertex v in G. If $n \neq 0$ then there is a relation $r \in \mathcal{R}$ such that $r = \{(n, v)\}$, and $\{(n, \alpha(v))\} = \alpha r \in \mathcal{S}$, so there are at least n loops in H at vertex $\alpha(v)$.

Suppose $s = \{(n+1, \alpha(v))\}$. If $s \in \mathcal{S}$, then $\{(n+1, v)\} = \{(n+1, \alpha^{-1}(\alpha(v)))\} = \alpha^{-1}s \in \mathcal{R}$, and thus there are $n+1$ loops at v, contradicting the assumption that there are exactly n loops at v in G. So $s \notin \mathcal{S}$, and thus there are not $n+1$ loops at $\alpha(v)$. So there are exactly n loops at $\alpha(v)$ in H.

Suppose $n \in \mathbb{N}$, and there are exactly n edges connecting vertices v and w in G. If $n \neq 0$ then there is a relation $r \in \mathcal{R}$ such that $r = \{(n, v), (n, w)\}$, and $\{(n, \alpha(v)), (n, \alpha(w))\} = \alpha r \in \mathcal{S}$, so there are at least n edges in H connecting vertices $\alpha(v)$ and $\alpha(w)$.

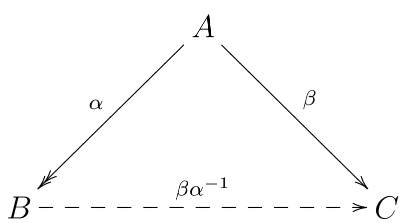
Suppose $s = \{(n+1, \alpha(v)), (n+1, \alpha(w))\}$. If $s \in \mathcal{S}$, then $\{(n+1, v), (n+1, w)\} = \{(n+1, \alpha^{-1}(\alpha(v))), (n+1, \alpha^{-1}(\alpha(w)))\} = \alpha^{-1}s \in \mathcal{R}$, and thus there are $n+1$ edges connecting v and w, contradicting the assumption that there are exactly n edges connecting v and w in G. So $s \notin \mathcal{S}$, and thus there are not $n+1$ edges connecting $\alpha(v)$ and $\alpha(w)$. So there are exactly n edges connecting $\alpha(v)$ and $\alpha(w)$ in H.

So α is a graph isomorphism and G and H are isomorphic. $\square$

Chapter 4

Fundamental (Co)Homomorphism Theorems

Lemma 4.1.1. *Suppose each of A, B, and C is a set, $\alpha : A \to B$ is a surjection, $\beta : A \to C$ is a function, and $\alpha^{-1}\alpha \subseteq \beta^{-1}\beta$. Then $\beta\alpha^{-1}$ is a unique function with domain B such that $(\beta\alpha^{-1})\alpha = \beta$. Moreover,* $\mathrm{im}(\beta\alpha^{-1}) = \mathrm{im}(\beta)$.



Proof:

$$
\begin{aligned}
&(b, c_1) \in \beta\alpha^{-1} \text{ and } (b, c_2) \in \beta\alpha^{-1} \\
\Longrightarrow\ &\exists a_1 \in A \text{ such that } (b, a_1) \in \alpha^{-1} \text{ and } (a_1, c_1) \in \beta \\
\text{and } &\exists a_2 \in A \text{ such that } (b, a_2) \in \alpha^{-1} \text{ and } (a_2, c_2) \in \beta \\
\Longrightarrow\ &(a_1, b) \in \alpha \text{ and } (b, a_2) \in \alpha^{-1},\ c_1 = \beta(a_1),\ c_2 = \beta(a_2) \\
\Longrightarrow\ &(a_1, a_2) \in \alpha^{-1}\alpha \subseteq \beta^{-1}\beta,\ c_1 = \beta(a_1),\ c_2 = \beta(a_2) \\
\Longrightarrow\ &\exists c \in C \text{ such that } (a_1, c) \in \beta \text{ and } (c, a_2) \in \beta^{-1},\ c_1 = \beta(a_1),\ c_2 = \beta(a_2) \\
\Longrightarrow\ &(a_1, c) \in \beta \text{ and } (a_2, c) \in \beta,\ c_1 = \beta(a_1),\ c_2 = \beta(a_2) \\
\Longrightarrow\ &c_1 = \beta(a_1) = c = \beta(a_2) = c_2
\end{aligned}
$$

So $\beta\alpha^{-1}$ is a function.

Note $1_A \subseteq \alpha^{-1}\alpha$ and $\beta\beta^{-1} \subseteq 1_C$.

$$(\beta\alpha^{-1})\alpha = \beta(\alpha^{-1}\alpha) \subseteq \beta(\beta^{-1}\beta) = (\beta\beta^{-1})\beta \subseteq 1_C\beta = \beta$$

$$\beta = \beta 1_A \subseteq \beta(\alpha^{-1}\alpha) = (\beta\alpha^{-1})\alpha$$

So $(\beta\alpha^{-1})\alpha = \beta$.

Suppose γ is a function with domain B such that $\gamma\alpha = \beta$. α is a surjection, so $\alpha\alpha^{-1} = 1_B$

$$\gamma\alpha = \beta$$

$$\implies \gamma = \gamma 1_B = \gamma\alpha\alpha^{-1} = \beta\alpha^{-1}$$

So $\beta\alpha^{-1}$ is the only function having domain B with the property that $(\beta\alpha^{-1})\alpha = \beta$.

$$\text{im}(\beta\alpha^{-1}) = \beta(\text{im}(\alpha^{-1})) = \beta(\text{dom}(\alpha)) = \beta(A) = \text{im}(\beta)$$

So $\text{im}(\beta\alpha^{-1}) = \text{im}(\beta)$. □

Corollary 4.1.1. *Suppose each of A, B, and C is a set, $\alpha : A \to B$ is a surjection, $\beta : A \to C$ is a function, and $\alpha^{-1}\alpha \subseteq \beta^{-1}\beta$. Then β is a surjection if and only if $\beta\alpha^{-1}$ is a surjection with respect to C.*

Proof: Suppose β is a surjection.

By Lemma 4.1.1, $\beta\alpha^{-1}$ is a function.

$C = \text{im}(\beta) = \text{im}(\beta\alpha^{-1})$, so $\beta\alpha^{-1}$ is a surjection with respect to C.

Suppose $\beta\alpha^{-1}$ is a surjection with respect to C.

$C = \mathrm{im}(\beta\alpha^{-1}) = \mathrm{im}(\beta)$, so β is a surjection. $\square$

Corollary 4.1.2. *Suppose each of A, B, and C is a set, $\alpha : A \to B$ is a surjection, $\beta : A \to C$ is a function, and $\alpha^{-1}\alpha \subseteq \beta^{-1}\beta$. Then $\alpha^{-1}\alpha = \beta^{-1}\beta$ if and only if $\beta\alpha^{-1}$ is an injection.*

Proof: Suppose $\alpha^{-1}\alpha = \beta^{-1}\beta$.

$$(\beta\alpha^{-1})^{-1}\beta\alpha^{-1} = \alpha\beta^{-1}\beta\alpha^{-1} = \alpha\alpha^{-1}\alpha\alpha^{-1} = 1_B 1_B = 1_B$$

So $\beta\alpha^{-1}$ is an injection.

Suppose $\beta\alpha^{-1}$ is an injection. Note $\beta\alpha^{-1}\alpha = \beta$

$$\beta^{-1}\beta = 1_A\beta^{-1}\beta \subseteq \alpha^{-1}\alpha\beta^{-1}\beta = \alpha^{-1}\alpha\beta^{-1}(\beta\alpha^{-1}\alpha) = \alpha^{-1}(\beta\alpha^{-1})^{-1}\beta\alpha^{-1}\alpha = \alpha^{-1}1_B\alpha = \alpha^{-1}\alpha$$

So $\beta^{-1}\beta \subseteq \alpha^{-1}\alpha$ and thus $\alpha^{-1}\alpha = \beta^{-1}\beta$. $\square$

Lemma 4.1.2. *Suppose each of $(A, \mathcal{R})$, $(B, \mathcal{S})$, and $(C, \mathcal{T})$ is an I-structure, $\alpha : A \to B$ is a saturating surjection, $\beta : A \to C$ is a preservative function, and $\alpha^{-1}\alpha \subseteq \beta^{-1}\beta$. Then $\beta\alpha^{-1}$ is preservative.*

Proof: Suppose $s \in \mathcal{S}$. $\mathrm{im}(s) \subseteq B = \mathrm{im}(\alpha)$, so since α is saturating, there is an $r \in \mathcal{R}$ such that $\alpha r = s$. β is preservative, so $\beta r \in \mathcal{T}$. By Lemma 4.1.1, $\beta\alpha^{-1}\alpha = \beta$, so $\beta\alpha^{-1}s = \beta\alpha^{-1}\alpha r = \beta r$.

Thus $\beta\alpha^{-1}s = \beta r \in \mathcal{T}$. So $\beta\alpha^{-1}$ is preservative. $\square$

Lemma 4.1.3. *Suppose each of $(A, \mathcal{R})$, $(B, \mathcal{S})$, and $(C, \mathcal{T})$ is an I-structure, $\alpha : A \to B$ is a preservative surjection, $\beta : A \to C$ is a saturating function, and $\alpha^{-1}\alpha \subseteq \beta^{-1}\beta$. Then $\beta\alpha^{-1}$ is saturating.*

Proof: Suppose $t \in \mathcal{T}$ and $\mathrm{im}(t) \subseteq \mathrm{im}(\beta\alpha^{-1}) = \mathrm{im}(\beta)$. β is saturating, so there is an $r \in \mathcal{R}$ such that $\beta r = t$. α is preservative, so $\alpha r \in \mathcal{S}$.

By Lemma 4.1.1, $\beta\alpha^{-1}\alpha = \beta$, so $(\beta\alpha^{-1})(\alpha r) = (\beta\alpha^{-1}\alpha)r = \beta r = t$.

So αr is a relation in $\mathcal{S}$ such that $(\beta\alpha^{-1})\alpha r = t$. So $\beta\alpha^{-1}$ is saturating. □

Theorem 4.1. *Suppose each of $(A, \mathcal{R})$, $(B, \mathcal{S})$, and $(C, \mathcal{T})$ is an I-structure, $\alpha : A \to B$ is an epimorphism, $\beta : A \to C$ is a homomorphism, and $\alpha^{-1}\alpha \subseteq \beta^{-1}\beta$. Then $\beta\alpha^{-1}$ is a unique homomorphism with domain B such that $\beta\alpha^{-1}\alpha = \beta$.*

Proof: By Lemma 4.1.1, $\beta\alpha^{-1}$ is a unique function with domain B such that $\beta\alpha^{-1}\alpha = \beta$.

α is a homomorphism and thus is saturating, and β is a homomorphism and thus is preservative, so by Lemma 4.1.2, $\beta\alpha^{-1}$ is preservative.

α is a homomorphism and thus is preservative, and β is a homomorphism and thus is saturating, so by Lemma 4.1.3, $\beta\alpha^{-1}$ is saturating.

Since $\beta\alpha^{-1}$ is both preservative and saturating, $\beta\alpha^{-1}$ is an I-structure homomorphism. □

Lemma 4.2.1. *Suppose each of $(A, \mathcal{R})$, $(B, \mathcal{S})$, and $(C, \mathcal{T})$ is an I-structure, $\alpha : A \to B$ is a conservative surjection, $\beta : A \to C$ is a continuous function, and $\alpha^{-1}\alpha \subseteq \beta^{-1}\beta$. Then $\beta\alpha^{-1}$ is continuous.*

Proof: Suppose $t \in \mathcal{T}$. β is continuous, so, $\beta^{-1}t \in \mathcal{R}$. α is conservative, so there is an $s \in \mathcal{S}$ such that $\mathrm{im}(s) \subseteq \mathrm{im}(\alpha)$ and $\alpha^{-1}s = \beta^{-1}t$. α is surjective, so $\alpha\alpha^{-1} = 1_B$.

$$(\beta\alpha^{-1})^{-1}t = \alpha\beta^{-1}t = \alpha\alpha^{-1}s = 1_B s = s$$

Thus $(\beta\alpha^{-1})^{-1}t = s \in \mathcal{S}$. So $\beta\alpha^{-1}$ is continuous. □

Lemma 4.2.2. *Suppose each of $(A, \mathcal{R})$, $(B, \mathcal{S})$, and $(C, \mathcal{T})$ is an I-structure, $\alpha : A \to B$ is a continuous surjection, $\beta : A \to C$ is a conservative function, and $\alpha^{-1}\alpha \subseteq \beta^{-1}\beta$. Then $\beta\alpha^{-1}$ is conservative.*

Proof: Suppose $s \in \mathcal{S}$. α is continuous, so $\alpha^{-1}s \in \mathcal{R}$. β is conservative, so there is a $t \in \mathcal{T}$ such that $\mathrm{im}(t) \subseteq \mathrm{im}(\beta)$ and $\beta^{-1}t = \alpha^{-1}s$. α is surjective, so $\alpha\alpha^{-1} = 1_B$.

$$(\beta\alpha^{-1})^{-1}t = \alpha\beta^{-1}t = \alpha\alpha^{-1}s = 1_B s = s$$

Thus $(\beta\alpha^{-1})^{-1}t = s$, and t is a relation in $\mathcal{T}$ such that $\mathrm{im}(t) \subseteq \mathrm{im}(\beta) = \mathrm{im}(\beta\alpha^{-1})$ and $(\beta\alpha^{-1})^{-1}t = s$. So $\beta\alpha^{-1}$ is conservative. □

Theorem 4.2. *Suppose each of $(A, \mathcal{R})$, $(B, \mathcal{S})$, and $(C, \mathcal{T})$ is an I-structure, $\alpha : A \to B$ is a coepimorphism, $\beta : A \to C$ is a cohomomorphism, and $\alpha^{-1}\alpha \subseteq \beta^{-1}\beta$. Then $\beta\alpha^{-1}$ is a unique cohomomorphism with domain B such that $\beta\alpha^{-1}\alpha = \beta$.*

Proof: By Lemma 4.1.1, $\beta\alpha^{-1}$ is a unique function with domain B such that $\beta\alpha^{-1}\alpha = \beta$.

α is a cohomomorphism and thus is conservative, and β is a cohomomorphism and thus is continuous, so by Lemma 4.2.1, $\beta\alpha^{-1}$ is continuous.

α is a cohomomorphism and thus is continuous, and β is a cohomomorphism and thus is conservative, so by Lemma 4.2.2, $\beta\alpha^{-1}$ is conservative.

Since $\beta\alpha^{-1}$ is both continuous and conservative, $\beta\alpha^{-1}$ is an I-structure cohomomorphism.

□

Lemma 4.3.1. *Suppose each of α and β is a function. Then $\alpha\alpha^{-1} \subseteq \beta\beta^{-1}$ if and only if $\mathrm{im}(\alpha) \subseteq \mathrm{im}(\beta)$.*

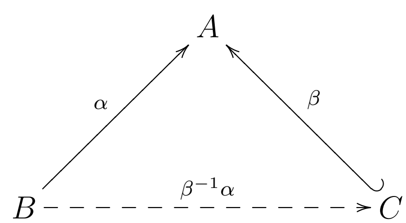
Proof: Suppose $\alpha\alpha^{-1} \subseteq \beta\beta^{-1}$.

$$\mathrm{im}(\alpha) = \mathrm{im}(1_{\mathrm{im}(\alpha)}) = \mathrm{im}(\alpha\alpha^{-1}) \subseteq \mathrm{im}(\beta\beta^{-1}) = \mathrm{im}(1_{\mathrm{im}(\beta)}) = \mathrm{im}(\beta)$$

Suppose $\mathrm{im}(\alpha) \subseteq \mathrm{im}(\beta)$.

$$\alpha\alpha^{-1} = 1_{\mathrm{im}(\alpha)} \subseteq 1_{\mathrm{im}(\beta)} = \beta\beta^{-1} \quad \square$$

Lemma 4.3.2. *Suppose each of A, B, and C is a set, $\alpha : B \to A$ is a function, $\beta : C \to A$ is an injection, and $\alpha\alpha^{-1} \subseteq \beta\beta^{-1}$. Then $\beta^{-1}\alpha$ is a unique function with image a subset of C such that $\beta(\beta^{-1}\alpha) = \alpha$. Moreover,* $\mathrm{dom}(\beta^{-1}\alpha) = \mathrm{dom}(\alpha)$.



Proof:

$$\begin{aligned}
&(b, c_1) \in \beta^{-1}\alpha \text{ and } (b, c_2) \in \beta^{-1}\alpha \\
&\implies \exists a_1 \in A \text{ such that } (b, a_1) \in \alpha \text{ and } (a_1, c_1) \in \beta^{-1} \\
&\text{and } \exists a_2 \in A \text{ such that } (b, a_2) \in \alpha \text{ and } (a_2, c_2) \in \beta^{-1} \\
&\implies a_1 = a_2 \text{ since } \alpha \text{ is a function} \\
&\implies (c_1, a_1) \in \beta \text{ and } (c_2, a_1) \in \beta \\
&\implies c_1 = c_2 \text{ since } \beta \text{ is an injection}
\end{aligned}$$

So $\beta^{-1}\alpha$ is a function.

Note $\beta\beta^{-1} \subseteq 1_A$ and $\alpha\alpha^{-1} = 1_{\text{im}(\alpha)}$.

$$\beta(\beta^{-1}\alpha) = (\beta\beta^{-1})\alpha \subseteq 1_A\alpha = \alpha$$
$$\alpha = 1_{\text{im}(\alpha)}\alpha = (\alpha\alpha^{-1})\alpha \subseteq (\beta\beta^{-1})\alpha = \beta(\beta^{-1}\alpha)$$

So $\beta(\beta^{-1}\alpha) = \alpha$.

Suppose γ is a function with image a subset of C such that $\beta\gamma = \alpha$. Note since β is an injection, $\beta^{-1}\beta = 1_C$.

$$\beta\gamma = \alpha$$
$$\implies \gamma = 1_C\gamma = \beta^{-1}\beta\gamma = \beta^{-1}\alpha$$

So $\beta^{-1}\alpha$ is the only function with image a subset of C with the property that $\beta(\beta^{-1}\alpha) = \alpha$.

$$b \in \text{dom}(\beta^{-1}\alpha)$$
$$\implies \exists c \in C \text{ such that } (b, c) \in \beta^{-1}\alpha$$
$$\implies \exists a \in A \text{ such that } (b, a) \in \alpha \text{ and } (a, c) \in \beta^{-1}$$
$$\implies b \in \text{dom}(\alpha)$$

So $\text{dom}(\beta^{-1}\alpha) \subseteq \text{dom}(\alpha)$.

$$b \in \text{dom}(\alpha)$$
$$\implies \exists a \in A \text{ such that } (b, a) \in \alpha = \beta(\beta^{-1}\alpha)$$
$$\implies \exists c \in C \text{ such that } (c, a) \in \beta \text{ and } (b, c) \in \beta^{-1}\alpha$$
$$\implies b \in \text{dom}(\beta^{-1}\alpha)$$

So $\mathrm{dom}(\alpha) \subseteq \mathrm{dom}(\beta^{-1}\alpha)$, and thus $\mathrm{dom}(\beta^{-1}\alpha) = \mathrm{dom}(\alpha)$. $\square$

Corollary 4.3.1. *Suppose each of A, B, and C is a set, $\alpha : B \to A$ is a function, $\beta : C \to A$ is an injection, and $\alpha\alpha^{-1} \subseteq \beta\beta^{-1}$. Then α is an injection if and only if $\beta^{-1}\alpha$ is an injection.*

Proof: Suppose α is an injection. $\mathrm{im}(\alpha) \subseteq \mathrm{im}(\beta)$.

$$(\beta^{-1}\alpha)^{-1}\beta^{-1}\alpha = \alpha^{-1}\beta\beta^{-1}\alpha = \alpha^{-1}1_{\mathrm{im}(\beta)}\alpha = \alpha^{-1}\alpha = 1_B$$

So $\beta^{-1}\alpha$ is an injection.

Suppose $\beta^{-1}\alpha$ is an injection.

$$\alpha^{-1}\alpha = \alpha^{-1}1_{\mathrm{im}(\beta)}\alpha = \alpha^{-1}\beta\beta^{-1}\alpha = (\beta^{-1}\alpha)^{-1}\beta^{-1}\alpha = 1_B$$

So α is an injection. $\square$

Corollary 4.3.2. *Suppose each of A, B, and C is a set, $\alpha : B \to A$ is a function, $\beta : C \to A$ is an injection, and $\alpha\alpha^{-1} \subseteq \beta\beta^{-1}$. Then $\alpha\alpha^{-1} = \beta\beta^{-1}$ if and only if $\beta^{-1}\alpha$ is a surjection with respect to C.*

Proof: Suppose $\alpha\alpha^{-1} = \beta\beta^{-1}$.

$$\beta^{-1}\alpha(\beta^{-1}\alpha)^{-1} = \beta^{-1}\alpha\alpha^{-1}\beta = \beta^{-1}\beta\beta^{-1}\beta = 1_C 1_C = 1_C$$

So $\beta^{-1}\alpha$ is a surjection with respect to C.

Suppose $\beta^{-1}\alpha$ is a surjection with respect to C.

$$\beta\beta^{-1} = \beta 1_C \beta^{-1} = \beta\beta^{-1}\alpha(\beta^{-1}\alpha)^{-1}\beta^{-1} = \beta\beta^{-1}\alpha\alpha^{-1}\beta\beta^{-1} = 1_{\mathrm{im}(\beta)}\alpha\alpha^{-1}1_{\mathrm{im}(\beta)} = \alpha\alpha^{-1} \quad \square$$

Lemma 4.3.3. *Suppose each of* $(A, \mathcal{R})$, $(B, \mathcal{S})$, *and* $(C, \mathcal{T})$ *is an* I*-structure,* $\alpha : B \to A$ *is a preservative function,* $\beta : C \to A$ *is a saturating injection, and* $\alpha\alpha^{-1} \subseteq \beta\beta^{-1}$. *Then* $\beta^{-1}\alpha$ *is preservative.*

Proof: Suppose $s \in \mathcal{S}$. α is preservative, so $\alpha s \in \mathcal{R}$. $\text{im}(\alpha s) \subseteq \text{im}(\alpha) \subseteq \text{im}(\beta)$, so since β is saturating, there is a $t \in \mathcal{T}$ such that $\beta t = \alpha s$.

Thus $\beta^{-1}\alpha s = \beta^{-1}\beta t = 1_A t = t \in \mathcal{T}$. So $\beta^{-1}\alpha$ is preservative. □

Lemma 4.3.4. *Suppose each of* $(A, \mathcal{R})$, $(B, \mathcal{S})$, *and* $(C, \mathcal{T})$ *is an* I*-structure,* $\alpha : B \to A$ *is a saturating function,* $\beta : C \to A$ *is a preservative injection, and* $\alpha\alpha^{-1} \subseteq \beta\beta^{-1}$. *Then* $\beta^{-1}\alpha$ *is saturating.*

Proof: Suppose $t \in \mathcal{T}$ such that $\text{im}(t) \subseteq \text{im}(\beta^{-1}\alpha)$. β is preservative, so $\beta t \in \mathcal{R}$. $\text{im}(\beta t) = \beta(\text{im}(t)) \subseteq \beta(\text{im}(\beta^{-1}\alpha)) = \text{im}(\beta\beta^{-1}\alpha) \subseteq \text{im}(1_A\alpha) = \text{im}(\alpha)$, so since α is saturating, there is an $s \in \mathcal{S}$ such that $\alpha s = \beta t$.

So s is a relation in $\mathcal{S}$ such that $\beta^{-1}\alpha s = \beta^{-1}\beta t = 1_A t = t$. So $\beta^{-1}\alpha$ is saturating. □

Theorem 4.3. *Suppose each of* $(A, \mathcal{R})$, $(B, \mathcal{S})$, *and* $(C, \mathcal{T})$ *is an* I*-structure,* $\alpha : B \to A$ *is a homomorphism,* $\beta : C \to A$ *is a monomorphism, and* $\alpha\alpha^{-1} \subseteq \beta\beta^{-1}$. *Then* $\beta^{-1}\alpha$ *is a unique homomorphism with image a subset of* C *such that* $\beta\beta^{-1}\alpha = \alpha$.

Proof: By Lemma 4.3.2, $\beta^{-1}\alpha$ is a unique function with image a subset of C such that $\beta\beta^{-1}\alpha = \alpha$.

α is a homomorphism and thus is preservative, and β is a homomorphism and thus is saturating, so by Lemma 4.3.3, $\beta^{-1}\alpha$ is preservative.

α is a homomorphism and thus is saturating, and β is a homomorphism and thus is preservative, so by Lemma 4.3.4, $\beta^{-1}\alpha$ is saturating.

Since $\beta^{-1}\alpha$ is both preservative and saturating, $\beta^{-1}\alpha$ is an I-structure homomorphism. $\square$

Lemma 4.4.1. *Suppose each of $(A, \mathcal{R})$, $(B, \mathcal{S})$, and $(C, \mathcal{T})$ is an I-structure, $\alpha : B \to A$ is a continuous function, $\beta : C \to A$ is a conservative injection, and $\alpha\alpha^{-1} \subseteq \beta\beta^{-1}$. Then $\beta^{-1}\alpha$ is continuous.*

Proof: Suppose $t \in \mathcal{T}$. β is conservative, so there is an $r \in \mathcal{R}$ such that $\operatorname{im}(r) \subseteq \operatorname{im}(\beta)$ and $\beta^{-1}r = t$. α is continuous, so $\alpha^{-1}r \in \mathcal{S}$.

$$(\beta^{-1}\alpha)^{-1}t = \alpha^{-1}\beta t = \alpha^{-1}\beta\beta^{-1}r = \alpha^{-1}1_{\operatorname{im}(\beta)}r = \alpha^{-1}r$$

Thus $(\beta^{-1}\alpha)^{-1}t = \alpha^{-1}r \in \mathcal{S}$. So $\beta^{-1}\alpha$ is continuous. $\square$

Lemma 4.4.2. *Suppose each of $(A, \mathcal{R})$, $(B, \mathcal{S})$, and $(C, \mathcal{T})$ is an I-structure, $\alpha : B \to A$ is a conservative function, $\beta : C \to A$ is a continuous injection, and $\operatorname{im}(\alpha) \subseteq \operatorname{im}(\beta)$. Then $\beta^{-1}\alpha$ is conservative.*

Proof: Suppose $s \in \mathcal{S}$. α is conservative, so there is an $r \in \mathcal{R}$ such that $\operatorname{im}(r) \subseteq \operatorname{im}(\alpha) \subseteq \operatorname{im}(\beta)$ and $s = \alpha^{-1}r$. β is continuous, so $\beta^{-1}r \in \mathcal{T}$.

$$(\beta^{-1}\alpha)^{-1}\beta^{-1}r = \alpha^{-1}\beta\beta^{-1}r = \alpha^{-1}1_{\operatorname{im}(\beta)}r = \alpha^{-1}r = s$$

Thus $(\beta^{-1}\alpha)^{-1}\beta^{-1}r = s$, and $\beta^{-1}r$ is a relation in $\mathcal{T}$ such that $\operatorname{im}(\beta^{-1}r) = \beta^{-1}(\operatorname{im}(r)) \subseteq \beta^{-1}(\operatorname{im}(\alpha)) = \operatorname{im}(\beta^{-1}\alpha)$ and $(\beta^{-1}\alpha)^{-1}\beta^{-1}r = s$. So $\beta^{-1}\alpha$ is conservative. $\square$

Theorem 4.4. *Suppose each of $(A, \mathcal{R})$, $(B, \mathcal{S})$, and $(C, \mathcal{T})$ is an I-structure, $\alpha : B \to A$ is a cohomomorphism, $\beta : C \to A$ is a comonomorphism, and $\alpha\alpha^{-1} \subseteq \beta\beta^{-1}$. Then $\beta^{-1}\alpha$ is a unique cohomomorphism with image a subset of C such that $\beta\beta^{-1}\alpha = \alpha$.*

Proof: By Lemma 4.3.2, $\beta^{-1}\alpha$ is a unique function with image a subset of C such that $\beta\beta^{-1}\alpha = \alpha$.

α is a cohomomorphism and thus is continuous, and β is a cohomomorphism and thus is conservative, so by Lemma 4.4.1, $\beta^{-1}\alpha$ is continuous.

α is a cohomomorphism and thus is conservative, and β is a cohomomorphism and thus is continuous, so by Lemma 4.4.2, $\beta^{-1}\alpha$ is conservative.

Since $\beta^{-1}\alpha$ is both continuous and conservative, $\beta^{-1}\alpha$ is an I-structure cohomomorphism.

$\square$

Chapter 5

First Isomorphism Theorems

Definition Suppose $M = (A, \mathcal{R})$ is an I-structure and $B \subseteq A$. The *I-substructure of M induced by B* is the I-structure $(B, \hat{\mathcal{R}})$ where $\hat{\mathcal{R}}$ is the set of relations to which a relation $\hat{r}$ belongs if and only if $\hat{r} \in \mathcal{R}$ and $\text{im}(\hat{r}) \subseteq B$. Denote the I-structure $(B, \hat{\mathcal{R}})$ by $M|B$. The statement that $(C, \mathcal{T})$ is an *I-substructure of M* means $C \subseteq A$ and $(C, \mathcal{T})$ is the I-substructure of M induced by C.

Definition Suppose $M = (A, \mathcal{R})$ is an I-structure and φ is a function with domain A. Suppose $\bar{\mathcal{R}}$ is the set of relations to which a relation $\bar{r}$ belongs if and only if there is a relation $r \in \mathcal{R}$ such that $\bar{r} = \pi_\varphi r$. Denote the I-structure $(A/\varphi, \bar{\mathcal{R}})$ by M/φ.

Lemma 5.1.1. *Suppose each of A and B is a set, and $\varphi : A \to B$ is a function. Then $\varphi\pi_\varphi^{-1}$ is a bijection with respect to* $\text{im}(\varphi)$.

Proof: By Theorem 2.4, π_φ is a surjection with respect to A/φ, and by Theorem 2.6, $\pi_\varphi^{-1}\pi_\varphi \subseteq \varphi^{-1}\varphi$, so by Lemma 4.1.1, $\varphi\pi_\varphi^{-1}$ is a function.

φ is a surjection with respect to $\text{im}(\varphi)$, so by Corollary 4.1.1, $\varphi\pi_\varphi^{-1}$ is a surjection with respect to $\text{im}(\varphi)$.

By Theorem 2.6 $\varphi^{-1}\varphi = \pi_\varphi^{-1}\pi_\varphi$, so by Corollary 4.1.2, $\varphi\pi_\varphi^{-1}$ is an injection.

Thus $\varphi\pi_\varphi^{-1}$ is a function which is both an injection and a surjection with respect to $\text{im}(\varphi)$, so $\varphi\pi_\varphi^{-1}$ is a bijection with respect to $\text{im}(\varphi)$. □

Lemma 5.1.2. *Suppose each of* $M = (A, \mathcal{R})$ *and* $(B, \mathcal{S})$ *is an* I*-structure, and* $\varphi : A \to B$ *is a function. Then* π_φ *is a* I*-structure epimorphism with respect to the* I*-structure* $M/\varphi = (A/\varphi, \bar{\mathcal{R}})$.

Proof:

1. π_φ is preservative: Suppose $r \in \mathcal{R}$, then $\pi_\varphi r \in \bar{\mathcal{R}}$ by definition of M/φ.

2. π_φ is saturating: Suppose $\bar{r} \in \bar{\mathcal{R}}$ such that $\mathrm{im}(\bar{r}) \subseteq \mathrm{im}(\pi_\varphi)$, then by the definition of M/φ there is an $r \in \mathcal{R}$ such that $\pi_\varphi r = \bar{r}$.

3. π_φ is a surjection: By Theorem 2.4.

So π_φ is an epimorphism. □

Lemma 5.1.3. *Suppose each of* $M = (A, \mathcal{R})$ *and* $N = (B, \mathcal{S})$ *is an* I*-structure, and* $\varphi : A \to B$ *is a function. Then* φ *is an* I*-structure homomorphism from* M *to* N *if and only if* φ *is an* I*-structure homomorphism from* A *to the* I*-substructure of* M *induced by* $\mathrm{im}(\varphi)$, $(\mathrm{im}(\varphi), \hat{\mathcal{S}})$.

Proof: Suppose φ is an I-structure homomorphism from M to N.

Suppose $r \in \mathcal{R}$. φ is preservative, so $\varphi r \in \mathcal{S}$.

Suppose $b \in \mathrm{im}(\varphi r)$. Then there is an $i \in I$ such that $(i, b) \in \varphi r$. There is an $a \in A$ such that $\varphi(a) = b$ and $(i, a) \in r$. $b = \varphi(a) \in \mathrm{im}(\varphi)$. So $\mathrm{im}(\varphi r) \subseteq \mathrm{im}(\varphi)$, and thus $\varphi r \in \hat{\mathcal{S}}$.

So φ is preservative between $(A, \mathcal{R})$ and $(\mathrm{im}(\varphi), \hat{\mathcal{S}})$.

Suppose $\hat{s} \in \hat{\mathcal{S}}$ such that $\mathrm{im}(\hat{s}) \subseteq \mathrm{im}(\varphi)$. φ is saturating and $\hat{s} \in \mathcal{S}$, so there is an $r \in \mathcal{R}$ such that $\varphi r = \hat{s}$.

So φ is saturating between $(A, \mathcal{R})$ and $(\text{im}(\varphi), \hat{\mathcal{S}})$.

φ is both preservative and saturating between $(A, \mathcal{R})$ and $(\text{im}(\varphi), \hat{\mathcal{S}})$, so φ is an I-structure homomorphism between $(A, \mathcal{R})$ and $(\text{im}(\varphi), \hat{\mathcal{S}})$. □

Suppose φ is an I-structure homomorphism from A to the I-substructure of M induced by $\text{im}(\varphi)$.

Suppose $r \in \mathcal{R}$. φ is preservative with respect to $(\text{im}(\varphi), \hat{\mathcal{S}})$, so $\varphi r \in \hat{\mathcal{S}}$, so $\varphi r \in \mathcal{S}$.

So φ is preservative between M and N.

Suppose $s \in \mathcal{S}$ such that $\text{im}(s) \subseteq \text{im}(\varphi)$. Then $s \in \hat{\mathcal{S}}$, and since φ is saturating with respect to $(\text{im}(\varphi), \hat{\mathcal{S}})$, there is an $r \in \mathcal{R}$ such that $\varphi r = s$.

So φ is saturating between M and N.

φ is both preservative and saturating between M and N, so φ is an I-structure homomorphism between M and N. □

Theorem 5.1. *Suppose each of $M = (A, \mathcal{R})$ and $N = (B, \mathcal{S})$ is an I-structure and $\varphi : A \to B$ is a function. Then φ is an I-structure homomorphism if and only if $\varphi\pi_\varphi^{-1}$ is an isomorphism from M/φ to the I-substructure of N induced by* $\text{im}(\varphi)$.

Proof: Suppose φ is an I-structure homomorphism.

$M/\varphi = (A/\varphi, \bar{\mathcal{R}})$ where $\bar{\mathcal{R}}$ is the set of relations to which a relation $\bar{r}$ belongs if and only if there is a relation $r \in \mathcal{R}$ such that $\pi_\varphi r = \bar{r}$.

The I-substructure of N induced by $\text{im}(\varphi)$ is $(\text{im}(\varphi), \hat{\mathcal{S}})$ where $\hat{\mathcal{S}}$ is the set of relations to which a relation $\hat{s}$ belongs if and only if $\hat{s} \in \mathcal{S}$ and $\text{im}(\hat{s}) \subseteq \text{im}(\varphi)$.

By Lemma 5.1.2, π_φ is an I-structure epimorphism, by Lemma 5.1.3, φ is an I-structure homomorphism between $(A, \mathcal{R})$ and $(\text{im}(\varphi), \hat{\mathcal{S}})$, and by Lemma 2.6, $\pi_\varphi^{-1}\pi_\varphi \subseteq \varphi^{-1}\varphi$. So by Theorem 4.1, $\varphi\pi_\varphi^{-1}$ is a homomorphism.

By Lemma 5.1.1, $\varphi\pi_\varphi^{-1}$ is a bijection, so by Theorem 1.19, $\varphi\pi_\varphi^{-1}$ is an isomorphism.

So M/φ is isomorphic to the I-substructure of N induced by $\text{im}(\varphi)$. $\square$

Suppose $\varphi\pi_\varphi^{-1}$ is an isomorphism from M/φ to the I-substructure of N induced by $\text{im}(\varphi)$.

By Lemma 5.1.2 π_φ is an epimorphism, and by assumption, $\varphi\pi_\varphi^{-1}$ is a homomorphism from M/φ to the I-substructure of N induced by $\text{im}(\varphi)$. So by Theorem 1.22, $(\varphi\pi_\varphi^{-1})\pi_\varphi$ is a homomorphism from M to the I-substructure of N induced by $\text{im}(\varphi)$.

By Lemma 4.1.1, $\varphi = (\varphi\pi_\varphi^{-1})\pi_\varphi$, so φ is a homomorphism from M to the I-substructure of N induced by $\text{im}(\varphi)$, and thus by Lemma 5.1.3, φ is a homomorphism from M to N. $\square$

Corollary 5.1.1. *Suppose each of $M = (A, \mathcal{R})$ and $(B, \mathcal{S})$ is a 1-structure, and $\varphi : A \to B$ is a homomorphism. Then if each of $(A, \mathcal{R})$ and $(B, \mathcal{S})$ is the structurization of a topological space, then $(A/\varphi, \bar{\mathcal{R}})$ is the structurization of a topological space.*

Proof:

1. $1 \times A/\varphi \in \bar{\mathcal{R}}$:

$(A, \mathcal{R})$ is the structurization of a topological space, so $1 \times A \in \mathcal{R}$.

π_φ is an epimorphism, so $1 \times A/\varphi = 1 \times \pi_\varphi[A] = \pi_\varphi(1 \times A) \in \bar{\mathcal{R}}$.

2. $\varnothing \in \bar{\mathcal{R}}$:

   $(A, \mathcal{R})$ is the structurization of a topological space, so $\varnothing \in \mathcal{R}$.

   π_φ is a homomorphism, so $\varnothing = \pi_\varphi \varnothing \in \bar{\mathcal{R}}$.

3. Suppose J is a set and for each $j \in J$, $\bar{r}_j \in \bar{\mathcal{R}}$, then there is an $\bar{r} \in \bar{\mathcal{R}}$ such that $\bar{r}[1] = \bigcup_{j \in J} \bar{r}_j[1]$:

   For each $j \in J$, there is an $r_j \in \mathcal{R}$ such that $\pi_\varphi r_j = \bar{r}_j$. Since $(A, \mathcal{R})$ is the structurization of a topological space, there is an $r \in \mathcal{R}$ such that $r[1] = \bigcup_{j \in J} r_j[1]$. $\pi_\varphi r \in \bar{\mathcal{R}}$.

$$\begin{aligned} & P \in \pi_\varphi r[1] \\ \iff & P \in \pi_\varphi[r[1]] \\ \iff & P \in \pi_\varphi(\bigcup_{j \in J} r_j[1]) = \bigcup_{j \in J} \pi_\varphi[r_j[1]] \text{ by Lemma 3.4.2} \\ \iff & P \in \bigcup_{j \in J} \pi_\varphi r_j[1] \\ \iff & P \in \bigcup_{j \in J} \bar{r}_j[1] \end{aligned}$$

   So $\pi_\varphi r \in \bar{\mathcal{R}}$ such that $\pi_\varphi r[1] = \bigcup_{j \in J} \bar{r}_j[1]$.

4. Suppose each of $\bar{r}_1$ and $\bar{r}_2$ is in $\bar{\mathcal{R}}$, then there is an $\bar{r} \in \bar{\mathcal{R}}$ such that $\bar{r}[1] = \bar{r}_1[1] \cap \bar{r}_2[1]$:

   By the definition of $\bar{\mathcal{R}}$, there is an $r_1 \in \mathcal{R}$ and an $r_2 \in \mathcal{R}$ such that $\pi_\varphi r_1 = \bar{r}_1$ and $\pi_\varphi r_2 = \bar{r}_2$.

Since φ is a homomorphism, $\varphi r_1 \in \mathcal{S}$ and $\varphi r_2 \in \mathcal{S}$, and since $(B, \mathcal{S})$ is the structurization of a topological space, there is an $s \in \mathcal{S}$ such that $s[1] = \varphi r_1[1] \cap \varphi r_2[1] \subseteq \operatorname{im}(\varphi)$. So there is an $r \in \mathcal{R}$ such that $\varphi r = s$.

$$\begin{aligned}
\pi_\varphi r[1] = \pi_\varphi[r[1]] =& \{\pi_\varphi(a) \mid a \in r[1]\} \\
=& \{\varphi^{-1}[\varphi[\{a\}]] \mid a \in r[1]\} \\
=& \{\pi_\varphi(a) \mid a \in \varphi^{-1}[\varphi[r[1]]]\} \\
=& \{\pi_\varphi(a) \mid a \in \varphi^{-1}[s[\{0\}]]\} \\
=& \{\pi_\varphi(a) \mid a \in \varphi^{-1}[\varphi r_1[1] \cap \varphi r_2[1]]\} \\
=& \{\pi_\varphi(a) \mid a \in \varphi^{-1}[\varphi[r_1[1]] \cap \varphi[r_2[1]]]\} \\
=& \{\pi_\varphi(a) \mid a \in \varphi^{-1}[\varphi[r_1[1]]] \cap \varphi^{-1}[\varphi[r_2[1]]]\} \\
=& \{\pi_\varphi(a) \mid a \in \varphi^{-1}[\varphi[r_1[1]]]\} \cap \{\pi_\varphi(a) \mid a \in \varphi^{-1}[\varphi[r_2[1]]]\} \\
=& \{\varphi^{-1}[\varphi[\{a\}]] \mid a \in r_1[1]\} \cap \{\varphi^{-1}[\varphi[\{a\}]] \mid a \in r_2[1]\} \\
=& \{\pi_\varphi(a) \mid a \in r_1[1]\} \cap \{\pi_\varphi(a) \mid a \in r_2[1]\} \\
=& \pi_\varphi[r_1[1]] \cap \pi_\varphi[r_2[1]] \\
=& \pi_\varphi r_1[1] \cap \pi_\varphi r_2[1] \\
=& \bar{r}_1[1] \cap \bar{r}_2[1]
\end{aligned}$$

So $\pi_\varphi r \in \bar{\mathcal{R}}$ such that $\pi_\varphi r[1] = \bar{r}_1[1] \cap \bar{r}_2[1]$.

So by the above properties, $(A/\varphi, \bar{\mathcal{R}})$ is the structurization of a topological space. □

Corollary 5.1.2. *Suppose $(G, *)$ is a group with identity e, $I = \{p_e, p_{-1}, 0_{-1}, p_*, 0_*, 1_*\}$, $M = (G, \mathcal{R})$ is the structurization of $(G, *)$, and φ is a function with domain G. Then M/φ is the structurization of a group under the operation induced by $*$ if and only if for all $(a_1, a_2) \in \varphi^{-1}\varphi$, $(a_1^{-1}, a_2^{-1}) \in \varphi^{-1}\varphi$, and for all $(a_1, b_1), (a_2, b_2)$ in $\varphi^{-1}\varphi$, $(a_1 * a_2, b_1 * b_2) \in \varphi^{-1}\varphi$.*

Proof: Suppose for all $(a_1, a_2) \in \varphi^{-1}\varphi$, $(a_1^{-1}, a_2^{-1}) \in \varphi^{-1}\varphi$, and for all $(a_1, b_1), (a_2, b_2)$ in $\varphi^{-1}\varphi$, $(a_1 * a_2, b_1 * b_2) \in \varphi^{-1}\varphi$.

Suppose each of $P = \varphi^{-1}\varphi(\{x\})$ and $Q = \varphi^{-1}\varphi(\{y\})$ is in G/φ.

$P * Q = \{p * q \mid p \in P \text{ and } q \in Q\}$. Consider the element of G/φ, $\varphi^{-1}\varphi(\{x * y\})$.

$$\begin{aligned}
&g \in P * Q \\
\Longrightarrow\ &\exists p \in P,\ q \in Q \text{ such that } g = p * q \\
\Longrightarrow\ &\exists p \in \varphi^{-1}\varphi(\{x\}),\ q \in \varphi^{-1}\varphi(\{y\}) \text{ such that } g = p * q \\
\Longrightarrow\ &\exists p, q \text{ such that } (p, x) \in \varphi^{-1}\varphi \text{ and } (q, y) \in \varphi^{-1}\varphi \text{ and } g = p * q \\
\Longrightarrow\ &\exists p, q \text{ such that } (p * q, x * y) \in \varphi^{-1}\varphi \text{ and } g = p * q \\
\Longrightarrow\ &g \in \varphi^{-1}\varphi(\{x * y\})
\end{aligned}$$

So $P * Q \subseteq \varphi^{-1}\varphi(\{x * y\})$.

$$\begin{aligned}
&g \in \varphi^{-1}\varphi(\{x * y\}) \\
\Longrightarrow\ &(g, x * y) \in \varphi^{-1}\varphi \text{ and } (y^{-1}, y^{-1}) \in \varphi^{-1}\varphi \\
\Longrightarrow\ &(g * y^{-1}, x) = (g * y^{-1}, x * e) = (g * y^{-1}, x * (y * y^{-1})) = (g * y^{-1}, (x * y) * y^{-1}) \in \varphi^{-1}\varphi \\
\Longrightarrow\ &g * y^{-1} \in \varphi^{-1}\varphi(\{x\}) = P \text{ and } y \in \varphi^{-1}\varphi(\{y\}) = Q \\
\Longrightarrow\ &g = g * e = g * (y^{-1} * y) = (g * y^{-1}) * y \in P * Q
\end{aligned}$$

So $\varphi^{-1}\varphi(\{x * y\}) \subseteq P * Q$.

Thus $\varphi^{-1}\varphi(\{x\}) * \varphi^{-1}\varphi(\{y\}) = P * Q = \varphi^{-1}\varphi(\{x * y\}) \in G/\varphi$.

So G/φ is closed under the operation $*$.

Consider the element $\varphi^{-1}\varphi(\{e\})$.

By the above, if $P = \varphi^{-1}\varphi(\{x\}) \in G/\varphi$, then:

$$P * \varphi^{-1}\varphi(\{e\}) = \varphi^{-1}\varphi(\{x\}) * \varphi^{-1}\varphi(\{e\}) = \varphi^{-1}\varphi(\{x * e\}) = \varphi^{-1}\varphi(\{x\}) = P$$

And

$$\varphi^{-1}\varphi(\{e\}) * P = \varphi^{-1}\varphi(\{e\}) * \varphi^{-1}\varphi(\{x\}) = \varphi^{-1}\varphi(\{e * x\}) = \varphi^{-1}\varphi(\{x\}) = P$$

So $\varphi^{-1}\varphi(\{e\})$ is an identity in G/φ with respect to the operation $*$.

Suppose $P = \varphi^{-1}\varphi(\{x\}) \in G/\varphi$. Consider $\varphi^{-1}\varphi(\{x^{-1}\})$.

$$P * \varphi^{-1}\varphi(\{x^{-1}\}) = \varphi^{-1}\varphi(\{x\}) * \varphi^{-1}\varphi(\{x^{-1}\}) = \varphi^{-1}\varphi(\{x * x^{-1}\}) = \varphi^{-1}\varphi(\{e\})$$

And

$$\varphi^{-1}\varphi(\{x^{-1}\}) * P = \varphi^{-1}\varphi(\{x^{-1}\}) * \varphi^{-1}\varphi(\{x\}) = \varphi^{-1}\varphi(\{x^{-1} * x\}) = \varphi^{-1}\varphi(\{e\})$$

So $\varphi^{-1}\varphi(\{x^{-1}\})$ is an inverse for P with respect to the identity $\varphi^{-1}\varphi(\{e\})$.

Suppose each of $P = \varphi^{-1}\varphi(\{x\})$, $Q = \varphi^{-1}\varphi(\{y\})$, and $R = \varphi^{-1}\varphi(\{z\})$ are members of

G/φ.

$$(P * Q) * R = (\varphi^{-1}\varphi(\{x\}) * \varphi^{-1}\varphi(\{y\})) * \varphi^{-1}\varphi(\{z\}) = \varphi^{-1}\varphi(\{x * y\}) * \varphi^{-1}\varphi(\{z\})$$
$$= \varphi^{-1}\varphi(\{(x * y) * z\}) = \varphi^{-1}\varphi(\{x * (y * z)\}) = \varphi^{-1}\varphi(\{x\}) * \varphi^{-1}\varphi(\{y * z\})$$
$$= \varphi^{-1}\varphi(\{x\}) * (\varphi^{-1}\varphi(\{y\}) * \varphi^{-1}\varphi(\{z\})) = P * (Q * R)$$

So G/φ is associative with respect to the operation $*$.

So $(G/\varphi, *)$ is a group. (Note that saying for all $(a_1, a_2) \in \varphi^{-1}\varphi$, $(a_1^{-1}, a_2^{-1}) \in \varphi^{-1}\varphi$, and for all $(a_1, b_1), (a_2, b_2)$ in $\varphi^{-1}\varphi$, $(a_1 * a_2, b_1 * b_2) \in \varphi^{-1}\varphi$ implies that $\varphi^{-1}\varphi(\{e\})$ is a normal subgroup of G).

Now I intend to prove that M/φ is the structurization of said group.

$M/\varphi = (G/\varphi, \bar{\mathcal{R}})$ where $I = \{p_e, p_{-1}, 0_{-1}, p_*, 0_*, 1_*\}$ and $\bar{\mathcal{R}} = \{\pi r \mid r \in \mathcal{R}\}$.

The strucuturization of $(G/\varphi, *) = (G/\varphi, \mathcal{S})$ where $\mathcal{S}$ is the set of relations to which a relation s belongs if and only if either $s = \{(p_e, \varphi^{-1}\varphi(e))\}$, there is a $P \in G/\varphi$ such that $s = \{(p_{-1}, P^{-1}), (0_{-1}, P)\}$, or there is a P and a Q each of which is in G/φ such that $s = \{(p_*, P * Q), (0_*, P), (1_*, Q)\}$.

Since $\mathcal{R}$ is the relation set for the structurization of a group, for each $r \in \mathcal{R}$ either $r = \{(p_e, e)\}$, there is an $x \in G$ such that $r = \{(p_{-1}, x^{-1}), (0_{-1}, x)\}$, or there is an x

and a y in G such that $r = \{(p_*, x * y), (0_*, x), (1_*, y)\}$.

$$
\begin{aligned}
& t \in \bar{\mathcal{R}} \\
\iff & \exists r \in \mathcal{R} \text{ such that } t = \pi r \\
\iff & t = \pi r \text{ where} \\
& r = \{(p_e, e)\} \\
& \text{or } r = \{(p_{-1}, x^{-1}), (0_{-1}, x)\} \text{ for some } x \in G \\
& \text{or } r = \{(p_*, x * y), (0_*, x), (1_*, y)\} \text{ for some } x, y \in G \\
\iff & t = \{(p_e, \pi(e))\} \\
& \text{or } t = \{(p_{-1}, \pi(x^{-1})), (0_{-1}, \pi(x))\} \text{ for some } x \in G \\
& \text{or } t = \{(p_*, \pi(x * y)), (0_*, \pi(x)), (1_*, \pi(y))\} \text{ for some } x, y \in G \\
\iff & t = \{(p_e, \varphi^{-1}\varphi(\{e\}))\} \\
& \text{or } t = \{(p_{-1}, \varphi^{-1}\varphi(\{x^{-1}\})), (0_{-1}, \varphi^{-1}\varphi(\{x\}))\} \text{ for some } x \in G \\
& \text{or } t = \{(p_*, \varphi^{-1}\varphi(\{x * y\})), (0_*, \varphi^{-1}\varphi(\{x\})), (1_*, \varphi^{-1}\varphi(\{y\}))\} \text{ for some } x, y \in G \\
\iff & t = \{(p_e, \varphi^{-1}\varphi(\{e\}))\} \\
& \text{or } t = \{(p_{-1}, (\varphi^{-1}\varphi(\{x\}))^{-1}), (0_{-1}, \varphi^{-1}\varphi(\{x\}))\} \text{ for some } x \in G \\
& \text{or } t = \{(p_*, \varphi^{-1}\varphi(\{x\}) * \varphi^{-1}\varphi(\{y\})), (0_*, \varphi^{-1}\varphi(\{x\})), (1_*, \varphi^{-1}\varphi(\{y\}))\} \text{ for some } x, y \in G \\
\iff & t = \{(p_e, \varphi^{-1}\varphi(\{e\}))\} \\
& \text{or } t = \{(p_{-1}, P^{-1}), (0_{-1}, P)\} \text{ for some } P \in G/\varphi \\
& \text{or } t = \{(p_*, P * Q), (0_*, P), (1_*, Q)\} \text{ for some } P, Q \in G/\varphi \\
\iff & t \in \mathcal{S}
\end{aligned}
$$

So $\bar{\mathcal{R}} = \mathcal{S}$, and thus M/φ is the structurization of $(G/\varphi, *)$.

Suppose M/φ is the structurization of a group under the operation induced by $*$.

Consider $\varphi^{-1}\varphi(\{e\})$. Suppose $P \in G/\varphi$. There is an $x \in G$ such that $P = \varphi^{-1}\varphi(x)$.

$$x \in \varphi^{-1}\varphi(x) \text{ and } x = e * x \in \varphi^{-1}\varphi(\{e\}) * \varphi^{-1}\varphi(\{x\}) = \varphi^{-1}\varphi(\{e\}) * P \in G/\varphi$$
$$\Longrightarrow \varphi^{-1}\varphi(\{e\}) * P = \varphi^{-1}\varphi(\{x\}) = P$$
$$\Longrightarrow \varphi^{-1}\varphi(\{e\}) \text{ is the identity element for } M/\varphi$$

Suppose $P \in G/\varphi$. There is an $x \in G$ such that $P = \varphi^{-1}\varphi(\{x\})$. Consider $\varphi^{-1}\varphi(\{x^{-1}\})$

$$e \in \varphi^{-1}\varphi(\{e\}) \text{ and } e = x * x^{-1} \in \varphi^{-1}\varphi(\{x\}) * \varphi^{-1}\varphi(\{x^{-1}\}) = P * \varphi^{-1}\varphi(\{x^{-1}\}) \in G/\varphi$$
$$\Longrightarrow P * \varphi^{-1}\varphi(\{x^{-1}\}) = \varphi^{-1}\varphi(\{e\}) \text{ (elements of a partition intersect if and only if they are equal)}$$
$$\Longrightarrow P^{-1} = \varphi^{-1}\varphi(\{x^{-1}\})$$

Suppose each of P and Q is in G/φ. There is an x and a y in G such that $P = \varphi^{-1}\varphi(\{x\})$ and $Q = \varphi^{-1}\varphi(\{y\})$. Consider $\varphi^{-1}\varphi(\{x * y\})$.

$$x * y \in \varphi^{-1}\varphi(\{x * y\}) \text{ and } x * y \in \varphi^{-1}\varphi(\{x\}) * \varphi^{-1}\varphi(\{y\}) = P * Q$$
$$\Longrightarrow P * Q = \varphi^{-1}\varphi(\{x * y\})$$

Now the main result follows.

$$(a_1, a_2) \in \varphi^{-1}\varphi$$
$$\Longleftrightarrow \varphi^{-1}\varphi(\{a_1\}) = \varphi^{-1}\varphi(\{a_2\})$$
$$\Longleftrightarrow (\varphi^{-1}\varphi(\{a_1\}))^{-1} = (\varphi^{-1}\varphi(\{a_2\}))^{-1}$$
$$\Longleftrightarrow \varphi^{-1}\varphi(\{a_1^{-1}\}) = \varphi^{-1}\varphi(\{a_2^{-1}\})$$
$$\Longleftrightarrow (a_1^{-1}, a_2^{-1}) \in \varphi^{-1}\varphi$$

$$
\begin{aligned}
&(a_1, b_1) \in \varphi^{-1}\varphi \text{ and } (a_2, b_2) \in \varphi^{-1}\varphi \\
\Longrightarrow\ & \varphi^{-1}\varphi(\{a_1\}) = \varphi^{-1}\varphi(\{b_1\}) \text{ and } \varphi^{-1}\varphi(\{a_2\}) = \varphi^{-1}\varphi(\{b_2\}) \\
\Longrightarrow\ & \varphi^{-1}\varphi(\{a_1\}) * \varphi^{-1}\varphi(\{a_2\}) = \varphi^{-1}\varphi(\{b_1\}) * \varphi^{-1}\varphi(\{b_2\}) \\
\Longrightarrow\ & \varphi^{-1}\varphi(\{a_1 * a_2\}) = \varphi^{-1}\varphi(\{b_1 * b_2\}) \\
\Longrightarrow\ & (a_1 * a_2, b_1 * b_2) \in \varphi^{-1}\varphi
\end{aligned}
$$

Thus for all $(a_1, a_2) \in \varphi^{-1}\varphi$, $(a_1^{-1}, a_2^{-1}) \in \varphi^{-1}\varphi$, and for all $(a_1, b_1), (a_2, b_2)$ in $\varphi^{-1}\varphi$, $(a_1 * a_2, b_1 * b_2) \in \varphi^{-1}\varphi$. □

Definition Suppose $M = (A, \mathcal{R})$ is an I-structure and $B \subseteq A$. The *I-understructure of M induced by B* is the I-structure $(B, \hat{\mathcal{R}})$ where $\hat{\mathcal{R}}$ is the set of relations to which a relation $\hat{r}$ belongs if and only if there is a relation $r \in \mathcal{R}$ such that $\hat{r} = r \cap (I \times B)$. Denote the structure $(B, \hat{\mathcal{R}})$ by $M \| B$. The statement that $(C, \mathcal{T})$ is an *I-understructure of M* means $C \subseteq A$ and $(C, \mathcal{T})$ is the I-understructure of M induced by C.

Definition Suppose $M = (A, \mathcal{R})$ is an I-structure. Suppose $\bar{\mathcal{R}}$ is the set of relations to which a relation $\bar{r}$ belongs if and only if $\bar{r} \subseteq I \times A/\varphi$ and $\pi_\varphi^{-1}\bar{r} \in \mathcal{R}$. Denote the I-structure $(A/\varphi, \bar{\mathcal{R}})$ by $M /\!\!/ \varphi$.

Lemma 5.2.1. *Suppose each of $M = (A, \mathcal{R})$ and $(B, \mathcal{S})$ is an I-structure, and $\varphi : A \to B$ is a cohomomorphism. Then π_φ is an I-structure coepimorphism with respect to the I-structure $M /\!\!/ \varphi = (A/\varphi, \bar{\mathcal{R}})$.*

Proof:

1. π_φ is continuous: Suppose $\bar{r} \in \bar{\mathcal{R}}$, then $\pi_\varphi^{-1}\bar{r} \in \mathcal{R}$ by definition of $M /\!\!/ \varphi$. So π_φ is continuous.

2. π_φ is conservative:

Suppose $r \in \mathcal{R}$. φ is conservative, so there is an $s \in \mathcal{S}$ such that $\mathrm{im}(s) \subseteq \mathrm{im}(\varphi)$ and $\varphi^{-1}s = r$. So $s = \varphi r$ and $\pi_\varphi^{-1}\pi_\varphi r = \varphi^{-1}\varphi r = \varphi^{-1}s = r$.

Thus $\pi_\varphi r \in \bar{\mathcal{R}}$, $\mathrm{im}(\pi_\varphi r) \subseteq \mathrm{im}(\pi_\varphi)$, and $\pi_\varphi^{-1}(\pi_\varphi r) = r$.

So π_φ is conservative.

3. π_φ is a surjection with respect to A/φ: By Theorem 2.4.

So π_φ is an coepimorphism with respect to $M /\!\!/ \varphi$. □

Lemma 5.2.2. *Suppose each of $M = (A, \mathcal{R})$ and $N = (B, \mathcal{S})$ is an I-structure, and $\varphi : A \to B$ is an I-structure cohomomorphism. Then φ is a cohomomorphism from A to the understructure of M induced by* $\mathrm{im}(\varphi)$, $(\mathrm{im}(\varphi), \hat{\mathcal{S}})$.

Proof: Suppose $\hat{s} \in \hat{\mathcal{S}}$. Then there is an $s \in \mathcal{S}$ such that $\hat{s} = s \cap (I \times \mathrm{im}(\varphi))$. φ is continuous, so $\varphi^{-1}s \in \mathcal{R}$. I intend to show that $\varphi^{-1}\hat{s} = \varphi^{-1}s$.

Suppose $(i, a) \in \varphi^{-1}\hat{s}$. Then there is a $b \in \mathrm{im}(\varphi)$ such that $\varphi(a) = b$ and $(i, b) \in \hat{s}$. $\hat{s} \subseteq s$, so $(i, b) \in s$, and thus $(i, a) \in \varphi^{-1}s$.

So $\varphi^{-1}\hat{s} \subseteq \varphi^{-1}s$.

Suppose $(i, a) \in \varphi^{-1}s$. Then there is an $b \in B$ such that $b = \varphi(a) \in \mathrm{im}(\varphi)$ and $(i, b) \in s$. $(i, b) \in I \times \mathrm{im}(\varphi)$, so $(i, b) \in \hat{s}$, and thus $(i, a) \in \varphi^{-1}\hat{s}$.

So $\varphi^{-1}s \subseteq \varphi^{-1}\hat{s}$.

So $\varphi^{-1}\hat{s} = \varphi^{-1}s \in \mathcal{R}$.

So φ is continuous between $(A, \mathcal{R})$ and $(\mathrm{im}(\varphi), \hat{\mathcal{S}})$.

Suppose $r \in \mathcal{R}$. φ is conservative, so there is an $s \in \mathcal{S}$ such that $\mathrm{im}(s) \subseteq \mathrm{im}(\varphi)$ and $\varphi^{-1}s = r$. Since $\mathrm{im}(s) \subseteq \mathrm{im}(\varphi)$, $s \cap (I \times \mathrm{im}(\varphi)) = s$, so $s \in \hat{\mathcal{S}}$.
So $s \in \hat{\mathcal{S}}$, $\mathrm{im}(s) \subseteq \mathrm{im}(\varphi)$, and $\varphi^{-1}s = r$.

So φ is conservative between $(A, \mathcal{R})$ and $(\mathrm{im}(\varphi), \hat{\mathcal{S}})$.

φ is both continuous and conservative between $(A, \mathcal{R})$ and $(\mathrm{im}(\varphi), \hat{\mathcal{S}})$, so φ is an I-structure cohomomorphism between $(A, \mathcal{R})$ and $(\mathrm{im}(\varphi), \hat{\mathcal{S}})$. □

Theorem 5.2. *Suppose each of $M = (A, \mathcal{R})$ and $N = (B, \mathcal{S})$ is an I-structure, and $\varphi : A \to B$ is a cohomomorphism. Then $M /\!\!/ \varphi$ is isomorphic to the understructure of N induced by $\mathrm{im}(\varphi)$.*

Proof: $M /\!\!/ \varphi = (A/\varphi, \bar{\mathcal{R}})$ where $\bar{\mathcal{R}}$ is the set of relations to which a relation $\bar{r}$ belongs if and only if $\bar{r} \subseteq I \times A/\varphi$ and $\pi_\varphi^{-1}\bar{r} \in \mathcal{R}$.

The understructure of N induced by $\mathrm{im}(\varphi)$ is $(\mathrm{im}(\varphi), \hat{\mathcal{S}})$ where $\hat{\mathcal{S}}$ is the set of relations to which a relation $\hat{s}$ belongs if and only if there is a relation $s \in \mathcal{S}$ such that $\hat{s} = s \cap (I \times \mathrm{im}(\varphi))$.

By Lemma 5.2.1, π_φ is an I-structure coepimorphism, by Lemma 5.2.2, φ is an I-structure cohomomorphism between $(A, \mathcal{R})$ and $(\mathrm{im}(\varphi), \hat{\mathcal{S}})$, and by Lemma 2.6, $\pi_\varphi^{-1}\pi_\varphi \subseteq \varphi^{-1}\varphi$. So by Theorem 4.2, $\varphi\pi_\varphi^{-1}$ is a cohomomorphism.

By Lemma 5.1.1, $\varphi\pi_\varphi^{-1}$ is a bijection, so by Theorem 1.21, $\varphi\pi_\varphi^{-1}$ is an isomorphism.

So $M /\!\!/ \varphi$ is isomorphic to the understructure of N induced by $\mathrm{im}(\varphi)$. □

Corollary 5.2.1. *Suppose $M = (A, \mathcal{R})$ is an I-structure and φ is a cohomomorphism between M and another structure. Then $M /\!\!/ \varphi$ is isomorphic to M/φ.*

Proof: By Theorem 1.20, φ is a homomorphism.

By Lemma 5.2.1, π_φ is a coepimorphism between M and $M /\!\!/ \varphi$ and (by Lemma 5.1.2) an epimorphism between M and M/φ. Moreover, $\pi_\varphi^{-1}\pi_\varphi \subseteq \pi_\varphi^{-1}\pi_\varphi$.

So by Theorem 4.1, $\pi_\varphi\pi_\varphi^{-1}$ is a homomorphism between $M /\!\!/ \varphi$ and M/φ.

π_φ is a surjection, so $\pi_\varphi\pi_\varphi^{-1} = 1_{A/\varphi}$, which is a bijection. So by Theorem 1.19, $1_{A/\varphi}$ is an isomorphism.

So $M /\!\!/ \varphi \cong M/\varphi$. □

Chapter 6

Second Isomorphism Theorem

Lemma 6.1.1. *Suppose A is a set, $B \subseteq A$, φ is a function with domain A, $\bar{B} = \varphi^{-1}\varphi(B)$, and $P \in \bar{B}/\varphi|_{\bar{B}}$. Then $P \cap B \in B/\varphi|_B$.*

Proof: Suppose $b \in P \cap B$.

Since $P \in \bar{B}/\varphi|_{\bar{B}}$ and $b \in P$, $P = (\varphi|_{\bar{B}})^{-1}((\varphi|_{\bar{B}})(\{b\}))$.

$b \in B$ so $b \in (\varphi|_B)^{-1}((\varphi|_B)(\{b\})) \in B/\varphi|_B$. So $P \cap B \subseteq (\varphi|_B)^{-1}((\varphi|_B)(\{b\})) \in B/\varphi|_B$.

Suppose $b' \in (\varphi|_B)^{-1}((\varphi|_B)(\{b\}))$. Then there is a c such that $(c, b') \in (\varphi|_B)^{-1}$ and $(b, c) \in \varphi|_B$, so $(b', c) \in \varphi|_B$, so $(b, c) \in \varphi$, $(b', c) \in \varphi$ and $b' \in B \subseteq \bar{B}$.

$(b, c) \in \varphi$ and $b \in \bar{B}$, so $(b, c) \in \varphi|_{\bar{B}}$. $(b', c) \in \varphi$ and $b' \in \bar{B}$, so $(b', c) \in \varphi|_{\bar{B}}$. So $b' \in (\varphi|_{\bar{B}})^{-1}((\varphi|_{\bar{B}})(\{b\})) = P$.

So $b' \in P \cap B$ and $P \cap B = (\varphi|_B)^{-1}((\varphi|_B)(\{b\})) \in B/\varphi|_B$. □

Lemma 6.1.2. *Suppose A is a set, $B \subseteq A$, φ is a function with domain A, $\bar{B} = \varphi^{-1}\varphi(B)$. Then the function $\psi : \bar{B}/\varphi|_{\bar{B}} \to B/\varphi|_B$ such that for each $P \in \bar{B}/\varphi|_{\bar{B}}$, $\psi(P) = P \cap B$, is a bijection.*

Proof: Suppose each of P and Q is in $\bar{B}/\varphi|_{\bar{B}}$, and $\psi(P) = \psi(Q)$.

$P \cap B = \psi(P) = \psi(Q) = Q \cap B$.

Since each of $P \cap B$ and $Q \cap B$ is in $B/\varphi|_B$, each is nonempty. So there is a $b \in P \cap B = Q \cap B$. $b \in P \cap B \subseteq P$ and $b \in Q \cap B \subseteq Q$. P and Q intersect, and $\bar{B}/\varphi|_{\bar{B}}$ is a partition, so $P = Q$.

So ψ is an injection.

Suppose $H \in B/\varphi|_B$. H is nonempty, so there is a $b \in H$ and $H = (\varphi|_B)^{-1}((\varphi|_B)(\{b\}))$.

$(b, \varphi(b)) \in \varphi$ and $b \in B \subseteq \bar{B}$, so $b \in (\varphi|_{\bar{B}})^{-1}((\varphi|_{\bar{B}})(\{b\})) \in \bar{B}/\varphi|_{\bar{B}}$.

$\psi((\varphi|_{\bar{B}})^{-1}((\varphi|_{\bar{B}})(\{b\}))) = (\varphi|_{\bar{B}})^{-1}((\varphi|_{\bar{B}})(\{b\})) \cap B$.

$b \in H = (\varphi|_B)^{-1}((\varphi|_B)(\{b\})) \subseteq B$ and $b \in (\varphi|_{\bar{B}})^{-1}((\varphi|_{\bar{B}})(\{b\}))$, so $b \in (\varphi|_{\bar{B}})^{-1}((\varphi|_{\bar{B}})(\{b\})) \cap B = \psi((\varphi|_{\bar{B}})^{-1}((\varphi|_{\bar{B}})(\{b\})))$

Since $B/\varphi|_B$ is a partition, and $p \in H \in B/\varphi|_B$ and $b \in \psi((\varphi|_{\bar{B}})^{-1}((\varphi|_{\bar{B}})(\{b\}))) \in B/\varphi|_B$, it must be the case that $H = \psi((\varphi|_{\bar{B}})^{-1}((\varphi|_{\bar{B}})(\{b\})))$.

So $(\varphi|_{\bar{B}})^{-1}((\varphi|_{\bar{B}})(\{b\}))$ is an element of $\bar{B}/\varphi|_{\bar{B}}$ such that $\psi((\varphi|_{\bar{B}})^{-1}((\varphi|_{\bar{B}})(\{b\}))) = H$.

So ψ is a surjection.

So ψ is a bijection. □

Theorem 6.1. *Suppose $M = (A, \mathcal{R})$ is an I-structure, $B \subseteq A$, φ is a function with domain A, $\bar{B} = \varphi^{-1}\varphi(B)$, $M|_{\bar{B}}/\varphi|_{\bar{B}} = (\bar{B}/\varphi_{\bar{B}}, \mathcal{S})$, $\psi : \bar{B}/\varphi|_{\bar{B}} \to B/\varphi|_B$ is the function such that for each $P \in \bar{B}/\varphi|_{\bar{B}}$, $\psi(P) = P \cap B$, and $N = (B/\varphi|_B, \mathcal{T})$ where $\mathcal{T}$ is the set of relations to* *which a relation t belongs if and only if there is a relation $s \in \mathcal{S}$ such that $\psi s = t$. Then $M|_{\bar{B}}/\varphi|_{\bar{B}} \cong N$.*

Proof:

1. ψ is preservative: By definition of N, if $s \in \mathcal{S}$, then $\psi s \in \mathcal{T}$.

2. ψ is saturating: By definition of N, if $t \in \mathcal{T}$, then there is a relation $s \in \mathcal{S}$ such that $\psi s = t$.

3. ψ is a bijection: By Lemma 6.1.2, ψ is a bijection.

So by Theorem 1.19, ψ is an isomorphism.

So $M|_{\bar{B}}/\varphi|_{\bar{B}} \cong N$. □

Chapter 7

Third Isomorphism Theorem

Definition Suppose A is a set, and each of α and β is a function with domain A such that $\alpha^{-1}\alpha \subseteq \beta^{-1}\beta$. Then define $\beta/\alpha : A/\alpha \to A/\beta$ such that if $P \in A/\alpha$, then $\beta/\alpha(P) = \pi_\beta \pi_\alpha^{-1}$.

Lemma 7.1.1. *Suppose $M = (A, \mathcal{R})$ is an I-structure, and each of α and β is a homomorphism with domain A such that $\alpha^{-1}\alpha \subseteq \beta^{-1}\beta$. Then β/α is an epimorphism between M/α and M/β.*

Proof: By Lemma 5.1.2, π_α is an epimorphism.

By Lemma 5.1.2, π_β is an epimorphism.

Suppose $P \in A/\alpha$.

$$\pi_\beta \pi_\alpha^{-1}(P) = \pi_\beta \pi_\alpha^{-1}(\pi_\alpha(P)) = \pi_\beta \pi_\alpha^{-1} \pi_\alpha(P) = \pi_\beta(P) = \beta^{-1}(\beta(P)) = \beta/\alpha(P)$$

So $\beta/\alpha = \pi_\beta \pi_\alpha^{-1}$.

So by Theorem 4.1 and Lemma 4.1.1, $\beta/\alpha = \pi_\beta \pi_\alpha^{-1}$ is an epimorphism between M/α and M/β. □

Lemma 7.1.2. *Suppose A is a set, and each of α and β is a function with domain A such that $\alpha^{-1}\alpha = \beta^{-1}\beta$. Then $A/\alpha = A/\beta$ and $\beta/\alpha = 1_{A/\alpha}$.*

Proof:

$$\begin{aligned} A/\alpha =& \{\alpha^{-1}(\alpha(\{a\})) \mid a \in A\} = \{\alpha^{-1}\alpha(\{a\}) \mid a \in A\} \\ =& \{\beta^{-1}\beta(\{a\}) \mid a \in A\} = \{\beta^{-1}(\beta(\{a\})) \mid a \in A\} = A/\beta \end{aligned}$$

So $A/\alpha = A/\beta$.

Suppose $P \in A/\alpha$.

$$\beta/\alpha(P) = \beta^{-1}(\beta(P)) = \beta^{-1}\beta(P) = \alpha^{-1}\alpha(P) = \alpha^{-1}(\alpha(P)) = P$$

So $\beta/\alpha = 1_{A/\alpha}$. □

Lemma 7.1.3. *Suppose each of A and B is a set, and each of α and $\beta : A \to B$ is a function with domain A such that $\alpha^{-1}\alpha \subseteq \beta^{-1}\beta$ and γ is a function with domain B. Then $\alpha^{-1}\alpha \subseteq (\gamma\beta)^{-1}\gamma\beta$.*

Proof:

$$\alpha^{-1}\alpha \subseteq \beta^{-1}\beta = \beta^{-1}1_B\beta \subseteq \beta^{-1}\gamma^{-1}\gamma\beta = (\gamma\beta)^{-1}\gamma\beta$$

So $\alpha^{-1}\alpha \subseteq (\gamma\beta)^{-1}\gamma\beta$. □

Lemma 7.1.4. *Suppose A is a set, and each of $\mathcal{P}$ and $\mathcal{Q}$ is a partition, and $\gamma : \mathcal{P} \to \mathcal{Q}$ is a surjection such that for each $P \in \mathcal{P}$, $P \subseteq \gamma(P)$. Suppose $\pi_1 : A \to \mathcal{P}$ is the function such that for each $a \in A$, $\pi_1(a)$ is the part in $\mathcal{P}$ to which a belongs, and $\pi_2 : A \to \mathcal{Q}$ is the function such that for each $a \in A$, $\pi_2(a)$ is the part in $\mathcal{Q}$ to which a belongs. Then $\pi_2 = \gamma\pi_1$.*

Proof: Suppose $a \in A$.

$$a \in \pi_1(a) \subseteq \gamma(\pi_1(a)) = \gamma\pi_1(a)$$

So $\gamma\pi_1(a)$ is the part in $\mathcal{Q}$ to which a belongs. So $\gamma\pi_1(a) = \pi_2(a)$.

Since this is true for all $a \in A$, $\pi_2 = \gamma\pi_1$. □

Lemma 7.1.5. *Suppose each of $M = (A, \mathcal{R})$, $N = (B, \mathcal{S})$, and $L = (\mathcal{P}, \mathcal{T})$ is an I-structure, where $\mathcal{P}$ is a partition of A, and $\alpha : A \to B$ is a homomorphism, and $\gamma : A/\alpha \to \mathcal{P}$ is an epimorphism between M/α and L such that for all $P \in A/\alpha$, $P \subseteq \gamma(P)$. Then there is a homomorphism β from M such that $\alpha^{-1}\alpha \subseteq \beta^{-1}\beta$ and $\gamma = \beta/\alpha$.*

Proof: Define $\beta : A \to \mathcal{P}$ such that $\beta = \gamma\pi_\alpha$.

Since α is a homomorphism, π_α is an epimorphism. γ is a homomorphism. $\beta = \gamma\pi_\alpha$ is the composition of a homomorphism with an epimorphism, so β is a homomorphism.

Suppose $(a_1, a_2) \in \alpha^{-1}\alpha$. By Lemma 2.6, $\alpha^{-1}\alpha = \pi_\alpha^{-1}\pi_\alpha$.

$$(a_1, a_2) \in \alpha^{-1}\alpha = \pi_\alpha^{-1}\pi_\alpha = \pi_\alpha^{-1}1_{A/\alpha}\pi_\alpha \subseteq \pi_\alpha^{-1}\gamma^{-1}\gamma\pi_\alpha = (\gamma\pi_\alpha)^{-1}\gamma\pi_\alpha = \beta^{-1}\beta$$

So $\alpha^{-1}\alpha \subseteq \beta^{-1}\beta$.

Define $\pi : A \to \mathcal{P}$ is the function such that for each $a \in A$, $\pi(a)$ is the part in $\mathcal{P}$ to which a belongs.

By Lemma 7.1.4, $\beta = \gamma\pi_\alpha = \pi$.

By Theorem 2.8, $\pi = \pi_\beta$.

So $\mathcal{P} = A/\beta$.

$\pi_\alpha^{-1}\pi_\alpha = \alpha^{-1}\alpha \subseteq \beta^{-1}\beta = \pi_\beta^{-1}\pi_\beta$.

So $\beta/\alpha = \pi_\beta\pi_\alpha^{-1}$ is the unique homomorphism such that $(\pi_\beta\pi_\alpha^{-1})\pi_\alpha = \pi_\beta = \beta$.

$\gamma\pi_\alpha = \beta$, so $\gamma = \pi_\beta\pi_\alpha^{-1} = \beta/\alpha$.

So β is a homomorphism from M such that $\alpha^{-1}\alpha \subseteq \beta^{-1}\beta$ and $\gamma = \beta/\alpha$. □

Theorem 7.1. *Suppose $M = (A, \mathcal{R})$ is an I-structure, and each of α and β is a homomorphism with domain A such that $\alpha^{-1}\alpha \subseteq \beta^{-1}\beta$. Then $M/\alpha \Big/ \beta/\alpha \cong M/\beta$.*

Proof: By Lemma 7.1.1 β/α is an epimorphism between M/α and M/β.

So by Theorem 5.1, $M/\alpha \Big/ \beta/\alpha \cong M/\beta$. □

Chapter 8

Correspondence Theorem

Definition Suppose A is a set, B is a subset of A, and φ is a function with domain A. The statement that B is φ *exact* means $B = \varphi^{-1}\varphi(B)$.

Definition Suppose $M = (A, \mathcal{R})$ is an I-structure, $N = (B, \mathcal{S})$ is an I-substructure of M, and φ is a function with domain A. The statement that N is φ *exact* means $B = \varphi^{-1}\varphi(B)$.

Lemma 8.1.1. *Suppose A is a set, f is a function with domain A, $B \subseteq A$ such that $B = f^{-1}f(B)$, and $b \in B$. Then $f|_B^{-1} f|_B(\{b\}) = f^{-1}f(\{b\})$.*

Proof:

$$\begin{aligned} & p \in f|_B^{-1} f|_B(\{b\}) \\ \iff & (b,p) \in f|_B^{-1} f|_B \\ \iff & p \in B = f^{-1}f(B) \text{ and } (b,p) \in f^{-1}f \\ \iff & p \in f^{-1}f(\{b\}) \quad \square \end{aligned}$$

Lemma 8.1.2. *Suppose A is a set, f is a function with domain A, $B \subseteq A$ such that $B = f^{-1}f(B)$, and $b \in B$. Then $\pi_{f|_B}(b) = \pi_f(b)$.*

Proof:

$$\pi_{f|_B}(b) = f|_B^{-1} f|_B(\{b\}) = f^{-1}f(\{b\}) = \pi_f(b) \quad \square$$

Lemma 8.1.3. *Suppose A is a set, f is a function with domain A, $B \subseteq A$ such that $B = f^{-1}f(B)$, r is a relation such that* $\operatorname{im}(r) \subseteq B/(\varphi|_B)$. *Then $\pi_{f|_B}^{-1} r = \pi_f^{-1} r$.*

Proof:

$$
\begin{aligned}
& (i,b) \in \pi_{f|_B}^{-1} r \\
\iff & \exists P \text{ such that } (i,P) \in r \text{ and } (P,b) \in \pi_{f|_B}^{-1} \\
\iff & (i, \pi_{f|_B}(b)) \in r \\
\iff & (i, \pi_f(b)) \in r \text{ and } b \in B \\
\iff & \exists P \text{ such that } (i,P) \in r \text{ and } (P,b) \in \pi_f^{-1} \text{ (so } P \in B/\varphi|_B) \\
\iff & (i,b) \in \pi_f^{-1} r
\end{aligned}
$$

So $\pi_{f|_B}^{-1} r = \pi_f^{-1} r$. □

Lemma 8.1.4. *Suppose $M = (A, \mathcal{R})$ is an I-structure, φ is a function with domain A, and $N = (B, \hat{\mathcal{R}})$ is a φ exact I-substructure of M. Then $N/(\varphi|_B) = (B/(\varphi|_B), \mathcal{T})$ is the I-substructure of $M/\varphi = (A/\varphi, \mathcal{S})$ induced by $B/(\varphi|_B)$.*

Proof: Suppose $P \in B/(\varphi|_B)$. Then $P = \varphi|_B^{-1}\varphi|_B(\{b\})$ for some b in B.

$$P = \varphi|_B^{-1}\varphi|_B(\{b\}) = \varphi^{-1}\varphi(\{b\}) \in A/\varphi$$

So $B/(\varphi|_B) \subseteq A/\varphi$.

Note $\mathcal{T}$ is the relation set to which a relation t belongs if and only if there is an $\hat{r} \in \hat{\mathcal{R}}$ such that $t = \pi_{\varphi|_B}\hat{r}$.

Suppose $(B/(\varphi|_B), \hat{\mathcal{S}})$ is the I-substructure of M/φ induced by $B/(\varphi|_B)$. Note $\hat{\mathcal{S}}$ is the relation set to which a relation $\hat{s}$ belongs if and only if $\hat{s} \in \mathcal{S}$ and $\operatorname{im}(\hat{s}) \subseteq B/\varphi_B$.

$$
\begin{aligned}
& k \in \mathcal{T} \\
\iff & \exists r \in \hat{\mathcal{R}} \text{ (so } \operatorname{im}(r) \subseteq B) \text{ such that } k = \pi_\varphi r = \pi_{\varphi|_B} r \\
\iff & \exists r \in \mathcal{R} \text{ such that } \pi_\varphi r = k \text{ and } \operatorname{im}(r) \subseteq B \\
\iff & \exists r \in \mathcal{R} \text{ such that } \pi_\varphi r = k \text{ and } \operatorname{im}(k) \subseteq B/(\varphi|_B) \\
& (\implies: \operatorname{im}(k) = \operatorname{im}(\pi_{\varphi|_B} r) = \pi_{\varphi|_B}(\operatorname{im}(r)) \subseteq \pi_{\varphi|_B}(B) = B/(\varphi|_B)) \\
& (\impliedby: \operatorname{im}(r) \subseteq \operatorname{im}(\pi_\varphi^{-1}\pi_\varphi r) = \operatorname{im}(\pi_\varphi^{-1} k) = \pi_\varphi^{-1}(\operatorname{im}(k)) \subseteq \pi_\varphi^{-1}(\pi_\varphi(B)) = \varphi^{-1}\varphi(B) = B) \\
\iff & k \in \mathcal{S} \text{ and } \operatorname{im}(k) \subseteq B/(\varphi|_B) \\
\iff & k \in \hat{\mathcal{S}}
\end{aligned}
$$

So $\mathcal{T} = \hat{\mathcal{S}}$, and $N/(\varphi|_B) = (B/(\varphi|_B), \mathcal{T}) = (B/(\varphi|_B), \hat{\mathcal{S}})$, the I-substructure of M/φ induced by $B/(\varphi|_B)$. □

Theorem 8.1. *Suppose $M = (A, \mathcal{R})$ is an I-structure, and φ is a function with domain A. Then there is a bijection between the set of φ exact I-substructures of M, and the set of I-substructures of M/φ.*

Proof: Suppose S is the set of φ exact I-substructures of M.

Suppose T is the set of I-substructures of M/φ.

Define $f : S \to T$ such that for each $N = (B, \hat{\mathcal{R}})$ in S, $f(N) = N/(\varphi|_B)$.

By Lemma 8.1.4, $f(N) \in T$.

1. f is an injection: Suppose each of $N_1 = (B_1, \mathcal{R}_1)$ and $N_2 = (B_2, \mathcal{R}_2)$ is in S, and $f(N_1) = f(N_2)$. Note each of N_1 and N_2 is φ exact, so $B_1 = \varphi^{-1}\varphi(B_1)$ and $B_2 =$

$\varphi^{-1}\varphi(B_2)$.

$$(B_1/(\varphi|_{B_1}), \hat{\mathcal{R}}_1) = N_1/(\varphi|_{B_1}) = f(N_1) = f(N_2) = N_2/(\varphi|_{B_2}) = (B_2/(\varphi|_{B_2}), \hat{\mathcal{R}}_2)$$

So $B_1/(\varphi|_{B_1}) = B_2/(\varphi|_{B_2})$.

$$\pi_\varphi(B_1) = \pi_{\varphi|_{B_1}}(B_1) = B_1/(\varphi|_{B_1}) = B_2/(\varphi|_{B_2}) = \pi_{\varphi|_{B_2}}(B_2) = \pi_\varphi(B_2)$$
$$\implies B_1 = \varphi^{-1}\varphi(B_1) = \pi_\varphi^{-1}\pi_\varphi(B_1) = \pi_\varphi^{-1}\pi_\varphi(B_2) = \varphi^{-1}\varphi(B_2) = B_2$$

$$\mathcal{R}_1 = \{r \mid r \in \mathcal{R} \text{ and } \operatorname{im}(r) \subseteq B_1\} = \{r \mid r \in \mathcal{R} \text{ and } \operatorname{im}(r) \subseteq B_2\} = \mathcal{R}_2$$

So $N_1 = (B_1, \mathcal{R}_1) = (B_2, \mathcal{R}_2) = N_2$.

So f is an injection.

2. f is a surjection: Suppose $(\mathcal{P}, \mathcal{T})$ is an I-substructure of $M/\varphi = (A/\varphi, \bar{\mathcal{R}})$ (namely the I-substructure of M/φ induced by $\mathcal{P}$).

   Then $\mathcal{P} \subseteq A/\varphi = \pi_\varphi(A)$ so $\pi_\varphi^{-1}(\mathcal{P}) \subseteq \pi_\varphi^{-1}(\pi_\varphi(A)) = A$.

   Define $B = \pi_\varphi^{-1}(\mathcal{P})$.

   Consider $L = (B, \mathcal{S})$ where $\mathcal{S}$ is the relation set to which a relation s belongs if and only if $s \in \mathcal{R}$ and $\operatorname{im}(s) \subseteq B$. So L is an I-substructure of M.

$$B = \pi_\varphi^{-1}(\mathcal{P}) = \pi_\varphi^{-1}\pi_\varphi\pi_\varphi^{-1}(\mathcal{P}) = \pi_\varphi^{-1}\pi_\varphi(\pi_\varphi^{-1}(\mathcal{P})) = \varphi^{-1}\varphi(\pi_\varphi^{-1}(\mathcal{P})) = \varphi^{-1}\varphi(B)$$

So L is φ exact.

So $L \in S$.

$$B/(\varphi|_B) = \pi_{\varphi|_B}(B) = \pi_\varphi(B) = \pi_\varphi(\pi_\varphi^{-1}(\mathcal{P})) = \mathcal{P}$$

By Lemma 8.1.4, $L/(\varphi|_B)$ is the I-substructure of M/φ induced by $B/(\varphi|_B) = \mathcal{P}$.

So $f(L) = L/(\varphi|_B) = (\mathcal{P}, \mathcal{T})$.

So f is a surjection.

Thus f is a bijection. □

Definition Suppose $M = (A, \mathcal{R})$ is an I-structure, $N = (B, \mathcal{S})$ is an understructure of M, and φ is a function with domain A. The statement that N is φ *exact* means $B = \varphi^{-1}\varphi(B)$.

Lemma 8.2.1. *Suppose each of A and B is a set, r is a relation, and $B \subseteq \mathrm{dom}(r)$. Then $r(A \times B) = A \times r(B)$.*

Proof:

$$\begin{aligned}
&(a, c) \in r(A \times B) \\
\iff &\exists b \text{ such that } (a, b) \in A \times B \text{ and } (b, c) \in r \\
\iff &a \in A \text{ and } \exists b \in B \subseteq \mathrm{dom}(r) \text{ such that } c \in r(\{b\}) \\
\iff &(a, c) \in A \times r(B)
\end{aligned}$$

So $r(A \times B) = A \times r(B)$. □

Lemma 8.2.2. *Suppose r is a relation, I is a set such that $\mathrm{dom}(r) \subseteq I$, φ is a function, and B is a set such that $B \subseteq \mathrm{dom}(\varphi)$ and B is φ exact. Then $\pi_\varphi(r \cap (I \times B)) = (\pi_\varphi r) \cap (\pi_\varphi(I \times B))$.*

Proof:

$$
\begin{aligned}
& (i,P) \in \pi_\varphi(r \cap (I \times B)) \\
\iff & \exists b \text{ such that } (b,P) \in \pi_\varphi \text{ and } (i,b) \in r \cap (I \times B) \\
\iff & \exists b \text{ such that } (b,P) \in \pi_\varphi \text{ and } (i,b) \in r \text{ and } (i,b) \in I \times B \\
\iff & \exists b_1, b_2 \text{ such that } (b_1,P) \in \pi_\varphi,\ (b_2,P) \in \pi_\varphi,\ (i,b_1) \in r, \text{and } (i,b_2) \in I \times B \\
& (\Longleftarrow: b_1 \in \varphi^{-1}\varphi(\{b_2\}) \subseteq \varphi^{-1}\varphi(B) = B \implies (i,b_1) \in I \times B) \\
\iff & (i,P) \in \pi_\varphi r \text{ and } (i,P) \in \pi_\varphi(I \times B) \\
\iff & (i,P) \in (\pi_\varphi r) \cap (\pi_\varphi(I \times B))
\end{aligned}
$$

So $\pi_\varphi(r \cap (I \times B)) = (\pi_\varphi r) \cap (\pi_\varphi(I \times B))$. □

Lemma 8.2.3. *Suppose each of f and g is a relation, I is a set such that $\mathrm{dom}(f) \subseteq I$ and $\mathrm{dom}(g) \subseteq I$, φ is a function such that $\mathrm{im}(f) \subseteq \mathrm{dom}(\varphi)/\varphi$ and $\mathrm{im}(g) \subseteq \mathrm{dom}(\varphi)/\varphi$, and B is a set such that $B \subseteq \mathrm{dom}(\varphi)$ and B is φ exact. Then $f = g \cap (I \times B/(\varphi|_B))$ if and only if $\pi_\varphi^{-1} f = (\pi_\varphi^{-1} g) \cap (I \times B)$.*

Proof: Suppose $f = g \cap (I \times B/(\varphi|_B))$.

$$
\begin{aligned}
\pi_\varphi^{-1} f &= \pi_\varphi^{-1}(g \cap (I \times B/(\varphi|_B))) = (\pi_\varphi^{-1} g) \cap (\pi_\varphi^{-1}(I \times \pi_{\varphi|_B}(B))) = (\pi_\varphi^{-1} g) \cap (\pi_\varphi^{-1}(I \times \pi_\varphi(B))) \\
&= (\pi_\varphi^{-1} g) \cap (I \times \pi_\varphi^{-1}\pi_\varphi(B)) = (\pi_\varphi^{-1} g) \cap (I \times \varphi^{-1}\varphi(B)) = (\pi_\varphi^{-1} g) \cap (I \times B)
\end{aligned}
$$

So $\pi_\varphi^{-1} f = (\pi_\varphi^{-1} g) \cap (I \times B)$. □

Suppose $\pi_\varphi^{-1} f = (\pi_\varphi^{-1} g) \cap (I \times B)$.

$$
\begin{aligned}
f &= \pi_\varphi \pi_\varphi^{-1} f = \pi_\varphi((\pi_\varphi^{-1} g) \cap (I \times B)) = (\pi_\varphi \pi_\varphi^{-1} g) \cap (\pi_\varphi(I \times B)) = g \cap (I \times \pi_\varphi(B)) \\
&= g \cap (I \times \pi_{\varphi|_B}(B)) = g \cap (I \times B/(\varphi|_B))
\end{aligned}
$$

So $f = g \cap (I \times B/(\varphi|_B))$. $\square$

Lemma 8.2.4. *Suppose* $M = (A, \mathcal{R})$ *is an* I*-structure,* φ *is a cohomomorphism from* M*, and* $N = (B, \hat{\mathcal{R}})$ *is a* φ *exact understructure of* M*. Then* $N /\!\!/ \varphi|_B = (B/(\varphi|_B), \mathcal{T})$ *is the understructure of* $M /\!\!/ \varphi = (A/\varphi, \mathcal{S})$ *induced by* $B/(\varphi|_B)$*.*

Proof: Suppose $P \in B/(\varphi|_B)$. Then $P = \varphi|_B^{-1}\varphi|_B(\{b\})$ for some b in B.

$$P = \varphi|_B^{-1}\varphi|_B(\{b\}) = \varphi^{-1}\varphi(\{b\}) \in A/\varphi$$

So $B/(\varphi|_B) \subseteq A/\varphi$.

Note $\mathcal{T}$ is the relation set to which a relation t belongs if and only if $t \subseteq I \times B/(\varphi|_B)$ and $\pi_{\varphi|_B}^{-1} t \in \hat{\mathcal{R}}$.

Suppose $(B/(\varphi|_B), \hat{\mathcal{S}})$ is the understructure of $M /\!\!/ \varphi$ induced by $B/(\varphi|_B)$. Note $\hat{\mathcal{S}}$ is the relation set to which a relation $\hat{s}$ belongs if and only if there is an $s \in \mathcal{S}$ such that $\hat{s} = s \cap (I \times B/\varphi_B)$.

$$\begin{aligned}
& k \in \hat{\mathcal{S}} \\
\iff & \exists s \in \mathcal{S} \text{ such that } k = s \cap (I \times B/(\varphi|_B)) \\
\iff & \exists s \in \mathcal{S} \text{ such that } \pi_\varphi^{-1} s \cap (I \times B) = \pi_\varphi^{-1} k \\
\iff & \exists r \in \mathcal{R} \text{ such that } r \cap (I \times B) = \pi_\varphi^{-1} k \\
\iff & \pi_\varphi^{-1} k \in \hat{\mathcal{R}} \text{ and } \pi_\varphi^{-1} k \subseteq I \times B \\
\iff & \pi_{\varphi|_B}^{-1} k \in \hat{\mathcal{R}} \text{ and } k \subseteq I \times B/(\varphi|_B) \\
\iff & k \in \mathcal{T}
\end{aligned}$$

So $\hat{\mathcal{S}} = \mathcal{T}$, and $N /\!\!/ \varphi|_B = (B/(\varphi|_B), \mathcal{T}) = (B/(\varphi|_B), \hat{\mathcal{S}})$, the understructure of $M /\!\!/ \varphi$ induced by $B/(\varphi|_B)$. $\square$

Example Suppose $M = (A, \mathcal{R})$ is an I-structure, φ is a function with domain A, and $N = (B, \hat{\mathcal{R}})$ is a φ exact understructure of M. Then $N /\!\!/ \varphi|_B = (B/(\varphi|_B), \mathcal{T})$ is not necessarily the understructure of $M /\!\!/ \varphi = (A/\varphi, \mathcal{S})$ induced by $B/(\varphi|_B)$.

Proof: Define the following:

$M = (\{a_1, a_2, b\}, \{0\}, \{\{(0, a_1), (0, b)\}\})$

$B = \{b\}$

$\varphi = \{(a_1, x), (a_2, x), (b, y)\}$

$N = (\{b\}, \{0\}, \{\{(0, b)\}\})$, a φ exact understructure of M

Then $\pi_\varphi = \{(a_1, \{a_1, a_2\}), (a_2, \{a_1, a_2\}), (b, \{b\})\}$

$M /\!\!/ \varphi = (\{\{a_1, a_2\}, \{b\}\}, \{0\}, \varnothing)$

$\varphi|_B = \{(0, b)\}$

$B/(\varphi|_B) = \{\{b\}\}$

$N /\!\!/ (\varphi|_B) = (\{\{b\}\}, \{0\}, \{\{(0, \{b\})\}\})$

$(\{\{b\}\}, \{0\}, \varnothing)$ is the understructure of $M /\!\!/ \varphi$ induced by $B/(\varphi|_B)$.

$N /\!\!/ \varphi|_B \neq (\{\{b\}\}, \{0\}, \varnothing)$ □

Theorem 8.2. *Suppose $M = (A, \mathcal{R})$ is an I-structure, and φ is a cohomomorphism from M. Then there is a bijection between the set of φ exact understructures of M, and the set of understructures of $M /\!\!/ \varphi$.*

Proof: Suppose S is the set of φ exact understructures of M.

Suppose T is the set of understructures of $M /\!\!/ \varphi$.

Define $f : S \to T$ such that for each $N = (B, \hat{\mathcal{R}})$ in S, $f(N) = N /\!\!/ (\varphi|_B)$.

By Lemma 8.2.4, $f(N) \in T$.

1. f is an injection: Suppose each of $N_1 = (B_1, \mathcal{R}_1)$ and $N_2 = (B_2, \mathcal{R}_2)$ is in S, and $f(N_1) = f(N_2)$. Note each of N_1 and N_2 is φ exact, so $B_1 = \varphi^{-1}\varphi(B_1)$ and $B_2 = \varphi^{-1}\varphi(B_2)$.

$$(B_1/(\varphi|_{B_1}), \hat{\mathcal{R}}_1) = N_1/\!\!/(\varphi|_{B_1}) = f(N_1) = f(N_2) = N_2/\!\!/(\varphi|_{B_2}) = (B_2/(\varphi|_{B_2}), \hat{\mathcal{R}}_2)$$

So $B_1/(\varphi|_{B_1}) = B_2/(\varphi|_{B_2})$.

$$\begin{aligned} \pi_\varphi(B_1) = \pi_{\varphi|_{B_1}}(B_1) = B_1/(\varphi|_{B_1}) &= B_2/(\varphi|_{B_2}) = \pi_{\varphi|_{B_2}}(B_2) = \pi_\varphi(B_2) \\ \implies B_1 = \varphi^{-1}\varphi(B_1) = \pi_\varphi^{-1}\pi_\varphi(B_1) &= \pi_\varphi^{-1}\pi_\varphi(B_2) = \varphi^{-1}\varphi(B_2) = B_2 \end{aligned}$$

$$\mathcal{R}_1 = \{\hat{r} \mid \exists r \in \mathcal{R} \text{ such that } \hat{r} = r \cap (I \times B_1)\} = \{\hat{r} \mid \exists r \in \mathcal{R} \text{ such that } \hat{r} = r \cap (I \times B_2)\} = \mathcal{R}_2$$

So $N_1 = (B_1, \mathcal{R}_1) = (B_2, \mathcal{R}_2) = N_2$.

So f is an injection.

2. f is a surjection: Suppose $(\mathcal{P}, \mathcal{T})$ is an understructure of $M/\!\!/\varphi = (A/\varphi, \bar{\mathcal{R}})$ (namely, the understructure of $M/\!\!/\varphi$ induced by $\mathcal{P}$).

Then $\mathcal{P} \subseteq A/\varphi = \pi_\varphi(A)$ so $\pi_\varphi^{-1}(\mathcal{P}) \subseteq \pi_\varphi^{-1}(\pi_\varphi(A)) = A$.

Define $B = \pi_\varphi^{-1}(\mathcal{P})$.

Consider $L = (B, \mathcal{S})$ where $\mathcal{S}$ is the relation set to which a relation s belongs if and only if there is an $r \in \mathcal{R}$ such that $s = r \cap (I \times B)$. So L is an understructure of

M.

$$B = \pi_\varphi^{-1}(\mathcal{P}) = \pi_\varphi^{-1}\pi_\varphi\pi_\varphi^{-1}(\mathcal{P}) = \pi_\varphi^{-1}\pi_\varphi(\pi_\varphi^{-1}(\mathcal{P})) = \varphi^{-1}\varphi(\pi_\varphi^{-1}(\mathcal{P})) = \varphi^{-1}\varphi(B)$$

So L is φ exact.

So $L \in S$.

$$B/(\varphi|_B) = \pi_{\varphi|_B}(B) = \pi_\varphi(B) = \pi_\varphi(\pi_\varphi^{-1}(\mathcal{P})) = \mathcal{P}$$

By Lemma 8.2.4, $L /\!\!/ (\varphi|_B)$ is the understructure of $M /\!\!/ \varphi$ induced by $B/(\varphi|_B) = \mathcal{P}$.

So $f(L) = L /\!\!/ (\varphi|_B) = (\mathcal{P}, \mathcal{T})$.

So f is a surjection.

Thus f is a bijection. $\square$

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