Space-Time Fractional Cauchy Problems and Trace Estimates for Relativistic Stable Processes

by

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Abstract

Fractional derivatives can be used to model time delays in a diffusion process. When the order of the fractional derivative is distributed over the unit interval, it is useful for modeling a mixture of delay sources. In some special cases distributed order derivative can be used to model ultra-slow diffusion. In the fist part of the thesis, we extend the results of Baeumer and Meerschaert [3] in the single order fractional derivative case to distributed order fractional derivative case. In particular, we develop the strong analytic solutions of distributed order fractional Cauchy problems.

In this thesis, we also study the asymptotic behavior of the trace of the semigroup of a killed relativistic α-stable process in any bounded $R-$smooth boundary open set. More precisely, we establish two-term estimates of the trace with an error bound of $e^{2mt}t^{(2-d)/\alpha}$. When $m = 0$, our result reduces to the result established by Bañuelos and Kulczycki for stable processes given in [7].

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I dedicate this thesis to

my wife, my mother, my father, my brothers and sisters

for their constant support and unconditional love.

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Chapter 1

Introduction

In this chapter, we give an introduction to Cauchy problems and relativistic $\alpha-$stable processes.

1.1 Cauchy problems

Cauchy problems $\frac{\partial u}{\partial t} = Lu$ model diffusion processes and have appeared as an essential tool for the study of dynamics of various complex stochastic processes arising in anomalous diffusion in physics [39, 50], finance [21], hydrology [11], and cell biology [46]. Complexity includes phenomena such as the presence of weak or strong correlations, different sub-or super-diffusive modes, and jump effects. For example, experimental studies of the motion of macromolecules in a cell membrane show apparent subdiffusive motion with several simultaneous diffusive modes (see [46]). When $L = \Delta = \sum_j \partial^2 u/\partial x_j^2$, Cauchy problem is a tradition diffusion equation.

Traditional diffusion represents the long-time limit of a random walk, where finite variance jumps occur at regularly spaced intervals. Eventually, after each particle makes a series of random steps, a histogram of particle locations follows a bell-shaped normal density. The central limit theorem of probability ensures that this same bell-shaped curve will eventually emerge from any random walk with finite variance jumps, so that this diffusion model can be considered universal. The random walk limit is a Brownian motion, whose probability densities solve the diffusion equation.

The fractional Cauchy problem $\partial^\beta u/\partial t^\beta = Lu$ with $0 < \beta < 1$ models anomalous sub-diffusion, in which a cloud of particles spreads slower than the square root of time. The "particles" might be pollutants in ground water, stock prices, sound waves, proteins crossing a cell boundary, or animals invading a new ecosystem. When $L = \Delta$, the solution $u(t,x)$ is the density of a time-changed Brownian motion $B(E(t))$, where the non-Markovian time change $E(t) = \inf\{\tau > 0; D(\tau) > t\}$ is the inverse, or first passage time of a stable subordinator $D(t)$ with index β.

The process $B(E(t))$ is the long-time scaling limit of a random walk [31, 32], when the random waiting times between jumps belong to the β-stable domain of attraction. Roughly speaking, a power-law distribution of waiting times leads to a fractional time derivative in the governing equation. Recently, Barlow and Černý [9] obtained $B(E(t))$ as the scaling limit of a random walk in a random environment. More generally, for a uniformly elliptic operator L on a bounded domain $D \subset \mathbb{R}^d$, under suitable technical conditions and assuming Dirichlet boundary conditions, the diffusion equation $\partial u/\partial t = Lu$ governs a Markov process $Y(t)$ killed at the boundary, and the corresponding fractional diffusion equation $\partial^\beta u/\partial t^\beta = Lu$ governs the time-changed process $Y(E(t))$ [36].

In some applications, waiting times between particle jumps evolve according to a more complicated process, which cannot be adequately described by a single power law. A mixture of power laws leads to a distributed-order fractional derivative in time [16, 28, 29, 30, 34, 41]. An important application of distributed-order diffusions is to model ultraslow diffusion where a plume of particles spreads at a logarithmic rate [34, 47]. This thesis considers the distributed-order time-fractional diffusion equations with the generator L of a uniformly bounded and strongly continuous semigroup in a Banach space. Hahn et al. [22] discussed the solutions of such equations on $\mathbb{R}^d$, and the connections with certain subordinated processes. Kochubei [25] proved strong solutions on $\mathbb{R}^d$ for the case $L = \Delta$. Luchko [27] proved the uniqueness and continuous dependence on initial conditions on bounded domains. Meerschaert et al. [37] established the strong solutions of distributed order fractional Cauchy problems in bounded domains with Dirichlet boundary conditions.

When L is the generator of a uniformly bounded and continuous semigroup on a Banach spaces, Baeumer and Meerschaert [3] showed that the solution of

$$\partial^\beta u/\partial t^\beta = Lu$$

is analytic in a sectorial region. A similar problem has been considered in the literature on a purely analytic level, without a probabilistic interpretation of the subordination representation: see, for example, Prüss [44, Corollary 4.5] . The case of a single fractional order was also considered by Bazhlekova [10]. In this thesis, we extend the results of Baeumer and Meerschaert [3] to distributed order fractional diffusion case. Our proofs work for operators L that are generators of uniformly bounded and continuous semigroups on Banach spaces. Our result for this case is given in chapter 3.

In the next few paragraphs we introduce basic facts about relativistic stable processes.

1.2 Relativistic stable processes

In Ryznar [45] Green function estimates of the Schödinger operator with the free Hamiltonian of the form

$$(-\Delta + m^{2/\alpha})^{\alpha/2} - m,$$

were investigated, where $m > 0$ and Δ is the Laplace operator acting on $L^2(\mathbb{R}^d)$. Some of these estimates (see Lemma 2.13 below) and (essentially the same) proof in Bañuelos and Kulczycki (2008) can be used to provide an extension of the asymptotics in [7] to the relativistic α stable processes for any $0 < \alpha < 2$.

An $\mathbb{R}^d$-valued process with independent, stationary increments having the following characteristic function

$$\mathbb{E}e^{i\xi\cdot X_t^{\alpha,m}} = e^{-t\{(m^{2/\alpha}+|\xi|^2)^{\alpha/2}-m\}}, \quad \xi \in \mathbb{R}^d$$

is called relativistic α-stable process with mass m. We assume that sample paths of $X_t^{\alpha,m}$ are right continuous and have left-hand limits a.s. If we put $m = 0$ we obtain the symmetric rotation invariant $\alpha-$stable process with the characteristic function $e^{-t|\xi|^\alpha}, \xi \in \mathbb{R}^d$. We refer to this process as standard $\alpha-$stable process. The infinitesimal generator of $X_t^{\alpha,m}$ is $m - (m^{2/\alpha} - \Delta)^{\alpha/2}$, which is a non-local operator. Note that when $m = 1$, this infinitesimal generator reduces to $1 - (1 - \Delta)^{\alpha/2}$. Thus the 1$-$resolvent kernel of the relativistic $\alpha-$stable process $X_t^{\alpha,1}$ on $\mathbb{R}^d$ is just the Bessel potential kernel. When $\alpha = 1$, the infinitesimal generator reduces to the so-called free relativistic Hamiltonian $m - \sqrt{-\Delta + m^2}$. The operator $m - \sqrt{-\Delta + m^2}$ is very important in mathematical physics due to its application to relativistic quantum mechanics. For the rest of the thesis we keep α, m and $d \geq 2$ fixed and drop α, m in the notation, when it does not lead to confusion. Hence from now on the relativistic α-stable process is denoted by X_t and its standard $\alpha-$ stable counterpart by $\tilde{X}_t$. We keep this notational convention consistently throughout the paper, e.g., if $p_t(x - y)$ is the transition density of X_t then $\tilde{p}_t(x - y)$ is the transition density of $\tilde{X}_t$.

Brownian motion has characteristic function

$$\mathbb{E}^0 e^{i\xi \cdot B_t} = e^{-t|\xi|^2}, \quad \xi \in \mathbb{R}^d$$

Let $\alpha = 2\beta$. Ryznar showed that X_t is subordinated to Brownian motion. Let $T_\beta(t),\ t > 0$, denote the strictly β-stable subordinator with the following Laplace transform

$$\mathbb{E}^0 e^{-\lambda T_\beta(t)} = e^{-t\lambda^\beta}, \quad \lambda > 0. \tag{1.1}$$

Let $\theta_\beta(t, u),\ u > 0$, denote the density function of $T_\beta(t)$. Then the process $B_{T_\beta(t)}$ is the standard symmetric α-stable process.

Ryznar [45, Lemma 1] showed that we can obtain $X_t = B_{T_\beta(t,m)}$, where $T_\beta(t, m)$ is a positive infinitely divisible process with stationary increments with probability density function

$$\theta_\beta(t,u,m) = e^{-m^{1/\beta}u+mt}\theta_\beta(t,u), \quad u>0.$$

Transition density of $T_\beta(t,m)$ is given by $\theta_\beta(t,u-v,m)$. Hence the transition density of X_t is given by

$$p(t,x) = e^{mt}\int_0^\infty \frac{1}{(4\pi u)^{d/2}} e^{\frac{-|x|^2}{4u}} e^{-m^{1/\beta}u}\theta_\beta(t,u)du \tag{1.2}$$

Then $p(t,x,y) = p(t,x-y)$. Since the transition density is obtained from the characteristic function by inverse Fourier transform, it follows that $p(t,x)$ is a radially symmetric decreasing function and that

$$p(t,x) \le p(t,0) \le e^{mt}\int_0^\infty \frac{1}{(4\pi u)^{d/2}}\theta_\beta(t,u)du = e^{mt}t^{-d/\alpha}\frac{\omega_d\Gamma(d/\alpha)}{(2\pi)^d\alpha} \tag{1.3}$$

where $\omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ is the surface area of the unit sphere in $\mathbb{R}^d$. For $A \subset \mathbb{R}^d$ we define the first exit time from A by $\tau_A = \inf\{t \ge 0 : \quad X_t \notin A\}$.

Let $D \subset \mathbb{R}^d$ be a domain.

We set

$$r_D(t,x,y) = \mathbb{E}^x[p(t-\tau_D, X_{\tau_D}, y); \tau_D < t] \tag{1.4}$$

and

$$p_D(t,x,y) = p(t,x,y) - r_D(t,x,y) \tag{1.5}$$

for any $x, y \in \mathbb{R}^d$, $t > 0$. For a nonnegative Borel function f and $t > 0$, let

$$P_t^D f(x) = \mathbb{E}^x[f(X_t) : \ t < \tau_D] = \int_D p_D(t,x,y)f(y)dy$$

be the semigroup of the killed process acting on $L^2(D)$, see, Ryznar [45, Theorem 1].

Let D be a bounded domain (or of finite volume). Then the operator P_t^D is a Hilbert-Schmidt operator mapping $L^2(D)$ into $L^\infty(D)$ for every $t > 0$. This follows from (1.3), (1.4), and the general theory of heat semigroups as described in [18]. It follows that there exists an orthonormal basis of eigenfunctions $\{\varphi_n: \ n = 1, \ 2, \ 3, \cdots\}$ for $L^2(D)$ and corresponding eigenvalues $\{\lambda_n: \ n = 1, \ 2, \ 3, \cdots\}$ of the generator of the semigroup P_t^D satisfying

$$\lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$$

with $\lambda_n \to \infty$ as $n \to \infty$. By definition, the pair $\{\varphi_n, \lambda_n\}$ satisfies

$$P_t^D \varphi_n(x) = e^{-\lambda_n t}\varphi_n(x), \quad x \in D, \ t > 0.$$

Under such assumptions we also have

$$p_D(t, x, y) = \sum_{n=1}^{\infty} e^{-\lambda_n t}\varphi_n(x)\varphi_n(y) \tag{1.6}$$

In this thesis we are interested in the behavior of the trace of this semigroup defined by

$$Z_D(t) = \int_D p_D(t, x, x)dx. \tag{1.7}$$

Because of (1.6) we can write (5.9) as

$$Z_D(t) = \sum_{n=1}^{\infty} e^{-\lambda_n t} \int_D \varphi_n^2(x)dx = \sum_{n=1}^{\infty} e^{-\lambda_n t}. \tag{1.8}$$

We denote d-dimensional volume of D by $|D|$. It is shown in [7] that for any open set D with finite volume, it holds that

$$\lim_{t\to 0} t^{d/\alpha}\tilde{Z}_D(t) = C_1|D|, \qquad C_1 = \frac{\omega_d \Gamma(d/\alpha)}{(2\pi)^d \alpha} \tag{1.9}$$

This is closely related to the growth of the eigenvalues of $\tilde{P}_t^D$. Let $N(\lambda)$ be the number of eigenvalues $\{\lambda_j\}$ of $\tilde{P}_t^D$ which do not exceed λ, it follows from the classical Tauberian theorem (see for example [20], p.445 Theorem 2) that

$$\lim_{\lambda\to\infty} \lambda^{-d/\alpha} N(\lambda) = \frac{C_1|D|}{\Gamma(1+d/\alpha)} \tag{1.10}$$

This is the analogue for killed stable processes of the celebrated Weyl's asymptotic formula for the eigenvalues of the Laplacian. We will prove later in (2.38) that similar formula is true for relativistic stable processes.

Definition 1.1. The boundary, ∂D, of an open set D in $\mathbb{R}^d$ is said to be $R-$smooth if for each point $x_0 \in \partial D$ there are two open balls B_1 and B_2 with radii R such that $B_1 \subset D, B_2 \subset \mathbb{R}^d \backslash (D \cup \partial D)$ and $\partial B_1 \cap \partial B_2 = x_0$.

The asymptotic for the trace of the heat kernel when $\alpha = 2$ (the case of the Laplacian with Dirichlet boundary condition in a domain of $\mathbb{R}^d$), have been extensively studied by many authors. The van den Berg [49] result states that under the $R-$ smoothness condition when $\alpha = 2$,

$$\left| Z_D(t) - (4\pi t)^{-d/2} \left(|D| - \frac{\sqrt{\pi t}}{2} |\partial D| \right) \right| \leq \frac{C_d |D| t^{1-d/2}}{R^2}, t > 0. \tag{1.11}$$

For domains with C^1 boundaries the result

$$Z_D(t) = (4\pi t)^{-d/2} \left(|D| - \frac{\sqrt{\pi t}}{2} |\partial D| + o(t^{1/2}) \right) \tag{1.12}$$

was proved by Brossard and Carmona [13]. The asymptotic behavior for the trace of killed symmetric $\alpha-$stable processes, $\alpha \in (0,2)$, for an open bounded set with $R-$smooth boundary was given in [7]

$$\left| \tilde{Z}_D(t) - \frac{C_1|D|}{t^{d/\alpha}} + \frac{C_2 |\partial D| t^{1/\alpha}}{t^{d/\alpha}} \right| \leq \frac{C_3 |D| t^{2/\alpha}}{R^2 t^{d/\alpha}} \tag{1.13}$$

where C_1, C_2, and C_3 are some constants depending on d, α, X. In [8], the authors proved that, for any bounded Lipschitz domain D, $\tilde{Z}_D(t)$ satisfies

$$t^{d/\alpha}\tilde{Z}_D(t) = C_1|D| - C_2\mathcal{H}^{d-1}(\partial D)t^{1/\alpha} + o(t^{1/\alpha}) \tag{1.14}$$

where C_1, C_2 are some constants depending on d, α and $\mathcal{H}^{d-1}(\partial D)$ denote the $(d-1)$ dimensional Hausdorff measure of ∂D.

In the second part of the thesis we obtained the second term in the asymptotics of $Z_D(t)$ for bounded open set with $R-$smooth boundary. Our result is inspired by result for Trace estimates for stable processes by Bañuelos and Kulczycki [7].

1.3 Outline of Dissertation

This dissertation is divided into two more or less independent parts. The first part is dedicated to finding strong analytic solution of distributed order fractional Cauchy problems. Whereas the second part is more concerned with finding two-term trace estimate of relativistic $\alpha-$stable processes on bounded $R-$smooth open set.

Chapter 2 consists of preliminary mathematical material, which serves the purpose of setting up the vocabulary and the framework for the rest of the dissertation. In Section 2.1 we discuss the basic facts of traditional diffusion model and shows that the solution of a diffusion equation is a density of a Brownian motion. Section 2.2 and Section 2.2.2 discusses fractional calculus. We give a generator form, Caputo, and Riemann-Liouville form of fractional derivative with some simple examples. In Section 2.4 we state a defining property of a semigroup. There we give all of the results on a semigroup we shall use in the proof our main results in chapter 3. Finally in Section 2.5 we present the basic facts about relativistic α-stable process. There we give all of the results on a relativistic stable process we shall use in the proof of our main result for such process in chapter 4. We also give our first result in proposition 2.9, which is the analogue for relativistic stable process of the celebrated Weyl's asymptotic formula for the eigenvalues of the Laplacian.

In chapter 3 we state and prove our two main results about distributed order time fractional Cauchy problems. Our proofs work for operators L that are generators of uniformly bounded and continuous semigroups on Banach spaces. This chapter describes work contained in paper [40].

Chapter 4 discusses a trace estimates for relativistic stable process on open bounded domain with $R-$smooth boundary. There we state and proof our main result. Similar result is obtained for stable process in [7].

Chapter 5 is the last chapter and discusses a basic facts about sum of two independent stable processes. So far we are only able to find the first term asymptotic expansion of trace of such process. I am still looking in finding the best estimate for the trace on $R-$smooth boundary domains and then extend to Lipschitz domains if possible.

Chapter 2

Preliminaries

In this chapter, we summarize some results from fractional calculus and relativistic $\alpha-$stable processes. We mainly focus on trandional diffusion model, fractional diffusion models and relativistic $\alpha-$stable processes. Results in the following sections are mainly adapted from [33], [42] and [43].

2.1 Diffusion Model

The traditional model for diffusion combines elements of probability, differential equations, and physics. A random walk provides the basic physical model of particle motion. The central limit theorem gives convergence to a Brownian motion, whose probability densities solves the diffusion equation. We start with a sequence of independent and identically distributed (iid) random variables $Y_1, Y_2, \cdots$ that represent the jumps of a randomly selected particle. The *random walk*

$$S_n = Y_1 + Y_2 + \cdots + Y_n$$

gives the location of that particle after n jumps. Next we recall the well-known *central limit theorem,* which shows that the probability distribution of S_n converges to a normal limit. Here we sketch the argument in the simplest case, using Fourier transforms. For complete proof of the central theorem and to see the same normal limit governs a somewhat broader class of random walk (see [33], Theorem 3.5 and 4.5).

Let $F(x) = \mathbb{P}[Y \leq x]$ denote the cumulative distribution function (cdf) of the jumps, and assume that the probability density function (pdf) $f(x) = F'(x)$ exists. Then we have

$$\mathbb{P}[a \leq Y \leq b] = \int_a^b f(x)\,dx$$

for any real numbers $a < b$. The moments of this distribution are given by

$$\mu_l = \int_{-\infty}^{\infty} x^l f(x)\,dx$$

The *Fourier transform* (FT) of the pdf is

$$\hat{f}(k) = \mathbb{E}[e^{-ikY}] = \int_{-\infty}^{\infty} e^{-ikx} f(x)\,dx$$

The FT is closely related to the *characteristic function* $\mathbb{E}[e^{ikY}] = \hat{f}(-k)$. If the first two moments exist, a Taylor series expansion $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ leads to

$$\hat{f}(k) = \int_{-\infty}^{\infty} \left(1 - ikx + \frac{1}{2!}(-ikx)^2 + \cdots\right) f(x)\,dx = 1 - ik\mu_1 - \frac{1}{2}k^2\mu_2 + o(k^2) \tag{2.1}$$

since $\int f(x)\,dx = 1$. Here $o(k^2)$ denotes a function that tends to zero faster than k^2 as $k \to 0$. For a formal proof of (2.1) (see [33], p.7).

Suppose $\mu_1 = 0$ and $\mu_2 = 2$, i.e., the jumps have mean zero and variance 2. Then we have

$$\hat{f}(k) = 1 - k^2 + o(k^2)$$

as $k \to 0$. The sum $S_n = Y_1 + Y_2 + \cdots + Y_n$ has FT

$$\begin{aligned}
\mathbb{E}[e^{-ikS_n}] &= \mathbb{E}\left[e^{-ik(Y_1+Y_2+\cdots+Y_n)}\right] \\
&= \mathbb{E}\left[e^{-ikY_1}\right]\mathbb{E}\left[e^{-ikY_2}\right]\cdots\mathbb{E}\left[e^{-ikY_n}\right] \\
&= \mathbb{E}\left[e^{-ikY}\right]^n = \hat{f}(k)^n
\end{aligned}$$

and so the normalized sum $n^{-1/2}S_n$ has FT

$$\begin{aligned}\mathbb{E}\left[e^{-ikn^{-1/2}S_n}\right] &= \hat{f}(n^{-1/2}k)^n \\ &= \left(1-\frac{k^2}{n}+o(n^{-1})\right)^n \to e^{-k^2}\end{aligned} \tag{2.2}$$

since the general fact that $(1+(r/n)+o(n^{-1}))^n \to e^r$ as $n\to\infty$ for any $r\in\mathbb{R}$ (see [33], p. 7). The limit

$$e^{-k^2} = \mathbb{E}\left[e^{-ikZ}\right] = \int_{-\infty}^{\infty} e^{-ikx}\frac{1}{\sqrt{4\pi}}e^{-x^2/4}\,dx$$

using the standard formula from FT tables [48, p. 524]. The continuity theorem for FT [33, p. 8] yields the traditional central limit theorem (CLT):

$$n^{-1/2}S_n = \frac{Y_1+\cdots+Y_n}{\sqrt{n}} \Rightarrow Z \tag{2.3}$$

where $\Rightarrow$ indicates convergence in distribution. The limit Z in (2.3) is normal with mean zero and variance 2.

An easy extension of this argument gives convergence of the rescaled random walk:

$$S_{[ct]} = Y_1+Y_2+\cdots+Y_{[ct]}$$

gives the particle location at time $t>0$ at any time scale $c>0$. Increasing the time scale c makes time to go faster. The long-time limit of the rescaled random walk is a Brownian motion: As $c\to\infty$ we have

$$\mathbb{E}\left[e^{-ikc^{-1/2}S_{[ct]}}\right] = \left(1-\frac{k^2}{c}+o(c^{-1})\right)^{[ct]} = \left[\left(1-\frac{k^2}{c}+o(c^{-1})\right)^c\right]^{\frac{[ct]}{c}} \to e^{-tk^2} \tag{2.4}$$

where the limit

$$e^{-tk^2} = \hat{p}(k,t) = \int_{-\infty}^{\infty} e^{-ikx}p(x,t)\,dx$$

is the FT of a normal density

$$p(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}$$

with mean zero and variance $2t$. Then the continuity theorem for FT implies that

$$c^{-1/2} S_{[ct]} \Rightarrow Z_t$$

where the Brownian motion Z_t is normal with mean zero and variance $2t$.

Clearly the FT $\hat{p}(k,t) = e^{-tk^2}$ solves a differential equation

$$\frac{d\hat{p}}{dt} = -k^2 \hat{p} = (ik)^2 \hat{p}. \tag{2.5}$$

If f' exists and if f, f' are integrable, then the FT of $f'(x)$ is $(ik)\hat{f}(k)$ [33, p. 8]. Using this fact, we can invert the FT on both sides of (2.5) to get

$$\frac{\partial p}{\partial t} = \frac{\partial^2 p}{\partial x^2} \tag{2.6}$$

This shows that the pdf of Z_t solves the diffusion equation (2.6). The diffusion equation models the spreading of a cloud of particles. The random walk S_n gives the location of a randomly selected particle, and the long-time limit density $p(x,t)$ gives the relative concentration of particles at location x at time $t > 0$.

More generally, suppose that $\mu_1 = \mathbb{E}[Y_n] = 0$ and $\mu_2 = \mathbb{E}[Y_n^2] = \sigma^2 > 0$. Then

$$\hat{f}(k) = 1 - \frac{1}{2}\sigma^2 k^2 + o(k^2)$$

leads to

$$\mathbb{E}\left[e^{-ikn^{-1/2}S_n}\right] = \left(1 - \frac{\sigma^2 k^2}{2n} + o(n^{-1})\right)^n \to \exp(-\frac{1}{2}\sigma^2 k^2)$$

and

$$\mathbb{E}\left[e^{-ikc^{-1/2}S_{[ct]}}\right] = \left(1 - \frac{\sigma^2 k^2}{2c} + o(c^{-1})\right)^{[ct]} \to \exp(-\frac{1}{2}t\sigma^2 k^2) = \hat{p}(k,t). \tag{2.7}$$

This FT inverts to a normal density

$$p(x,t) = \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-x^2/(2\sigma^2 t)}$$

with mean zero and variance $\sigma^2 t$. The FT solves

$$\frac{d\hat{p}}{dt} = -\frac{\sigma^2}{2}k^2\hat{p} = \frac{\sigma^2}{2}(ik)^2\hat{p}$$

which inverts to

$$\frac{dp}{dt} = \frac{\sigma^2}{2}\frac{\partial^2 p}{\partial x^2} \tag{2.8}$$

This form of the diffusion equation shows the relation between the *dispersivity* $D = \sigma^2/2$ and the particle jump variance. Apply the continuity theorem for FT to (2.7) to get random walk convergence:

$$c^{-1/2}S_n \Rightarrow Z_t$$

where Z_t is a Brownian motion, normal with mean zero and variance $\sigma^2 t$.

In many applications, it is useful to add a drift: $vt + Z_t$ has FT

$$\mathbb{E}\left[e^{-ik(vt+Z_t)}\right] = \exp(-ikvt - \frac{1}{2}t\sigma^2 k^2) = \hat{p}(k,t),$$

which solves

$$\frac{d\hat{p}}{dt} = \left(-ikv + \frac{\sigma^2}{2}(ik)^2\right)\hat{p}$$

Invert the FT to obtain the diffusion equation with drift:

$$\frac{\partial p}{\partial t} = -v\frac{\partial p}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 p}{\partial x^2} \tag{2.9}$$

This represents the long-time limit of a random walk whose jumps have a non-zero mean $v = \mu_1$ [33, p. 9].

2.2 Fractional Derivatives

The concept of differentiation operator $D = d/dx$ is familiar to all who have studied elementary calculus. And for suitable function f, the nth derivative of f, denoted by $D^n f(x) = d^n f(x)/dx^n$ is well defined-provided n is a positive integer. In 1695, L' Hôpital inquired of Leibniz what meaning could be ascribed to $D^n f(x)$ is n were a fraction. Since that time the fractional calculus has drawn the attention of many famous mathematicians, such as Euler, Laplace, Fourier, Abel, Liouville, and Laurent. But it was not until 1884 that the theory of generalized operators achieved a level in its development suitable as a point of departure for the modern mathematician. By then the theory had been extended to include operators D^v, where v could be rational or irrational, positive or negative, real or complex. In the past few years fractional calculus appeared as an important tool to deal with anomalous diffusion processes. An anomalous diffusion process can be visualized as an ant in a labyrinth where the average square of the distance covered by the ant is $\langle x^2(t)\rangle \propto t^{2\beta}$ where β is a phenomenological constant; for $\beta = 1/2$ we have the ordinary diffusion processes. A more physical approach of anomalous diffusion processes has several applications in many field such as diffusion in porous media or long range correlation of DNA sequence.

2.2.1 Generator Form

The *generator form* of the fractional derivative for $0 < \beta < 1$ is given by

$$\frac{d^\beta f(x)}{dx^\beta} = \int_0^\infty [f(x) - f(x-y)] \frac{\beta}{\Gamma(1-\beta)} y^{-\beta-1}\, dy. \tag{2.10}$$

[33, see p. 30], where $\Gamma(\beta) = \displaystyle\int_0^\infty e^{-x} x^{\beta-1}\, dx$ is a gamma function.

For continuously differentiable and bounded function integrate by parts with $u = f(x) - f(x-y)$ to get the *caputo form*

$$\frac{d^\beta f(x)}{dx^\beta} = \frac{1}{\Gamma(1-\beta)} \int_0^\infty f'(x-y) y^{-\beta}\, dy = \frac{1}{\Gamma(1-\beta)} \int_0^\infty \frac{d}{dx} f(x-y) y^{-\beta}\, dy \tag{2.11}$$

Take the derivative outside the integral to get the *Riemann-Liouville* form

$$\left(\frac{d}{dt}\right)^\beta f(x) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dx} \int_0^\infty f(x-y) y^{-\beta}\, dy \tag{2.12}$$

For $1 < \beta < 2$ we can write the generator form

$$\frac{d^\beta f(x)}{dx^\beta} = \frac{\beta(\beta-1)}{\Gamma(2-\beta)} \int_0^\infty [f(x) - f(x-y) + y f'(x)] y^{-\beta-1}\, dy. \tag{2.13}$$

Integrate by parts twice to get the Caputo form for $1 < \beta < 2$:

$$\frac{d^\beta f(x)}{dx^\beta} = \frac{1}{\Gamma(2-\beta)} \int_0^\infty \frac{d^2}{dx^2} f(x-y) y^{1-\beta}\, dy \tag{2.14}$$

Move the derivative outside to get the Riemann-Liouville form for $1 < \beta < 2$:

$$\left(\frac{d}{dt}\right)^\beta f(x) = \frac{1}{\Gamma(2-\beta)} \frac{d^2}{dx^2} \int_0^\infty f(x-y) y^{1-\beta}\, dy \tag{2.15}$$

In general, Caputo's definition can be written as

$$\frac{d^\beta f(x)}{dx^\beta} = \frac{1}{\Gamma(n-\beta)} \int_0^\infty \frac{d^n}{dx^n} f(x-y) y^{n-1-\beta}\, dy, \quad (n-1 < \beta < n). \tag{2.16}$$

and the Riemann-Liouville form as

$$\left(\frac{d}{dt}\right)^\beta f(x) = \frac{1}{\Gamma(n-\beta)} \frac{d^n}{dx^n} \int_0^\infty f(x-y) y^{n-1-\beta}\, dy \quad (n-1 < \beta < n). \tag{2.17}$$

Example 2.1. Let $f(x) = e^{\lambda x}$ for some $\lambda > 0$, so that $f'(x) = \lambda e^{\lambda x}$. Using the Caputo form for $0 < \beta < 1$, a substitution $u = \lambda y$, and the definition of gamma function $\Gamma(\beta) = \int_0^\infty e^{-x} x^{\beta-1}\, dx$, we get

$$\frac{d^\beta}{dx^\beta}[e^{\lambda x}] = \lambda^\beta e^{\lambda x}$$

which agrees with the integer order case. Using the Riemann-Liouville form we get

$$\left(\frac{d}{dt}\right)^\beta [e^{\lambda x}] = \lambda^\beta e^{\lambda x}$$

which agrees with the Caputo. In this case, both forms lead to the same result.

2.2.2 Distributed Order Fractional Derivatives

In this section, we give an equivalent form of Caputo and Riemann-Liouville form for a function defined on a nonnegative real line.

For a function $u(t, x)$, the Caputo fractional derivative [15] for $t \geq 0$, is defined as

$$\frac{\partial^\beta u(t,x)}{\partial t^\beta} = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{\partial u(r,x)}{\partial r} \frac{dr}{(t-r)^\beta} \quad \text{for} \quad 0 < \beta < 1. \tag{2.18}$$

For $0 < \beta < 1$, its Laplace transform is given by

$$\int_0^\infty e^{-st} \frac{\partial^\beta u(t,x)}{\partial t^\beta}\, ds = s^\beta \tilde{u}(s,x) - s^{\beta-1} u(0,x) \tag{2.19}$$

where $\tilde{u}(s,x) = \int_0^\infty e^{-st} u(t,x) dt$. For $1 < \beta < 2$, $\frac{\partial^\beta u(t,x)}{\partial t^\beta}$ has Laplace transform $s^\beta \tilde{u}(s,x) - s^{\beta-1} u(0,x) - \frac{\partial}{\partial t} u(0,x)$ and incorporates the initial condition in the usual way as the regular derivative. The distributed order fractional derivative is

$$\mathbb{D}^{(\mu)} u(t,x) := \int_0^1 \frac{\partial^\beta u(t,x)}{\partial t^\beta} \mu(d\beta), \tag{2.20}$$

where μ is a finite Borel measure with $\mu(0,1) > 0$.

For a function $u(t,x)$ continuous in $t \geq 0$, the Riemann-Liouville fractional derivative of order $0 < \beta < 1$ is defined by

$$\left(\frac{\partial}{\partial t}\right)^{\beta} u(t,x) = \frac{1}{\Gamma(1-\beta)} \frac{\partial}{\partial t} \int_0^t \frac{u(r,x)}{(t-r)^{\beta}} dr. \tag{2.21}$$

Its Laplace transform

$$\int_0^{\infty} e^{-st} \left(\frac{\partial}{\partial t}\right)^{\beta} u(t,x)\, ds = s^{\beta} \tilde{u}(s,x). \tag{2.22}$$

The main advantage of Caputo's approach is that the initial conditions for fractional differential equations with Caputo derivates take on the same form as for integer-order differential equations, i.e. contain the limit values of integer-order derivatives of unknown functions at the lower terminal $t = 0$.

Example 2.2. Let $f(t) = 1$ for $t \geq 0$ and $f(t) = 0$ for $t < 0$. Then $f'(t) = 0$ for $t \neq 0$, so the Caputo fractional derivative is zero. In fact, the Caputo fractional derivative of a constant function is always zero, just like the integer order derivative. But the Riemann-Liouville derivative is not. For $t > 0$ and $0 < \beta < 1$, use (2.21) to get

$$\begin{aligned}
\left(\frac{d}{dt}\right)^{\beta} f(t) &= \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t 1(t-y)^{-\beta}\, dy \\
&= \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t u^{-\beta}\, du \\
&= \frac{x^{-\beta}}{\Gamma(1-\beta)} \neq 0.
\end{aligned}$$

If $u(\cdot, x)$ is absolutely continuous on bounded intervals (e.g., if the derivative exists everywhere and is integrable) then the Riemann-Liouville and Caputo derivatives are related by

$$\frac{\partial^{\beta} u(t,x)}{\partial t^{\beta}} = \left(\frac{\partial}{\partial t}\right)^{\beta} u(t,x) - \frac{t^{-\beta} u(0,x)}{\Gamma(1-\beta)}. \tag{2.23}$$

The Riemann-Liouville fractional derivative is more general, as it does not require the first derivative to exist. It is also possible to adopt the right-hand side of (2.23) as the definition of the Caputo derivative, see for example Kochubei [25]. Hence we adopt this as our definition of Caputo derivative in this paper. Then the (extended) distributed order derivative is

$$\mathbb{D}_1^{(\mu)} u(t,x) := \int_0^1 \left[\left(\frac{\partial}{\partial t}\right)^{\beta} u(t,x) - \frac{t^{-\beta} u(0,x)}{\Gamma(1-\beta)} \right] \mu(d\beta), \tag{2.24}$$

which exists for $u(t,x)$ continuous, and agrees with the usual definition (2.20) when $u(t,x)$ is absolutely continuous.

2.3 Time-fractional Diffusion

In this section we will outline the stochastic model for time-fractional diffusion. For additional details and precise mathematical proofs see [33, chapter 4].

Distributed order fractional derivatives are connected with random walk limits. For each $c > 0$, take a sequence of i.i.d. waiting times (J_n^c) and i.i.d. jumps (Y_n^c). Let $X^c(n) = Y_1^c + \cdots + Y_n^c$ be the particle location after n jumps, and $T^c(n) = J_1^c + \cdots + J_n^c$ the time of the nth jump. Suppose that $X^c(ct) \Rightarrow A(t)$ and $T^c(ct) \Rightarrow D_\psi(t)$ as $c \to \infty$, where the limits $A(t)$ and $D_\psi(t)$ are independent Lévy processes. The number of jumps by time $t \geq 0$ is $N_t^c = \max\{n \geq 0 : T^c(n) \leq t\}$, and [35, Theorem 2.1] shows that the continuous time random walk (CTRW) $X^c(N_t^c) \Rightarrow A(E_\psi(t))$, where

$$E_\psi(t) = \inf\{\tau : D_\psi(\tau) > t\}. \tag{2.25}$$

A specific mixture model from [34] gives rise to distributed order fractional derivatives: Let (B_i), $0 < B_i < 1$, be i.i.d. random variables such that $P\{J_i^c > u | B_i = \beta\} = c^{-1} u^{-\beta}$, for $u \geq c^{-1/\beta}$. Then $T^c(ct) \Rightarrow D_\psi(t)$, a subordinator with $\mathbb{E}[e^{-sD_\psi(t)}] = e^{-t\psi(s)}$, where

$$\psi(s) = \int_0^\infty (e^{-sx} - 1)\phi(dx). \tag{2.26}$$

Then the associated Lévy measure is

$$\phi(t,\infty) = \int_0^1 t^{-\beta}\nu(d\beta), \tag{2.27}$$

where ν is the distribution of B_i. An easy computation gives

$$\psi(s) = \int_0^1 s^{\beta}\Gamma(1-\beta)\nu(d\beta) = \int_0^1 s^{\beta}\mu(d\beta). \tag{2.28}$$

Here we define $\mu(d\beta) = \Gamma(1-\beta)\nu(d\beta)$. Then, Theorem 3.10 in [34] shows that $c^{-1}N_t^c \Rightarrow E_\psi(t)$, where $E_\psi(t)$ is given by (2.25). The Lévy process $A(t)$ defines a strongly continuous convolution semigroup with generator L, and $A(E_\psi(t))$ is the stochastic solution to the distributed order-fractional diffusion equation

$$\mathbb{D}_1^{(\mu)}u(t,x) = Lu(t,x), \tag{2.29}$$

where $\mathbb{D}_1^{(\mu)}$ is given by (2.24) with $\mu(d\beta) = \Gamma(1-\beta)\nu(d\beta)$. The condition

$$\int_0^1 \frac{1}{1-\beta}\,\nu(d\beta) < \infty \tag{2.30}$$

is imposed to ensure that $\mu(0,1) < \infty$. Since $\phi(0,\infty) = \infty$ in (2.26), Theorem 3.1 in [35] implies that $E_\psi(t)$ has a Lebesgue density

$$g_{E_\psi(t)}(x) = \int_0^t \phi(t-y,\infty)P_{D_\psi(x)}(dy). \tag{2.31}$$

Note that $E_\psi(t)$ is almost surely continuous and nondecreasing.

The CTRW model provides a physical explanation for fractional diffusion. A power law jump distribution with $\mathbb{P}[Y_n^c > x] = Cx^{-\alpha}$ leads to a fractional derivative in space $\partial^\alpha/\partial x^\alpha$ of the same order. A power law waiting time distribution $\mathbb{P}[J_n^c > t] = Bt^{-\beta}$ leads to a fractional time derivative $\partial^\beta/\partial t^\beta$ of the same order. Long power-law jumps reflect a heavy tailed velocity distribution, which allows particles to make occasional long jumps, leading to anomalous super-diffusion. Long waiting times model particle sticking and trapping, leading to anomalous sub-diffusion:

$$B(E_{ct}) \simeq B(c^{\beta} E_t) \simeq c^{\beta/2} B(E_t).$$

where B is a Brownian motion. Since $\beta < 1$, the density of this process spreads slower that a Brownian motion. For detailed discussion see for example [33].

2.4 Semigroup

This section will serve as a basic introduction to semigroups of linear operators. In general, semigroups can be used to solve a large class of problems commonly known as evolution equations. These types of equations appear in many disciplines including physics, chemistry, biology, engineering, and economics. A semigroup is a family of linear operator on a Banach space. A *Banach space* X is a complete normed vector space. That is, if $f_n \in X$ is a Cauchy sequence in this vector space, such that $||f_n - f_m|| \to 0$ as $m, n \to \infty$, then there exists some $f \in X$ such that $||f_n - f|| \to 0$ as $n \to \infty$ in the Banach space norm.

Definition 2.3. Let X be a Banach space. A family of linear operators $\{T_t : t \geq 0\}$ from X into X is called a *semigroup* if
(i) $T(0) = I$, (I is the identity operator on X).
(ii) $T(t + s) = T(t)T(s)$ for all $t, s \geq 0$ (the semigroup property).

We say that $T(t)$ is *uniformly bounded* if $||T(t)f|| \leq M||f||$ for all $f \in X$ and all $t \geq 0$. If $T(t_n)f \to T(t)f$ in X for all $f \in X$ whenever $t_n \to t$ then the operator T is strongly continuous. It is easy to check that $\{T(t); t \geq 0\}$ is strongly continuous if $T(t)f \to f$ in X for all $f \in X$ as $t \to 0$. A strongly continuous, bounded semigroup is also called a C_0 semigroup.

For any strongly continuous semigroup $\{T(t); t > 0\}$ on a Banach space X we define the *infinitesimal generator* as

$$Lf = \lim_{t \to 0^+} \frac{T(t)f - f}{t} \text{ in } X \tag{2.32}$$

meaning that $||t^{-1}(T(t)f - f) - Lf|| \to 0$ in the Banach space norm. The domain $D(L)$ of this linear operator is the set of all $f \in X$ for which the limit in (2.32) exists. The domain $D(L)$ is dense in X, and L is closed, meaning that if $f_n \to f$ and $Lf_n \to g$ in X then $f \in D(L)$ and $Lf = g$ (see, for example Corollary I.2.5 in [42]).

Theorem 2.4. *Let $T(t)$ be a C_0 semigroup. There exists a constant $a \geq 0$ and $M \geq 1$ such that*

$$||T(t)|| \leq Me^{at} \quad \textit{for} \;\; 0 \leq t < \infty.$$

Proof. See, for example, Pazy [42, Theorem I.2.2]. □

Corollary 2.5. *If $T(t)$ is a C_0 semigroup then for every $f \in X$, $t \to T(t)f$ is a continuous function from $\mathbb{R}^+$ (the nonnegative real line) into X.*

Proof. See, for example, Pazy [42, Theorem I.2.3]. □

Theorem 2.6. *Let $T(t)$ be a C_0 semigroup and let L be its infinitesimal generator. Then*
(i) For $f \in X$, $\int_0^t T(s)f ds \in D(L)$ and

$$L\left(\int_0^t T(s)f ds\right) = T(t)f - f$$

(ii) For $f \in D(L)$, $T(t)f \in D(L)$ and

$$\frac{d}{dt}T(t)f = LT(t)f = T(t)Lf$$

Proof. See, for example, Pazy [42, Theorem I.2.4]. □

2.5 Basic Facts of Relativistic Stable Process

In this section we assemble the basic notation and facts of relativistic stable process that will be used in the sequel. We also give a proof to some simple lemmas and propositions.

Next we introduce some notations. For $x \in \mathbb{R}^d$, let $\delta_D(x)$ denote the Euclidean distance between x and ∂D and the ball in $\mathbb{R}^d$ center at x and radius $r, \{y : |y-x| < r\}$ will be denoted by $B(x,r)$. Define

$$\psi(\theta) = \int_0^\infty e^{-v} v^{p-1/2} (\theta + v/2)^{p-1/2} dv, \theta \geq 0,$$

where $p = (d+\alpha)/2$. We put $\mathcal{R}(\alpha, d) = \mathcal{A}(-\alpha, d)/\psi(0)$, where $\mathcal{A}(v,d) = (\Gamma((d-v)/2))/(\pi^{d/2} 2^v |\Gamma(v/2)|)$. Let $\nu(x), \tilde{\nu}(x)$ be the densities of the Lévy measures of the relativistic $\alpha-$stable process and the standard $\alpha-$stable process, respectively. These densities, are given by

$$\nu(x) = \frac{\mathcal{R}(\alpha, d)}{|x|^{d+\alpha}} e^{-m^{1/\alpha}|x|} \psi(m^{1/\alpha}|x|) \tag{2.33}$$

$$\tilde{v}(x) = \frac{\mathcal{A}(-\alpha, d)}{|x|^{d+\alpha}} \tag{2.34}$$

We need the following estimate of the transition probabilities of the process X_t which is given in ([26], Lemma 2.2): For any $x, y \in \mathbb{R}^d$ and $t > 0$ there exist constants $c_1 > 0$ and $c_2 > 0$,

$$p(t,x,y) \leq c_1 e^{mt} \min\left\{ \frac{t}{|x-y|^{d+\alpha}} e^{-c_2|x-y|}, t^{-d/\alpha} \right\} \tag{2.35}$$

We will use the fact([14], Lemma 6) that if $D \subset \mathbb{R}^d$ is an open bounded set satisfying a uniform outer cone condition, then $P^x(X(\tau_D) \in \partial D) = 0$ for all $x \in D$. For the open bounded set D we will be denoted by $G_D(x,y)$ the Green function for the set D equal to

$$G_D(x,y) = \int_0^\infty p_D(t,x,y) dt, x, y \in \mathbb{R}^d$$

and for any such D the expectation of the exit time of the processes X_t from D is given by the integral of the Green function over the domain. That is,

$$E^x(\tau_D) = \int_D G_D(x,y)dy.$$

Now we state a simple lemma about the upper bound of $r_D(t,x,y)$, which is an analogue of [7, Lemma 2.1.] for stable processes.

Lemma 2.7. *Let $D \subset \mathbb{R}^d$ be an open set. For any $x, y \in D$ we have*

$$r_D(t,x,y) \le c_1 e^{mt} \left(\frac{t}{\delta_D^{d+\alpha}(x)} e^{-c_2 \delta_D(x)} \wedge t^{-d/\alpha} \right)$$

Proof. Using (1.4) and (2.35) we have

$$\begin{aligned} r_D(t,x,y) &= E^y(p(t-\tau_D, X(\tau_D), x); \tau_D < t) \\ &\le c_1 e^{mt} E^y \left(\frac{t}{|x - X(\tau_D)|^{d+\alpha}} e^{-c_2|x-X(\tau_D)|} \wedge t^{-d/\alpha} \right) \\ &\le c_1 e^{mt} \left(\frac{t}{\delta_D^{d+\alpha}(x)} e^{-c_2 \delta_D(x)} \wedge t^{-d/\alpha} \right) \end{aligned}$$

□

We need the following result for to get analogue to the celebrated Weyl's asymptotic formula for the eigenvalues of the Laplacian.

Lemma 2.8.

$$\lim_{t \to 0} p(t,0) e^{-mt} t^{d/\alpha} = C_1 \tag{2.36}$$

where

$$C_1 = (4\pi)^{d/2} \int_0^\infty u^{-d/2} \theta_\beta(1,u) du = \frac{\omega_d \Gamma(d/\alpha)}{(2\pi)^d \alpha}$$

Proof. By (1.2) we have

$$p(t,x,x) = p(t,0) = e^{mt}\int_0^\infty \frac{1}{(4\pi u)^{d/2}} e^{-m^{1/\beta}u}\theta_\beta(t,u)du$$

Now using the scaling of stable subordinator $\theta_\beta(t,u) = t^{-1/\beta}\theta_\beta(1, ut^{-1/\beta})$ and a change of variables we get

$$p(t,0) = \frac{e^{mt}}{(4\pi)^{d/2}t^{d/\alpha}}\int_0^\infty z^{-d/2}e^{-m^{1/\beta}t^{1/\beta}z}\theta_\beta(1,z)dz$$

then by dominated convergence theorem we obtain

$$\lim_{t\to 0} p(t,0)e^{-mt}t^{d/\alpha} = \frac{1}{(4\pi)^{d/2}}\int_0^\infty z^{-d/2}\theta_\beta(1,z)dz$$

and this last integral is equal to the density of α-stable process at time 1 and $x=0$ which was calculated in [7] to be

$$\frac{\omega_d\Gamma(d/\alpha)}{(2\pi)^d\alpha}.$$

□

Now we state a proposition which gives the Weyl's asymtotic for the eigenvalues of the relativistic Laplacian.

Proposition 2.9.

$$\lim_{t\to 0} t^{d/\alpha}e^{-mt}Z_D(t) = C_1|D| \tag{2.37}$$

where $C_1 = \frac{\omega_d\Gamma(d/\alpha)}{(2\pi)^d\alpha}$.

Let $N(\lambda)$ be the number of eigenvalues $\{\lambda_j\}$ which do not exceed λ, it follows from (2.37) and the classical Tauberian theorem (see for example [20], p.445 Theorem 2) where $L(t) = C_1|D|e^{m/t}$ is our slowly varying function at infinity that

$$\lim_{\lambda\to\infty} \lambda^{-d/\alpha}e^{-m/\lambda}N(\lambda) = \frac{C_1|D|}{\Gamma(1+d/\alpha)} \tag{2.38}$$

This is the analogue for relativistic stable process of the celebrated Weyl's asymptotic formula for the eigenvalues of the Laplacian.

We next give the proof of Proposition 2.9.

Proof of Porposition 2.9. By (1.4) we see that

$$\frac{p_D(t,x,x)}{C_1 e^{mt} t^{-d/\alpha}} = \frac{p(t,0)}{C_1 e^{mt} t^{-d/\alpha}} - \frac{r_D(t,x,x)}{C_1 e^{mt} t^{-d/\alpha}}. \tag{2.39}$$

Since the limit of the first term tend to 1 as $t \to 0$ by (2.36), in order to prove (2.37), we must show that

$$\frac{t^{d/\alpha}}{C_1 e^{mt}} \int_D r_D(t,x,x)dx \to 0, \quad \text{as } t \to 0. \tag{2.40}$$

For $0 < t < 1$, consider the subdomains $D_t = \{x \in D : \ \delta_D(x) \geq t^{1/2\alpha}\}$ and its complement $D_t^C = \{x \in D : \ \delta_D(x) < t^{1/2\alpha}\}$. By Lebesgue dominated convergence Theorem and recalling that $|D| < \infty$ we get $|D_t^C| \to 0$ as $t \to 0$. Since $p_D(t,x,x) \leq p(t,x,x)$, by (1.3) we see that

$$\frac{r_D(t,x,x)}{C_1 e^{mt} t^{-d/\alpha}} \leq 1,$$

for all $x \in D$. It follows that

$$\frac{t^{d/\alpha}}{C_1 e^{mt}} \int_{D_t^C} r_D(t,x,x)dx \to 0, \quad \text{as } t \to 0. \tag{2.41}$$

On the other hand, by Lemma 2.2 in [26] we obtain

$$\begin{aligned}
\frac{r_D(t,x,x)}{C_1 e^{mt} t^{-d/\alpha}} &= \frac{\mathbb{E}^x[p(t-\tau_D, X_{\tau_D}, x); t \geq \tau_D]}{C_1 e^{mt} t^{-d/\alpha}} \\
&\leq c\mathbb{E}^y \min\left\{ \frac{t^{1+d/\alpha}}{|x - X(\tau_D)|^{d+\alpha}} e^{-c_2|x - X(\tau_D)|}, 1 \right\} \\
&\leq c \min\left\{ \frac{t^{1+d/\alpha}}{\delta_D(x)^{d+\alpha}} e^{-c_2 \delta_D(x)}, 1 \right\}.
\end{aligned} \tag{2.42}$$

For $x \in D_t$ and $0 < t < 1$, the right hand side of (2.42) is bounded above by $ct^{d/2\alpha+1/2}$ and hence

$$\frac{t^{d/\alpha}}{C_1 e^{mt}} \int_{D_t} r_D(t,x,x)dx \le ct^{d/2\alpha+1/2}|D| \tag{2.43}$$

and this last quantity goes to 0 as $t \to 0$. $\square$

For an open set $D \subset \mathbb{R}^d$ and $x \in \mathbb{R}^d$, the distribution $P^x(\tau_D < \infty, X(\tau_D) \in \cdot)$ will be called the relativistic $\alpha-$harmonic measure for D. The following Ikeda-Watanabe formula recovers the relativistic $\alpha-$harmonic measure for the set D from the Green function.

Proposition 2.10 ([26]). *Assume that D is an open, nonempty, bounded subset of $\mathbb{R}^d$, and A is a Borel set such that dist$(D,A) > 0$. Then*

$$P^x(X(\tau_D) \in A, \tau_D < \infty) = \int_D G_D(x,y) \int_A v(y-z)dzdy, x \in D \tag{2.44}$$

Here we need the following generalization already stated and used in [7].

Proposition 2.11. *[26, Proposition 2.5] Assume that D is an open, nonempty, bounded subset of R^d, and A is a Borel set such that $A \subset D^c \backslash \partial D$ and $0 \le t_1 < t_2 < \infty, x \in D$. Then we have*

$$P^x(X(\tau_D) \in A, t_1 < \tau_D < t_2) = \int_D \int_{t_1}^{t_2} p_D(s,x,y)ds \int_A v(y-z)dzdy.$$

Now we need the following proposition that holds for a large class of Lévy processes. The result is about the difference $p_F(t,x,y) - p_D(t,x,y)$ where D and F are open sets and $D \subset F$. The proof given in [7], given for stable processes, mainly uses the strong Markov property it works for all strong Markov processes with transition densities.

Proposition 2.12. *[7, Proposition 2.3] Let D and F be open sets in $\mathbb{R}^d$ such that $D \subset F$. Then for any $x, y \in \mathbb{R}^d$ we have*

$$p_F(t,x,y) - p_D(t,x,y) = E^x(\tau_D < t, X(\tau_D) \in F \backslash D; p_F(t-\tau_D, X(\tau_D), y))$$

Lemma 2.13. *[45, Lemma 5] Let $D \subset \mathbb{R}^d$ be an open set. For any $x, y \in D$ and $t > 0$ the following estimates hold;*

$$\begin{aligned} p_D(t,x,y) &\leq e^{mt}\tilde{p}_D(t,x,y) \\ r_D(t,x,y) &\leq e^{2mt}\tilde{r}_D(t,x,y) \end{aligned} \tag{2.45}$$

We need the following lemma given by Van den Berg in [49].

Lemma 2.14. *[49, Lemma 5] Let D be an open bounded set in R^d with R-smooth boundary ∂D and define for $0 \leq q < R$*

$$D_q = \{x \in D : \delta_D(x) > q\}$$

and denote the area of its boundary ∂D_q by $|\partial D_q|$. Then

$$\left(\frac{R-q}{R}\right)^{d-1} |\partial D| \leq |\partial D_q| \leq \left(\frac{R}{R-q}\right)^{d-1} |\partial D|, 0 \leq q < R. \tag{2.46}$$

Corollary 2.15. *([7], Corollary 2.14) Let D be an open bounded set in $\mathbb{R}^d$ with R-smooth boundary. For any $0 < q \leq R$ we have*

(i)

$$2^{-d+1}|\partial D| \leq |\partial D_q| \leq 2^{d-1}|\partial D|,$$

(ii)

$$|\partial D| \leq \frac{2^d|D|}{R},$$

(iii)

$$\Big||\partial D_q| - |\partial D|\Big| \leq \frac{2^d dq|\partial D|}{R} \leq \frac{2^{2d}dq|D|}{R^2}$$

Now we will introduce the following notation. Since D has R-smooth boundary, for any point $y \in \partial D$ there are two open balls B_1 and B_2 both of radius R such that $B_1 \subset D, B_2 \subset \mathbb{R}^d \backslash (D \cup \partial D), \partial B_1 \cap \partial B_2 = y$. For any $x \in D_{R/2}$ there exist a unique point $x_* \in \partial D$ such that $\delta_D(x) = |x - x_*|$. Let $B_1 = B(z_1, R), B_2 = B(z_2, R)$ be the balls for the point x_*. Let $H(x)$ be the half-space containing B_1 such that $\partial H(x)$ contains x_* and is perpendicular to the segment $\overline{z_1 z_2}$.

We will need the following very important proposition in the proof of our main result. Such a proposition has been proved for the stable process in [7, Proposition 3.1].

Proposition 2.16. *Let $D \subset R^d, d \geq 2$, be an open bounded set with R-smooth boundary ∂D. Then for any $x \in D \backslash D_{R/2}$ and $t > 0$ such that $t^{1/\alpha} \leq R/2$ we have*

$$|r_D(t, x, x) - r_{H(x)}(t, x, x)| \leq \frac{ce^{2mt} t^{1/\alpha}}{R t^{d/\alpha}} \left(\left(\frac{t^{1/\alpha}}{\delta_D(x)} \right)^{d+\alpha/2-1} \wedge 1 \right) \tag{2.47}$$

Proof. Exactly as in [7], let $x_* \in \partial D$ be a unique point such that $|x - x_*| = \operatorname{dist}(x, \partial D)$ and B_1 and B_2 be balls with radius R such that $B_1 \subset D, B_2 \subset \mathbb{R}^d \backslash (D \cup \partial D), \partial B_1 \cap \partial B_2 = x_*$. Let us also assume that $x_* = 0$ and choose an orthonormal coordinate system $(x_1, x_2, ..., x_d)$ so that the positive axis $0x_1$ is in the direction of $\overrightarrow{0p}$ where p is the center of the ball B_1. Note that x lies on the interval $0p$ so $x = (|x|, 0, 0, ..., 0)$. Note also that $B_1 \subset D \subset (\overline{B_2})^c$ and $B_1 \subset H(x) \subset (\overline{B_2})^c$. For any open sets A_1, A_2 such that $A_1 \subset A_2$ we have $r_{A_1}(t, x, y) \geq r_{A_2}(t, x, y)$ so

$$|r_D(t, x, x) - r_{H(x)}(t, x, x)| \leq r_{B_1}(t, x, x) - r_{(\overline{B_2})^c}(t, x, x).$$

So in order to prove the proposition it suffices to show that

$$r_{B_1}(t, x, x) - r_{(\overline{B_2})^c}(t, x, x) \leq \frac{ce^{2mt} t^{1/\alpha}}{R t^{d/\alpha}} \left(\left(\frac{t^{1/\alpha}}{\delta_D(x)} \right)^{d+\alpha/2-1} \wedge 1 \right)$$

for any $x = (|x|, 0, ..., 0), |x| \in (0, R/2]$. Such an estimate was proved for the case $m = 0$ in [7]. In order to complete the proof it is enough to prove that

$$r_{B_1}(t, x, x) - r_{(\overline{B_2})^c}(t, x, x) \leq ce^{2mt} \left\{ \tilde{r}_{B_1}(t, x, x) - \tilde{r}_{(\overline{B_2})^c}(t, x, x) \right\}.$$

But this follows from Propositions 2.11, 2.12 and 2.13.

Given the ball B_2, we set $U = (\overline{B_2})^c$. Now using the generalized Ikeda-Watanabe formula, Proposition (2.12) and Lemma 2.4 in [26] we have

$$
\begin{aligned}
& r_{B_1}(t,x,x) - r_U(t,x,x) \\
= \;& E^x\left[t > \tau_{B_1}, X(\tau_{B_1}) \in U \backslash B_1; p_U(t-\tau_{B_1}, X(\tau_{B_1}), x)\right] \\
= \;& \int_{B_1}\int_0^t p_{B_1}(s,x,y)ds \int_{U\backslash B_1} v(y-z)p_U(t-s,z,x)dzdy \\
\leq \;& e^{2mt}\int_{B_1}\int_0^t \tilde{p}_{B_1}(s,x,y)ds \int_{U\backslash B_1} \tilde{v}(y-z)\tilde{p}_U(t-s,z,x)dzdy \\
\leq \;& ce^{2mt}E^x\left[t > \tilde{\tau}_{B_1}, \tilde{X}(\tilde{\tau}_{B_1}) \in U\backslash B_1; \tilde{p}_U(t-\tilde{\tau}_{B_1}, \tilde{X}(\tilde{\tau}_{B_1}), x)\right] \\
= \;& ce^{2mt}\left(\tilde{r}_{B_1}(t,x,x) - \tilde{r}_U(t,x,x)\right) \\
\leq \;& \frac{ce^{2mt}t^{1/\alpha}}{Rt^{d/\alpha}}\left(\left(\frac{t^{1/\alpha}}{\delta_D(x)}\right)^{d+\alpha/2-1} \wedge 1\right)
\end{aligned}
$$

□

The last inequality follows by Proposition 3.1 in [7].

Chapter 3

Strong Analytic Solution to Distributed Order Time Fractional Cauchy problems

In this chapter we give strong analytic solution to distributed order time fractional Cauchy problems. Our proofs work for operators L that are generators of uniformly bounded and continuous semigroups on Banach spaces. In our first main result the Lévy subordinator is written as the sum of n independent stable subordinators of index $0 < \beta_1 < \beta_2 < \cdots < \beta_n < 1$ and Theorem 3.4 provides an extension with subordinator $D_\mu(t)$ as the weighted average of an arbitrary number of independent stable subordinators.

Let $D_\psi(t)$ be a strictly increasing Lévy process (subordinator) with $\mathbb{E}[e^{-sD_\psi(t)}] = e^{-t\psi(s)}$, where the Laplace exponent

$$\psi(s) = bs + \int_0^\infty (e^{-sx} - 1)\phi(dx), \tag{3.1}$$

$b \geq 0$, and ϕ is the Lévy measure of D_ψ. Then we must have either

$$\phi(0, \infty) = \infty, \tag{3.2}$$

or $b > 0$, or both, see [35]. Let

$$E_\psi(t) = \inf\{\tau \geq 0 : \ D_\psi(\tau) > t\} \tag{3.3}$$

be the inverse subordinator.

Let T be a uniformly bounded, strongly continuous semigroup on a Banach space. Let

$$S(t)f = \int_0^\infty (T(l)f) g_{E_\psi(t)}(l) dl \tag{3.4}$$

where $g_{E_\psi(t)}(l)$ is a Lebesgue density of $E_\psi(t)$.

Using (2.31), it is easy to show that

$$\int_0^\infty e^{-st} g_{E_\psi(t)}(l)dt = \frac{1}{s}\psi(s)e^{-l\psi(s)}.$$

Using Fubini's Theorem, we get

$$\int_0^\infty \psi(s)e^{-l\psi(s)}T(l)fdl = s\int_0^\infty e^{-st}S(t)fdt. \tag{3.5}$$

We define a *sectorial region* of the complex plane $\mathbb{C}(\alpha) = \{re^{i\theta} \in \mathbb{C} : \ r > 0, |\theta| < \alpha\}$. Note that $\mathbb{C}(\pi/2) = \mathbb{C}_+ = \{\mathrm{Re}(Z) > 0\}$. We call a family of linear operators on a Banach space X *strongly analytic in a sectorial region* if for some $\alpha > 0$ the mapping $t \to T(t)f$ has an analytic extension to the sectorial region $\mathbb{C}(\alpha)$ for all $f \in X$ (see, for example, section 3.12 in [23]).

Next we state two theorem that is very important in proving our main results. The first theorem is about Bochner intergal and the next gives analytic representation of an operators.

Theorem 3.1. *(**Bochner**). A function $f : I \to X$ is Bochner integrable if and only if f is measurable and $|f|$ is integrable. If f is Bochner integrable, then*

$$\left\| \int_I f(t)\, dt \right\| \le \int_I \|f(t)\|\, dt$$

Proof. See, for example, [1, Theorem 1.1.4]. □

Theorem 3.2. *(**Analytic Representation**). Let $0 < \alpha \le \frac{\pi}{2}, \omega \in \mathbb{R}$ and $q : (\omega, \infty) \to X$. The following are equivalent:*

i) There exists a holomorphic function $f : \mathbb{C}(\alpha) \to X$ such that $sup_{z\in\mathbb{C}(\beta)}||e^{-\omega z}f(z)|| < \infty$ for all $0 < \beta < \alpha$ and $q(\lambda) = \tilde{f}(\lambda)$ for all $\lambda > \omega$.

ii) The function q has a holomorphic extension $\bar{q} : \omega + C(\alpha + \pi/2) \to X$ such that $sup_{\lambda\in\omega+\mathbb{C}(\gamma+\pi/2)}||(\lambda - \omega)\bar{q}(\lambda)|| < \infty$ for all $0 < \gamma < \alpha$.

Proof. See, for example, [1, Theorem 2.6.1]. □

Now we give our main result for distributed order fractional Cauchy problems. Our solution works for operators L that are generators of uniformly bounded and continuous semigroups on Banach spaces.

Let $0 < \beta_1 < \beta_2 < \cdots < \beta_n < 1$. In the next theorem we consider the case where

$$\psi(s) = c_1 s^{\beta_1} + c_2 s^{\beta_2} + \cdots + c_n s^{\beta_n}.$$

In this case the Lévy subordinator can be written as

$$D_\psi(t) = (c_1)^{1/\beta_1} D^1(t) + (c_2)^{1/\beta_2} D^2(t) + \cdots + (c_n)^{1/\beta_n} D^n(t)$$

where $D^1(t), D^2(t), \cdots, D^n(t)$ are independent stable subordinators of index $0 < \beta_1 < \beta_2 < \cdots < \beta_n < 1$.

Theorem 3.3. *Let $(X, ||.||)$ be a Banach space and L be the generator of a uniformly bounded, strongly continuous semigroup $\{T(t) : t \geq 0\}$. Then the family $\{S(t) : t \geq 0\}$ of linear operators from X into X given by (3.4) is uniformly bounded and strongly analytic in a sectorial region. Furthermore, $\{S(t) : t \geq 0\}$ is strongly continuous and $h(x,t) = S(t)f(x)$ is a solution of*

$$\sum_{i=1}^{n} c_i \frac{\partial^{\beta_i} h(x,t)}{\partial t^{\beta_i}} = Lh(x,t);\ \ h(x,0) = f(x).$$

for $\beta_1 < \beta_2 < \cdots < \beta_n \in (0,1)$

Proof. We adapt the methods of Baeumer and Meerschaert [3, Theorem 3.1] with some very crucial changes in the following. For the purpose of completeness of the arguments we included some parts verbatim from Baeumer and Meerschaert [3].

Since $\{T(t) : t \geq 0\}$ is uniformly bounded we have $||T(t)f|| \leq M||f||$ for all $f \in X$. Theorem 3.1 implies that a function $F : \mathbb{R}^1 \to X$ is integrable if and only if $F(s)$ is measurable and $||F(s)||$ is integrable, in which case

$$\left\| \int F(l)\, dl \right\| \leq \int ||F(l)||dl.$$

For fixed $f \in X$ and applying Bochner's Theorem with $F(l) = (T(l)f)g_{E_\psi(t)}(l)$ we have that

$$\begin{aligned} ||S(t)f|| &= \left\| \int_0^\infty (T(l)f)g_{E_\psi(t)}(l)dl \right\| \\ &\leq \int_0^\infty ||(T(l)f)g_{E_\psi(t)}(l)||dl \\ &= \int_0^\infty ||T(l)f||g_{E_\psi(t)}(l)dl \\ &\leq \int_0^\infty M||f||g_{E_\psi(t)}(l)dl = M||f|| \end{aligned}$$

since $g_{E_\psi(t)}(l)$ is the Lebesgue density for $E_\psi(t)$. This shows that $\{S(t) : t \geq 0\}$ is well defined and uniformly bounded family of linear operators on X.

The definition of $T(t)$ and dominated convergence theorem implies

$$\begin{aligned} ||S(t)f - f|| &= \left\| \int_0^\infty (T(l)f - f)g_{E_\psi(t)}(l)\, dl \right\| \\ &\leq \int_0^\infty ||T(l)f - f||g_{E_\psi(t)}(l)dl \\ &\to ||T(0)f - f|| = 0 \end{aligned}$$

as $t \to 0^+$. This shows $\lim_{t \to 0^+} S(t)f = f$. Now if $t, h > 0$ then we have

$$||S(t+h)f - S(t)f|| \leq \int_0^\infty ||T(l)f|||g_{E_\psi(t+h)}(l) - g_{E_\psi(t)(l)}|dl \to 0$$

as $h \to 0^+$ since $E_\psi(t+h) \implies E_\psi(t)$ as $h \to 0$.

This shows that $\{S(t) : t > 0\}$ is strongly continuous.

Let $q(s) = \int_0^\infty e^{-st}T(t)fdt$ and $r(s) = \int_0^\infty e^{-st}S(t)fdt$ for any $s > 0$, so that we can write (3.5) in the form

$$\psi(s)q(\psi(s)) = sr(s) \tag{3.6}$$

for any $s > 0$. Now we want to show that this relation holds for certain complex numbers. Fix $s \in \mathbb{C}_+ = \{z \in \mathbb{C} : \mathcal{R}(z) > 0\}$, and let $F(t) = e^{-st}T(t)f$. Since F is continuous, it is measurable, and we have $||F(t)|| \leq |e^{-st}|M||f|| = e^{-t\mathcal{R}(s)}M||f||$ since $||T(t)f|| \leq M||f||$, so that the function $||F(t)||$ is integrable. Then Bochner's Theorem implies that $q(s) = \int_0^\infty F(t)dt$ exists for all $s \in \mathbb{C}_+$, with

$$||q(s)|| = \left\| \int_0^\infty F(t)\, dt \right\| \leq \int_0^\infty ||F(t)||dt \leq \int_0^\infty e^{-t\mathcal{R}(s)}M||f||dt = \frac{M||f||}{\mathcal{R}(s)}. \tag{3.7}$$

Since $q(s)$ is the Laplace transform of the bounded continuous function $t \mapsto T(t)f$, Theorem 1.5.1 of [1] shows that $q(s)$ is an analytic function on $s \in \mathbb{C}_+$.

Now we carry out the details of the proof for only in the case $n = 2$. We want to show that $r(s)$ is the Laplace transform of an analytic function defined on a sectorial region. Theorem 3.2 implies that if for some real x and some $\alpha \in (0, \pi/2]$ the function $r(s)$ has an analytic extension to the region $x + \mathbb{C}(\alpha + \pi/2)$ and if $\sup\{||(s-x)r(s)|| : s \in x + \mathbb{C}(\alpha' + \pi/2)\} < \infty$ for all $0 < \alpha' < \alpha$, then there exists an analytic function $\overline{r}(t)$ on $t \in \mathbb{C}(\alpha)$ such that $r(s)$ is the Laplace transform of $\overline{r}(t)$. We will apply the theorem with $x = 0$. It follows from (3.6) that $r(s) = \frac{1}{s}\psi(s)q(\psi(s))$ for all $s > 0$, but the right hand side here is well defined and analytic on the set of complex s that are not on the branch cut and are such that $\mathcal{R}(\psi(s)) = \mathcal{R}(c_1 s^{\beta_1} + c_2 s^{\beta_2}) > 0$, since $\beta_1 < \beta_2$, it suffices to consider $\mathcal{R}(s^{\beta_2}) > 0$, so if $1/2 < \beta_2 < 1$, then $r(s)$ has a unique analytic extension to the sectorial region $\mathbb{C}(\pi/2\beta_2) = \{s \in \mathbb{C} : \mathcal{R}e(s^{\beta_2}) > 0\}$ (e.g., [23, 3.11.5]), and note that $\pi/2\beta_2 = \pi/2 + \alpha$

for some $\alpha > 0$. If $\beta_2 < 1/2$ then $r(s)$ has an analytic extension to the sectorial region $s \in \mathbb{C}(\pi/2+\alpha)$ for any $\alpha < \pi/2$ and $\mathcal{R}(s^{\beta_2}) > 0$ for all such s. Now for any complex $s = re^{i\theta}$ such that $s \in \mathbb{C}(\pi/2+\alpha')$ for any $0 < \alpha' < \alpha$, we have in view of (3.6) and (3.7) that

$$\begin{aligned}
||sr(s)|| &= ||\psi(s)q(\psi(s))|| \\
&= |c_1 s^{\beta_1} + c_2 s^{\beta_2}|||q(c_1 s^{\beta_1} + c_2 s^{\beta_2})|| \\
&= \left|\frac{c_1 r^{\beta_1} e^{i\beta_1\theta} + c_2 r^{\beta_2} e^{i\beta_2\theta}}{c_1 r^{\beta_1}\cos(\beta_1\theta) + c_2 r^{\beta_2}\cos(\beta_2\theta)}\right| \\
&\quad \times ||\mathcal{R}(c_1 s^{\beta_1} + c_2 s^{\beta 2})q(c_1 s^{\beta_1} + c_2 s^{\beta_2})|| \\
&\le \left|\frac{c_1 r^{\beta_1} e^{i\beta_1\theta}}{c_1 r^{\beta_1}\cos(\beta_1\theta) + c_2 r^{\beta_2}\cos(\beta_2\theta)}\right| M||f|| \\
&\quad + \left|\frac{c_2 r^{\beta_2} e^{i\beta_2\theta}}{c_1 r^{\beta_1}\cos(\beta_1\theta) + c_2 r^{\beta_2}\cos(\beta_2\theta)}\right| M||f|| \\
&\le \left(\frac{1}{cos(\beta_1\theta)} + \frac{1}{\cos(\beta_2\theta)}\right) M||f|| \qquad (3.8)
\end{aligned}$$

which is finite since $|\beta_1\theta| < |\beta_2\theta| < \pi/2$. Hence Theorem 2.6.1 of [1] implies there exists an analytic function $\overline{r}(t)$ on $t \in \mathbb{C}(\alpha)$ with Laplace transform $r(s)$. Using the uniqueness of the Laplace transform (e.g., [1, Thm. 1.7.3]), if follows that $t \mapsto S(t)f$ has an analytic extension (namely $t \mapsto \overline{r}(t)$) to the sectorial region $t \in \mathbb{C}(\alpha)$. Next we wish to apply Theorem 2.6.1 of [1] again to show that for any $0 < \beta_1 < \beta_2 < 1$ the function

$$t \mapsto \int_0^t \frac{(t-u)^{-\beta_i}}{\Gamma(1-\beta_i)} S(u) f du \qquad i = 1,2 \qquad (3.9)$$

has an analytic extension to the same sectorial region $t \in \mathbb{C}(\alpha)$. It is easy to show that

$$\int_0^\infty \frac{t^{-\beta_i}}{\Gamma(1-\beta_i)} e^{-st} dt = s^{\beta_i - 1} \qquad (3.10)$$

for any $0 < \beta_i < 1$ and any $s > 0$. Since $r(s)$ is the Laplace transform of $t \mapsto S(t)f$, it follows from the convolution property of the Laplace transform (e.g. property 1.6.4 [1]) that the function (3.9) has Laplace transform $s^{\beta_i - 1} r(s)$ for all $s > 0$. Since $r(s)$ has an analytic extension to the sectorial region $s \in \mathbb{C}(\pi/2+\alpha)$, so does $s^{\beta_i-1}r(s)$. For any $x>0$, if $s = x + re^{i\theta}$ for some $r>0$ and $|\theta| < \pi/2+\alpha'$ for any $0<\alpha'<\alpha$ then in view of (3.8) we have

$$\begin{aligned} ||(s-x)s^{\beta_i-1}r(s)|| &= ||(s-x)s^{\beta_i-2}sr(s)|| \\ &\leq r||s^{\beta_i-2}||\left(\frac{1}{\cos(\beta_1\theta)}+\frac{1}{\cos(\beta_2\theta)}\right)M||f|| \end{aligned}$$

where $||s||$ is bounded away from zero, $||s|| \leq r+x$ and $\beta_i - 2 < -1$, so that $||(s-x)s^{\beta_i-1}r(s)||$ is bounded on the region $x+\mathbb{C}(\alpha'+\pi/2)$ for all $0<\alpha'<\alpha$. Then it follows as before that the function (3.9) has an analytic extension to the sectorial region $t \in \mathbb{C}(\alpha)$.

Since $\{T(t) : t \geq 0\}$ is a strongly continuous semigroup with generator L, Theorem 2.6 implies that $\int_0^t T(s)fds$ is in the domain of the operator L and

$$T(t)f = L\int_0^t T(s)fds + f.$$

Since the Laplace transform $q(s)$ of $t \mapsto T(t)f$ exists, Corollary 1.6.5 of [1] show that the Laplace transform of $t \mapsto \int_0^t T(s)fds$ exists and equals $s^{-1}q(s)$. Corollary 1.2.5 [42] shows that L is closed. Fix s and let $g = q(s) = \int_0^\infty e^{-st}T(t)fdt$ and let g_n be a finite Riemann sum approximating this integral, so that $g_n \to g$ in X. Let $h_n = s^{-1}g_n$ and $h = s^{-1}g$. Then g_n, g are in the domain of $L, g_n \to g$ and $h_n \to h$. Since h_n is a finite sum we also have $L(h_n) = s^{-1}L(g_n) \to s^{-1}L(g)$. Since L is closed, this implies that h is in the domain of L and that $L(h) = s^{-1}L(g)$. In other words, the Laplace transform of $t \mapsto L\int_0^t T(s)fds$ exists and equals $s^{-1}Lq(s)$. Then we have by taking the Laplace transform of each term

$$\int_0^\infty e^{-sl}T(l)fdt = s^{-1}L\int_0^\infty e^{-sl}T(l)fdl + s^{-1}f$$

for all $s > 0$. Multiply through by s to obtain

$$s \int_0^\infty e^{-sl} T(l) f dl = L \int_0^\infty e^{-sl} T(l) f dl + f$$

and substitute $c_1 s^{\beta_1} + c_2 s^{\beta_2}$ for s to get

$$(c_1 s^{\beta_1} + c_2 s^{\beta_2}) \int_0^\infty e^{-(c_1 s^{\beta_1} + c_2 s^{\beta_2}) l} T(l) f dl = L \int_0^\infty e^{-(c_1 s^{\beta_1} + c_2 s^{\beta_2}) l} T(l) f dl + f$$

for all $s > 0$. Now use (3.5) twice to get

$$s \int_0^\infty e^{-sl} S(l) f dl = L \biggl(\frac{s}{c_1 s^{\beta_1} + c_2 s^{\beta_2}} \int_0^\infty e^{-sl} S(l) f dl \biggr) + f$$

and multiplying both sides by $c_1 s^{\beta_1 - 2} + c_2 s^{\beta_2 - 2}$ we get

$$(c_1 s^{\beta_1 - 1} + c_2 s^{\beta_2 - 1}) \int_0^\infty e^{-sl} S(l) f dl = L s^{-1} \int_0^\infty e^{-st} S(l) f dl + c_1 s^{\beta_1 - 2} f + c_2 s^{\beta_2 - 2} f. \quad (3.11)$$

where we have again used the fact that L is closed. The term on the left hand side of (3.11) is $c_1 s^{\beta_1 - 1} r(s) + c_2 s^{\beta_2 - 1} r(s)$ which was already shown to be the Laplace transform of the function $c_1 \int_0^t \frac{(t-u)^{-\beta_1}}{\Gamma(1-\beta_1)} S(u) f du + c_2 \int_0^t \frac{(t-u)^{-\beta_2}}{\Gamma(1-\beta_2)} S(u) f du$, which is analytic in a sectorial region. Equation (3.10) also shows that $s^{\beta_i - 2}$ is the Laplace transform of $t \mapsto \frac{t^{1-\beta_i}}{\Gamma(2-\beta)}$. Now take the term $c_1 s^{\beta_1 - 2} f + c_2 s^{\beta_2 - 2} f$ to the other side and invert the Laplace transforms. Using the fact that $\{S(t) : t \geq 0\}$ is uniformly bounded, we can apply the Phragmen-Mikusinski Inversion formula for the Laplace transform (see [2, Corollary 1.4]) to obtain

$$c_1 \left(\int_0^t \frac{(t-u)^{-\beta_1}}{\Gamma(1-\beta_1)} S(u) f du - \frac{t^{1-\beta_1}}{\Gamma(2-\beta_1)} f \right) + c_2 \left(\int_0^t \frac{(t-u)^{-\beta_2}}{\Gamma(1-\beta_2)} S(u) f du - \frac{t^{1-\beta_2}}{\Gamma(2-\beta_2)} f \right)$$
$$= \lim_{n \to \infty} L \sum_{j=1}^{N_n} \alpha_{n,j} \frac{e^{c_{nj} l}}{c_{nj}} \int_0^\infty e^{-c_{nj} l} S(l) f dl$$

where the constants $N_n, \alpha_{n,j}$, and c_{nj} are given by the inversion formula and the limit is uniform on compact sets. Using again the fact that L is closed we get

$$
\begin{aligned}
& c_1 \left(\int_0^t \frac{(t-u)^{-\beta_1}}{\Gamma(1-\beta_1)} S(u) f du - \frac{t^{1-\beta_1}}{\Gamma(2-\beta_1)} f \right) \\
& + c_2 \left(\int_0^t \frac{(t-u)^{-\beta_2}}{\Gamma(1-\beta_2)} S(u) f du - \frac{t^{1-\beta_2}}{\Gamma(2-\beta_2)} f \right) \\
& = L \int_0^t S(l) f dl
\end{aligned} \tag{3.12}
$$

and since the function (3.9) is analytic in a sectorial region, the left hand side of (3.12) is differentiable for $t > 0$ Corollary 1.6.6 of [1] shows that

$$
\frac{d}{dt} \int_0^t \frac{(t-u)^{-\beta_i}}{\Gamma(1-\beta_i)} S(u) f du \tag{3.13}
$$

has Laplace transform $s^{\beta_i} r(s)$ and hence (3.13) equals $\frac{d^{\beta_i} S(t) f}{dt^{\beta_i}}$. Now take the derivative with respect to t on both sides of (3.12) to obtain

$$
c_1 \left(\frac{d^{\beta_1}}{dt^{\beta_1}} S(t) f - \frac{t^{-\beta_1}}{\Gamma(1-\beta_1)} f \right) + c_2 \left(\frac{d^{\beta_2}}{dt^{\beta_2}} S(t) f - \frac{t^{-\beta_2}}{\Gamma(1-\beta_2)} f \right) = L S(t) f
$$

for all $t > 0$, where we use the fact that L is closed to justify taking the derivative inside. Using the relation (2.23) between the Rieman-Liuoville and Caputo fractional derivatives we proved the theorem $\square$

The next theorem provides an extension with subordinator $D_\mu(t)$ as the weighted average of an arbitrary number of independent stable subordinators. Let $E_\mu(t)$ be the inverse of the subordinator $D_\mu(t)$ with Laplace exponent $\psi(s) = \int_0^1 s^\beta d\mu(\beta)$ where $\operatorname{supp} \mu \subset (0,1)$.

Theorem 3.4. *Let $(X, ||\cdot||)$ be a Banach space and μ be a positive finite measure with* $\operatorname{supp}\mu \subset (0,1)$. *Then the family $\{S(t) : t \geq 0\}$ of linear operators from X into X given by $S(t)f = \displaystyle\int_0^\infty (T(l)f) g_{E_\mu(t)}(l)dl$, is uniformly bounded and strongly analytic in a sectorial region. Furthermore, $\{S(t) : t \geq 0\}$ is strongly continuous and $h(x,t) = S(t)f(x)$ is a solution of*

$$\mathbb{D}_1^{(\mu)} h(x,t) = \int_0^1 \partial_t^\beta h(x,t)\mu(d\beta) = Lh(x,t); \; h(x,0) = f(x). \tag{3.14}$$

Proof. Since $\operatorname{supp}\mu \subset (0,1)$, the density $g_{E_\mu(t)}(l), l \geq 0$, exists and since $||T(l)f|| \leq M||f||$, then $S(t)f$ exists and $||S(t)f|| \leq M||f||$. Also, $S(t)f$ is strongly continuous as in Theorem 3.3.

Let $q(s) = \int_0^\infty e^{-st}T(t)f dt$ and $r(s) = \int_0^\infty e^{-st}S(t)f dt$ for any $s > 0$, then by (3.5) we have

$$\psi(s)q(\psi(s)) = sr(s) \quad \text{where} \quad \psi(s) = \int_0^1 s^\beta \mu(d\beta) \tag{3.15}$$

for any $s > 0$. Now we want to show that this relation holds for certain complex numbers s. In Theorem 3.3, we have shown that $q(s)$ is an analytic function on $s \in \mathbb{C}_+$ and $||q(s)|| \leq \frac{M||f||}{\mathcal{R}(s)}$.

Now we want to show that $r(s)$ is the Laplace transform of an analytic function defined on a sectorial region. It follows from equation (3.15) that

$$r(s) = \left(\int_0^1 s^{\beta-1}\mu(d\beta)\right) q\left(\int_0^1 s^\beta \mu(d\beta)\right)$$

for all $s > 0$, but the right hand side here is well defined and analytic on the set of complex s such that $\mathcal{R}\left(\int_0^1 s^\beta\mu(d\beta)\right) > 0$. Let $\beta_1 = \sup\{\operatorname{supp}\ \mu\}$ and fix $\epsilon > 0$ small such that $\frac{\pi/2-\epsilon}{\beta_1} > \pi/2$. So if $1/2 < \beta_1 < 1$, then $r(s)$ has a unique analytic extension to the sectorial region $\mathbb{C}(\frac{\pi/2-\epsilon}{\beta_1}) \subset \{s \in \mathbb{C} : \mathcal{R}(\int_0^1 s^\beta\mu(d\beta)) > 0\}$ and note that $\frac{\pi/2-\epsilon}{\beta_1} = \pi/2 + \alpha$ for some $\alpha > 0$. If $0 < \beta_1 < 1/2$ then $r(s)$ has an analytic extension to the sectorial region $s \in \mathbb{C}(\pi/2 + \alpha)$ for any $\alpha < \pi/2$, and $\mathcal{R}(\int_0^1 s^\beta\mu(d\beta)) > 0$ for all such s. Now for any complex $s = re^{i\theta}$ such that

$s \in \mathbb{C}(\pi/2 + \alpha')$ for any $0 < \alpha' < \alpha$ we have that

$$
\begin{aligned}
||sr(s)|| &= \left|\left|\left(\int_0^1 s^\beta \mu(d\beta)\right) q \left(\int_0^1 s^\beta \mu(d\beta)\right)\right|\right| \\
&\leq \left| \frac{\displaystyle\int_0^1 s^\beta \mu(d\beta)}{\mathcal{R}\left(\displaystyle\int_0^1 s^\beta \mu(d\beta\right)} \right| M||f|| \\
&= \left| \frac{\int_0^1 r^\beta \cos(\beta\theta)\mu(d\beta) + i \int_0^1 r^\beta \sin(\beta\theta)\mu(d\beta)}{\int_0^1 r^\beta \cos(\beta\theta)\mu(d\beta)} \right| M||f|| \\
&\leq \left(1 + \left| \frac{\displaystyle\int_0^1 r^\beta \sin(\beta\theta)\mu(d\beta)}{\displaystyle\int_0^1 r^\beta \cos(\beta\theta)\mu(d\beta)} \right|\right) M||f|| \\
&\leq \left(1 + \frac{\displaystyle\int_0^1 r^\beta (d\beta)}{\cos(\pi/2 - \epsilon)\displaystyle\int_0^1 r^\beta \mu(d\beta)}\right) M||f|| \\
&= \left(1 + \frac{1}{\cos(\pi/2 - \epsilon)}\right) M||f|| < \infty.
\end{aligned}
$$

Hence Theorem 2.6.1 of [1] implies that there exists an analytic function $\overline{r}(t)$ on $t \in \mathbb{C}(\alpha)$ with Laplace transform $r(s)$. Using the uniqueness of the Laplace transform it follows that $t \mapsto S(t)f$ has an analytic extension $\overline{r}(t)$ to the sectorial region $t \in \mathbb{C}(\alpha)$.

As in Theorem 3.3 for any $\beta \in \operatorname{supp} \mu$ the function

$$t \mapsto \int_0^t \frac{(t-u)^{-\beta}}{\Gamma(1-\beta)} S(u) f du \tag{3.16}$$

has analytic extension to the sectorial region $t \in \mathbb{C}(\alpha)$.

Next we wish to apply Theorem 2.6.1 of [1] again to show that for any $0 < \beta < 1$ the function

$$t \mapsto \int_0^t \left(\int_0^1 \frac{(t-u)^{-\beta}}{\Gamma(1-\beta)} \mu(d\beta)\right) S(u) f du \tag{3.17}$$

has analytic extension to the sectorial region $t \in \mathbb{C}(\alpha)$.

Since $\int_0^\infty \left(\int_0^1 \frac{t^{-\beta}}{\Gamma(1-\beta)} \mu(d\beta) \right) e^{-st} dt = \int_0^1 s^{\beta-1} \mu(d\beta)$

for any $0 < \beta < 1$ and any $s > 0$ and $r(s)$ is the Laplace transform of $t \mapsto S(t)f$ it follows from convolution property of the Laplace transform that the function (3.17) has Laplace transform $s^{-1}\psi(s)r(s)$ for all $s > 0$. Since $r(s)$ has an analytic extension to the sectorial region $s \in \mathbb{C}(\pi/2 + \alpha)$, so does $s^{-1}\psi(s)r(s)$. For any $x > 0$, if $s = x + re^{i\theta}$ for some $r > 0$ and $|\theta| < \pi/2 + \alpha'$ for any $0 < \alpha' < \alpha$ then we have

$$\begin{aligned}
\left\| (s-x) \left(\int_0^1 s^{\beta-1} \mu(d\beta) \right) r(s) \right\| &= \left\| (s-x) \int_0^1 s^{\beta-2} s.r(s) \mu(d\beta) \right\| \\
&\leq r \left\| \int_0^1 s^{\beta-2} \mu(d\beta) \right\| (1 + \frac{1}{\cos(\pi/2 - \epsilon)}) M||f|| \\
&\leq r \left(\int_0^1 ||s^{\beta-2}|| \mu(d\beta) \right) \left(1 + \frac{1}{\cos(\pi/2 - \epsilon)} \right) M||f|| \\
&\leq r \left(\int_0^1 x^{\beta-2} \mu(d\beta) \right) \left(1 + \frac{1}{\cos(\pi/2 - \epsilon)} \right) M||f|| \\
&= \frac{r}{x^2} \left(\int_0^1 x^{\beta} \mu(d\beta) \right) \left(1 + \frac{1}{\cos(\pi/2 - \epsilon)} \right) M||f||.
\end{aligned}$$

Since μ positive finite measure and $x > 0$, so that $||(s-x)s^{-1}\psi(s)r(s)||$ is bounded on the region $x + \mathbb{C}(\alpha' + \pi/2)$ for all $0 < \alpha' < \alpha$. Then it follows as before that the function (3.17) has an analytic extension to the sectorial region $\mathbb{C}(\alpha)$.

Since $\{T(t) : t \geq 0\}$ is a strongly continuous semigroup with generator L, Theorem 1.2.4 (b) in [42] implies that $\int_0^t T(l) f dl$ is in the domain of the operator L and

$$T(t)f = L \int_0^t T(l) f dl + f.$$

Then by taking Laplace transform of both sides we have

$$\int_0^\infty e^{-st}T(t)f\,dt = s^{-1}L\int_0^\infty e^{-st}T(t)f\,dt + s^{-1}f$$

for all $s > 0$. Multiply both sides by s to obtain

$$s\int_0^\infty e^{-st}T(t)f\,dt = L\int_0^\infty e^{-st}T(t)f\,dt + f$$

and substitute $\psi(s) = \int_0^1 s^\beta \mu(d\beta)$ for s to obtain

$$\psi(s)\int_0^\infty e^{-\psi(s)t}T(t)f\,dt = L\int_0^\infty e^{-\psi(s)t}T(t)f\,dt + f$$

for all $s > 0$. Now use (3.15) twice to get

$$s\int_0^\infty e^{-st}S(t)f\,dt = L\frac{s}{\psi(s)}\int_0^\infty e^{-st}S(t)f\,dt + f$$

and multiplying through by $s^{-2}\psi(s)$ to get

$$s^{-1}\psi(s)\int_0^\infty e^{-st}S(t)f\,dt = Ls^{-1}\int_0^\infty e^{-st}S(t)f\,dt + \psi(s)s^{-2}f$$

since L is closed. Invert the Laplace transform to get

$$\begin{aligned}&\int_0^t \left(\int_0^1 \frac{(t-u)^{-\beta}}{\Gamma(1-\beta)}\mu(d\beta)\right) S(u)f\,du - \int_0^1 \frac{t^{1-\beta}}{\Gamma(2-\beta)} f\mu(d\beta)\\ &= \lim_{n\to\infty} L\sum_{j=1}^{N_n} \alpha_{n,j}\frac{e^{c_{n_j}t}}{c_{n_j}}\int_0^\infty e^{-C_{n_j}t}S(t)f\,dt\end{aligned} \tag{3.18}$$

where the constant $N_n, \alpha_{n,j}$, and c_n are given by the inversion formula.

Next using Fubini's theorem we show that $\int_0^1 \int_0^t \frac{(t-u)^{-\beta}}{\Gamma(1-\beta)} S(u) f du \mu(d\beta)$ have same Laplace transform $s^{-1}\psi(s)r(s)$. This is true because

$$\begin{aligned}
&\int_0^1 \int_0^\infty \left\| e^{-st} \int_0^t \frac{(t-u)^{-\beta}}{\Gamma(1-\beta)} S(u) f \right\| dudt\mu(d\beta) \\
&\leq M||f|| \int_0^1 \int_0^\infty e^{-st} \int_0^t \frac{(t-u)^{-\beta}}{\Gamma(1-\beta)} dudt\mu(d\beta) \\
&= M||f|| \int_0^1 \int_0^\infty \frac{e^{-st} t^{1-\beta}}{\Gamma(2-\beta)} dt\mu(d\beta) \\
&\leq M||f|| \int_0^1 s^{\beta-2} \mu(d\beta) < \infty.
\end{aligned} \tag{3.19}$$

Since μ is positive finite measure and $S(t)f$ is uniformly bounded then using Fubini's theorem and the uniqueness of the Laplace transform for functions in $L^1_{loc}(\mathbb{R}^d)$ (Theorem 1.7.3 in [1]) we have

$$\begin{aligned}
&\int_0^1 \left[\int_0^t \frac{(t-u)^{-\beta}}{\Gamma(1-\beta)} S(u) f du - \frac{t^{1-\beta}}{\Gamma(2-\beta)} f \right] \mu(d\beta) \\
&= \lim_{n\to\infty} L \sum_{j=1}^{N_n} \alpha_{n,j} \frac{e^{c_{n_j} t}}{c_{n_j}} \int_0^\infty e^{-C_{n_j} t} S(t) f dt.
\end{aligned} \tag{3.20}$$

Using again the fact that L is closed we get

$$\int_0^1 \left[\int_0^t \frac{(t-u)^{-\beta}}{\Gamma(1-\beta)} S(u) f du - \frac{t^{1-\beta}}{\Gamma(2-\beta)} f \right] \mu(d\beta) = L \int_0^t S(u) f du$$

and now take the derivative with respect to t on both sides to obtain

$$\int_0^1 \left(\frac{d^\beta}{dt^\beta} S(t) f - \frac{t^{-\beta}}{\Gamma(1-\beta)} f \right) \mu(d\beta) = LS(t) f$$

for all $t > 0$, where we use the fact that L is closed to justify taking the derivative inside. $\square$

Corollary 3.5. *Let $0 < \gamma \leq 2$. Let $-(-\Delta)^{\gamma/2}$ be fractional Laplacian on $L^1(\mathbb{R}^d)$ corresponding to the semigroup $T(t)$ on $L^1(\mathbb{R}^d)$. Let $Y(t)$ be the corresponding symmetric stable process (i.e. $T(t)f(x) = \mathbb{E}_x(f(Y(t)))$). Then the family $\{S(t) : t \geq 0\}$ of linear operators from X*

into X *given by* $S(t)f = \int_0^\infty (T(l)f)g_{E_\mu(t)}(l)dl = \mathbb{E}(f(Y(E_\mu(t))))$, *is uniformly bounded and strongly analytic in a sectorial region. Furthermore,* $\{S(t) : t \geq 0\}$ *is strongly continuous and* $h(x,t) = S(t)f(x)$ *is a solution of*

$$\int_0^1 \partial_t^\beta h(x,t)\mu(d\beta) = -(-\Delta)^{\gamma/2} h(x,t); \;\; h(x,0) = f(x). \tag{3.21}$$

Chapter 4

Two-term trace estimates

In this chapter we state and proof our main result about trace of relativistic stable processes for $R-$smooth boundary domains. The asymptotic behavior for the trace of killed symmetric $\alpha-$stable processes, $\alpha \in (0,2)$, for an open bounded set with $R-$smooth boundary was given in [7]. When $m = 0$ our result reduces to the result for $\alpha-$stable processes as given in [7].

Theorem 4.1. *Let $D \subset \mathbb{R}^d, d \geq 2$, be an open bounded set with $R-$ smooth boundary. Let $|D|$ denote the volume (d$-$dimensional Lebesgue measure) of D and ∂D denote its surface area ((d $-$ 1)$-$dimensional Lebesgue measure) of its boundary. Suppose $\alpha \in (0,2)$. Then*

$$\left| Z_D(t) - \frac{C_1(t)e^{mt}|D|}{t^{d/\alpha}} + C_2(t)|\partial D| \right| \leq \frac{C_3 e^{2mt}|D|t^{2/\alpha}}{R^2 t^{d/\alpha}},\ t > 0, \tag{4.1}$$

where

$$C_1(t) = \frac{1}{(4\pi)^{d/2}} \int_0^\infty z^{-d/2} e^{-(mt)^{1/\beta} z} \theta_\beta(1,z) dz \to C_1 = \frac{\omega_d \Gamma(d/\alpha)}{(2\pi)^d \alpha} \quad \text{as } t \to 0,$$

$$C_2(t) = \int_0^\infty r_H(t,(x_1,0,\cdots,0),(x_1,0,\cdots,0))dx_1 \leq \frac{C_4 e^{2mt} t^{1/\alpha}}{t^{d/\alpha}}, \quad t > 0$$

$$C_4 = \int_0^\infty \tilde{r}_H(1,(x_1,0,\cdots,0),(x_1,0,\cdots,0))dx_1$$

$C_3 = C_3(d,\alpha), H = (x_1,\cdots,x_d) \in \mathbb{R}^d : x_1 > 0$ *and* r_H *is given by* (1.4)

When $m = 0$, $0 < \alpha \leq 2$ our result becomes for bounded domains with $R-$smooth boundary

$$\left| Z_D(t) - \frac{C_5|D|}{t^{d/\alpha}} + \frac{C_4|\partial D|t^{1/\alpha}}{t^{d/\alpha}} \right| \leq \frac{C_7|D|t^{2/\alpha}}{R^2 t^{d/\alpha}} \tag{4.2}$$

where $c_5 = \frac{\omega_d \Gamma(d/\alpha)}{(2\pi)^d \alpha}$, C_4 as in Theorem 4.1. This was established by Bañuelos and Kulczycki [7] for stable processes.

Proof of Theorem 4.1. For the case $t^{1/\alpha} > R/2$ the theorem holds trivially. This is true because for such $t's$ we have by Equation (1.3)

$$Z_D(t) \leq \int_D p(t,x,x)dx \leq \frac{c_1 e^{mt}|D|}{t^{d/\alpha}} \leq \frac{c_1 e^{mt}|D| t^{2/\alpha}}{R^2 t^{d/\alpha}}$$

By Corollary 2.15 and Lemma 2.13 we also have

$$C_2(t)|\partial D| \leq \frac{C_4 e^{2mt}|\partial D| t^{1/\alpha}}{t^{d/\alpha}} \leq \frac{2^d C_4 e^{2mt}|D| t^{1/\alpha}}{R t^{d/\alpha}} \leq \frac{2^{d+1} C_4 e^{2mt}|D| t^{2/\alpha}}{R^2 t^{d/\alpha}}$$

$$\frac{C_1(t) e^{mt}|D|}{t^{d/\alpha}} \leq \frac{C_1 e^{mt}|D| t^{2/\alpha}}{R^2 t^{d/\alpha}}$$

Therefore for $t^{1/\alpha} > R/2$ (4.1) holds. Here and in sequel we consider the case $t^{1/\alpha} \leq R/2$. From (1.5) and the fact that $p(t,x,x) = \frac{C_1(t) e^{mt}}{t^{d/\alpha}}$, we have that

$$\begin{aligned} Z_D(t) - \frac{C_1(t) e^{mt}|D|}{t^{d/\alpha}} &= \int_D p_D(t,x,x)dx - \int_D p(t,x,x)dx \\ &= -\int_D r_D(t,x,x)dx, \end{aligned} \tag{4.3}$$

where $C_1(t)$ is as stated in the theorem. Therefore we must estimate (4.3). We break our domain into two pieces, $D_{R/2}$ and its complement. We will first consider the contribution of $D_{R/2}$.

Claim 1:

$$\int_{D_{R/2}} r_D(t,x,x)dx \leq \frac{c e^{2mt}|D| t^{2/\alpha}}{R^2 t^{d/\alpha}} \tag{4.4}$$

for $t^{1/\alpha} \leq R/2$.

Proof of Claim 1: By Lemma 2.13 we have

$$\int_{D_{R/2}} r_D(t,x,x)dx \le e^{2mt} \int_{D_{R/2}} \tilde{r}_D(t,x,x)dx, \tag{4.5}$$

and by scaling of the stable density the right hand side of (4.5) equals

$$\frac{e^{2mt}}{t^{d/\alpha}} \int_{D_{R/2}} \tilde{r}_{D/t^{1/\alpha}}(1, \frac{x}{t^{1/\alpha}}, \frac{x}{t^{1/\alpha}})dx. \tag{4.6}$$

For $x \in D_{R/2}$ we have $\delta_{D/t^{1/\alpha}}(x/t^{1/\alpha}) \ge R/(2t^{1/\alpha}) \ge 1$. By [7, Lemma 2.1], we get

$$\tilde{r}_{D/t^{1/\alpha}}\left(1, \frac{x}{t^{1/\alpha}}, \frac{x}{t^{1/\alpha}}\right) \le \frac{c}{\delta^{d+\alpha}_{D/t^{1/\alpha}}(x/t^{1/\alpha})} \le \frac{c}{\delta^{2}_{D/t^{1/\alpha}}(x/t^{1/\alpha})} \le \frac{ct^{2/\alpha}}{R^2}.$$

Using the above inequality, we get

$$\int_{D_{R/2}} r_D(t,x,x)dx \le \frac{e^{2mt}}{t^{d/\alpha}} \int_{D_{R/2}} \frac{ct^{2/\alpha}}{R^2} dx \le \frac{ce^{2mt}|D|t^{2/\alpha}}{R^2 t^{d/\alpha}},$$

which proves (4.4).

Now using proposition 2.16 we estimate the contribution from $D \backslash D_{R/2}$ to the integral of $r_D(t,x,x)$ in (4.3).

Claim 2:

$$\left| \int_{D\backslash D_{R/2}} r_D(t,x,x)dx - \int_{D\backslash D_{R/2}} r_{H(x)}(t,x,x)dx \right| \le \frac{ce^{2mt}|D|t^{2/\alpha}}{R^2 t^{d/\alpha}} \tag{4.7}$$

for $t^{1/\alpha} \le R/2$.

Proof of Claim 2: By Proposition 2.16 the left hand side of (4.7) is bounded above by

$$\frac{ce^{2mt}t^{1/\alpha}}{Rt^{d/\alpha}} \int_0^{R/2} |\partial D_q| \left(\left(\frac{t^{1/\alpha}}{q} \right)^{d+\alpha/2-1} \wedge 1 \right) dq.$$

By Corollary 2.15, (i), the last quantity is smaller than or equal to

$$\frac{ce^{2mt}(e^{mt}t^{1/\alpha}|\partial D|}{Rt^{d/\alpha}}\int_0^{R/2}\left(\left(\frac{t^{1/\alpha}}{q}\right)^{d+\alpha/2-1}\wedge 1\right)dq.$$

The integral in last quantity is bounded above by $ct^{1/\alpha}$. To see this observe that since $t^{1/\alpha}\le R/2$ the above integral is equal to

$$\begin{aligned}
& \frac{ce^{2mt}t^{1/\alpha}|\partial D|}{Rt^{d/\alpha}}\left[\int_0^{t^{1/\alpha}}\left(\left(\frac{t^{1/\alpha}}{q}\right)^{d+\alpha/2-1}\wedge 1\right)dq+\int_{t^{1/\alpha}}^{R/2}\left(\left(\frac{t^{1/\alpha}}{q}\right)^{d+\alpha/2-1}\wedge 1\right)dq\right. \\
= & \frac{ce^{2mt}t^{1/\alpha}|\partial D|}{Rt^{d/\alpha}}\left[\int_0^{t^{1/\alpha}}1dq+\int_{t^{1/\alpha}}^{R/2}\left(\frac{t^{1/\alpha}}{q}\right)^{d+\alpha/2-1}dq\right] \\
\le & \frac{ce^{2mt}t^{2/\alpha}|\partial D|}{Rt^{d/\alpha}}.
\end{aligned}$$

Using this and Corollary (2.15), (ii), we get (4.7).

Recall that $H=\{(x_1,\cdots,x_d)\in\mathbb{R}^d : x_1>0\}$. For abbreviation let us denote

$$f_H(t,q)=r_H(t,(q,0,\cdots,0),(q,0,\cdots,0)),\quad t,q>0.$$

Of course we have $r_{H(x)}(t,x,x)=f_H(t,\delta_H(x))$. In the next step we will show that

$$\left|\int_{D\setminus D_{R/2}} r_{H(x)}(t,x,x)dx-|\partial D|\int_0^{R/2} f_H(t,q)dq\right|\le\frac{ce^{2mt}|D|t^{2/\alpha}}{R^2t^{d/\alpha}} \tag{4.8}$$

We have

$$\int_{D\setminus D_{R/2}} r_{H(x)}(t,x,x)dx=\int_0^{R/2}|\partial D_q|f_H(t,q)dq$$

Hence the left hand side of (4.8) is bounded above by

$$\int_0^{R/2}||\partial D_q|-|\partial D||\,f_H(t,q)dq$$

By Corollary 2.15, (iii), this is smaller than

$$
\begin{aligned}
& \frac{c|D|}{R^2}\int_0^{R/2} q f_H(t,q)dq \\
\leq\ & \frac{c|D|e^{2mt}}{R^2}\int_0^{R/2} q\tilde{f}_H(t,q)dq \\
=\ & \frac{c|D|e^{2mt}}{R^2}\int_0^{R/2} qt^{-d/\alpha}\tilde{f}_H(1,qt^{-1/\alpha})dq \\
=\ & \frac{c|D|e^{2mt}}{R^2 t^{d/\alpha}}\int_0^{R/2t^{1/\alpha}} qt^{2/\alpha}\tilde{f}_H(1,q)dq \\
\leq\ & \frac{c|D|e^{2mt}t^{2/\alpha}}{R^2 t^{d/\alpha}}\int_0^{\infty} q\left(q^{-d-\alpha}\wedge 1\right)dq \leq \frac{c|D|e^{2mt}t^{2/\alpha}}{R^2 t^{d/\alpha}}
\end{aligned}
$$

This shows (4.8). Finally, we have

$$
\begin{aligned}
& \left| |\partial D|\int_0^{R/2} f_H(t,q)dq - |\partial D|\int_0^{\infty} f_H(t,q)dq \right| \\
\leq\ & |\partial D|\int_{R/2}^{\infty} f_H(t,q)dq \\
\leq\ & \frac{c|D|}{R}\int_{R/2}^{\infty} f_H(t,q)dq \quad \text{by Corollary 2.15, (ii)} \\
\leq\ & \frac{c|D|e^{2mt}}{Rt^{d/\alpha}}\int_{R/2}^{\infty} \tilde{f}_H(1,qt^{-1/\alpha})dq \\
=\ & \frac{c|D|e^{2mt}t^{1/\alpha}}{Rt^{d/\alpha}}\int_{R/2t^{1/\alpha}}^{\infty} \tilde{f}_H(1,q)dq
\end{aligned}
$$

Since $R/2t^{1/\alpha} \geq 1$, so for $q \geq R/2t^{1/\alpha} \geq 1$ we have $\tilde{f}_H(1,q) \leq cq^{-d-\alpha} \leq cq^{-2}$. Therefore,

$$
\int_{R/2t^{1/\alpha}}^{\infty} \tilde{f}_H(1,q)dq \leq c\int_{R/2t^{1/\alpha}}^{\infty} \frac{dq}{q^2} \leq \frac{ct^{1/\alpha}}{R}.
$$

Hence,

$$
\left| |\partial D|\int_0^{R/2} f_H(t,q)dq - |\partial D|\int_0^{\infty} f_H(t,q)dq \right| \leq \frac{c|D|e^{2mt}t^{2/\alpha}}{R^2 t^{d/\alpha}} \tag{4.9}
$$

Note that the constant $C_2(t)$ which appears in the formulation of Theorem (4.1) satisfies $C_2(t) = \int_0^\infty f_H(t,q)dq$. Now equations (4.3), (4.4), (4.7), (4.8), (4.9) give (4.1). $\square$

Chapter 5

Mixed Stable Processes

In this chapter we explore the basic general properties of the sum of two independent stable processes and give a first term asymptotic expansion of the trace. I'm still working on finding a better trace estimate for such processes for a domain with $R-$smooth boundary. Most of the notations and results of this chapter are adapted from [17].

Let X be a Lévy process that is the independent sum of an α-stable process Y and a β-stable process W in bounded open subset of $\mathbb{R}^d$. The infinitesimal generator of the Lévy process X is $\Delta^{\alpha/2}+\Delta^{\beta/2}$. Let $p_D^1(t,x,y)$ and $G_D^1(x,y)$ denote the transition density function and the Green function of the subprocess X_D of X killed upon exiting an open set $D \subset \mathbb{R}^d$. Let $p_D(t,x,y)$ and $G_D(x,y)$ denote the transition density function and Green function of the subprocess Y_D of Y killed upon exiting D. Intuitively, one expects the following Duhamel's formulas (or Trotter-Kato formula) hold:

$$p_D^1(t,x,y) = p_D(t,x,y) + \int_0^t \int_D p_D^1(s,x,z)\Delta_z^{\beta/2} p_D(t-s,z,y)\, dz \tag{5.1}$$

$$G_D^1(x,y) = G_D(x,y) + \int_D G_D^1(x,z)\Delta_z^{\beta/2} G_D(z,y)\, dz \tag{5.2}$$

The Lévy process X runs on two different scales: on the small spatial scale, the α component dominates, while on the large spatial scale the β component takes over. Both components play essential roles, and so in general this process can not be regarded as a perturbation of the $\alpha-$stable process or of the $\beta-$stable process.

Let us first recall some basic facts about the independent sum of stable processes and state our main result. Throughout the remainder of this paper, we assume that $d \geq 1$ and $0 < \beta < \alpha < 2$. The Euclidean distance between x and y will be denoted as $|x-y|$.

Suppose X is a symmetric $\alpha-$stable process and Y is a symmetric $\beta-$stable process on $\mathbb{R}^d$ and that X and Y are independent. For any $a \geq 0$, we define X^a by $X^a_t := X_t + aY_t$. We will call the process X^a the independent sum of the symmetric $\alpha-$stable process X and the symmetric $\beta-$stable process Y with weight a. The infinitesimal generator of X^a is $\Delta^{\alpha/2} + a^\beta \Delta^{\beta/2}$. Let $p^a(t, x, y)$ denote the transition density of X^a (or equivalently the heat kernel of $\Delta^{\alpha/2} + a^\beta \Delta^{\beta/2}$) with respect to the Lebesgue measure on $\mathbb{R}^d$. We will use $p(t, x, y) = p^0(t, x, y)$ to denote the transition density of $X = X^0$. Recently it is proven in [17] that

$$p^1(t, x, y) \asymp \left(t^{-d/\alpha} \wedge t^{-d/\beta}\right) \wedge \left(\frac{t}{|x-y|^{d+\alpha}} + \frac{a^\beta t}{|x-y|^{d+\beta}}\right) \quad \text{on } (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \tag{5.3}$$

Here and in the sequel, for $a, b \in \mathbb{R}, a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$; for any two positive functions f and g, $f \asymp g$ means that there is a positive constant $c \geq 1$ so that $c^{-1} g \leq f \leq cg$ on their common domain of definition.

For every open subset $D \subset \mathbb{R}^d$, we denote by $X^{a,D}$ the subprocess of X^a killed upon leaving D. The infinitesimal generator of $X^{a,D}$ is $\Delta^{\alpha/2} + a^\beta \Delta^{\beta/2}|_D$, the sum of two fractional Laplacians in D with zero exterior condition. It is known [17] that $X^{a,D}$ has a Hölder continuous transition density $p^a_D(t, x, y)$ with respect to the Lebesgue measure.

Unlike the case of the symmetric $\alpha-$stable process $X := X^0, X^a$ does not have the stable scaling for $a > 0$. Instead, the following approximate scaling property is true and will be used several times in the rest of this paper: If $\{X^{a,D}, t \geq 0\}$ is the subprocess of X^a killed upon t leaving D, then $\{\lambda^{-1} X^{a,D}_{\lambda^\alpha t}, t \geq 0\}$ is the subprocess of $\{X^{a\lambda^{(\alpha-\beta)/\beta}}_t, t \geq 0\}$ killed upon leaving $\lambda^{-1} D := \{\lambda^{-1} y : y \in D\}$. Consequently, for any $\lambda > 0$, we have

$$p^{a\lambda^{(\alpha-\beta)/\beta}}_{\lambda^{-1}D}(t, x, y) = \lambda^d p^a_D(\lambda^\alpha t, \lambda x, \lambda y) \quad \text{for } t > 0 \text{ and } x, y \in \lambda^{-1} D \tag{5.4}$$

In particular, letting $a = 1, \lambda = a^{\beta/(\alpha-\beta)}$ and $D = \mathbb{R}^d$, we get

$$p^a(t,x,y) = a^{d\beta/(\alpha-\beta)} p^1(a^{\alpha\beta/(\alpha-\beta)}t, a^{\beta/(\alpha-\beta)}x, a^{\beta/(\alpha-\beta)}y) \quad \text{for} \quad t > 0 \text{ and } \ x, y \in \mathbb{R}^d. \quad (5.5)$$

So we deduce from (5.3) that there exists a constants $C > 1$ depending only on d, α and β such that for every $a > 0$ and $(t, x, y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$

$$C^{-1} f^a(t,x,y) \leq p^a(t,x,y) \leq C f^a(t,x,y), \quad (5.6)$$

where

$$f^a(t,x,y) = \left(t^{-d/\alpha} \wedge (a^\beta t)^{-d/\beta}\right) \wedge \left(\frac{t}{|x-y|^{d+\alpha}} + \frac{a^\beta t}{|x-y|^{d+\beta}}\right).$$

For a domain $D \subset \mathbb{R}^d$, we define the first exit time from D by $\tau_D^a = \inf\{t \geq 0 : X_t^a \notin D\}$.

We set

$$r_D^a(t,x,y) = \mathbb{E}^x[p^a(t - \tau_D, X^a_{\tau^a_D}, y); \tau_D^a < t] \quad (5.7)$$

and

$$p_D^a(t,x,y) = p^a(t,x,y) - r_D^a(t,x,y) \quad (5.8)$$

for any $x, y \in \mathbb{R}^d$, $t > 0$.

We are interested in the behavior of the trace of this semigroup defined by

$$Z_D^a(t) = \int_D p_D^a(t,x,x)dx. \quad (5.9)$$

Lemma 5.1. *Let $D \subset \mathbb{R}^d$ be an open set. For any $x, y \in D$ we have*

$$r_D^a(t,x,y) \leq c \left(\left(t^{-d/\alpha} \wedge (a^\beta t)^{-d/\beta}\right) \wedge \left(\frac{t}{(\delta_D^a(x))^{d+\alpha}} + \frac{a^\beta t}{(\delta_D^a(x))^{d+\beta}} \right) \right)$$

Proof.

$$\begin{aligned}
r_D^a(t,x,y) &= E^y\left[\tau_D < t; p^a(t-\tau_D, X^a(\tau_D), x)\right] \\
&\leq cE^y\left[\left(t^{-d/\alpha} \wedge (a^\beta t)^{-d/\beta}\right) \wedge \left(\frac{t}{|x - X^a(\tau_D)|^{d+\alpha}} + \frac{a^\beta t}{|x - X^a(\tau_D)(x)|^{d+\beta}}\right)\right] \\
&\leq c\left(\left(t^{-d/\alpha} \wedge (a^\beta t)^{-d/\beta}\right) \wedge \left(\frac{t}{|\delta_D^a(x)|^{d+\alpha}} + \frac{a^\beta t}{|\delta_D^a(x)|^{d+\beta}}\right)\right)
\end{aligned}$$

□

Proposition 5.2 gives the asymptotic of $Z_D^a(t)$ near $t = 0$. So far we are only able find the first term asymptotic expansion of $Z_D^a(t)$.

Proposition 5.2.

$$\lim_{t\to 0} t^{d/\alpha} Z_D^a(t) = C_1|D| \tag{5.10}$$

where $C_1 = \frac{\omega_d \Gamma(d/\alpha)}{(2\pi)^d \alpha}$.

Proof. By the definition of r_D^a we see that

$$t^{d/\alpha} Z_D^a(t) = \int_D t^{d/\alpha} p_D^a(t,x,x)dx = \int_D t^{d/\alpha} p^a(t,x)dx - \int_D t^{d/\alpha} r_D^a(t,x,x) \tag{5.11}$$

Let us first consider the first term on the right hand side of (5.11), by scaling property, we have

$$\begin{aligned}
\int_D t^{d/\alpha} p^a(t,x)dx = \int_D p^{at^{(\alpha-\beta)/\alpha\beta}}(1,x)dx &= |D| p^{at^{(\alpha-\beta)/\alpha\beta}}(1,0) \\
\to p(1,0)|D| = C_1|D| \text{ as } t \to 0.
\end{aligned}$$

In order to prove the proposition (5.2), we must show that

$$\int_D t^{d/\alpha} r_D^a(t,x,x)dx \to 0, \quad \text{as} \quad t \to 0 \tag{5.12}$$

By Lemma (5.1) we have $t^{d/\alpha} r_D^a(t,x,x) \le c$ and consider the sub-domains $D_t = \{x \in D : \delta_D^a(x) \ge t^{1/2\alpha}\}$ and its complement $D_t^c = \{x \in D : \delta_D^a(x) < t^{1/2\alpha}\}$. Since the indicator function of the set D_t^c tends to zero pointwise, the Lebesgue dominated convergence theorem implies, assuming $|D| < \infty$, that $|D_t^c| \to 0$ as $t \to 0$. It follows that

$$\int_{D_t^c} t^{d/\alpha} r_D^a(t,x,x)dx \to 0, \quad as \quad t \to 0.$$

On the other hand, by Lemma (**??**) we have

$$t^{d/\alpha} r_D^a(t,x,x) \le c t^{d/\alpha} \left(\frac{t}{|\delta_D^a(x)|^{d+\alpha}} + \frac{a^\beta t}{|\delta_D^a(x)|^{d+\beta}} \right) \tag{5.13}$$

For $x \in D_t$ and $0 < t < 1$, the right hand side of (5.13) is bounded by

$$c t^{d/\alpha} \left(\frac{t}{\left(t^{1/2\alpha}\right)^{d+\alpha}} + \frac{a^\beta t}{\left(t^{1/2\alpha}\right)^{d+\beta}} \right)$$

and therefore

$$\int_{D_t} t^{d/\alpha} r_D^a(t,x,x)dx \le c|D| \left(t^{d/2\alpha+1/2} + a^\beta t^{d/2\alpha+1-\beta/2\alpha} \right) \tag{5.14}$$

and this last quantity goes to 0 as $t \to 0$ since $0 < \beta < \alpha < 2$. This proves the proposition (5.2). □

Let $N(\lambda)$ be the number of eigenvalues $\{\lambda_j\}$ which do not exceed λ, it follows from (5.10) and the classical Tauberian theorem (see for example [20], p.445 Theorem 2) that

$$\lim_{\lambda \to \infty} \lambda^{-d/\alpha} N(\lambda) = \frac{C_1 |D|}{\Gamma(1 + d/\alpha)} \tag{5.15}$$

This is the analogue for the sum of two independent stable process of the celebrated Weyl's asymptotic formula for the eigenvalues of the Laplacian.

Chapter 6

Open Problems and Future Work

In this chapter we discuss some open problems and my future work. The first paragraph talks about extending our result for time-fraction Cauchy problems to more general time operator. I'm also trying to extend our result for relativistic stable process to more general processes like sum of two independent stable processes.

The table below contains different time operator for sub diffusion, ultraslow diffusion, and intermediate between sub-diffusion and diffusion. Our result in this thesis gives ultraslow diffusion in special case when $\mu \in RV_0(\theta - 1)$ for some $\theta > 0$ [34, Theorem 3.9]. Currently, I'm working on finding strong analytic solution for Tempered fractional Cauchy problems and then extending all this time fractional result to more general time operator.

Laplace symbol: $\psi(s)$	inverse subordinator	time operator
$\int_0^\infty (1 - e^{-sy})\nu(dy)$	$E_\psi(t)$	$\psi(\partial_t) - \delta(0)\nu(t, \infty)$
s^β	$E(t)$	∂_t^β, Caputo
$\int_0^1 s^\beta \Gamma(1-\beta)\mu(d\beta)$	$E_\mu(t)$	$\int_0^1 \partial_t^\beta \Gamma(1-\beta)\mu(d\beta)$
$(s+\lambda)^\beta - \lambda^\beta$	$E_\lambda(t)$	$\partial_t^{\beta,\lambda}$ in (6.1)

$$\begin{aligned} \partial_t^{\beta,\lambda} g(t) &= \psi_\lambda(\partial_t)g(t) - g(0)\phi_\lambda(t,\infty) \\ &= e^{-\lambda t}\frac{1}{\Gamma(1-\beta)} d_t \left[\int_0^t \frac{e^{\lambda s} g(s)\, ds}{(t-s)^\beta} \right] - \lambda^\beta g(t) \\ &- \frac{g(0)}{\Gamma(1-\beta)} \int_t^\infty e^{-\lambda r} \beta r^{-\beta-1}\, dr. \end{aligned} \tag{6.1}$$

Subdiffusion: $0 < \beta < 1$, $\mathbb{E}_x(W(E(t)))^2 = \mathbb{E}(E(t)) \approx t^\beta$.

Ultraslow diffusion: For special $\mu \in RV_0(\theta - 1)$ for some $\theta > 0$: $\mathbb{E}_x(W(E_\mu(t)))^2 = \mathbb{E}(E_\mu(t)) \approx (\log t)^\theta$ [34, Theorem 3.9].

Intermediate between subdiffusion and diffusion: Tempered fractional diffusion

$$\mathbb{E}_x(W(E_\lambda(t)))^2 \approx \begin{cases} t^\beta/\Gamma(1+\beta), & t << 1 \\ t/\beta, & t >> 1. \end{cases}$$

$W(E_\lambda(t))$ occupies an intermediate place between subdiffusion and diffusion (Stanislavsky et al., 2008)

I am also working on estimating the trace of general processes like sum of two independent stable process over a bounded domains with $R-$smooth boundary and Lipschitz domains. So far I am able find the first term asymptotic for the trace of such processes over a bounded domains with $R-$smooth boundary.

Bibliography

[1] Arendt, W., Batty, C., Hieber, M. and Neubrander, F. (2001). *Vector-Valued Laplace Transforms and Cauchy Problems.* Monographs in Mathematics **96**, Birkhäuser, Basel.

[2] Baeumer, B. (2003). On the inversion of the convolution and Laplace transform. Trans. Amer. Math. Soc. **355**, no. 3, 1201–1212 (electronic).

[3] Baeumer, B. and Meerschaert, M. M. (2001). Stochastic solutions for fractional Cauchy problems. *Fract. Calc. Appl. Anal.* **4**, 481–500.

[4] Baeumer, B. and Meerschaert, M. M. (2010). Tempered stable Lévy motion and transient super-diffusion. *J. Comput. Appl. Math.* **233**, 2438–2448.

[5] Bañuelos, R. (2009). Trace asymptotics for stable heat semigroups Schrd́ingier Institute Lecture, Vienna.

[6] Bañuelos, R., Mijena, J.B. and Nane, E. Two-term trace estimates for relativistic stable processes. (*Submitted*). Journal of Mathematical Analysis and Applications.

[7] Bañuelos, R. Kulczycki, T. (2008). Trace estimates for stable processes. Probab. Theory and Rel. Fields 142 no. 3-4 313-338.

[8] Bañuelos, R., Kulczycki, T. and Siudeja, B. (2009). "On the Trace of symmetric stable processes on Lipschitz domains", J. Funct. Anal. 257.

[9] Barlow, M.T. and Černý, J. (2009). Convergence to fractional kinetics for random walks associated with unbounded conductances. *Probab. Theory Related Fields* Doi:10.1007/s00440-009-0257-z.

[10] Bazhlekova, E. (2000). Subordination principle for fractional evolution equations. *Fract. Calc. Appl. Anal.* **3**, 213-230.

[11] Benson, D. A., Wheatcraft, S.W., and Meerschaert, M. M.(2000). Application of a fractionaladvection-dispersion equation. . *Water Resour. Res* **36(6)**, 1403-1412.

[12] Blumenthal, R.M., Getoor, R.K., and Ray, D.B. (1961). On the distribution of first hits for the symmetric stable processes. Trans. Amer. Math. Soc. 99.

[13] Brossard, J. and Carmona, R. (1986). Can one hear the dimension of a fractal? Comm. Math. Phys. **104,** 103-122.

[14] Bogdan, K. (1986). The boundary Harnack principle for the fractional Laplacian. Stud. Math. **123,** 103-122

[15] Caputo, M. (1967). Linear models of dissipation whose Q is almost frequency independent, Part II. *Geophys. J. R. Astr. Soc.* **13** 529–539.

[16] Chechkin, A. V., Gorenfo, R., and Sokolov, I. M. (2002). Retarding subdiffusion and accelerating superdiffusion governed by distributed order fractional diffusion equations. *Phys. Rev. E* **66** 1–7.

[17] Chen, Z.Q. and Kumagai, T. (2008). Heat kernel estimates for jump processes of mixed types on metric measure spaces. Probab. Theory Relat. Fields, 140, 277-317.

[18] Davies, E.B. (1989). *Heat Kernels and Spectral Theory.* Cambridge Univ. Press, Cambridge.

[19] Einstein, A. (1906). On the theory of the Brownian movement. *Annalen der Physik*, **4** 171-180.

[20] Feller, W. (1971). An Introduction to Probability Theory and its Applications. Vol. II. Wiley, New York.

[21] Gorenflo, R., Mainardi, F., Scalas, E. and Raberto, M. Fractional calculus and continuous-time finance. III (2001). *Trends Math., Birhauser, Basel.*, 171-180.

[22] Hahn, M. G., Kobayashi, K. and Umarov, S. (2009). SDEs driven by a time-changed Lévy process and their associated time-fractional order pseudo-differential equations. Journal of Theoretical Probability. (to appear). DOI: 10.1007/s10959-010-0289-4

[23] Hille, E. and Phillips, R. S. (1957). Functional Analysis and Semi-Groups. Amer. Math. Soc. Coll. Publ. **31**, American Mathematical Society, Providence.

[24] Ikeda, N. and Watanabe, S. (1962). On some relations between the harmonic measure and the Lévy measure for a certain class of Markov processes, J. Math. Kyoto Univ. **2,** 79-95.

[25] Kochubei, A. N. (2008). Distributed order calculus and equations of ultraslow diffusion. *J. Math. Anal. Appl.* **340** 252–281.

[26] Kulczycki, T., Siudeja, B. (2006). Intrinsic Ultracontractivity of the Feynman-Kac semigroup for relativistic stable processes. Trans. Am. Math. Soc. **358** 5025-5057.

[27] Luchko, Y. (2009). Boundary value problems for the generalized time-fractional diffusion equation of distributed order. *Fract. Calc. Appl. Anal.* **12**, 409–422.

[28] Mainardi, F., Mura, A., Pagnini, G. and Gorenflo, R. (2008). Time-fractional diffusion of distributed order, Journal of Vibration and Control, **14**, 1267–1290.

[29] Mainardi, F., Mura, A., Gorenflo, R. and Stojanovic, M. (2007). The two forms of fractional relaxation of distributed order, Journal of Vibration and Control,**13**, 1249–1268.

[30] Mainardi, F., Pagnini, G. and Gorenflo, R.(2007). Some aspects of fractional diffusion equations of single and distributed order, Applied Mathematics and Computation, **187**, 295–305.

[31] Meerschaert, M. M., Benson, D. A., Scheffler, H. P. and Baeumer, B. (2002). Stochastic solution of space-time fractional diffusion equations. *Phys. Rev. E* **65** 1103–1106.

[32] Meerschaert, M. M. and Scheffler, H. P. (2004). Limit theorems for continuous time random walks with infinite mean waiting times. *J. Appl. Probab.* **41** 623–638.

[33] Meerschaert, M. M. and Sikorskii, A. (2012). Stochastic Models for Fractional Calculus. De Gruyter Studies in Mathematics.

[34] Meerschaert, M. M. and Scheffler, H. P. (2006). Stochastic model for ultraslow diffusion. *Stochastic Processes Appl.* **116** 1215–1235.

[35] Meerschaert, M. M. and Scheffler, H. P. (2008). Triangular array limits for continuous time random walks. *Stochastic Processes Appl.* **118** 1606–1633.

[36] Meerschaert, M. M., Nane, E. and Vellaisamy, P. (2009). Fractional Cauchy problems on bounded domains. *Ann. Probab.* **37** 979–1007.

[37] Meerschaert, M. M., Nane, E. and Vellaisamy, P. (2011). Distributed-order fractional diffusions on bounded domains. J. Math. Anal. Appl. **379** 216-228.

[38] Meerschaert, M. M., Nane, E. and Vellaisamy, P. (2013). Transient Anomalous Sub-diffusion on bounded domains. Proc. Amer. Math. Soc. Vol. 141, No. 2, pp. 699-710.

[39] Metzler, R. and Klafter, J.(2000). The random walk's guide to anomalous diffusion: a fractionaldynamics approach . *Phys. Rep. 339* **1**, 1-77.

[40] Mijena, J and Nane, E. Strong analytic solutions of fractional Cauchy problems. (*To appear*). Proceedings of the American Mathematical Society.

[41] Naber, M. (2004). Distributed order fractional sub-diffusion. *Fractals* **12** 23–32.

[42] Pazy, A. (1983). Semigroups of Linear Operators and Applications to Partial Differential equations. Applied Mathematical Sciences **44**, Springer-Verlag, New York.

[43] Podlubny, I. (1999). Fractional Differential Equations. Mathematics in Science and Engineering, Volume 198, Academic Press, San Diego.

[44] Prüss, J. (1993). Evolutionary Integral Equations and Applications, Birkhäuser, Basel.

[45] Ryznar, M. (2002). Estimates of Green function for relativistic α-stable process, *Pot. Anal.* **17** 1-23.

[46] Saxton, M. J. and Jacobson, K. (1997). Single-particle tracking: application to membrane dynamics. . *Annu. Rev.Biophys. Biomol. Struct.* **26**, 373-399.

[47] Sinai, Y.G. (1982). The limiting behavior of a one-dimensional random walk in a random medium. *Theor. Probab. Appl.* **27**, 256–268.

[48] Sneddon, I.N. (1995). Fourier Transforms, Dover, New York.

[49] Van den Berg, M. (1987). On the asymptotics of the heat equation and bounds on traces associated with Dirichlet Laplacian. J. Funct. Anal. **71** 279-293

[50] Zaslavsky, G. M. Choas (2000). Fractional kinetics, and anomalous transport. *Phys. Rep. 371* **6**, 461-580.

[51] Zhen-Qing Chen, Panki Kim, Renming Song (2011). *Green function estimates for relativistic stable processes in half-space-like open sets.* Stochastic Process. Appl. **121**.