Generating Renewal Functions of Uniform, Gamma, Normal and Weibull Distributions for Minimal and Non Negligible Repair by Using Convolutions and Approximation Methods

by

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Abstract

This dissertation explores renewal functions for minimal repair and non-negligible repair for the most common reliability underlying distributions Weibull, gamma, normal, lognormal, logistic, loglogistic and the uniform. The normal, gamma and uniform renewal functions and the renewal intensities are obtained by the convolution method. In the uniform distribution case complexity becomes immense as the number of convolutions increases. Therefore, after obtaining twelve convolutions of the uniform distribution, we applied the normal approximation. The exact Weibull convolutions, except in the case of shape parameter $\beta = 1$, as far as we know are not attainable.

Unlike the gamma and the normal underlying failure distributions, the Weibull base-line distribution does not have a closed-form expression for the *n*-fold convolution. Since the Weibull is the most important and common base-line distribution in reliability analyses and its renewal and intensity functions cannot be obtained analytically, we used the time-discretizing method. Most calculations have been done with the aid of MATLAB Programming Language.

When MTTR (Mean Time to Repair) is not negligible and that TTR has a pdf denoted as $r(t)$, the expected number of failures, expected number of cycles and the resulting availability were obtained by taking the Laplace transforms of renewal functions. Finally, the approximation method for obtaining the expected number of cycles, number of failures and availability using raw moments of failure and repair distribution is provided.

Keywords: Reliability, Renewal Function, Renewal Intensity, Convolutions

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List of Acronyms

TTF = T	Time to Failure
MTTF	Mean Time to Failure
MTTR	Mean Time to Repair or to Restore
MTBF	Mean Time Between Failures
CDF	Cumulative Distribution Function
FR	Failure Rate
pdf	Probability Density Function
pmf	Probability Mass Function
rv	Random Variable
HZF	Hazard Function
CHF	Cumulative Hazard Function
Pr	Probability
RHS	Right Hand Side
CPU	Central Processing Unit
IFR	Increasing Failure Rate

CFR	Constant Failure Rate
DFR	Decreasing Failure Rate
MO	Modal Point
NID	Normally Independently Distributed
RF	Renewal Function
RNIF	Renewal Intensity Function

Nomenclature

t	A specified value in the range space of T
$H(t)$	Cumulative hazard function
$E(T)$	Expected Value of TTF
$f(t)$	Probability density function (pdf)
$Q_T(t)$	Unreliability function at time t
$F(t)$	Cumulative distribution function
$h(t)$	Instantaneous hazard function
$R(t)$	Reliability function
$N_s(t)$	Number of survivors at time t
$N_f(t)$	Number of failed items by time t
σ	Population standard deviation
μ	Population mean
σ^2	Population variance
$\delta = \mathrm{t}_0$	Minimum or guaranteed life
λ	The Exponential failure rate
β	Weibull shape parameter (or slope)

θ	Weibull characteristic life
$\rho(t)$	Renewal intensity function at time t
$M(t)$	Renewal function within the interval [0, t]
Δt	Time Increment
$A(t)$	Availability Function
TTF_i	Time to the i^{th} failure
$U(a,b)$	Uniform distributions over the real interval [a, b]
$\Gamma(n)$	Gamma function at $n > 0$
α	The Gamma distribution shape parameter
β	The Gamma distribution scale parameter
$f_{(n)}(t)$	n-fold convolution at time t
$\mathcal{L}$	Laplace transformation symbol

CHAPTER 1

Introduction

This chapter introduces the history of reliability, outlines the objectives that the research intends to achieve, and then gives the research methods adopted. The introduction provides the framework for the research that follows. The chapter concludes by describing the layout of the dissertation.

1.1 History of Reliability

"During the expansion after World War I, the aircraft industry was the first to use reliability concepts. Initially everything was qualitative. As the number of aircraft grew during the 1930's, reliability was slowly being quantified as function of mean failure rate and average number of failures of an airship or airplane" [1].

Before World War II, reliability studies were mostly intuitive, qualitative and subjective [2]. The early development of mathematical reliability models began in Germany during World War II, where "a group led by Wernher Von Braun was developing the V-1 missile" [3].

During the 1950's, nuclear industry started to develop and to use reliability concepts in nuclear power plants and control systems [1]. Further, this last decade witnessed the initial stages of the use of component reliability in terms of "failure rate, life expectancy, design adequacy and success prediction" [3].

Since the turn of the 20th century, much of the reliability research results started to transfer to industry and academia. In the past eight decades universities have been teaching reliability theory and applications. There has also been steady growth on reliability publications. Furthermore, in order to compete in today's global economy, manufacturing and other industries should consider reliability as a primary concern [4].

1.2 Research Objectives

The research objectives are a combination of mathematical methods along with the some approximations. This work focuses on the renewal function (RF), whose definition will follow, and has mainly four objectives. The RF is simply the mathematical expectation of number of renewals in a stochastic process.

The first objective explores the RF, $M(t)$, for minimal repair by using convolutions of gamma, normal and uniform distributions. The closed-form expressions for the RFs are provided in the case of normal, gamma (which includes the exponential) and the uniform distributions. Both renewal functions and the renewal intensities $\rho(t) = dM(t)/d(t)$ are provided for these distributions. Further, the exact uniform convolutions through the eighth have been known since 1983[5].

The second objective is discretizing time in order to approximate the fundamental renewal equation. Unlike the gamma and normal underlying failure distributions, the Weibull base-line distribution (except when the shape parameter $\beta = 1$) does not have a closed form expression for the *n*-fold convolution $f_{(n)}(t)$. Therefore, we cannot obtain the Weibull renewal and intensity functions by using the convolution method. Since the Weibull distribution is the most important of all underlying distributions in reliability analyses, this dissertation uses an approximation method by discretizing time in order to estimate the RF, which can be applied to any baseline distribution [see also E. A. Elsayed, (pp. 428-432)].

The third objective is obtaining renewal and availability functions for some time to failure and time to repair distributions by using Laplace transforms when repair time is not negligible.

The fourth objective is obtaining an approximation for expected number of failures, number of cycles and availability when repair is not negligible for some common reliability distributions by using the first four raw moments of failure and repair distributions.

1.3 Research Methods

The dissertation presented so far generates the renewal functions for some common distributions under the case of minimal repair and non-minimal repair. It develops and enhances an overarching process based on multiple mathematical and statistical theories that will generate renewal and intensity functions, and will also provide approximation methods.

For the exponential distribution obtaining a closed-form expression for the RF was documented for well over 100 years ago. Unfortunately, for other distributions such as Weibull and uniform this is very difficult. We used the convolution method for the uniform distribution in order to obtain its renewal and intensity functions. For the uniform distribution, as the number of convolutions increases the problem becomes more complex both geometrically and mathematically. Therefore, after convoluting twelve uniforms we applied the normal approximation.

Unlike the gamma and the normal underlying failure distributions, the Weibull base-line distribution (except when the shape parameter $\beta = 1$) does not have a closed-form expression for the *n*-fold cumulative convolution $F_{(n)}(t)$, and hence $M(t) = E[N(t)] = \sum_{\mathrm{n}=1}^{\infty} F_{(\mathrm{n})}(t)$ cannot be used to obtain the value of the RF $M(t)$. Since the Weibull distribution is the most important baseline distribution in reliability analyses, we used the time-discretizing approximation method [6] described in Chapter 4. Further, most calculations are done with the aid of MATLAB Programming Language. We also used moment based method and Laplace transforms method for non-negligible repair.

1.4 Dissertation Layout

The dissertation is divided into eight chapters including this first chapter entitled "Introduction". The layout and organization of remaining chapters are as follows.

Chapter 2 presents the literature review on reliability. This includes the concept of reliability in mathematical details. It also explains the terms availability and maintainability and underlines the differences of these two concepts. Moreover, it identifies the differences between quality and reliability. Chapter 2 continues on renewal stochastic processes and it discusses reliability applications. Finally, it explains the importance of renewal functions, contributions to the literature and the general previous works that have been done.

In Chapter 3, the reliability methodology is presented. The principles, mathematical development, and structure are identified and introduced. Reliability measures like, MTTF (Mean Time to Failure), hazard function are explained both mathematically and conceptually. It explains the mathematical concepts of convolution method. Chapter 3 continues by some distribution functions such as uniform, gamma, normal and Weibull that was studied in this work. Chapter 3 concludes with a brief summary table.

Chapter 4 describes the renewal processes for minimal repair in detail. It gives the renewal and intensity functions for the normal and gamma distributions. It also provides convolutions of the uniform through order $n = 12$ for the case of minimal repair. By using these convolutions it calculates the renewal and intensity functions for the interval $[0, t]$ when the underlying distribution is uniform. Chapter 4 continues by explaining the time-discretizing method in order to approximate the fundamental renewal equation, which can be applied to any base-line distribution. It concludes with approximating the Weibull distribution's RF.

In Chapters 5 and 6, unlike the previous chapters, we assume that MTTR (Mean Time to Repair) is not negligible and that TTR has a pdf denoted as $r(t)$. Chapter 5 gives the expected number of failures, expected number of cycles and availability by taking the Laplace transforms of renewal functions, and Chapter 6 gives the approximate number of renewals for most common distributions by using raw moments of failure and repair distributions.

Chapter 7 introduces the Matlab program, describes the inputs and outputs of the program for minimal and non-minimal repair cases. Finally, Chapter 8 gives the conclusion and possible future work.

CHAPTER 2

Literature Review

This chapter introduces the literature review on reliability. It defines reliability, availability and maintainability. It also discusses differences between quality and reliability. Furthermore, it briefly discusses the applications of reliability which abound. Then, it continues with importance of renewal functions and previous work on renewal functions. Finally, it summarizes the contribution of this study.

2.1 Reliability

The concept of reliability is not new. Both manufacturers and customers have long been concerned with the reliability of products they produce and use [7]. Basically the general perception about reliability is functioning without any problem. Stating something is reliable implies it can be depended on to work satisfactorily. However, the real definition of reliability involves quantifying measures.

"The reliability of an item is the probability that it will adequately perform its specific purpose for a given period of time under specified stressed conditions" [8]. Another definition is as follows: "Reliability is defined to be the probability that a component or system will perform a required function for a given period of time when used under stated operating conditions" [4]. As it is seen from the two above definitions, reliability is a probability. Therefore, this implies reliability can never be negative or greater than one. Since reliability is a probability, probability axiom results are used in reliability [8]. Reliability studies deal with different complexity levels of units or systems [9].

Unreliability induces heavy losses to organizations. It also causes them to lose reputation. Failure of products before reaching their warranty period is costly. Therefore, today we have reached a point where reliability is considered as a major performance measure [10].

In summary, reliability is the survival probability of an item or system beyond a certain point in time. Since the cost of unreliability is high, reliability is considered as a major performance parameter. Unlike most classical statistical parameters, reliability is always a function of time (t). Although in cases of very short mission time, reliability can be considered as static merely as an approximation.

2.2 Availability and Maintainability

Availability includes both failure and repair rates of a system, and therefore, it is considered to be one of the most important reliability performance measures [6]. "Availability is the probability that a system or component is performing its required function at a given point in time or over a stated period of time when operated and maintained in a prescribed manner" [4]. There are generally four types of availability measures [8]. These are:

1) Point (or instantaneous) availability $A(t)$: Instantaneous availability at time t is the probability that a repairable unit is functioning reliably at time t. Therefore, if there is no repair, the availability, $A(t)$, is equal to the reliability function $R(t)$.

2) Limiting availability: As time increases, instantaneous availability will approach a constant value, once an item has stationary times to failure T_i, stationary times to repair TTR_i, and this value is known as the limiting availability [8].

3) Average availability on (0, *t*]: This can be explained as the "expected fraction of time" that a system is operational during (0, *t*], [8].

4) Limiting average availability on (0, *t*]: This is also known as "steady-state availability". It is the availability of a component or system when the time interval is very large [6].

Maintainability is the probability that a failed item can be repaired or restored to become operationally effective within a specified period of time when repair action is performed in accordance with prescribed procedures [11]. Maintainability is important for eliminating defects, correcting defects and their causes, meet new requirements and adapt to a changing world. A design process should start with defining system maintainability objectives.

Both maintainability and availability are very important measures in reliability theory and have a wide range of applicability. However, in the context of this work the brief amount of information given above on maintainability and availability should perhaps be satisfactory.

Maintainability directly affects availability because time for repairs or preventive maintenance can change a system from available to unavailable state. Therefore, there is a close relationship between reliability and maintainability. Reliability affects maintainability, maintainability affects reliability and they both impact availability [12]. Further, like reliability, availability and maintainability are also probabilities. Therefore, probability theory rules can be applied to both availability and maintainability.

2.3 Reliability versus Quality

Quality may be defined in many different ways. It can be conceptual, perceptual or conditional and it may be interpreted differently. Basically quality means "fitness to use" and it has several dimensions such as "performance", "reliability", "durability", "serviceability", "conformance to standards", etc. [13], [14]. Therefore, reliability which is closely associated with product quality is one of several quality dimensions. Quality can be defined qualitatively and can be achieved through a satisfactory quality assurance program [4]. "Quality assurance is the set of activities that ensures quality levels of products and services properly maintained and that supplier and customer quality issues are properly resolved" [14] .

On the other hand, reliability is largely concerned with how long a product can continue to operate under specified conditions once it becomes functional. "Reliability may be viewed as the quality of a product's operational performance over time, and as such it extends quality into the time domain" [4]. Improving reliability is an important part of improving product quality [15].

2.4 Reliability Applications

Examples of high-reliability systems abound worldwide, such as aircraft systems([16], [17] etc.), electric power generating stations ([18], [19] etc.), chemical plants ([20] etc.), power systems to telephone and communication systems ([21], [22], [23], [24], etc.), computer systems and networks ([25] etc.) [1].

Reliability studies are conducted at either the component or system level. Generally, reliability calculations are easier at the component than system level. If it is the system level, then there are two different classifications. A system can be static or dynamic. Further, in both cases it can be serial, parallel or mixed. Explanations of each follow.

In static reliability component or subsystem reliabilities are considered to be approximately constant for a specified duration of time [26]. In this case, the mission time is sufficiently short so that the assumption of constant reliability is almost tenable. Whereas, in dynamic models there will be continual reliability degradation of subcomponents with respect to time.

In series system-models all subsystems must operate reliably in order for the system to function properly. As soon as one subsystem fails, the system fails [27]. There is also another model type called "chain model" or "weakest link model" that is described in the literature under the title of series systems [26]. Based on this model, a system will fail as soon as the weakest component (or link) fails. Therefore, system reliability is equal to reliability of the weakest among all components or subsystems [26].

However, a pure parallel system is composed of subsystems or components that the success of any one of which results in system success [28]. Therefore, such a system is reliable if at least one component is reliable. There are many studies in the literature that deal with system reliabilities such as [29], [30], [31], [32], etc.

2.5 Renewal Processes

Renewal processes are stochastic events such that their nth stage value is the sum of n independent random variables of common distributions with nonnegative ranges [33]. As an example, consider a machine component that is replaced as soon as it fails with a new one. Let

$N(t)$ be the number of replacements during the interval [0, *t*] of length *t*. Then, $N(t)$ is called a renewal counting process. The study of renewal processes focus on the following topics:

(1) The pmf (Pr mass function) of $N(t)$,

(2) The expected number of renewals during [0, *t*] or [t_0, t_0+t], $E[N(t)]$, denoted by $M(t) = E[N(t)]$, *M* for mean, is called the RF. Henceforth, the symbol *E* will represent the Expected-Value operator. (Note that this case also includes the negligible repair-time.),

(3) The occurrence Pr mass or density function of a renewal at specific epochs of time, and

(4) The time needed for the occurrence of *n* events (such as failures that are followed by a replacements) to occur [34]. [For more details see U. N. Bhat (1984), *Elements of Applied Stochastic Processes*, 2nd Ed., Chapter 8.].

2.6 Importance of Renewal Functions

Renewal functions, gives the expected number of failures of a system or a component during a time interval and this is used to determine the optimal preventive maintenance schedule of a system [35]. Renewal functions are quantities that have particular importance in analysis of warranty ([36], [37], [38], [39] etc.) [40]. Expected cost of warranty estimation is closely related to the RF estimation [36]. For example, consider the case that cell phone manufacturer has a two year free replacement warranty which means that if a cell phone fails manufacturer agrees to replace it with a brand new one without any charge. Then , suppose *M*(2) is the expected number of failures(replacements) during two years warranty period and C(2) expected warranty cost is $C(2) = c*M(2)$, where c is assumed as the fixed cost per replacement [36]. It is obvious that the cost of warranty is greatly affected by the number of replacements [35]. In today`s competitive environment product`s warranty policy is important to attract customer. Offering longer warranty terms usually attract more customers but it means more cost [41]. So, warranty has two important roles protection and promotion [42]. Therefore, it is very important to determine an optimal warranty time and this means obtaining *M*(*t*) with greater accuracy is very essential especially if manufacturer produces large number of units or very expensive items.

Renewal functions play an important role and have wide variety of applications in decision making such as inventory theory ([33]), supply chain planning ([43], [44]), continuous sampling plans ([45],[46]), insurance application and sequential analysis ([47][48]) [36],[43].

2.7 Previous Work on Renewal Functions

As we have seen in the previous section renewal functions play an important role in many applications. Therefore it is important to obtain renewal functions analytically. Based on analytic method, *M*(*t*) is the inverse Laplace transform of $\bar{M}(s)$ where $\bar{M}(s) = \frac{\bar{f}(s)}{s[1-\bar{f}(s)]}$ ([35]), where Laplace transforms will be defined later. "The advantage of analytical method is one can carry out parametric studies of the RF, i.e., the behavior of *M*(*t*) as a function of the parameters of the distribution" [40]. However, for most distribution functions obtaining the RF analytically is complicated and even impossible [43]. Therefore development of computational techniques and approximations for renewal functions has attracted researchers [49].

One of the well-known approximations is $M(t) \approx t/\mu_1' + \mu_2'/2\mu_1'^2 - 1$ which is generally known as asymptotic approximation and was generated by Tacklind,S (1945)[50] and also cited in numerous papers such as [51]. The asymptotic expression has a closed-form expression thus it is easy to apply optimization problems that involve renewal process [43]. However since asymptotic expansion is not accurate for small values of *t*, Parsa&Jin (2013) [43] propose better approximation by keeping the positive features of asymptotic approximations such as simplicity, closed-form expression, and independence from the distribution. Jiang (2010) [52], proposes an approximation for the RF with an IFR which is also useful in areas such as optimization where renewal function needs to be evaluated.

There are series methods available in the literature to approximate renewal functions such as, Smith and Leadbetter (1963) [53] who developed a method to compute the RF for Weibull by using power series expansion of t^β where β is the shape parameter of the Weibull. On the other hand instead of using power series expansion, Lomnicki (1966) [54] proposes another method by using the infinite series of appropriate Poissonian functions of t^β. There are also many other approximations methods available such as Xie (1989) [55], Smeiticnk & Dekker (1990) [56], Baxter et al (1982) [57], Gang&Kalagnaman (1998) [58], From (2001) [59]etc. For example Xie (1989) [55] proposed RS-method for solving renewal-type integral equations based on direct numerical Riemann-Stieltjes integration. There are usually three criteria: model simplicity, applicability and approximation accuracy to evaluate the value of the analytical RF approximation [52]. Increasing the complexity may lead more accurate approximation but may make the process complicated and difficult to implement in practice [60].

Studies on renewal functions for some particular underlying renewal functions such as Weibull have been done. Jiang (2009) [61], proposes an approximation for the RF of Weibull distribution with an IFR which is accurate for time *t* up to a certain value of larger than the characteristic life. On the other hand, Jin & Gonigunta (2010) [62], proposes an approximation method for Weibull RF with DFR. Sinha (1985) [63] obtains Bayes estimation of the survivor function of the s-normal distribution. Papdopoulos &Tsokos (1975) [64] obtain confidence bounds for the Weibull failure model. Many others are also available like [65], [66], [67], etc.

Furthermore, in the literature bounds on renewal functions have been discussed. Since they provide upper and lower bounds on warranty costs bounds on $M(t)$ are very useful for many warranty models [40]. Ross (1996) [68], shows that if a distribution has DFR then the RF is bounded as $\frac{t}{\mu_1'} \le M\left(t\right) \le \frac{t}{\mu_1'} + \frac{\mu_2'}{2\mu_1'^2} - 1$ where μ_1' and μ_2' are the first and second raw moments. Marshall(1973) [69] provides lower and upper linear bounds on the RF of an ordinary renewal process. Ayhan et.al. (1999) [70] provide tight lower and upper bounds for the RF which are based on Riemann-Stieljes integration. There are also many other studies available about bounds on RF such as [71], [72], [73], [74], [75], [76] and etc.

Finally, simulation can be considered as an alternate approach to estimate the value of renewal function. Brown et al (1981) [77] use the Monte Carlo simulation to estimate the RF for a renewal process with known interarrival time distribution. Papadopoulos & Tsokos (1975) [64], perform Monte Carlo simulation to obtain 90% and 95% Bayes confidence bounds for the random scale parameter and reliability function to illustrate their results. “The simulation approach offers an alternate method for obtaining the solution to the problem without the need of solving complicated mathematical formulations" [40].

2.8 Contribution of this work to the Literature

The renewal and intensity functions with minimal repair for the most common lifetime underlying distributions normal, gamma, uniform and Weibull are explored. The exact normal, gamma, and uniform renewal and intensity functions are derived by the convolution method. Unlike these last three failure distributions, the Weibull distribution, except at shape $\beta = 1$, does not have a closed-form function for the *n*-fold convolution. Since the Weibull is the most important failure distribution in reliability analyses, its approximate renewal and intensity functions were obtained by the time-discretizing method. And also for non-minimal repair moment based approximation method was generated. Table 1 summarizes the contributions of this dissertation.

Table 1: Gaps and Contributions

<table>
<tr><th>Gaps in the Literature</th><th>Contributions</th></tr>
<tr><td>Obtaining the renewal functions for the most baseline distributions is not possible.</td><td>Obtaining renewal functions and renewal intensities for normal and gamma distribution by using convolution method.
Approximating Weibull renewal function and renewal intensity by using time discretizing method.</td></tr>
<tr><td>¨The renewal function of uniform distribution using the convolution method is not available.</td><td>We used geometrical mathematical statistics method to obtain uniform convolutions from n= 2 through n = 12 and then applied the normal approximation for convolutions beyond 12 to obtain renewal function of uniform distribution.</td></tr>
<tr><td rowspan="3">For the case of non-negligible repair only the closed-form renewal functions exist in the case of exponential TTF and TTR.</td><td>We obtained the closed form renewal function for gamma TTF and exponential TTR when α (shape parameter of gamma) is an exact positive integer from 2 to 7.</td></tr>
<tr><td>We obtained approximations for expected number of cycles
• When TTF is Weibull and TTR is uniform
• When TTR is Weibull and TTF is Weibull
• When TTF is gamma and TTR is uniform
• When TTR is gamma and TTF is uniform
by first obtaining the convolution density functions and then using the time discretizing method.</td></tr>
<tr><td>We obtained approximations for two parameter exponential, three parameter Weibull, gamma, normal, lognormal, logistic and loglogistic distributions by using the first four moments of failure and repair distributions.</td></tr>
</table>

2.9 Literature Review Summary

The literature describes a multitude of research in the area of reliability. Reliability has a wide variety of applications from aircraft to power systems. The literature supports increasing applications of reliability in today's competitive global economy. In this chapter, we explained some important concepts such as availability and maintainability and described the differences and association between quality and reliability. This chapter also explains the importance of renewal functions and the general previous work that has been done about them.

CHAPTER 3

Methodology

This chapter describes the mathematical concepts and relationships that are needed in reliability and renewal theory presented throughout this dissertation.

3.1 Reliability Measure

There are three measures of reliability:

(1) The reliability function $R(t)$,

(2) The mean time to failure (MTTF), and

(3) The hazard (rate) function $h(t)$.

If either function (1) or (3) is known, then the other 2 measures can be uniquely determined, but the knowledge of (2), i.e., μ = MTTF, is not sufficient to obtain unique functions for $R(t)$ and $h(t)$. In fact, for the same MTTF of an underlying distribution, there are uncountably infinite other base-line distributions that have identically the same MTTF.

3.1.1 The Reliability (or Survivor) Function *R*(t)

The reliability of a component is the probability (Pr) that the device will perform without failure during the mission time *t*, under specified stress conditions. For example,

$$R(\text{of a new passenger tire for } t = 500 \text{ interstate miles}) \cong 100\% = 1.$$

However, the reliability of the same passenger tire under racing conditions at Indianapolis 500 would be almost zero. Note that the terminology survivor function, $S(t)$, is also used for non-repairable items, such as light bulbs, transistors, or rocket-motor of an unmanned spacecraft. Let *T* = the random variable lifetime, or time to failure (TTF), with Pr density function $f(t)$. Then, the reliability function at time *t*, or the survival Pr for a mission of length *t*, is given by (the Pr that a component lifetime exceeds time *t*)

$$R(t) = \Pr(T > t) = \int_t^{\infty} f(x)\mathrm{d}x = 1 - Pr(T \le t) = 1 - F_T(t) = 1 - Q_T(t)\,, \qquad (3.1)$$

where $F(t)$ = the cdf of *T* at *t*, and $Q_T(t) = F_T(t)$ represents the unreliability function at time *t*, or the cumulative failure Pr by time *t*. The pdf (Pr density function), $f(t)$, is also referred to as the failure (or mortality) density function. Some authors use the notation $S(t)$ for the reliability or survivor function at time *t* to imply survival probability beyond *t*; however, the notation $R(t)$ is a bit more prevalent in engineering applications. We now obtain the relationship between $R(t)$ and $f(t)$ by differentiating equation (3.1) with respect to *t*, recalling that

$f(t) = dF_T(t)/dt = dQ_T(t)/dt$ because $F_T(t) = \int_{-\infty}^{t} f(T)dT$. Due to the fact that time cannot be negative, and thus the lower limit in this last integral must be zero instead of $-\infty$, i.e.

$F_T(t) = \int_0^t f(T)dT$ = Failure Probability by time *t*. The developments below show the relationship between $R(t)$ and $f(t)$.

$$\frac{dR(t)}{dt} = \frac{d}{dt}\left[1 - F(t)\right] = -\frac{dF(t)}{dt} = -f(t) \quad \rightarrow dR(t) = -dF(t)$$

and $f(t) = -dR(t)/dt \quad \rightarrow f(t)dt = -dR(t)$.

3.1.2 The Mean Time to Failure

We are now in a position to obtain the relationship between $R(t)$ and the mean time to failure (MTTF=μ), which is defined as the mathematical expectation of T = TTF. Henceforth, the symbol V will represent the variance operator, and the reader must be cognizant of the fact that anytime the operator E or V is applied to any rv (random variable), the end-result will always be a population parameter. The Mean Time to Failure (MTTF) is given by;

$$\text{MTTF} = E(T) = \int_0^\infty tf(t)dt = \int_0^\infty t(-dR)$$

$$\text{MTTF} = -tR(t)\Big]_0^\infty + \int_0^\infty R(t)dt = 0 + \int_0^\infty R(t)dt = \int_0^\infty R(t)dt \qquad . \qquad (3.2)$$

The above equation clearly shows that the unconditional mean-life, $E(T) = \text{MTTF}$, starting at age zero, of any device or system is given by the integral of its reliability function evaluated always from zero to infinity so that the MTTF is the total area under $R(t)$ and the abscissa t (i.e., from 0 to ∞ even if the minimum life is greater than zero).

Note that if a component or system is repairable (or renewable, i.e., failed units are almost immediately replaced), then $E(T)$ is called the mean time between failures (MTBF). The MTBF of any system also can be found by integrating its reliability function from zero to infinity [78] (a simple technique of obtaining MTBF for complex systems). Again, the lower limit of the integral must always be zero even if the minimum life $t_0 = \delta > 0$.

3.1.3 The Hazard (or the Failure Rate) Function $h(t)$

By definition the failure rate (FR) of any device is defined as the failure rate during $(t, t+\Delta t)$ given that the device age is t, i.e.,

$$FR(t) = \frac{P\left(t \le T \le t+\Delta t \middle| T > t\right)}{\Delta t} = \frac{Pr\left(t \le T \le t+\Delta t\right)}{\Pr(T > t)\Delta t} = \frac{R(t) - R(t+\Delta t)}{R(t)\Delta t} \qquad (3.3)$$

This implies that the failure rate of a component at time t is the probability that it will fail in the interval $(t, t+\Delta t)$ given that its life has exceeded time t (i.e., given that the age of the component is t). Put differently, the failure rate of a population of identical items at time t is the proportion of the units failing per unit of time in the interval $(t, t+\Delta t)$ amongst all survivors at time t. The hazard function (HZF), $h(t)$, is simply the instantaneous FR, i.e.,

$$h(t) = \lim_{\Delta t \to 0} [-\frac{1}{R(t)}]\left[\frac{R\left(t+\Delta t\right) - R(t)}{\Delta t}\right] = -\frac{dR(t)/dt}{R(t)} = \frac{f(t)}{R(t)} \qquad (3.4)$$

In other words, if we have a system with N_0 identical items on test at time 0, $N_s(t)$ survivors at time t and $N_f(t)$ failed items by time t, then by the above definition $h(t)$ is the rate of failure, $dN_f(t)/dt$ (this derivative is not quite appropriate because $N_f(t)$ is discrete), only amongst the survivors $N_s(t)$ beyond time *t*, i.e.,

$$h(t)=\frac{\frac{dN_f}{dt}}{N_s(t)}=\frac{\frac{d\left[N_0-N_s(t)\right]}{dt}}{N_s(t)}=\frac{-\frac{dN_s(t)}{dt}}{N_s(t)}=\frac{-\frac{d\left[\frac{N_s(t)}{N_0}\right]}{dt}}{\frac{N_s(t)}{N_0}}=\frac{-dR(t)/dt}{R(t)}=\frac{f(t)}{R(t)}$$

as before. The quantity

$$h(\mathrm{t})dt=-dR/R=dF(\mathrm{t})/R(\mathrm{t})=P\left(t\le T\le t+dt\right)/R(\mathrm{t})=P\left(t\le T\le t+dt|T>t\right),$$

gives the proportion of items that will fail within $(t,t+dt)$ amongst those that are still functioning at time *t*. From the cumulative of hazard function, $CHF=H(t)$, we can obtain the relationship amongst the reliability measures $f(t)$, $R(t)$, and $h(t)$ as shown below:

$$H(t)=\int_0^t h(x)dx=\int_0^t -dR/R=-\ln\left[R(x)\right]_0^{\mathrm{t}}=-\ln\left[R(t)\right]\rightarrow R(t)=e^{-H(t)}$$

$$R(t)=\mathrm{e}^{-\int_0^{\mathrm{t}} h(\mathrm{x})\mathrm{dx}}\;;\; h(t)=\frac{f(t)}{R(t)}\;\rightarrow f(t)=h(t)R(t)=h(t)e^{-H(t)} \qquad (3.5)$$

Properties of the Hazard function $h(t)$

(a) $h(t)\ge 0$ for all *t*.

(b) $h(0) = \frac{f(0)}{R(0)} = f(0) \to h(0)$ must be finite at t = 0 unless f(0) = ∞

(c) $\lim_{t \to \infty} h(t) = \infty$ iff $h(t)$ is an IFR, simply implying that any man-made system must have a finite life (or a finite TTF), i.e., no man-made system can last forever!

(d) $\int_0^\infty f(t)dt = \int_0^\infty h(t)e^{-H(t)}dt = 1$. Note that the above relationships imply that the assumption of almost constant reliability during a short interval (0, t) leads to an almost zero HZF value.

3.2 Renewal Intensity Function

The renewal intensity, $\rho(t)$, gives the instantaneous renewal rate at time t, i.e.,

$$\rho(t) = \lim_{\Delta t \to 0} \frac{M\left(t + \Delta t\right) - M(t)}{\Delta t} = \frac{dM(t)}{dt}, \tag{3.6}$$

so that $\rho(t) \times \Delta t$ gives the unconditional probability element of a renewal during the interval $(t,\, t + \Delta t)$, and in the case of negligible repair time, $\rho(t)$ also represents the instantaneous failure intensity function; hence, $M(t) = \int_0^t \rho(x)\mathrm{d}x$. Note that nearly authors in Stochastic Processes and some in reliability literature refer to $\rho(t)$ as the renewal density because $\rho(t) \times \Delta t$ gives the probability element of a renewal during the interval $(t,\, t + \Delta t)$; however $\rho(t)$ is never a pdf. Hence, we have chosen to refer to $\rho(t)$ as the renewal intensity in lieu of renewal density. The renewal intensity, $\rho(t)$, should not be confused with the hazard rate function $h(t)$ [79], because $h(t) \times \Delta t$ is the conditional probability of a failure during time interval (t, t+ Δt) given that the unit's age is t, whereas $\rho(t) \times \Delta t$ is the unconditional probability of a failure during Δt. The hazard rate function is a relative rate pertaining only to the first failure, whereas the intensity function is an absolute rate of failure for also repairable systems [4], including minimal repair. Only in the case of exponential base-line distribution (CFR) both $h(t)$ and $\rho(t)$ are identically equal to the constant failure rate λ.

3.3 Convolutions

The *n*-fold convolution of a statistical distribution arises in a wide variety of applications of probabilistic models such as reliability theory, renewal theory, inventory theory, queuing theory, continuous sampling plans, insurance risk analysis, and sequential analyses [80].

Mathematically, a convolution of two density functions $f_1(t)$ and $f_2(t)$ denoted $f_1 * f_2$, gives the density of sum of two variates $T_1 + T_2$. It can be proven that the convolution of f_1 with f_2 is given by [81],

$$f_1(t) * f_2(t) = \int_0^t f_2(t-u) f_1(u) du = \int_0^t f_1(t-u) f_2(u) du \qquad (3.7)$$

Note that in general the lower limit would be $-\infty$ but in reliability theory, the lower limit is always zero. In the literature review there are some studies about uniform convolutions [82], [83], [81], but we used the geometrical approach to obtain the precise uniform convolutions through order 12 that are presented in Chapter 4. Maghsoodloo and Hool [5] obtained the uniform convolutions for orders 2, 3, 4, 5, 6 and 8.

3.4 The Weibull Distribution

It is well known that the underlying distribution of almost any manufactured dimension by man can be approximately modeled by a normal (or Laplace-Gaussian) pdf. The Weibull pdf plays the exact same important role for the underlying distribution of TTF (or lifetime) of most mechanical and electrical components or systems. The key events in the derivation of what is now known as the Weibull distribution took place between the years 1922 and 1943. There were three groups of scientists working independently for different aims. Waloddi Weibull (1887-1979) was one of the three working on this distribution. The reason that the distribution bears his name is the fact that he propagated it internationally and interdisciplinary [84]. To arrive at a Weibull pdf, consider an exponential pdf at the constant failure rate $\lambda = 1$ failure per unit of time. Note that the symbol λ is used throughout this dissertation if and only if $h(t)$ is a constant failure rate (CFR). Clearly,

$$\int_0^\infty e^{-x} dx = 1. \tag{3.8}$$

Then, for convenience letting $t_0 = \delta$, we make the transformation $x = \left(\frac{t-\delta}{\theta-\delta}\right)^\beta$, $\beta > 0$. As a result, $\frac{dx}{dt} = \beta\left(\frac{t-\delta}{\theta-\delta}\right)^{\beta-1}\left(\frac{1}{\theta-\delta}\right)$, and substitution into (3.8) yields

$$\int_\delta^\infty e^{-\left(\frac{t-\delta}{\theta-\delta}\right)^\beta} \beta\left(\frac{t-\delta}{\theta-\delta}\right)^{\beta-1} \frac{dt}{\theta-\delta} = \int_\delta^\infty \frac{\beta}{\theta-\delta} e^{-\left(\frac{t-\delta}{\theta-\delta}\right)^\beta} \left(\frac{t-\delta}{\theta-\delta}\right)^{\beta-1} dt = 1. \tag{3.9}$$

Since the value of the integral in Eq. (3.9) is equal to 1 (or 100%), the integrand

$$\left(\frac{\beta}{\theta-\delta}\right)\left(\frac{t-\delta}{\theta-\delta}\right)^{\beta-1} e^{-\left(\frac{t-\delta}{\theta-\delta}\right)^{\beta}}$$

must be a probability density function (pdf) over the range $[\delta, \infty)$. The pdf,

$$f(t)=\left(\frac{\beta}{\theta-\delta}\right)\left(\frac{t-\delta}{\theta-\delta}\right)^{\beta-1} e^{-\left(\frac{t-\delta}{\theta-\delta}\right)^{\beta}}, \; t \geq \delta = t_0 \tag{3.10}$$

and $f(t)=0$ for $0 \leq t < \delta = t_0$, is called the Weibull model, denoted $W(\delta = t_0, \theta, \beta)$, with minimum (or guaranteed) life $t_0 = \delta$ (the location parameter), the characteristic life θ, and slope (or the shape parameter) β; $(\theta - \delta)$ is called the scaling parameter. Different authors tend to use different symbols for the three parameters of a Weibull pdf, but β is the most common symbol for the slope. Figure 1 shows the Weibull density based on different shape (β) values.

The reliability function for Weibull distribution is,

$$R(t)=\Pr\left(T>t\right)=\int_t^{\infty}\frac{\beta}{\theta-\delta}e^{-\left(\frac{x-\delta}{\theta-\delta}\right)^{\beta}}\left(\frac{x-\delta}{\theta-\delta}\right)^{\beta-1}dx=\int_{\left(\frac{t-\delta}{\theta-\delta}\right)^{\beta}}^{\infty} e^{-u}du.$$

$$R(t)=\begin{cases}1, & 0 \leq t \leq \delta \\ e^{-\left(\frac{t-\delta}{\theta-\delta}\right)^{\beta}}, & \delta \leq t < \infty\end{cases} \tag{3.11}$$

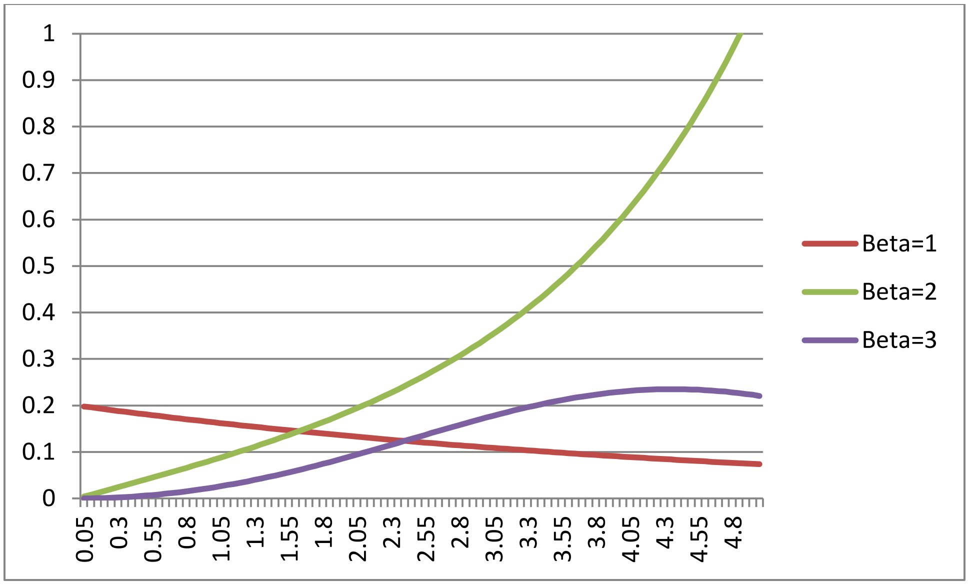

Figure 1: Weibull Graph Based on Different Beta Values

The MTTF of Weibull distribution,

$E(T)=\int_0^\infty R(t)dt=\int_0^\delta dt+\int_\delta^\infty e^{-\left(\frac{t-\delta}{\theta-\delta}\right)^\beta} dt$; letting $x=\left(\frac{t-\delta}{\theta-\delta}\right)^\beta$ in the second integral results in,

$$E(T)=\delta+\frac{\theta-\delta}{\beta}\int_0^\infty e^{-x}\left(x^{1/\beta}\right)^{1-\beta} dx=\delta+\frac{\theta-\delta}{\beta}\int_0^\infty x^{(1/\beta)-1}e^{-x}dx .$$

Since by definition, $\Gamma(n)=\int_0^\infty x^{n-1}e^{-x}dx$, and $n\Gamma(n)=\Gamma(n+1)$, we obtain

$$\text{MTTF}=E(T)=\delta+(\theta-\delta)\frac{\Gamma\left(1/\beta\right)}{\beta}=\delta+(\theta-\delta)\Gamma\left[\left(1/\beta\right)+1\right] \qquad (3.12)$$

The above $\text{MTTF}\le\theta$ iff $\beta\ge1$ (i.e., only for the CFR = constant failure rate, and IFR =

increasing failure rate cases). The modal point of Weibull is given by

$$MO = \delta + (\theta - \delta)\left[(\beta - 1)/\beta\right]^{1/\beta} \tag{3.13}$$

The variance of Weibull is given by

$$\sigma_T^2 = V(T) = (\theta - \delta)^2 \left[\Gamma\left(1 + \frac{2}{\beta}\right) - \Gamma^2\left(1 + \frac{1}{\beta}\right)\right] \tag{3.14}$$

Special cases of the Weibull model

(i) When $\beta = 1$, the Weibull reduces to the exponential pdf with minimum life $t_0 = \delta$ and mean-life (or characteristic life t_c) equal to θ. The Weibull pdf with slope $\beta = 1$ can be used to model the TTF of a component during its useful life (constant failure rate = CFR).

(ii) When $\beta = 2$ and $\delta = 0$, the Weibull becomes the Rayleigh density function

$f(t) = (2/\theta)(t/\theta)e^{-(t/\theta)^2} = \lambda t e^{-\lambda t^2/2}$, where $\theta - t_c = \sqrt{2/\lambda}$.

(iii) When $0 < \beta < 1$, it can be shown that the hazard function of the Weibull is a decreasing function of time (DFR = decreasing failure rate) so that the Weibull may be used to model the TTF during the burn-in (or debugging, or infant-mortality) period of a component (see Ebeling pp. 31-32).

(iv) When $\beta > 1$, the hazard function is increasing (i.e., increasing failure rate = IFR) and the Weibull density can be used to model the TTF during the wear-out period of a component.

(v) When $1 < \beta < 2$, $h(t)$ is an IFR concave function, and for $\beta > 2$, $h(t)$ is an IFR convex function because $\frac{d^2 h(t)}{dt^2} > 0$ when $\beta > 2$.

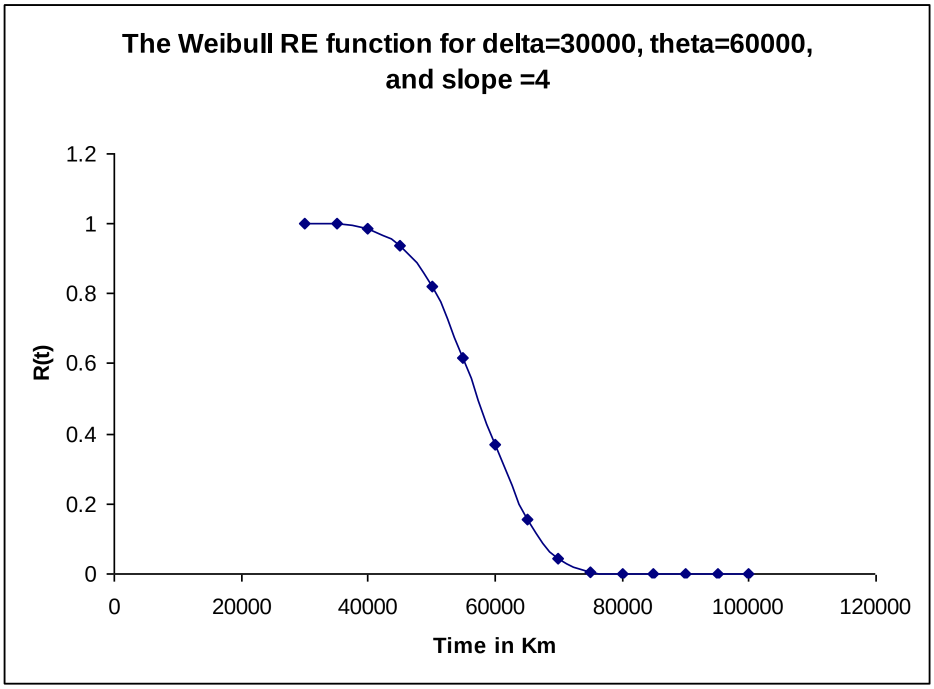

Figure 2: The Graph of Weibull Reliability Function

3.5 The Normal Distribution

The normal distribution which is also called Laplace-Gaussian is the most commonly used distribution in the field of statistics [85]. The reasons are related to its mathematical properties, central limit theorem, and various experimental responses often have distributions that are approximately normal [86]. The normal density was first discovered by Abraham de Moivre (a French mathematician) in 1738 as the limiting distribution of the binomial pmf and was named the "normal" by Karl Pearson in 1894 [87]. It can be proven that the points of inflection occur at $\mu \pm \sigma$.

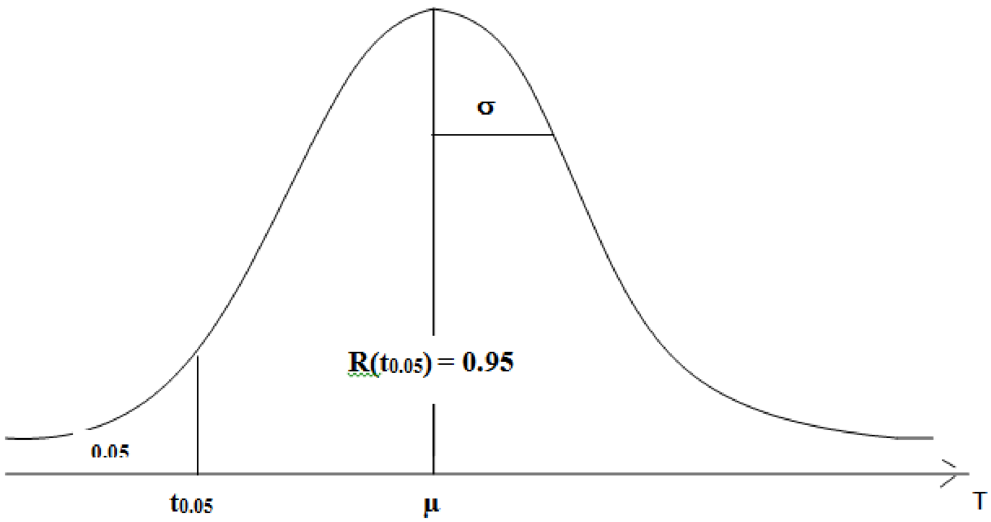

Figure 3: Normal Distribution Graph When $R(t_{0.05}) = 0.95$

A continues random variable T is said to have a normal distribution with parameters μ and σ (or μ and σ^2), where $-\infty < \mu = \text{MTTF} < \infty$ and $0 < \sigma$, iff the pdf of T is [88]

$$f\left(t;\mu,\sigma\right) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\left(t-\mu\right)^2 / \left(2\sigma^2\right)} \quad -\infty < t < \infty \tag{3.15}$$

The reliability function is derived from

$$R(t) = \int_t^{\infty} \frac{1}{\sigma\sqrt{2\pi}} exp\left[-\frac{1}{2}\frac{\left(x-\mu\right)^2}{\sigma^2}\right] dx = 1 - \Phi\left(\frac{t-\mu}{\sigma}\right), \tag{3.16}$$

where Φ represents the cdf of the N(0,1). There is no closed-form solution to the above integral, and it must be evaluated numerically. The hazard function cannot be written in closed form either and is provided below.

$$h(t)=\frac{f(t)}{R(t)}=\frac{\frac{1}{\sigma\sqrt{2\pi}}e^{-(t-\mu)^2/(2\sigma^2)}}{1-\Phi\left(\frac{t-\mu}{\sigma}\right)} \tag{3.17}$$

It must be noted that the normal failure law is applicable only if its MTTF = μ is more than 10 standard deviations to the right of the origin (or zero) because $\Phi(-10) < 7.62\times10^{-24}$.

3.6 The Gamma Distribution

By definition $\Gamma(n)=\int_0^\infty x^{n-1}e^{-x}dx$. Dividing both sides of this last definition we obtain:

$1=\int_0^\infty \frac{1}{\Gamma(n)}x^{n-1}e^{-x}dx$, and per force the integrand $f(t;n)=\frac{1}{\Gamma(n)}t^{n-1}e^{-t}$ must be a density function called the standard gamma pdf. When *n* is not an integer, the common notation for the shape parameter *n* is α, η, or κ.

A single integration by parts of $\Gamma(n)$ will show that $\Gamma(n)=(n-1)\Gamma(n-1)$. After $(n-1)$ integration by parts, we obtain $\Gamma(n)=(n-1)(n-2)...\Gamma(1)$. Inserting $n=1$ into the definition of $\Gamma(n)$ yields $\Gamma(1)=\int_0^\infty t^{1-1}e^{-t}dt=\int_0^\infty e^{-t}dt=1$; therefore,

$\Gamma(n) = (n-1)(n-2)...1 = (n-1)! \rightarrow 0! = \Gamma(1) = 1$. Further, this last result also implies that $\Gamma(n+1) = (n)\Gamma(n)$; it can be proven that $\Gamma(1/2) = \sqrt{\pi}$ and hence $\Gamma(3/2) = (1/2)\Gamma(1/2)$, $\Gamma\left(\frac{5}{2}\right) = \Gamma\left(\frac{3}{2}+1\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right)$, etc. For example, $\Gamma(10) = 9! = 362880$. To obtain the gamma pdf, we make the transformation $x = \lambda t$ in the definition of gamma function:

$$\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}dx = \int_0^\infty (\lambda t)^{n-1}e^{-\lambda t}\lambda dt \;\rightarrow\; 1 = \frac{1}{\Gamma(n)}\int_0^\infty (\lambda t)^{n-1}e^{-\lambda t}\lambda dt \text{, or}$$

$$\int_0^\infty \frac{1}{\Gamma(n)}(\lambda t)^{n-1}e^{-\lambda t}\lambda dt = 1. \tag{3.18}$$

The above equation clearly shows that the integrand must be a pdf over the range [0, ∞) because its integral over [0, ∞) yields 100%. The function under the integral is called the gamma pdf, as shown below, in statistical literature with rate λ (or β = 1/λ the scale parameter) and the shape parameter *n*.

$$f(t) = \frac{\lambda}{\Gamma(n)}(\lambda t)^{n-1}e^{-\lambda t}. \tag{3.19}$$

When *n* is a positive integer the above density is called Erlang and has extensive applications in Queuing Processes. When *n* is not a positive integer, the most common notion for shape is α; thus, a better representation for the general gamma pdf is $f(t) = \frac{\lambda}{\Gamma(\alpha)}(\lambda t)^{\alpha-1}e^{-\lambda t}$

.

The meaning of the parameters n and λ will be made clear in the application example provided below, where $1/\lambda$ is also called the scale parameter. The major application of the gamma pdf occurs from the fact that the sum of n independent and identical exponential rvs, $T_{System} = T_1 + T_2 + ... + T_n$, has a gamma time to failure density function with parameters n and λ, where each T_i is distributed according to $\lambda e^{-\lambda t}$. Therefore, an n-unit standby system with quiescent failure rates of almost zero for standby units and perfect switching, has a lifetime that has the gamma density with parameters n and λ. Further, the gamma density has applications in maintenance scheduling where the amount of deterioration during an interval $[t_1, t_2]$ has a gamma pdf with scale parameter $1/\lambda$ and shape $n = \gamma(t_2 - t_1)$, where $\gamma > 0$ is a constant of proportionality.

The first four moments of the gamma pdf are given by

$$E\left(T_{System}\right) = E\left(T_1 + T_2 + \ldots + T_n\right) = n / \lambda$$

$$V\left(T_{System}\right) = V\left(T_1 + T_2 + \ldots + T_n\right) = n / \lambda^2$$

the standardized third moment $\alpha_3 = 2/\sqrt{n}$, and $\alpha_4 = 3 + (6/n)$; hence the kurtosis is equal to $\beta_4 = 6/n$, and it can be verified that $MO = (n-1)/\lambda$. Note that the values of $\alpha_3 = 2/\sqrt{n}$ and $\alpha_4 = 3 + (6/n)$, clearly show that the limiting distribution (i.e., as $n \to \infty$) of the gamma density is the Gaussian $N(n/\lambda,\ n/\lambda^2)$. Unfortunately, n must exceed 200 (because the exponential is highly skewed) before the normal approximation to gamma becomes fairly adequate.

To obtain the reliability function for the gamma pdf, we make use of its relationship with the Poisson pmf as described below, where X(t) describes the number of failures during a mission time of length t and T_n represents time to the nth failure.

$R(t) = \Pr(T_n > t) = Pr\left[X(t) \le (n-1)\ failures\right] = \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!} e^{-\lambda t}$. So far, we know the expressions for the gamma pdf, the reliability function, and the $\mathrm{MTTF} = n/\lambda$. The hazard function can be obtained from $f(t)/R(t)$.

3.7 The Uniform Distribution

If $a < b$, the random variable T is said to have a continuous uniform probability density on the interval [a, b] if and only if the density function of T is given by [89]

$$f(t) = \begin{cases} 1/(b-a), & a \le t \le b \\ 0, & elsewhere \end{cases} \tag{3.20}$$

where a is the minimum-life and b is the maximum-life. The graph of the above pdf is shown atop the next page. Note that all values of t from a to b are equally likely in the sense that the probability that t lies in an interval of width Δt entirely contained in the interval from a to b is equal to $\Delta t/(b - a)$ regardless of the exact location of the interval [90]. Its reliability measures are listed below.

The MTTF is: $\mu = \frac{1}{2}(b+a)$. (3.21)

Median of the uniform distribution is: $t_{0.5} = \frac{1}{2}(b+a)$. (3.22)

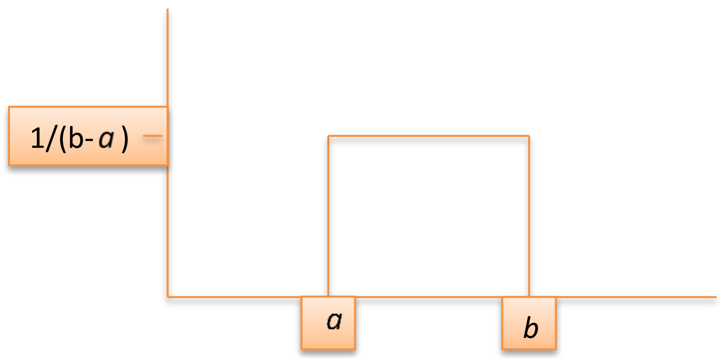

Figure 4: Uniform Distribution Graph

Variance of the distribution is: $\sigma^2 = \frac{1}{12}(b-a)^2$ (3.23)

Because $\int_t^{\infty} \frac{1}{b-a} dx = \int_t^{b} \frac{1}{b-a} dx = \frac{b-t}{b-a}$, the reliability function is given by:

$$R(t) = \begin{cases} 1 \ , & 0 \le t \le a \\ \frac{b-t}{b-a}, & a \le t \le b \\ 0, & b < t \end{cases} \tag{3.24}$$

The hazard function is: $h(t) = \frac{f(t)}{R(t)} = \frac{1/(b-a)}{(b-t)/(b-a)} = \begin{cases} 0 \ , & 0 \le t \le a \\ \frac{1}{b-t}, & a \le t \le b \end{cases}$, which is an IFR.

Hence the CHF is given by H(t) = $\begin{cases} 0 \ , & 0 \le t \le a \\ \ln\left[\frac{b-a}{b-t}\right], & a \le t \le b \\ \infty, & b \le t \end{cases}$ (3.25)

3.8 Laplace Transforms

If $f(t)$ is a known function of t for values of $0 \leq t < \infty$, its Laplace transform $\bar{f}(s)$ is defined by the equation,

$$\mathcal{L}\left\{ f(t)\right\} = \int_0^\infty e^{-st} f(t)dt, \tag{3.26}$$

and abbreviated as,

$$\bar{f}\left(s\right) = \mathcal{L}\left\{ f(t)\right\} \tag{3.27}$$

Thus, the above operation transforms the function $f(t)$ of the real variable t into a new function $\bar{f}(s)$ of the subsidiary variable s, which may be real or complex [91]. Often, it is easier to work in the s-space than the t-space to obtain solutions to mathematical problems.

3.9 Chapter Summary

This chapter introduced the mathematical concept of the dissertation. The table below shows the summary of reliability measures of most important base-line distributions. Only the Logistic and Loglogistic density reliability measures, both of which have also reliability applications, are not yet provided.

Table 2: Reliability Measures of Most Common Base-line Distributions

Lifetime Distribution	Failure Density $f(t)$	Survival Function R(t)	Hazard Function	MTTF
Exponential	$\lambda e^{-\lambda t}$	$e^{-\lambda t}$	λ	$1/\lambda$
Weibull	$\frac{\beta}{\theta-\delta}(\frac{t-\delta}{\theta-\delta})^{\beta-1}e^{-(\frac{t-\delta}{\theta-\delta})^{\beta}}$	$e^{-(\frac{t-\delta}{\theta-\delta})^{\beta}}$	$\frac{\beta}{\theta-\delta}(\frac{t-\delta}{\theta-\delta})^{\beta-1}$	$\delta+(\theta-\delta)x\ \Gamma[(1/\beta)+1]$
Gamma	$\frac{\lambda}{\Gamma(n)}(\lambda t)^{n-1}e^{-\lambda t}$	$\sum_{k=0}^{n-1}\frac{(\lambda t)^k}{k!}e^{-\lambda t}$, when n is a pos. integer	$f(t)/R(t)$	n/λ
Lognormal	$\frac{1}{\sigma t\sqrt{2\pi}}e^{-(\frac{\ln t-\mu}{\sigma})^2}$	$\Phi[\frac{\mu-\ln(t)}{\sigma}]$	$f(t)/R(t)$	$e^{\mu+\sigma^2/2}$
Beta, t= δ + $(U-\delta)$ x , $0 \le x \le 1$	$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}x^{a-1}(1-x)^{b-1}$ Standard Beta pdf	$1-betadist\left(\frac{t-\delta}{U-\delta},a,b\right)$	$f(t)/R(t)$	$\delta+\frac{(U-\delta)a}{(a+b)}$
Normal	$\frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/(2\sigma^2)}$	$1-\Phi\left(\frac{t-\mu}{\sigma}\right)$	$f(t)/R(t)$	μ
Uniform	$\begin{cases}1/(b-a), & a\le t\le b\\ 0, & elsewhere\end{cases}$	$\begin{cases}1, & 0\le t\le a\\ \frac{b-t}{b-a}, & a\le t\le b\\ 0, & b<t\end{cases}$	$\begin{cases}0, & 0\le t\le a\\ \frac{1}{b-t}, & a\le t\le b\end{cases}$	$\frac{1}{2}(b+\alpha)$

CHAPTER 4

Renewal Processes with Minimal Repair

Suppose that failures occur at times T_n (n = 1, 2, 3, 4, …) measured from zero and assuming that replacement (or restoration time) is negligible relative to operational time, then T_n represents the operating time (measured from zero) until the n^{th} failure, where $T_0=0$. Because the *pdf* of T_1 may be different from the intervening times $X_2 = (T_2 - T_1)$, $X_3 = (T_3 - T_2)$, $X_4 = (T_4 - T_3)$, ..., we consider only the simpler case of probability density function (*pdf*) of time to first failure $f_1(t)$ being identical to those of intervening times X_2, X_3, X_4, … as depicted below.

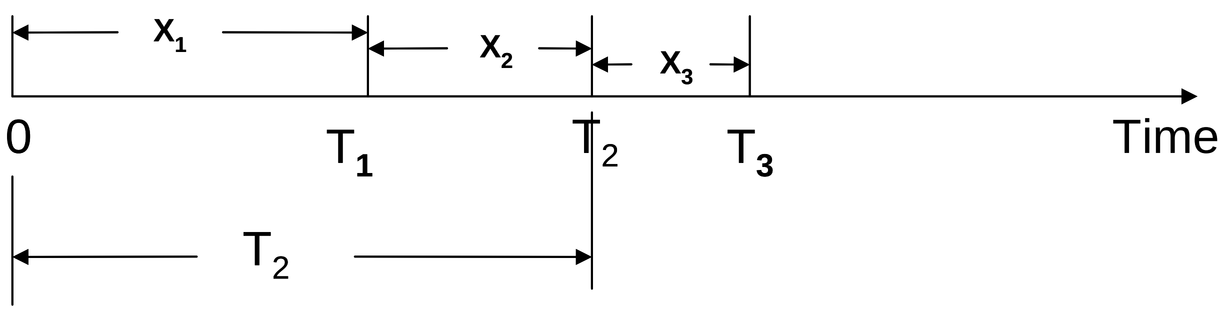

Figure 5: The Intervening Times of Two Successive Renewals

Note that X_1, X_2, X_3 … represent intervening times between failures, while T_i represents time to the i^{th} renewal measured from zero. Further, all X_i's are assumed iid (independently and identically distributed). The above figure clearly shows that T_n (Time to the n^{th} renewal) = $\sum_{i=1}^{n} X_i$ = sum of the times to the 1st failure from zero plus the intervening times of 2nd failure until the n^{th} failure. If $n > 60$, then the Central Limit Theorem (CLT) states that the distribution of T_n

approaches normality with mean $n\mu$, where $\mu = E(X_i)$ = the mean time between successive renewals, i = 1, 2, 3, 4, … and with variance $n\sigma^2$, where $\sigma^2 = V(X_i)$. However, if the *pdf* of X_i, $f(x)$, X being the parent variable, is highly skewed and/or n is not sufficiently large, then the exact *pdf* of $T_n = \sum_{i=1}^{n} X_i$ is given by the n-fold convolution of $f(x)$ with itself denoted as $f_{T_n}(t) = f_1(t) * f_{(n-1)}(t)$, where $f_{(n-1)}(t)$ is the pdf of the sum $X_2 + X_3 + \ldots + X_n$, or the $(n-1)$ convolution of $f(t)$ with itself. Note that Figure 5 is also approximately valid for the case of Minimal-Repair, i.e., the case when MTTR ≅ 0, or MTTR = Mean Time to Repair is negligible relative to MTTF. Therefore, in this section by renewal we mean either the replacement of a failed component with a brand-new one, or the case when the failed component can almost immediately be repaired and consequently be put back on-line.

The simplest and most common renewal process is the homogeneous Poisson process (HPP), where the intervening times are exponentially distributed at the constant inter-renewal (or failure) rate λ. Because λ is a constant and intervening times are iid, a Poisson process is also referred to as a homogeneous renewal process. Throughout this dissertation, we will establish that only in the case of exponential failure Pr (Probability) law with CFR (Constant Failure-Rate), the RNIF is identical to the constant instantaneous hazard function $h(t) = \lambda$. Further, it is also well-known that for a HPP the $V[N(t)] = \lambda t$, and hence the coefficient variation of $N(t)$ is given by $CV_{N(t)} = 1/\sqrt{\lambda t}$.

4.1 The Renewal Function $M(t)$ during [0, t] with Minimal Repair

Because the two events $\{N(t) \geq n\}$ and $\{T_n \leq t\}$ are equivalent, it follows that $P[N(t) \geq n] = P(T_n \leq t) = F_{(n)}(t)$, where $F_{(n)}(t) = F_{(1)}(t) * F_{(n-1)}(t)$ is the n-fold convolution representing the cdf of $T_n = \sum_{i=1}^{n} X_i$. Thus,

$$P[N(t) = n] = P[N(t) \geq n] - P[N(t) \geq n+1] = F_{(n)}(t) - F_{(n+1)}(t)$$

It has been proven by many authors both in Stochastic Processes and Reliability Engineering that the RF for the duration [0, t] is given by

$$M(t) = E[N(t)] = \sum_{n=1}^{\infty} F_{(n)}(t), \quad (4.1a)$$

and

$$V[N(t)] = \sum_{n=1}^{\infty} (2n-1) F_{(n)}(t) - [M(t)]^2 \quad (4.1b)$$

where the random variable $N(t)$ represents the number of renewals that occur during the time interval [0, t], and $F_{(n)}(t)$ is the *cdf* (*cumulative distribution function*) of the n-fold convolution of f(t) with itself, i.e., $F_{(n)}(t) = \int_0^t f_{(n)}(\tau) d\tau$, where $f_{(n)}(t)$ is the n-fold convolution density of $f(t)$.

It is also widely known that the RNI (Renewal Intensity) of $F(t)$ by definition is given by $\rho(t) = dM(t)/dt$. Authors in Stochastic Processes refer to $\rho(t)$ as the renewal density, while some authors in Reliability Engineering refer to $\rho(t)$ as the RNIF (Renewal Intensity Function);

because it is also well known that $\rho(t)$ is never a *pdf*, throughout this dissertation we refer to it as the RNIF. The reader should distinguish between the RNIF $\rho(t)$ and the generally constant and unit-less traffic intensity parameter in Queuing Theory defined as ρ = (average arrival rate of customers)/ (average service rate) = expected fraction of time a single server is busy. Further, an approximate expression for the third raw moment of $N(t)$ is given by Kambo et al (2012) [92]. Their expression for $E[N(t)^3]$ can be used to approximate the skewness of $N(t)$.

It is also well-known that for a homogeneous renewal process the *pdf* of interarrivals X_i's is given by $f(t) = \lambda e^{-\lambda t}$, and that of the time to n^{th} failure (or arrival, or renewal) is given by

$f_{T_n}(t) = f_{(n)}(t) = \dfrac{\lambda}{\Gamma(n)}(\lambda t)^{n-1} e^{-\lambda t}$ (the gamma density with shape n and scale $\beta = 1/\lambda$). As a result, the use of Eq. (4.1a) for the interval [0, t] leads to the RF $M(t) = E\left[N(t)\right] = \sum_{n=1}^{\infty} F_{(n)}(t) =$

$$\sum_{n=1}^{\infty}\int_{x=0}^{t}\frac{\lambda}{\Gamma(n)}(\lambda x)^{n-1}e^{-\lambda x}dx = \int_{\mathrm{x}=0}^{t}\lambda \mathrm{e}^{-\lambda \mathrm{x}}\sum_{\mathrm{n}=1}^{\infty}\frac{(\lambda \mathrm{x})^{n-1}}{(n-1)!}\mathrm{dx} = \int_{\mathrm{x}=0}^{t}\lambda \mathrm{e}^{-\lambda \mathrm{x}}\sum_{n=0}^{\infty}\frac{(\lambda \mathrm{x})^{n}}{n!}\mathrm{dx} = \int_{\mathrm{x}=0}^{t}\lambda\,\mathrm{dx} = \lambda t$$, a fact

that has been known for more than 100 years. Further, the RNIF for a HPP is a constant and is given by $\rho(t) = dM(t)/dt = d(\lambda t)/dt = \lambda$. It has also been proven in the theory of stochastic processes [see D. R. Cox and H. D. Miller (1968), pp. 340-347][93] that the $\lim_{t\to\infty} M(t) = t/\mu$, where μ = MTTF. In the case of CFR (Constant Failure-Rate), because the mean of the exponential base-line distribution is $\mu = 1/\lambda$, then it follows that this last limiting result is an exact identity only for the exponential density $\lambda e^{-\lambda t}$, $0 \le t < \infty$. Further, when MTTR (Mean

Time-to-Restore) is almost zero, then the renewal intensity is practically the same as the failure intensity. In the case of exponential base-line distribution, with minimal repair, the point availability function $A(t)$, is simply equal to its reliability function $R(t) = e^{-\lambda t}$.

4.2 The Renewal Functions for Known Convolutions

Unfortunately, obtaining a closed-form expression for the RF for all distributions is not as simple as the case of exponential interarrival times. This is due to the fact that the *n*-fold convolution of most baseline distributions used in reliability analyses is not either known or attainable.

4.2.1 The Renewal Function for a Normal Distribution

For the sake of illustration, suppose that the time between failures, X_i , i = 1,2, 3, 4, …, are NID(μ = MTBF, σ^2), i.e., normally & independently distributed with MTBF (Mean Time Between Failures), and process variance σ^2. Then, Statistical Theory dictates that time to the n^{th} failure (measured from zero) is the *n*-fold convolution of N(μ, σ^2) with itself, i.e., Time-to-the-n^{th}-Failure T_n = TTF_n ~ N($n\mu$, $n\sigma^2$). Hence, in the case of minimal-repair, from Eq. (4.1a) the RF is given by

$$M(t) = \sum_{n=1}^{\infty} F_{(n)}(t) = \sum_{n=1}^{\infty} \Phi\left(\frac{t - n\mu}{\sigma\sqrt{n}}\right) \quad (4.2)$$

where Φ universally stands for the *cdf* (cumulative distribution function) of the standardized normal deviate N(0, 1), and $F_{(n)}(t)$ gives the Pr of at least *n* renewals by time *t*. Xie et al (2003)

[94] gives the same exact expression for the normal RF as in Eq. (4.2), which they used as an approximation for the Weibull renewal with shape $\beta \geq 3$. It should be born in mind that the normal failure law is approximately applicable in reliability analyses only if the coefficient variation of T, denoted CV_T, ≤ 0.15625 = 15.625% because the support for the normal density is $(-\infty,\ \infty)$, while TTF can never be negative (this assures that the size of left-tail below zero is less than 1×10^{-10}). If the CV is not sufficiently small, then the truncated-normal can qualify as a failure distribution; From (2001) [59] discusses the RF for the truncated-normal. As an example, suppose a cutting tool's TTF has the lifetime distribution N(μ = MTBF = 15 operating hours, σ^2 = 2.25) with minimal-repair (or replacement-time), where CV_T = 0.10. Then, Eq. (4.2) shows that the expected number of renewals (or replacements) during 42 hours of use is given by

$M(42) = \sum_{n=1}^{\infty} \Phi(\frac{42-15n}{1.5\sqrt{n}}) = 2.124107$ while M(62 *hours)* = 3.747561 expected renewals. Using the limiting result $M(t) \cong t/\mu$, we obtain M(42 *hours*) ≅ 42/15 = 2.80 (% relative-error = 31.82, while M(62) ≅ 62/15 = 4.133333 with % relative-error = 10.29).

There is a more accurate approximation for $M(t)$, through the second moment, given by $M(t) \cong t/\mu + (\sigma^2 - \mu^2)/(2\mu^2) = t/\mu + 0.50(CV_T^2 - 1)$ [51]. For the above normal nonhomogeneous Process, M(62 *hours)* = 4.1333333 + (2.25–225)/450 = 3.638333 (a much closer approximation). We attempted to obtain a more accurate approximation through the third moment for RF $M(t)$, but the corresponding approximate Laplace transform had complex roots, consistent with other findings. Further, for a Laplace-Gaussian process, the renewal intensity by direct differentiation of (4.2) is given by

$$\rho(t) = \frac{dM(t)}{dt} = \sum_{n=1}^{\infty} \varphi(n\mu, n\sigma^2) = \sum_{n=1}^{\infty} \frac{1}{\sigma\sqrt{2n\pi}} e^{-[(t-n\mu)/\sqrt{n\sigma^2}]^2/2} \qquad (4.3)$$

where the symbol φ stands for the standard normal density. The value of renewal intensity at 42 hours for the N(15, 2.25) baseline distribution, from Eq. (4.3), is ρ(at 42 hours) = 0.07883674 failures/hour. Note that the value of the hazard-rate at 42 hours is given by $h(42) = f(42)/R(42)$ = 10.358138 failures/hour, where $R(t)$ is the reliability function at time t. Because the normal failure Pr law always has an IFR (Increasing Failure-Rate) h(t), then $h(t) > \rho(t)$. (Sheldon M. Ross, 1996, pp.426-427) [68] proves that $t/\mu \le M(t) \le t/\mu + 0.50(\mathrm{CV}_T^2 - 1)$ if $h(t)$ is a DFR (Decreasing Failure-Rate), and he further proves when $h(t)$ is a DFR, then $h(t) \le \rho(t)$ for all $t \ge 0$, and as a result $R(t) \ge \mathrm{e}^{-M(t)}$, and we add that equalities can occur only at $t = 0$.

4.2.2 The Renewal Function for a Gamma Baseline Distribution

Suppose that the TTF of a hot-water heater has a gamma failure density with shape parameter $\alpha = 1.5$ and scale $\beta = 1/\lambda = 3.5$ years [this is quite similar to the Example 9.6 on p. 226 of Ebeling (2010) [4]]. When the heater fails, it is replaced with a new one with the same identical shape and scale (i.e., minimal replacement-time). Our objective is to obtain the RF $M(t)$ for t years of operation. In order to obtain the RF and RNIF of a gamma NHPP, we first resort to Laplace-transforms (LTs).

It has been proven in theory of stochastic processes that the LTs of $\rho(t)$ and $M(t)$ are, respectively, given by [for a proof see (Bhat,1984) [34], pp. 277-280)]

$$\mathcal{L}\{\rho(t)\} = \bar{\rho}(s) = \int_0^\infty e^{-st}\rho(t)dt = \frac{\bar{f}(s)}{1-\bar{f}(s)}, \ s > 0, \tag{4.4a}$$

and

$$\bar{M}(s) = \mathcal{L}\{M(t)\} = \int_0^\infty e^{-st}M(t)dt = \frac{\bar{f}(s)}{s[1-\bar{f}(s)]} = \frac{\bar{\rho}(s)}{s} \tag{4.4b}$$

Further, it is also widely known that the LT of the gamma density $f(t) = \frac{\lambda}{\Gamma(\alpha)}(\lambda t)^{\alpha-1}e^{-\lambda t}$ is given by $\bar{f}(s) = \frac{\lambda^\alpha}{(\lambda+s)^\alpha}$, $\bar{\rho}(s) = \mathcal{L}\{\rho(t)\} = \frac{\lambda^\alpha}{(\lambda+s)^\alpha-\lambda^\alpha}$, and the gamma $\bar{M}(s) = \mathcal{L}\{M(t)\} = \frac{\lambda^\alpha}{s[(\lambda+s)^\alpha-\lambda^\alpha]}$. We used Matlab's *ilaplace* at λ = 1/3 and α =1.5, but Matlab(R2012a,64bit) could not invert $\bar{\rho}(s)$ to the *t*-space at α = 1.5. It seems that when the shape parameter α is not a positive integer, there exists no closed-form inverse-Laplace transform for the gamma density; however, when the underlying failure distribution is Erlang (i.e., gamma with positive integer shape), then there exists a closed-form inverse Laplace-transform for α = 2, 3, and 4; for positive integers beyond 4 there does exist complicated closed-form expressions. In fact, for the specific Erlang density with shape $\alpha \equiv 2$ and scale $\beta = 1/\lambda$, it is well known that $\bar{M}(s) = \int_0^\infty e^{-st}M(t)dt$ =

$\frac{\lambda^2}{s^2(s+2\lambda)} = \frac{-1}{4s} + \frac{\lambda}{2s^2} + \frac{1}{4(s+2\lambda)} = \frac{-1}{4s} + \frac{\lambda}{2s^2} + \frac{1}{4(s+2\lambda)}$. Upon inversion to the *t*-space, we

obtain the well-known $M(t) = E[N(t)] = \mathcal{L}^{-1}\{\bar{M}(s)\} = \mathcal{L}^{-1}\left\{\frac{-1}{4s} + \frac{\lambda}{2s^2} + \frac{1}{4(s+2\lambda)}\right\} =$

$-\frac{1}{4} + \frac{\lambda}{2}t + \frac{1}{4}e^{-2\lambda t}$. Hence, at $\alpha \equiv 2$, the RNIF is given by $\rho(t) = \frac{dM(t)}{dt} = \frac{\lambda}{2} - \frac{\lambda}{2}e^{-2\lambda t}$, which is

quite different from the corresponding gamma (at $\alpha \equiv 2$) IFR HZF

$h(t) = \lambda(\lambda t)e^{-\lambda t} \Big/ \sum_{k=0}^{\alpha-1} \frac{(\lambda t)^k}{k!} e^{-\lambda t} = \lambda(\lambda t)/(1+\lambda t) > \rho(t)$ for $t > 0$. We used Matlab's *ilaplace*

function as an aid in order to obtain the RF and RNIF for the Erlang at shapes $\alpha \equiv 3$ and 4

which are, respectively, given below.

$$M(t) = \mathcal{L}^{-1}\left\{\frac{\lambda^3}{s[(\lambda+s)^3 - \lambda^3]}\right\} = \lambda t/3 + e^{-(3\lambda/2)t}[\cos(\lambda t\sqrt{3/4}) + \sin(\lambda t\sqrt{3/4})/\sqrt{3}]/3 - 1/3,$$

$$\rho(t) = \mathcal{L}^{-1}\left\{\frac{\lambda^3}{(\lambda+s)^3 - \lambda^3}\right\} = \frac{\lambda}{3}\{1 - e^{-(3\lambda/2)t}[\cos(\lambda t\sqrt{3/4}) + \sqrt{3}\sin(\lambda t\sqrt{3/4}]\},$$

$$M(t) = \lambda t/4 + e^{-2\lambda t}/8 + e^{-\lambda t}[\cos(\lambda t) + \sin(\lambda t)]/4 - 3/8,$$

$$\rho(t) = \mathcal{L}^{-1}\left\{\frac{\lambda^4}{(\lambda+s)^4 - \lambda^4}\right\} = \frac{\lambda}{4}[1 - e^{-2\lambda t} - 2e^{-\lambda t}\sin(\lambda t)]$$

The gamma HZF at $\alpha \equiv 3$ is given by $h(t) = [\lambda(\lambda t)^2/\Gamma(3)]/(1+\lambda t+\lambda^2 t^2/2)$, which is not the

same function as the first $\rho(t)$ above. At $\alpha = 5, 6, 7 \ldots$ Matlab(R2012a) provides an expression for $\rho(t)$ only in terms of roots of a polynomial of at least order 5. The user has to find the roots in order to obtain $\rho(t)$.

Referring back to our example where λ = 1/3 and α =1.5, we use the *cdf* $F_{(n)}(t)$ = $\Pr(T_n \le t)$ in order to obtain the RF directly from Eq. (4.1a):

$$M(t) = \sum_{n=1}^{\infty} F_{(n)}(t) = \sum_{n=1}^{\infty} \int_0^t \frac{\lambda}{\Gamma(n\alpha)} (\lambda x)^{n\alpha-1} e^{-\lambda x} dx = \sum_{n=1}^{\infty} \int_0^{\lambda t} \frac{u^{n\alpha-1}}{\Gamma(n\alpha)} e^{-u} du = \sum_{n=1}^{\infty} \Gamma(\lambda t, n\alpha), \quad (4.5)$$

where $\Gamma(\lambda t, n\alpha)$ = Matlab's gammainc $(\lambda t, n\alpha) = \dfrac{1}{\Gamma(n\alpha)} \int_0^{\lambda t} u^{n\alpha-1} e^{-u} du$ represents the incomplete-gamma function at point λt and shape $n\alpha$. In fact, $\Gamma(\lambda t, n\alpha)$ gives the *cdf* of the standard gamma density at point λt and shape $n\alpha$. Thus, for the Water-heater example $M(t = 12 \; years) = \sum_{n=1}^{\infty} \Gamma(12/3.5, 1.5n)$ = 2.11934672 expected failures. Using the 2nd-order approximation we obtain $M(t) \cong t/(\alpha/\lambda) + (\alpha/\lambda^2 - \alpha^2/\lambda^2)/[2(\alpha/\lambda)^2]$ = $\lambda t/\alpha + 0.50(1-\alpha)/\alpha$ = 2.1190476 (which yields a –0.0141% relative error). We next directly differentiate Eq. (4.5) in order to obtain the gamma RNIF. That is,

$$\rho(t) = \frac{dM(t)}{dt} = \frac{\partial}{\partial t} \sum_{n=1}^{\infty} \int_0^t \frac{\lambda}{\Gamma(n\alpha)} (\lambda x)^{n\alpha-1} e^{-\lambda x} dx = \sum_{n=1}^{\infty} \frac{\partial}{\partial t} \int_0^t \frac{\lambda}{\Gamma(n\alpha)} (\lambda x)^{n\alpha-1} e^{-\lambda x} dx$$

$$= \sum_{n=1}^{\infty} \frac{\lambda}{\Gamma(n\alpha)} (\lambda t)^{n\alpha-1} e^{-\lambda t} \quad (4.6)$$

However, the function under the summation in Eq. (4.6) is simply the gamma density with shape $n\alpha$ and scale $\beta = 1/\lambda$. Then, we used our Matlab program, to obtain the value of Eq. (4.6) at t = 12 years, at shape $\alpha = 1.5$ and scale $\beta = 3.5$ years, which yielded $\rho(t = 12) = 0.190348$ renewals/ year. The value of the hazard-function at 12 years is $h(12) = f(12)/R(12) = 0.0193613/0.0765932 = 0.252781$ failures/year. Because the gamma density is an IFR model iff (if and only if) the shape $\alpha > 1$, then $\rho(t) < h(t)$ for $t > 0$. Only at $\alpha = 1$, the gamma baseline failure distribution reduces to the exponential with CFR, the only case for which $\rho(t) \equiv h(t) = \lambda$. In order to check the validity of $\rho(t = 12) = 0.190348$, we resort to the limiting form $\lim_{t\to\infty} \rho(t) = 1/\mu = 1/\text{MTBF}$. Because the expected TTF of the gamma density is $\mu = \alpha\times\beta$, then for the Water-heater example $\mu = \text{MTBF} = 1.5\times3.5 = 5.25$, which yields $\rho(12 \text{ years}) \cong 1/5.25 = 0.190476$/year. Note that since the renewal-type equation for the RNIF $\rho(t)$ is given by $\rho(t) = f(t) + \int_0^t \rho(t-x) f(x)dx$, this last equation clearly shows that $\rho(0) = f(0)$; further $h(t) = f(t)/R(t)$ for certain yields $h(0) = f(0)$, and hence $\rho(0) = f(0) = h(0)$ for all baseline failure distributions. Moreover, if the minimum-life $\delta > 0$, then $\rho(\delta) = f(\delta) = h(\delta)$.

Some authors in Reliability Engineering, such as Ebeling (2010) [4], use the expression $e^{-M(0,t)} = e^{-\int_0^t \rho(x)dx}$ to represent the reliability function $R(t)$ for a NHPP. Clearly, for the case of gamma baseline distribution, which represents a NHPP, the above Ebeling’s expression is an approximation because the exact unconditional reliability is always given by

$R(t) = \mathrm{e}^{-\int_0^t h(\mathrm{x})\mathrm{dx}} = \mathrm{e}^{-H(t)}$, $H(t)$ being the CHF (Cumulative Hazard Function). Leemis (2009) [8] defines the RF as the *cumulative intensity function* using his notation "$\Lambda(t) = \int_0^t \lambda(\tau)\mathrm{d}\tau$", which is identical to the RF $M(t)$, where $\lambda(t)$ is his notation for the RNIF. It seems that he is also using, atop p. 146, the notation $\lambda(t)$ as the hazard function for the Weibull. In section 4.3 it will also be established that the HZF $h(t)$ and the RNIF $\rho(t)$ of the Weibull are not the same, except at $t = 0$.

It is well known from statistical theory that the skewness of gamma density is given by $\beta_3 = 2/\sqrt{\alpha}$ and its kurtosis is $\beta_4 = 6/\alpha$, both of these clearly showing that their limiting values, in terms of shape α, is zero, which are those of Laplace-Gaussian $N(\alpha/\lambda, \alpha/\lambda^2)$. We compared our gamma program at $\alpha = 70$, $\beta = 15$, and $t = 5000$ which yielded $M(5000) \equiv 4.186155$, while the corresponding normal program yielded $M(5000) \cong 4.185793$ expected renewals.

4.3 The Renewal Function when the Underlying TTF Distribution is Uniform

"Uniform distribution is used to model the time of occurrence of events that are equally likely to occur at any time during an interval" [95]. Kececioglu (2002)[95] states that "the most frequently used distributions in Reliability Engineering are exponential, Weibull, normal, lognormal, extreme value, Rayleigh (the Rayleigh being a special case of the Weibull with minimum-life $\delta = 0$ and shape $\beta = 2$), and uniform". Electrical bulbs, stress of mechanical component which has lower and upper psi limits, network systems are some examples in which uniform distribution is used in reliability models. Zhao and Duan [96] propose a reliability estimation model of IC`s interconnect based on uniform distribution of defects on a chip. And also "the uniform distribution is used in Bayesian estimation as a prior reliability distribution" ([97], [95]) as an example please see [64].

Accordingly, suppose the TTF of a component or system (such as a network) is uniformly distributed over the real interval [*a*, *b* weeks]; then $f(t) = 1/c$, a ≥ 0, *b* > *a*, c = *b*–*a* > 0, and the *cdf* is *F*(*t*) = (*t*–*a*)/*c*, *a* ≤ *t* < *b* weeks. Further, succeeding failures have identical failure distributions as U(*a*, *b*). From a practical standpoint, the common value of minimum-life *a* = 0. Then, the fundamental renewal equation is given by $M(t) = F_1(t) + \int_0^t M(t-\tau) f(\tau) \mathrm{d}\tau$ [[34], pp.277-280]. Since we are considering the simpler case of time to first failure distribution being identical to those of succeeding times to failure, then

$$M(t) = F(t) + \int_0^t M(t - \mathrm{x}) f(\mathrm{x}) \mathrm{dx}, \tag{4.7a}$$

whereas before *F*(*t*) represents the *cdf* of *T* = TTF. However, Hildebrand (1962) [98] proves that $\bar{\mathrm{M}}(\mathrm{s}) \times \bar{\mathrm{f}}(\mathrm{s}) = \mathcal{L}\left\{ \int_0^t M(t-\mathrm{x}) f(\mathrm{x}) d\mathrm{x} \right\} = \mathcal{L}\left\{ \int_0^t M(t-\mathrm{x}) dF(\mathrm{x}) \right\}$, the integral inside the first brackets representing the convolution of *M*(*t*) with *f*(*t*). Conversely we can conclude that $\bar{\mathrm{f}}(\mathrm{s}) \times \bar{\mathrm{M}}(\mathrm{s}) =$

$\mathcal{L}\left\{\int_0^t F(t-\mathrm{x})dM(\mathrm{x})\right\}$. Upon inversion of this last LT we obtain $\mathcal{L}^{-1}\{\,\bar{\mathrm{f}}(\mathrm{s})\times\bar{\mathrm{M}}(\mathrm{s})\,\}$ =

$\int_0^t F(t-\mathrm{x})dM(\mathrm{x}) = \int_0^t F(t-\mathrm{x})\rho(\mathrm{x})d(\mathrm{x})$. Hence, Eq. (4.7a) can also be represented as

$$M(t) = F(t) + \int_0^t F(t-\mathrm{x})dM(\mathrm{x}) = F(t) + \int_0^t F(t-\mathrm{x})\rho(\mathrm{x})d\mathrm{x} \qquad (4.7b)$$

The renewal-equation of the type (4.7b) has been given by many authors such as [55], [99], [8], and other notables.

In order to obtain the RF for the uniform density, we substitute into Eq. (4.7a), for the specific uniform U(0, *b*) baseline distribution, for which *a* = 0, in order to obtain

$M(t) = E[N(t)] = \frac{\mathrm{t}}{b} + \int_0^t M(t-\mathrm{x})\mathrm{dx}/b$; letting $t-\mathrm{x}=\tau$ in this last equation yields

$$M(t) = \frac{t}{b} + \frac{1}{b}\int_t^0 M(\tau)(-\mathrm{d}\tau) = \frac{t}{b} + \frac{1}{b}\int_0^t M(\tau)\mathrm{d}\tau. \qquad (4.8)$$

The above Eq. (4.8) shows that the RNIF is given by $\rho(t) = \frac{dM(t)}{dt} = \frac{1}{b} + \frac{M(t)}{b} \rightarrow$

$\frac{\mathrm{d}M(t)}{\mathrm{dt}} - \frac{M(t)}{b} = \frac{1}{b} \rightarrow \frac{\mathrm{d}M(t)}{\mathrm{dt}}\mathrm{e}^{-\mathrm{t}/b} - \frac{M(t)}{b}\mathrm{e}^{-\mathrm{t}/b} = \frac{1}{b}\mathrm{e}^{-\mathrm{t}/b} \rightarrow \frac{\mathrm{d}}{\mathrm{dt}}[M(t)e^{-\mathrm{t}/b}] = \frac{1}{a}\mathrm{e}^{-\mathrm{t}/b} \rightarrow$

$M(t)e^{-\mathrm{t}/b} = \int \frac{1}{b}\mathrm{e}^{-\mathrm{t}/b}\mathrm{d\,t} + \mathrm{C} = -\mathrm{e}^{-\mathrm{t}/b} + \mathrm{C}$, where C is the constant of integration. Applying the boundary condition $M(t = 0) = 0$, we obtain $M(t) = E[N(t)] = \mathrm{e}^{t/b} - 1$, where time must start at zero, i.e., this last expression is valid only for $0 \le t \le b$, $b > 0$. Note that Ross (1996) [68] gives, without proof, the same identical $M(t)$ only for the standard U(0, 1) underlying failure density. For example, if the time to down-state of a network is U(0, 8 weeks), then the expected number of down-states during the interval (0, 8 weeks) is given by $M(8\,weeks) = e^{8/8} - 1 = 1.718282$. In order to calculate the RF for the same uniform distribution during the interval (2, 4) we may use the above result:

$$M(2, 4\,weeks) = \int_2^4 \rho(t)dt = \int_2^4 \frac{1}{8}e^{t/8}dt = e^{t/8}]_2^4 = 0.364696$$

Bartholomew (1963) [100] describes $\rho(t) \times \Delta t$ as the (unconditional) Pr element of a renewal during the interval (t, $t+\Delta t$), and in the case of negligible repair-time, $\rho(t)$ also represents the instantaneous failure intensity function. However, as described by nearly every author in Reliability Engineering, the HZF $h(t)$ gives the instantaneous conditional hazard-rate at time t only amongst survivors of age t, i.e., $h(t) \times \Delta t = \Pr(t \le T \le t + \Delta t)/R(t)$. The hazard function for the U(0, b) baseline distribution is given by $h(t) = \frac{1}{b-t}$, $0 \le t \le b$, $b > 0$, which is infinite at the end of life-interval b, as expected. Because the uniform HZF is an IFR, then for the uniform density it can be proven, using the infinite series for $\frac{1/b}{1-t/b}$ and the Maclaurin series for $e^{t/b}$, that $h(t) > \rho(t)$ for all $0 < t \le b$.

Next in order to obtain the RF for the U(a, b), we transform the origin from zero to minimum-life = $a > 0$ by letting $\tau = a + (b-a)t/b = a + ct/b$ in the RF $M(t) = E[N(t)] =$

$e^{t/b} - 1$. This yields, $M(\tau) = e^{(\tau - a)/c} - 1$, and hence $M(t) = E[N(t)] = e^{(t-a)/c} - 1$, $0 \le a \le t < b$, and $c = b - a > 0$. The corresponding RNIF is given by $\rho(t) = \left[e^{(t-a)/c} \right] / c$, $0 \le a \le t < b$.

Because the uniform renewal function is valid only for the interval [*a*, *b*], we will obtain the *n*-fold convolution of the U(*a*, *b*)-distribution which in turn will enable us to obtain *M*(*t*) for *t* > *b* by making use of Eq. (4.1a) that uses the infinite-sum of convolution *cdf*s, $F_{(n)}(t)$. As stated by Olds (1952) [101], the convolutions of uniform density of equal bases, *c*, have been known since Laplace. The specific convolutions of the uniform density with itself over the interval [−1/2, 1/2] were obtained in [5] only for *n* = 2-fold, 3, 4, 5, 6, and 8-folds. There are other articles on the uniform convolutions such as [102], [103], and [81]. We used the procedure in Maghsoodloo & Hool (1983) [5] but re-developed each of the *n* = 2 through *n* = 8 convolutions of U(*a*, *b*) by a geometrical mathematical statistics method. Further, we programmed this last geometric method in Matlab in order to obtain the exact 9 through 12-fold convolutions of the U(*a*, *b*) with itself. Convolution-densities of U(*a*, *b*) are given at the appendix.

The Matlab program, uses the exact *n* = 2 through 12 convolutions $F_{(n)}(t)$ and then applies the normal approximation for convolutions beyond 12. The question now arises how accurate is the normal approximation to $F_{(n)}(t)$, *n* = 13, 14, 15, …? We used our 12-fold convolution of the standard uniform U(0, 1) to determine the accuracy. Clearly, the partial sum $T_{12} = \sum_{i=1}^{12} X_i$, each $X_i \sim U(0, 1)$ and mutually independent, has a mean of 6 and variance $12(b-a)^2/12 = 1$, where *a* = 0, and maximum-life *b* = 1. Table 3 shows the normal approximation to $F_{(12)}(t)$ for intervals of

0.50-Stdev. The table clearly shows that the worst relative-error occurs at ½ StDev, and that the normal approximation improves as Z moves toward the right-tail. The accuracy is within 2 decimals up to one StDev and 3 decimals beyond 1.49 StDevs. Therefore, we conclude that the normal approximation to each of $F_{(n)}(t)$, n = 13, 14, 15, … , due to the CLT, should not have a relative error at Z = 0.50 exceeding 0.002960.

Table 3: Normal Approximation to the 12-fold convolution of U(0,1)

Z	0.5	1	1.5	2	2.5	3	3.5	4
$F_{(12)}$(z)	0.689422	0.839273	0.932553	0.977724	0.994421	0.998993	0.999879	0.999991
Normal	0.691462	0.841345	0.933193	0.97725	0.99379	0.99865	0.999767	0.999968
Rel-Error	0.002960	0.002469	0.000686	-0.00049	-0.00063	-0.00034	-0.00011	-2.3E-05

It should also be noted that the normal approximation to the uniform $F_{(n)}(t)$, n = 13, 14, 15, … must be very accurate from the standpoint of the first 4 moments. Because the skewness of n-fold convolution of U(a, b) with itself is identically zero, which is identical to the Laplace-Gaussian $N(n(a+b)/2, n(b-a)^2/12)$, and hence a perfect match between the first 3 moments of $f_{T_n}(t) = f_{(n)}(t)$ with those of normal. It can be proven (the proof is at the appendix) that the skewness of the partial-sum $T_n = \sum_{i=1}^{n} X_i$, X_i's being iid like X, is given by

$$\beta_3(T_n) = \mu_3(T_n) \big/ [\mu_2(T_n)]^{3/2} = n\mu_3(X) \big/ (n\sigma^2)^{3/2} = \mu_3(X) \big/ (\sigma^3\sqrt{n}) = \beta_3(X) \big/ \sqrt{n} \qquad (4.9a)$$

Further, the kurtosis of T_n is given by

$$\beta_4(T_n) = \alpha_4(X)/n + 3(n-1)/n - 3 \rightarrow \beta_4(T_n) = [\alpha_4(X) - 3]/n = \beta_4(X)/n, \qquad (4.9b)$$

where μ_i (i = 2, 3, 4) are the universal notation for the i^th central moments. Eq. (4.9b) clearly shows that the kurtosis of the uniform $f_{(n)}(t)$, for n = 13, 14, and 15, respectively, are β_4 = –1.20/13 = –0.09231, –1.20/14 = –0.08571, and –1.20/15 = –0.08000, the amount –1.20 being the kurtosis of a U(*a*, *b*) underlying failure density. Thus, an *n* = 120 is needed in order for the kurtosis of $f_{(n)}(t)$ to be within 0.01 of Laplace-Gaussian $N(n(a+b)/2, n(b-a)^2/12)$. Fortunately, the previous summary table clearly indicates that the normal approximation is superior at the tails, where kurtosis plays a more important role, than the middle of $f_{(n)}(t)$ density.

In order to compute the RNIF $\rho(t)$ for $t > b$, we used two different approximate procedures, one of which will be detailed in section 4.5. Eq. (4.1a) clearly shows that

$\rho(t) = \frac{d}{dt}\sum_{n=1}^{\infty} F_{(n)}(t) = \sum_{n=1}^{\infty} f_{(n)}(t)$; the Matlab program knows the exact $f_{(1)}(t) = f(t)$ and uniform convolutions $f_{(n)}(t)$, for $n = 2, 3, \ldots, 12$. For $n > 12$, it uses the ordinate of normal density $N(n\mu, n\sigma^2)$ approximation, where $\mu = (a+b)/2$ and $\sigma^2 = (b-a)^2/12$.

4.4 Approximating the Renewal Function with Unknown Convolutions

Unlike the gamma, normal and uniform underlying failure distributions, the Weibull base-line distribution (except when the shape parameter β = 1) does not have a closed-form expression for the *n*-fold *cdf* convolution $F_{(n)}(t)$, and hence Eq. (4.1a) cannot directly be used to calculate the renewal function $M(t)$ for all $\beta > 0$. When minimum-life = 0 and shape β = 2, the Weibull specifically is called the Rayleigh *pdf*; we do have a closed-form function for the Rayleigh $\bar{\rho}(s)$ but it cannot be inverted to yield a closed-form expression for its $\rho(t)$. Because the

Weibull is the most important underlying mortality density in reliability analyses, we will use the discretizing method to approximate the renewal function for three parameter Weibull distribution. The parameter δ is the minimum-life or threshold (a location parameter), θ is called the characteristic-life ($\alpha = \theta - \delta$ being the scale parameter), and β is the slope (or shape) parameter. Further, it is well-known that the Weibull's HZF $h(t) = \begin{cases} 0, & t < \delta \\ \frac{\beta}{\alpha}(\frac{t-\delta}{\alpha})^{\beta-1}, & t \geq \delta \end{cases}$ is an IFR (with CV < 100%) iff the slope $\beta > 1$, $h(t) = \lambda = 1/\alpha$ is a CFR iff $\beta \equiv 1$ (with CV = 100%), and it is a DFR iff $0 < \beta < 1$ (with CV > 100%). It must be highlighted that there have been many articles on approximating the Weibull RF such as Jiang (2007)[66], From (2001) [59], and other notables. Note that Murthy et al (2004) [99] provide an extensive treatise on Weibull Models, referring to the Weibull with zero minimum-life as the standard model. These last three authors also highlight the confusion and misconception resulting from the terminologies of intensity and hazard function for the Weibull. Jin & Gonigunta (2008) [60] first approximated the *cdf* of the 2-parameter Weibull (i.e., threshold $\delta = 0$) by an optimum generalized exponential function; then they obtained the LT of the corresponding generalized exponential, which could be inverted to yield their actual Weibull RF.

4.5 Discretizing Time in Order to Approximate the Renewal Equation

Because $M(t) = F(t) + \int_0^t M(t-\tau) f(\tau) \mathrm{d}\tau$ and the underling distributions are herein specified, the first term on the RHS (Right-Hand Side) of $M(t)$, $F(t)$, can be easily computed.

However, the convolution integral on the RHS, $\int_0^t M(t-\tau)\,f(\tau)\mathrm{d}\tau$, except for rare cases, cannot in general be computed and has to be approximated. The discretization method was first applied by Xie [55], where he called his procedure "THE RS-METHOD", RS for Riemann-Stieltjes. However, Xie [55] used renewal-type Eq. (7b) in his RS-METHOD.

The first step in the discretization is to divide the specified interval (0, *t*) into equal-length subintervals, and only for the sake of illustration we consider the interval (0, *t* = 5 weeks) and divide it into 10 subintervals (0, 0.50), (0.50, 1), …, (4.5, 5). Note that Xie's method does not require equal-length subintervals. Thus, the length of each subinterval in this example is $\Delta t = 0.50$ weeks. As a result, $\int_0^5 M(5-\tau)\,f(\tau)\mathrm{d}\tau \equiv \sum_{\mathrm{i}=1}^{10} \int_{(i-1)/2}^{i/2} M(5-\tau)\,f(\tau)\mathrm{d}\tau$, where the index *i* = 1 pertains to the subinterval (0, 0.50), and *i* = 10 pertains to the last subinterval (4.5, 5). We now make use of the *Mean-value Theorem for Integrals*, which states**:** if a function f(x) is continuous over the real closed interval [a, b], then for certain there exists a real number x_0 such that $\int_a^b$ f(x)dx ≡ f(x_0)×(b–a), a ≤ x_0 ≤ b, f(x_0) being the ordinate of the integrand at x_0. Because both *M*(*t*–*τ*) and the density *f*(*t*) are continuous, applying the above *Mean-value Theorem for Integrals* to the 4$^{\text{th}}$ subinterval, there exists for certain a real number τ_4 such that $\int_{3/2}^{2} M(5-\tau)\,f(\tau)\mathrm{d}\tau \equiv M(5-\tau_4)\,f(\tau_4)(2-3/2)$, 3/2 ≤ τ_4 ≤ 2. As a result, $\int_0^5 M(5-\tau)\,f(\tau)\mathrm{d}\tau \equiv$

$\sum_{i=1}^{10} M(5-\tau_i)\, f(\tau_i)(1/2)$, where $0 \le \tau_1 \le 0.50$, $0.5 \le \tau_2 \le 1$, …, $4.5 \le \tau_{10} \le 5$. Clearly the exact values of $M(5-\tau_i)$, i = 1, …, 10 cannot in general be determined, and because in this example $f(\tau_i)(1/2) = \int_{(i-1)/2}^{i/2} f(\tau)\mathrm{d}\tau =$ Pr[$(i-1)/2 \le \text{TTF} \le i/2$], it follows that $\int_0^5 M(5-\tau)\, f(\tau)\mathrm{d}\tau \equiv \sum_{i=1}^{10} M(5-\tau_i) \int_{(i-1)/2}^{i/2} f(\tau)\mathrm{d}\tau$. As proposed by Elsayed [35] who used the end of each subinterval, we will approximate this function in the same manner by $M(5-0.50i)$ which results in

$$M(5) \equiv \text{F}(5) + \int_0^5 M(5-\tau)\, f(\tau)\mathrm{d}\tau \cong F(5) + \sum_{\text{i}=1}^{10} M(5-0.50i) \int_{(i-1)/2}^{i/2} f(\tau)\mathrm{d}\tau = \sum_{\text{i}=1}^{10} \int_{(i-1)/2}^{i/2} f(\tau)\mathrm{d}\tau \; +$$

$$\sum_{\text{i}=1}^{10} M(5-0.50i) \int_{(i-1)/2}^{i/2} f(\tau)\mathrm{d}\tau = \sum_{\text{i}=1}^{10} \left\{ [1+M(5-0.50i)] \times \int_{(i-1)/2}^{i/2} f(\tau)\mathrm{d}\tau \right\} \tag{4.10}$$

The above Eq. (10) is similar to that of (7.10) of Elsayed [35], where his subintervals are of length $\Delta t = 1$. We first used the information $M(0) \equiv 0$ at i = 10 to calculate the last term of Eq. (10); further, at i = 9, Eq. (10) yields $[1+M(5-4.5)] \int_{8/2}^{(8+1)/2} f(\tau)\mathrm{d}\tau = [1+M(0.50)] \int_4^{4.5} f(\tau)\mathrm{d}\tau$.

However, $M(0.50)$ represents the expected number of renewals during an interval of length $\Delta t =$ 0.50. Assuming that Δt is sufficiently small relative to t such that $N(t)$ is approximately Bernoulli, then $M(\Delta t = 0.50) \cong 1 \times F(0.50) + 0 \times R(0.50)$. Hence, at i = 9, the value of the term before last in Eq. (10) reduces approximately to $[1+F(0.50)]\times\int_{4}^{4.5} f(\tau)\mathrm{d}\tau$. At i = 8, the value of Eq. (10) is given by $[1+M(1)]\int_{3.5}^{4} f(\tau)\mathrm{d}\tau$, where M (1) = $\sum_{\mathrm{i}=1}^{2}\{[1+M(1-0.50i)]$ $\times\int_{(i-1)/2}^{i/2} f(\tau)\mathrm{d}\tau\}$, where M(0.50) has been approximated. Continuing in this manner, we backward recursively solved Eq. (4.10) to approximate M (t). The smaller Δt always leads to a better approximation of M (t).

In order to check the accuracy of this approximation method, we first used it to approximate $M(t)$ at β =1 (which is the exponential failure law with $M(t) = \lambda t$), t = 10000, minimum-life $\delta = 0$, $\theta = \alpha$ = 1000 = 1/λ, and Δt = 50 = 0.005t, approximation yielded M(10000) ≅ 9.754115099857199 compared to the exact λt = 0.001×10000 = 10, a percent relative error of –2.459 with *cpu*-time = 65.112829 seconds. While, at the same exact parameters, our Matlab function at Δt = 40 = 0.004t, M(10000) ≅ 9.802640211919197 (a % relative error of -1.97360) with *cpu*-time = 567.432046. We ran the same program with same parameters by just changing Δt. We observed that for smaller values of Δt we really approach the exact value. The table below depicts the results.

Table 4: Time Discretizing Approximation Method Results

Δt	M(t)	App_M(t)	Relative Error	Elapsed Time (seconds)
10	10.0000000000	9.9501662508319	-0.49834%	40238.61714
25	10.0000000000	9.8760351886669	-1.23965%	2018.707614
40	10.0000000000	9.8026402119192	-1.97360%	567.432046
50	10.0000000000	9.7541150998572	-2.45885%	82.738732
100	10.0000000000	9.5162581964040	-4.83742%	65.112829
200	10.0000000000	9.0634623461009	-9.36538%	32.320588
250	10.0000000000	8.8479686771438	-11.52031%	21.066944
500	10.0000000000	7.8693868057473	-21.30613%	16.510933
1000	10.0000000000	6.3212055882856	-36.78794%	16.347242

Figure 6 shows relationship between Δt and Relative Error. As Δt increases Relative Error increases. This implies that smaller Δt leads to more accurate result.

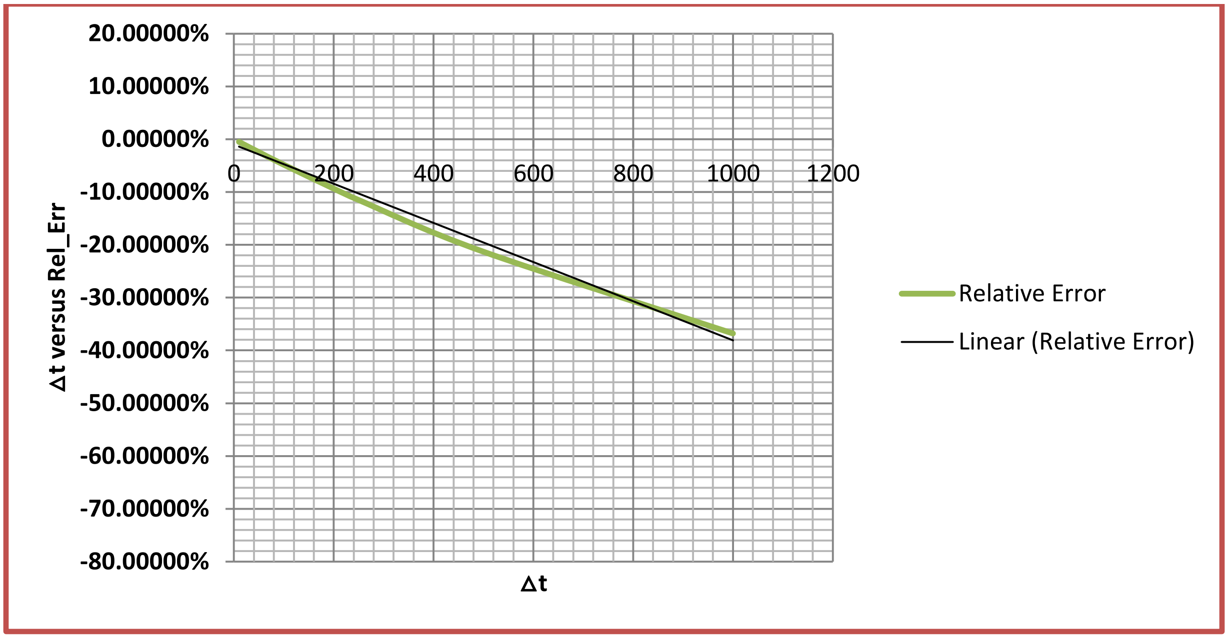

Figure 6: Relative Error versus Δt

Next, after approximating the Weibull RF, how do we use its $M(t)$ to obtain a fairly accurate value of Weibull RNIF $\rho(t)$? Because $\rho(t) \equiv \frac{dM(t)}{dt} \equiv \lim_{\Delta t \to 0} \frac{M(t+\Delta t)-M(t)}{\Delta t}$, then for sufficiently small $\Delta t > 0$ the approximate $\rho(t) \approx \frac{M(t+\Delta t)-M(t)}{\Delta t}$, which uses the right-hand derivative, and $\rho(t) \approx \frac{M(t)-M(t-\Delta t)}{\Delta t}$, using the left-hand derivative. Because the RF is not linear but strictly increasing, our Matlab program computes both the left-and right-hand expressions and approximates $\rho(t)$ by averaging the two, where t and Δt are inputted by the user. It is recommended that the user inputs $0 < \Delta t \leq 0.10t$.

As a result, all available approaches have merits and demerits in terms of complexity, computational time and accuracy. The method here is easy to implement basically for any lifetime distribution and it gives fairly accurate results for smaller subintervals. However, the downside is for smaller subintervals the computational time increases. And also it is not a closed-form approximation. Based on given parameters the program calculates both renewal and reliability measures. Therefore, this approach cannot be used in some maintenance optimization models if is desired to have closed-form expression.

4.6 Chapter Summary

This chapter provided the RF and RNIF for the gamma and uniform underlying failure densities. We also devised Matlab programs that output all the renewal and reliability measures of a 3-parameter Weibull, normal, gamma, and uniform. We have highlighted that the RNIF $\rho(t)$ is different from the HZF $h(t)$ for $t > 0$, except in the case of CFR.

CHAPTER 5

Expected Number of Renewals for Non-Negligible Repair

Unlike the previous chapters, we now assume that MTTR (Mean Time to Repair) is not negligible and that TTR has a pdf denoted as $r(t)$. This chapter gives expected number of failures, number of cycles and availability by taking the Laplace transforms of renewal functions.

5.1 Expected Number of Failures and Cycles

Let the variates $X_1, X_2, X_3, \ldots$ represent TTF_i be iid with the underlying failure density $f(x)$ having means MTBF = μ_x and variance σ_x^2; further, let $Y_1, Y_2, Y_3, \ldots$ represent the ith Time-to-Repair (TTRi), i = 1, 2, 3, 4,… with the same pdf $r(y)$ having means MTTR = μ_y and variance σ_y^2. Then, $T_i = X_i + Y_i$ represents the time between cycles (TBCs) which are also iid whose density is given by the convolution $g(t) = f(t) * r(t)$, and whose Laplace transform (LT) is given by $\bar{g}(s) = \bar{f}(s) \times \bar{r}(s)$. Clearly the mean and variance of the cycle-times T_i's are $\mu_x + \mu_y$ and $\sigma_x^2 + \sigma_y^2$. As described by U. N. Bhat (1984) [34] there will be two types of renewals:

(1) A transition from a Y-state (i.e., when system is under repair) to an X-state (at which the system is operating reliably),

(2) A transition from an X-state (or operating-reliably-state) to a Y-state (where system will go under repair).

Let $M_1(t)$ represent the expected number of cycles (or number of renewals of type 1), and $M_2(t)$ represent the expected number of failures (or renewals of type 2). Then, as stated by Bhat (1984) [34] and E. A. Elsayed (2012), the LTs (Laplace-Transforms) of the two renewal functions, respectively, are given by

$$\bar{M}_1(s) = \frac{\bar{g}(s)}{s[1-\bar{g}(s)]} = \frac{\bar{f}(s)\times\bar{r}(s)}{s[1-\bar{f}(s)\times\bar{r}(s)]} \tag{5.1a}$$

$$\bar{M}_2(s) = \frac{\bar{f}(s)}{s[1-\bar{g}(s)]} = \frac{\bar{f}(s)}{s[1-\bar{f}(s)\times\bar{r}(s)]} \tag{5.1b}$$

The corresponding LTs of RNIFs (Renewal-Intensity Functions) are given by

$$\bar{\rho}_1(s) = \frac{\bar{f}(s)\times\bar{r}(s)}{1-\bar{f}(s)\times\bar{r}(s)}, \quad \text{and } \bar{\rho}_2(s) = \frac{\bar{f}(s)}{1-\bar{f}(s)\times\bar{r}(s)} \tag{5.2}$$

As an example, suppose $TTF_i \sim Exp(\lambda)$ and $TTR_i \sim Exp(r)$; then as has been documented by numerous other authors, $\bar{f}(s) = \int_0^\infty \lambda e^{-\lambda t} e^{-st} dt = \lambda/(\lambda+s)$ and $\bar{r}(s) = \int_0^\infty re^{-rt} e^{-st} dt = r/(r+s)$. On substituting these last 2 LTs into Eq. (5.1a), we obtain

$$\bar{M}_1(s) = \frac{\lambda r}{s[(\lambda+s)(r+s)-\lambda r]} = \frac{-\lambda r}{s\xi^2} + \frac{\lambda r}{s^2\xi} + \frac{\lambda r}{\xi^2(s+\xi)}$$, where $\xi = \lambda + r$

$M_1(t) = \mathcal{L}^{-1}\{\bar{M}_1(s)\} = \mathcal{L}^{-1}\left\{\frac{-\lambda r}{s\xi^2} + \frac{\lambda r}{s^2\xi} + \frac{\lambda r}{\xi^2(s+\xi)}\right\} = \frac{-\lambda r}{\xi^2} + \frac{\lambda r}{\xi} t + \frac{\lambda r}{\xi^2} e^{-\xi t}$, which gives the expected number of transitions from a repair-state to an operational-state (or expected number of cycles). Similarly,

$$\bar{M}_2(s) = \frac{\bar{f}(s)}{s[1-\bar{f}(s)\times\bar{r}(s)]} = \frac{\lambda^2}{s\xi^2} + \frac{\lambda r}{s^2\xi} - \frac{\lambda^2}{\xi^2(s+\xi)}$$, which upon inversion yields

$$M_2(t) = \frac{\lambda^2}{\xi^2} + \frac{\lambda r}{\xi}t - \frac{\lambda^2}{\xi^2}e^{-\xi t}$$, representing the expected number of failures during an interval of length t. Note that the limit of both renewal functions $M_1(t)$ and $M_2(t)$ as $r \to \infty$ (i.e., MTTR $\to$ 0) is exactly equal to λt, as expected. Further, a comparison of $M_2(t)$ with $M_1(t)$ reveals that $M_2(t) > M_1(t)$ for all $t > 0$, which is intuitively meaningful because the expected number of failures must exceed the expected number of cycles for all $t > 0$. As an example, if λ = 0.0005/hour and repair-rate = 0.05, then $M_1(t$ =1000 hours) = 0.485246544456426, while $M_2(1000)$ = 0.495147534555436, and hence the availability will be shown below that A(at t = 1000 hours) = 1+0.485246544456426–0.495147534555436 = 0.990099009900990.

We now obtain a general expression for the RF s M_1 and M_2 by inverting equations (5.1). Eq. (5.1a) shows that $\bar{M}_1(s)[1-\bar{g}(s)] = \frac{\bar{g}(s)}{s} \longrightarrow \bar{M}_1(s) = \frac{\bar{g}(s)}{s} + \bar{M}_1(s)\bar{g}(s) \longrightarrow$

$M_1(t) = G(t) + \int_0^t M_1(t-x)\,g(x)dx$, where G($t$) is the cdf of $f(x)*r(x) = g(x)$. Eq. (5.1b) now

shows that $\bar{M}_2(s)[1-\bar{g}(s)] = \frac{\bar{f}(s)}{s} \longrightarrow \bar{M}_2(s) = \frac{\bar{f}(s)}{s} + \bar{M}_2(s)\bar{g}(s) \longrightarrow$

$M_2(t) = F(t) + \int_0^t M_2(t-x)\,g(x)dx$; thus, in general the expected number of cycles is given by

$$M_1(t) = G(t) + \int_0^t M_1(t - x)\, g(x) dx \qquad (5.3a)$$

While the expected number of failures

$$M_2(t) = F(t) + \int_0^t M_2(t - x)\, g(x) dx \qquad (5.3b)$$

For example, suppose TBFs ~ N(μ_x = MTBF, σ_x^2) and TTR is also N(μ_y, σ_y^2); then TBCs ~ N(μ_x+μ_y, $\sigma_x^2 + \sigma_y^2$). Then, $M_1(t) = \sum_{n=1}^{\infty} \Phi(\frac{t - n\mu}{\sigma\sqrt{n}})$, where $\mu = \mu_x + \mu_y$, $\sigma = \sqrt{\sigma_x^2 + \sigma_y^2}$, and $M_1(t)$ gives the expected number of renewals of the first type, i.e., the expected number of cycles. However, because the system is under repair a fraction of the times, then $M_2(t) \neq \sum_{n=1}^{\infty} \Phi(\frac{t - n\mu_x}{\sigma_x\sqrt{n}})$. In order to obtain a good approximation for $M_2(t)$ and the resulting $A(t)$, we may argue that the expected duration of time the system is under repair is given by $M_1(t)$ ×MTTR; letting $t_2 = t - M_1(t)$ ×MTTR, then Eq. (5.3b) shows that the expected number of failures is approximately given by $M_2(t) \cong \Phi(\frac{t - \mu_x}{\sigma_x}) + \sum_{n=2}^{\infty} \Phi(\frac{t_2 - n\mu_x}{\sigma_x\sqrt{n}})$. Clearly, $M_2(t) \geq M_1(t)$ for all $t \geq 0$.

5.2 Availability

Because we are assuming that a system can be either in an operational-state, or under repair, then the reliability function must be replaced by the instantaneous (or point) availability function at time *t*, denoted *A*(*t*), which represents the Pr that a repairable unit or system is functioning reliably at time *t*. Thus, if there is no repair, the availability function is simply *A*(*t*) = *R*(*t*), the reliability function . However, if the component (or system) is repairable, then there are two mutually exclusive possibilities:

(1) The system is reliable at *t*, in which case $A_1(t) = R(t)$,

(2) The system fails at time x, 0 < x < *t*, gets renewed (or restored to almost as-good-as-new) in the interval (x, x+Δx) with Pr element ρ(x) dx, and then is reliable from time x to time *t* Trivedi (1982) [104].

This second Pr is given by $A_2(t) = \int_0^t \rho(\mathrm{x})\mathrm{dx}\, R(t-\mathrm{x})$. Because the above two cases are mutually exclusive, then

$$A(t) = A_1(t) + A_2(t) = R(t) + \int_0^t R(t-\mathrm{x})\rho(\mathrm{x})\mathrm{dx} \qquad (5.4).$$

Taking Laplace transform of the above Eq. (5.4) [and observing that the integral is the convolution of *R*(*t*) with ρ(t)] yields

$$\bar{\mathrm{A}}(\mathrm{s}) = \bar{\mathrm{R}}(\mathrm{s}) + \bar{\mathrm{R}}(\mathrm{s})\bar{\rho}(\mathrm{s}) = \bar{\mathrm{R}}(\mathrm{s})\,[1+\bar{\rho}(\mathrm{s})] =$$

$$\bar{\mathrm{R}}(\mathrm{s})\,[1 + \frac{\bar{\mathrm{f}}(\mathrm{s})\times\bar{\mathrm{r}}(\mathrm{s})}{1-\bar{\mathrm{f}}(\mathrm{s})\times\bar{\mathrm{r}}(\mathrm{s})}\,] = \frac{\bar{\mathrm{R}}(\mathrm{s})}{1-\bar{\mathrm{f}}(\mathrm{s})\times\bar{\mathrm{r}}(\mathrm{s})}\,, \qquad (5.5)$$

where *r*(*t*) is the density of repair-time. For the case when the TTF (of a component or a system) has a constant failure rate λ and time to repair is also exponential at the rate r,

$\bar{R}(s) = \int_0^\infty e^{-\lambda t} e^{-st} dt = \lambda / (\lambda + s)$; hence, the Laplace-transform of availability from Eq. (5.5) is

given by $\bar{A}(s) = \dfrac{1/(\lambda+s)}{1-[\lambda/(\lambda+s)][r/(r+s)]} = \dfrac{r+s}{s[s+(\lambda+r)]} = \dfrac{r}{\xi s} + \dfrac{\lambda/\xi}{s+\xi} \rightarrow$

$A(t) = \mathcal{L}^{-1}\{\bar{A}(s)\} = \dfrac{r}{\xi} + \dfrac{\lambda}{\xi} e^{-\xi t}$ where $\xi = \lambda + r$, which is provided by many other authors in Reliability Engineering such as C. E. Ebeling (2010) [4], E. A. Elsayed (2012) [6], etc. For example, given that $\lambda = \lambda_{\text{failure-rate}} = 0.0005$ and $r = \lambda_{\text{repair-rate}} = 0.05$ per hour, then $\xi = \lambda + r = 0.0505$ and the Pr that the unit is available (i.e., not under repair) at t = 1000 hours is given by

$A(1000) = \dfrac{0.05}{0.0505} + \dfrac{0.0005}{0.0505} e^{-0.0505(1000)} = 0.990099009901$, while R(1000 hours W/O Repair) = $e^{-0.5} = 0.60653066 < A(1000) = 0.9901$. Note that in the exponential case, we can also obtain the availability function $A(t)$ directly from Eq. (5.4) as follows:

$A(t) = R(t) + \int_0^t R(t-x)\rho_1(x)dx = e^{-\lambda t} + \int_0^t e^{-\lambda(t-x)} \rho_1(x)dx$, where

$\rho_1(x) = dM_1(x)/dx = \dfrac{d}{dx}\left(\dfrac{-\lambda r}{\xi^2} + \dfrac{\lambda r}{\xi} x + \dfrac{\lambda r}{\xi^2} e^{-\xi x}\right) = \dfrac{\lambda r}{\xi} - \dfrac{\lambda r}{\xi} e^{-\xi x}$. Upon substitution of this

RNIF into the expression for $A(t)$, we obtain $A(t) = e^{-\lambda t} + \int_0^t e^{-\lambda(t-x)} \dfrac{\lambda r}{\xi}(1-e^{-\xi x})dx = \dfrac{r}{\xi} + \dfrac{\lambda}{\xi} e^{-\xi t}$,

as before. As pointed out by E. A. Elsayed (2012, pp. 466-467) [6], we also observe that

$\bar{R}(s) = \int_0^\infty e^{-st}R(t)dt \ = \int_0^\infty e^{-st}[(1-F(t)]dt = \frac{1}{s} - \int_0^\infty e^{-st}F(t)dt = \frac{1}{s} - \bar{F}(s)$. Hildebrand (1962) [98]

proves that $\bar{F}(s) = \bar{f}(s)/s$ so that $\bar{R}(s)=\frac{1-\bar{f}(s)}{s}$; on substitution into Eq. (5.5) we obtain

$$\bar{A}(s) = \frac{1-\bar{f}(s)}{s[1-\bar{f}(s)\times\bar{r}(s)]} = \frac{1}{s[1-\bar{f}(s)\times\bar{r}(s)]} - \frac{\bar{f}(s)}{s[1-\bar{f}(s)\times\bar{r}(s)]} =$$

$\frac{1}{s} + \frac{\bar{f}(s)\times\bar{r}(s)}{s[1-\bar{f}(s)\times\bar{r}(s)]} - \frac{\bar{f}(s)}{s[1-\bar{f}(s)\times\bar{r}(s)]}$. Inverting these 3 LTs, we obtain

$$A(t) \ = 1 + M_1(t) - M_2(t) \qquad (5.6)$$

for all underlying failure distribution $f(t)$ and TTR-distribution $r(t)$. Eq. (5.6) is the same as that of E.A. Elsayed (2012) [6] on his page 467.

5.3 Markov Analysis

Note that we can also use Markov analysis, as has been done by many authors in stochastic processes, in the case of constant failure and repair rates to obtain the availability of a simple on & off- system as depicted in the following Figure:

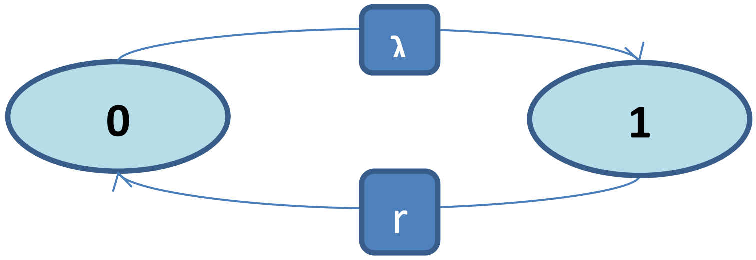

Figure 7: On & off system

where state "0" represents a system in the reliable-state and "1" represents the same system under repair. The above figure clearly shows that $dP_0(t)/dt = -\lambda P_0(t) + rP_1(t)$, where $P_0(t)$ represents the unconditional Pr of finding the systems in the operational state "0" at time *t*, and similarly for $P_1(t)$. Because $P_1(t) = 1 - P_0(t)$ for all *t*, we obtain $dP_0(t)/dt = -\lambda P_0(t) + r(1 - P_0)$ and hence $dP_0(t)/dt + (\lambda + r)P_0(t) = r$, or $dP_0(t)/dt + \xi P_0(t) = r$. This is a simple differential equation with the integrating factor $e^{\xi t}$. Solving and applying the boundary condition $P_0(t = 0) = 1$ results in $P_0(t) = \frac{r}{\xi} + \frac{\lambda}{\xi}e^{-\xi t}$, which is identical to the *A*(*t*) obtained above in 2 other different methods.

It should be noted that, although the solution $P_0(t) = \frac{r}{\xi} + \frac{\lambda}{\xi}e^{-\xi t}$ is valid exactly iff both failure and repair rates are constants, it can be used to obtain a rough approximate solution for the *A*(*t*) when only the MTBF and MTTR are available. For example, suppose that a system's TTF ~ U(0, 2000 hours), while TTR is also U(10, 30 hours); then, the MFR (Mean Failure Rate) ≈ 1/2000 = 0.0005, and the MRR = 1/20 = 0.05. Thus, a rough approximate solution for *A*(*t*) in this case of uniform TTF and TTR is also given by $P_0(t) \approx \frac{r}{\xi} + \frac{\lambda}{\xi}e^{-\xi t} = 0.99009901$, while this last availability value is exact only for the exponential cases of TTF and TTR.

5.4 Renewal and Availability Functions when TTF is Gamma and TTR is Exponential

It is well known that the LT of an underlying gamma failure density with shape α and scale β = 1/λ is given by $\bar{f}(s) = \lambda^{\alpha} / (\lambda + s)^{\alpha}$; note that only when α is a positive integer this last closed-form is valid. When α is not an exact positive integer, there is no closed-form solution for the LT of a gamma density because the integration-by-parts never terminates. Thus, in the case of shape being an exact positive integer, i.e., Erlang underlying failure-density, we have:

$$\bar{A}(s) = \frac{1-\bar{f}(s)}{s[1-\bar{f}(s)\times\bar{r}(s)]} = \frac{1-\dfrac{\lambda^{\alpha}}{(\lambda+s)^{\alpha}}}{s[1-\dfrac{\lambda^{\alpha}}{(\lambda+s)^{\alpha}}\times\dfrac{r}{r+s}]} = \frac{(\lambda+s)^{\alpha}(s+r)-\lambda^{\alpha}(s+r)}{s[(\lambda+s)^{\alpha}(r+s)-\lambda^{\alpha}r]}$$

. At α = 2 this last LT reduces to $\bar{A}(s) = \dfrac{s^2+(2\lambda+r)s+2\lambda r}{s[s^2+(2\lambda+r)s+\lambda^2+2\lambda r]} = \dfrac{c_1}{s}+\dfrac{c_2}{s-r_1}+\dfrac{c_3}{s-r_2}$, where r_1 & r_2 are the roots of the polynomial $s^2+(2\lambda+r)s+\lambda^2+2\lambda r = 0$. Thus, $r_1 = -(\lambda+r/2)-\sqrt{(r/2)^2-\lambda r}$,

$r_2 = -(\lambda+r/2)+\sqrt{(r/2)^2-\lambda r}$, $c_1 = 2r/(2r+\lambda)$, $c_2 = \dfrac{-\lambda(2\lambda+r+r_1)}{(\lambda+2r)\sqrt{r^2-4\lambda r}}$, and

$c_3 = \dfrac{\lambda(2\lambda+r+r_2)}{(\lambda+2r)\sqrt{r^2-4\lambda r}}$. Inverting back to the *t*-space we obtain

$A(t) = 2r/(2r+\lambda)+c_2e^{r_1t}+c_3e^{r_2t}$. This last availability function clearly shows that as $t\to\infty$,

$A(t)\to 2r/(2r+\lambda) = r/(r+\lambda/2)$ = MTBF/(MTBF+MTTR), and further, $A(0)\equiv 1$, as expected.

For example, if a system has an underlying gamma failure distribution with shape α = 2, scale

β = 1/λ = 1000 hours and TTR has a constant repair-rate r = 0.05, then the availability at 500 hours is given by $A(500)$ = 0.993855509027565; while the same system with minimal repair has an $A\left(t=500\right)=R\left(500\right)=\int_t^{\infty}\lambda(\lambda x)e^{-\lambda x}dx=\mathrm{e}^{-\lambda t}(1+\lambda t)=0.909795989568950$. That is, repair will improve availability by 9.24%. The same system has an A(1000 hours) = 0.991466622031406, and R(1000 hours, no repair) = 0.735758882342885; now repair will improve availability by 34.75%. Thus, the steady-state (or long-term) availability of such a system as discussed by many other authors is A = 0.05/(0.05+0.0005) = 0.99009901.

At α = 2 the LT of expected number of cycles reduces to

$\bar{\mathrm{M}}_1(\mathrm{s})=\dfrac{\mathrm{r}\lambda^2}{\mathrm{s}^2[\mathrm{s}^2+(2\lambda+\mathrm{r})\mathrm{s}+\lambda^2+2\lambda\mathrm{r}]} = \dfrac{\mathrm{c}_4}{\mathrm{s}}+\dfrac{\mathrm{c}_5}{\mathrm{s}^2}+\dfrac{\mathrm{c}_6}{\mathrm{s}-\mathrm{r}_1}+\dfrac{\mathrm{c}_7}{\mathrm{s}-\mathrm{r}_2}$, where r_1 & r_2 are the same roots, $\mathrm{c}_4=\dfrac{-\mathrm{r}(2\lambda+\mathrm{r})}{(\lambda+2\mathrm{r})^2}$, $\mathrm{c}_5=\lambda\mathrm{r}/(\lambda+2\mathrm{r})$, $\mathrm{c}_6=\dfrac{\mathrm{c}_5-\mathrm{c}_4\mathrm{r}_2}{\sqrt{\mathrm{r}^2-4\lambda\mathrm{r}}}$, and $\mathrm{c}_7=\dfrac{\mathrm{c}_4\mathrm{r}_1-\mathrm{c}_5}{\sqrt{\mathrm{r}^2-4\lambda\mathrm{r}}}$. Upon inversion, we obtain $M_1\left(t\right)=\mathrm{c}_4+\mathrm{c}_5t+\mathrm{c}_6\mathrm{e}^{\mathrm{r}_1t}+\mathrm{c}_7\mathrm{e}^{\mathrm{r}_2t}$. For the same parameters as above, we obtain $M_1(t$ = 10,000 hours) = 4.695618077086354 expected cycles. Similarly, it can be shown that the LT of the expected number of failures is given by $\bar{\mathrm{M}}_2(\mathrm{s})=\dfrac{\lambda^2(\mathrm{r}+\mathrm{s})}{\mathrm{s}^2[\mathrm{s}^2+(2\lambda+\mathrm{r})\mathrm{s}+\lambda^2+2\lambda\mathrm{r}]}$ = $\dfrac{\mathrm{c}_8}{\mathrm{s}}+\dfrac{\mathrm{c}_9}{\mathrm{s}^2}+\dfrac{\mathrm{c}_{10}}{\mathrm{s}-\mathrm{r}_1}+\dfrac{\mathrm{c}_{11}}{\mathrm{s}-\mathrm{r}_2}$, where $\mathrm{c}_8=\dfrac{(\lambda^2-\mathrm{r}^2)}{(\lambda+2\mathrm{r})^2}$, $\mathrm{c}_9=\lambda\mathrm{r}/(\lambda+2\mathrm{r})$, $\mathrm{c}_{10}=\dfrac{(2\lambda+\mathrm{r}+\mathrm{r}_1)\mathrm{c}_8+\mathrm{c}_9}{\sqrt{\mathrm{r}^2-4\lambda\mathrm{r}}}$, and $\mathrm{c}_{11}=\dfrac{-(2\lambda+\mathrm{r}+\mathrm{r}_2)\mathrm{c}_8-\mathrm{c}_9}{\sqrt{\mathrm{r}^2-4\lambda\mathrm{r}}}$. Upon inversion to the t-space we obtain

$M_2(t) = c_8 + c_9 t + c_{10}e^{r_1 t} + c_{11}e^{r_2 t}$. The value of expected number of failures during a mission of length 10,000 hours is $M_2(t = 10000) = 4.705519067168098$, which exceeds $M_1(10000) = 4.695618077086354$, as expected. Further, $M_1(10000) - M_2(10000) + 1 = 0.990099009918256$, which is identical to the value availability function obtained from

$A(t) = 2r/(2r+\lambda) + c2e^{r_1 t} + c3e^{r_2 t}$ at $t = 10000$.

Unfortunately, when TTF is Erlang at $\alpha = 3$, 4, 5 & 6 and a specified constant repair rate r, the corresponding denominators $D(s) = s[1 - \bar{f}(s) \times \bar{r}(s)]$ has at least 2 complex roots, which are generally complex conjugate pairs. Yet, after partial-fractioning, the LT's can be inverted to yield real-valued $M_1(t)$ and $M_2(t)$, as demonstrated below.

$$\text{At } \alpha = 3,\ \bar{M}_1(s) = \frac{\bar{f}(s) \times \bar{r}(s)}{s[1 - \bar{f}(s) \times \bar{r}(s)]} = \frac{\frac{\lambda^3}{(\lambda+s)^3} \times \frac{r}{r+s}}{s[1 - \frac{\lambda^3}{(\lambda+s)^3} \times \frac{r}{r+s}]} = \frac{\lambda^3 r}{s[(\lambda+s)^3(r+s) - r\lambda^3]} =$$

$$\frac{\lambda^3 r}{s^2[s^3 + (3\lambda + r)s^2 + (3\lambda r + 3\lambda^2)s + \lambda^3 + 3r\lambda^2]} = \frac{c_1}{s} + \frac{c_2}{s^2} + \frac{c_3}{s - r_1} + \frac{c_4}{s - r_2} + \frac{c_5}{s - r_3}$$

, where the root r_1 will be real, while r_2 and r_3 will be complex conjugates, i.e., both $r_2 + r_3$ and $r_2 \times r_3$ will be real numbers. In order to maintain equality in the above PFRAC (Partial Fraction), it can be shown that $c_2 = \frac{-\lambda^3 r}{r_1 r_2 r_3}$, $c_1 = c_2 \times \frac{r_1 r_2 + r_1 r_3 + r_2 r_3}{r_1 r_2 r_3}$; further, letting the constants

$a_1 = c_2(r_1 + r_2 + r_3) - c_1(r_1 r_2 + r_1 r_3 + r_2 r_3)$, $a_2 = c_2 - c_1(r_1 + r_2 + r_3)$, then c_3, c_4, and c_5

are the unique solution given by C = $\begin{bmatrix} c_3 & c_4 & c_5 \end{bmatrix}' = A^{-1} \times b$, where C is the 3×1 solution vector, b

is a 3×1 vector b=$\begin{bmatrix} a_1 \\ a_2 \\ -c_1 \end{bmatrix}$ and the 3×3 matrix A = $\begin{bmatrix} r_2 r_3 & r_1 r_3 & r_1 r_2 \\ r_2 + r_3 & r_1 + r_3 & r_1 + r_2 \\ 1 & 1 & 1 \end{bmatrix}$. A Matlab program

was devised to obtain the expected number of cycles $M_1(t)$ as outlined above. The program also uses similar procedure as above to compute $M_2(t)$ and the resulting $A(t)$. The Matlab program has the capability to compute the 3 renewal measures $M_1(t)$, $M_2(t)$, and $A(t)$ for $\alpha = 2, 3, 4, 5, 6$ and 7.

CHAPTER 6

The Approximate Expected Number of Renewals for Non-Negligible Repair

As in chapter 5, we assume that MTTR (Mean Time to Repair) is not negligible and that TTR (Time to Restore, or repair) has a pdf denoted as $r(t)$ but this chapter gives the approximate number of cycles, number of failures and the resulting availability for particular distributions.

Availability was explained in the previous chapter. The inverse Laplace transform of Equation 5.5 results in the point availability $A(t)$. If the underlying distributions are not exponential, problems arise in inverting the Laplace transform [105]. Therefore numerical solutions and approximations become the only alternatives for obtaining $A(t)$ [35]. There are numerous approximation techniques in the literature such as Sarkar & Chaudhuri (1999) [105] uses Fourier transform technique to determine the availability of a maintained system under continuous monitoring and with perfect repair policy. They also obtain closed-form expressions when the system has gamma life distribution and exponential repair time. Ananda and Gamage (2004) [106] consider statistical inference for the steady state availability of a system when repair distribution is two-parameter lognormal and failure distributions are Weibull, gamma and lognormal. There are also other papers in the literature that work on confidence limits for steady state availability of a system like [107], [108] etc.

In this chapter in order to approximate availability and renewal functions two different approximation techniques are discussed. First for some cases like Weibull TTF and uniform TTR we managed to obtain the convolution of failure density $f(t)$ and repair density $r(t)$. Then we used these convolution densities to approximate $M_1(t)$, $M_2(t)$ and $A(t)$ by using time discretizing approximation method that was discussed in Chapter 4.

However, obtaining the convolutions of $f(t)$ with $r(t)$ for the general classes of failure and repair distributions is not always tractable, such is the case of both TTF and TTR being Weibull. In these cases we used moment based approximation which only requires knowing the first four row moments of failure and repair distributions. “There are a number of cases where the moments of a distribution easily obtained, but theoretical distributions are not available in closed form” [109]. And also, efficient estimators for the various moments of the underlying distribution could be calculated from the observed sample data [92]. Kambo et. al. (2012) [92], uses first three moments of failure distribution in order to approximate the renewal function for negligible repair and they conclude that the method produces exact results of the renewal function for certain important distributions like mixture of two exponential and Coxian-2.

In this chapter, we propose an approximation for the evaluation of expected number of cycles, number of failures and availability based on first four row moments of failure and repair distributions where convolution of $f(t)$ and $r(t)$ is intractable. We conclude that the method produces very accurate results for especially large values of time t.

6.1 Weibull TBF and Uniform TTR

Let the variates $X_1, X_2, X_3, \ldots$ represent TTF_i be iid with the underlying failure density $f(x)$ having means MTBF = μ_x and variance σ_x^2; further, let $Y_1, Y_2, Y_3, \ldots$ represent the ith Time-to-Repair (TTR_i), i = 1, 2, 3, 4,… with the same pdf $r(y)$ having means MTTR = μ_y and variance σ_y^2 . Then, $T_i = X_i + Y_i$ represents the time between cycles (TBCs) which are also iid whose density is given by the convolution $g(t) = f(t) * r(t)$, and whose Laplace transform (LT) is given by $\bar{g}(s) = \bar{f}(s) \times \bar{r}(s)$. Clearly the mean and variance of the cycle-times T_i's are $\mu_x + \mu_y$ and $\sigma_x^2 + \sigma_y^2$.

Suppose the TBFs of a component (or a system) has the Weibull distribution with minimum life $\delta = t_0 \geq 0$, characteristic life $\theta > \delta = t_0$, and shape (or slope) $\beta > 0$, i.e., TTF ~ W(δ, θ, β). Letting $\lambda = 1/(\theta - t_0)$, the density of X = TBFs (Time Between Failures) is given by

$$f(x) = \beta\lambda[\lambda(x - t_0)]^{\beta-1} e^{-[\lambda(x-t_0)]^{\beta}}, \quad t_0 \leq x < \infty$$

Further, we assume that the time-to-restore (TTR) has the uniform U(a, b), $0 \leq a < b < \infty$ density function. Then, the repair-density $r(y) = 1/c$, $a \leq y < b$ and $c = b - a > 0$. We are considering only the simpler case of the TTFF (Time to first Failure) and TTF_i, i = 2, 3, 4, … having the same Weibull distributions, and also succeeding repairs have the same identical U(a, b) distributions. Then, the Time-Between-Cycles is given by TBCs = TBF + TTR; we used a geometrical mathematical statistics method to obtain the exact convolution of f(x) with r(y), denoted g(t). The corresponding pdf of TBCs, g(t), is given below.

$$g(t) = f(t) * r(t) = \begin{cases} \{1 - e^{-[\lambda(t-t_0-a)]^{\beta}}\}/c, & a + t_0 \leq t \leq b + t_0 \\ \{e^{-[\lambda(t-t_0-b)]^{\beta}} - e^{-[\lambda(t-t_0-a)]^{\beta}}\}/c, & b + t_0 \leq t < \infty \end{cases} \qquad (6.1a)$$

The above density has no closed-form (or explicit) antiderivative, except when $\beta \equiv 1$, but Matlab can integrate $g(t)$ within any desired limits (t_1, t_2), $a+t_0 \leq t_1 < t_2 < \infty$.

6.2 Uniform TBF and Weibull TTR

Conversely, suppose that the TBFs of a component or system has the U(a, b) density function and its TTR has the W(δ, θ, β) density. Thus, the repair-rate function is given by $\beta r[r(t-\delta)]^{\beta}$, where δ represents the minimum repair-time. Only when β =1, the repair-rate is constant and is denoted by r, and at β =1 the TTR has the exponential distribution. Because most of TTR distributions in Reliability Engineering are positively-skewed, it is recommended that the value of shape β not to exceed 3. Then, the failure density is $f(x) = 1/c$, $a \leq x < b$, $c = b-a$, and repair density is given by

$$r(y) = \beta r[r(y-\delta)]^{\beta-1} e^{-[r(y-\delta)]^{\beta}}, \quad \delta \leq y < \infty$$

As in the previous case, assuming that the TTFF and TTF_i, i = 2, 3, 4, … have the same identical U(a, b) distributions, and also succeeding repairs have the same Weibull distributions, then it can be proven that the TBCs = TTF + TTR has the following density, which is the convolution of $f(t)$ with $r(t)$.

$$g(t) = f(t) * r(t) = \begin{cases} \{1 - e^{-[r(t-\delta-a)]^{\beta}}\} / c, & a+\delta \leq t \leq b+\delta \\ \{e^{-[r(t-\delta-b)]^{\beta}} - e^{-[r(t-\delta-a)]^{\beta}}\} / c, & b+\delta \leq t < \infty \end{cases} \tag{6.1b}$$

It should be noted that the convolution in Eq. (6.1a) is common, while in (6.1b) is not. We then used the same procedure as in section 4.5, to devise a Matlab program, to approximate the expected number of cycles $M_1(t)$ using the density in (6.1a), and also the same procedure to approximate the renewal function $M_2(t)$, and the resulting approximate availability $A(t)$.

6.3 Gamma TBF and Uniform TTR

Secondly, suppose the TBFs of a component (or a system) has the gamma distribution with shape $\alpha > 0$ and scale $\beta = 1/\lambda$. The density of X = TBF (or uptime) is given by

$$f(x) = \frac{\lambda}{\Gamma(\alpha)}(\lambda x)^{\alpha-1}e^{-\lambda x}, \quad 0 \le x < \infty$$

Further, we assume that the time-to-restore (TTR) has the uniform U(a, b), $0 \le a < b < \infty$ density function. Then, the repair (or downtime) density $r(x) = 1/c$, $a \le x < b$ and $c = b - a > 0$. We are considering only the simpler case of the TTFF (Time to first Failure) and TTF_i, i = 2, 3, 4, … having the same gamma distributions, and also succeeding repairs have the same identical U(a, b) distributions. Then, the Time-Between-Cycles is given by TBCs = TBF + TTR, and it can be proven that TBCs has the following density, which is the convolution of f(t) with r(t), and is denoted by $g(t)$.

$$g(t) = f(t) * r(t) = \begin{cases} \Gamma[\lambda(t-a),\alpha]/c\ , & a \le t \le b \\ \{\Gamma[\lambda(t-a),\alpha] - \Gamma[\lambda(t-b),\alpha]\}/c, & b \le t < \infty \end{cases} \qquad (6.2a)$$

where $\Gamma[\lambda(t-a),\alpha] = \frac{1}{\Gamma(\alpha)}\int_0^{\lambda(t-a)} x^{\alpha-1}e^{-x}dx$ is the *cdf* of the standard gamma density, at $\lambda(t-\alpha)$. The above density has no closed-form antiderivative but Matlab can integrate $g(t)$ for any interval within $a \le t < \infty$.

6.4 Uniform TTF and Gamma TTR

Conversely, suppose that the TTF of a component or system has the U(a, b) density function and its TTR has the gamma density. Then, the failure density is $f(x) = 1/c$, $a \le x < b$, $c = b - a$, and repair density is given by

$$r(x) = \frac{r}{\Gamma(\alpha)}(rx)^{\alpha-1}e^{-rx}, \quad 0 \le x < \infty$$

Note that when the shape $\alpha \equiv 1$, the repair-density reduces to the Exponential in which case the parameter r becomes the constant repair-rate. As in the previous case, assuming that the TTFF and TTF_i, i = 2, 3, 4 … have the same identical U(a, b) distributions, and also succeeding repairs have the same gamma distributions, then it can be proven that the TBCs = TTF + TTR has the following density, which is the convolution of $f(t)$ with $r(t)$.

$$g(t) = f(t) * r(t) = \begin{cases} \Gamma[r(t-a), \alpha]/c\ , & a \le t \le b \\ \{\Gamma[r(t-a), \alpha] - \Gamma[r(t-b), \alpha]\}/c, & b \le t < \infty \end{cases} \tag{6.2b}$$

A Matlab program was devised to obtain the renewal measure $M_1(t)$, $M_2(t)$, and availability $A(t)$ only for the prevalent case of Eq. (6.2a).

6.5 Intractable Convolutions of $f(t)$ with $r(t)$

It is clear by now that obtaining the convolutions of $f(t)$ with $r(t)$ for the general classes of failure and repair distributions is not always tractable, such is the case of both TTF and TTR being Weibull, then $g(t)$ cannot be obtained as above. Therefore, below we will develop an approximate method based on raw moments that will yield approximations for the three functions $M_1(t)$, $M_2(t)$, and the resulting $A(t)$ for any failure and repair distributions.

It has been well documented since Pierre Laplace that his LT of any density function is given by $\bar{f}(s) = \int_0^\infty e^{-st} f(t)dt$, where the dummy-variable s must exceed zero. Such a Laplace-transformation is quite often necessary because the solution in the s-space is lot easier to obtain than the direct solution from the t-space. As discussed in Chapter 5, we obtained $\bar{M}_1(s) = \frac{\bar{f}(s) \times \bar{r}(s)}{s[1 - \bar{f}(s) \times \bar{r}(s)]}$, and $\bar{M}_2(s) = \frac{\bar{f}(s)}{s[1 - \bar{f}(s) \times \bar{r}(s)]}$ so that we can easily observe these 2 renewal function LTs have identical denominators, and is given by $D(s) = s[1 - \bar{f}(s) \times \bar{r}(s)]$ in the s-space, then we can obtain approximations for $M_1(t)$ and $M_2(t)$, and the resulting $A(t) \cong 1 + M_1(t) - M_2(t)$. We start our procedure with the definition of LT, the method having also been applied by other authors.

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t)dt = \int_0^\infty (1 - st + s^2t^2/2! - s^3t^3/3! + s^4t^4/4! - ...)f(t)dt \ , \qquad (6.3a)$$

where we have made use of the Maclaurin series for e^{-st}. Using the definition of statistical raw moments, Eq. (6.3a) yields

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t)dt = 1 - \mu_1' s + \mu_2' s^2/2! - \mu_3' s^3/3! + \mu_4' s^4/4! - .... \ , \qquad (6.3a)$$

where $\mu_k' = \int_0^\infty t^k f(t)dt = E(T^k)$. Similarly, for the repair-density, $r(t)$, we have its LT as:

$$\bar{r}(s) = 1 - m_1 s + m_2 s^2/2! - m_3 s^3/3! + m_4 s^4/4! - .... , \qquad (6.3b)$$

where $m_k = \int_0^\infty t^k r(t)dt$ is the k[th] raw moment for TTR (Time to Restore). Therefore, in terms of raw moments the denominators of $M_1(s)$ and $M_2(s)$ are given by

$$D(s) = s[1-(1-\mu_1's+\mu_2's^2/2!-\mu_3's^3/3!+\mu_4's^4/4!-....))\times$$

$$(1-m_1s+m_2s^2/2!-m_3s^3/3!+m_4s^4/4!-....)]$$

$$= s[(\mu_1'+m_1)s-(\mu_2'+m_2+2\mu_1'm_1)s^2/2 + (\mu_3'+m_3+3\mu_1'm_2+3\mu_2'm_1)s^3/6 -$$

$$(\mu_4'+m_4+4\mu_3'm_1+6\mu_2'm_2+4\mu_1'm_3)s^4/24+(\mu_4'm_1+2\mu_3'm_2+2\mu_2'm_3+\mu_1'm_4)s^5/24-$$

$$(3\mu_4'm_2+4\mu_3'm_3+3\mu_2'm_4)s^6/144+(\mu_3'm_4+\mu_4'm_3)s^7/144-\mu_4'm_4s^8/576 + \ldots]$$

$$= s^2[(\mu_1'+m_1)-(\mu_2'+m_2+2\mu_1'm_1)s/2 + (\mu_3'+m_3+3\mu_1'm_2+3\mu_2'm_1)s^2/6 -$$

$$(\mu_4'+m_4+4\mu_3'm_1+4\mu_1'm_3+6\mu_2'm_2)s^3/24+(\mu_4'm_1+2\mu_3'm_2+2\mu_2'm_3+\mu_1'm_4)s^4/24-$$

$$(3\mu_4'm_2+4\mu_3'm_3+3\mu_2'm_4)s^5/144 + (\mu_3'm_4+\mu_4'm_3)s^6/144-\mu_4'm_4s^7/576 + \ldots]$$

(6.4a)

Note that the inclusion of higher exponents s^8, s^9, etc. in the brackets inside Eq. (6.4a) will require the 5[th], 6[th], etc. raw moments in the above D(s) which we will not consider. Thus, the 4[th]-order approximation for D(s) is given by

$$D(s) \approx s^2[(\mu_1'+m_1)-(\mu_2'+m_2+2\mu_1'm_1)s/2 + (\mu_3'+m_3+3\mu_1'm_2+3\mu_2'm_1)s^2/6 -$$

$$(\mu_4'+m_4+4\mu_3'm_1+4\mu_1'm_3+6\mu_2'm_2)s^3/24+(\mu_4'm_1+2\mu_3'm_2+2\mu_2'm_3+\mu_1'm_4)s^4/24-$$

$$(3\mu_4'm_2+4\mu_3'm_3+3\mu_2'm_4)s^5/144 + (\mu_3'm_4+\mu_4'm_3)s^6/144-\mu_4'm_4s^7/576.$$

(6.4b)

Before setting up partial fractions for $\bar{M}_1(s) = \dfrac{\bar{f}(s) \times \bar{r}(s)}{s[1 - \bar{f}(s) \times \bar{r}(s)]}$, it will be judicious to first multiply the numerator and denominator by 576 and then divide both by $b_7 = -\mu'_4 m_4$ so that $\bar{M}_1(s)$ will take the following form:

$$\bar{M}_1(s) =$$

$$\frac{576(1 - \mu'_1 s + \mu'_2 s^2/2! - \mu'_3 s^3/3! + \mu'_4 s^4/4!) \times (1 - m_1 s + m_2 s^2/2! - m_3 s^3/3! + m_4 s^4/4!)/b_7}{s^2(s^7 + b_6 s^6 + b_5 s^5 + b_4 s^4 + b_3 s^3 + b_2 s^2 + b_1 s + b_0)} \tag{6.5a}$$

where the coefficients $b_6 = 4(\mu'_3 m_4 + \mu'_4 m_3)/b_7$, $b_5 = -4(3\mu'_4 m_2 + 4\mu'_3 m_3 + 3\mu'_2 m_4)/b_7$,

$b_4 = 24(\mu'_4 m_1 + 2\mu'_3 m_2 + 2\mu'_2 m_3 + \mu'_1 m_4)/b_7$,

$b_3 = -24(\mu'_4 + m_4 + 4\mu'_3 m_1 + 4\mu'_1 m_3 + 6\mu'_2 m_2)/b_7$,

$b_2 = 96(\mu'_3 + m_3 + 3\mu'_1 m_2 + 3\mu'_2 m_1)/b_7$, $b_1 = -288(\mu'_2 + m_2 + 2\mu'_1 m_1)/b_7$, and

$b_0 = 576(\mu'_1 + m_1)/b_7$. The above $\bar{M}_1(s)$ can now be partial-fractionated as follows:

$$\bar{M}_1(s) = \frac{c_1}{s} + \frac{c_2}{s^2} + \frac{c_3}{s - r_3} + \frac{c_4}{s - r_4} + \frac{c_5}{s - r_5} + \frac{c_6}{s - r_6} + \frac{c_7}{s - r_7} + \frac{c_8}{s - r_8} + \frac{c_9}{s - r_9} \tag{6.5b}$$

where r_i, i = 3, 4, 5, 6, 7, 8 and 9 are the 7 real and complex-conjugate pairs of roots of the polynomial $P_7 = s^7 + b_6 s^6 + b_5 s^5 + b_4 s^4 + b_3 s^3 + b_2 s^2 + b_1 s + b_0$. A comparison of Eq. (6.5b) with (6.5a) shows that $c_2 = -576/(R_7 b_7)$, where $R_7 = \prod_{i=3}^{9} r_i$ is the product of all the 7 roots and will be real-valued. Similarly, $c_1 = c_2 \times R_6/R_7 + 576(\mu'_1 + m_1)/(b_7 R_7)$, where $R_6 = \sum \prod_{i=3} r_i$ is the sum of products of any distinct 6 roots out of 7. Once the 7 roots are obtained, then the values of c_1 and c_2 can be computed, and a good approximation to $M_1(t)$ is given by $M_1(t) \cong c_1 + c_2 t$. In order to solve the partial-fraction coefficients $c_3, c_4, \ldots, c_9$, we must obtain 7 equations in 7 unknowns by comparing (6.5b) with (6.5a) such that the expression for $\bar{M}_1(s)$ in Eq. (6.5b) will exactly equal to the one in Eq. (6.5a). The coefficient for c_3 will be obtained by equation the coefficient of s^2 in the numerator of (6.5b) with that of (6.5a). This yields the first equation in 7 unknowns, as shown below:

$$c_1 R_6 - c_2 R_5 + c_3 \prod_{i \neq 3}^{9} r_i + c_4 \prod_{i \neq 4}^{9} r_i + c_5 \prod_{i \neq 5}^{9} r_i + c_6 \prod_{i \neq 6}^{9} r_i + c_7 \prod_{i \neq 7}^{9} r_i + c_8 \prod_{i \neq 8}^{9} r_i + c_9 \prod_{i=3}^{8} r_i =$$

$576(m_2/2 + \mu_1' m_1 + \mu_2'/2)/b_7$, where $R_5 = \sum \prod_{i=3} r_i$ is the sum of products of any 5 distinct roots. Thus the first equation is

$$c_3 \prod_{i \neq 3}^{9} r_i + c_4 \prod_{i \neq 4}^{9} r_i + c_5 \prod_{i \neq 5}^{9} r_i + c_6 \prod_{i \neq 6}^{9} r_i + c_7 \prod_{i \neq 7}^{9} r_i + c_8 \prod_{i \neq 8}^{9} r_i + c_9 \prod_{i=3}^{8} r_i =$$

$288(m_2 + 2\mu_1' m_1 + \mu_2')/b7 + c_2 R_5 - c_1 R_6$. The comparison of s^3 coefficients will give rise to the second equation in the 7 unknowns $c_3, c_4, \ldots, c_9$. Letting $R_{j5} = \sum \prod_{i \neq j} r_i$, j = 3, 4… 9 be the sum of products of any 5 distinct roots out of 7, excluding the j[th] root, and hence it will have exactly ${}_6C_5$ = 6 terms. Using these notations, the second equation by comparing the coefficients of s^3 will be as follows:

$$c_3 R_{35} + c_4 R_{45} + c_5 R_{55} + c_6 R_{65} + c_7 R_{75} + c_8 R_{85} + c_9 R_{95} =$$

$-c_1R_5 + c_2R_4 + 96(m_3 + 3\mu_1'm_2 + 3\mu_2'm_1 + \mu_3')/b_7$, where $R_4 = \sum_{i=3} \prod r_i$ has ${}_7C_4$ = 35 terms of products of any distinct 4 out of 7 roots.

Similarly, letting $R_{j4} = \sum_{i \neq j} \prod r_i$, j = 3, 4, …, 9 be the sum of products of any 4 distinct roots out of 7 excluding the jth roots and equating the coefficients of s^4, we obtain the third equation

$$c_3R_{34} + c_4R_{44} + c_5R_{54} + c_6R_{64} + c_7R_{74} + c_8R_{84} + c_9R_{94} =$$

$$-c_1R_4 + c_2R_3 + 24(m_4 + 4\mu_1'm_3 + 6\mu_2'm_2 + 4\mu_3' m_1 + \mu_4')/b_7,$$

where $R_3 = \sum_{i=3} \prod r_i$ is the sum of products of any 3 distinct roots out of 7. Note that in the definition of R_{ji}, the second index i always indicates the number of roots in the product $\prod r_i$.

Next equating the coefficients of s^5 results in

$$c_3R_{33} + c_4R_{43} + c_5R_{53} + c_6R_{63} + c_7R_{73} + c_8R_{83} + c_9R_{93} =$$

$$-c_1R_3 + c_2R_2 + 24(\mu_1'm_4 + 2\mu_2'm_3 + 2\mu_3'm_2 + \mu_4' m_1)/b_7$$

Next equating the coefficients of s^6 results in:

$$c_3R_{32} + c_4R_{42} + c_5R_{52} + c_6R_{62} + c_7R_{72} + c_8R_{82} + c_9R_{92} =$$

$-c_1R_2 + c_2R_1 + 4\left(3\mu_2'm_4 + 4\mu_3'm_3 + 3\mu_4'm_2\right)/b_7$, where $R_1 = \sum_{i=3}^{9} r_i$ is the sum of all 7 roots.

Equating the coefficients of s^7 results in:

$$c_3R_{31} + c_4R_{41} + c_5R_{51} + c_6R_{61} + c_7R_{71} + c_8R_{81} + c_9R_{91} = -c_1R_1 + c_2 + 4(\mu_3'm_4 + \mu_4'm_3)/b_7,$$

where $R_{j1} = \sum_{i \neq j}^{9} r_i$. Finally, equating the coefficients of s^8 will yield the last equation.

$$c_3 + c_4 + c_5 + c_6 + c_7 + c_8 + c_9 = -c_1 + \mu'_4 m_4 / b_7 = -c_1 - 1$$

Unfortunately, the above 4th-order approximation is sometimes only for small duration of time t is not quite adequate, such is the case of normal TTF and TTR. The only option left is include the 5th raw moment in the expression for $\bar{M}_1(s)$, i.e.,

$$\bar{M}_1(s) = \frac{2880(1-\mu'_1 s + \mu'_2 s^2/2! - \mu'_3 s^3/3! + \mu'_4 s^4/4! - \mu'_5 s^5/5!) \times (1 - m_1 s + m_2 s^2/2! - m_3 s^3/3! + m_4 s^4/4!)/b_8}{s^2(s^8 + b_7 s^7 + b_6 s^6 + b_5 s^5 + b_4 s^4 + b_3 s^3 + b_2 s^2 + b_1 s + b_0)} \quad (6.6)$$

where, $b_8 = \mu'_5 m_4$, $b_7 = -(5\mu'_4 m_4 + 4\mu'_5 m_3)/b_8$, $b_6 = (20\mu'_3 m_4 + 20\mu'_4 m_3 + 12\mu'_5 m_2)/b_8$,

$b_5 = -(24\mu'_5 m_1 + 60\mu'_4 m_2 + 80\mu'_3 m_3 + 60\mu'_2 m_4)/b_8$,

$b_4 = (24\mu'_5 + 60\mu'_4 m_1 + 240\mu'_3 m_2 + 240\mu'_2 m_3 + 60\mu'_1 m_4)/b_8$, ,

$b_3 = -(120\mu'_4 + 480\mu'_3 m_1 + 720\mu'_2 m_2 + 480\mu'_1 m_3 + 120 m_4)/b_8$,

$b_2 = (480\mu'_3 + 1440\mu'_2 m_1 + 1440\mu'_1 m_2 + 480 m_3)/b_8$ and

$b_1 = -(1440\mu'_2 + 2880\mu'_1 m_1 + 1440 m_2)/b_8$, and $b_0 = 2880(\mu'_1 + m_1)/b_8$. Similarly, it can be shown that $c_2 = +2880/(R_8 b_8)$, where $R_8 = \prod_{i=3}^{10} r_i$ is the product of all the 8 roots and will be real-valued. Similarly, $c_1 = c_2 \times R_7 / R_8 - 2880(\mu'_1 + m_1)/(b_8 R_8)$, where $R_7 = \sum \prod_{i=3} r_i$ is the sum of products of any distinct 7 roots out of 8, which also will be real-valued. Once the 8 roots of the polynomial $P_8 = s^8 + b_7 s^7 + b_6 s^6 + b_5 s^5 + b_4 s^4 + b_3 s^3 + b_2 s^2 + b_1 s + b_0$ are obtained, then the values of c_1 and c_2 can be computed, and a good approximation to $M_1(t)$ is given by $M_1(t) \cong c_1 + c_2 t$. In order to solve the partial-fraction coefficients c_3, c_4, …, c_{10}, we must obtain 8 equations in 8 unknowns similar to the previous case. Unfortunately, even the above partial-5th order approximation does not improve the value of $M_1(t)$ for small t for the case of normal TTF and TTR. We determined, however, that the exact $M_1(t) = \sum_{n=1}^{\infty} \Phi(\frac{t - n\mu}{\sigma\sqrt{n}})$ for μ_x =1000 hours, μ_y = 90, σ_x^2 = 5625, and σ_y^2 = 81 gave a value of M_1(6541.938667 hours) = 5.504179908259676, while the approximation $M_1(t) \cong c_1 + c_2 \times 6541.938667$ = 5.5041799075162, where c_1 = – 0.4975986869792105, and c_2 = 0.00091743119266055051.

We now use the equation $\bar{M}_2(s) = \dfrac{\bar{f}(s)}{s[1 - \bar{f}(s) \times \bar{r}(s)]}$ in order to obtain the 4th-order approximation for $M_2(t)$. Because $\bar{M}_2(s)$ has the same denominator as $\bar{M}_1(s)$, then

$$\bar{M}_2(s) = \frac{576(1 - \mu_1' s + \mu_2' s^2/2! - \mu_3' s^3/3! + \mu_4' s^4/4!)/b_7}{s^2(s^7 + b_6 s^6 + b_5 s^5 + b_4 s^4 + b_3 s^3 + b_2 s^2 + b_1 s + b_0)}, \qquad (6.7a)$$

where the 7 denominator roots r_3, …, r_9 will be the same as those of $\bar{M}_1(s)$. On a comparison of (6.7a) with (6.5b) will show that $c_2 = -576/(R_7 b_7)$ will stay intact, but $c_1 = c_2 \times R_6 / R_7 + 576\mu_1' / (b_7 R_7)$. The 7 equations in 7 unknowns can easily be obtained from the 7 equations for $\bar{M}_1(s)$ and replacing m_1, m_2, m_3, and m_4 therein by zeros in every term. Once $M_2(t)$ is approximated, then $A(t) \approx 1 + M_1(t) - M_2(t)$.

6.6 Approximation Results

The moment based approximation for expected number of cycles $M_1(t)$, number of failures $M_2(t)$ and availability $A(t)$ were obtained for the three parameter Weibull, normal, lognormal, exponential, logistic, loglogistic and gamma distribution as failure and repair distributions. As we discussed in Chapter 5, the exact results of $M_1(t)$, $M_2(t)$ and $A(t)$ when TTF and TTR are exponentially distributed have been known. So, we used those results at $\lambda = 0.001$ and r = 0.05 to compare the approximation method that we have developed. Based on these results relative errors were calculated and concluded that the method produces very accurate results for especially large values of t versus small values of t. The figures and tables below explain the results better.

Table 5 shows the error of approximation relative to the exact value. As it is seen from the table Relative error is almost 95% when time is 20, but as the time increases relative error decreases dramatically. And when time is 5000 and relative error is zero on the six decimals.

Table 5: Moment Based Approximation Results for $M_1(t)$

Time	Exact	Approximated	Rel-Err
20	0.007316319	0.0003845	-94.745069%
50	0.031297225	0.0297962	-4.795931%
100	0.07893304	0.0788158	-0.148480%
250	0.225874719	0.2258747	-0.000026%
500	0.470972703	0.4709727	-0.000001%
1000	0.961168781	0.9611688	0.000000%
2000	1.941560938	1.9415609	-0.000002%
5000	4.882737409	4.8827374	0.000000%

Table 6 shows the percent error of approximation of $M_2(t)$ relative to the exact value. As it is seen from the table relative error is almost 0.698% when time is 20, but as the time increases relative error decreases dramatically. When time is 5000 and relative error is almost zero. Further, relative error is much higher for $M_1(t)$ then $M_2(t)$ for smaller values of t.

Table 6: Moment Based Approximation Results for $M_2(t)$

Time	Exact	Approximated	Rel-Err
20	0.019853674	0.0199923	0.698296%
50	0.049374055	0.0494041	0.060800%
100	0.098421339	0.0984237	0.002381%
250	0.245482506	0.2454825	0.000002%
500	0.490580546	0.4905806	0.000001%
1000	0.980776624	0.9807766	0.000000%
2000	1.961168781	1.9611688	0.000001%
5000	4.902345252	4.9023453	0.000001%

Table 7 shows the percent error of approximation of $A(t)$ relative to the exact $A(t)$. Same conclusion can be made for availability also.

Table 7: Moment Based Approximation Results for $A(t)$

Time	Exact	Approximated	Rel-Err
20	0.987462646	0.9803920	-0.716042%
50	0.98192317	0.9803920	-0.155936%
100	0.980511701	0.9803920	-0.012208%
250	0.980392214	0.9803920	-0.000022%
500	0.980392157	0.9803920	-0.000016%
1000	0.980392157	0.9803920	-0.000016%
2000	0.980392157	0.9803920	-0.000016%
5000	0.980392157	0.9803920	-0.000016%

Figure 8 is just graphical representation of Table 5, 6 and 7. It also shows that the approximation method works well for large values of *t*.

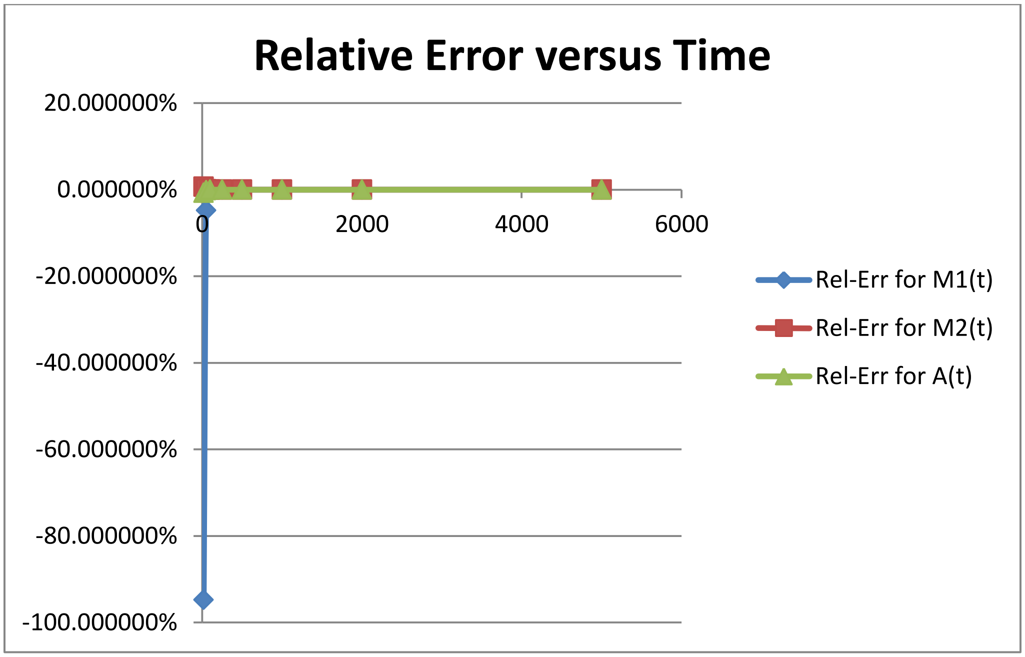

Figure 8: Relative Error versus Time Graph for $M_1(t)$, $M_2(t)$ and $A(t)$

CHAPTER 7

MATLAB Program

This chapter introduces the MATLAB based program, describes the input, processing and outputs of the program for minimal and non-minimal repair that were explained in previous chapters. It also describes how the program was verified.

7.1 Minimal Repair

This section explains the Matlab code for gamma, normal, Weibull and uniform distribution for negligible repair that was covered in Chapter 4. All the code in this section is combined in one Graphical User Interface (GUI). The figure below depicts the MATLAB Minimal Repair GUI.

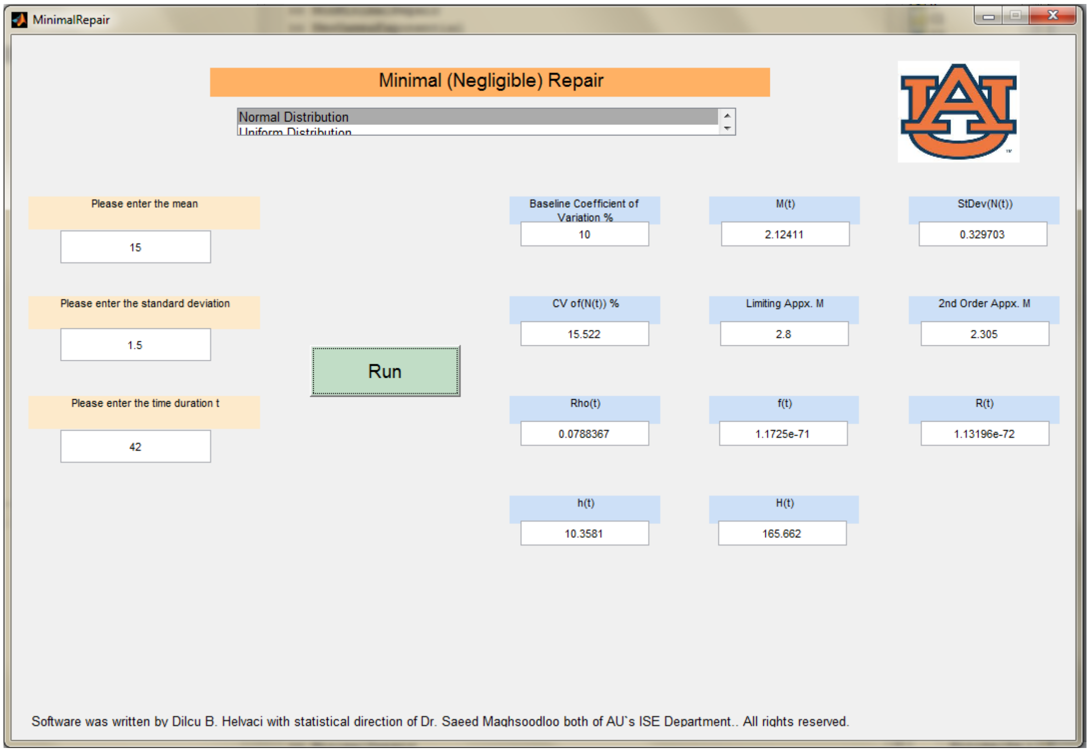

Figure 9: MATLAB Based Minimal Repair GUI

7.1.1 Gamma Distribution

Matlab output for the gamma distribution gives the MTTF, Standard Deviation, Coefficient of Variation, Skewness, Kurtosis, Value of RF at time t, Variance of $N(t)$, First Order Limiting Approximation, Second Order Approximation of $M(t)$, Renewal Intensity Function $\rho(t)$ and reliability measures $f(t)$, $R(t)$, $h(t)$, $H(t)$ based on the shape parameter, scale parameter and time duration that are inputted by a user.

In some cases such as the exponential TTF, it is possible to calculate the exact $M(t)$ and gamma distribution converts to exponential distribution when the shape parameter is one. Therefore in order to verify the code it was run when the shape parameter is one which gave exactly the same RF that is calculated from the exponential RF. Further, in order to check the accuracy of the Matlab program, we ran our program for nearly all *t*-values in Table 5 of [60] at their scale $\lambda = 1$ and their shape parameter $k = 1.5, 3, 5, 7$ and 9. Our gamma RF, $M(t)$, matches their actual RF "$H(t)$" to 3 decimals.

7.1.2 Normal Distribution

The Matlab output for normal distribution gives Coefficient of Variation, Value of RF at time *t*, Variance of $N(t)$, First Order Limiting Approximation, Second Order Approximation of $M(t)$, Renewal Intensity Function $\rho(t)$ and the reliability measures $f(t)$, $R(t)$, $h(t)$, $H(t)$ based on the MTTF, Standard Deviation, MTTR and time duration that are inputted by a user. Unlike gamma distribution it doesn`t calculate skewness and kurtosis because it is well known that skewness and kurtosis for normal distribution is zero.

7.1.3 Weibull Distribution

The Matlab program for Weibull distribution outputs the MTTF, Standard Deviation, Coefficient of Variation, Skewness, Kurtosis, Value of RF at time *t*, First Order Limiting Approximation, Second Order Approximation of $M(t)$, Renewal Intensity Function $\rho(t)$ and reliability measures $f(t)$, $R(t)$, $h(t)$, $H(t)$. Unlike gamma, normal and uniform distributions, future research is required an approximation for the $V[N(t)]$ because we do not know the exact $F_{(n)}(t)$ for the Weibull, except at $\beta = 1$ for which we know the exact variance. Besides possibly approximating the Weibull $F_{(n)}(t)$, one has to simulate $N(t)$ from a Weibull process in order to obtain a rough approximation for the standard deviation of $N(t)$. However, this is not a simple task because the variate $N(t)$ is clearly time dependent, and hence no single approximation is possible for all $0 \le t < \infty$.

The user must specify the parameters of Weibull distribution which are shape parameter (β), scale parameter (θ) and minimum guaranteed life (δ). Next the program asks for time duration t and Δt increment which will specify the number of subintervals. The value of Δt should divide the time duration into equal subintervals by considering the minimum guaranteed life. For example, if δ is zero and time duration is 6000, then Δt can be 300, but if δ is 200 and then Δt cannot be 300 because 6000−200 =5800 and 300 doesn't divide 5800 without any remainder.

It has been verified that at $\beta = 3.439541$, the Weibull mean, median and mode become almost identical at which the Weibull skewness is $\beta_3 = 0.0405259532$ and Weibull kurtosis is $\beta_4 = -0.288751313$, these last 2 shape parameters being very close to those of Laplace-Gaussian of identically zero. The Weibull portion of the program at $\beta = 3.439541$, $\theta = \alpha = 2000$ and $\Delta t = 50$ yielded M(4000) ≅1.753638831, while the corresponding normal program (i.e., MTTF = $\delta + (\theta - \delta) \times \Gamma[(1/\beta) + 1] = 1797.84459964$ and StDev = $\sqrt{(\theta-\delta)^2 \times [\Gamma[(2/\beta)+1] - \Gamma^2[(1/\beta)+1)]} =$ 577.9338342) resulted in M(4000) = 1.774397152.

Moreover, Sheldon M. Ross (1996, pp. 426-427) [68] proves that $t/\mu \le M(t) \le t/\mu + 0.50(CV_T - 1)$ if h(t) is a DFR, and the author further proves when $h(t)$ is a DFR, then $h(t) \le \rho(t)$ for all $t \ge 0$. The program has been checked when $\beta < 1$(for which failure is decreasing) at different cases and observed that $M(t)$ is always smaller than second order approximation and larger than the first order approximation.

7.1.4 Uniform Distribution

The uniform distribution part has been built little differently than the gamma, Weibull and normal portion of the program. In this case, a function to obtain convolutions called uniform convolution as addition to the main uniform code has been generated. It obtains 12 uniform convolutions and for succeeding convolutions it uses the normal approximation. Inputs for the code are the minimum value a, maximum value b, time duration t and Δt increment for $\rho(t)$. As an output it gives the mean, standard deviation, coefficient of variation, kurtosis, $M(t)$, Standard deviation of $N(t)$, first order approximation, second order approximation, $\rho(t)$, $h(t)$ and $R(t)$.

The expected number of renewals for uniform distribution when t is between minimum value a and maximum value b is $M(t) = e^{t/a} - 1 \quad a \leq t \leq b$. As an example, the code was run for a = 0, b =100 and t=90 and $M(t)$= 1.459603158715127 was obtained where the exact value is 1.459603111156950. Therefore the exact same result was obtained through six decimals.

7.2 Non-Minimal Repair

This section explains the non-negligible repair part of the program that was covered in chapters 5 and 6.

7.2.1 The Exact Non-Minimal Repair

This part contains the code for gamma TTF and exponential TTR when the shape parameter α is an integer from 1 to 7. When α is not a positive integer there is no closed-form solution for the Laplace transform of the density. Section 5.4 explains the procedure in detail.

Inputs are shape parameter α, 1/scale parameter, repair rate and time duration t. The outputs are the expected number of cycles $M_1(t)$, expected number of failures $M_2(t)$ and availability $A(t)$.

In order to verify this code we compared the output for the simplest cases α =1 and α = 2 for which the exact values of $M_1(t)$, $M_2(t)$ and $A(t)$ are obtained in Chapter 5.

7.2.2 Approximate Non-Minimal Repair

This section explains the code of approximation methods that were discussed in Chapter 6. The Matlab code for Weibull TTF and uniform TTR explained in section 6.1 asks the user to enter the minimum value a and maximum value b for uniform distribution, shape parameter, characteristic life and minimum life for Weibull distribution, time duration t and Δt . The output gives the approximate expected number of cycles $M_1(t)$, approximate expected number of failures and approximate availability. After the convolution of Weibull TTF and uniform TTR were obtained, the approximation procedure that is explained in section 4.5 was used to approximate the $M_1(t)$ when TTF is Weibull and TTR is uniform. The code for uniform TTF and Weibull TTR and gamma TTF and uniform TTR were generated the same manner.

As it is explained in Section 6.5 the convolutions of $f(t)$ and $r(t)$ for any repair and failure distributions are not always tractable as when TTF is normal and TTR is Weibull. Therefore an approximate method based on raw moments of TTF and TTR was developed. The first four moments of both TTF and TTR distribution were used. Matlab code was generated based on the procedure that is explained in Section 6.5. The user must select the TTR and TTF distribution and inputs the parameters for these distributions. It outputs the $M_1(t)$, $M_2(t)$ and $A(t)$ along with the skewness, kurtosis, MTTF and standard deviation of the baseline failure distribution. The figure below depicts the MATLAB based Non-Minimal Repair GUI.

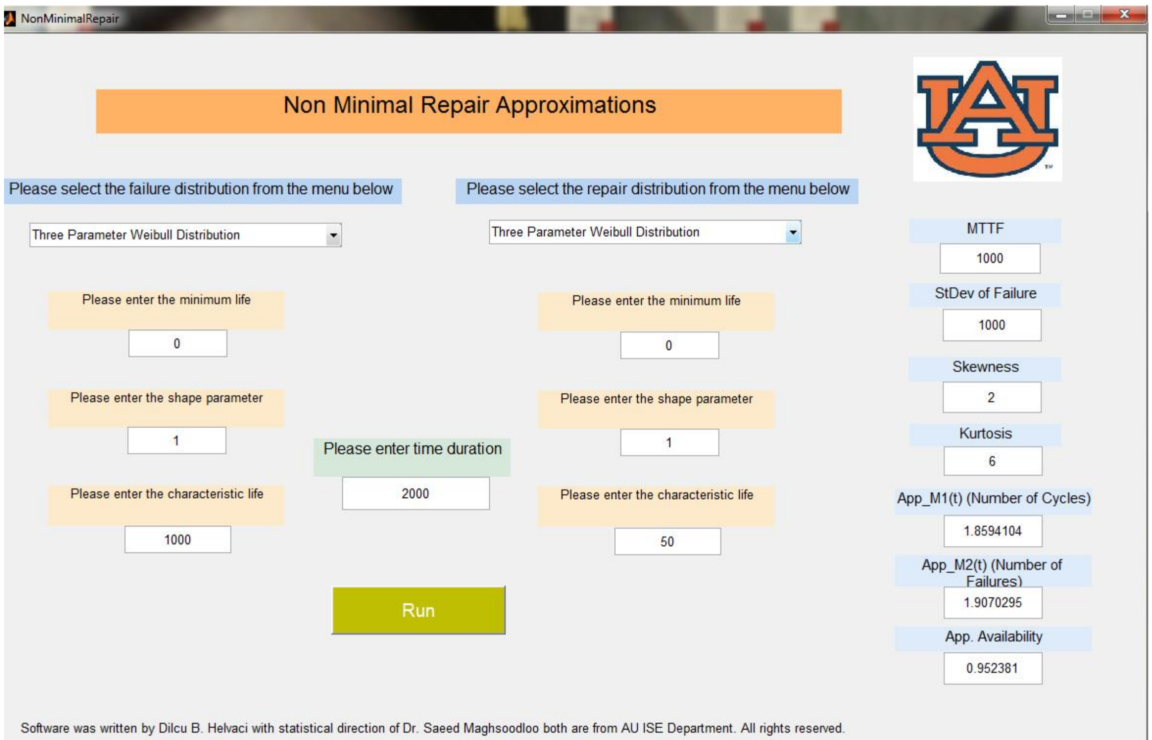

Figure 10: MATLAB Based Non Minimal Repair GUI

Since the closed-form expression of $M_1(t)$, $M_2(t)$ and $A(t)$ are known when both TTF and TTR are exponential, the code was verified by comparing results with exponential-exponential case.

CHAPTER 8

Conclusions and Proposed Future Research

Renewal Theory is an expanding field of research with many different applications. This dissertation investigated the renewal and intensity functions for minimal repair and non-minimal repair for most common distributions. Summarized details of our conclusions are found in the following section.

8.1 Summary and Conclusions

The renewal and intensity functions with minimal repair for the most common lifetime underlying distributions normal, gamma, uniform and Weibull were explored. The exact normal, gamma, and uniform renewal and intensity functions were derived by the convolution method. In the uniform distribution case complexity becomes immense as the number of convolutions increases. Therefore, after obtaining twelve convolutions of the uniform distribution, we applied the normal approximation. Unlike these last three failure distributions, the Weibull distribution, except at shape $\beta = 1$, does not have a closed-form function for the *n*-fold convolution. Since the Weibull is the most important failure distribution in reliability analyses, its approximate renewal and intensity functions were obtained by the time-discretizing method.

The expected number of failures, number of cycles and availability by taking the Laplace transforms of renewal functions were obtained when MTTR is not negligible and that TTR has a pdf denoted as $r(t)$. Finally, the raw moments of failure and repair distributions were used to approximate the expected number of cycles, expected number of failures and the corresponding availability.

8.2 Future Work

The work herein has opened avenues for future research as listed below. We used time discretizing method to approximate the RF for Weibull distribution when repair time was negligible. However, based on the value of Δt the accuracy of the results may change. Smaller Δt yields more accurate result but more processing times. Therefore, a correction factor may be considered to apply that yields smaller CPU time and more accurate result. Also we provided expected value of $N(t)$ which is $M(t)$ for uniform, normal, gamma and Weibull distributions and standard deviation of $N(t)$ for uniform, normal and gamma distribution. However, we didn`t provide skewness and kurtosis for $N(t)$ which are also can be provided as an output. An approximate expression for the third raw moment of $N(t)$ is given by Kambo et al (2012) [92]. Their expression for $E[N(t)^3]$ can be used to approximate the skewness of $N(t)$.

Moreover, for the uniform distribution we obtained the RF from twelve uniform convolutions and after the twelfth convolution we used normal approximation which gave fairly accurate result. On the other hand, as a future work more convolutions for uniform distribution can be obtained.

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Appendix

Appendix A: Uniform Convolution Equations, where $c = b\text{-}a > 0$

$$f_{(2)}(t)=\begin{cases} -(2a-t)/c^2 & 2a\le t\le a+b \\ (2b-t)/c^2 & a+b\le t\le 2b \end{cases}$$

$$f_{(3)}(t)=\begin{cases} (3a-t)^2/2c^3 & 3a\le t\le 2a+b \\ -\left((3a-t)^2+3c(3a-t)+\left(3c^2\right)/2\right)/c^3 & 2a+b\le t\le a+2b \\ (3b-t)^2/(2c^3) & a+2b\le t\le 3b \end{cases}$$

$$f_{(4)}(t)=\begin{cases} -(4a-t)^3/\left(6c^4\right) & 4a\le t\le 3a+b \\ (3(4a-t)^3+12c(4a-t)^2+12c^2(4a-t)+4c^3)/\left(6c^4\right) & 3a+b\le t\le 2a+2b \\ -(3(4a-t)^3+24c(4a-t)^2+60c^2(4a-t)+44c^3)/\left(6c^4\right) & 2a+2b\le t\le a+3b \\ (4b-t)^3/(6c^4) & a+3b\le t\le 4b \end{cases}$$

$$f_{(5)}(t)=\begin{cases} (5a-t)^4/\left(24c^5\right) & 5a\le t\le 4a+b \\ -(4(5a-t)^4+20c(5a-t)^3+30c^2(5a-t)^2+20c^3(5a-t)+5c^4)/\left(24c^5\right) & 4a+b\le t\le 3a+2b \\ (6(5a-t)^4+60c(5a-t)^3+210c^2(5a-t)^2+300c^3(5a-t)+155c^4)/\left(24c^5\right) & 3a+2b\le 2a+3b \\ -(4(5a-t)^4+60c(5a-t)^3+330c^2(5a-t)^2+780c^3(5a-t)+655c^4)/\left(24c^5\right) & 2a+3b\le t\le a+4b \\ (5b-t)^4/(24c^5) & a+4b\le t\le 5b \end{cases}$$

$$
f_{(6)}(t)=\begin{cases}
-(6a-t)^5/\left(120c^6\right) & 6a\le t\le 5a+b\\
[5(6a-t)^5+30c(6a-t)^4+60c^2(6a-t)^3+60c^3(6a-t)^2+30c^4(6a-t)+6c^5]/\left(120c^6\right) & 5a+b\le t\le 4a+2b\\
-[5(6a-t)^5+60c(6a-t)^4+270c^2(6a-t)^3+570c^3(6a-t)^2+585c^4(6a-t)+237c^5]/\left(60c^6\right) & 4a+2b\le t\le 3a+3b\\
[5(6a-t)^5+90c(6a-t)^4+630c^2(6a-t)^3+2130c^3(6a-t)^2+3465c^4(6a-t)+2193c^5]/\left(60c^6\right) & 3a+3b\le t\le 2a+4b\\
-[5(6a-t)^5+120c(6a-t)^4+1140c^2(6a-t)^3+5340c^3(6a-t)^2+12270c^4(6a-t)+ \\ \qquad 10974c^5]/\left(120c^6\right) & 2a+4b\le t\le a+5b\\
(6b-t)^5/(120c^6) & a+5b\le t\le 6b
\end{cases}
$$

$$
f_{(7)}(t)=\begin{cases}
(7a-t)^6/\left(720c^7\right), & 7a\le t\le 6a+b\\
-[6(7a-t)^6+42c(7a-t)^5+105c^2(7a-t)^4+140c^3(7a-t)^3+ \\ \qquad 105c^4(7a-t)^2+42c^5(7a-t)+7c^6]/(720c^7), & 6a+b\le t\le 5a+2b\\
[15(7a-t)^6+210c(7a-t)^5+1155c^2(7a-t)^4+3220c^3(7a-t)^3+ \\ \qquad 4935c^4(7a-t)^2+3990c^5(7a-t)+1337c^6]/(720c^7), & 5a+2b\le t\le 4a+3b\\
-[10(7a-t)^6+210c(7a-t)^5+1785c^2(7a-t)^4+7840c^3(7a-t)^3+ \\ \qquad 18795c^4(7a-t)^2+23520c^5(7a-t)+12089c^6]/(360c^7), & 4a+3b\le t\le 3a+4b\\
[15(7a-t)^6+420c(7a-t)^5+4830c^2(7a-t)^4+29120c^3(7a-t)^3+ \\ \qquad 96810c^4(7a-t)^2+168000c^5(7a-t)+119182c^6]/(720c^7), & 3a+4b\le t\le 2a+5b\\
-[6(7a-t)^6+210c(7a-t)^5+3045c^2(7a-t)^4+23380c^3(7a-t)^3+ \\ \qquad 100065c^4(7a-t)^2+225750c^5(7a-t)+208943c^6]/(720c^7), & 2a+5b\le t\le a+6b\\
(7b-t)^6/(720c^7), & a+6b\le t\le 7b
\end{cases}
$$

$$f_{(8)}(t)=\begin{cases} -(8a-t)^7/(5040c^8) & 8a\le t\le 7a+b \\ [7(8a-t)^7+56c(8a-t)^6+168c^2(8a-t)^5+280c^3(8a-t)^4+ \\ 280c^4(8a-t)^3+168c^5(8a-t)^2+56c^6(8a-t)+8c^7]/(5040c^8) & 7a+b\le t\le 6a+2b \\ -[21(8a-t)^7+336c(8a-t)^6+2184c^2(8a-t)^5+7560c^3(8a-t)^4+ \\ 15400c^4(8a-t)^3+18648c^5(8a-t)^2+12488c^6(8a-t)+3576c^7]/(5040c^8) & 6a+2b\le t\le 5a+3b \\ [(8a-t)^7/144+c(8a-t)^6/6+\left(5c^2(8a-t)^5\right)/3+9c^3(8a-t)^4+ \\ \left(256c^4(8a-t)^3\right)/9+53c^5(8a-t)^2+\left(488c^6(8a-t)\right)/9+\left(2477c^7\right)/105]/\left(c^8\right) & 5a+3b\le t\le 4a+4b \\ -[(8a-t)^7/144+2c(8a-t)^6/9+3c^2(8a-t)^5+\left(199c^3(8a-t)^4\right)/9+ \\ 96c^4(8a-t)^3+\left(737c^5(8a-t)^2\right)/3+344c^6(8a-t)+\left(64249c^7\right)/315]/\left(c^8\right) & 4a+4b\le t\le 3a+5b \\ [(8a-t)^7/240+c(8a-t)^6/6+\left(17c^2(8a-t)^5\right)/6+\left(53c^3(8a-t)^4\right)/2+ \\ \left(2647c^4(8a-t)^3\right)/18+\left(967c^5(8a-t)^2\right)/2+\left(15683c^6(8a-t)\right)/18+\left(139459c^7\right)/210]/\left(c^8\right) & 3a+5b\le t\le 2a+6b \\ -[7(8a-t)^7+336c(8a-t)^6+6888c^2(8a-t)^5+78120c^3(8a-t)^4+ \\ 528920c^4(8a-t)^3+2135448c^5(8a-t)^2+4753336c^6(8a-t)+4491192c^7]/(5040c^8) & 2a+6b\le t\le a+7b \\ (8b-t)^7/(5040c^8) & a+7b\le t\le 8b \end{cases}$$

$$
f_{(9)}(t)=\begin{cases}
(9a-t)^8/\left(40320c^9\right) & 9a\le t\le 8a+b\\
-[\left(2(9a-t)^4+12c(9a-t)^3+18c^2(9a-t)^2+12c^3(9a-t)+3c^4\right)* \\ (4(9a-t)^4+12c(9a-t)^3+18c^2(9a-t)^2+12c^3(9a-t)+3c^4)]/\left(40320c^9\right) & 8a+b\le t\le 7a+2b\\
[28(9a-t)^8+504c(9a-t)^7+780c^2(9a-t)^6+315624c^3(9a-t)^5+ \\ 39690c^4(9a-t)^4+64008c^5(9a-t)^3+64260c^6(9a-t)^2+ \\ 36792c^7(9a-t)+9207c^8]/\left(40320c^9\right) & 7a+2b\le t\le 6a+3b\\
-[56(9a-t)^8+1512c(9a-t)^7+17388c^2(9a-t)^6+ \\ 111384c^3(9a-t)^5+436590c^4(9a-t)^4+1079064c^5(9a-t)^3+ \\ 1650348c^6(9a-t)^2+1432872c^7(9a-t)+541917c^8]/\left(40320c^9\right) & 6a+3b\le t\le 5a+4b\\
[70(9a-t)^8+2520c(9a-t)^7+39060c^2(9a-t)^6+ \\ 340200c^3(9a-t)^5+1821330c^4(9a-t)^4+6146280c^5(9a-t)^3+ \\ 12800340c^6(9a-t)^2+15082200c^7(9a-t)+7715619c^8]/\left(40320c^9\right) & 5a+4b\le t\le 4a+5b\\
-[56(9a-t)^8+2520c(9a-t)^7+49140c^2(9a-t)^6+ \\ 541800c^3(9a-t)^5+3691170c^4(9a-t)^4+15903720c^5(9a-t)^3+ \\ 42324660c^6(9a-t)^2+63667800c^7(9a-t)+41503131c^8]/\left(40320c^9\right) & 4a+5b\le t\le 3a+6b\\
[28(9a-t)^8+1512c(9a-t)^7+35532c^2(9a-t)^6+ \\ 474264c^3(9a-t)^5+3929310c^4(9a-t)^4+20674584c^5(9a-t)^3+ \\ 67410252c^6(9a-t)^2+124449192c^7(9a-t)+99584613c^8]/\left(40320c^9\right) & 3a+6b\le t\le 2a+7b\\
-[\left(2(9a-t)^4+60c(9a-t)^3+666c^2(9a-t)^2+3228c^3(9a-t)+5727c^4\right)* \\ (4(9a-t)^4+132c(9a-t)^3+1638c^2(9a-t)^2 \\ +9060c^3(9a-t)+18849c^4)]/\left(40320c^9\right) & 2a+7b\le t\le a+8b\\
(9b-t)^8/\left(40320c^9\right) & a+8b\le t\le 9b
\end{cases}
$$

$$f_{(10)}(t)=\begin{cases}
-(10a-t)^9/\left(362880c^{10}\right) & 10a\le t\le a+9b \\[2ex]
\begin{aligned}&[9(10a-t)^9+90c(10a-t)^8+360c^2(10a-t)^7+840c^3(10a-t)^6+\\ &1260c^4(10a-t)^5+1260c^5(10a-t)^4+840c^6(10a-t)^3\\ &+360c^7(10a-t)^2+90c^8(10a-t)+10c^9]/\left(362880c^{10}\right)\end{aligned} & a+9b\le t\le 2a+8b \\[2ex]
\begin{aligned}&-[18(10a-t)^9+360c(10a-t)^8+3060c^2(10a-t)^7+14700c^3(10a-t)^6+\\ &44730c^4(10a-t)^5+90090c^5(10a-t)^4+120540c^6(10a-t)^3\\ &+103500c^7(10a-t)^2+51795c^8(10a-t)+11515c^9]/\left(181440c^{10}\right)\end{aligned} & 2a+8b\le t\le 3a+7b \\[2ex]
\begin{aligned}&[42(10a-t)^9+1260c(10a-t)^8+16380c^2(10a-t)^7+121380c^3(10a-t)^6+\\ &567630c^4(10a-t)^5+1746990c^5(10a-t)^4+3553620c^6(10a-t)^3\\ &+4620420c^7(10a-t)^2+3491145c^8(10a-t)+1169465c^9]/\left(181440c^{10}\right)\end{aligned} & 3a+7b\le t\le 4a+6b \\[2ex]
\begin{aligned}&-[63(10a-t)^9+2520c(10a-t)^8+44100c^2(10a-t)^7+443100c^3(10a-t)^6+\\ &2819250c^4(10a-t)^5+11800530c^5(10a-t)^4+32573100c^6(10a-t)^3\\ &+57311100c^7(10a-t)^2+58440375c^8(10a-t)+26355655c^9]/\left(181440c^{10}\right)\end{aligned} & 4a+6b\le t\le 5a+5b \\[2ex]
\begin{aligned}&[63(10a-t)^9+3150c(10a-t)^8+69300c^2(10a-t)^7+879900c^3(10a-t)^6+\\ &7103250c^4(10a-t)^5+37811970c^5(10a-t)^4+132801900c^6(10a-t)^3\\ &+297063900c^7(10a-t)^2+384528375c^8(10a-t)+219738095c^9]/\left(181440c^{10}\right)\end{aligned} & 5a+5b\le t\le 4a+6b \\[2ex]
\begin{aligned}&-[42(10a-t)^9+2520c(10a-t)^8+66780c^2(10a-t)^7+1025220c^3(10a-t)^6+\\ &10042830c^4(10a-t)^5+65064510c^5(10a-t)^4+278704020c^6(10a-t)^3\\ &+761094180c^7(10a-t)^2+1202708745c^8(10a-t)+838419985c^9]/\left(181440c^{10}\right)\end{aligned} & 4a+6b\le t\le 3a+7b \\[2ex]
\begin{aligned}&[18(10a-t)^9+1260c(10a-t)^8+39060c^2(10a-t)^7+703500c^3(10a-t)^6+\\ &8108730c^4(10a-t)^5+61996410c^5(10a-t)^4+314246940c^6(10a-t)^3\\ &+1017758700c^7(10a-t)^2+1910283795c^8(10a-t)+1582796435c^9]/\left(181440c^{10}\right)\end{aligned} & 3a+7b\le t\le 2a+8b \\[2ex]
\begin{aligned}&-[9(10a-t)^9+720c(10a-t)^8+25560c^2(10a-t)^7+528360c^3(10a-t)^6+\\ &7006860c^4(10a-t)^5+61801740c^5(10a-t)^4+362410440c^6(10a-t)^3\\ &+1361868840c^7(10a-t)^2+2974204890c^8(10a-t)+2874204890c^9]/\left(362880c^{10}\right)\end{aligned} & 2a+8b\le t\le a+9b \\[2ex]
(10b-t)^9/\left(362880c^{10}\right) & a+9b\le t\le 10b
\end{cases}$$

$$
f_{(11)}(t)=\begin{cases}
(11a-t)^{10}/\left(3628800c^{11}\right) & 11a\le t\le 10a+b \\[2ex]
\begin{array}{c}
-[110c(11a-t)^{9}+110c^{9}(11a-t)+10(11a-t)^{10}+495c^{2}(11a-t)^{8}+ \\
1320c^{3}(11a-t)^{7}+2310c^{4}(11a-t)^{6}+2772c^{5}(11a-t)^{5}+ \\
2310c^{6}(11a-t)^{4}+1320c^{7}(11a-t)^{3}+495c^{8}(11a-t)^{2}+11c^{10}]/\left(3628800c^{11}\right)
\end{array} & 10a+b\le t\le 9a+2b \\[2ex]
\begin{array}{c}
[45(11a-t)^{10}+990c(11a-t)^{9}+9405c^{2}(11a-t)^{8}+ \\
51480c^{3}(11a-t)^{7}+182490c^{4}(11a-t)^{6}+440748c^{5}(11a-t)^{5}+736890c^{6}(11a-t)^{4}+ \\
843480c^{7}(11a-t)^{3}+633105c^{8}(11a-t)^{2}+281490c^{9}(11a-t)+56309c^{10}]/\left(3628800c^{11}\right)
\end{array} & 9a+2b\le t\le 8a+3b \\[2ex]
\begin{array}{c}
-[30(11a-t)^{10}+990c(11a-t)^{9}+14355c^{2}(11a-t)^{8}+20780c^{3}(11a-t)^{7}+1656040c^{4}(11a-t)^{6} \\
+2415798c^{5}(11a-t)^{5}+6130740c^{6}(11a-t)^{4}+10614780c^{7}(11a-t)^{3}+ \\
12020580c^{8}(11a-t)^{2}+8048865c^{9}(11a-t)+2421694c^{10}]/\left(907200c^{11}\right)
\end{array} & 8a+3b\le t\le 7a+4b \\[2ex]
\begin{array}{c}
[105(11a-t)^{10}+4620c(11a-t)^{9}+90090c^{2}(11a-t)^{8}+ \\
1025640c^{3}(11a-t)^{7}+7558320c^{4}(11a-t)^{6}+37746324c^{5}(11a-t)^{5}+ \\
129664920c^{6}(11a-t)^{4}+303173640c^{7}(11a-t)^{3} \\
+462563640c^{8}(11a-t)^{2}+416439870c^{9}(11a-t)+168171652c^{10}]/\left(1814400c^{11}\right)
\end{array} & 7a+4b\le t\le 6a+5b \\[2ex]
\begin{array}{c}
-[126(11a-t)^{10}+6930c(11a-t)^{9}+169785c^{2}(11a-t)^{8}+ \\
2439360c^{3}(11a-t)^{7}+22760430c^{4}(11a-t)^{6}+144166176c^{5}(11a-t)^{5}+ \\
628303830c^{6}(11a-t)^{4}+1862451360c^{7}(11a-t)^{3}+ \\
3597983235c^{8}(11a-t)^{2}+4095278880c^{9}(11a-t)+2087687723c^{10}]/\left(1814400c^{11}\right)
\end{array} & 6a+5b\le t\le 5a+6b \\[2ex]
\begin{array}{c}
[105(11a-t)^{10}+6930c(11a-t)^{9}+204435c^{2}(11a-t)^{8}+ \\
3548160c^{3}(11a-t)^{7}+40108530c^{4}(11a-t)^{6}+308490336c^{5}(11a-t)^{5}+ \\
1634978730c^{6}(11a-t)^{4}+5897374560c^{7}(11a-t)^{3}+13861625085c^{8}(11a-t)^{2} \\
+19184198880c^{9}(11a-t)+11879998933c^{10}]/\left(1814400c^{11}\right)
\end{array} & 5a+6b\le t\le 4a+7b \\[2ex]
\begin{array}{c}
-[30(11a-t)^{10}+2310c(11a-t)^{9}+79695c^{2}(11a-t)^{8}+ \\
1621620c^{3}(11a-t)^{7}+21543060c^{4}(11a-t)^{6}+195172362c^{5}(11a-t)^{5}+ \\
1220779560c^{6}(11a-t)^{4}+5204388420c^{7}(11a-t)^{3}+14471011170c^{8}(11a-t)^{2} \\
+17364208576c^{10}]/\left(907200c^{11}\right)
\end{array} & 4a+7b\le t\le 3a+8b \\[2ex]
\begin{array}{c}
[45(11a-t)^{10}+3960c(11a-t)^{9}+156420c^{2}(11a-t)^{8}+ \\
3651120c^{3}(11a-t)^{7}+55754160c^{4}(11a-t)^{6}+581803992c^{5}(11a-t)^{5}+ \\
4200171360c^{6}(11a-t)^{4}+20706055920c^{7}(11a-t)^{3}+66686784120c^{8}(11a-t)^{2} \\
+126660745860c^{9}(11a-t)+107710566656c^{10}]/\left(3628800c^{11}\right)
\end{array} & 3a+8b\le t\le 2a+9b \\[2ex]
\begin{array}{c}
-[10(11a-t)^{10}+990c(11a-t)^{9}+44055c^{2}(11a-t)^{8}+ \\
1160280c^{3}(11a-t)^{7}+20025390c^{4}(11a-t)^{6}+236615148c^{5}(11a-t)^{5}+ \\
1937972190c^{6}(11a-t)^{4}+10861539480c^{7}(11a-t)^{3}+39853850355c^{8}(11a-t)^{2} \\
+86420523090c^{9}(11a-t)+84062575399c^{10}]/\left(3628800c^{11}\right)
\end{array} & 2a+9b\le t\le a+10b \\[2ex]
(11a+11c-t)^{10}/\left(362880c^{11}\right) & a+10b\le t\le 11b
\end{cases}
$$

$$
f_{(12)}(t)=\begin{cases}
-(12a-t)^{11}/\left(39916800c^{12}\right) & 12a\le t\le 11a+b \\[2ex]
\begin{aligned}&[11(12a-t)^{11}+132c(12a-t)^{10}+660c^{2}(12a-t)^{9}+1980c^{3}(12a-t)^{8}\\
&+3960c^{4}(12a-t)^{7}+5544c^{5}(12a-t)^{6}+5544c^{6}(12a-t)^{5}+3960c^{7}(12a-t)^{4}\\
&+1980c^{8}(12a-t)^{3}+660c^{9}(12a-t)^{2}+132c^{10}(12a-t)+12c^{11}]/\left(39916800c^{12}\right)\end{aligned} & 11a+b\le t\le 10a+2b \\[2ex]
\begin{aligned}&-[55(12a-t)^{11}+1320c(12a-t)^{10}+13860c^{2}(12a-t)^{9}+\\
&85140c^{3}(12a-t)^{8}+344520c^{4}(12a-t)^{7}+970200c^{5}(12a-t)^{6}+\\
&1945944c^{6}(12a-t)^{5}+2783880c^{7}(12a-t)^{4}+2785860c^{8}(12a-t)^{3}\\
&+1857900c^{9}(12a-t)^{2}+743292c^{10}(12a-t)+135156c^{11}]/\left(39916800c^{12}\right)\end{aligned} & 10a+2b\le t\le 9a+3b \\[2ex]
\begin{aligned}&[55(12a-t)^{11}+1980c(12a-t)^{10}+31680c^{2}(12a-t)^{9}+\\
&298320c^{3}(12a-t)^{8}+1845360c^{4}(12a-t)^{7}+7909440c^{5}(12a-t)^{6}+\\
&24049872c^{6}(12a-t)^{5}+51997440c^{7}(12a-t)^{4}+78459480c^{8}(12a-t)^{3}\\
&+78768800c^{9}(12a-t)^{2}+47385096c^{10}(12a-t)+12945728c^{11}]/\left(13305600c^{12}\right)\end{aligned} & 9a+3b\le t\le 8a+4b \\[2ex]
\begin{aligned}&-[55(12a-t)^{11}+2640c(12a-t)^{10}+56760c^{2}(12a-t)^{9}+\\
&722040c^{3}(12a-t)^{8}+6046920c^{4}(12a-t)^{7}+35075040c^{5}(12a-t)^{6}+\\
&144094104c^{6}(12a-t)^{5}+420055680c^{7}(12a-t)^{4}+852879060c^{8}(12a-t)^{3}\\
&+1150094000c^{9}(12a-t)^{2}+927890172c^{10}(12a-t)+339557216c^{11}]/\left(6652800c^{12}\right)\end{aligned} & 8a+4b\le t\le 7a+5b \\[2ex]
\begin{aligned}&[77(12a-t)^{11}+4620c(12a-t)^{10}+124740c^{2}(12a-t)^{9}+\\
&2000460c^{3}(12a-t)^{8}+21178080c^{4}(12a-t)^{7}+155499960c^{5}(12a-t)^{6}+\\
&808780896c^{6}(12a-t)^{5}+2983069320c^{7}(12a-t)^{4}+7654933440c^{8}(12a-t)^{3}\\
&+13029593500c^{9}(12a-t)^{2}+13251797328c^{10}(12a-t)+6105755284c^{11}]/\left(6652800c^{12}\right)\end{aligned} & 7a+5b\le t\le 6a+6b
\end{cases}
$$

$$
f_{(12)}(t)=\begin{cases}
\begin{array}{l}
-[77(12a-t)^{11}+5544c(12a-t)^{10}+180180c^{2}(12a-t)^{9}+\\
3488100c^{3}(12a-t)^{8}+44684640c^{4}(12a-t)^{7}+397746888c^{5}(12a-t)^{6}+\\
2510700192c^{6}(12a-t)^{5}+11243278200c^{7}(12a-t)^{4}+35024109120c^{8}(12a-t)^{3}\\
+72328491620c^{9}(12a-t)^{2}+89177904816c^{10}(12a-t)+49764991340c^{11}]/\left(6652800c^{12}\right)
\end{array} & 6a+6b\le t\le 5a+7b\\[2ex]
\begin{array}{l}
[55(12a-t)^{11}+4620c(12a-t)^{10}+175560c^{2}(12a-t)^{9}+3982440c^{3}(12a-t)^{8}\\
+59902920c^{4}(12a-t)^{7}+627211200c^{5}(12a-t)^{6}+4664006424c^{6}(12a-t)^{5}\\
+24630254880c^{7}(12a-t)^{4}+90533256660c^{8}(12a-t)^{3}+220638695200c^{9}(12a-t)^{2}\\
+320976156732c^{10}(12a-t)+211242138736c^{11}]/\left(6652800c^{12}\right)
\end{array} & 5a+7b\le t\le 4a+8b\\[2ex]
\begin{array}{l}
-[55(12a-t)^{11}+5280c(12a-t)^{10}+229680c^{2}(12a-t)^{9}\\
5974320c^{3}(12a-t)^{8}+103221360c^{4}(12a-t)^{7}+1243482240c^{5}(12a-t)^{6}\\
+10655224272c^{6}(12a-t)^{5}+64929416640c^{7}(12a-t)^{4}\\
+275693192280c^{8}(12a-t)^{3}+7767484491200c^{9}(12a-t)^{2}\\
1306889097096c^{10}(12a-t)+994854930208c^{11}]/\left(13305600c^{12}\right)
\end{array} & 4a+8b\le t\le 3a+9b\\[2ex]
\begin{array}{l}
[55(12a-t)^{11}+5940c(12a-t)^{10}+291060c^{2}(12a-t)^{9}\\
8539740c^{3}(12a-t)^{8}+166664520c^{4}(12a-t)^{7}+2271293640c^{5}(12a-t)^{6}+\\
22049990424c^{6}(12a-t)^{5}+152455299480c^{7}(12a-t)^{4}\\
+735516395460c^{8}(12a-t)^{3}+2357542443300c^{9}(12a-t)^{2}\\
4517350959132c^{10}(12a-t)+3919268323356c^{11}]/\left(39916800c^{12}\right)
\end{array} & 3a+8b\le t\le 2a+10b\\
\begin{array}{l}
-[11(12a-t)^{11}+1320c(12a-t)^{10}+71940c^{2}(12a-t)^{9}+2350260c^{3}(12a-t)^{8}\\
+51135480c^{4}(12a-t)^{7}+777906360c^{5}(12a-t)^{6}+\\
8442009576c^{6}(12a-t)^{5}+65344700520c^{7}(12a-t)^{4}\\
+353483604540c^{8}(12a-t)^{3}+1272457556700c^{9}(12a-t)^{2}\\
2742649040868c^{10}(12a-t)+2680731676644c^{11}]/\left(39916800c^{12}\right)
\end{array} & 2a+10b\le t\le a+11b\\[2ex]
(12a+12c-t)^{11}/\left(39916800c^{12}\right) & a+11b\le t\le 12b
\end{cases}
$$

Appendix B: 3rd and 4th Moment of Sums of iid rvs

The Skewness of the Sum of n independent and Identically Distributed (iid) Variates

Suppose $X_1, X_2, \ldots, X_n$ are independently and identically distributed random variables each with means μ and variances $\mu_2 = V(X_i) = \sigma^2 = \sigma_X^2$, where each X_i is identically distributed like X. Let the n^{th} partial sum $S_n = \sum_{i=1}^{n} X_i$; then, clearly $E(S_n) = n\mu$ and $V(S_n) = \sigma^2_{S_n} = n\sigma^2$. It is well known that the $a_3(S_n) = \mu_3(X_i)/(\sigma^3 x\sqrt{n}) = a_3(X)/\sqrt{n}$, where $a_3(X) = \mu_3/\sigma^3$, and $\mu_3 = E[(X-\mu)^3]$, the 3^{th} central moment of X. The proof follows.

$$\mu_3(S_n) = E[(S_n - n\mu)^3] = E[(\sum_{i=1}^{n} X_i - n\mu)^3] = E[(\sum_{i=1}^{n} X_i - \mu)^3]$$

$$= E\left\{\sum_{i=1}^{n}(X_i - \mu_i)^3 + 6\sum_{i=1}^{n}\sum_{j>1}^{n}(X_i - \mu)^2(X_j - \mu)\right\}$$

$$= E[\sum_{i=1}^{n}(X_i - \mu_i)^3] = \sum_{i=1}^{n} E[(X_i - \mu_i)^3] = nE[(X_i - \mu_i)^3 = n\mu_3(X_i)$$

The corresponding skewness of S_n is given by;

$$\alpha_3(S_n) = n\mu_3(X_i)\big/(\mu_2(S_n))^{3/2} = n\mu_3(X_i)\big/(n\sigma^2)^{3/2} = \mu_3(X)\big/(\sigma^3\sqrt{n}) = \alpha_3(X)\big/\sqrt{n}$$

The Kurtosis of the Sum of n iid Variants

Suppose $X_1, X_2, \ldots, X_n$ are independently and identically distributed random variables with means μ and variances $\mu_2 = V(X_i) = \sigma^2 = \sigma_X^2$, where each X_i is identically distributed like X. It is well known that the kurtosis of each x_i is equal to $\beta_4 = \alpha_4(X) - 3$, where $\alpha_4(X) = \mu_4/\sigma^4$, and $\mu_4 = E[(X-\mu)^4]$, the 4^{th} central moment of X.

Now consider the partial sum $S_n = \sum_{i=1}^{n} X_i$; our objective is to compute the 4th central moment of S_n from the known central moment of each identical X_i, i =1, 2,…, n. Clearly, the mean of S_n is given by $E(S_n) = n\mu$, the variance is given by $V(S_n) = nV(X_i) = n\sigma^2$, and

thus $\mu_4(S_n) = E\left[\sum_{i=1}^{n} X_i - (n\mu)\right]^4 = E\left\{\left[\sum_{i=1}^{n}(X_i - \mu)\right]^4\right\}$

$$= E\left[\sum_{i=1}^{n}(X_i-\mu)^4 + {}_4C_2 \times \sum_{i=1}^{n-1}\sum_{j>1}^{n}(X_i-\mu)^2(X_j-\mu)^2\right]$$

$$= n\mu_4(X_i) + 6 \times {}_nC_2 V(X_i) \times V(X_j)$$

Note that in the binomial expansion of $\left[\sum_{i=1}^{n}(X_i - \mu)\right]^4$, the expectation of odd products such as $E[(X_1-\mu)(X_2-\mu)^3]$ vanish due to mutual independence of X_i and X_j for all $i \neq j$.

Hence,

$$\mu_4(S_n) = n\mu_4(X) + 3n(n-1)\sigma^4$$

Thus, $\alpha_4(S_n) = \frac{\mu_4(S_n)}{V(S_n)} = \frac{n\mu_4(X) + 3n(n-1)\sigma^4}{(n\sigma^2)^2} = \alpha_4(X)/n + 3(n-1)/n$

The corresponding kurtosis of S_n is given by,

$\beta_4(S_n) = \alpha_4(X)/n + 3(n-1)/n - 3 \rightarrow \beta_4(S_n) = [\alpha_4(X) - 3]/n = \beta_4(X)/n.$

Appendix C: Moments of the Most Common Base-line Distributions in Reliability

Table 8: Parameters and Density Functions of Most Common Baseline Distributions in Reliability

Lifetime Distribution	Failure Density *f*(*t*)	Threshold Or: Minimum-life	Location Or: Shape	Scale
Exponential	$\lambda e^{-\lambda(t-\delta)}$	δ		1/λ
Three Parameter Weibull	$f(t)=\frac{\beta}{\theta-\delta}(\frac{t-\delta}{\theta-\delta})^{\beta-1}e^{-(\frac{t-\delta}{\theta-\delta})^{\beta}}$	δ	β	θ–δ
Gamma	$f(t)=\frac{\lambda}{\Gamma(\alpha)}[\lambda(t-\delta)]^{\alpha-1}e^{-\lambda(t-\delta)}$	δ	α	β =1/λ
Lognormal	$f(t:\delta,\theta,\sigma)=\frac{1}{\sigma(t-\delta)\sqrt{2\pi}}e^{-[\frac{\ln(t-\delta)-\theta}{\sigma}]^2}$	δ	θ	σ
Normal	$f(t)=\frac{1}{\sigma\sqrt{2\pi}}e^{-(t-\mu)^2/(2\sigma^2)}$		μ	σ
Logistic	$g(t;\mu,\beta)=\frac{1}{\beta}\times\frac{e^{-(t-\mu)/\beta}}{[1+e^{-(t-\mu)/\beta}]^2}$		μ	β
Loglogistic	$f(t)=\frac{1}{\beta(t-\delta)}\times\frac{e^{-[\ln(t-\delta)-\mu]/\beta}}{[1+e^{-[\ln(t-\delta)-\mu)]/\beta}]^2}$	δ	μ	β

The Normal N(μ, σ^2); $\mu'_1 = \mu$; $\mu'_2 = \sigma^2 + \mu^2$; $\mu'_3 = 3\sigma^2\mu + \mu^3$; $\mu'_4 = 3\sigma^4 + 6\sigma^2\mu^2 + \mu^4$; $\mu'_5 = 15\sigma^4\mu + 5\sigma^2\mu^3 + \mu^5$.

Two-Parameter Exponential pdf: μ'_1 = MTTF = $\delta + 1/\lambda$; $\mu'_2 = 2/\lambda^2 + 2\delta/\lambda + \delta^2$

The skewness $\alpha_3 = 2$, while the kurtosis $\beta_4 = 6$. $\mu'_3 = 6/\lambda^3 + 6\delta/\lambda^2 + 3\delta^2/\lambda + \delta^3$

$\mu'_4 = 9/\lambda^4 + 4\mu'_3\mu'_1 - 6\mu'_2\mu'^2_1 + 3\mu'^4_1$.

Three-Parameter Weibull pdf: Shape =β; μ'_1 = MTTF = $\delta + (\theta - \delta)\times\Gamma[(1/\beta) + 1]$;

Note that when $\beta = 1$, then $(\theta - \delta) = 1/\lambda$; the 2nd raw moment is given by

$\mu'_2 = (\theta - \delta)^2\times\Gamma[(2/\beta) + 1] + 2\delta(\theta - \delta)\times\Gamma[(1/\beta) + 1] + \delta^2$; $\mu_2 = V = (\theta - \delta)^2\times[\Gamma[(2/\beta) + 1] - \Gamma^2[(1/\beta) + 1)] = \sigma^2$

$$\alpha_3 = \frac{\Gamma(1+\frac{3}{\beta}) - 3\,\Gamma(1+\frac{2}{\beta})\,\Gamma(1+\frac{1}{\beta}) + 2\,\Gamma^3(1+\frac{1}{\beta})}{[\Gamma(1+\frac{2}{\beta}) - \Gamma^2(1+\frac{1}{\beta})]^{3/2}}$$

, and the kurtosis is

$$\beta_4 = \frac{\Gamma(1+\frac{4}{\beta}) - 4\,\Gamma(1+\frac{3}{\beta})\,\Gamma(1+\frac{1}{\beta}) + 6\,\Gamma(1+\frac{2}{\beta})\Gamma^2(1+\frac{1}{\beta}) - 3\,\Gamma^4(1+\frac{1}{\beta})}{[\Gamma(1+\frac{2}{\beta}) - \Gamma^2(1+\frac{1}{\beta})]^2} - 3 = \alpha_4 - 3$$

The 3rd raw moment is

$\mu'_3 = (\theta - \delta)^3\times\Gamma[(3/\beta) + 1] + 3\delta(\theta - \delta)^2\times\Gamma[(2/\beta) + 1] + 3\delta^2(\theta - \delta)\times\Gamma[(1/\beta) + 1] + \delta^3$

$\mu'_4 = \sigma^4\alpha_4 + 4\mu'_3\mu'_1 - 6\mu'_2\mu'^2_1 + 3\mu'^4_1$

Lognormal pdf: $f(t:\delta,\theta,\sigma) = \dfrac{1}{\sigma(t-\delta)\sqrt{2\pi}} e^{-[\frac{\ln(t-\delta)-\theta}{\sigma}]^2}$, where δ =Threshold, or Min-Life,

θ is a location parameter, and σ is scale. The MTTF = $\mu_1' = \delta+\exp(\theta+\sigma^2/2)$ and

$V(T) = e^{2\theta+2\sigma^2} - e^{2\theta+\sigma^2} = \exp(2\theta+2\sigma^2) - \exp(2\theta+\sigma^2) = e^{2\theta+\sigma^2}(e^{\sigma^2}-1)$, and

$\mu_2' = e^{2\theta+2\sigma^2} + 2\delta e^{\theta+\sigma^2/2} + \delta^2$, the skewness is $\alpha_3 = \mu_3/\sigma^3 = \dfrac{e^{3\sigma^2} - 3e^{\sigma^2} + 2}{(e^{\sigma^2}-1)^{1.5}} > 0.$

$\alpha_4 = \mu_4/\sigma^4 = \dfrac{e^{6\sigma^2} - 4e^{3\sigma^2} + 6e^{\sigma^2} - 3}{(e^{\sigma^2}-1)^2}$, and the kurtosis is

$\beta_4 = \mu_4/\sigma^4 - 3 = \dfrac{e^{6\sigma^2} - 4e^{3\sigma^2} - 3e^{2\sigma^2} + 12e^{\sigma^2} - 6}{(e^{\sigma^2}-1)^2}$, and the 3rd origin-moment is given by

$$\mu_3' = e^{3\theta+4.5\sigma^2} + 3\delta e^{2\theta+2\sigma^2} + 3\delta^2 e^{\theta+\sigma^2/2} + \delta^3$$

$$\mu_4' = \sigma^4 a_4 + 4\mu_3'\mu_1' - 6\mu_2'\mu_1'^2 + 3\mu_1'^4$$

Gamma pdf: $f(t) = \dfrac{\lambda}{\Gamma(\alpha)}[\lambda(t-\delta)]^{\alpha-1} e^{-\lambda(t-\delta)}$, δ = Threshold, α = Shape, and $\beta = 1/\lambda$ = scale.

$\mu_1' = \delta+\alpha/\lambda$; $V(T) = \alpha/\lambda^2 = \sigma^2$ $\longrightarrow$ $\mu_2' = \alpha/\lambda^2 + (\delta+\alpha/\lambda)^2$; $\alpha_3 = 2/\sqrt{\alpha}$, $\alpha_4 = 3+6/\alpha$

and the kurtosis is $\beta_4 = 6/\alpha$,

$\mu_3' = 2\alpha/\lambda^3 + \delta^3 + 3\delta\alpha^2/\lambda^2 + \alpha^3/\lambda^3 + 3\alpha\delta/\lambda^2 + 3\alpha^2/\lambda^3 + 3\delta^2\alpha/\lambda$, and μ_4' =

$\sigma^4 a_4 + 4\mu_3'\mu_1' - 6\mu_2'\mu_1'^2 + 3\mu_1'^4$.

Logistic pdf: $g(t;\mu,\beta) = \frac{1}{\beta} \times \frac{e^{-(t-\mu)/\beta}}{[1+e^{-(t-\mu)/\beta}]^2}$; $E(T) = \mu_1' = \mu$ and $V(T) = \sigma^2 = (\pi\beta)^2/3$

$E(T) = \mu_1'$ = location = μ, β = Scale; $\mu_2' = (\pi\beta)^2/3 + (\mu_1')^2$, $-\infty < t < \infty$; therefore, for RE-analyses μ must exceed $10(\pi\beta/\sqrt{3})$; the cdf is

$$G(t;\ \mu,\ \beta) = \frac{1}{[1+e^{-(t-\mu)/\beta}]} = [1+e^{-(t-\mu)/\beta}]^{-1} \qquad -\infty < t < \infty.$$

The skewness is zero due to symmetry about μ and the kurtosis is $\beta_4 = \frac{7\pi^4/15}{(\pi^2/3)^2} - 3 = 1.20000$, i.e., the Logistic distribution has thicker tails than the corresponding normal with mean μ and variance $\sigma^2 = \pi^2\beta^2/3$, i.e., the N(μ, $\sigma^2 = \pi^2\beta^2/3$). For example, if the location of the mean is μ = 200 hours and scale β = 4.5 hours, then the cdf G(t = 185; 200, 4.5) = 0.034445195666211, while Φ[(185-200)/ 8.162097139054] = Pr(Z≤ −1.837762984739307) = 0.033048669103432; so, it seems the left-tail of the Logistic is a bit heavier than that of the corresponding normal. At t = 180, the respective cdfs are 0.011607316445305 (Logistic), and 0.007135857827108 for the corresponding normal. So, as we move further on the tail, the Logistic seems to become even heavier than the normal. The 3^rd^ raw moment is given by:

$\mu_3' = \mu(\pi\beta)^2 + \mu^3$, and $\mu_4' = 7(\pi\beta)^4/15 + 2(\pi\beta)^2\mu^2 + \mu^4$.

Loglogistic pdf: A rv T has a Loglogistic pdf with threshold δ iff X = ln(T−δ) has logistic pdf .

$T = e^X + \delta$, where X is logistic. The cdf of Loglogistic is obtained as follows:

$F_T(t) = \Pr(T \le t) = \Pr(e^X + \delta \le t) = \Pr(e^X \le t-\delta) = \Pr[X \le \ln(t-\delta)] = F_X[\ln(t-\delta)]$; thus,

$f(t) = dF_T(t)/dt = dF_X[\ln(t-\delta)]/dt = dF_X[\ln(t-\delta)]/dx] \times (dx/dt) = f_X[\text{at } \ln(t-\delta)] \times (t-\delta)^{-1} =$

$$= \frac{1}{\beta} \times \frac{e^{-[\ln(t-\delta)-\mu]/\beta}}{[1+e^{-[\ln(t-\delta)-\mu)]/\beta}]^2} \times (t-\delta)^{-1} = \frac{1}{\beta(t-\delta)} \times \frac{e^{-[\ln(t-\delta)-\mu]/\beta}}{[1+e^{-[\ln(t-\delta)-\mu)]/\beta}]^2}, \ t > \delta, \ \beta > 0.$$

Thus, cdf is $F(t) = \dfrac{1}{[1+e^{-[\ln(t-\delta)-\mu)]/\beta}]} = [1+e^{-[\ln(t-\delta)-\mu)]/\beta}]^{-1}$, $\delta \le t < \infty$, where μ is the Location-parameter and β is the scale of the Loglogistic pdf. It seems that the 1st two moments do not generally exist from Johnson & et al. on p. 152, where the rth raw moment is given by $E(T^r) = e^{-r\gamma/\delta}(r\pi/\delta)\operatorname{cosec}(r\pi/\delta)$, where on a comparison of their Eq. (23. 89) atop their p. 152 with the above cdf of Loglogistic we must have $1/\delta = \beta$, and $-\gamma = \mu/\beta$; then $\csc(r\pi/\delta)$ = $1/\sin(r\pi/\delta) = 1/\sin(r\pi)$; for the 1st raw moment r = 1, and $1/\sin(\pi) = 1/0$, which does not exist. Similarly none of the raw moments for r = 2, 3, 4, 5, …, n exist.

Note that Johnson, Kotz & Balakrishnan define on their p. 151 the log-logistic variate, X, as $Y = \gamma + \delta\ln(X)$, where Y is the standard logistic, and X is Log-logistic according to their definition. The pdf of their standard logistic Y is given by: $f_Y(y) = \dfrac{e^{-y}}{(1+e^{-y})^2}$,

$$f_Y(y) = \frac{e^{y}}{(1+e^{y})^2} \quad -\infty < y < \infty \qquad \text{(Eq. 23.8 J. K.,\& B [110])}$$

In our above notation, we have $X = \gamma + \delta\ln(T)$, where now X is logistic and T is Loglogistic. Then, $T = e^{(X-\gamma)/\delta}$; this last expression clearly shows that the 3 authors (J. K. and B) [110] are using γ as location and δ as scale, and no minimum-life. Proceeding as before, we have:

$F_T(t)$ = Pr(T ≤ t) = Pr[$e^{(X-\gamma)/\delta} \le t$] = Pr[(X−γ)/δ ≤ ln(t) = Pr[X ≤ γ+ δln(t)] = $F_X[\delta \ln(t) + \gamma]$; thus,

f(t) = $dF_T(t)/dt$ = $dF_X[\delta \ln(t) + \gamma]/dt$ = $dF_X[\delta \ln(t) + \gamma]/dx] \times (dx/dt)$ = f_X[at δln(t) + γ]×(δ/ T) =

$\frac{\delta e^{-[\delta \ln(t)+\gamma]}}{t\{1+e^{-[\delta \ln(t)+\gamma]}\}^2}$; However, $e^{-[\delta \ln(t)+\gamma]} = e^{-\delta \ln(t)} \times e^{-\gamma} = e^{\ln(t^{-\delta})} \times e^{-\gamma} = t^{-\delta} \times e^{-\gamma}$;

Thus, $f(t) = \frac{\delta t^{-\delta} e^{-\gamma}}{t[1+t^{-\delta} e^{-\gamma}]^2} = \frac{(\delta t^{-\delta} e^{-\gamma}) t^{2\delta}}{t \times t^{2\delta}[1+t^{-\delta} e^{-\gamma}]^2} = \frac{\delta t^{\delta} e^{-\gamma}}{t \times [t^{\delta} + e^{-\gamma}]^2} = \frac{\delta t^{\delta} e^{\gamma}}{t \times [e^{\gamma} t^{\delta} + 1]^2}$

$$f(t) = \frac{\delta t^{\delta-1} e^{\gamma}}{(e^{\gamma} t^{\delta} + 1)^2} \quad t \ge 0 \text{ and scale } \delta > 0.$$ (Eq. 23.88, p.151 of J. K. &B) [110]

The cdf of T is given by F(t) = $\frac{1}{1+t^{-\delta} e^{-\gamma}} = (1+t^{-\delta} e^{-\gamma})^{-1}$.

The raw moments of T are given atop p. 152 of [110] as follows:

$E(T^r) = e^{-r\gamma/\delta} \times \frac{r\pi}{\delta} \times \csc(\frac{r\pi}{\delta})$, where $\csc(\frac{r\pi}{\delta}) = \text{cosec}(\frac{r\pi}{\delta}) = 1/\sin(\frac{r\pi}{\delta})$. (Eq. 23.90 p.152 of [110])

From the above Eq. of [110] we obtain $E(T) = e^{-\gamma/\delta} \times \frac{\pi}{\delta} / \sin(\frac{\pi}{\delta})$; this last clearly shows that the mean exists iff $\sin(\frac{\pi}{\delta}) \neq 0$. Further, as $\delta \to \infty$, $\sin(\frac{\pi}{\delta}) \to 0$ and then E(T) does not exist.

Further, for some 0 < δ < 1, $\sin(\frac{\pi}{\delta})$ < 0 leading to a negative MTTF, which is not admissible.

We now compare the cdf of Loglogistic from Minitab, F(t) = $\dfrac{1}{[1+e^{-[\ln(t-\delta)-\mu)]/\beta}]}$

$[1+e^{-[\ln(t-\delta)-\mu)]/\beta}]^{-1}$, $\delta \le t < \infty$, with the form provided by [110] as F(t) = $\dfrac{1}{1+t^{-\delta}e^{-\gamma}}$ =

$\dfrac{1}{1+e^{\ln(t^{-\delta})}e^{-\gamma}} = \dfrac{1}{1+e^{\ln(t^{-\delta})\ -\gamma}} = \dfrac{1}{1+e^{-\delta\times\ln(t)-\gamma}} = \dfrac{1}{1+e^{-[\delta\times\ln(t)+\gamma]}}$. Because, [110] do not have

minimum life, then comparing $\delta \times \ln(t) + \gamma$ against $[\ln(t\text{–}0) - \mu]/\beta$ shows that $\beta = 1/\delta$, and $\mu/\beta =$

$-\gamma$. Hence, the raw moments of Minitab's Loglogistic at zero-min-life are E(T^r) =

$e^{-r\gamma/\delta}(r\pi/\delta)\csc(r\pi/\delta) = e^{r\mu}(r\pi\beta)\operatorname{cosec}(r\pi\beta) = e^{r\mu}(r\pi\beta)/\sin(r\pi\beta)$; this again shows that when

scale β =1, 2, 3, 4, 5, …. E(T) and E(T^2) do not exist. When β = 0.50, E(T) exists but E(T^2) does

not.

We now derive the first four moments of Loglogistic as follows.

E(T) = $\displaystyle\int_{\delta}^{\infty} \frac{t}{\beta(t-\delta)} \times \frac{e^{-[\ln(t-\delta)-\mu]/\beta}\, dt}{[1+e^{-[\ln(t-\delta)-\mu)]/\beta}]^2}$; letting z= $[\ln(t-\delta)-\mu)]/\beta$ results in

dz = $[(t-\delta)^{-1}/\beta]dt$ $\longrightarrow$ E(T) = $\displaystyle\int_{-\infty}^{\infty} \frac{t\,e^{-z}\,dz}{[1+e^{-z}]^2}$; however, $\beta z + \mu = \ln(t-\delta)$

$t = \delta + e^{\beta z+\mu}$; thus E(T) = $\displaystyle\int_{-\infty}^{\infty} \frac{(\delta + e^{\beta z+\mu})e^{-z}\,dz}{[1+e^{-z}]^2} = \delta + e^{\mu}\int_{-\infty}^{\infty} \frac{e^{-z(1-\beta)}\,dz}{[1+e^{-z}]^2}$.

We now let u = e^{-z} in the above last integral; then, du = $-e^{-z}dz$, and du $(-e^{z}) = dz$

$\longrightarrow$ $u^{-1} = e^{z}$ and $dz = -u^{-1}du$ $\longrightarrow$

$$E(T) = \delta + e^{\mu} \int_{+\infty}^{0} \frac{u^{(1-\beta)}\ du(-e^{z})}{[1+u]^2} = \delta + e^{\mu} \int_{0}^{+\infty} \frac{u^{(1-\beta)}\ du(1/u)}{[1+u]^2} = \delta + e^{\mu} \int_{0}^{+\infty} \frac{u^{-\beta}\ du}{[1+u]^2} .$$

Hildebrand (1962, p. 91) [98] proves that $\int_{0}^{+\infty} \frac{x^c\ dx}{(1+x)^2} = \Gamma(1+c)\times\Gamma(1-c)$, for all c within the open interval (−1, +1), i.e., c must lie within $-1 < c < +1$, or else the integral does not exist.

Therefore, from Hildebrand's formula, we obtain

$$\mu_1' = E(T) = \delta + e^{\mu}\Gamma(1-\beta)\times\Gamma(1+\beta) = \delta + e^{\mu}\,\mathrm{Beta}[(1-\beta),(1+\beta)] =$$

$$\delta + e^{\mu} B[(1-\beta),(1+\beta)]$$, where $0 < \beta < 1$, where *B* represents the Beta-function.

The above Eq. clearly shows that the $\mu_1' = E(T)$ of a Loglogistic exists iff $0 < \beta < 1$.

The Beta-function is defined as $\mathrm{Beta}(a,b) = B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx =$

$\Gamma(a)\times\Gamma(b)/\Gamma(a+b)$, a & $b > 0$. Next we derive the 2nd raw moment of the Loglogistic density.

$$E(T^2) = \int_{\delta}^{\infty} \frac{t^2}{\beta(t-\delta)} \times \frac{e^{-[\ln(t-\delta)-\mu]/\beta}\ dt}{[1+e^{-[\ln(t-\delta)-\mu)]/\beta}]^2}$$; The same z-transformation yields

$$\mu_2' = E(T^2) = \int_{-\infty}^{\infty} t^2 \times \frac{e^{-z}\ dz}{[1+e^{-z}]^2} = \int_{-\infty}^{\infty} (\delta + e^{\beta z+\mu})^2 \times \frac{e^{-z}\ dz}{[1+e^{-z}]^2} =$$

$$\delta^2 + 2\delta \int_{-\infty}^{\infty} e^{\beta z+\mu} \times \frac{e^{-z}\ dz}{[1+e^{-z}]^2} + \int_{-\infty}^{\infty} e^{2\beta z+2\mu} \times \frac{e^{-z}\ dz}{[1+e^{-z}]^2}$$

$$\mu_2' = \delta^2 + 2\delta e^{\mu} \int_{-\infty}^{\infty} \frac{e^{-z(1-\beta)}\ dz}{[1+e^{-z}]^2} + e^{2\mu} \int_{-\infty}^{\infty} \frac{e^{-z(1-2\beta)}\ dz}{[1+e^{-z}]^2}$$

Again, letting u = e^{-z} results in $u^{-1} = e^{z}$, $du = -e^{-z}dz$, or $dz = -e^{z}du = -u^{-1}du$, we obtain

$$\int_{-\infty}^{\infty} \frac{e^{-z(1-\beta)}\ dz}{[1+e^{-z}]^2} = \int_{\infty}^{0} \frac{u^{(1-\beta)}\ (-u^{-1}du)}{(1+u)^2} = \int_{0}^{\infty} \frac{u^{-\beta}\ du}{(1+u)^2} = \Gamma(1-\beta)\times\Gamma(1+\beta) = B(1-\beta, 1+\beta) .$$

Similarly, $\int_{-\infty}^{\infty} \frac{e^{-z(1-2\beta)}\ dz}{[1+e^{-z}]^2} = B(1-2\beta, 1+2\beta)$, $0 < \beta < 0.50$. Hence,

$$\mu_2' = E(T^2) = \delta^2 + 2\delta e^{\mu} B(1-\beta, 1+\beta) + e^{2\mu} B(1-2\beta, 1+2\beta) ,\ 0 < \beta < 0.50$$

The variance of the Loglogistic is given by

$$V(T) = \delta^2 + 2e^{\mu} B(1-\beta, 1+\beta) + e^{2\mu} B(1-2\beta, 1+2\beta) - \{\delta + e^{\mu} B[(1-\beta), (1+\beta)]\}^2$$

$$= e^{2\mu} B(1-2\beta, 1+2\beta) - e^{2\mu} B^2[(1-\beta), (1+\beta)]$$

$\mu_2 = e^{2\mu}[B(1-2\beta, 1+2\beta) - B^2(1-\beta, 1+\beta)]$, $0 \le \beta < 0.50$; thus the variance does not exist outside the range $0 \le \beta < 0.50$; it does not exist at $\beta = 0.50$, and is identically equal to zero at $\beta = 0$, which is not permissible. On a comparison with J. K. B. Eq.(23.90, p. 152), we may deduce that $V(T) = e^{2\mu}[2\pi\beta \times \csc(2\pi\beta) - \pi^2\beta^2 \csc^2(\pi\beta)]$, where $\csc(\pi\beta) = 1/\sin(\pi\beta)$.

To determine the skewness, we proceed as follows:

$\mu_3' = E(T^3) = \int_{-\infty}^{\infty} t^3 \times \frac{e^{-z}\ dz}{[1+e^{-z}]^2}$; because we know that minimum life does not impact the variance, then for simplicity we obtain the 3rd raw moment for the case of $\delta = 0$; thus $t = e^{\beta z + \mu}$, and hence E(T³) = $\int_{-\infty}^{\infty} e^{3\beta z+3\mu} \times \frac{e^{-z}\ dz}{[1+e^{-z}]^2} = e^{3\mu}\int_{-\infty}^{\infty} \frac{e^{-z(1-3\beta)}\ dz}{(1+e^{-z})^2} = e^{3\mu}B(1-3\beta,1+3\beta)$, $0 < \beta <$ 1/3. The 3rd central moment is given by

$$\mu_3 = e^{3\mu}B(1-3\beta,1+3\beta) - 3e^{2\mu}B(1-2\beta,1+2\beta)\times e^{\mu}B(1-\beta,1+\beta) + 2\left\{e^{\mu}B(1-\beta,1+\beta)\right\}^3 =$$

$$e^{3\mu}B(1-3\beta,1+3\beta) - 3e^{3\mu}B(1-2\beta,1+2\beta)\times B(1-\beta,1+\beta) + 2e^{3\mu}B^3(1-\beta,1+\beta)$$ Hence, the skewness is

$$\alpha_3 = \frac{B(1-3\beta,1+3\beta) - 3B(1-2\beta,1+2\beta)\times B(1-\beta,1+\beta) + 2B^3(1-\beta,1+\beta)}{[B(1-2\beta,1+2\beta) - B^2(1-\beta,1+\beta)]^{1.50}}, 0 < \beta < 1/3$$

The 3rd raw moment is given by

$$\mu_3' = E(T^2) = \delta^3 + 3\delta^2 e^{\mu}B(1-\beta,1+\beta) + 3\delta e^{2\mu}B(1-2\beta,1+2\beta) + e^{3\mu}B(1-3\beta,1+3\beta)$$

Similarly, the 4th standardized moment is

α_4 =

$$\frac{B(1-4\beta,1+4\beta) - 4B(1-3\beta,1+3\beta)B(1-\beta,1+\beta) + 6B(1-2\beta,1+2\beta)\times B^2(1-\beta,1+\beta) - 3B^4(1-\beta,1+\beta)}{[B(1-2\beta,1+2\beta) - B^2(1-\beta,1+\beta)]^2},$$

Letting $C = B(1-\beta,1+\beta) = \text{Beta}(1-\beta,1+\beta)$, the above reduces to

$$\alpha_4 = \frac{B(1-4\beta,1+4\beta) - 4CB(1-3\beta,1+3\beta) + 6C^2B(1-2\beta,1+2\beta) - 3C^4}{[B(1-2\beta,1+2\beta) - C^2]^2}$$; thus, the kurtosis is $\beta_4 = \alpha_4 - 3$, only for $0 < \beta < 1/4 = 0.25$.