Concerning a problem of K. Kuratowski

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Thesis Abstract

Concerning a problem of K. Kuratowski

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Suppose (S, T) is a topological space and A is any subset of S. Then the functions f and g from the power set of S, $P(S)$, into $P(S)$ are defined as: $f(A)$ is the closure of A and $g(A)$ is the complement of A. In this thesis, our goal is proving that there are at most fourteen type of image from any subset of S by using finite compositions of the closure function f and the complement function g, including the null composition.

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Style manual or journal used Journal of Approximation Theory (together with the style known as "aums"). Bibliograpy follows van Leunen's *A Handbook for Scholars*.

Computer software used The document preparation package TeX (specifically LaTeX) together with the departmental style-file `aums.sty`.

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Chapter 1

INTRODUCTION

Suppose (S, T) is a topological space and A is any subset of S, then the functions f and g from the power set of S, $P(S)$ into $P(S)$, are defined as: $f(A)$ is the closure of A and $g(A)$ is the complement of A. There is referenced in Munkres "Topology" [1] a problem created by K. Kuratowski. He asked how many images these two functions can generate starting with a subset A of the real line. By using finite compositions of the closure function f and the complement function g, we can generate a fourteen different images of A. This includes the image of the null composition, A itself. We also prove that no more than 14 images can ever be generated

For preparation of the proofs in chapters 4 and 5, in chapter 2, we want to introduce to you the list of theorems that come from my class notes. For some of them, I did the proofs and I presented them on the board in class. But these theorems are still not enough. In chapter 3, we need to prove some more theorems that are necessary for my proofs in chapters 4 and 5. Especially, theorems 3.5, 3.6a, and 3.6b; they help us to understand that for any finite composition with any exponent n on f or g (for any positive integer n), we can reduce these exponents to 1 or 0.

Example: $f^{4454}(g^3[f^{33}(g^4(A))]) = f(g[f(g^0(A))]) = f(g[f(A)])$.

In chapter 4, we do research on a co-finite topological space: If (S, T) is a topological space with the co-finite topology and A is a subset of S, then the collection of all images of A using finite compositions of the closure function f and the complement function g, including the null composition, has at most four elements.

Finally, in chapter 5, by using the information in the previous chapter, we want to show you that for any topological space (S, T) with A is any subset of S, and by using finite compositions of the closure function f and the complement function g, we are always able to generate a maximum of fourteen different images of A, including the image of the null composition. Then, to demonstrate that this theorem cannot be improved, we build an example on the real line that has fourteen different images.

Chapter 2

BASIC DEFINITIONS AND THEOREMS

In this chapter, without proofs, we include the definitions, theorems and notations that come from the class notes, and they are necessary for the proofs in the next chapters.

Definition 2.1: Suppose (S, T) is a topological space and A is a subset of S. If the set of limit points of A is denoted A', then $\overline{A}$ is defined to be $A \cup A'$.

Definition 2.2: Suppose (S, T) is a topological space, and suppose M is subset of S. The *interior* of M, denoted $Int(M)$, is defined to be the set of all points x in M such that there is an open set $O(x)$ satisfying $x \in O(x) \subseteq M$. The *boundary* of M, denoted $Bd(M)$, is defined to the set of all points x such that every open set containing x contains a point in M and a point not in M.

Definition 2.3: A subset M of a topological space (S, T) is said to be *closed* if the set $S \setminus M$ is open.

Theorem 2.4-2.12. Suppose (S, T) is a topological space and M is a subset of S. From the class notes we have the following reference theorems:

Theorem 2.4: $\overline{M} = \overline{\overline{M}}$

Theorem 2.5: $M \subseteq Int(M) \cup Bd(M)$

Theorem 2.6: $Int(M)$ is open.

Theorem 2.7: M is open iff $M = Int(M)$

Theorem 2.8: M is closed iff $Bd(M) \subset M$

Theorem 2.9: $\overline{S \setminus M} = S \setminus Int(M)$

Theorem 2.10: $Bd(M) = \overline{M} \cap \overline{S \setminus M}$

Theorem 2.11: $\overline{M} = M \cup Bd(M)$

Theorem 2.12: M is closed iff $M = \overline{M}$

Theorem 2.13: If (S, T) is a topological space, M is a closed set in S, and p is a limit point of M, then p is an element of M.

Notation 2.14: N will denote the set of all positive integers.

Notation 2.15: Q will denote the set of all rational numbers.

Notation 2.16: I will denote the set of all irrational numbers.

Notation 2.17: R will denote the set of all real numbers.

Definition 2.18: Suppose S is any set and T consists of the empty set and any subset of S whose complement is finite. Then (S, T) is a *co-finite* topological space.

Theorem 2.19: Suppose (S, T) is a topological space with the co-finite topology. If A is a finite subset of S, then the closure of A is A, and if A is an infinite subset, then the closure of A is S.

Definition 2.20: Suppose (S, T) is the topological space. Then the functions f and g from the power set of S, $P(S)$, into $P(S)$ are defined as follows: if A is any subset of S, then $f(A)$ is the closure of A and $g(A)$ is the complement of A.

Chapter 3

SUPPORTING LEMMAS AND THEOREMS

Beside the definitions and theorems that were introduced in chapter 2, in this chapter, we will prove some more theorems to support the proofs of our results .

<u>Lemma 3.1:</u> If (S, T) is a topological space and B is a subset of S, then $\overline{B}$ is closed.

<u>Proof:</u> Since from theorem 2.4, $\overline{B} = \overline{\overline{B}}$, and by theorem 2.12 with $\overline{B}$ substituted for M and $\overline{\overline{B}}$ for $\overline{M}$, we will get $\overline{B}$ is closed.

Theorem 3.2-3.4: Suppose (S, T) is a topological space and A is a subset of S. Then we will prove the following theorems.

<u>Theorem 3.2:</u> $\overline{Int(A)} = \overline{Int(\overline{Int(A)})}$

<u>Proof:</u> First we will show that $\overline{Int(A)} \subseteq \overline{Int(\overline{Int(A)})}$.

By theorem 2.6, with A substituted for M, then $Int(A)$ is open. And by theorem 2.7 with $Int(A)$ substituted for M, we have $Int(A) = Int(Int(A))$.

This implies $\overline{Int(A)} = \overline{Int(Int(A))}$.

By theorem 2.5 with $Int(A)$ substituted for M, we have $Int(A) \subseteq [Int(Int(A)) \cup Bd(Int(A))]$.

Therefore, $\overline{Int(Int(A))} \subseteq \overline{Int[Int(Int(A)) \cup Bd(Int(A))]}$.

Since $Int(Int(A)) = Int(A)$, we can remove one "Int" in $Int(Int(A))$.

So we get $\overline{Int(A)} \subseteq \overline{Int[Int(A) \cup Bd(Int(A))]}$.

And by theorem 2.11 with $Int(A)$ substituted for M, we have $\overline{Int(A)} = Int(A) \cup Bd[Int(A)]$.

This means that $\overline{Int[Int(A) \cup Bd(Int(A))]} = \overline{Int(\overline{Int(A)})}$.

Hence, $\overline{Int(A)} \subseteq \overline{Int(\overline{Int(A)})}$ (Inclusion I).

Second we will show that $\overline{Int(\overline{Int(A)})} \subseteq \overline{Int(A)}$.

Let $x \in \overline{Int(\overline{Int(A)})}$; then we will have 2 cases that can happen: $x \in Int(\overline{Int(A)})$ or $x \notin Int(\overline{Int(A)})$.

Case 1: If $x \in Int(\overline{Int(A)})$, then $x \in \overline{Int(A)}$.

Case 2: If $x \notin Int(\overline{Int(A)})$, then x must be a limit point of $Int(\overline{Int(A)})$. This means that $x \in [S \setminus Int(\overline{Int(A)})]$ and is a limit point of $Int(\overline{Int(A)})$.

By theorem 2.9 with $\overline{Int(A)}$ substituted for M, we have $x \in \overline{S \setminus (\overline{Int(A)})}$.

Again, we have 2 cases that can happen: $x \in S \setminus \overline{Int(A)}$ or $x \notin (S \setminus \overline{Int(A)})$.

Case 2a: If $x \in S \setminus \overline{Int(A)}$, since x is a limit point of $Int(\overline{Int(A)})$, then every open set containing x contains at least one point in $Int(\overline{Int(A)})$ which is different from x. But $[S \setminus \overline{Int(A)}] \cap [Int(\overline{Int(A)})] = \phi$.

This gives us a contradiction since, by lemma 3.1 with $Int(A)$ substituted for B, we have $\overline{Int(A)}$ is closed, so $S \setminus \overline{Int(A)}$ is open.

Case 2b: Therefore, x must be in $[S \setminus S \setminus \overline{Int(A)}]$.

This implies $x \in \overline{Int(A)}$.

Therefore, $\overline{Int(\overline{Int(A)})} \subseteq \overline{Int(A)}$ (Inclusion II)

From inclusions I and II, we proved that $\overline{Int(A)} = \overline{Int(\overline{Int(A)})}$.

Theorem 3.3: $S \setminus Int(\overline{A}) = S \setminus Int(\overline{Int(\overline{A})})$

Proof: To start to show that $S \setminus Int(\overline{A}) = S \setminus Int(\overline{Int(\overline{A})})$, we will show that $Int(\overline{A}) = Int(\overline{Int(\overline{A})})$.

First we will show that $Int(\overline{A}) \subseteq Int(\overline{Int(\overline{A})})$.

By theorem 2.6, with $\overline{A}$ substituted for M, then $Int(\overline{A})$ is open, and by theorem 2.7 with $Int(\overline{A})$ substituted for M, we have $Int(\overline{A}) = Int(Int(\overline{A}))$.

By theorem 2.5 with $Int(\overline{A})$ substituted for M, we have $Int(\overline{A}) \subseteq Int(Int(\overline{A})) \cup Bd(Int(\overline{A}))$.

Therefore, $Int(Int(\overline{A})) \subseteq Int[Int(Int(\overline{A})) \cup Bd(Int(\overline{A}))]$.

Since $Int(Int(\overline{A})) = Int(\overline{A})$, we can remove one "Int" in $Int(Int(\overline{A}))$.

This means that $Int(\overline{A}) \subseteq Int[Int(\overline{A}) \cup Bd(Int(\overline{A}))]$.

But by theorem 2.11 with $Int(\overline{A})$ substituted for M, we have $\overline{Int(\overline{A})} = Int(\overline{A}) \cup Bd(Int(\overline{A}))$.

This implies $Int[Int(\overline{A}) \cup Bd(Int(\overline{A}))] = Int(\overline{Int(\overline{A})})$.

Hence, $Int(\overline{A}) \subseteq Int(\overline{Int(\overline{A})})$ (Inclusion I)

Second we will show that $Int(\overline{Int(\overline{A})}) \subseteq Int(\overline{A})$.

Let x be any limit point of $Int(\overline{A})$. This means that x is a limit point of $\overline{A}$.

By lemma 3.1 with A substituted for B, we have $\overline{A}$ is closed.

And by theorem 2.13 with $\overline{A}$ substituted for M, and x substituted for p, we will get $x \in \overline{A}$.

Hence $(Int(\overline{A}))' \subseteq \overline{A}$, and obviously, $Int(\overline{A}) \subseteq \overline{A}$, too.

Notice that $[Int(\overline{A})]' \cup Int(\overline{A}) = \overline{Int(\overline{A})}$.

Therefore $\overline{Int(\overline{A})} \subseteq \overline{A}$.

This implies $Int(\overline{Int(\overline{A})}) \subseteq Int(\overline{A})$ (Inclusion II)

From inclusions I and II, we have $Int(\overline{A}) = Int(\overline{Int(\overline{A})})$.
This means that $S \setminus Int(\overline{A}) = S \setminus Int(\overline{Int(\overline{A})})$.

Theorem 3.4: $\overline{S \setminus \overline{Int(\overline{A})}} = S \setminus Int(\overline{A})$
Proof: By theorem 2.9 with $\overline{Int(\overline{A})}$ substituted for M, we have $\overline{S \setminus \overline{Int(\overline{A})}} = S \setminus Int(\overline{Int(\overline{A})})$.
By theorem 3.3, we have $S \setminus Int(\overline{Int(\overline{A})}) = S \setminus Int(\overline{A})$.
Therefore, $\overline{S \setminus \overline{Int(\overline{A})}} = S \setminus Int(\overline{A})$.

Theorem 3.5: Suppose (S, T) is a topological space and A is a subset of S. Then for any positive integer n, the collection of all images of A using only finite compositions of the closure function f is $f^n(A) = f(A)$.

Proof: We will use induction to show that for any $n \in N$, $f^n(A) = f(A)$.

Clearly, if $n = 1$, then $f^n(A) = f^1(A) = f(A)$.

If $n = 2$, then $f^n(A) = f^2(A) = f(f(A))$.
But $f(f(A))$ is defined to be $\overline{(\overline{A})} = \overline{\overline{A}}$.
By theorem 2.4 with A substituted for M, we have $\overline{\overline{A}} = \overline{A}$.
Therefore, $f(f(A)) = \overline{A} = f(A)$.

Assume $f^k(A) = f(A)$ (for some $k \in N$). Then we need to prove that $f^{k+1}(A) = f(A)$.

Since $f^{k+1}(A) = f(f^k(A))$, by substituting $f(A)$ for $f^k(A)$, we will have $f^{k+1}(A) = f(f(A))$.

This means that $f^{k+1}(A) = f(A)$, as shown in the case $n = 2$.

Hence, $f^n(A) = f(A)$ (for any $n \in N$).

Theorem 3.6a: Suppose (S, T) is a topological space and A is a subset of S. Then for any non-negative integer n, the collection of all images of A using only odd finite compositions of the complement function g is $g^{2n+1}(A) = g(A)$.

Proof: By using induction:

If $n = 0$, then $g^{2n+1}(A) = g^1(A) = g(A)$.

If $n = 1$, then $g^{2n+1}(A) = g^3(A) = g(g[g(A)]) = g(g[S \setminus A]) = g(S \setminus [S \setminus A]) = g(A)$.

Assume $g^{2k+1}(A) = g(A)$ (for some $k \in N$). Then we need to prove that $g^{2(k+1)+1}(A) = g(A)$.

Note that $g^{2(k+1)+1}(A) = g^{2k+3}(A)) = g^{2k+1}(g[g(A)])$.

But $g(g(A)) = g(S \setminus A) = S \setminus (S \setminus A) = A$.

Therefore $g^{2(k+1)+1}(A) = g^{2k+1}(g[g(A)]) = g^{2(k+1)}(A) = g(A)$.
This means that $g^{2(k+1)+1}(A) = g(A)$.

Hence, $g^{2n+1}(A) = g(A)$ [for any $n \in (N \cup \{0\})$].

<u>Theorem 3.6b:</u> Suppose (S, T) is a topological space and A is a subset of S. Then for any n is a positive integer, the collection of all images of A using only even finite compositions of the complement function g is $g^{2n}(A) = A$.

<u>Proof:</u> By using induction:

If $n = 1$, then $g^{2n}(A) = g^2(A) = g(g(A)) = g(S \setminus A) = S \setminus (S \setminus A) = A$.

Assume $g^{2k}(A) = A$ (for some $k \in N$). Then we need to prove that $g^{2(k+1)}(A) = A$. Since $g^{2(k+1)}(A) = g^{2k+2}(A) = g^{2k}(g[g(A)])$. By substituting A for $g[g(A)]$, we will have $g^{2k}(g[g(A)]) = g^{2k}(A) = A$.
This means that $g^{2(k+1)}(A) = A$.

Hence, $g^{2n}(A) = A$ (for any $n \in N$).

<u>Corollary 3.7</u>: The images of A generated by all finite composition of f and g are also generated by the compositions that alternate f and g; i.e. compositions of the form $(fg)^n$,

$g(fg)^n$, $(gf)^n$, or $f(gf)^n$, for non-negative integers n.

Proof: We want to prove that any image of A is always generated by the finite compositions that alternate f and g.

By theorem 3.5, we have all finite compositions of f and g are also generated by the finite compositions that alternate f and the finite compositions of the complement function g.

We have 2 cases that can happen: when the finite compositions of the complement function g is odd *or* when the finite compositions of the complement function g is even

Case 1: Where any of the finite compositions of the complement function g is odd, then by theorem 3.6a, we have the images of A are generated by the compositions that alternate f and g.

Case 2: Where any of the finite compositions of the complement function g is even. Again, we have 2 cases:

- *Case 2a:* It is clear that if the function starts with g, then by theorem 3.6b, this means that the function starts with f.
- *Case 2b:* If some finite compositions of the complement function g are even, which are in between two closure functions f. Then by theorem 3.6b, we have the images of A are generated by some compositions that alternate ff and g, by theorem 3.5, this implies that the images of A are generated by the compositions that alternate f and g.

Therefore, the images of A generated by all finite composition of f and g are also generated by the compositions that alternate f and g.

CHAPTER 4

THE CO-FINITE CASE

In this chapter, we will see that for any co-finite topological space (S, T) with A a subset of S, the collection of all images of A using finite compositions of the closure function f and the complement function g, including the null composition, always has at most four elements, A, $S \setminus A$, S, and ϕ.

Example 4.1: Suppose (N, T) is the positive integers with the co-finite topology, and A is the finite set $\{1, 2, 3\}$. By using the finite compositions of the closure function f and the complement function g, we will get :

$f(A) = \overline{A} = A = \{1, 2, 3\}$ (by theorem 2.19).

$g(A) = N \setminus A = N \setminus \{1, 2, 3\}$.

This implies that $f(g(A)) = N$ (by theorem 2.19).

Hence, $g(f(g(A))) = N \setminus N = \phi$.

From corollary 3.7, observe that for any composition of the closure f and the complement function g, an image of A is always either A or $N \setminus A$ or N or ϕ.

Theorem 4.2: If (S, T) is a topological space with the co-finite topology and A is a subset of S, then the collection of all images of A using finite compositions of the closure function f and the complement function g, including the null composition, has at most four elements, A, $S \setminus A$, S, and ϕ.

Proof: By proving this theorem, we will get a maximum of four images from any subset A of S, those images are: A, $S \setminus A$, S, and ϕ.

We have 2 cases that can happen: A is a finite subset of S *or* A is an infinite subset of S.

Case 1: If A is a finite subset of S, then again, we have 2 cases: $S \setminus A$ is a finite subset of S or $S \setminus A$ is an infinite subset of S.

Case 1a: If $S \setminus A$ is a finite subset of S then:

We have $f(A) = \overline{A} = A$ (by theorem 2.19).

This implies that $g(f(A)) = S \setminus A$.

Now, starting with g, $g(A) = S \setminus A$.

Therefore $f(g(A)) = S \setminus A$ (by theorem 2.19).

Since A and $S \setminus A$ are finite subsets of S, there are only two types of finite compositions and all of them alternate f and g. From corollary 3.7, this means that the images of A generated by all finite compositions of f and g will be either A or $S \setminus A$ (by theorem 2.19).

Case 1b: If $S \setminus A$ is an infinite subset of S then:

We have $f(A) = \overline{A} = A$ (by theorem 2.19).

This implies that $g(f(A)) = S \setminus A$ and $f(g(f(A))) = S$.

Now, starting with g, we have $g(A) = S \setminus A$.

Hence, $f(g(A)) = \overline{S \setminus A} = S$ (by theorem 2.19).

This implies that $g(f(g(A))) = \phi$.

By corollary 3.7, there are only four types of finite compositions and all of them alternate f and g. This means that the images we can generate will be either A, $S \setminus A$, S or ϕ.

<u>Case 2:</u> If A is an infinite subset of S. Then again, we have 2 cases: $S \setminus A$ is a finite subset of S or $S \setminus A$ is an infinite subset of S.

<u>Case 2a:</u> If $S \setminus A$ is a finite subset of S then:

We have $f(A) = \overline{A} = S$ (by theorem 2.19).

Therefore $g(f(A)) = \phi$.

Now, starting with g, we get $g(A) = S \setminus A$.

This implies that $g(g(A)) = A$.

By theorem 3.6b, it is clear to see that the finite complements of A are $g(A) = S \setminus A$ or null composition, A itself.

Hence, $f(g(A)) = S \setminus A$ (by theorem 2.19), and $g(f(g(A))) = A$.

From corollary 3.7, observe that there are only four types of finite compositions and all of them alternate f and g. This means that the images we can generate will be either A, $S \setminus A$, S or ϕ.

<u>Case 2b:</u> If $S \setminus A$ is an infinite subset of S then:

We have $f(A) = \overline{A} = S$ (by theorem 2.19).

Therefore $g(f(A)) = \phi$.

Now, starting with g, we get $g(A) = S \setminus A$.

This implies that $g(g(A)) = A$.

By theorem 3.6b, it is clear to see that the finite complements of A are $g(A) = S \setminus A$ or null composition, A itself.

Also, $f(g(A)) = \overline{S \setminus A} = S$ (by theorem 2.19).

From corollary 3.7, we also observe that there are only four types of finite compositions and all of them alternate f and g, this means that the images those we can generate will be either A, $S \setminus A$, S or ϕ.

Chapter 5

MAIN RESULTS

Now, in this chapter, we want to show you that for any topological space (S, T), there are at most fourteen type of image from any subset of S by using finite compositions of the closure function f and the complement function g, including the null composition.

We begin with an example of a subset of R that has fourteen different images.

Example 5.1: Let $A = (0,1) \cup (1,2) \cup \{3,4\} \cup [5,6) \cup \{7\} \cup (8,9) \cup (Q \cap [9,10])$, which is in the reals with the usual topology. We will find seven different images using compositions that start with g (steps 2 to 8) and another six different images using compositions that start with f (steps 9 to 14). Then, counting A itself, we will have fourteen different images.

Facts: We will assume without proof that if a and b are two (unequal) real numbers then the rational numbers, Q, and the irrational numbers, I, are both dense in the interval $[a, b]$. Therefore, by using these facts, we can get:

1. $A = (0,1) \cup (1,2) \cup \{3,4\} \cup [5,6) \cup \{7\} \cup (8,9) \cup (Q \cap [9,10])$
2. $g(A) = (-\infty, 0] \cup \{1\} \cup [2,3) \cup (3,4) \cup (4,5) \cup [6,7) \cup (7,8] \cup (I \cap [9,10]) \cup (10, \infty)$
3. $f(g(A)) = (-\infty, 0] \cup \{1\} \cup [2,5] \cup [6,8] \cup [9,\infty)$
4. $g(f(g(A))) = (0,1) \cup (1,2) \cup (5,6) \cup (8,9)$
5. $(fg)^2(A) = [0,2] \cup [5,6] \cup [8,9]$
6. $g(fg)^2(A) = (-\infty, 0) \cup (2,5) \cup (6,8) \cup (9,\infty)$
7. $(fg)^3(A) = (-\infty, 0] \cup [2,5] \cup [6,8] \cup [9,\infty)$

8. $g(fg)^3(A) = (0,2) \cup (5,6) \cup (8,9)$

Now we compute the images using compositions that start with f.

9. $f(A) = [0,2] \cup \{3,4\} \cup [5,6] \cup \{7\} \cup [8,10]$

10. $g(f(A)) = (-\infty,0) \cup (2,3) \cup (3,4) \cup (4,5) \cup (6,7) \cup (7,8) \cup (10,\infty)$

11. $f(g(f(A))) = (-\infty,0] \cup [2,5] \cup [6,8] \cup [10,\infty)$

12. $(gf)^2(A) = (0,2) \cup (5,6) \cup (8,10)$

13. $f(gf)^2(A) = [0,2] \cup [5,6] \cup [8,10]$

14. $(gf)^3(A) = (-\infty,0) \cup (2,5) \cup (6,8) \cup (10,\infty)$

Theorem 5.2: Suppose (S,T) is a topological space, and A is any subset of S. Then the collection of all images of A using finite compositions of the closure function f and the complement function g, including the null composition, has at most fourteen elements.

Proof: From corollary 3.7, we know in the co-finite case, there are only four types of finite compositions and all of them alternate f and g. We will start with g and compute the seven images up to $g(fg)^3(A)$ (steps 2 to 8). These images may or may not be distinct, depending on A. But when we go one more factor and compute $(fg)^4(A)$ we generate an image we already have so the process repeats from that point on. Then we do the same for the first six compositions that start with f (steps 9 to 14) and find that the seventh composition $f(gf)^3(A)$ repeats. Thus, together with A itself, there are at most fourteen different images of A.

Those fourteen possibly different images of A are:

First, we find the compositions that start with g.

1. A

2. $g(A) = S \setminus A$

3. $f(g(A)) = \overline{S \setminus A}$

4. $g(f(g(A))) = S \setminus (\overline{S \setminus A}) = S \setminus [S \setminus Int(A)]$ (by theorem 2.9 with A substituted for M)

This implies $g(f(g(A))) = Int(A)$.

5. $(fg)^2(A) = \overline{Int(A)}$

6. $g(fg)^2(A) = S \setminus \overline{Int(A)}$

7. $(fg)^3(A) = \overline{S \setminus \overline{Int(A)}} = S \setminus Int(\overline{Int(A)})$ (by theorem 2.9 with $\overline{Int(A)}$ substituted for M)

8. $g(fg)^3(A) = S \setminus (S \setminus Int(\overline{Int(A)})) = Int(\overline{Int(A)})$

The seven images above (from 2 to 8) are the most that can be generated by composition functions starting with g since $(fg)^4(A) = \overline{Int(\overline{Int(A)})} = \overline{Int(A)}$ (by theorem 3.2). This means that the solution of $(fg)^4(A)$ repeats the image in step 5.

Second, we find the compositions that start with f.

9. $f(A) = \overline{A}$

10. $g(f(A)) = S \setminus \overline{A}$

11. $f(g(f(A))) = \overline{S \setminus \overline{A}} = S \setminus Int(\overline{A})$ (by theorem 2.9 with $\overline{A}$ substituted for M)

12. $(gf)^2(A) = S \setminus (S \setminus Int(\overline{A})) = Int(\overline{A})$

13. $f(gf)^2(A) = \overline{Int(\overline{A})}$

14. $(gf)^3(A) = S \setminus \overline{Int(\overline{A})}$

These six images (from 9 to 14) are the most that can be generated by composition functions starting with f since $f(gf)^3(A) = \overline{S \setminus \overline{Int(\overline{A})}} = S \setminus Int(\overline{A})$ (by theorem 3.4).

This mean that the solution of $f(gf)^3(A)$ repeats the image in step 11.

Bibliography

[1] James R. Munkres "Topology" Second edition, Pearson Education, 2000.