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Iterative processes generating dense point sets

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A Thesis

Submitted to

the Graduate Faculty of

Auburn University

in Partial Fulfillment of the

Requirements for the

Degree of

Master of Science

Auburn, Alabama
May 11, 2006

Iterative processes generating dense point sets

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Gergely Ambrus, son of Imre Ambrus and Judit (Bálint) Ambrus was born on May 1, 1983, in Budapest, Hungary. He graduated from the Radnóti Miklós Experimental Grammar School in Szeged, Hungary with Summa Cum Laude in 2001. He has been attending the University of Szeged, Hungary on Mathematics major since 2001. He will graduate from the University of Szeged with a Diploma in Mathematics in June, 2006. In September, 2004 he entered the Graduate School at Auburn University on Mathematics major.

Thesis Abstract

Iterative processes generating dense point sets

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Master of Science, May 11, 2006

44 Typed Pages

Directed by András Bezdek

The central problem of the thesis is the question of denseness of certain sets of points in the plane. All of the following results are joint with A. Bezdek. We consider point sets $\mathcal{P}$, not a subset of a line, having the property that for every three noncollinear points in $\mathcal{P}$, a specific triangle center (incenter (IC), circumcenter (CC), orthocenter (OC) resp.) is also in the set $\mathcal{P}$. In Chapter 2, we generalize and solve the CC problem in higher dimensions. We prove that if a point set $\mathcal{P} \subset \mathbb{E}^n$ has the property that for each simplex of $\mathcal{P}$ the circumcenter of the simplex also belongs to $\mathcal{P}$, then it is dense in the whole space. Chapter 3 contains the solution of the OC problem in the plane, essentially proving that $\mathcal{P}$ is either a dense point set of the plane or it is a subset of a rectangular hyperbola. In the latter case, it is either a dense subset or it is a special discrete subset of a rectangular hyperbola, for which we give both an algebraic and a geometric characterization.

Style manual or journal used <u>Journal of Approximation Theory (together with the style known as "aums"). Bibliograpy follows van Leunen's *A Handbook for Scholars.*</u>

Computer software used <u>The document preparation package TeX (specifically LaTeX) together with the departmental style-file `aums.sty`.</u>

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INTRODUCTION AND RESEARCH OVERVIEW

During the period spent at Auburn University two new publications have been written jointly with my supervisors András Bezdek (Auburn University) and Ferenc Fodor (University of Szeged). Both of the investigated questions belong to Discrete Geometry. The first article [1] was prepared in the Fall Semester of 2004 as a joint work with A. Bezdek and F. Fodor, and later it won a 3rd Prize at the Hungarian Student Paper Contest in 2005. It was also awarded as the Best Student Paper Presentation of the 12th SCMA Conference in 2005. The second paper [2] was written in the Fall Semester of 2005, jointly with A. Bezdek. This second paper provides the frame of the thesis.

In the rest of this chapter we shall give a brief outline of the above mentioned two papers together with a small historical overview.

1.1 Transversal problem for higher dimensional unit balls

A line l is a *transversal* to a family $\mathcal{F}$ of sets if l intersects every element of $\mathcal{F}$. If there is a line that intersects every member of the family $\mathcal{F}$, then we say that $\mathcal{F}$ has the property T. If every k or fewer members of $\mathcal{F}$ have a transversal then $\mathcal{F}$ has property $T(k)$.

In 1958, Grünbaum [9] conjectured that for a family of pairwise disjoint translates of a convex disk $T(5) \Rightarrow T$. The special case of circular disks was settled by Danzer [7]. The conjecture was proved in full generality by Tverberg [16] in 1989.

Higher dimensional generalizations for families of balls were initiated by Problem 107 [11] of Hadwiger, which states that for any family of *thinly distributed* balls in $\mathbb{E}^n$ the property $T(n^2)$ implies T. A family of balls is *thinly distributed* if the distance between the centers of any two balls is at least twice the sum of their radii. In his solution, Hadwiger reduced the problem to proving a 3-dimensional lemma (Hilfssatz), essentially claiming that for two thinly distributed balls in $\mathbb{E}^3$ the cone consisting of the direction vectors of the common transversals of the balls is convex. However, the proof of this Hilfssatz was not included in the published solution, although at that time it was available from the editor. A special case, when the two balls are congruent, has been proved rigorously in [12]. In 1960, Grünbaum [10] improved the n^2 in Hadwiger's statement to $2n - 1$ using the Topological Helly Theorem and Hadwiger's Hilfssatz.

Recent progress concerning tranversals of balls has been made by Holmsen, Katchalski and Lewis [12], who proved that there exists a positive integer $n_0 \leq 46$ such that $T(n_0)$ implies T for any family of pairwise disjoint unit balls in $\mathbb{E}^3$. This bound was improved by Cheong, Goaoc and Holmsen [5] to 11. On the other hand, a result of Holmsen and Matoušek [13] states that there is no such Helly-number if one considers families of pairwise disjoint translates of an arbitrary convex body in $\mathbb{E}^3$.

In the article [1], we generalize the main result of [12]. Our statement also strengthens Hadwiger's theorem [11] for congruent balls. We prove the following theorem.

Theorem 1.1.1. *Let $n \geq 2$ and $\mathcal{F}$ be a family of unit balls in $\mathbb{E}^n$ with the property that the mutual distances of the centers are at least $2\sqrt{2 + \sqrt{2}}$. If $\mathcal{F}$ has property $T(n^2)$, then $\mathcal{F}$ has a transversal.*

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Our method of proof avoids Hadwiger's Hilfssatz. The core part is proving that for a set of unit balls satisfying the above conditions, the direction vectors of the common transversals of all the balls from a convex set. We manage to prove this by reducing the problem for the three-dimensional case. Then the convexity allows us to finish the proof with the repeated use of Helly's Theorem and the Spherical Helly Theorem.

Since then, our result has been strengthened by Cheong, Goaoc, Holmsen and Petitjean [6], who proved that for a family of disjoint unit balls in $\mathbb{E}^n$, $T(4n-1)$ implies T.

1.2 Problems on iterative processes

History and terminology. There are several results in the literature which deal with iterative processes in the plane. A typical problem starts with the description of a geometric construction, which when applied to an initial point set generates additional points and thus expands it. The problem usually is to prove that repeated expansions lead to an everywhere dense point set in the plane.

For example K. Bezdek and J. Pach [4] answering a question of L. Fejes Tóth proved that if one starts with two points (whose distance is ≤ 2 and different from 1 and $\sqrt{3}$) then repeated use of the construction "add to the figure all intersection points of those unit circles whose centers belong to the existing points set" leads to a dense point set. Recently D. Ismailescu and R. Radoičić [14] (and earlier B. Grünbaum) showed that the repeated use of the construction "add to the figure all the intersection points of lines which connect pairs of already existing points" also leads to a dense point set (with the exception of a few particular starting configurations). Ismailescu also suggested to study

similar problems where one uses the construction "add the circumcenters (incenters resp.) of all nondegenerate triangles formed by the existing points". It was proved in [15] that in case of the circumcenters the iterative process always leads to a dense point set of the plane, and in case of the incenters it always leads to a dense point set in the convex hull of the initial point set.

Higher dimensional circumcenter problem. In Chapter 2 we present a new approach to the problem of iterative processes. Instead of analyzing the elementary geometric properties of the particular iterative processes we prove a key theorem on n-dimensional closed point sets. Essentially we prove that if a point set in $\mathbb{E}^n$ is not dense in its convex hull, then there exists a ball which contains $n + 1$ affinely independent points of the set on the boundary, and all the other points of the set contained in the ball are close to one of these points. As an application of this theorem we obtain an affirmative answer for the higher dimensional version of the circumcenter problem. In fact we show that more is true, namely the construction "add a point anywhere close to the circumcenter of each existing simplex" also implies denseness in the whole $\mathbb{E}^n$.

Planar orthocenter problem. Planar iterative processes can be defined with respect of any triangle center. The case of orthocenters (intersection of the three altitudes of a triangle) was recognized in [15] as a particularly interesting one due to the following peculiar behavior of the orthocenters: if a point set belongs to a rectangular hyperbola (a hyperbola for which the asymptotes are perpendicular), then the orthocenters of the triangles formed by triplets of this point set are also on this hyperbola. We will say that a noncollinear point set $\mathcal{P}$ is *orthocentrically closed* if for every three noncollinear points in $\mathcal{P}$, the orthocenter of the triangle determined by these three points is also

in $\mathcal{P}$. Observe that the vertices of a triangle together with their orthocenter form an orthocentrically closed set. These four point sets will be called *orthocentric quadruples*. It was conjectured by Iorio, Ismailescu, Radoičić and Silva in [15] that orthocentrically closed sets with the exception of orthocentric quadruples are either dense sets in the plane or are dense subsets of a rectangular hyperbola. We prove in Chapter 3 that it is also possible that the set is a special infinite discrete subset of a rectangular hyperbola. Furthermore, we give an exact characterization of these infinite discrete subsets.

In this section we give an affirmative answer for the higher dimensional circumcenter problem. First we prove the following useful theorem.

Theorem 2.0.1. *Let $\mathcal{P}$ be a closed set in $\mathbb{E}^n$ which is not dense in its convex hull and contains $n+1$ affinely independent points. For each $0 < \lambda < 1$ there exists an n-dimensional ball B which contains $n+1$ affinely independent points of $\mathcal{P}$ on its boundary and is essentially empty, meaning that the interior of the concentric ball λB does not contain any points from $\mathcal{P}$.*

As an application of Theorem 2.0.1 we obtain the answer for the circumcenter problem as stated in the following theorem.

Theorem 2.0.2. *Let $\mathcal{P}$ denote a point set in $\mathbb{E}^n$ containing a subset of $n+1$ affinely independent points, with $n \geq 1$. If there exists a number $0 \leq \varrho < 1$ such that for any ball B determined by $n+1$ affinely independent points of $\mathcal{P}$ the ball ϱB contains at least one point from $\mathcal{P}$, then $\mathcal{P}$ is dense in $\mathbb{E}^n$.*

Now we introduce some terminology. A ball determined by $n+1$ affinely independent points means the ball of dimension n circumscribed about the simplex determined by the points. For any ball B with radius r and for any nonnegative number ϱ, the ball ϱB denotes the ball concentric to B with radius ϱr.

We will say for sets $R, S \subset \mathbb{E}^n$ that R is S-disjoint provided that the interior of R does not contain any point of S. The convex hull of $\mathcal{P}$ will be denoted by $\operatorname{conv} \mathcal{P}$, and $\partial(.)$ stands for the boundary operator.

Proof of Theorem 2.0.1. The basic idea of the proof is the following. First we find a $\mathcal{P}$-disjoint ball in $\operatorname{conv} \mathcal{P}$. Then we inductively find essentially empty balls containing $1, 2, \ldots, n + 1$ points of $\mathcal{P}$ on their boundary. The last ball found by this algorithm will have the required properties. At finding the next ball, the concept of transforming one ball into the other will play the key role:

Suppose that S is a subset of $\mathbb{E}^n$ containing the balls A_0 and A_1 of dimension n. We will say that A_0 *can be transformed to* A_1 *within* S if for every $t \in [0, 1]$ there exists a ball $B(t)$ in S, such that $A_0 = B(0)$, $A_1 = B(1)$, and both the radii and the position of the centers of the balls $B(t)$ are changing continuously in t.

We distinguish two cases according to the $\mathcal{P}$-disjointness of the interior of $\operatorname{conv} \mathcal{P}$.

Case 1. There is a point of $\mathcal{P}$ in the interior of the convex hull of $\mathcal{P}$.

First we find a $\mathcal{P}$-disjoint ball in the convex hull of $\mathcal{P}$ with at least one point of $\mathcal{P} \setminus \partial(\operatorname{conv} \mathcal{P})$ on the surface.

Consider the closed inner δ-parallel sets of $\operatorname{conv} \mathcal{P}$ denoted by P_δ:

$$P_\delta = \{x \in \operatorname{conv} \mathcal{P} \mid d(x, \partial(\operatorname{conv} \mathcal{P})) \geq \delta\}.$$

Note that P_δ is convex. According to the conditions, the interior of $\operatorname{conv} \mathcal{P}$ contains a $\mathcal{P}$-disjoint ball and also a point of $\mathcal{P}$. Hence there exists a $\delta > 0$ such that the interior of P_δ is not $\mathcal{P}$-disjoint while it contains a $\mathcal{P}$-disjoint ball B. This yields that there exists

a ball A contained entirely in P_δ which contains a point of $\mathcal{P}$. Since P_δ is convex, B can be transformed into A within P_δ. The closedness of $\mathcal{P}$ yields that there exists a first ball (say B_1) along the transformation which contains a point p_1 of $\mathcal{P}$ on the boundary while it is still $\mathcal{P}$-disjoint. Since B_1 is contained in P_δ, the point p_1 must belong to $\mathcal{P} \setminus \partial(\operatorname{conv} \mathcal{P})$, what we wanted.

The ball B_1 of radius r will serve as the starting ball in our forthcoming algorithm. Introduce $\varepsilon = \min\{(1 - \lambda)r, (1 - \lambda)\delta/2\}$.

In the following process we will choose a sequence of balls $B_2, \ldots, B_{n+1}$ and a sequence of points $p_2, \ldots, p_{n+1}$ of $\mathcal{P}$ such that for each $2 \le k \le n + 1$ the following four conditions hold:

- the radius of B_k is at least as big as the radius of B_{k-1}

- $p_1, \ldots, p_k$ are affinely independent

- $p_1, \ldots, p_k$ belong to the boundary of B_k

- all points of $\mathcal{P}$ which are contained in the interior of B_k belong to the ε-neighborhood of one of the points $p_1, \ldots, p_k$.

Note that the first and last conditions guarantee that B_k is essentially empty. Also notice that the above conditions hold for the ball B_1 and the point p_1.

We will choose the points and the balls by induction on k. Assume that we already selected the first k members of the sequences satisfying all four of the above conditions. Let A_k be the $k - 1$ dimensional affine subspace (i. e. a $k - 1$-dimensional hyperplane) spanned by $p_1, \ldots, p_k$. Let E be the union of the open ε-neighborhoods of $p_1, \ldots, p_k$. We know that the interior of B_k does not contain points $\mathcal{P}$ outside of E. Consider the

closed set $\mathcal{S} = \mathcal{P} \setminus E$, which is the set of the points of $\mathcal{P}$ not contained in the open ε-neighborhood of any of the points $p_1, \ldots, p_k$.

Note that our choice of ε provides that $\mathcal{S}$ cannot be a subset of A_k. Indeed, that would mean that all the points of $\mathcal{P}$ are of a distance at most ε from A_k, hence the convex hull of $\mathcal{P}$ is contained in the ε-neighborhood of A_k. Therefore the distance between p_1 (contained in A_k) and conv $\mathcal{P}$ could be at most ε, which contradicts to the assumption $\varepsilon \leq (1 - \lambda)\delta/2 < \delta$.

Define C as the maximal affine subspace containing the center of B_k and orthogonal to A_k (hence the linear subspace corresponding to C is the orthogonal complement of the linear subspace corresponding to A). C is the set of the centers of the spheres passing through the points $p_1, \ldots, p_k$. Let r be the radius of B_k, and consider among these balls which, as an additional property, have radius not less than the radius of B_k, and let C_r denote the set of their centers. Then $C_r \subset C$, and it is easy to see that C_r is obtained from C by subtracting an open ball. C_r may be considered as the set of all possible centers of candidates for B_{k+1}.

Clearly, the dimension of C is $n - k + 1$, and therefore the set C_r is path-wise connected whenever $k \leq n - 1$ holds. Moreover, the union of all balls centered in C_r and containing $p_1, \ldots, p_k$ on the boundary cover $\mathbb{E}^n \setminus A_k$.

Since S is not a subset of A_k, there is a non $\mathcal{S}$-disjoint ball A centered in C_r, containing $p_1, \ldots, p_k$ on the boundary. On the other hand, B_k is also centered in C_r, while it is $\mathcal{S}$-disjoint. The path-connectedness of C_r enables us to transform B_k into A via balls centered in C_r, and the closedness of $\mathcal{S}$ yields that - just as in the beginning of the proof - there is an $\mathcal{S}$-disjoint ball B along the transformation which contains a point

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p of $\mathcal{S}$. Since the point p is affinely independent from the points $p_1, \ldots, p_k$ (remember that it is not contained in A_k), the choices $p_{k+1} = p$ and $B_{k+1} = B$ satisfy the inductive conditions.

If $k = n$ then C_r consists of two half-lines, which are distinct or their union is the whole line (this holds when the center of B_n is contained in the hyperplane A_n). If there exists a non $\mathcal{S}$-disjoint ball containing $p_1, \ldots, p_k$ on its boundary centered in the same half-line as the center of B_k, then the same process works as in the $k \leq n - 1$ case. If this does not hold, then the $(n-1)$-dimensional hyperplane A_n separates $\mathcal{S}$ and the center of B_n, yielding that there is no point of $\mathcal{P} \setminus E$ in the half-space bounded by A_n containing the center of B_n. Hence the intersection of this half-space and the convex hull of $\mathcal{P}$ is contained in the ε-neighborhood of the hyperplane A_n, and therefore the distance between p_1 and the boundary of $\operatorname{conv} \mathcal{P}$ is less then ε, a contradiction.

Case 2. There is no point of $\mathcal{P}$ in the interior of the convex hull of $\mathcal{P}$.

Choose a point Q in the interior of the convex hull of $\mathcal{P}$ and consider the point set $\mathcal{P}' = \mathcal{P} \cup \{Q\}$. Applying the process described in the solution of Case 1 for $\mathcal{P}'$ we obtain an essentially empty ball B satisfying the conditions of the inductive process.

Observe that when selecting the $n + 1$ special affinely independent points, the first one is Q, and the other points are in $\mathcal{P}$, labeled as $p_2, \ldots, p_{n+1}$.

Now we drop the point Q, and try to find a ball B_{n+1} containing $n + 1$ points of $\mathcal{P}$ on the surface with the desired conditions. Observe that the ε-neighborhood of Q is $\mathcal{P}$-disjoint, and apply the last step of Case 1 with B in place of B_n. The process can fail only in one case, if the hyperplane A determined by the points $p_2, \ldots, p_{n+1}$ separates

the center of B and the part of $\mathcal{P}$ not contained in the ε-neighborhoods of the points $p_2, \ldots, p_{n+1}$. In the rest of the proof we assume this holds.

First we prove that the radius of the ball circumscribed about $p_2, \ldots, p_{n+1}$ is greater than $\varepsilon/(1-\lambda)$. By the choice of ε, $\varepsilon/(1-\lambda) \le \delta/2$, therefore it suffices to show that the considered radius is at least $\delta/2$. Remember that δ is smaller than the distance between the additional point Q and the boundary of conv $\mathcal{P}$. Hence the ball with center Q and radius δ is contained in conv $\mathcal{P}$. On the other hand, the convex hull is not further from A_n than ε on one side of A_n. Comparing these two facts yields that the distance between Q and A_n is at least $\delta - \varepsilon$, what is greater than $\delta/2$ since $\varepsilon < \delta/2$. This also yields that A_n separates Q and the center of B.

Now an elementary geometric observation yields that since Q is on the surface of B, and A_n separates Q from the center of B, the distance between Q and A_n is strictly less than the radius of the $(n-1)$-dimensional ball obtained by taking the intersection of B and A_n, what is the same as the radius of the ball circumscribed about $p_2, \ldots, p_{n+1}$. Since the distance between Q and A_n is greater than $\delta/2$, the radius is strictly greater than $\delta/2$, what we wanted.

Notice that each ball containing $p_2, \ldots, p_{n+1}$ on the boundary has radius at least $\varepsilon/(1-\lambda)$. Thus for any such ball G the distance between G and λG is at least ε. Hence the first of the four conditions of the choosing process can be omitted, the last of the four conditions guarantees essentially emptiness. Therefore in order to find the ball B_{n+1} we can search among all balls centered on the line C instead of restricting ourselves to C_r.

Since the starting ball is contained in $\operatorname{conv}\mathcal{P}$, and ε is smaller than the radius of this ball, 2ε is strictly smaller than the width of $\mathcal{P}$. This yields that $\mathcal{S}$ can not be a subset of the hyperplane A.

The union of all balls centered on the line C and containing $p_2,\ldots,p_{n+1}$ cover $\mathbb{E}^n\backslash A$, therefore there exist such a ball which is not $\mathcal{S}$-disjoint. Now the usual transformation reasoning can be applied, leading to the ball B_{n+1}. $\qquad\square$

Now we are able to prove Theorem 2.0.2.

Proof of Theorem 2.0.2. First we will show that set $\mathcal{P}$ is dense in its convex hull, the we prove that the convex hull is the whole space $\mathbb{E}^n$.

Suppose on the contrary that $\mathcal{P}$ is not dense in the convex hull, and apply Theorem 2.0.1 to the closed point set $\overline{\mathcal{P}}$ with the constant $\lambda = (1+\delta)/2$. We obtain an essentially empty ball B with points $p_1,p_2,\ldots,p_{n+1}$ of $\overline{\mathcal{P}}$ on the boundary. Choose the points $q_1,q_2,\ldots,q_n$ of $\mathcal{P}$ close enough to the points $p_1,p_2,\ldots,p_n$, respectively, such that for the ball B' determined by them, $\varrho B' \subset \lambda B$ holds. It is easy to check that this is valid if the distance between p_i and q_i is less than $r(1-\delta)/(4+2\delta)$ for every $i = 1,2,\ldots,n+1$, where r is the radius of B. By the assumption of Theorem 2.0.2 there is a point of $\mathcal{P}$ in the ball $\varrho B'$, but this contradicts the essential emptiness of B. Therefore $\mathcal{P}$ is dense in its convex hull.

Now suppose that the convex hull is not $\mathbb{E}^n$. Since $\operatorname{conv}\mathcal{P}$ is closed, there is an open n-dimensional ball in $\mathbb{E}^n \setminus \operatorname{conv}\mathcal{P}$, and therefore there is a closed ball of this type as well. The existence of a separating hyperplane between any two disjoint closed convex sets of $\mathbb{E}^n$ yields that $\operatorname{conv}\mathcal{P}$ must be contained in a closed half-space determined by a

hyperplane. Let $\mathcal{H}$ denote this half-space, and let $\mathcal{G} = \mathbb{E}^n \setminus \mathcal{H}$. We can also suppose that the distance between $\mathcal{G}$ and $\operatorname{conv} \mathcal{P}$ is 0.

Choose a point Q in $\mathcal{G}$ and a ball B centered at Q such that B intersects $\operatorname{conv} \mathcal{P}$ while the ball λB is contained in $\mathcal{G}$ with the constant λ defined above. Since the convex hull is not degenerated, we can also suppose that the intersection $B \cap \operatorname{conv} \mathcal{P}$ is not one point. Choose $n+1$ affinely independent points of $\overline{\operatorname{conv} \mathcal{P}} \cap \partial B$ and points of $\mathcal{P}$ close enough to them such that for the ball B' determined by the points chosen from $\mathcal{P}$, $\varrho B' \subset \lambda B$ holds as above. Then the existence of a point of $\mathcal{P}$ in $\varrho B'$ yields that the intersection of $\mathcal{P}$ and $\mathcal{G}$ is not empty, a contradiction. $\qquad \square$

Remarks. The assumptions of Theorem 2.0.1 do not imply the existence of a ball which has at least $n+1$ affinely independent points of the given set on its boundary and whose interior is $\mathcal{P}$-disjoint. To see this, consider the union of a hyperplane and a point outside of it.

The next example shows that the condition $\varrho < 1$ in Theorem 2.0.2 cannot be weakened. Let $\mathcal{P}$ be a dense subset of the open ring $R = \{x \in \mathbb{E}^2 \mid 1 < \|x\| < 2\}$. Then $\mathcal{P}$ satisfies the condition with $\delta = 1$, but it is not dense in its convex hull.

The following statement is an immediate corollary of Theorem 2.0.1:

Corollary 2.0.1. *Let $n \geq 1$, and fix the positive numbers $a_0, a_1, \ldots, a_n$. Let $\mathcal{P}$ be a set in $\mathbb{E}^n$ containing a subset of $n+1$ affinely independent points. If for any $n+1$ affinely independent points $p_0, p_1, \ldots, p_n$ of $\mathcal{P}$ there exists another point $p \in \mathcal{P}$ for which the ratio-equality $p_0 p : p_1 p : \cdots : p_n p = a_0 : a_1 : \cdots : a_n$ hold, then $\mathcal{P}$ is dense in $\mathbb{E}^n$.*

Yielding a complete answer for the orthocenter problem in the plane, we shall prove the following theorem:

Theorem 3.0.3. *If $\mathcal{P}$ contains at least three noncollinear points and is an orthocentrically closed set of the plane, then exactly one of the following holds true:*

i) $\mathcal{P}$ consists of the vertices of a triangle together with its orthocenter.

ii) $\mathcal{P}$ is a discrete infinite subset of a rectangular hyperbola $\mathcal{H}$, parameterizable by two parameters.

iii) $\mathcal{P}$ is a dense subset of a rectangular hyperbola $\mathcal{H}$.

iv) $\mathcal{P}$ is a dense point set in the plane.

The proof will be divided into several parts. In Section 3.1 we review geometrical facts known about rectangular hyperbolas. Section 3.2 describes specific subproblems and show how to fit them together to prove Theorem 3.0.3. The subsequent Sections 3.4-3.6 will contain detailed solutions of these subproblems.

3.1 Review of elementary properties of hyperbolas

The curve with equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

is called a *hyperbola* in standard position. The lines $y = (b/a)x$ and $y = -(b/a)x$ are its *asymptotes*. If $b = a$, then the asymptotes are perpendicular lines and the hyperbola is

called *rectangular.* If a rectangular hyperbola is rotated by $\pi/4$, the asymptotes become the x- and y-axis, and it turns out that the new equation of the hyperbola is $xy = \alpha$, where α is a constant. A very straight and simple computation using coordinates shows the following two properties:

Hyperbola Property 3.1.1. *If a rectangular hyperbola passes through the vertices of a given triangle, then it also passes through the orthocenter of that triangle.*

Hyperbola Property 3.1.2. *If one extends any chord (connecting to points of the same branch) of a rectangular hyperbola at both ends until the line of extension intersects the coordinate axis, then these two segments are of equal length.*

Equivalently to the last property we have the following:

Hyperbola Property 3.1.3. *Assume a point P and a rectangular coordinate system is given. One can construct point-by-point the unique rectangular hyperbola (in this coordinate system) passing through point P in the following way: Take any line ℓ containing P and label by X and Y its intersection points with the x and y-axes, respectively. Then the reflection of point P around the midpoint of segment XY gives a point P' on the hyperbola.*

A theorem called *Feuerbach's Conic Theorem* (proved by Brianchon and Poncelet in [3] in 1821, see e. g. Problem 46 in [8]) says that

Hyperbola Property 3.1.4. (Feuerbach Conic Theorem) *The locus of the centers of those rectangular hyperbolas which pass through three given noncollinear points (and thus, according to Hyperbola Property 3.1.1, also pass through the orthocenter of this triangle) is the so called* nine point circle *(i.e. the circle which contains the midpoints*

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of the sides, the points half way from the orthocenter to the vertices, and the feet of the
altitudes).

As an immediate consequence of the Feuerbach Conic Theorem we have:

Hyperbola Property 3.1.5. *If four points P, Q, R and S are given such that no three of them are collinear and point S is different from the orthocenter of the triangle PQR, then there is a unique rectangular hyperbola passing through them.*

Indeed, consider the Feuerbach circles of the triangles PQR and PQS. Since S is not the orthocenter of the triangle PQR, the two Feuerbach circles are different and thus they either intersect each other in two points or they are tangent to each other. The midpoint Y of PQ is a common point of the Feuerbach circles. Let X be the other common point (in case of tangency let $X = Y$). X and Y are the only candidates for the center of a perpendicular hyperbola containing the given four points. There is an important difference between X and Y: In case of X the axis of the two rectangular hyperbolas containing the triplets P, Q, R and P, Q, S coincide and thus the two hyperbolas are identical. This does not hold for Y, unless $X = Y$.

3.2 Outline of the proof of Theorem 3.0.3

According to the Introduction a set $\mathcal{P}$ is *orthocentrically closed* if it contains the orthocenters of every three noncollinear points of $\mathcal{P}$. It was also mentioned that the vertices of a triangle together with their orthocenter form an orthocentrically closed set. Case i) in Theorem 3.0.3 is the case of orthocentric quadruples.

Let $\mathcal{P}$ be an orthocentrically closed point set and assume $\mathcal{P}$ contains four points, such that no three of them is collinear and the four points do not form an orthocentric

quadruple. It is known (see Hyperbola Property 3.1.5) that there exist a rectangular hyperbola $\mathcal{H}$ passing through these four points. According to Hyperbola Property 3.1.1 the intersection $P \cap \mathcal{H}$ is also an orthocentrically closed point set. Moreover, it was proved in [15] that $\mathcal{P}$ is unbounded. Any subset of a hyperbola is either a discrete point set or it has at least one limit point. We will show in Section 3.3 that the existence of one single limit point of $P \cap \mathcal{H}$ implies that $P \cap \mathcal{H}$ is a dense subset of the hyperbola $\mathcal{H}$, i.e. it is of type iii) in Theorem 3.0.3. The case when $\mathcal{P}$ is an infinite discrete point set is listed as case ii) in Theorem 3.0.3. We will address the existence of sets of type ii) in Sections 3.4 and 3.5 by giving both the geometric and the algebraic characterizations.

We will show in Section 3.6 that the existence of one additional point of $\mathcal{P}$ outside the rectangular hyperbola $\mathcal{H}$ (which already contains infinite many points of $\mathcal{P}$) implies that $\mathcal{P}$ is a dense point set of the plane. Thus we have that iv) is the only additional possibility besides i), ii) and iii) in Theorem 3.0.3.

3.3 If $\mathcal{P}$ has a convergent subsequence on the hyperbola $\mathcal{H}$, then $\mathcal{P}$ is a dense set of $\mathcal{H}$.

Again let $\mathcal{H}$ be a rectangular hyperbola with equation $xy = \alpha$. Assume on the contrary that the existence of a convergent subsequence of $\mathcal{P}$ on the hyperbola $\mathcal{H}$ does not imply that $\mathcal{P}$ is a dense set of $\mathcal{H}$. Without loss of generality we may assume that there is a convergent sequence of points $P_i \in \mathcal{P}(i = 1, \dots)$, whose x-coordinates form a positive, increasing, convergent sequence. Let P be the limit point of this sequence (Figure 3.1). Furthermore, assume that there is a point Q on the same branch of $\mathcal{H}$ such

that the open hyperbola arc (PQ) does not contain points from $\mathcal{P}$ (note that P and Q do not necessarily belong to $\mathcal{P}$).

Starting with P_1, for a sufficiently small $\epsilon > 0$ we are going to choose subsequently four points P_1, P_i, P_j and P_k from the above sequence so that each succeeding point is significantly closer (say ϵ times closer) to P than the previous one.

Let O be the orthocenter of the triangle $P_1 P_j P_k$. Notice that O belongs to the other branch of hyperbola $\mathcal{H}$. Finally consider the orthocenter O^* of the triangle $P_1 P_i O$. Obviously O^* is a point on the first branch of $\mathcal{H}$ again. Moreover, if ϵ is sufficiently small, then both P_j and O^* are arbitrarily close to P, while they are separated by P, therefore O^* belongs to the open hyperbola arc (PQ), a contradiction.
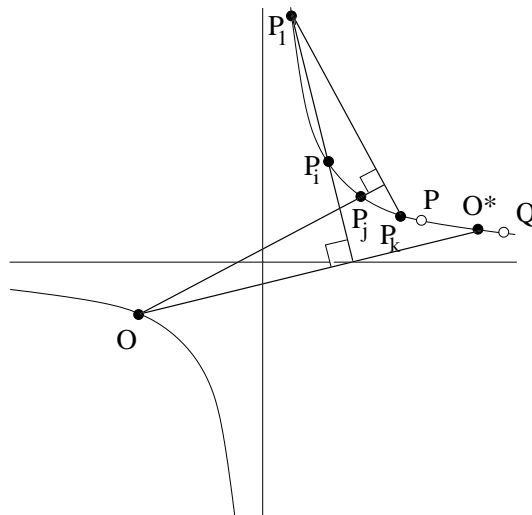

Figure 3.1: The case of a convergent sequence

3.4 Geometric characterization of infinite, discrete and orthocentrically closed sets of a rectangular hyperbola

In this section, we characterize all infinite and discrete orthocentrically closed subsets of a rectangular hyperbola. It turns out that discreteness implies a symmetrical structure for the diagonals (segments connecting pairs of points of the point set) which best can be compared to the diagonal structure of the even-sided regular polygons: for each diagonal there are both parallel and perpendicular sets of diagonals covering all vertices of the polygon (except at most two vertices). In order to give the precise characterization we need the following notations.

We will refer to the branches of the given rectangular hyperbola $\mathcal{H}$ as positive and negative branches (denoted by $\mathcal{H}^+$ and $\mathcal{H}^-$) according to the signs of the x-coordinates of their points.

Assume that set $\mathcal{P}$ is a discrete, infinite, orthocentrically closed subset of a rectangular hyperbola $\mathcal{H}$. Denote by $\mathcal{P}^+$ (and $\mathcal{P}^-$) the set of those points of $\mathcal{P}$ which belong to the positive branch (negative branch respectively) of $\mathcal{H}$. It was proved in [15] that $\mathcal{P}$ is an unbounded set. In fact, it is easy to prove, that both $\mathcal{P}^+$ and $\mathcal{P}^-$ are unbounded sets, moreover, they are unbounded sets along both direction of the hyperbola branches. Let us label the points of the set $\mathcal{P}^+$ using all integers in the order which matches the order of the x-coordinates of the points.

Let i, j and k be three different points of $\mathcal{P}^+$. We already know that the intersection point of the negative branch $\mathcal{P}^-$ with the line perpendicular to ij through k belongs to the set $\mathcal{P}$, since it is the orthocenter of the triangle ijk. It is not automatic that similar

perpendicular projections of the points i and j also give points of $\mathcal{P}^-$. Before we state this property together with some additional ones we introduce the following notations:

We define the $\mathcal{P}$-length of a diagonal ij in $\mathcal{P}^+$ (or in $\mathcal{P}^-$) by the number of points of set $\mathcal{P}^+$ (of set $\mathcal{P}^-$ respectively) contained by the open hyperbola arc (ij).

Let O_{ijk} (where i, j, k are different points of $\mathcal{P}$) be the orthocenter of the triangle ijk.

Denote by π_{ij} (where $i, j \in \mathcal{P}^+$) the projection from the branch $\mathcal{P}^+$ to the branch $\mathcal{P}^-$ along the direction which is perpendicular to the line ij.

Denote by "Shift" the map within the set $\mathcal{P}^-$ (or $\mathcal{P}^+$), where each point of $\mathcal{P}^-$ (of $\mathcal{P}^+$) is mapped to its neighbor along the same direction of the hyperbola branch $\mathcal{H}^-$ ($\mathcal{H}^+$ resp.).

We will prove the following characterization:

Theorem 3.4.1. *An infinite discrete point set $\mathcal{P}$ of the rectangular hyperbola $\mathcal{H}$ is orthocentrically closed if and only if it satisfies the following three conditions:*

1. For each $i, j \in \mathcal{P}^+$ the projection π_{ij} gives a one-to-one correspondence between the points of $\mathcal{P}^+$ and $\mathcal{P}^-$.

2. For each diagonal d in $\mathcal{P}^+$ it is true that each point of $\mathcal{P}^+$ is the endpoint of a diagonal parallel to d (this condition includes tangent lines as degenerate diagonals).

3. For each diagonal in $\mathcal{P}^-$ there is a parallel diagonal in $\mathcal{P}^+$.

Proof of Theorem 3.4.1. Since this theorem is an "if and only if" theorem, we need to show that the listed conditions are both necessary and sufficient conditions. We start addressing the "necessary" part by proving a chain of elementary facts:

1. First we prove Condition 1 for diagonals of $\mathcal{P}$-length equal to two: Without loss of generality consider the diagonal 14 of $\mathcal{P}^+$, which has length three. The only nontrivial part of Condition 1 is to show that the lines perpendicular to the diagonal 14 through its endpoints intersect $\mathcal{H}^-$ at points of $\mathcal{P}$. Consider the perpendicular line through endpoint 1, and assume indirectly that the intersection point F is not a point of $\mathcal{P}^-$. We indicate this on Figure 3.2.a by an "empty circle" as a point F. Denote the orthocenter O_{123} by E, which is a point of $\mathcal{P}^-$ different from F. We distinguish two cases:

Case 1: *The x-coordinate of E is greater than that of F.* Consider the triangle $E14$ and its orthocenter (Figure 3.2.a). The ray connecting E to the orthocenter must lie between rays $\overrightarrow{E2}$ and $\overrightarrow{E1}$. This means that the orthocenter of triangle $E14$ is contained by the open hyperbola arc (12), a contradiction.
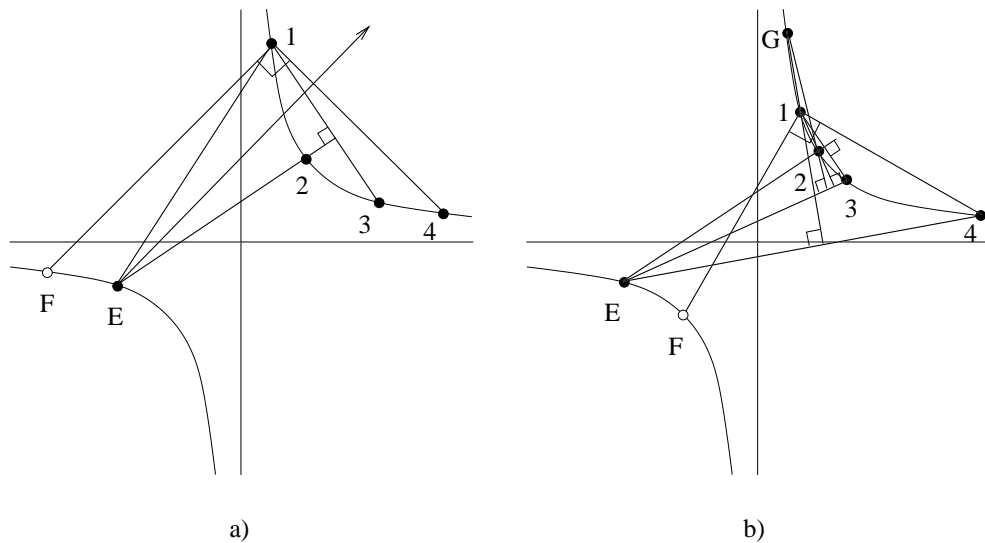

Figure 3.2: The set is closed under the projection π_{ij}

Case 2: *The x-coordinate of E is smaller than that of F.* Denote the orthocenter of the triangle $E14$ by G (Figure 3.2.b). G is a point of $\mathcal{H}^+$. The x-coordinate of G

must be less than that of point 1. Consider the triangle $EG2$ and its orthocenter. The ray connecting E to this orthocenter must be between rays $\overrightarrow{E4}$ and $\overrightarrow{E2}$. This means that the orthocenter of the triangle $EG2$ is point 3, but then there are two different perpendiculars to this line through point 2, a contradiction.

2. We prove Condition 1 for diagonals of $\mathcal{P}$-length equal to one: Without loss of generality consider the diagonal 13 of $\mathcal{P}^+$, which has length one. The only nontrivial part of Condition 1 is to show, that the lines perpendicular to diagonal 13 through the endpoints intersect $\mathcal{H}^-$ at points of $\mathcal{P}^-$. It is enough to show this for one of the endpoints, say for endpoint 3. We already know that the images of points $1, 2, 3$ and 4 under the perpendicular projection π_{14} belong to $\mathcal{P}^-$ and they are consecutive points in that set. Denote them by A, B, C and D (Figure 3.3.a). We also know that A is the orthocenter of the triangle 123, thus $\pi_{12}(3) = A$. Since points 3 and 4 are consecutive points in $\mathcal{P}$ we have that $\pi_{12}(4) = B$. The orthocenter of the triangle 134 must be between B and D on $\mathcal{H}^-$, thus it is equal to C. This leaves only one option for the orthocenter of the triangle $13B$ and that is point 3, which we wanted to prove.

3. We prove Condition 1 for diagonals of $\mathcal{P}$-length zero: Without loss of generality consider the diagonal 23 of $\mathcal{P}^+$, which has length zero. It is enough to show that the diagonals 23 and 14 are parallel.

Denote, again, by A the image of point 1 under the perpendicular projection π_{14}. From the paragraph labeled as 1. above we already know that the perpendicular projection π_{13} maps point 2 to point A, i.e. A is the orthocenter of the triangle 123, thus 23 is perpendicular to $A1$. Since $A1$ is perpendicular to 14, we get that 23 and 14 are parallel.
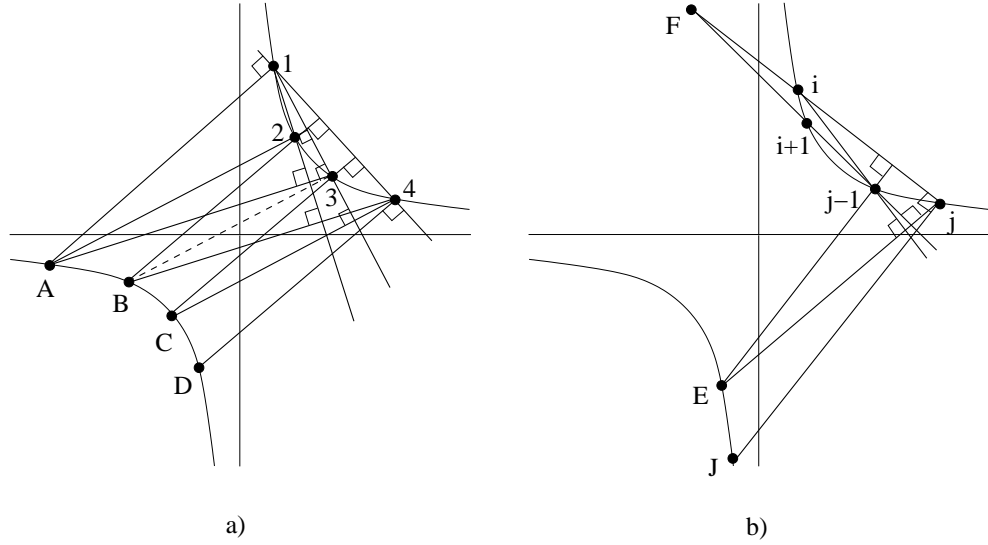
Figure 3.3: Inductive steps

4. We prove Condition 1 and 2 for diagonals of $\mathcal{P}$-length $5, 6, \ldots$: We are going to use induction on the length of the diagonals of $\mathcal{P}^+$. Assume that we already proved Condition 1 and 2 for all diagonals of $\mathcal{P}$-length less than N. Consider a diagonal ij of length N, i.e. $j = i + N + 1$ (see Figure 3.3.b). All we need to prove is that diagonal ij is parallel to diagonal $(i+1)(j-1)$. On the contrary assume that these two diagonals intersect each other, say at F, so that i separates F and j. According to Condition 1 there is a point $J \in \mathcal{P}^-$ so that the $\pi_{ij}(j) = J$. Denote $\pi_{ij}(j-1) \in \mathcal{P}^-$ by E. Since $j-1$ and j are consecutive points of $\mathcal{P}^-$ both triangles $i(j-1)j$ and $(i+1)(j-1)j$ must have their orthocenter adjacent to J, therefore at E. Thus both diagonals $i(j-1)$ and $(i+1)(j-1)$ are perpendicular to the line jJ, a contradiction.

5. We prove Condition 3: Consider a diagonal ij of $\mathcal{P}^+$. According to Condition 1 there are points, say I and $J \in \mathcal{P}^-$ so that the $\pi_{ij}(i) = I$ and $\pi_{ij}(j) = J \in \mathcal{P}^-$. Project the point j onto $\mathcal{P}^-$ perpendicularly to each of the diagonals $ij, i(j-1), i(j-2), \ldots (i,i)$.

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The first image, as we know, is J, and the last image is I. Notice that the images at the projections are different and the number of diagonals we consider equals the number of points on the hyperbola arc IJ. Thus we have that "moving one endpoint of the diagonal ij to its neighbor implies that the perpendicular projection of that endpoint moves to its neighbor on the other hyperbola branch". This means that if we start by two diagonals belonging to different hyperbola branches, then we can change gradually one of them (by moving one of its endpoint to a neighboring point) until the perpendicular direction of the changed diagonal coincides with the perpendicular direction of the unchanged diagonal.

6. We prove that Conditions $1, 2$ and 3 are sufficient in Theorem 3.4.1: We need to verify that such sets are orthocentrically closed. Take three points from $\mathcal{P}$. If they belong to the same branch of the hyperbola then Condition 1 implies that the orthocenter of this triangle belongs to $\mathcal{P}$. Without loss of generality assume that two of the vertices, say i and j belong to $\mathcal{H}^+$ and the third one, say O, belongs to $\mathcal{H}^-$. According to Condition 3, $\mathcal{P}^-$ has a diagonal, say d, which is parallel to ij. According to Condition 1, the projection of O onto $\mathcal{H}^+$ in the direction perpendicular to the diagonal d, belongs to $\mathcal{P}^+$, which we wanted to show. □

3.5 Existence and algebraic characterization of the infinite, discrete and orthocentrically closed sets of the rectangular hyperbola

Let $\mathcal{H}$ be the rectangular hyperbola defined by the equation $xy = \alpha$. The points of $\mathcal{H}$ can be identified with their x-coordinates, and this gives an isomorphism between $\mathcal{H}$

and the x-axis except the origin. Throughout the section $h(x)$ will denote the point of $\mathcal{H}$ with the given abscissa x. In other notations we follow Section 3.4.

We will show that there exists a discrete subset of $\mathcal{H}$ which is orthocentrically closed. So far we have proved a geometric necessary and sufficient condition for such a point set detailed in Theorem 3.4.1; easy algebraic arguments will yield that there really exist such a point set. In fact we give an algebraic characterization for these sets, also yielding that there are infinitely many of them on the same rectangular hyperbola.

Suppose that $\mathcal{P}$ is a discrete subset of $\mathcal{H}$ with the given property. Let $h(x_0)$ be an arbitrary point of $\mathcal{P}^+$. Let the points of $\mathcal{P}^+$ be $\{h(x_i)\}_{i=-\infty}^{\infty}$, labeled in accordance with the ordering of the abscissas, so $0 < x_j < x_k$ holds for any integers $j < k$.

Condition 2 of Theorem 3.4.1 yield the following parallelism property:

$$h(x_j)\, h(x_k) \parallel h(x_{j-d})\, h(x_{k+d})$$

for any integers j, k and d with $j < k$.

Using the equation of the hyperbola this condition turns into the following algebraic equality:

$$x_j\, x_k = x_{j-d}\, x_{k+d} \tag{3.1}$$

With the help of (3.1) we can express all x_i's in terms of x_0 and x_1. Introducing the notations $a := x_0$ and $b := x_1$ will simplify the formulae.

We shall prove by induction on $|i|$ that the following formula holds for every integer i.

$$x_i = b^i\, a^{1-i} \tag{3.2}$$

Clearly (3.2) is true for $i = 0, 1$. First we show it for $i = -1$. Applying (3.1) for $j = 0$, $k = 1$ and $d = 1$ yields that $x_2 = ab/x_{-1}$. On the other hand, the choice $j = -1$, $k = 0$ and $d = 1$ gives $x_{-2} = x_{-1}a/b$. Now in the case $j = -1$, $k = 1$ and $d = 1$ we obtain the equality $x_{-1} = a^2/b$, and this was our goal.

Suppose now that (3.2) is proved for all i with $|i| < n$. Applying (3.1) for $j = 0$, $k = 1$ and $d = n - 1$ we obtain

$$x_n = ab/(b^{1-n} a^n) = b^n a^{1-n}.$$

Furthermore, the choice $j = -1$, $k = 0$ and $d = n - 1$ yields

$$x_{-n} = b^{-1} a^3/(b^{n-1} a^{2-n}) = b^{-n} a^{1+n},$$

finishing the proof of (3.2).

So far we have described the points of $\mathcal{P}^+$. Now turn to the points with negative abscissa. We have proved in Condition 1. of Theorem 3.4.1 that those can be obtained by taking the orthogonal projection of $\mathcal{P}^+$ onto $\mathcal{H}^-$ with respect to an arbitrarily chosen diagonal of $\mathcal{P}^+$.

Let choose the diagonal to be $h(a) h(b)$. For any x_i let $h(x'_{i+1})$ denote the projection of x_i onto $\mathcal{H}^-$. The following holds:

$$x'_{i+1} = -\alpha^2/(x_i a b) = -\alpha^2 b^{-i-1} a^{i-2} = -\alpha^2 b^{-(i+1)} a^{-3+(i+1)}.$$

Therefore the negative part of $\mathcal{P}$ consists of all the points with x-coordinates of the form

$$x_i' = \alpha^2(-b^i\, a^{-3-i}), \tag{3.3}$$

where i is any integer.

According to Theorem 3.4.1 for the sufficiency we have to guarantee that for any given diagonal of $\mathcal{P}^-$, there exists a diagonal in $\mathcal{P}^+$ parallel to the given one. Translating to algebra we get the following statement: for any different integers i and j there exist integers $k \neq l$ such that

$$\alpha^4 b^{i+j} a^{-6-(i+j)} = b^{k+l} a^{2-(k+l)},$$

or, equivalently,

$$\exists s \in \mathbb{Z} : (b/a)^s = (a/\sqrt{\alpha})^8. \tag{3.4}$$

Since all the algebraic transformations done so far are reversible, we have proved the following theorem:

Theorem 3.5.1. *An infinite discrete subset of the rectangular hyperbola $\mathcal{H}$ is orthocentrically closed if and only if it is described by the forms (3.2) and (3.3) where $a < b$ are positive real numbers satisfying the condition (3.4).*

We note that for the choice $a = \sqrt{\alpha}$ condition (3.4) holds for any positive number b with $s = 0$. In this case the point set $\mathcal{P}$ is symmetric to the line $x = y$. Also notice that for any given real number $a > 1$ and integer s, the equation (3.4) has a solution for b; in other words, for any fixed point of the hyperbola there are infinitely many different, discrete, orthocentrically closed subsets containing that point.

3.6 If $\mathcal{P}$ is not on the hyperbola, then it is a dense set of the plane.

We shall now show that if an orthocentrically closed point set is not contained in a rectangular hyperbola, then it must be dense in the whole plane. We are going to use the notations of Section 3.4. We will also label the points of $\mathcal{P}^-$ by the accented integers (n') with the order which matches the order of the x-coordinates.

We can assume that the point set $\mathcal{P}$ is closed. Indeed, if a planar point set is orthocentrically closed, then the same property holds for its closure: taking any nondegenerate triplet of the closure, and point sequences converging to the vertices, the orthocenters of the triplets of the sequences will converge to the orthocenter of the given triangle. This yields that we should prove that a closed and orthocentrically closed set is either a subset of a rectangular hyperbola or equals to the whole plane itself.

Our argument is divided into two steps: the first step is to show that if the closed set $\mathcal{P}$ contains an infinite orthocentrically closed discrete subset of a rectangular hyperbola, and also a point outside of the hyperbola, then it contains the whole hyperbola. The second step is to prove that if the closed set $\mathcal{P}$ contains an entire rectangular hyperbola and one additional point, then it is the whole plane.

Lemma 3.6.1. *Suppose the $\mathcal{P}$ is an infinite discrete orthocentrically closed point of the rectangular hyperbola $\mathcal{H}$. Then for any point P outside of $\mathcal{H}$, the set $\mathcal{P} \cup \{P\}$ is generating a dense subset of $\mathcal{H}$ by the orthocenter transformation.*

Proof. As an extension of our former definition, let $\pi_{ij}(P)$ denote the intersection points of the hyperbola $\mathcal{H}$ and the line perpendicular to ij through P. First we show that there exists a diagonal ij of $\mathcal{P}^+$ such that the two points of $\pi_{ij}(P)$ do not belong to $\mathcal{P}$. Indeed, choose the diagonals $n(2n)$ and let n tend to infinity. Then the direction vectors

of the diagonals are tending to the horizontal direction, and therefore unless the given point P is on the y-axis, the projections have a limit point on the hyperbola. Since the set $\mathcal{P}$ is discrete, there must be a diagonal of the chosen ones for which the projection does not belong to $\mathcal{P}$. We can use the same argument for the diagonals $(-n)(-2n)$ yielding the same result for any point P outside of the x-axis. Therefore the statement holds for any point except for the origin. Finally, if P is the origin, and $\pi_{ij}(P)$ belongs to $\mathcal{P}$ then it is easy to see that $\pi_{i(j+1)}(P)$ does not belong to $\mathcal{P}$: Theorem 3.4.1 yields that the projections of $\mathcal{P}^+$ are shifted by one when taking $\pi_{i(j+1)}$ instead of π_{ij}, and therefore two projection lines cannot intersect in the region between the two branches of the hyperbola.

We can also assume that the diagonal ij has the property that the projection of P onto the diagonal is a point of $\mathcal{H}$ between i and j. Moreover, we can choose a diagonal $k'l'$ of $\mathcal{P}^-$ parallel to ij and satisfying the same condition. Finally, let e denote the line perpendicular to ij and passing through P (see Figure 3.4.a).

We will define an infinite point set on e by taking the orthocenters of certain triplets, and we shall show that the intersections of e and the hyperbola $\mathcal{H}$ belong to the closure of our infinite point set, yielding a contradiction.

Let D_p and D_n denote the Thales discs of the segments ij and $k'l'$, respectively. We claim that $D_p \cap D_n$ is empty. Indeed, if we take points Q and R as the intersection points of the diagonal ij with the y-axis and the x-axis, then the segment ij is strictly contained in the segment QR. We obtain the segment ST in the same manner from $k'l'$. Now, the Thales discs D_p and D_n are strictly contained in the Thales discs of QR and

ST, while an easy geometric observation yields that those are touching each other at the origin, hence the intersection of D_p and D_n must be empty.

Orient e such that the points of $e \cap D_n$ are "below" the points of $e \cap D_p$. We note that if we take a point M of e outside of D_p, then O_{ijM} (the orthocenter of the triangle ijM) is in the intersection of e and D_p. Moreover, if we take two points M_1 and M_2 from the same component of $e \setminus D_p$ such that $M_1 < M_2$, then $O_{ijM_1} > O_{ijM_2}$ holds as well. The same observations hold for D_n in place of D_p.

Let $P_0 = P$ and define the sequence $\{P_n\}_{n=0}^{\infty} \subset e$ in the following way: if P_n is not in D_p then let $P_{n+1} = O_{P_n ij}$, otherwise let $P_{n+1} = O_{P_n k'l'}$. It is easy to see that for $n \geq 2$ the point P_n is on the open segment bounded by the intersection points of e with ij and $k'l'$. Without loss of generality suppose that P_2 is outside of D_p, therefore P_3 is in D_p, and it is easy to see that P_{2n} is in D_n while P_{2n+1} is in D_p for any $n \geq 2$.

If $P_2 \geq P_4$ holds, then by induction on n we obtain easily that $P_{2n-1} \leq P_{2n+1}$ and $P_{2n} \geq P_{2n+2}$ holds true for any $n \geq 2$, while if $P_2 \leq P_4$, then $P_{2n-1} \geq P_{2n+1}$ and $P_{2n} \leq P_{2n+2}$ for any $n \geq 2$. This means that the subsequences $\{P_{2n+1}\}_{n=1}^{\infty}$ and $\{P_{2n}\}_{n=1}^{\infty}$ are monotonic and bounded, therefore their limits exist. Let L_1 and L_2 denote the limit points, respectively. Then $L_1 \in D_p$ and $L_2 \in D_n$.

Note that the following property holds for the limit points: L_2 is the orthocenter of the triangle $k'l'L_1$ and L_1 is the orthocenter of the triangle ijL_2. We claim that L_1 and L_2 must be the intersection points of the hyperbola with the line e.

We refer to Hyperbola Property 3.1.5. Since the points i, j, k' and l' do not form an orthocentric system (remember that $ij \parallel k'l'$), it suffices to show that there is a rectangular hyperbola containing all the six points i, j, k', l', L_1 and L_2. Take a rectangular

hyperbola through i, j, L_2 and k'. Then this hyperbola also contains L_1, since this is the orthocenter of the triangle ijL_2. Moreover, l' is also on the hyperbola, being the orthocenter of the triangle $k'L_1L_2$. Therefore such a hyperbola exists, and by uniqueness it is $\mathcal{H}$. Hence the intersection points of e and $\mathcal{H}$ belong to the closure of the orthocentrically generated point set, contradicting the choice of the diagonal ij. $\qquad\square$
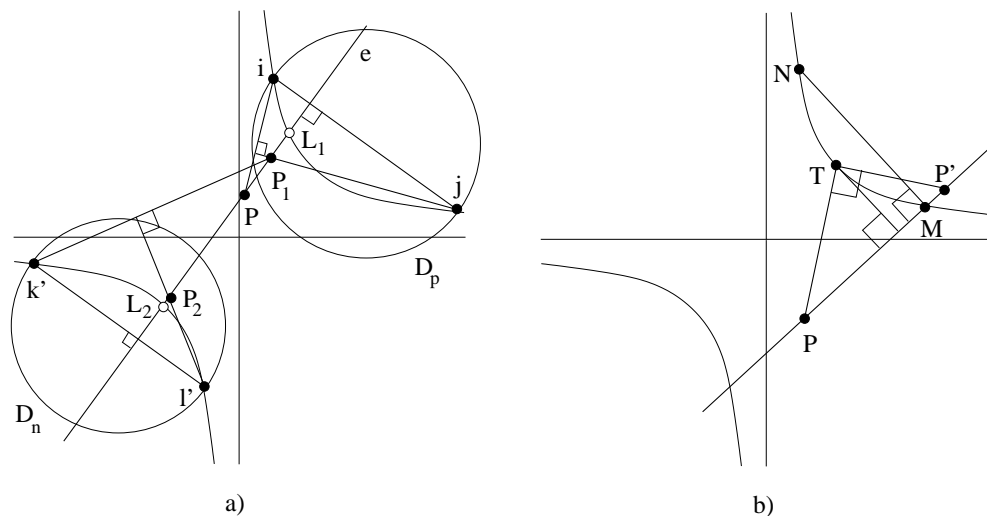

Figure 3.4: A point outside the hyperbola is generating a dense set

The final step in the proof of Theorem 3.0.3 is to show that

Lemma 3.6.2. *Suppose that $\mathcal{H}$ is a rectangular hyperbola. For any point P outside of $\mathcal{H}$, the set $\mathcal{H}\cup\{P\}$ is generating an everywhere dense subset of the plane by the orthocenter transformation.*

Proof. Let us choose a line ℓ through P intersecting $\mathcal{H}^+$ in the point M. Moreover, assume that the angle between ℓ and the tangent of $\mathcal{H}$ at M is not $\pi/2$. Let m denote the slope of a line perpendicular to ℓ. Consider a tangent with slope m of the positive branch of $\mathcal{H}$, and let T be the common point of this tangent line and the hyperbola. Let

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N denote the second intersection point of the positive branch of $\mathcal{H}$ and the line through M with slope m (see Figure 3.4.b).

Consider the family of triangles with common vertex P and with two other vertices on $\mathcal{H}^+$ so that their connecting segment has slope m. The orthocenters of these triangles are on line ℓ. If the vertices different from P converge to point T, then the orthocenters of these triangles converge to point P' of line ℓ for which $\angle P'TP = \pi/2$. On the other hand, if we choose the two variable vertices of the triangle as M and N, then the resulted orthocenter is M. If we change continuously the two vertices of the triangle between these limits, the orthocenters are changing continuously between M and P'. Therefore the whole segment MP' belongs to the closure of the set of the orthocentrically generated points.

The same argument with P' in the place of P yields that the segment PM is also generated. Now let the point Q run along the segment PP', what we already know to be orthocentrically generated. It is easy to see that the points $Q' \in \ell$ for which $\angle QTQ' = \pi/2$ cover the whole line ℓ, hence ℓ is orthocentrically generated.

The same argument works if we replace the positive branch of the hyperbola with the negative branch. Moreover it is easy to check that except for the line $y = x$ no line can be perpendicular to both branches of the hyperbola (where the angle between and the line are understood to the angle between the line and the tangent to the hyperbola at the intersection point). Therefore all the lines with positive slope passing through P, except (optionally) for the line $y = x$, are generated. The union of these lines is the union of two infinite domains bounded by the horizontal and vertical lines through P, what we call a *wedge with vertex P*. Now replace P by any point Q in the wedge

with vertex P, the same arguments yield that the wedge with vertex Q is also generated. Since the union of all of these wedges cover the plane, we obtain that the orthocentrically generated points form a dense subset of the plane. $\qquad\square$

[1] G. Ambrus, A. Bezdek and F. Fodor, A Helly-type transversal theorem for n-dimensional unit balls, to appear in *Archiv der Math.*

[2] G. Ambrus, A. Bezdek, On iterative processes generating dense point sets, to appear in *Period. Math. Hung.*

[3] C.-J. Brianchon and J.-V. Poncelet, Recherches sur la détermination d'une hyperbole équilatére, au moyen de quatre conditions données, , *Ann. des Math.* **1** (1821), no.4, 205-220.

[4] K. Bezdek and J. Pach, A point set everywhere dense in the plane, *Elem. Math.* **40** (1985), no.4, 81-84.

[5] O. Cheong, X. Goaoc, A. Holmsen, Helly type theorems for line transversals to unit balls in $\mathbb{R}^3$, (2004).

[6] O. Cheong, X. Goaoc, A. Holmsen and S. Petitjean, Helly-type theorems for line transversals to disjoint unit balls, to appear.

[7] L. Danzer, Über ein Problem aus der kombinatorischen Geometrie. *Arch. Math.* **8** (1957), 347–351.

[8] H. Dörrie, 100 Great Problems of Elementary Mathematics: The History and Solutions, New York: Dower,1965.

[9] B. Grünbaum, On common transversals, *Arch. Math.* **9** (1958), 465–469.

[10] B. Grünbaum, Common transversals for families of sets, *J. Lond. Math.Soc.* **35** (1960), 408–416.

[11] H. Hadwiger, Problem 107, *Nieuw Arch. Wiskunde* (3) **4** (1956), 57; Solution, *Wiskundige Opgaven* **20** (1957), 27–29.

[12] A. Holmsen, M. Katchalski, and T. Lewis, A Helly-type theorem for line transversals to disjoint unit balls, *Discrete Comput. Geom.* **29** (2003), no. 4, 595–602.

[13] A. Holmsen, and J. Matoušek, No Helly theorem for stabbing translates by lines in $\mathbb{R}^3$, *Discrete Comput. Geom.* **31** (2004), no. 3, 405–410.

[14] D. Ismailescu and R. Radoičić, A dense planar point set from iterated line intersections, *Computational Geometry*, **27** (2004), 257-267.

[15] M. Iorio, D. Ismailescu, R. Radoičić and M. Silva, On point sets containing their triangle centers, to appear.

[16] H. Tverberg, Proof of Grünbaum's conjecture on common transversals for translates, *Discrete Comput. Geom.* **4** (1989), 191–203.