An Exploratory Study of High School Geometry Teachers' Applied Content Knowledge

by

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Abstract

Results from the 2007 TIMSS showed the grim reality that the United States did not measure up to other industrialized countries in both fourth- and eighth-grade results in geometry (Mullis, Martin, & Foy, 2008). Furthermore, Clements (2003) concluded from literature on teaching and learning geometry in kindergarten through twelfth grade (K-12) that the U.S.'s curriculum and teaching are weak. Since teacher knowledge impacts student achievement, it is pertinent to study teacher knowledge as a subset of the solution to improving student achievement in geometry.

Case studies were conducted to see what connections there are between two high school geometry teachers' specialized content knowledge, knowledge of content and students, and knowledge of content and teaching (Hill, Ball, & Schilling, 2008) shown during the planning of lessons and the teachers' actual execution of the lessons. The three types of knowledge were collectively called Teachers' Applied Content Knowledge (TACK) for this study.

The two teachers chosen for the case study were dubbed as exemplary geometry teachers by the researcher. Each teacher participated in interviews and observations involving two units of her choice. One teacher was observed for 7 lessons, and the other, 6; all lessons were also of their choosing. Interviews and classroom observations were video taped and audio recorded. Classroom observations were recorded with a video camera and with a voice recorder that the teachers carried with them. Qualitative data

analysis was done through case study and grounded theory. These exemplary teachers' ability to execute what was planned with additional geometry TACK shown during observations was based on knowledge accumulated from many sources, but the most commonly referenced were professional development and reflections from previously taught lessons.

Acknowledgements

一隻蜂釀不成蜜，一粒米熬不成粥

(One bee makes no honey, one grain makes no rice soup.)

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欲窮千里目，更上一層樓

(To enjoy a grander sight, climb to a greater height)

Table of Contents

List of Tables

List of Figures

Chapter 1: Introduction

In 2010, President Obama highlighted the need for education in science, technology, engineering, and mathematics (STEM) in school systems in the United States in order for future generations to compete in a global marketplace. In the United States alone, Lacey and Wright (2009) projected 785,700 new jobs in computer and mathematical occupations for 2008 to 2018. As a group, these jobs “will grow more than twice as fast as the average for occupations in the economy” (p. 85). The notion that math is a necessary occupation requirement is not new. According to the National Research Panel (1989), many jobs require basic knowledge in algebra and geometry. A closer look at the 46 careers mentioned on WeUseMath.org (2011) showed 24 careers, which specifically named geometry as one of the types of mathematics required in careers like high school math teacher, urban planner, attorney, political scientist, and animator.

However, results from the 2007 TIMSS showed the grim reality that the United States did not measure up to other industrialized countries in both fourth- and eighth-grade results in geometry (Mullis, Martin, & Foy, 2008). On the national level, Main National Assessment of Education Progress (NAEP) results from 1990 to 2009, as presented in Figure 1-1, showed that geometry and measurement are still areas of need.

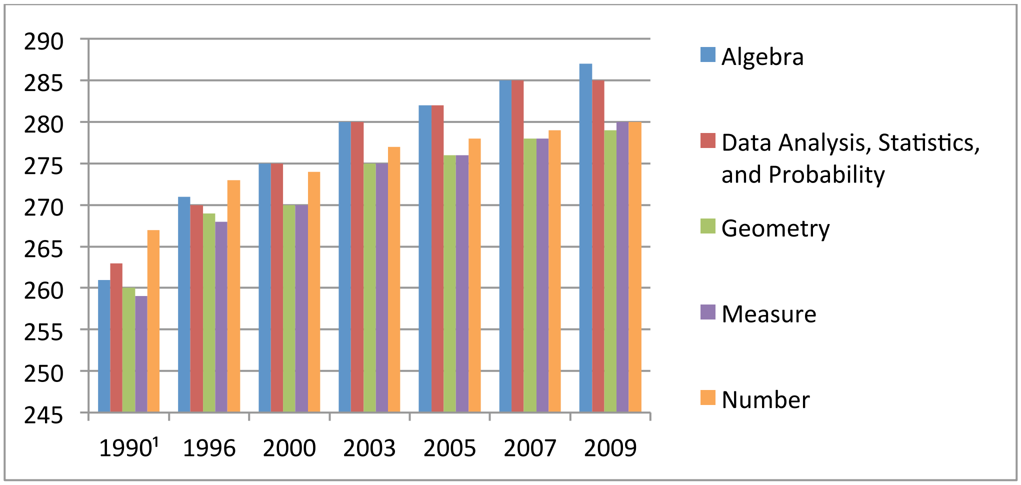


Figure 1-1 Main NAEP Average Scores of Topics in Mathematics by Years Tested (National Center for Education Statistics, 2010)

Michael Shaughnessy, in the October 2011 President's Message in the twice monthly newsletter *Summing Up* of the National Council of Teachers of Mathematics (NCTM), highlighted a concern that algebra is getting more attention in standards than geometry. "When states or national organizations develop sample assessment tasks, they usually begin the process with tasks that involve arithmetic operations or algebraic concepts and procedures. Geometry tasks are often lower on their priority list" (Shaughnessy, 2011, ¶ 1). The Common Core State Standards, a set of mathematics and English standards adopted by 45 states and four territories in the United States, emphasizes topics in mathematics such as operations on numbers, algebra, and functions, "putting geometry a tad off to the side" (Shaughnessy, 2011, ¶ 1). "Geometry is a crucial part of the mathematical education of our students and our citizens" (Shaughnessy, 2011, ¶ 6). Shaughnessy's (2011) concerns for geometry's place in school mathematics are not new. Nearly a decade prior, Glenda Lappan, in her (1999) President's Message for the National Council of Teachers of Mathematics, cited Trends in International Mathematics and Science Study (TIMSS) results up to 1995 as well as National Assessment of

Education Progress (NAEP) results up to 1996 as reasons for the need to enrich the geometry strand across all grades. Smith et al. (2005) made the observation that "geometry and measurement are the topic areas on which U.S. students have exhibited their worst performance on national and international assessments during and since the 1990s" (p. xi).

Clements (2003) concluded from literature on teaching and learning geometry in kindergarten through twelfth grade (K-12) that "U.S. curriculum and teaching in the domain of geometry is generally weak, leading to unacceptably low levels of achievement" (p. 152). Thus, one way of ensuring students get the education they need to "compete in an age where knowledge is capital, and the marketplace is global" (President Obama in the document from Office of Science and Technology Policy, 2010, p. vii) is to make sure that teachers have the content knowledge necessary.

Student achievement is linked to teacher knowledge (Hill, Rowan, & Ball, 2005; Monk, 1994). However, different aspects of teacher knowledge have been used to assess relationships between teacher knowledge and student achievement. For example, Hill, Rowan, and Ball, (2005) assessed teachers' knowledge through a multiple choice test based on common and specialized content knowledge and compared their results with student achievement; whereas Monk (1994) used the number of mathematics courses a high school teacher took as the level of subject matter knowledge and compared that to student achievement (further details of those studies will be expounded upon in chapter two).

Teacher Knowledge

Teacher knowledge has been a subject of concern in the past twenty years (Hill, Schilling, & Ball, 2004). In order to study teachers' knowledge, teachers' knowledge has been assessed; but why and how teachers are assessed have varied among the different assessments (Hill, Sleep, Lewis, & Ball, 2007). The political environment, that need to show what makes teaching a profession, and improving teacher input and output were some reasons for the necessity of building assessments of teacher knowledge (Hill, Sleep, Lewis, & Ball, 2007). Teachers' knowledge of teaching a topic should not be identical to an average educated person's knowledge of that topic since the teaching of a topic requires a deeper understanding of the topic (Ball, 2003; Hill, Sleep, Lewis, & Ball, 2007; Shulman, 1987). Improving teacher input and output refers to establishing "evidence on the effects of teacher education on teachers' capacity, and of teachers' knowledge and skill on their students' learning" (Hill, Sleep, Lewis, & Ball, 2007, pp. 111-112). It is now possible and necessary to develop a coherent approach to assessing teachers' knowledge, specifically, knowledge of teaching mathematics (Hill, Sleep, Lewis, & Ball, 2007).

"The past several years have seen ambitious policy making in the area of teacher quality and qualifications" (Hill, Sleep, Lewis, & Ball, 2007). For example, the No Child Left Behind (NCLB) Act of 2001

> require[d] states to set standards for designating all public school teachers as highly qualified and require[d] districts to notify parents of students in Title I programs if their child's teacher d[id] not meet these standards. The requirements appl[ied] to all teachers of core academic subjects—English, reading or language

> arts, mathematics, science, foreign languages, civics and government, economics, arts, history, and geography—and to teachers who provide[d] instruction in these subjects to students with limited English proficiency (LEP) and students with disabilities (Birman, Boyle, Le Floch, et al. 2009, p.xix)

Teachers' knowledge is assessed to determine whether or not teachers are "highly qualified" (Hill, Sleep, Lewis, & Ball, 2007). In September 2010, the Office of Science and Technology, from the President's Council of Advisors on Science and Technology made public a report called "Prepare and Inspire: K-12 Education in Science, Technology, Engineering, and Math (STEM) for America's Future". One of the recommendations in this report was to "Recruit and train 100,000 great STEM teachers over the next decade who are able to prepare and inspire students" (Office of Science and Technology, 2010, p. viii). According to the Office of Science and Technology (2010), "The most important factor in ensuring excellence is great STEM teachers, with both deep content knowledge in STEM subjects and mastery of the pedagogical skills required to teach these subjects well" (p. viii).

Knowledge of the skills and concepts of the mathematics being taught, although important, is not enough to teach it effectively (Ball, 2003; Suzuka, et al., 2009). Ball (2003) gave an example of teaching 0.3×0.7.

> Knowing how to multiply 0.3×0.7, and being able to produce efficiently the answer of 0.21, is not sufficient to explain and justify the algorithm to students. In teaching fifth graders, a student will likely ask why, in multiplication, you count the number of decimal places in the numbers you are multiplying and "count over" the same number of places in the product to place the decimal point

correctly. The student may point out that when you add two decimals, you simply line the numbers up:

$$\begin{array}{r} 0.3 \\ \times\ 0.7 \\ \hline 0.21 \end{array} \qquad \begin{array}{r} 0.3 \\ +0.7 \\ \hline 1.0 \end{array}$$

(p. 2)

Being able to do the calculations oneself is insufficient for being able to respond well. Even understanding the procedure in the formal terms that one might learn in a mathematics course may not equip one to explain it in ways that are both mathematically valid and accessible to fifth graders. The capacity to do this is a form of mathematical work that has been overlooked in the current discussions of improving teaching quality (Ball, 2003, p. 2).

From this example, the teacher needed to: 1. "Interpret and make mathematical and pedagogical judgments about students' questions, solutions, problems, and insights (both predictable and unusual)" (Ball, 2003, p. 6), and 2. "Be able to respond productively to students' mathematical questions and curiosities" (Ball, 2003, p. 6). Thus, mathematical knowledge for teaching not only needs to be built upon a firm foundation of understanding of mathematics content, but also incorporates instructional aspects specific to the teaching of mathematics (Ball, 2003).

Ball's (2003) example of teaching 0.3×0.7 illustrates how teacher knowledge can influence classroom instruction. Studies like Swafford, Jones, and Thornton (1997) and Hill et al. (2008) looked at teacher knowledge at the same time as observing classroom instruction, and they found evidence that teacher knowledge influenced classroom instruction. With regards to classroom instruction and student learning, according to Hiebert and Grouws (2007), one can logically claim that student learning is linked to classroom instruction. Jacobson and Lehrer (2000) and Pesek and Kirshner (2000) are just two examples of studies that explored classroom instruction and its impact on student achievement. Data from these studies substantiated Heibert and Grouws (2007) logical assertion that student learning is linked to classroom instruction.

Conceptual Framework

"A conceptual framework is an argument that the concepts chosen for investigation, and any anticipated relationships among them, will be appropriate and useful given the research problem under investigation" (Lester, 2010, p. 73). Summing up what has been discussed so far: student achievement in the area of geometry is an area of concern (Lappan, 1999; Mullis, Martin, & Foy, 2008); teacher knowledge influences student achievement (Hill, Rowan, & Ball, 2005; Monk, 1994); teacher knowledge impacts classroom instruction (Swafford, Jones, & Thornton, 1997; Hill et al., 2008); and classroom instruction impacts student achievement (Jacobson & Lehrer, 2000; Pesek & Kirshner, 2000). Figure 1-2 shows the summary of implications that have been discussed that are between teacher knowledge, classroom instruction, and student achievement.

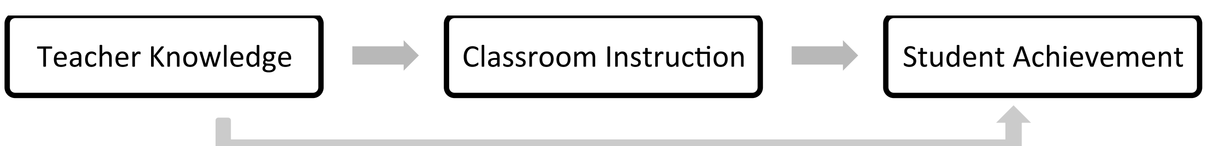


Figure 1-2 Teacher Knowledge, Classroom Instruction, and Student Achievement

Literature on student learning of geometry show that improving student achievement in geometry is necessary. This dissertation addresses this necessity by adding to existing literature on teacher knowledge; specifically by examining teachers' mathematical knowledge for teaching geometry in planning for and executing classroom instruction. Learning Mathematics for Teaching (LMT) Project defined common content knowledge, knowledge at the mathematical horizon, specialized content knowledge, knowledge of content and students, knowledge of content and teaching, and knowledge of curriculum as categories of mathematical knowledge for teaching. Shulman (1987) proposed many categories for types of knowledge needed to teach, and of those, subject matter knowledge and pedagogical content knowledge were two categories. Hill, Ball, and Schilling (2008) of Learning Mathematics for Teaching (LMT) Project published a domain map showing mathematical knowledge for teaching as it relates to Shulman's (1987) subject matter knowledge and pedagogical content knowledge.

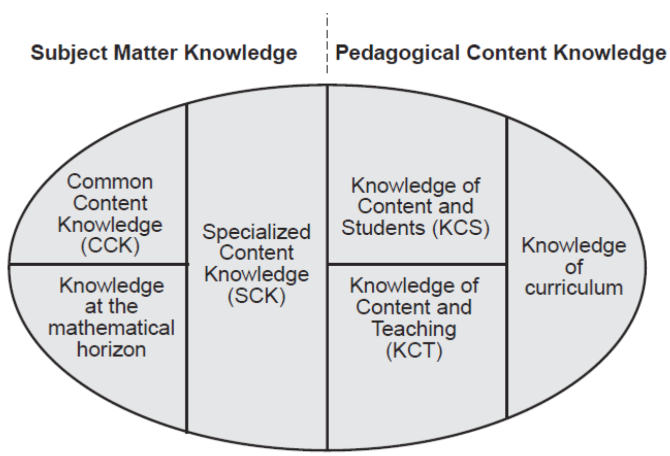


Figure 1-3 Domain map for mathematical knowledge for teaching as it relates to Shulman's (1987) pedagogical content knowledge**.** From "Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students." by H.C. Hill, D.L. Ball, and S.G. Schilling, 2008, *Journal for Research in Mathematics Education, 39*(4), p. 377. Copyright 2008 by *Journal for Research in Mathematics Education*. Reprinted with permission.

In this dissertation, I examined the middle slice of the domain map, that is – specialized content knowledge, knowledge of content and students, and knowledge of content and teaching. Specialized content knowledge is teachers' knowledge of mathematics used in teaching (Ball, Hill, & Bass, 2005) “but not directly taught to students” (Hill, Sleep, Lewis, & Ball, 2007). Knowledge of content and teaching is the knowledge of teaching the content with emphasis on teaching; e.g. how to correct student mistakes so that they will understand and how to build on what the students already know (Hill, Ball, & Schilling, 2008). Separate from both knowledge of content and teaching and specialized content knowledge; knowledge of content and students is the subset of pedagogical content knowledge that enables teachers to identify common student errors, interpret students' understanding of content, identify students' developmental sequences, and identify common student computational strategies (Hill, Ball, & Schilling, 2008).

Specialized content knowledge, knowledge of content and students, and knowledge of content and teaching are seen by the author of this dissertation as applications of content knowledge. Teachers are applying their content knowledge for explanations in specialized content knowledge. In knowledge of content and students, teachers are applying their knowledge of content in relation to how students learn; and similarly in knowledge of content and teaching, teachers are applying their knowledge of content in relation to how to teach it. Therefore, for the purposes of this dissertation; specialized content knowledge, knowledge of content and students, and knowledge of content and teaching, collectively will be named Teachers’ Applied Content Knowledge (TACK).

Purpose of Study

Students need to learn mathematics. Teachers’ Applied Content Knowledge (TACK) is an important component of mathematical knowledge for teaching, and mathematical knowledge for teaching is a factor of students’ opportunities to learn mathematics. For a teacher to know the content does not imply that the teacher can teach the content; in fact, in Borko et al. (1992) and Thompson and Thompson's (1994) studies, the teachers had sufficient knowledge of the content they were teaching, but could not explain it in a way for students to understand. However, from what a teacher knows about teaching, how does the knowledge of how to teach connect with what is actually taught?

An exhaustive search using databases such as Academic Search Premier, Education Research Complete, ERIC, Professional Development Collection, PsycARTICLES, and PsycINFO; professional orgainzations' websites for journal articles such as nctm.org and amte.net; and general search engines such as Google Scholar yielded many articles on mathematics teacher knowledge. However, in looking at studies on teacher knowledge and student achievement, studies with meaningful descriptions of what the teachers did in the classroom to foster student achievement were few – and the majority of those studies focused on elementary teachers, with some in middle school, and very few at the high school level. Existing research of high school teachers' knowledge for teaching mathematics nearly echoed Michael Shaughnessy's (2011) concerns that algebra is getting more attention than geometry, as there are few studies that shed light on high school teachers' mathematical knowledge for teaching.

Based on my interest in the area of high school geometry teachers' knowledge, the research question for this dissertation was: What connections are there between the high school geometry Teachers' Applied Content Knowledge shown during the planning of the lesson and the teachers' actual executions of the lesson?

Chapter 2: Literature Review

An element of quality mathematics education research is "building on the work of others" (Simon, 2004, p. 161). This chapter has three main parts: defining what it means to learn and teach with understanding; studies relating teacher knowledge, classroom instruction, and student achievement; and assessing teacher knowledge. The first part of this chapter lays the foundation of "learning and teaching" and "learning and teaching geometry" with studies of student learning and suggestions for teaching. The second part focuses on studies that look at the relationships between teacher knowledge, classroom instruction, and student achievement. Since teacher knowledge is the focus of this dissertation, the last section focuses on how other studies assessed teacher knowledge.

Learning

The "Learning Principle" in NCTM's (2000) *Principles and Standards for School Mathematics* stated that "students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge" (p. 20). NCTM (2000) suggested that learning with understanding "makes subsequent learning easier" (p. 20). However, what does learning with understanding mean? What does it mean to understand? The following section discusses what it means to learn with understanding and students' zone of proximal development.

Knowledge and Understanding. "For much of this century, most psychologists accepted the traditional thesis that a newborn's mind is a blank slate (*tabula rasa*) on which the record of experience is gradually impressed" (Bransford, Brown, & Cocking, 2000, p. 79). As time progressed and more studies were done about learning, "challenges to this view arose" (Bransford, Brown, & Cocking, 2000, p. 79). Studies showed that "very young children are competent, active agents of their own conceptual development" (Bransford, Brown, & Cocking, 2000, p. 79). The move away from *tabula rasa* perspective of the infant mind was attributed to Jean Piaget (Bransford, Brown, & Cocking, 2000, p. 79), whose theory of development proposed that cognitive development happens in invariant stages (Cobb, 2007). The knowledge attained at the various cognitive stages can be categorized as different types of knowledge; conceptual and procedural (Hiebert & Lefevre, 1986) and relational and instrumental understanding (Skemp, 1967).

Conceptual and Procedural Knowledge. Hiebert and Lefevre (1986) proposed two types of knowledge, conceptual and procedural. Procedural knowledge has two parts, formal language of mathematics and rules of sequences of actions to complete a mathematical task (Hiebert & Lefevre, 1986). Hiebert and Lefevre (1986) gave an example of procedural knowledge of multiplying 3.82 by .43, where "one usually applies three subprocedures: one to write the problem in appropriate vertical form, a second to calculate the numerical part of the answer, and a third to place the decimal point in the answer" (p. 7). This example highlighted the two parts of procedural knowledge; familiarity with the multiplication symbol and knowing the steps to solve the problem.

Conceptual knowledge differs from procedural knowledge in that it is a "connected web of knowledge (Hiebert & Lefevre, 1986, p. 5) where "the new material becomes part of an existing network" (Hiebert & Lefevre, 1986, p. 6). Conceptual knowledge is defined as the procedures to solve the problem as well as the understanding of what the procedures mean and the principles behind them (Hiebert & Lefevre, 1986; Rittle-Johnson, Siegler, & Alibali, 2001). “Relationships between items of knowledge cannot be constructed if the knowledge does not exist” (Hiebert & Lefevre, 1986, p. 17). Hiebert’s and Lefevre’s (1986) example of incorrectly adding $\frac{1}{2} + \frac{1}{3} = \frac{2}{5}$ showed lack of conceptual knowledge of fractions. Procedurally in adding whole numbers, 1+1=2, and 2+3=5, but lacking the conceptual knowledge of fractions caused the incorrect answer of $\frac{2}{5}$ (Hiebert & Lefevre, 1986, p. 17). Conceptual knowledge of fraction equivalencies would lead one to see that $\frac{2}{5}$ is less than $\frac{1}{2} + \frac{1}{3}$ since $\frac{2}{5}$ is already less than $\frac{1}{2}$. “Deficiencies in concepts or procedures, although sometimes hidden, can be a source of weak or missing connections” (Hiebert & Lefevre, 1986, p. 17).

Relational and Instrumental Understanding. “Understanding” and “meaningful learning” have been terms used to describe the phenomenon of new information properly being attached to existing knowledge (Hiebert & Lefevre, 1986). Skemp (1967) described two types of understanding—relational and instrumental. Students that learn mathematics relationally understand the concepts and the connections that exist between them; while students that learn instrumentally learn skills and ideas about mathematics, but may not grasp the connections that exist between concepts (Skemp, 1967). Learning mathematics relationally means that students learn it with understanding, and the knowledge has permanence in their minds (Skemp, 1967). However, whether students understand relationally or instrumentally is connected to how the teacher teaches to support a particular type of understanding (Pesek & Kirshner, 2000). Pesek and Kirshner (2000) called this relational and instrumental instruction.

Pesek and Kirshner (2000) studied the impact of instrumental and relational teaching on 12 fifth grade students in mathematics. The length of this study was “three 1-hour lessons over a 3-day period” (Pesek & Kirshner, 2000, p. 529). Relational instruction for area and perimeter of squares, rectangles, triangles, and parallelograms was done in such a way that:

> Area and perimeter for each shape were presented together to help students contrast and compare these constructs. The shapes were discussed in the following order: squares, rectangles, parallelograms, and triangles. Connections were developed through concrete materials, questioning, student communication, and problem solving (Pesek & Kirshner, 2000, p. 529).

Instrumental instruction focused on “memorization and routine application of the formulas” (Pesek & Kirsher, 2000, p. 529). Pesek and Kirshner (2000) created written pre- and post-tests and results of those showed that students that learned relationally scored better than students that learned instrumentally. Interviews of the students showed that those who learned concepts relationally better applied the mathematics concepts they learned. In contrast, some students who learned instrumentally attempted to apply the rules that they learned but misused the appropriate one for the task (Pesek & Kirshner, 2000).

Summary. Conceptual knowledge includes some procedural knowledge, but not all elements of procedural knowledge are in conceptual knowledge (Hiebert & Lefevre, 1986); thus relational understanding is having both conceptual and procedural knowledge and knowing the connections between the two. Instrumental understanding is just the rote memorization of procedures (Pesek & Kirshner, 2000). In Pesek and Kirshner's (2000) study, the term for how students learned was how the teachers taught; students of teachers that taught relationally learned relationally. Teaching relationally yielded students' relational understanding, and teaching instrumentally yielded students' instrumental understanding (Pesek & Kirshner, 2000). Thus in order for students to have relational understanding of a topic, teachers need to teach that topic relationally. What and how teachers teach impact what and how students learn (Darling-Hammond, 2006).

Zone of Proximal Development. Students possess "a range of prior knowledge, skills, beliefs and concepts" that impact learning in school (Bransford, Brown, & Cocking, 2000, p. 10), and new knowledge is created from interaction with prior knowledge. However, there can be a limit to what students can do, what Vygotsky (1978) calls the "Zone of Proximal Development". The zone of proximal development is the area bounded by the student's ability to solve problems independently and the students' ability to solve problems with the help of someone more capable (Vygotsky, 1978). The upper boundary changes with the student's competence (Vygotsky, 1978). "In the context of mathematics task involving new content for a learner, the zone can represent the level of success a student can achieve on that task with and without adult guidance or in collaboration with more capable peers" (Bay-Williams & Herrera, 2007, p. 29). Once the student can solve a particular problem independently, the zone of proximal development shifts to a new upper boundary (Vygotsky, 1978).

As stated earlier, "No one questions the idea that what a teacher knows is one of the most important influences on what is done in classrooms and ultimately on what students learn" (Fennema & Franke, 1992, p. 147). According to NCTM (2000) students are capable of learning mathematics with understanding, so the responsibility rests on the teacher to teach in a way that provides opportunities for students to learn with understanding. This requires that teachers have a deep understanding of the mathematics they teach and how to teach it (NCTM, 2000).

Teaching

The "Teaching Principle" in NCTM's (2000) *Principles and Standards for School Mathematics* states that "Effective mathematics teaching requires understanding what students know and need to learn and then challenging and supporting them to learn it well" (p. 16). However, NCTM (2000) went further and highlighted three requirements of effective teaching.

- Effective teaching requires knowing and understanding mathematics, students as learners, and pedagogical strategies.
- Effective teaching requires a challenging and supportive classroom learning environment.
- Effective teaching requires continually seeking improvement. (NCTM, 2000, pp. 17-19).

NCTM (2000) insisted that there is "no one 'right way' to teach" (p.18), as there are many strategies and methods of teaching students. However, in order to teach effectively, knowledge of mathematics, students, and teaching is a must (NCTM, 2000). Also, teachers need to be aware of students' zone of proximal development and keep them challenged and set a classroom environment that supports student growth (NCTM, 2000).

Further, teachers need to be constantly improving to meet the needs of their students, since all students need to learn, and no two students are exactly the same (NCTM, 2000).

Knowledge in Teaching. Silverman and Thompson (2008) stated that "it is axiomatic that teachers' knowledge of mathematics alone is insufficient to support [teachers'] attempts to teach for understanding" (p. 499). Ball's (2003) example of teaching 0.3×0.7 showed that not only do teachers need to know the mathematics of what they are teaching, but also abilities specific to the profession of teaching mathematics. Conceptions of teacher knowledge for teaching mathematics (e.g. An, Klum, & Wu, 2004; Fennema & Franke, 1992; Hill, Ball, & Schilling, 2008) were built, in part, upon general conceptions of teacher knowledge like Shulman (1987) and Elbaz (1983). Shulman (1987) discussed categories of teachers' knowledge base and sources of teachers' knowledge base, and Elbaz (1983) focused on "practical knowledge", knowledge specific to teachers.

Elbaz (1983) categorized practical knowledge (knowledge specific to teachers) of the teacher (Sarah) she studied into five parts; knowledge of self, milieu, subject matter, curriculum, and instruction. However, the structure of teacher's knowledge has three dimensions, rules of practice, principles of practice, and images—what directs decision making (Elbaz, 1983). When looking at Sarah's knowledge of herself, three aspects were apparent; her knowledge of her skills and abilities, relationships with others (colleagues, administration, students…etc.), and personality traits and limitations (Elbaz, 1983). Knowledge of milieu comprised of beliefs about the milieu and "by the way she structures her social experience in the school" (Elbaz, 1983, p. 50). Since Sarah was an English teacher, evidence of this category "subject matter knowledge" was specific to

English; but the categories of subject matter that appeared were "the conceptions which underlie the different facets of content, the ways in which content from different subject matter areas is selected and combined, and how this content changed as Sarah used it in teaching" (Elbaz, 1983, p. 55). Knowledge of curriculum involved "both the development process and the underlying approach to curriculum development" (Elbaz, 1983, p. 69). Lastly, knowledge of instruction referred to theory of learning, students as learners, organizing instruction, interacting with students, and assessments of student learning (Elbaz, 1983).

Similar to Elbaz (1983), Shulman (1987) used excerpts of an English teacher conducting a class, but Elbaz's (1983) book focused on what the English teacher showed to support each category of teacher knowledge and how the categories came about; whereas Shulman's (1987) article used excerpts from an English teacher's class as examples of the many categories of teacher knowledge. However, Shulman's (1987) teacher knowledge categories were based on teachers from different disciplines; English literature, science, mathematics and history. Shulman (1987) expanded what was presented earlier in Shulman (1986) by providing more categories, content knowledge; general pedagogical knowledge; curriculum knowledge; pedagogical content knowledge; knowledge of learners and their characteristics; knowledge of educational contexts; and "knowledge of educational ends, purposes, and values, and their philosophical and historical grounds" (p. 8). Again, the categories of teacher knowledge stated in Shulman (1986) were content knowledge, curriculuar knowledge, and pedagogical content knowledge.

Both Shulman (1987) and Elbaz's (1983) contributions to defining teacher knowledge impacted conceptions of teacher knowledge for teaching mathematics (An, Klum, & Wu, 2004; Fennema & Franke, 1992; Hill, Ball, & Schilling, 2008). An, Klum, and Wu's (2004) conceptual framework for pedagogical content knowledge cited Shulman (1985) and Elbaz's (1983) notion of the complex and extensive nature of teachers' knowledge and its importance to student learning. An, Klum, and Wu (2004) later cited Shulman's (1987) article for his definition of pedagogical content knowledge as a foundation for their framework. Related to Shulman's (1987) article in *Harvard Education Review*, Shulman's (1986) presidential address focused on three areas of teacher knowledge; subject matter knowledge, pedagogical content knowledge, and curriculum knowledge. Fennema and Franke (1992) attributed Shulman's (1986) categories of teacher knowledge as the foundation of Peterson's (1988) concepts of mathematics teacher knowledge framing teacher knowledge, while Elbaz (1983) was discussed as an aspect of teacher knowledge, practical and personal. Fennema and Franke (1992) stated that although Elbaz's (1983) conceptions were based on an English teacher and the distinction between knowledge and beliefs were not made clear, Elbaz's (1983) study would "provide a fertile ground for investigation" (Fennema & Franke, 1992, p. 159). Table 2-1 summarizes Elbaz (1983) and Shulman (1987).

Table 2-1 Elbaz (1983) and Shulman's (1987) conceptions of teacher knowledge

	Elbaz (1983)	Shulman (1987)
Teacher Knowledge	Practical knowledge: • knowledge of self • milieu • subject matter knowledge • curriculum • teaching	Categories of knowledge base: • general pedagogical knowledge • curriculum knowledge • pedagogical content knowledge • knowledge of learners and their characteristics

	• knowledge of educational contexts • "knowledge of educational ends, purposes, and values, and their philosophical and historical grounds" (Shulman, 1987, p. 8)

Knowledge in Teaching Mathematics. In discussing frameworks of teacher knowledge, Fennema and Franke (1992) cited Shulman's (1986) framework as the basis of Peterson's (1988) framework for mathematics teacher knowledge; knowledge of "how students think in specific content areas, how to facilitate growth in students' thinking, and self-awareness of their own cognitive processes" (Fennema & Franke, 1992, p.157). Peterson (1988) proposed that content knowledge is held within each of the three categories mentioned previously. Ball's (2003) sentiment about more coursework in mathematics was similar to Peterson's (1988) stance on the impact of content knowledge to teaching mathematics, that is, content knowledge is meaningful when applied to decoding how students think in specific content areas, facilitating growth in students' thinking, and understanding their own thinking of mathematics. However, teachers have content knowledge that may not directly impact teaching the content (Ball, Thames, & Phelps, 2008; Fennema & Franke, 1992).

Since 2000, Deborah Ball and colleagues of the Learning Mathematics for Teaching (LMT) Project designed and piloted surveys that measure teachers' knowledge for teaching mathematics. Components of mathematical knowledge for teaching were based on the fundamental question of "What mathematical knowledge is needed to help students learn mathematics?" (Hill, Ball, & Schilling, 2004, p. 10). The domain map, in Figure 2-1 is a model of mathematical knowledge for teaching with relationships to pedagogical content knowledge and subject matter knowledge from Shulam's conception (Hill, Ball, Schilling, 2008). There are many conceptions of what knowledge teachers need to teach mathematics (e.g. Fennema & Franke, 1992; Peterson, 1988); however, what constitutes as mathematical knowledge for teaching "remains underconceptualized and understudied" (Hill, Ball, & Schilling, 2008, p. 395). There are many studies that focus on teacher pedagogical content knowledge, but "[m]ost definitons are perfunctory and often broadly conceived" (Ball, Thames, & Phelps, 2008, p. 394).

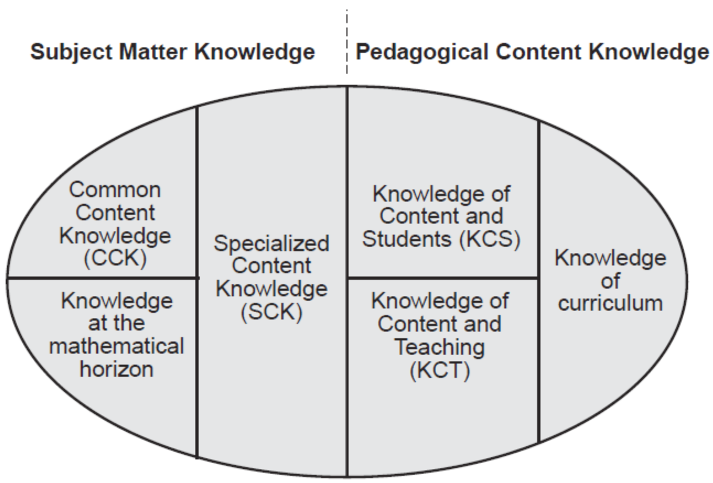


Figure 2-1 Domain map for mathematical knowledge for teaching as it relates to Shulman's (1987) pedagogical content knowledge**.** From "Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students." by H.C. Hill, D.L. Ball, and S.G. Schilling, 2008, *Journal for Research in Mathematics Education, 39*(4), p. 377. Copyright 2008 by *Journal for Research in Mathematics Education*. Reprinted with permission.

The Learning Mathematics for Teaching (LMT) Project aimed to be more specific about aspects of pedagogical content knowledge in defining mathematical knowledge for teaching (Ball, Thames, Phelps, 2008). Common content knowledge is the subject knowledge in teaching mathematics that is the same as mathematics in other jobs or disciplines in which mathematics is also used. Specialized content knowledge is teachers' knowledge of mathematics used in teaching (Ball, Hill, & Bass, 2005) “but not directly taught to students” (Hill, Sleep, Lewis, & Ball, 2007). Knowledge of content and teaching is the knowledge of teaching the content with emphasis on teaching; e.g. how to correct student mistakes so that they will understand and how to build on what the students already know (Hill, Ball, & Schilling, 2008). Separate from both knowledge of content and teaching and specialized content knowledge; knowledge of content and students is the subset of pedagogical content knowledge that enables teachers to identify common student errors, interpret students' understanding of content, identify students' developmental sequences, and identify common student computational strategies (Hill, Ball, & Schilling, 2008). Knowledge of content and students is also separate from knowledge of curriculum and knowledge of content and teaching because elements of knowledge of content and students appear in the classroom independent from teaching methods and curriculum choice (Hill, Ball, & Schilling, 2008).

Table 2-2 lists the definitions of the types of mathematical knowledge that Hill, Ball, and Schilling (2008) defined.

Table 2-2 Definitions of select types of mathematical knowledge for teaching

Type of Knowledge	Definition
Common Content Knowledge	subject knowledge in teaching mathematics (Hill, Ball, & Schilling, 2008)
Specialized Content Knowledge	“mathematical knowledge that is used in teaching, but not directly taught to students” (Hill, Sleep, Lewis, & Ball, 2007, p. 132)
Knowledge of Content and Students	knowledge that enables teachers to identify common student errors, interpret students' understanding of content, identify students' developmental sequences, and identify common student computational strategies (Hill, Ball, & Schilling, 2008)

Knowledge of Content and Teaching	knowledge of teaching the content with emphasis on teaching (Hill, Ball, & Schilling, 2008)

Hill, Ball, and Schilling (2008) described their efforts to conceptualize and develop teachers' knowledge of content and teachers' knowledge of content and students. They wrote, piloted, and analyzed results from multiple-choice items. Since, Hill, Ball, and Schilling (2008) based their multiple choice items on empirical data from literature, and the first step was to test the multiple-choice items validity as a measurement of knowledge of students and teaching; they did not measure how knowledge of content and students related to improving student learning in mathematics. They confirmed that "teachers have skills, insights, and wisdom beyond that of other mathematically well-educated adults" (Hill, Ball, & Schilling, 2008, p. 395). Although most scholars, teachers, and teacher educators would agree that teachers' knowledge of students' thinking in particular domains is likely to matter, what constitutes such 'knowledge' has yet to be understood" (Hill, Ball, & Schilling, 2008, p. 395).

Professional Standards for Teaching Mathematics. "Understanding learning as a process of individual and social construction provides teachers with a conceptual framework with which to understand the learning of their students" (Simon, 1993, p. 7). Based on how students learn and what type of knowledge they need to learn, the National Council of Teachers of Mathematics (NCTM) (1991) produced *Professional Standards for Teaching Mathematics*. NCTM (1991) listed five major shifts that need to occur in practice at the time to improve mathematics teaching:

- toward classrooms as mathematical communities—away from classrooms as simply a collection of individuals;

- toward logic and mathematical evidence as verification—away from the teacher as the sole authority for right answers;
- toward mathematical reasoning—away from merely memorizing procedures;
- toward conjecturing, inventing, and problem solving—away from an emphasis on mechanistic answer-finding;
- toward connecting mathematics, its ideas, and its applications—away from teaching mathematics as a body of isolated concepts and procedures (p. 3).

From these suggested shifts, four essentials of teaching were emphasized:

- Setting goals and selecting or creating mathematical tasks to help students achieve these goals;
- Stimulating and managing classroom discourse so that both the students and the teacher are clearer about what is being learned;
- Creating a classroom environment to support teaching and learning mathematics;
- Analyzing student learning, the mathematical tasks, and the environment in order to make ongoing instructional decisions (NCTM, 1991, p. 5).

From what is suggested of teachers to do and based on what is known about learning, the teacher needs to act as the facilitator of the exchange of knowledge in the classroom and not as the sole dispenser of knowledge (NCTM, 1991). Teachers create the environment in which students learn. An element of creating a productive learning environment is to provide students with the opportunity to learn multiple strategies, so that they may have an arsenal with which to approach a problem (Bransford, Brown, & Cocking, 2000). Once students have enough knowledge of multiple strategies, students can depend on themselves to solve problems or enlist the help of peers (Bransford, Brown, & Cocking, 2000). This must be accomplished with minimal help from the teacher so that they may then take charge of their own learning (Bransford, Brown, & Cocking, 2000); self-reliance.

Summary. The section "Knowledge and Teaching" compared two widely cited early conceptions of teacher knowledge, Elbaz (1983) and Shulman (1987). "Professional Standards for Teaching Mathematics" funneled the different types of teacher knowledge into shifts teachers needed to make to improve mathematics teaching. Although not explicitly stated, in order to comply with the five suggested shifts and four essentials of teaching, teachers need to have a firm knowledge base; including the categories mentioned by Elbaz (1983), Shulman (1986), Shulman (1987), Peterson (1988), and mathematical knowledge for teaching by the Deborah Ball and colleagues in the Learning Mathematics for Teaching (LMT) Project. The next section will delve deeper into what it means to learn and teach geometry.

Learning and Teaching Geometry

"Years ago an informal definition [of geometry] might have been that it was a branch of mathematics devoted to the study of shapes and space. Now, however, a more apt definition might be 'the branch of mathematics that studies visual phenomena'" (Malkevitch, 2009, p. 14). All four of NCTM's (2000) standards for geometry, from kindergarten to 12th grade, show attention to studying visual phenomena. They are:

- Analyze characteristics and properties of two- and three- dimensional geometric shapes and develop mathematical arguments about geometric relationships
- Specify locations and describe spatial relationships using coordinate geometry and other representational systems
- Apply transformations and use symmetry to analyze mathematical situations

- Use visualization, spatial reasoning, and geometric modeling to solve problems (p. 308).

Apparent from the standards, "Geometry is more than definitions; it is about describing relationships and reasoning" (NCTM, 2000, p.41). This section looks at theories, standards, proposed practices, and studies on teaching and learning geometry.

Learning. This section explores theories on the learning of geometry, expounding on the van Hiele levels and Driscoll's (2007) geometric habits of mind. The van Hiele level theory was developed by wife and husband, Dina van Hiele-Geldof and Pierre Marie van Hiele, in separate doctoral dissertations in 1957 (Usiskin, 1982). Soon after the completion of their dissertations, Dina passed away and further work with the van Hiele levels was done by Pierre Marie van Hiele (Usiskin, 1982).

Van Hiele Levels. Level 0 is the recognition or visualization level, where students learn the names of figures (van Hiele, 1986). Level 1 is where students identify properties of figures, called the analysis level (van Hiele, 1986). In the informal deduction level, or Level 2, students order figures and relationships; this is not where students formally prove yet, but begin to use simple deduction (van Hiele, 1986). In Level 3, the formal deduction level, students formally use deduction as a means of understanding the connections between postulates and theorems and using them to prove other theorems and postulates (van Hiele, 1986). Finally, Level 4 is rigor, where students understand "the necessity for rigor and are able to make abstract deductions (van Hiele, 1986). Students cannot skip levels, as according to the theory, they attain levels sequentially (Battista, 2007a; van Hiele, 1986; Usiskin, 1982). Table 2-3 shows the van Hiele levels by name and description.

Table 2-3 The van Hiele theory. (Breyfogle & Lynch, 2010, p. 234)

Level	Name	Description
0	Visualization	See geometric shapes as a whole; do not focus on their particular attributes.
1	Analysis	Recognize that each shape has different properties; identify the shape by that property.
2	Informal Deduction	See the interrelationships between figures.
3	Formal Deduction	Construct proofs rather than just memorize them; see the possibility of developing a proof in more than one way.
4	Rigor	Learn that geometry needs to be understood in the abstract; see the "construction" of geometric systems.

The van Hiele level theory "has three aspects; the existence of levels, properties of the levels, and movement from one level to the next" (Usiskin, 1982, p. 4). Burger and Shaughnessy's (1986) study contributed to confirming the existence of levels by adding examples of student thought at the various van Hiele levels. Burger and Shaughnessy (1986) cited Fuys et al. (1985) and Mayberry's (1983) results which supported the hierarchal nature of the van Hiele Levels as the foundation of the ability to assign levels. Burger and Shaughnessy's (1986) interview study involved 45 students from grades 1 to 12 and university mathematics majors. "The tasks included drawing shapes, identifying and defining shapes, sorting shapes, determining a mystery shape, establishing properties of parallelograms, and comparing components of a mathematical system" (Burger & Shaughnessy, 1986, p. 31). To show how Burger and Shaughnessy's (1986) data supported the van Hiele levels, a summary of data supporting Level 1 reasoning is shown below.

Level 1

1. Comparing shapes explicitly by means of properties of their components.
2. Prohibiting class inclusions among general types of shapes, such as quadrilaterals.
3. Sorting by single attributes, such as properties of sides, while neglecting angles, symmetry, and so forth.
4. Application of a litany of necessary properties instead of determining sufficient properties when identifying shapes, explaining identifications, and deciding on a mystery shape.
5. Descriptions of types of shapes by explicit use of their properties, rather than by type names, even if known. For example, instead of rectangle, the shape would be referred to as a four-sided figure with all right angles.
6. Explicit rejection of textbook definitions of shapes in favor of personal characterizations.
7. Treating geometry as physics when testing the validity of a proposition; for example, relying on a variety of drawings and making observations about them,
8. Explicit lack of understanding of mathematical proof (p. 44).

The list above was the level indicator for van Hiele Level 1. Burger and Shaughnessy (1986) created level indicators for levels 0-3. A complete listing of levels 0-3 is in Burger and Shaughnessy (1986) pages 43-45. “No attempt was made to investigate van Hiele Level 4 with these subjects, a level that requires the ability to compare different geometries” (Burger & Shaughnessy, 1986, p. 34). Students’ answers to the clinical interview tasks in the study reinforced the discreteness of the van Hiele Levels 0-3 (Burger & Shaughnessy, 1986).

Geometric Habits of Mind. Driscoll (2007) suggested that teachers of students in grades 5-10 foster geometric habits of mind in preparation for students to take high school geometry because what students should be learning in high school geometry is built upon geometric ideas learned in the middle school. The four geometric habits of mind are: reasoning with relationships, generalizing geometric ideas, investigating invariants, and balancing exploration and reflection (Driscoll, 2007).

Reasoning with relationships is "actively looking for relationships (e.g., congruence, similarity, and parallelism) within and between geometric objects in one, two, and three dimensions" (Driscoll, 2007, p. 12). Küchemann and Hoyles' (2006) 3-year longitudinal study highlighted the need for students to be actively looking for relationships, emphasis on the word "actively". Results from Küchemann and Hoyles' (2006) study showed that students that did not make progress were also students that did not actively explore other connections or relationships. Lawson and Chinnappan's (2000) qualitative study of 36 male students taking geometry showed that 10^{th} grade male students were more successful in problem solving when they had more content connections than other 10^{th} grade male students that did not. Without the connectedness of content, studies showed that it was difficult for students to use something that is not there (Herbst, 2006; Lawson & Chinnappan, 2000). Generalizing geometric ideas is "wanting to understand and describe the 'always' and the 'every' related to geometric phenomena" (Driscoll, 2007, p. 12). Investigating invariants is students paying attention to what is not changing, or staying the same (Driscoll, 2007). Balancing exploration and reflection "is trying various ways to approach a problem and regularly stepping back to take stock" (Driscoll, 2007, p. 14). The four habits of mind are not separate, as "a problem solver is likely to draw on several conceptual tools while approaching a problem" (Driscoll, 2007, p. 15). Table 2-4 below gives indicators of students' geometric habits of mind.

Table 2-4 Geometric Habits of Mind and Their Indicators (Driscoll, 2007, pp.12-15)

Geometric Habit of Mind	Indicators
Reasoning with Relationships	Basic: Identification of figures presented in a problem and correct enumeration of their properties Advanced: Relating multiple figures in a problem through proportional reasoning and reasoning through symmetry
Generalizing Geometric Ideas	Basic: Uses one problem situation to generate another, or when the solver intuits that he or she hasn't found all the solutions Advanced: Generate all solutions and make a convincing argument as to why there are no more; or wondering what happens if a problem's context is changed
Investigating Invariants	Basic: Decides to try a transformation of figures in a problem without being prompted, and considers what has changed and what has not changed Advanced: Consider extreme cases for what is being asked by a problem
Balancing Exploration and Reflection	Basic: Drawing, playing, and/or exploring with occasional (though maybe not be consistent) stock-taking Advanced: approaching a problem by imagining what a final solution would look like, then reasoning backward; or making what Herbst (2006) calls "reasoned conjectures" about solutions with strategies for testing the conjectures

Teaching. This section nearly mirrors the previous section in that this section focuses on teaching to move students through the van Hiele levels and principles for fostering geometric thinking. Teaching requires more than just the knowledge of the subject (Ball, 2003). An, Klum, and Wu (2004) highlighted a Chinese saying, if the teacher is to give students a cup of water, the teacher must have a pail of water of their own.

Van Hiele Levels. The phases of learning from the van Hiele model as described in van Hiele (1984) synthesized by Mistretta (2000) is in Table 2-5 below.

Table 2-5 Phases of Learning from the van Hiele Model (Mistretta, 2000, p. 367)

Phase	Description
Information	Discussions are held where the teacher learns of the students' prior knowledge and experience with the subject matter at hand.
Directed Orientation	The teacher provides activities that allow students to become more acquainted with the material being taught.
Explication	A transition between reliance on the teacher and students' self-reliance is made.
Free Orientation	The teacher is attentive to the inventive ability of the students. Tasks that can be approached in numerous ways are presented to the students.
Integration	The students summarize what was learned during the lesson.

Breyfogle and Lynch (2010) suggested that since "movement through [the van Hiele] model depends on experiences of the learner[,] [w]ell-devised tasks, [such as assessments], help move students through the levels" (p. 238). The assessments that teachers should use must “allow students to demonstrate their acquired knowledge in meaningful way[s] and helps them continue to learn as they go through the process” (Breyfogle & Lynch, 2010, p. 234). Table 2-6 below provides examples of what teachers should do for students to move from one level to the next.

Table 2-6 The van Hiele model of geometric understanding. (Breyfogle & Lynch, 2010, p. 234)

Level	Name	Description	Example	Teacher Activity
	Visualization	See geometric shapes as a whole; do not focus on their particular attributes.	A student would identify a square but would be unable to articulate that it has four congruent sides with right angles.	Reinforce this level by encouraging students to group shapes according to their similarities.
1	Analysis	Recognize that each shape has different properties; identify the shape by that property.	A student is able to identify that a parallelogram has two pairs of parallel sides, and that if a quadrilateral has two pairs of parallel sides it is identified as a parallelogram.	Play the game “guess my rule,” in which shapes that “fit” the rule are placed inside the circle and those that do not are outside the circle (see Russell and Economopoulos 2008).
2	Informal Deduction	See the interrelationships between figures.	Given the definition of a rectangle as a quadrilateral with right angles, a student could identify a square as a rectangle.	Create hierarchies (i.e., organizational charts of the relationships) or Venn diagrams of quadrilaterals to show how the attributes of one shape imply or are related to the attributes of others.
3	Formal Deduction	Construct proofs rather than just memorize them; see the possibility of	Given three properties about a quadrilateral, a student could logically deduce	Provide situations in which students could use a variety of different angles depending on what

		developing a proof in more than one way.	which statement implies which about the quadrilateral (see 2.7).	was given (e.g., alternate interior or corresponding angles being congruent, or same-side interior angles being supplementary).
4	Rigor	Learn that geometry needs to be understood in the abstract; see the "construction" of geometric systems.	Students should understand that other geometries exist and that what is important is the structure of axioms, postulates, and theorems.	Study non-Euclidean geometries such as Taxi Cab geometry (Krause 1987).

The following is an example where students should be able to answer this question using formal deduction. (Breyfogle & Lynch, 2010, p. 235)

> Three Properties of a Quadrilateral
> Property D: It has diagonals of equal length.
> Property S: It is a square.
> Property R: It is a rectangle.
> Which is true?
> a. D implies S, which implies R.
> b. D implies R, which implies S.
> c. S implies R, which implies D.
> d. R implies D, which implies S.
> e. R implies S, which implies D.
> *Source:* Usiskin (1992)

Principles for Fostering Geometric Habits of Mind. Driscoll (2007) proposed three principles for fostering geometric thinking, the first of which is "Geometric thinking develops with the help of regular problem-solving opportunities" (p. 95). In order for students to learn with understanding; new knowledge needs to form connections with prior knowledge (Bransford, Brown, & Cocking, 2000), and Driscoll (2007) cited Bransford, Brown, and Cocking (2000) as theoretical basis for the first principle. Driscoll

(2007) proposed that from his experienced, “problem solving provides a rich context for fostering understanding, active organization, and connections to prior knowledge” (p. 96). One of the examples Driscoll (2007) gave was: “Two vertices of a triangle are located at (0, 6) and (0, 12). The triangle has area 12. What are all possible positions for the third vertex? And how do you know you have them all?” (p. 98). This example shows the connections between geometry ideas and the coordinate system from algebra (Driscoll, 2007).

The second principle is “Geometry in the middle grades demands special attention to teacher-student communication” (Driscoll, p. 95). Two main components of this principle are language and geometry and the power of teacher questioning (Driscoll, 2007). “Geometric problem solving invites drawn, spoken, and given gestured representation of understanding, along with written verbal and written symbolic representations, and so it invites multimodal mathematical communication” (Driscoll, 2007, p. 100). Thus, “teacher questioning can be a powerful tool” (Driscoll, 2007, p. 101) for students to learn geometry with understanding. Teachers need to be purposeful in the type of questions they are asking (Driscoll, 2007).

Table 2-7 Driscoll’s (2007) framework for questioning (p. 102)

Question Type	Purpose	Examples
Orienting	To focus students’ attention on the problem and/or on a particular way to approach the problem.	What is the problem asking? Would tangrams be useful here? Do you think comparing these two sides might help?
Assessing	To gauge students’ understanding of their statements and actions while	What do you think *congruent* means in the problem statement? Why did you fold the patty paper like that?

	problem solving.	How did you arrive at this answer? How did you know this is a rectangle?
Advancing	To help students extend their thinking toward a deeper understanding of the problem	How could you convince a skeptic that the figure you've made is a parallelogram? What if you didn't know the measure of this angle? How would you solve the problem without graphing paper? What types of triangles will this work for and why?

The third and final principle is "Middle-grades geometry is groundwork for high school geometry" (p. 95). Since the focus of high school geometry is proving (Battista, 2009; NCTM, 2000), middle school educators need to place emphasis on convincing explanations, "cognitive demand of tasks, development of a 'geometric eye', and emphasiz[e] metacognitive development" (Driscoll, 2007, p. 109). In collecting student work samples for the book, Driscoll (2007) found three main problems:

- Middle graders appear to have little experience in putting together convincing mathematical explanations—for geometry problems in particular.
- Sometimes, language appears to be a barrier to writing an explanation…
- In geometry problems, middle graders often use perception as a warrant for their claims (e.g., the line "looks straight," or the angle is a right angle because it "has an L-shape")… (p. 110).

Therefore, to mitigate these problems, middle grades teachers need to:

> provide more opportunities for students to construct convincing explanations…[provide opportunities] accessible to students for whom language may be a barrier…[and] ask questions that both orient students to places in their

claims where they may be relying too much on perception, and advance their thinking to go beyond perception (Driscoll, p. 110).

Middle grades teachers also need to keep in mind the cognitive demand of tasks given to students (Driscoll, 2007). Teachers need to keep in mind that certain questions may lower the cognitive demand of the task and hamper the development of geometric habits of mind. Students need to develop the “geometric eye”, that is, the visual thought that “helps in geometric problem solving” (Driscoll, 2007, p. 113). This ability can be developed through practice with carefully selected tasks (Driscoll, 2007).

Even though the research question of this study focused on high school geometry teachers, literature on teaching and learning geometry at the middle school is important as well. From Vygotsky’s Zone of Proximal Development (Bransford, Brown, & Cocking, 2000; & Vygotsky, 1978) and van Hiele Levels (Usiskin, 1982; & van Hiele, 1986), teachers need to meet students at students’ levels of understanding in order to foster learning; and middle school is prior to high school.

Studies on Teaching and Learning Geometry. As mentioned in chapter one, a search using databases such as Academic Search Premier, Education Research Complete, ERIC, Professional Development Collection, PsycARTICLES, and PsycINFO; professional orgainzations’ websites for journal articles such as nctm.org and amte.net; and general search engines such as Google Scholar—yielded many articles on mathematics teacher knowledge, but few that described what teachers did in the classroom in relation to student achievement. Of those articles that focused on geometry, the majority of them were of the elementary grades, some at the middle school level, and few at the high school level. Also, the geometry articles could logically be organized into two categories regarding technology; those that focused on the use of technology—dynamic geometry software, and those that did not. In the next sections, studies related to teaching geometry are presented.

Dynamic Geometry Software. There are many instances in mathematics education literature where students benefited from using dynamic geometry software like Geometer's Sketchpad (Battista, 2007a). Some benefits of dynamic geometry software over paper and pencil methods are its efficiency (faster than paper and pencil) (Oner, 2008) and freedom to experiment (not limited by paper and pencil) (Marrades & Gutiérrez, 2000). Some studies such as Ubuz, Üstün, and Erbas, (2009) also touted the wonders of dynamic geometry software as it improved retention of seventh grade students. However, they compared the results of a class that used dynamic geometry software to a class where the teacher taught instrumentally, thus complicating results for the usefulness of dynamic geometry software with teaching philosophy. Referring back to the beginning of this chapter, the use of dynamic geometry software in this case was for relational understanding and positive results cannot be fully attributed to dynamic geometry software. In contrast, Han (2007) compared two classes as well, and also one with dynamic geometry software and the other without; but both were taught relationally. Han (2007) found that students that had the opportunity to use Geometer's Sketchpad showed higher van Hiele levels for knowledge of quadrilaterals than students that used paper and pencil methods

From classroom observations, Marrades and Gutiérrez (2000) found dynamic geometry software (Cabri version 1.7) to be more useful in teaching students to prove than paper and pencil methods. A feature of Cabri was "dragging", and this "lets students see as many examples as necessary in a few seconds, and provides them with immediate feedback that cannot be obtained from paper-pencil teaching” (Marrades & Gutiérrez, 2000, pp.119-120). The study involved two pairs of fourth grade students over 30 weeks with two 55 minute classes per week (Marrades & Gutiérrez, 2000). Data from three lessons (beginning, middle, and end) were chosen to illuminate how the students were progressing in their reasoning ability (Marrades & Gutiérrez, 2000).

The main objectives of the teaching unit observed for this study were:

- To facilitate the teaching of concepts, properties and methods usually found in the school plane geometry curriculum …
- To facilitate a better understanding by students of the need for and function of justification in mathematics.
- To facilitate and induce the progress of students toward types of justification closer to formal mathematical proofs … (Marrades & Gutiérrez, 2000, p. 97).

Activities were generally structured in three phases; create and/or explore, generate and/or verify conjectures, and justify conjectures (some activities did not have this stage). In the first stage, sometimes the teacher provided the figure and students had access to it to use it to explore properties instead of creating their own (Marrades & Gutiérrez, 2000). Teachers prompted students with questions such as “why is the construction valid?” and “why is the conjecture true?” (Marrades & Gutiérrez, 2000, p. 98). “As part of the didactical contract defined in the class, pupils knew that requirements like ‘justify your conjecture’ carried the implicit meaning of ‘justify why your conjecture or construction is true’” (Marrades & Gutiérrez, 2000, p. 98). At the beginning of class, students were presented with worksheets by the teacher where they had to "write their observations, comments, conjectures, and justifications" (Marrades & Gutiérrez, 2000, p. 98).

Answers from the worksheets, history from Cabri software, and semi-structured interviews were the basis of conclusions of the study. Both pairs of students improved in their justification abilities (Marrades & Gutiérrez, 2000). Marrades and Gutiérrez (2000) concluded that students were able to "progress toward more elaborated types of justifications" (p. 120) and showed confidence in their ability of forming deductive justifications and formal proofs. Marrades and Gutiérrez (2000) agreed with other studies on the benefits of dynamic geometry software (e.g., Battista, 2007a; Han, 2007), and in addition, showed improvement in student proving abilities when using Cabri.

Battista (2007b) documented "progress and difficulties in constructing meaning for a spatial property" (p. 69) of a pair of 5th grade students, Matt and Tom. Matt and Tom's class had been working in pairs on computers using a feature of Geometer's Sketchpad called "Shape Makers" (Battista, 2007b). Shape Makers allows students to make certain shapes with different sizes and orientations (Battista, 2007b). For example, the Parallelogram Maker allows students to manipulate sides and angles of a parallelogram to any parallelogram so long as it fits on the screen (Battista, 2007b). One of the goals of Shape Makers is to help students move from van Hiele levels 0 and 1 to 2 (Battista, 2007b). In the task "What shapes can a rectangle maker make?" (Battista, 2007b, p. 69) the class was given six different shapes and were asked to decide which were possible (Figure 2-2). Matt and Tom correctly predicted which shapes could be created (Battista, 2007b).

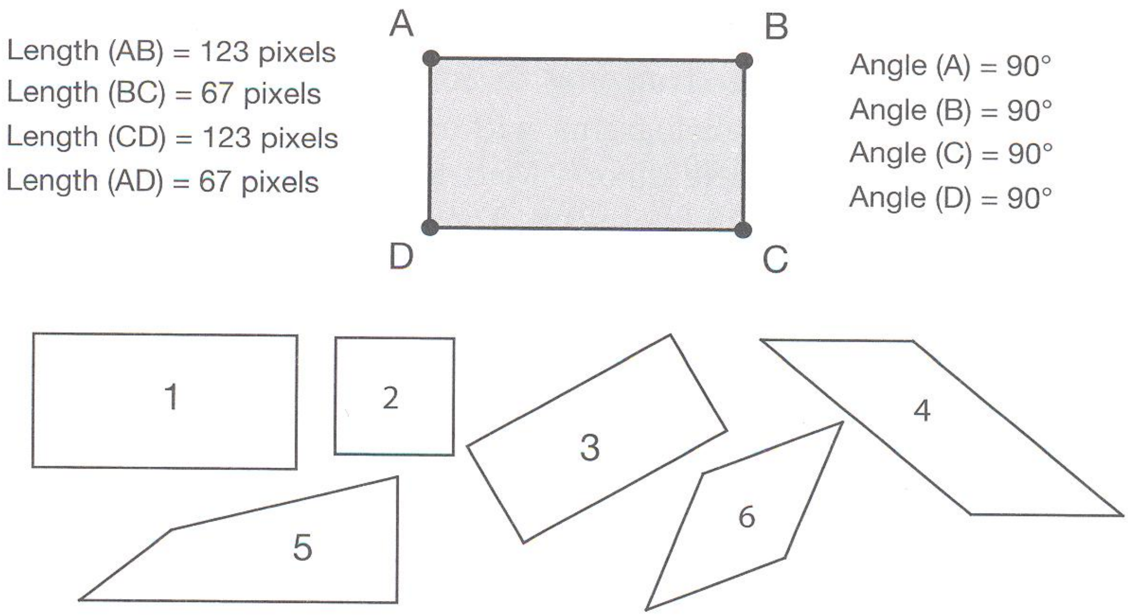


Figure 2-2 Rectangle Maker Task**.** From "Learning with understanding: Principles and processes in the construction of meaning for geometric ideas." by M. Battista, 2007, *The Learning of Mathematics, 69th Yearbook of the National Council of Teachers of Mathematics,* p. 69. Copyright 2007 by *The Learning of Mathematics, 69th Yearbook of the National Council of Teachers of Mathematics*. Reprinted with permission.

"After using the Rectangle Maker to check Shapes 1 and 2, Matt and Tom had some initial difficulty manipulating the Rectangle Maker to make Shape 3" (Battista, 2007b, p.70) because of what Tom called a "slant". The teacher challenged Matt and Tom by asking them to define what Tom called a "slant" (Battista, 2007b). It was evident that Tom and Matt tried again and made Shape 3 (Battista, 2007b). Tom and Matt had "sufficiently accurate mental models for rectangles and how the Rectangle Maker moves" due to their correct predictions and sufficient manipulation to show subsequent shapes were not able to be made by the Rectangle Maker (Battista, 2007b).

The following is an excerpt between Tom, Matt, and the teacher as to why Shape 4 cannot be made by the Rectangle Maker (Battista, 2007b).

Tom: I'm positive it can't do this one.

Matt: It [the Rectangle Maker] has no slants… It can't make a slant… The Rectangle Maker can't make something like that slanted here [*motioning along the bottom then left side of Shape 4*].

Teacher: You mean the angle there?

Matt: Yeah, it has an angle. [Shape] 4 is no because the Rectangle Maker can't make a slant.

Tom: Cause the Rectangle Maker can only make a square and a rectangle and that's not a square or a rectangle… (Battista, 2007b, p. 70).

Even though Matt's "reasoning was still extremely imprecise" (Battista, 2007b, p.70), being able to describe spatial relationships between parts of shapes shows that he was moving from van Hiele level 1 towards level 2. Battista (2007b) also made the connection between what Matt was doing to Bransford, Brown, and Cocking, 2000; where new knowledge is constructed based on existing knowledge. Matt created new concepts based on the common words "slant" and "angle" (Battista, 2007b). In contrast, "Tom's last comment indicat[ed] Level 1 reasoning—he was thinking about shapes strictly as wholes" (Battista, 2007b, p. 70). Explaining why the Rectangle Maker could not make Shape 4 was difficult for Matt and Tom (Battista, 2007b).

Thus, to facilitate shifting their focus from parts of a shape to the whole shape, the teacher asked Matt and Tom "What is different about the angle here [*motioning to the upper left angle on the Rectangle Maker*] and the angle here [*motioning to the upper left angle on Shape 4*]?" (Battista, 2007b, p. 71). This got Matt and Tom to start noticing parts more, but not to where they needed to be to understand why the Rectangle Maker could not make Shape 4 (Battista, 2007b). So the teacher drew on their screen Shape 3 and Shape 4 next to each other and asked Matt and Tom what the differences were between Shape 3 and Shape 4 (Battista, 2007b). Battista (2007b) saw that the teacher knew exactly what the differences were to situate the question in that manner, but Matt and Tom still did not see what the teacher saw (Battista, 2007b). The teacher recognized that Matt and Tom needed some more time with Shape Makers and pointed them to looking at the measures of the sides and angles that were on the screen of the shapes 3 and 4; and when they looked at that, the "aha" moment happened (Battista, 2007b). The teacher's ability to facilitate learning was an important aspect of moving students through the van Hiele Levels (Battista, 2007b). In order for learning to occur, according to Bransford, Brown, and Cocking, 2000; Tom and Matt of Battista's (2007b) study "had to mentally construct a new way of thinking about shapes" (p. 77). Within the interaction between the teacher and Matt and Tom, the teacher had to determine the students' difficulties and help them through the difficulty within a reasonable amount of time in order for them to learn.

Han's (2007) dissertation investigated the use of Geometer's Sketchpad with 97 8th grade students in Minneapolis, of which 57 students used Geometer's Sketchpad (GSP) and the other 40 by paper and pencil methods. The researcher set up both activities where students learned about properties of quadrilaterals by observing characteristics of different quadrilaterals, and the only difference was that one group had paper and pencil, and the other group used GSP (Han, 2007). Both activities required students to "explore geometric concepts with hands-on experiences, to develop conceptual understanding, and to stimulate higher level of thinking and mathematical reasoning ability" (Han, 2007, p. 53). Teachers facilitated learning through questioning to get students to come to their own correct conjectures of quadrilateral properties (Han, 2007). Both groups of students were taught relationally, and students that had the opportunity to use Geometer's

Sketchpad showed higher van Hiele levels for knowledge of quadrilaterals than students that used paper and pencil methods (Han, 2007).

A qualitative segment from a bigger study by Jones (2000) showed that 12 year old students in middle school using Cabri I (a dynamic geometry software similar to Geometer's Sketchpad), through an instruction sequence on classifying quadrilaterals, indicated upward movement in van Hiele levels. A sample of initial dialogue between the teacher and the students is as follows:

> Teacher: What sort of shape is that?
>
> Karol: It's a rhombus.
>
> Teacher: How do you know it's a rhombus?
>
> Karol: Our old maths teacher used to call a rhombus a drunken square, because it's like a square, only sick.
>
> Teacher: What do you know about a rhombus, from what you have done?
>
> Heather: It's got a centre.
>
> Karol: It's like a diamond . . .. But it's not a square.
>
> Teacher: What can you say about the sides or the angles . . . or the diagonals?
>
> Karol: Those two angles [indicating the angles at one pair of opposite vertices] are the same, and those two are the same . . . [indicating the other pair of opposite angles] But they are not all the same [indicating adjacent angles] And . . . the sides are all the same length . . . I think.
>
> Heather: It's the same distance across each side.
>
> Teacher: What can you say about how the diagonals cross?
>
> Karol: A right angle.
>
> Teacher: How do you know?
>
> Karol: It looks straight (Jones, 2000, pp. 74-75).

A sample of the dialogue between student and teacher at the end of the lesson is as follows:

> Teacher: Why that arrow? [indicating that a rhombus is a special form of parallelogram].
>
> Karol: It's just like the rhombus and the square because . . . because all the sides . . . the sides are . . . the opposite sides are of equal length, but there [in the rhombus] they [the diagonals] cross at 90 degrees and there [indicating a parallelogram that is not a rhombus] they don't.

Initially, students used everyday language for description and relied on perception, but by the end of the unit, were able to explain what they were doing mathematically and used mathematical reasoning to justify claims (Jones, 2000). Jones (2000) stated that this shift was "mediated by the software environment" (p. 80). However, the teacher set up the environment with computer software to facilitate learning (Jones, 2000).

The teacher set up "carefully designed" (Jones, 2000, p. 81) problem based tasks for students to work in pairs or small groups of 3-6. The teacher questioned students and redirected them to pertinent features of the geometry software, encouraged conjecturing, and focused students towards forming mathematical explanations (Jones, 2000). "Without such factors, the meditational impact of the software could be such that it may distract students from the geometry of the problem situation or possibly *reduce* the perceived need for deductive proof" (Jones, 2000, p. 81).

Using dynamic geometry software in conjunction with teachers teaching for understanding has shown to benefit students' van Hiele levels on a topic (Han, 2007), as well as improved abilities to verbalize and reason mathematically (Jones, 2000). Technology is certainly useful for students to learn geometry; however, not all students have access to technology (Bishop & Forgasz, 2007).

Non Dynamic Geometry Software studies related to the van Hiele Levels. Senk (1989) assessed van Hiele Levels of reasoning and proof writing abilities of 241 high school students at the beginning and at the end of their geometry course. The assessment for proofs had two short answer items and four formal proofs (Senk, 1989). The assessment for van Hiele Levels of reasoning was with Usiskin's (1982) Van Hiele Geometry Test. The Van Hiele Geometry Test had 25 multiple-choice questions, with 5 questions for each level. "High school students' achievement in writing geometry proofs is positively related to van Hiele levels of geometric thought and to achievement on standard nonproof geometry content" (Senk, 1989, p. 318). Further, Senk's (1989) findings aligned to the van Hiele theory that students who were not at levels 3 or 4 could not produce proofs on the proofs assessment.

A closer look at the relationship between teaching and learning proofs showed that certain teacher moves corresponded to student actions that eventually resulted in students being able to prove on their own (Martin, McCrone, Wallace, Bower, & Dindyal, 2005). The diagram in Figure 2-3 shows teacher and student verbal interactions for one of the cases studied provided by Martin et al. (2005, p. 116).

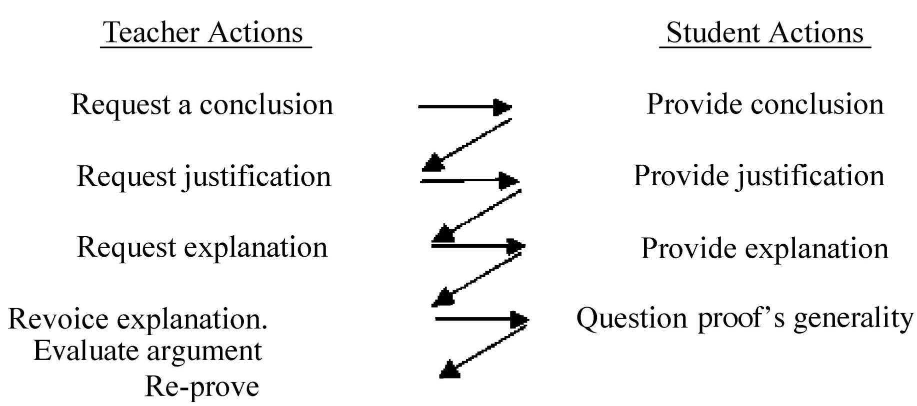


Figure 2-3 Teacher and student verbal interactions for one of the cases**.** From "The interplay of teacher and student actions in the teaching and learning of geometric proof." by T. Martin, S. McCrone, M. Bower, and J. Dindyal, 2005, *Educational Studies in Mathematics, 60*(1), p. 116. Copyright 2005 by *Springer*. Reprinted with permission.

The “choices and expectations [of the teacher] often led to discussion of student reasoning, proof modeling, and refinement of an argument” (Martin, et al. 2005, p. 106) for this particular case. For each of the four cases, an interaction flow chart was made (Martin, et al. 2005). From those four cases, Martin et al. (2005) concluded that when the teacher posed an open-ended task, it led to “engaging students in verbal reasoning, this provides a landscape for proving” (Martin, et al. 2005). When students were engaged in verbal reasoning the teacher “analyz[ed], coach[ed], and revoic[ed] questions back to students, [therefore] the teacher [could] monitor and influence students’ developing ability to construct chains of reasoning” (Martin, et al. 2005, p.121).

Herbst (2006) used a geometry problem to see what a teacher did to engage students, and how what the teacher did to engage students impacted the activity. The question posed to students was “Say you have a triangle ABC, how can you find a point O inside ABC so that the three new triangles AOB, BOC, and AOC have the same area?” (Herbst, 2006, p. 322). That geometry problem is an area problem that also called for knowledge of the relationship between the medians and area of a triangle (Herbst, 2006). Only one teacher was observed to see what the teacher did to engage students and what the teacher did to engage students impacted the activity.

This teacher, Megan, liked teaching geometry, and “had an extensive background in mathematics and education” (Herbst, 2006, p. 321). “She was a strong presence in the room and led a rich and logical development of the subject, constantly seeking student input in the form of focused questions, oriented toward making connections and anticipating problems” (Herbst, 2006, p. 321). Megan engaged students in the activity, and students worked on aspects of the activity, but never on making a conjecture about the relationship between medians and area of a triangle (Herbst, 2006). In introducing the problem; Megan was about to draw a triangle, but stopped and asked students to draw a triangle, and reminded them that they knew how to divide a triangle into two equal areas from the previous day; and then asked students to divide a triangle into three equal areas (Herbst, 2006). Even though there were slight differences between different classes in how Megan stated the task for the day, students understood what was expected of them and worked on the problem (Herbst, 2006). During the lesson, Megan was cognizant of the lesson's objectives and moved students towards that goal through questioning (Herbst, 2006). Some students answered her initial question with "in the middle", and Megan questioned students to be more specific with what they meant by "in the middle" (Herbst, 2006). From what Herbst (2006) observed, what the students were willing to do while still engaged in the task made it very difficult for the teacher to push students towards making a conjecture on their own without prompting.

Swafford, Jones, and Thornton (1997) studied the impact that a four week intervention program had on 4^{th} – 8^{th} grade teachers' content knowledge and instruction of geometry. The focus of the intervention was on content knowledge of geometry and student cognition in geometry. All 49 teachers that participated were given a pretest and posttest, to determine pre-intervention and post-intervention van Hiele levels. Three tasks to determine van Hiele levels at the pretest and posttest were hour long structured interview, 25 question multiple choice test designed for students adapted from the Cognitive Development and Achievement in Secondary School Geometry (CDASSG) project, and a lesson plan task where teachers had 20 minutes to write a lesson plan on a given topic for their grade level. All teachers increased van Hiele levels by the posttest from the pretest, and 50% of the teachers increased by two levels. Determined from achievement through intervention, four of the least improved and four of the most improved teachers, eight total, were chosen for follow-up observations. Observations were to confirm the van Hiele level shown in their posttest lesson plan. There was improvement in what the teachers taught and how it was taught. Within the group of eight that participated in the follow-up, some key changes in what was taught were: increase time in classroom teaching geometry; increase in quality of geometric tasks students were asked to engage in; and a shift from role of teacher is to give answers to teacher as a facilitator, “generating questions that probed students’ thinking and engaging them to discuss” (Swafford, Jones, & Thornton, 1997, p. 480). “Change in the way geometry was taught was further reflected in the higher expectations teachers had for their students’ thinking in geometry” (Swafford, Jones, & Thornton, 1997, p. 480).

Herbst and Kosko (2012) documented efforts in developing an instrument to measure mathematical knowledge for teaching high school geometry (MKT-g). MKT-g focused on four domains from Ball et al. (2008), common content knowledge (CCK), specialized content knowledge (SCK), knowledge of content and teaching (KCT), and knowledge of content and students (KCS) (Herbst & Kosko, 2012). Scores from items piloted showed no statistical relationship with total years of experience teaching in general (Herbst & Kosko, 2012). However, scores from collections of items in domains showed statistically significant correlations with years of experience teaching high school geometry (Herbst & Kosko, 2012). Table 2-8 summarizes articles involving geometry.

Table 2-8 Articles involving geometry

Author(s)	Grade	Teacher Attributes	Student Achievement
Marrades	4^{th}	- Activities were generally structured in	students were able to

& Gutiérrez (2000)	grade	three phases; create and/or explore, generate and/or verify conjectures, and justify conjectures - provided the figure and students had access to it to use it to explore properties instead of creating their own - prompted students with questions such as "why is the construction valid?" and "why is the conjecture true?" (Marrades & Gutiérrez, 2000, p. 98).	"progress toward more elaborated types of justifications" (p. 120) and showed confidence in their ability of forming deductive justifications and formal proofs
Martin, et al. (2005)	High School	- Posed an open-ended tasks - Engaged students in verbal reasoning - Analyzed, coached, and re-voiced questions back to students	Developed ability to construct chains of reasoning
Herbst, (2006)	High School	cognizant of the lesson's objectives and moved students towards that goal through questioning	students were willing to reason when engaged in the task
Jones (2000)	12 year old students	- Used Cabri I - teacher questioned students and redirected them to pertinent features of the geometry software, encouraged conjecturing, and focused students towards forming mathematical explanations	upward movement in van Hiele levels
Han (2007)	Middle School	- Questioning to get students to come to their own correct conjectures of quadrilateral properties - Both groups of students were taught relationally	Students that learned using dynamic geometry software scored higher on author created assessment
Herbst & Kosko (2012)	In-service teachers	- Looked at CCK, SCK, KCT, KCS - General scores were not statistically significant to years taught - Scores on domains were significant to years taught high school geometry	

Teacher Knowledge, Classroom Instruction, and Student Achievement

To organize existing teacher knowledge literature for the purpose of framing the problem for this dissertation, studies are classified into four categories: first, teachers' knowledge and its impact on student achievement; second, teachers' knowledge and how it relates to instruction; classroom instruction and how that impacts student achievement; and finally, studies of teacher knowledge without classroom instruction or student achievement.

Teacher Knowledge and Student Achievement. Many studies linking teacher knowledge and student achievement are of elementary grades (e.g. Carpenter, Fennema, Peterson, Chi-Pang, & Loef, 1989, first grade teachers teaching problem solving skills in addition and subtraction; Hill, Rowan, & Ball, 2005, first and third grade teachers; Saxe, Gearhart, & Nasir, 2001, upper elementary teachers teaching fractions). Some studies made the connection between teacher knowledge to student achievement to show positive effects of professional development (e.g. Jacobson & Lehrer, 2000; Saxe, Gearhart, & Nasir, 2001). Others (e.g. Darling Hammond, 2000), like Monk (1994) looked at different sources of teacher knowledge and/or other possible factors in student achievement to see what predicted student achievement better. However, Monk's (1994) study will be expounded upon in this dissertation over Darling-Hammond's (2000) study due to its relevancy to mathematics. The following studies expounded upon either related to this dissertation more than others in its group mentioned and/or were major studies in mathematics education.

Data on 483 secondary science teachers and 608 mathematics teachers' subject knowledge and knowledge of pedagogy were collected to see the effect on student achievement in Monk's (1994) study. The instruments used for collecting data were researcher generated surveys and achievement tests (Monk, 1994). As a part of a larger study, surveys were filled out by teachers, administrators, parents, and students; while the achievement tests were administered to the students (Monk, 1994). Coursework completed in the subject area represented the teachers' subject knowledge while courses in pedagogy represented teacher knowledge of pedagogy. Monk (1994) sampled 2,829 tenth grade public school students during the base year from schools randomly selected all over the nation over the course of three years; 1987, 1988, and 1989. In regards to results from student achievement in comparison to characteristics found on teacher surveys, Monk (1994) found that student achievement positively related to the number of courses the teacher took in the subject. However, teacher college coursework in pedagogy also had positive effects on student achievement, and at times, had a greater impact on student achievement than subject matter courses (Monk, 1994).

Hill, Rowan, and Ball's (2005) study of 1,190 first graders and 1,773 third graders; and 334 first and 365 third grade teachers in 115 elementary schools during the 2000–2001 through 2003–2004 school years. Hill, Rowan, and Ball's (2005) aim was to see if teachers' mathematical knowledge for teaching could positively predict student achievement. Teacher data was gathered from teacher self-report logs, self-report questionnaires, and a 25-question multiple choice test (Hill, Rowan, & Ball, 2005). The questionnaires gave Hill, Rowan, and Ball (2005) teacher background as well as teacher knowledge for teaching. Their measure for teacher knowledge of teaching represents the "knowledge that teachers use in classrooms, rather than general mathematical knowledge" (Hill, Rowan, & Ball, 2005, p. 387). Two examples of items on the multiple choice test measuring teacher content knowledge of teaching mathematics is shown in Figure 2-4 (Hill, Rowan, & Ball, 2005).

1. Mr. Allen found himself a bit confused one morning as he prepared to teach. Realizing that 10 to the second power equals 100 ($10^2 = 100$), he puzzled about what power of 10 equals 1. He asked Ms. Berry, next door. What should she tell him? (Mark [X] ONE answer.)

a) 0
b) 1
c) Ten cannot be raised to any power such that 10 to that power equals 1
d) −1
e) I'm not sure

2. Imagine that you are working with your class on multiplying large numbers. Among your students' papers, you notice that some have displayed their work in the following ways:

Student A	Student B	Student C
35	35	35
× 25	× 25	× 25
125	175	25
+ 75	+ 700	150
875	875	100
		+ 600
		875

Which of these students would you judge to be using a method that could be used to multiply any two whole numbers?

	Method would work for all whole numbers	Method would NOT work for all whole numbers	I'm not sure
Method A	1	2	3
Method B	1	2	3
Method C	1	2	3

Figure 2-4 Two Sample Questions on the Teacher Questionnaire**.** From "Effects of teachers' mathematical knowledge for teaching on student achievement." by H.C. Hill, B. Rowan, and D.L. Ball, 2005, *American Educational Research Journal, 42*(2), p. 401-402. Copyright 2005 by *SAGE Publications*. Reprinted with permission.

These two items exemplified “two key elements of content knowledge for teaching mathematics: ‘common’ knowledge of content…. and ‘specialized’ knowledge used in teaching students mathematics” (Hill, Rowan, & Ball, 2005, p. 387). Due to the two key elements of the test, items where the correct answer indicated a teaching orientation were eliminated.

Hill, Ball, and Rowan (2005) did not create the assessments for the students like they did for the teachers. Instead,

> The measures of student achievement used here were drawn from CTB/ McGraw-Hill’s Terra Nova Complete Battery (for spring of kindergarten), the Basic Battery (in spring of first grade), and the Survey (in third and fourth grades). Students were assessed in the fall and spring of each grade by project staff, and scores were computed by CTB via item response theory (IRT) scaling procedures. (Hill, Rowan, & Ball, 2005, p. 382).

Hill, Rowan and Ball (2005) found that teachers’ mathematical knowledge for teaching positively predicted student gains in mathematics achievement for first and third grade students. An improvement that Hill, Rowan, and Ball (2005) made on studies involving teacher knowledge is that this one involved not just computational fluency, but actual knowledge specific to teaching.

Summary. Studies in the past quantified teachers’ subject area knowledge, like Monk (1994), by looking at the coursework of teachers in college, college degree, and to some extent, teaching years; and quantifying teacher knowledge through assessments of current knowledge has only become mainstream in research starting in the past decade or so (Hill, Rowan, Ball, 2005). Coursework, degree(s), and teaching years were found to be insufficient to determine a teacher's content knowledge (Hill, Rowan, Ball, 2005). Thus, Hill, Rowan, and Ball (2005) used a questionnaire aimed at assessing teachers for their knowledge of content and knowledge for teaching students. However, Hill, Rowan, and Ball's (2005) participants took part in a larger study, thus information like courses taken and years teaching has been gathered, but was not a part of this particular study using the questionnaire for teacher knowledge. With a large sample size and collection of both qualitative and quantitative data, both qualitative and quantitative questions can be answered. Table 2-9 summarizes the two studies expounded upon in this section linking teacher knowledge and student achievement.

Table 2-9 Teacher Knowledge and Student Achievement

Author(s)	Grade	Teacher Knowledge	Student Achievement
Monk (1994)	Secondary	Content and pedagogy: Coursework completed in subject area and pedagogy	Positively related to the number of subject area courses, but courses in pedagogy had similar or greater positive effects
Hill, Rowan, & Ball (2005)	1st & 3rd grade	Mathematical knowledge for teaching: Logs, questionnaires, and multiple choice test	Positively predicted student achievement

Classroom Instruction and Student Achievement. Hill, Rowan, and Ball (2005) attributed origins of research in predicting student achievement from teacher characteristics to process-product literature on teaching; citing Brophy and Good (1986), Gage (1978), and Doyle (1977). Brophy's (1986) review of research "summarize[d] the findings of research linking teacher behavior to student achievement" (p. 1069). Brophy (1986) addressed both quantity and quality measures of teaching behaviors that positively influenced student achievement. Aspects of teaching behaviors that were documented quantitatively that positively impacted student achievement were opportunity to learn and content covered; role definition, expectations, and time allocation; classroom management and student-engagement time; consistent success and academic learning time; and active teaching (Brophy, 1986). In addition to how much teachers performed the above tasks, Brophy (1986) asserted that how well teachers performed instructional tasks were also addressed; the qualitative measures. Those are, giving information, questioning the students, and reacting to student responses (Brophy, 1986). One critique by Shulman (1986) of process-product research naming Brophy and Good (1986) and Gage (1978) among others was the lack of reference to subject specific knowledge and classroom and student contexts. Shulman (1986) attributed this void to "the quest for general principles of effective teaching" (p. 6).

Mindful of Shulman's (1986) criticisms of process-product research, I chose to expound upon studies like Jacobson and Lehrer's (2000) study as the scope of what the teacher does in the classroom is not limited to general teaching behaviors. Their approach to studying teacher knowledge and student achievement was not solely based on teacher behaviors, but on specific teachers' mathematical knowledge for teaching geometry (Jacobson & Lehrer, 2000). Unlike both Monk (1994) and Hill, Rowan, and Ball (2005) which had large numbers of participants; Jacobson and Lehrer (2000) studied only four teachers.

This study, on a smaller scale allowed the researchers to examine on a deeper level what teachers did in the classroom to promote discourse for student learning in conjunction to student achievement as a result of the discourse. Jacobson and Lehrer's (2000) study involved second grade teachers and their "knowledge about students' thinking" (p. 90). This case study was of four teachers and their classrooms, two of which are in the third year as participants in a research program on teaching and learning of geometry, and the other two were not (Jacobson & Lehrer, 2000). All four teachers had gone through Cognitively Guided Instruction (CGI) training, where instructional decisions are based on research of student thinking. The two teachers that participated in the research program were more knowledgeable in the area of "students' thinking about space and geometry" (Jacobson & Lehrer, 2000, p. 90) than the other two.

At the end of the 4 week unit, all students "comprehended and produced rotations, reflections, and compositions of motions," but students of the two more student-knowledgeable teachers "learned more than did their counterparts, and this difference in learning was maintained over time" (Jacobson & Lehrer, 2000, p.90). "Although all teachers elicited students' thinking, the two teachers who were more knowledgeable about students' thinking about space orchestrated classroom talk in ways that refined, elaborated, and extended students' thinking, albeit in different ways" (Jacobson & Lehrer, 2000, p.86). One of the two teachers that were in the research program on teaching and learning of geometry facilitated discourse "By posing questions and re-voicing students' comments that focused, refined, or 'lifted out' important ideas" (Jacobson & Lehrer, 2000, p. 86). Having participated in professional development, the teachers that were more knowledgeable about student thinking of geometry tailored their classroom discourse accordingly (Jacobson & Lehrer, 2000). Evidence in the case study showed that effective discourse was happening often in the two experimental classes and not in the other two control classes (Jacobson & Lehrer, 2000).

Han (2007); Marrades and Gutiérrez (2000); and Pesek and Kirsher (2000) (described earlier in this chapter) were also studies where instruction in the classroom impacted student achievement. Han (2007) was a dissertation that focused on one unit where both classes were taught relationally, but the one class that used dynamic geometry software scored higher on author created assessment than the one without using dynamic geometry software. Marrades and Gutiérrez (2000) students were able to "progress toward more elaborated types of justifications" (p. 120) and showed confidence in their ability of forming deductive justifications and formal proofs when the teacher thoughtfully created activities, scaffolded, and asked questions that promoted student learning. Pesek and Kirsher (2000) compared students who were taught relationally as opposed to instrumentally, and found that students that were taught relationally were better able to apply what was learned. Table 2-10 summarizes the studies that discuss classroom instruction and student achievement.

Table 2-10 Classroom Instruction and Student Achievement

Author(s)	Grade	Teacher Knowledge	Student Achievement
Han (2007)	Middle School	- Questioning to get students to come to their own correct conjectures of quadrilateral properties - Both groups of students were taught relationally	Students that learned using dynamic geometry software scored higher on author created assessment
Jacobson & Lehrer (2000)	2nd grade	Knowledge about students' thinking: Observation – more knowledgeable teachers orchestrated classroom talk in	Learned more with permanence

		ways that refined, elaborated, and extended students' thinking	
Marrades & Gutiérrez (2000)	4th grade	- Activities were generally structured in three phases; create and/or explore, generate and/or verify conjectures, and justify conjectures - provided the figure and students had access to it to use it to explore properties instead of creating their own - prompted students with questions such as "why is the construction valid?" and "why is the conjecture true?" (Marrades & Gutiérrez, 2000, p. 98).	students were able to "progress toward more elaborated types of justifications" (p. 120) and showed confidence in their ability of forming deductive justifications and formal proofs
Pesek & Kirshner (2000)	5th grade	Taught relationally	Able to better apply what was learned

Teacher Knowledge and Classroom Instruction. Some studies like Cohen (1990) and Heaton (1992) observed teachers during instruction in the classroom and based teacher knowledge off of what was observed; while other studies like Borko et al. (1992), Thompson and Thompson (1994), and Hill et al. (2008) both assessed teacher knowledge separately from instruction, and then assessed level of instruction. Cohen (1990) and Heaton (1992) were two studies among many in the literature that helped shape Hill et al.'s (2008) Mathematical Quality of Instruction observation protocol about significant mathematical errors in instruction. The following paragraphs of this section will expound on Borko et al.'s (1992), Cohen's (1990), Heaton's (1992), Hill et al.'s (2008), and Thompson and Thompson's (1994) articles.

Cohen's (1990) article served the purpose of looking at the relationship between educational policy and teaching practice. Mrs. Oublier, the elementary teacher in the study, believed that she was teaching what educational policy had suggested, teaching for understanding, but Cohen (1990) found that Mrs. Oublier taught very traditionally. Further, she did not promote discourse, accepted inaccurate student answers to her questions, and did not take opportunities that Cohen (1990) deemed necessary to develop student understanding. One example of this occurring was on a lesson about estimation, and she accepted inaccurate answers and failed to follow up on why those answers were wrong. In addition, she did not lead students to the correct answer based off of what the students talked about; she provided the rule for why estimation was the way it was.

Another study that helped shape Hill et al.'s (2008) Mathematical Quality of Instruction observation protocol about significant mathematical errors in instruction was Heaton's (1992) case study of a fifth grade teacher. Heaton's (1992) study found that there were significant problems in instruction in the areas of mathematics content, facilitation of student learning, lesson choice and focus, and learning outcomes. However, students were engaged in learning and were working on real-world problems (Heaton, 1992). On one observation, the lesson was a part of a 5-day unit that had students designing a park on a $5000 budget. It was not explicitly stated in the article what the teacher thought the goal of the lesson was, but Heaton (1992) stated that the goal should have been reasonableness of answers. From observation of the lesson, it was clear to Heaton (1992), that this was not the case. The teacher had students measure a sandbox at the school to get a sense of scaling, but two students measured in yards and the other student, that measured the height, measured it in feet. The sandbox was 46 yards x 10 yards x 1 foot. The teacher instructed students to multiply 46x10x1 and did not catch this error throughout the lesson. In this case, teacher self-assessment of teaching ability did not match reality (Heaton, 1992). Hill et al. (2008) attributed "policymakers' concerns about the quality of classroom work" (p. 433) to studies like Cohen (1990) and Heaton (1992).

Borko et al. (1992) studied a student teacher's failed attempt to teach division of fractions conceptually from many different vantage points. Borko et al. (1992) looked at coursework that Ms. Daniels, the sixth grade student teacher in the study, as basis for teacher knowledge. Ms. Daniels was a math major, with her math credits in courses modern algebra and computer science courses, but did not take many elementary mathematics courses; especially one in elementary number concepts and operations. When different aspects like coursework, beliefs about teaching, and beliefs about learning were considered; Borko et al. (1992) suggested that the reason for Ms. Daniels' failed attempt to teach division of fractions conceptually was because of the missed opportunity to explore number concepts and operations at the elementary level.

Bill, a middle school teacher, in an article by Thompson and Thompson (1994) "was very adept at reasoning proportionally, whether relationships were direct or inverse" (p. 282). In a paper and pencil test developed by the Rational Number Project (Post, Harel, Behr, & Lesh, 1991), Bill "meaningfully and creatively solved each proportional reasoning item" (Thompson & Thompson, 1994, p. 282). An example given by Thompson and Thompson (1992) was:

> "A pan balance is off center. An object put on one side weighs 10 lb. The same object put on the other side weighs 40 lb. How much does the object weigh?" Bill saw easily that the two pans' distances from the fulcrum had to be inversely proportional to the ratio of the weights. So, if $d1/d2$ is the ratio of the distances from the fulcrum, where $d1$ is the shorter distance, and if we let x stand for the

object's weight, then the ratios $x/40$ and $10/x$ must be equal to $d1/d2$, and therefore must be equal to each other (p. 282).

However, in a tutoring session where the student was confused about making sense of what the numbers mean when rates were represented in fraction form and what to do with them, Bill was not able to facilitate conceptually, what the student needed to know. Bill had conceptual knowledge of the topic, but was not able to facilitate the student to learn conceptually.

An exploratory study conducted by Hill et al. (2008) of mathematical knowledge for teaching and the mathematical quality of instruction showed that "there is a significant, strong, and positive association between levels of MKT [mathematical knowledge for teaching] and the mathematical quality of instruction" (Hill, et al., 2008, p. 430). This sample was of ten teachers, between 2nd and 6th grade, of which 5 of the strongest cases were chosen to be illustrated in the article (Hill, et al., 2008). They completed pencil and paper assessment of mathematical knowledge for teaching, were videotaped for nine lessons, and participated in post observation debriefings and interviews. The mathematical knowledge for teaching instrument assessed teachers' common and specialized content knowledge (Hill, et al., 2008). The observation protocol used was the Mathematical Quality of Instruction (MQI) developed by the Learning Mathematics for Teaching Project. Areas of focus for Mathematical Quality of Instruction when observing the teachers were:

- Mathematics errors—the presence of computational, linguistic, representational, or other mathematical errors in instruction;
    - Contains subcategory specifically for errors with mathematical language

• Responding to students inappropriately—the degree to which teacher either misinterprets or, in the case of student misunderstanding, fails to respond to student utterance;

• Connecting classroom practice to mathematics—the degree to which classroom practice is connected to important and worthwhile mathematical ideas and procedures as opposed to either non-mathematical focus, such as classroom management, or activities that do not require mathematical thinking, such as students following directions to cut, color, and paste, but with no obvious connections between these activities and mathematical meaning(s);

• Richness of the mathematics—the use of multiple representations, linking among representations, mathematical explanation and justification, and explicitness around mathematical practices such as proof and reasoning;

• Responding to students appropriately—the degree to which the teacher can correctly interpret students' mathematical utterances and address student misunderstandings;

• Mathematical language—the density of accurate mathematical language in instruction, the use of language to clearly convey mathematical ideas, as well as any explicit discussion of the use of mathematical language (Hill, et al., 2008, p. 437).

Further, Hill, et al. (2008) looked at other possible factors that influence teacher knowledge and quality of instruction as well: "teacher beliefs about how mathematics should be learned and how to make it enjoyable by students; teacher beliefs about curriculum materials and how they should be used; and the availability of curriculum materials to teachers" (Hill, et al., 2008, p. 497); and found teacher knowledge was itself, a factor of the factors that influence teacher knowledge and quality of instruction. Thus "the inescapable conclusion of this study is that there *is* a powerful relationship between what the teacher knows, how [the teacher] knows it, and what [the teacher] can do in the context of instruction" (Hill, et al., 2008, p. 496). Table 2-11 below summarizes articles on teacher knowledge and classroom instruction.

Table 2-11 Teacher Knowledge and Classroom Instruction

Author(s)	Grade Level	Purpose
Borko et al. (1992)	6th grade	Different vantage points of why a student teacher failed to teach fraction division conceptually
Cohen (1990)	elementary	Educational policy and classroom instruction
Heaton (1992)	5th grade	Importance of subject matter knowledge in teaching mathematics and what happens when there is an apparent lack of knowledge from observations
Hill, et al. (2008)	2nd -6th grade teachers	Exploratory study of mathematical knowledge for teaching and the mathematical quality of instruction
Thompson & Thompson	Middle School	Documented a teacher's difficulty conceptually explaining rates when the teacher understands rates conceptually

Teacher Knowledge. Articles like Garet, Porter, Desimone, Birman, and Yoon, (2002); Hill et al. (2007); and Hill, Schilling, and Ball (2004); looked at professional development and its effect on teacher knowledge. Hill, Ball, and Schilling's (2008) study took data from teachers that were going through professional development, but the data taken was to help define an aspect of mathematical knowledge for teaching. Studies like An, Klum, and Wu (2004); Ma (1996); Schmidt, Tatto, Bankov, Blömeke, Cedillo, Cogan, Han, Houang, Hsieh, Paine, Santillan, and Schwille, (2007); compared different teachers' knowledge in different countries. Most of the work that has been done to assess teacher content knowledge in geometry using the Van Hiele Geometry Test by Usiskin (1982) was by Erdoğan Halat in Turkey. Herbst and Kosko (2012) published a paper that documented efforts in developing an instrument that measured mathematical knowledge for teaching high school geometry (MKT-G). Details of that study were expounded upon earlier in this chapter on page 48.

Hill, Ball and Schilling (2008) used assessment documents to assess teachers' mathematical knowledge for teaching. Part of the data was from California's Mathematics Professional Development Institutes for their effectiveness. There were more than 370 teachers who responded for each of their three forms. The three forms were "number and operations common and specialized content knowledge; number and operation KCS [knowledge of content and students]; and patterns functions, and algebra common content knowledge" (Hill, Ball, & Schilling, 2008, p. 382). They found that knowledge of content and students is distinct form pure pedagogical or content knowledge (Hill, Ball, & Schilling, 2008).

In a comparison study, An, Klum, and Wu (2004) used an author-constructed teaching questionnaire, beliefs questionnaire, interviews, and observations to compare pedagogical content knowledge of middle school teachers in China and the United States. The sample size for this study at the questionnaire phase was 28 mathematics teachers in Texas and 33 mathematics teachers in China; five teachers from each country were observed to represent a diverse range of educational background and teaching experience. An, Klum, and Wu (2004) reported that they followed the Instructional Criteria Observation Checklist adapted from a framework set forth by the American Association for the Advancement of Science for analyzing instructional quality of mathematics textbooks. According to An, Klum, and Wu (2004).

> The observation criteria included specific activities in the categories: building on student ideas in mathematics, being alert to students' ideas, identifying student ideas, addressing misconceptions, engaging students in mathematics, providing first-hand experiences, promoting student thinking about mathematics, guiding

interpretation and reasoning, and encouraging students to think about what they had learned (pp. 151-153).

As An, Klum, and Wu (2004) stated in their methodology section, the use of interviews and observations was to provide validity and reliability for the data collected from questionnaires. A sample pedagogical content problem from the questionnaire stated that "Adam is a 10-year-old student in 5th grade who has average ability. His grade on the last test was an 82 percent" (An, Klum, & Wu, 2004, p. 152). Then the prompt asked teachers to look at Adam's work on two problems:

$$\frac{3}{4}+\frac{4}{5}=\frac{7}{9} \qquad 2\frac{1}{2}+1\frac{1}{2}=3\frac{2}{5}$$

Then teachers were asked three questions regarding the given situation:

a. What prerequisite knowledge might Adam not understand or be forgetting?
b. What questions or tasks would you ask Adam in order to determine what he understands about the meaning of fraction addition?
c. What real world example of fractions is Adam likely to be familiar with that you could use to help him? (An, Klum, & Wu, 2004, p. 152)

For this problem situation, 46% of the responses of the U.S. teachers thought that Adam forgot the prerequisite knowledge of finding common denominators, while 55% of the responses of the Chinese teachers thought that Adam did not understand the prerequisite knowledge of finding common denominators (An, Klum, & Wu, 2004). "Forgetting" and "not understanding" do not have the same meaning (An, Klum, & Wu, 2004). "Forgetting" implies that the teacher did not know of the challenges students face in learning addition of fractions, whereas "not understanding" implied that the teacher is aware of the student misconception (An, Klum, & Wu, 2004).

Results from the teaching questionnaire and beliefs questionnaire showed that teachers sampled from the two countries varied greatly, and the results from the observations and interviews showed consistency in the results from the questionnaire – pedagogical content knowledge has an impact on teaching practice (An, Klum, & Wu, 2004). Teachers sampled in the United States gave students a lot of different activities using manipulatives to promote creative thinking to develop concept mastery but lacked making the connection between what the students were doing and the abstract concept (An, Klum, & Wu, 2004). "The Chinese system emphasizes gaining correct conceptual knowledge by reliance on traditional, more rigid development of procedures, which has been the practice of teaching and learning mathematics content for many years" (An, Klum, & Wu, 2004, p. 169).

Cankoy (2010) is another example of teacher in teacher knowledge literature. Unlike An, Klum, and Wu (2004); Cankoy's (2010) study focused on a specific topic. Cankoy (2010) aimed to explore teacher topic-specific pedagogical content knowledge in asking for what the teacher thought was the best teaching strategy for teaching $a^0 = 1$, $0! = 1$ and $a \div 0$ where $a \neq 0$ to high school student. Of the 58 teachers in this study, the range of years of experience was from 1 – 17 years with the mean of 5.9 and standard deviation of 4.65. Cankoy (2010) categorized the teachers into two groups of teaching experience, "novice" having taught 1-4 years (n=33) and "experienced" having taught 5-17 (n=25). Qualitative data analysis through coding was used to sort the different teacher responses, and χ^2 test was used to determine if experience was related to strategy. For each $a^0 = 1$, $0! = 1$ and $a \div 0$ where $a \neq 0$, there were three main approaches; rule, pattern, and algebraic or limit for $a \div 0$ where $a \neq 0$. Surprisingly, for $0! = 1$ and $a \div 0$ where $a \neq 0$, some teachers had no answer, and for a÷0 where a≠0, there were some undeterminable approaches. An average of 27% of experienced teachers suggested pattern approach for all three statements, and only an average of 1% of novice teachers preferred that approach. However, most teachers in both groups relied heavily on the procedure and memorization to teach students. The separation between novice and experienced teachers was not as clear cut as expected. Cankoy's (2010) teaching strategies for the topic were literature based.

In several studies, Erodgan Halat used the Van Hiele Geometry Test (VGHT) to quantatively analyze differences in geometric understanding of teachers of different gender, school level (Halat, 2006; Halat, 2008, May), and pre-service compared to in-service (Halat, 2008; Halat & Sahin, 2008). When the reasoning stages of in-service middle and high school mathematics teachers in geometry were investigated, Halat (2008) found that the in-service middle and high school mathematics teachers represented all the van Hiele levels, visualization, analysis, ordering, deduction, and rigor, and that there was no difference in terms of mean reasoning stage between in-service middle and high school mathematics teachers. Results of which were contrary to preconceived notions of the differences between middle school and high school teachers. Moreover, from Halat's 2008 results, there was no significant gender difference found regarding the geometric thinking levels. In this study, there was a total of 148 in-service middle and high school mathematics teachers. Since the focus of this dissertation is on the relationships between knowledge exhibited before and during classroom instruction, and not geometry teachers' overall knowledge of geometry; the VGHT will not be used to assess teacher knowledge.

An interesting study outside mathematics education comes from science education, where teacher pedagogical content knowledge from the perspective of experienced science teachers was studied (Lee & Luft, 2008). The teachers in the study were chosen from a program where experienced teachers were paired up with beginning teachers in a mentorship program – four teachers volunteered to participate in this study, and all four had at least 10 years of teaching experience and at least three years of mentoring experience in this program. Over a period of two years, the four teachers participated in semi-structured interviews, classroom observations, a collection of lesson plans, and monthly reflective summaries. Coding teachers' responses of what science teachers need to know for teaching yielded seven components under pedagogical content knowledge, knowledge of science, goals, students, curriculum and organization, teaching, assessment, and resources. Teachers were then asked to organize the seven components in a concept map showing how each is related to the other, and all four had different conceptualizations, each placing importance on different components. The results from this study (Lee & Luft, 2008) showed that experienced teachers' conceptions of pedagogical content knowledge was not that different from those proposed by Shulman (1987) content knowledge; general pedagogical knowledge; curriculum knowledge; pedagogical content knowledge; knowledge of learners and their characteristics; knowledge of educational contexts; and "knowledge of educational ends, purposes, and values, and their philosophical and historical grounds" (p. 8).

Since Shulman (1987), researchers, when conducting research, (e.g., Cankoy, 2010; Hill, Rowan, & Ball, 2005; Lee & Luft, 2008; Monk, 1994) focused on different aspects of pedagogical content knowledge. For example, Cankoy (2010) studied what high school mathematics teachers' believed are best instructional strategies for teaching $a^0 = 1$, $0! = 1$ and $a \div 0$ where $a \neq 0$ and named that "topic specific pedagogical content knowledge". On a broader note, Li Ping Ma's (1996) dissertation looked at teachers in China and teachers in the United States to compare their profound understanding of fundamental mathematics (PUFM) and how PUFM is attained. Similar to An, Klum, and Wu's (2004) study, findings showed that teacher knowledge is very important, and the way teachers conduct class largely depends on teacher pedagogical content knowledge. Table 2-12 summarizes the studies that elucidate teacher in teacher knowledge literature.

Table 2-12 Teacher in the Teacher Knowledge Literature

Author(s)	Grade Level	Purpose
Hill, Ball, & Schilling (2008)	Elementary	Measuring the domain of mathematical knowledge for teaching, specifically knowledge of content and students
Hill, et al. (2008)	2^{nd} -6^{th} grade teachers	Exploratory study of mathematical knowledge for teaching and the mathematical quality of instruction
An, Wu, & Klum (2004)	Middle School	Compare pedagogical content knowledge of middle school teachers in China and the United States
Cankoy (2010)	High School	What the teacher thought was the best teaching strategy for teaching $a^0 = 1$, $0! = 1$ and $a \div 0$ where $a \neq 0$ to high school student
Halat (2008)	Secondary	Compare van Hiele levels of in-service middle school and high school teachers
Lee & Luft (2008)	Secondary	"look at how mentor science teachers conceptualized their own PCK that impacted their teaching practice" (p. 1348).
Ma (1996)	Elementary	Compared teachers in China and the United States on their profound understanding of fundamental

		mathematics
Herbst & Kosko (2012)	High School	Measured mathematical knowledge for teaching high school geometry, specifically common content knowledge, specialized content knowledge, knowledge of content and teaching, and knowledge of content and students.

Assessing Teacher Knowledge

Part of the goal of this dissertation is to explore teachers' mathematical knowledge for teaching. Different assessments of different teacher knowledge types will be necessary (Hill, Ball, & Schilling, 2008). As mentioned in the introduction, according to Hill, Sleep, Lewis, and Ball (2007); there are three contemporary pressures on researchers to develop more coherent approaches to assess teachers' knowledge: political environment, evidence of teacher competence, and to help define what makes teaching a profession. Studies in the past quantified teacher's subject area knowledge by looking at the coursework of teachers in college, college degree, and to some extent, teaching years; and quantifying teacher knowledge through assessments of current knowledge has only become mainstream in research starting in the past decade or so (Ball, & Rowan, 2004; Hill, Rowan, Ball, 2005). Usiskin's (1982) Van Hiele Geometry Test tested students' content knowledge in geometry, and several researchers have used it to assess current content knowledge of pre-service and in-service teachers. So far, in terms of high school geometry and assessing teacher knowledge, Usiskin's (1982) van Hiele Geometry Test is the only one. In this section, various types of teacher assessment will be discussed.

Observation Protocols. In order to assess teacher knowledge in selecting and using representations and actual emphasis of key concepts; interviews were proven to be insufficient, as what teachers believe that they are doing might not equate to what they actually do in the classroom (An, Klum, & Wu, 2004). An, Klum, and Wu (2004) used an author-constructed teaching questionnaire, beliefs questionnaire, interviews, and observations to compare pedagogical content knowledge of middle school teachers in China and the United States. According to An, Klum, and Wu (2004)

> The observation criteria included specific activities in the categories: building on student ideas in mathematics, being alert to students' ideas, identifying student ideas, addressing misconceptions, engaging students in mathematics, providing first-hand experiences, promoting student thinking about mathematics, guiding interpretation and reasoning, and encouraging students to think about what they had learned (pp. 151-153).

The sample size for this study at the questionnaire phase was 28 mathematics teachers in Texas, and 33 mathematics teachers in China; five teachers from each country were observed to represent a diverse range of educational background and teaching experience. An, Klum, and Wu (2004) reported that they followed the Instructional Criteria Observation Checklist adapted from a framework set forth by the American Association for the Advancement of Science for analyzing instructional quality of mathematics textbooks.

For a large-scale study focusing on measuring teachers' quality of instruction Matsumura, Garnier, Slater, and Boston (2008) based their rubrics in the mathematics framework on "the Mathematical Task Framework developed by Mary Kay Stein, Margaret Smith and their colleagues" (p. 276). Matsumura et al. (2008) chose this framework because it considered "the potential of a task to support higher level, conceptual thinking including students' opportunity to engage with a range of representations in their responses, and the implementation of the task in practice (as enacted)." (p. 277). In the Mathematical Tasks Framework were three major areas; analyzing mathematics instructional tasks, using cognitively complex tasks in the classroom, and learning from cases (Smith et al. 2005). Under "analyzing mathematics instructional tasks" were "defining levels or cognitive demand of mathematical tasks", "matching tasks with goals for student learning", "differentiating levels of cognitive demand", "gaining experience in analyzing cognitive demands", and "moving beyond task selection and creation" (Smith et al. 2005). Development of "tasks during a lesson" and "patterns of task setup and implementation" fell under the umbrella of "Using cognitively complex tasks in the classroom" (Smith et al. 2005).

Matsumura et al. (2008) inferred from their research that the minimum number of assignments from a teacher to assess their beliefs of level of work expected of students is four, and minimum number of observations to get an accurate picture of how the teacher facilitates discourse is also four. Teachers filled out a form for each assignment explaining their reasoning behind the assignment, where the lesson fit in the content of the course, and the belief of level of difficulty of questions; as well as provided four student samples of graded work for each assignment at the teachers’ perception of two levels, high and medium. Matsumura’s et al. (2008) analysis showed strong correlation between items overlapping assignments and observable traits in class, yielding a dependability coefficient of .80 for four assignments; thus some observable traits in class can also be assessed by collecting assignments.

Learning Mathematics for Teaching (LMT) created the mathematical quality of instruction (MQI Instrument) to look at the predictive validity of their test for mathematical knowledge for teaching (Hill, et al. 2008). The five elements of mathematical quality of instruction observed in MQI were; mathematics errors, responding to students inappropriately, connecting classroom practice to mathematics, richness of the mathematics, responding to students appropriately, and mathematical language (Hill, et al., 2008). Hill et al., (2008) used MQI to measure mathematical quality of instruction to compare to teacher scores of mathematical knowledge for teaching. A more detailed explanation of the study is on page 62.

An et al. (2004) and Matsumura et al. (2008) used established observation protocols to gather the data necessary for their respective studies. Another observation protocol widely used in reform mathematics and science classrooms is called Reformed Teaching Observation Protocol (RTOP) (Sawada & Pilburn, 2000). RTOP should only be used by trained observers, and a method in which to be trained is to read through the training guide provided by the creators of RTOP. The theoretical and philosophical rationale behind the reform mathematics movement was that of constructivism – where attainment of knowledge is through active participation of the learner; and "RTOP was designed to capture the current reform movement, and especially those characteristics that define 'reform teaching'" (Sawada & Pilburn, 2000, p. 2). There are many observation protocols available, but for the purpose of this dissertation, I created an observation protocol specific to capturing data for the types of teacher knowledge. Observation alone is not sufficient to gather data on teacher knowledge (Hill, Sleep, Lewis, & Ball, 2007).

Concept Maps. Concept maps are tools that researchers may use to assess conceptual understanding of a concept (Novak & Cañas, 2008; Williams, 1998).

Concepts are typically enclosed in planar geometric shapes with few words that signify the concept, sometimes called semantic units or units of meaning (Novak & Cañas, 2008). Then between concepts are arrows that signify two concepts are related, and relation may be written on that arrow; with the option of cross-links, or arrows crossing in a concept map (Novak & Cañas, 2008). Concept maps are also hierarchical in nature, depending on the directions in which the arrows point, some concepts may be a few concepts away from another concept; thus creating an order in which concepts are related (Novak & Cañas, 2008). Below, Figure 2-5, is a concept map that shows key features of concept maps. “Concept maps tend to be read progressing from the top downward" (Novak & Cañas, 2008, p. 2).

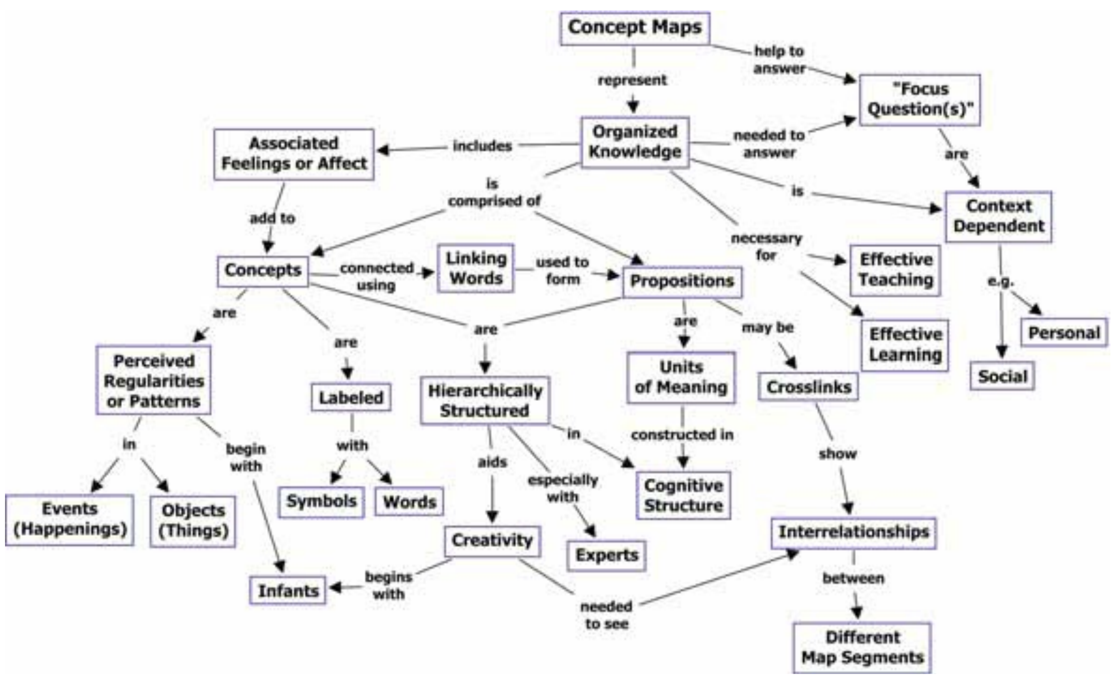


Figure 2-5 Sample concept map (Novak & Cañas, 2008, p. 2).

Carol G. Williams used concept maps to assess students' conceptual knowledge of function (Williams, 1998). Her subjects were 28 students in third quarter calculus; 14 took all three semesters of reformed calculus, and 14 took all three semesters traditional calculus; and eight professors of mathematics. There was a special session on how to construct a concept map. The main goal of the study was to use concept maps as a research tool to reflect conceptual understanding. " The general homogeneity of the experts' concept maps and their distinct variance from students' maps lend credibility to the conclusion that concept maps do capture a representative sample of conceptual knowledge and can differentiate among fairly disparate levels of understanding" (Williams, 1998, p. 420).

The purpose of Jin and Wong's (2010) study was to explore several different training durations to help secondary level students acquire sufficient concept mapping skills to use when learning mathematics. The time increments were 10 minutes, 15 minutes, and 50 minutes. They found that teaching how to concept map meaningfully took 50 minutes and is best done alongside current curriculum (Jin & Wong, 2010). Jin and Wong (2010) defined "meaningfully" as meaningful placement of given concepts, labeled connections, and accuracy of descriptions. However, a document like a teachers' concept map of the topic without the teacher explaining gives limited insight into what teachers know about teaching; even with classroom observations of the teachers teaching the topic.

Chinnappan and Lawson (2005) interviewed two experienced teachers who were involved in a larger study of five experienced teachers and five novice teachers in Australia. The purpose of this study was to test the framework in with Chinnappan and Lawson (2005) proposed to study teacher knowledge for teaching. These two teachers each had taught for at least 15 years at the high school level and were recommended by peers or other sources as exemplary teachers (Chinnappan & Lawson, 2005). These two teachers did not teach at the same school (Chinnappan & Lawson, 2005). Subjects were assessed through structured interviews where making concept maps were the interview tasks to gauge geometric content knowledge and geometric knowledge for teaching (Chinnappan & Lawson, 2005). Results showed that the mapping procedure was useful for purposes where teacher content knowledge and content knowledge for teaching are both required for comparison (Chinnappan & Lawson, 2005).

Interviews. Chinnappan and Lawson (2005) conducted three (one hour long) interviews with each teacher. The first interview focused on what the teacher knew about, and how to teach the content (Chinnappan & Lawson, 2005). Typical question during the first interview were “Tell me about what you know about squares” and “Tell me how you would teach square to your students” (Chinnappan & Lawson, 2005, p. 205). Teachers responded to four problems for the second interview (Chinnappan & Lawson, 2005). These problems were within the content area of what they taught; and in this case, about squares (Chinnappan & Lawson, 2005). Some questions asked during the second interview were “How would you expect your students to tackle this problem” and “What type of difficulties would you expect your students to experience if they are given these problem, Why?” (Chinnappan & Lawson, 2005, p. 205). The purpose of the final interview was to gather any more pertinent information not gathered in previous interviews (Chinnappan & Lawson, 2005). A sample statement to get the teacher to talk was “You haven’t said anything about the symmetry of a square yet” (Chinnappan & Lawson, 2005, p. 205). Chinnappan and Lawson used interviews with and without concept maps as an interview task to gather evidence of teacher knowledge (Chinnappan & Lawson, 2005).

An Klum and Wu (2004) used interviews to triangulate data from other data collection sources. As mentioned in the previous section, An Klum and Wu (2004) used interviews in conjunction with surveys and observations to compare pedagogical content knowledge of middle school teachers in China and the U.S. The set of interview questions examined teachers' beliefs of how teacher beliefs impact teaching practices, teaching approaches, preparation for instruction, and interpretation of student thinking. These interviews followed classroom observations "explored further the teachers' pedagogical content knowledge and its importance in their teaching" (An, Klum, & Wu, 2004, p. 153).

Some tasks in interviews for teachers are the same tasks given to students (Hill, Sleep, Lewis, & Ball, 2007). For example, Graeber, Tirosh, and Glover (1989) assessed prospective elementary teachers' misconceptions about multiplication and division with the same assessment for students. Burger and Shaughnessy (1986) devised tasks to interview students and were able to assess their van Hiele Levels, but did not assess in-service teachers. However, two college mathematics majors were included in the Burger and Shaughnessy (1986) study.

Burger and Shaughnessy (1986) determined van Hiele Levels of reasoning of 45 students from grades 1 to 12 and university mathematics majors through experimental interviews. "Experimental tasks were administered to each student by one of the four researchers in an audiotaped clinical interview" (Burger & Shaughnessy, 1986, p. 33). There were eight problems about geometric shapes for each individual (Burger & Shaughnessy, 1986). One of the tasks was for students to identify and define different quadrilaterals (Burger & Shaughnessy, 1986). Students were given a sheet of paper with many different types of quadrilaterals on it, and they were asked to write an “S” by the square, “P” by the parallelogram, and so forth (Burger & Shaughnessy, 1986). Researchers recorded and analyzed tapes to gather evidences of levels of thought (Burger & Shaughnessy, 1986). From those, van Hiele level of reasoning was determined based on the individual’s predominate level of reasoning (Burger & Shaughnessy, 1986). Table 2-13 summarizes literature on assessing teacher knowledge by type of assessment; observation protocols, concept maps, and interviews.

Table 2-13 Summary of Literature Assessing Teacher Knowledge

Type	Researcher	Purpose
Observation protocols	An, Klum, & Wu (2004)	Compare teachers in China to teachers in United States
	Matsumura, Garnier, Slater, & Boston (2008)	Efficient number of observations and assignments to collect from teachers for a large scale study
	Hill, et al. (2008)	Exploratory study of mathematical knowledge for teaching and the mathematical quality of instruction
Concept Maps	Williams (1998)	Assess students' conceptual knowledge of function
	Chinnappan & Lawson (2005)	Determine effectiveness of concept maps to assess teacher knowledge
	Jin Wong	Determine appropriate amount of time to teach effective creation of concept maps
Interviews	Burger & Shaughnessy (1986)	Determine student van Hiele Level of reasoning
	Chinnappan & Lawson (2005)	Determine effectiveness of concept maps to assess teacher knowledge
	An, Klum, & Wu (2004)	Compare teachers in China to teachers in United States

Summary of Literature Review

Chapter one gives reasons for why it would be pertinent to conduct this research study, and chapter two gives the literature background. Specifically, this chapter started with defining what it means for students to learn and teachers to teach. In discussing knowledge in teaching mathematics, I discussed Peterson's (1988) framework and Hill, Ball, and Schilling's (2004) domain map for mathematical knowledge for teaching. Figure 2-6 is a domain map for mathematical knowledge for teaching as it relates to Shulman's (1987) pedagogical content knowledge (Hill, Ball, Schilling, 2008, p. 377).

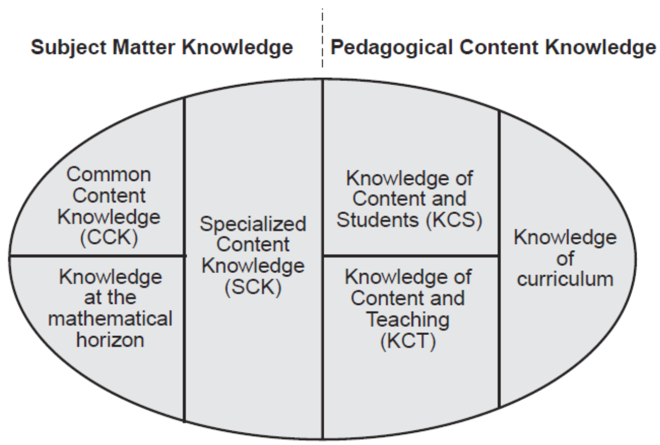


Figure 2-6 Domain map for mathematical knowledge for teaching as it relates to Shulman's (1987) pedagogical content knowledge. From "Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students." by H.C. Hill, D.L. Ball, and S.G. Schilling, 2008, *Journal for Research in Mathematics Education, 39*(4), p. 377. Copyright 2008 by *Journal for Research in Mathematics Education*. Reprinted with permission.

Specialized content knowledge is "mathematical knowledge that is used in teaching, but not directly taught to students" (Hill, Sleep, Lewis, & Ball, 2007, p. 132). Hence, specialized content knowledge is not categorized under pedagogical content knowledge. Hill, Sleep, Lewis, and Ball (2007) gave Figure 2-7, on the next page, as an example to illustrate specialized content knowledge.

Imagine that you are working with your class on multiplying large numbers. Among your students' papers, you notice that some have displayed their work in the following ways:

Student A	**Student B**	**Student C**
35	35	35
×25	×25	×25
125	175	25
+75	+700	150
875	875	100
		+600
		875

Which of these students would you judge to be using a method that could be used to multiply any two whole numbers?

	Method would work for all whole numbers	Method would *not* work for all whole numbers	I'm not sure
Method A	1	2	3
Method B	1	2	3
Method C	1	2	3

Figure 2-7 Sample SCK question for Multiplication**.** From "Effects of teachers' mathematical knowledge for teaching on student achievement." by H.C. Hill, B. Rowan, and D.L. Ball, 2005, *American Educational Research Journal, 42*(2), p. 401-402. Copyright 2005 by *SAGE Publications*. Reprinted with permission.

In answering the question in Figure 2-7, "If the teacher understands these explanations for the students' methods, she then must make a determination about whether each method generalizes to the multiplication of other whole numbers, perhaps by referencing the commutative or distributive properties of multiplication" (Hill, Sleep, Lewis, & Ball, 2007, pp. 132-133). Knowledge of content and students is the subset of pedagogical content knowledge that enables teachers to identify common student errors, interpret students' understanding of content, identify students' developmental sequences, and identify common student computational strategies (Hill, Ball, & Schilling, 2008). In general, knowledge of content and students is “the amalgamated knowledge that teachers possess about how students learn content” (Hill, et al., 2007, p. 133). Knowledge of content and teaching is the knowledge of teaching the content with emphasis on teaching; e.g. how to correct student mistakes so that they will understand and how to build on what the students already know (Hill, Ball, & Schilling, 2008). Hill et al. (2007) summarizes knowledge of content and teaching as “mathematical knowledge of the design of instruction” (p. 133). Specifically, knowledge of content and teaching “includes how to choose examples and representations, and how to guide student discussions toward accurate mathematical ideas” (Hill, et al., 2007, p. 133).

Specialized content knowledge, knowledge of content and students, and knowledge of content and teaching; the middle slice of Ball and colleagues’ domain map (Hill, Ball, & Schilling, 2008) of mathematical knowledge for teaching, are the three areas of mathematical knowledge for teaching that are the focus of this dissertation. I renamed these three areas collectively as Teacher Applied Content Knowledge (TACK) because I see them as applications of content knowledge.

TACK. Teachers’ Applied Content Knowledge (TACK) is based from a subset of Ball and colleagues’ categories of mathematical knowledge for teaching; knowledge of content and teaching, knowledge of content and students, and specialized content knowledge. The following subsections of this section describe these three categories of mathematical knowledge for teaching in reference to the data that will be collected for each.

Knowledge of Content and Teaching. Knowledge of content and teaching (KCT) is the knowledge of teaching content with emphasis on teaching (Hill, Ball, & Schilling, 2008). For example, an aspect of KCT is how to correct student mistakes so that teachers understand how to build on what students already know (Hill, Ball, & Schilling, 2008). Hill et al. (2007) summarized knowledge of content and teaching as "mathematical knowledge of the design of instruction" (p. 133). Specifically, knowledge of content and teaching "includes how to choose examples and representations, and how to guide student discussions toward accurate mathematical ideas" (Hill, et al., 2007, p. 133). Based off of literature from Ball and colleagues on knowledge of content and teaching, I simplified it to four categories to look at for this dissertation: 1) sequence of information or concepts taught, 2) time it takes to cover each item of the sequence of information, 3) explanations to teach, and 4) questions to teach.

Knowledge of Content and Students. Knowledge of content and students is knowledge of content specifically involving students (Ball, Thames, & Phillips, 2008). Hill, Ball, and Schilling (2008) defined knowledge of content and students as knowledge that enables teachers to identify common student errors, interpret students' understanding of content, identify students' developmental sequences, and identify common student computational strategies. Specifically for this study, knowledge of content and students is demonstrated by the teacher in three ways: modifications to what is being taught based on student difficulties; student misconceptions and how to help students overcome them; and knowledge of student prerequisite skills.

Specialized Content Knowledge. Specialized content knowledge, "mathematical knowledge that is used in teaching, but not directly taught to students" (Hill, Sleep, Lewis, & Ball, 2007, p. 132) and was assessed by Hill, et al. (2008) in a multiple choice paper and pencil test. For the purpose of specificity in coding, sections of transcript data were coded as specialized content knowledge if it exemplified any of the mathematical tasks of teaching from Ball, Thames, and Phelps (2008), with the exclusion of "Explaining mathematical goals and purposes to parents" as shown in Table 2-14 below.

Table 2-14 Mathematical Tasks of Teaching

Presenting mathematical ideas
Responding to students' "why" questions
Finding an example to make a specific mathematical point
Recognizing what is involved in using a particular representation
Linking representations to underlying ideas and to other representations
Connecting a topic being taught to topics from prior or future years
Explaining mathematical goals and purposes to parents
Appraising and adapting the mathematical content of textbooks
Modifying tasks to be either easier or harder
Evaluating the plausibility of students' claims (often quickly)
Giving or evaluating mathematical explanations
Choosing and developing useable definitions
Using mathematical notation and language and critiquing its use
Asking productive mathematical questions
Selecting representations for particular purposes
Inspecting equivalencies (p. 10)

In short, I condensed specialized content knowledge as the knowledge of concepts, definitions, and procedures for teaching. However, specialized content knowledge seems to overlap with knowledge of content and students and knowledge of content and teaching. Thus the next section aims to clarify the relationships.

SCK, KCT, and KCS Relationships. Ball, Thames, and Phelps (2008) noted that even though distinguishing between common content knowledge, specialized content knowledge, knowledge of content and students, and knowledge of content and teachers; may be subtle, it is possible given this example:

> recognizing a wrong answer is common content knowledge (CCK), while sizing up the nature of the error may be either specialized content knowledge (SCK) or knowledge of content and students (KCS) depending on whether a teacher draws predominantly from her knowledge of mathematics and her ability to carry out a kind of mathematical analysis or instead draws from experience with students and familiarity with common student errors. Deciding how best to remediate the error may require knowledge of content and teaching (KCT) (p.11).

However, I contend, for the purpose of this study that the kind of difference, highlighted above, between SCK and KCS cannot be determined within this study and is not necessary to determine the difference. In the way that TACK is defined, KCS, SCK, and KCT are not necessarily mutually distinct in their manifestations in the data for data analysis, but were defined separately as a way to explicate the origins of TACK.

Teaching and Learning Geometry. After discussion of literature on learning and teaching, literature regarding teaching and learning of geometry were expounded upon. Specific to teaching and learning geometry, separate from other topics in mathematics, are the van Hiele Levels (van Hiele, 1986), phases of learning based on the van Hiele Model (Mistretta, 2000), and geometric habits of mind (Driscoll, 2007).

Through discussing literature with evidence of student achievement, certain attributes of teachers surfaced. Many teachers used technology in the classroom, in the form of dynamic geometry software (Han, 2007; Jones, 2000; Marrades & Gutiérrez, 2000). In Marrades and Gutiérrez (2000) activities were generally structured in three phases; create and/or explore, generate and/or verify conjectures, and justify conjectures. The teacher provided the figure and students had access to it to use it to explore properties instead of creating their own (Marrades & Gutiérrez, 2000). Teachers questioned students with questions such as "why is the construction valid?" and "why is the conjecture true?" (Han, 2007; Jones, 2000; Marrades & Gutiérrez, 2000, p. 98). Also, teachers analyzed, coached, and re-voiced questions back to students (Martin, et al., 2006).

Teacher Knowledge, Classroom Instruction, and Student Achievement.

Studies discussed in reference to general teaching and learning, teaching and learning of geometry, and other studies involving teacher knowledge, classroom instruction, and student achievement was further detailed and synthesized into four categories. These four categories were: teacher knowledge and student achievement, classroom instruction and student achievement, teacher knowledge and classroom instruction, and teacher knowledge. From research on literature of studies in the four categories, attributes of teachers of students that showed achievement were similar to those described in teaching and learning geometry. However, in additional teacher attributes that surfaced were group work for students, persistence in student asking students questions, and knowledge related to teaching mathematics.

Assessing Teacher Knowledge. Discussing studies involving teacher knowledge and student achievement, classroom instruction and student achievement, teacher knowledge and classroom instruction, and teacher knowledge exposed methods in which those researchers assessed teacher knowledge. The "Assessing Teacher Knowledge" section emphasized three different assessments of teacher knowledge to expound upon; observation protocols, concept maps, and interviews. Some studies from the previous chapter were further discussed in relation to the methods in which they assessed teacher knowledge for each assessment method, while other studies that supported the use of those assessment methods were discussed as well.

The focus of this dissertation is on teacher knowledge during planning phase and execution of the lesson. Simon and Tzur (1999) stated that researchers need rich data to assess the complex relationships of teacher's beliefs, knowledge, values, intuitions, feelings, and practices. This chapter showed the literature behind choice of the aspects of teacher knowledge, teaching and learning geometry, area of need in literature on geometry teacher knowledge and classroom instruction, and literature on assessing teacher knowledge. The next chapter describes the theoretical perspective and methodologies used for this dissertation.

Chapter 3: Theoretical Perspectives and Methodologies

In the first chapter of this dissertation, I presented evidence to support that student achievement in geometry is an area of concern (Lappan, 1999; Mullis, Martin, & Foy, 2008) and why improving student achievement in geometry is necessary (National Research Panel, 1989; Shaughnessy, 2011). Thus, in order to improve student achievement in geometry, I chose to look at teacher knowledge of teaching geometry. In the second chapter I expounded on theories of learning and teaching; in general and specific to geometry; before delving into literature on teacher knowledge. In transition to this chapter, how to conduct this study of teacher knowledge of teaching geometry, the second chapter ended with a summary of different ways to assess teacher knowledge.

Since this study aims to add to the body of research pertaining to teacher knowledge, this paragraph describes the alignment that I see between this study and Silver and Herbst's (2007) idea of scholarship as a theory-making endeavor. From the standpoint of scholarship as a theory-making endeavor, there exist ongoing relationships between research, problems, theory making, and practices (Silver & Herbst, 2007). Although many relationships between research, problems, theory making, and practices have been addressed in chapters 1 and 2, the focus of this dissertation's place in the theory-making endeavor is the bidirectional relationship between research and practices. Data collection of teachers' knowledge while in practice is based on research. Findings from data collection of teachers' knowledge while in practice will hopefully inform research; which in turn, improves practice. In order to unite the various aspects of the statement of the problem of this dissertation, multiple frameworks will be used.

The first section of this chapter sets the background for the frameworks and methodologies of this study by addressing the theoretical perspective and philosophical assumptions. Then research design of this study is detailed; specifically, participants, sources of data, and procedures for conducting this research study. Once the background for the research design is explained, analysis of TACK and analysis with the geometry filters are discussed.

Theoretical Perspective

This study looked at Teachers' Applied Content Knowledge, a subset of Hill, Ball, and Schilling's (2008) Mathematical Knowledge for Teaching. Hill, Ball, and Schilling's (2008) measures of a subset of Mathematical Knowledge for Teaching, knowledge of content and students, were based "on empirical evidence regarding how students learn" (p. 375). They stated that they did not base it on theory because "several theories of student learning legitimately compete" (p. 375). Since this study looked at more of Hill, Ball, and Schilling's (2008) Mathematical Knowledge for Teaching than knowledge of content and students, giving this dissertation a theoretical perspective based on a theory of learning is pertinent (Lester, 2010).

The theoretical perspective of this dissertation is grounded in constructivism as a learning theory.

> Constructivism derives from a philosophical position that we as human beings have no access to an objective reality, i.e. a reality independent of our way of knowing it. Rather, we construct our knowledge of our world from our

> perceptions and experiences which are themselves mediated through our previous knowledge (Simon, 1993, p. 5).

A tenet of constructivism is that knowledge is constructed by an active participant, when existing knowledge interacts with new experiences (Simon, 1993; Steffe & D'Ambrosio, 1995). Constructivism is a theory of learning (Simon, 1993), and thus, as stated above, does not prescribe a particular model of teaching (Steffe & D'Ambrosio, 1995). Models of teaching based on constructivism are constructed from tenets of constructivism (Steffe & D'Ambrosio, 1995). Teachers that view student learning from the constructivist perspective "study the mathematical constructions of students and interact with students in a learning space whose design is based, at least in part, on a working knowledge of students' mathematics" (Steffe & D'Ambrosio, 1995, p. 148). D'Ambrosio and Steffe (1995) called these teachers "constructivist teachers".

In Herbst (2006), Megan, a high school teacher who taught geometry, was "powered by an interest in connecting students' mathematical experiences to other aspects of life" (p. 321). She often started class with open-ended problems, and then a whole class discussion before students worked in groups to solve the problem. In the whole class discussion, Herbst (2006) observed Megan move students towards the objective of the lesson through questioning. For example, some students answered her initial question with "in the middle", and Megan questioned students to be more specific with what they meant by "in the middle" (Herbst, 2006). Describing Megan's general design for each day of her class showed that she was a constructivist teacher since she designed her lesson in order for students to come to their own conclusions through interactions with the teacher and other students.

Philosophical Assumptions

"The research design process in qualitative research begins with philosophical assumptions that the inquirers make in deciding to undertake a qualitative study" (Creswell, 2007, p. 15). I referred to Cresswell's (2007) philosophical assumptions with implications for practice for my stance on nature of reality (ontological), relationships with this study's participants (epistemological), the language I use to discuss this research (rhetorical), the process of research (methodological), and the role of my values in this study (axiological).

The nature of reality (Creswell, 2007) for this study was subjective. The implication for practice from this ontological assumption (Creswell, 2007) was the use of quotes and themes in the words of the teachers, and evidence of teacher knowledge from multiple perspectives. I collaborated with teachers through interviews and observations to study their geometry TACK. From the epistemological assumption (Creswell, 2007), time in the schools with the teachers before, during, and after observed lessons was my attempt to lessen the distance between me and the teachers. The language I used the language of qualitative research to describe each aspect of this study. My methodological approach (Creswell, 2007) looked at details before making generalizations, described details in the context of the study, and continually revised interview questions from experiences observing. The axiological assumption (Creswell, 2007) was that I acknowledge that this study is value-laden and that biases are present. Thus, this chapter served to discuss the values that shape this study and includes disclosure of my biases.

Research Design

At the beginning of this chapter, I described how this study fits within research as a theory-making endeavor (Silver & Herbst, 2007), and the theoretical perspective and philosophical assumptions that undergirds this study. In this section, I will describe the research design. Since this study aimed to look at connections of geometry Teachers' Applied Content Knowledge, planned and executed; methodology is based on mathematics education literature for studying teachers knowledge and the nature of the research question. Simon and Tzur (1999) stated that researchers need rich data to assess the complex relationships of teacher's beliefs, knowledge, values, intuitions, feelings, and practices. To study the connections between teacher knowledge shown during the planning and execution of a lesson required data that is "well grounded, rich descriptions and explanations of processes in identifiable local contexts" (Miles & Huberman, 1994, p. 1). Thus, the appropriate research design is qualitative. Since this study is on teacher knowledge, I decided that the best course of action would be to study each teacher as a case in a multiple case study. The next paragraph details further the choice of case study as an approach to collecting and analyzing data.

Creswell (2007) suggested the following procedures for conducting a case study: determine if the approach is appropriate, identify the case or cases, determine data collection, determine type of analysis, and interpreting the findings. Since this study is about high school geometry Teachers' Applied Content Knowledge (TACK) shown during the planning of the lesson and the teachers' actual executions of the lesson, there are two components to the question: first, TACK planning and the other TACK execution. It makes sense to explore the two components to this question within a teacher as a case. The research question "seeks to provide an in-depth understanding of the cases" (Creswell, 2007, p. 74). Since there were two teachers; this case study has two cases, making this a multi case study. The next section describes each teacher in further detail.

Participants. Case studies "can look at many individuals connected with the case or at just a few or even one individual" (Lichtman, 2011, p. 109). For this study, each participant, or teacher, is a case. A key element of case studies is the "focus on an extensive examination of a particular group, program, or project" (Lichtman, 2011, p. 109). This study focused on geometry teacher knowledge of two teachers. In choosing teachers for this study, I needed teachers who were currently teaching high school geometry, who have taught geometry multiple times recently, and who I have observed teach in a manner for students to learn with relational understanding. I decided that these exemplary teachers would give me a greater base of teacher knowledge to look at the connections between teacher knowledge shared during planning and observed during execution of the lesson. Initially there were three teachers chosen for the study, with the hope that I would be able to study all three, but with the understanding that two teachers for the study would be sufficient. One of the teachers did not respond with permission despite prompting. Two teachers participated in this study.

My involvement with professional development in a southeastern state, supervising student teachers, and supervising practicum students allowed for interaction with a vast network of teachers in a southeastern state. From this network of teachers, two with whom I have closely worked satisfied these criteria and were able to participate in the study. According to Creswell (2007), this is called purposeful sampling. Types of sampling include theory based, criterion, and convenience (Miles & Huberman, 1994). Both of these teachers are examples of constructivist teachers, which also means they were selected based on theory – theory based (Miles & Huberman, 1994). Each teacher met the set of criterion listed above – criterion, which is "useful for quality assurance" (Miles & Huberman, 1994, p. 28). Also, as previously stated, these teachers were selected from a group of teachers I already knew, so this is also a convenience sample with the above conditions.

Mrs. Lotus* and Mrs. Orchid* both teach in the same district, but Mrs. Lotus teaches at the junior high school, and Mrs. Orchid at the high school. Given the timing in relation to the implementation of the Common Core State Standards and textbook revisions within the district, both teachers stated that they had to change from what they taught last year. Table 3-1 displays information about their college major and years taught.

Table 3-1 Teacher information

Teacher	Major/Degrees	Total Years Taught, of those how many geometry	How many years taught geometry in the past 10 years? 5 Years?
Mrs. Orchid	BS, M.Ed., and Ed.S Mathematics Education	10, 10	10, 5
Mrs. Lotus	BS Education Major in Math, minor in physical science	33, 15	10, 5

Both Mrs. Lotus and Mrs. Orchid completed bachelor's degrees that required coursework in education and mathematics. At the time of data collection, Mrs. Orchid recently completed an educational specialist degree in mathematics education, and was currently working on a doctorate in mathematics education. Both teachers taught at least one high school geometry class of students for the most recent 10 years of their teaching. Mrs. Orchid taught the high school geometry course for 10 years, which was her entire teaching career. Mrs. Lotus taught mathematics for 33 years, and of those 33 years, she taught 15 years of geometry. Thus, both teachers had recent experience teaching high school geometry.

Mrs. Orchid and Mrs. Lotus was each considered a case. One class was observed throughout the study for each teacher. The content observed was high school geometry. The classroom observation length of time was determined by the preset class schedule. The data collection time period was set within a semester, and each teacher was at a different school, so there were two different locations. Creswell (2007) suggested that cases in case studies need to be within a bounded system, clearly defined boundaries for a case that form a whole. “Boundaries [are] often bounded by time and place” (Creswell, 2007, p. 244). For this study, class of students, content, length of time for data collection, and location set the boundaries for each case (Creswell, 2007). During data collection, there were a variety of sources of data. The next section describes the sources of data.

Sources of Data. At the end of chapter two, I summarized literature on different ways to assess teacher knowledge; observation protocols, concept maps, and interviews. In collecting data for the two components of the question for this study, TACK planned and the TACK executed; I needed to know what the teacher planned to conduct class and what actually happened in class. Thus, sources of data were collected in the form of interviews and observations. Concept maps were drawn during interviews.

Interviews occurred before and after observations as well as before and after units. “Interviewing is an active process where interviewer and interviewee through their relationship produce knowledge" (Kvale & Brinkmann, 2009, p. 17). For this study, aspects of knowledge of Teachers' Applied Content Knowledge (TACK), was created from the interaction between interviewer and interviewee (Kvale & Brinkmann, 2009). So much of what a teacher plans to teach and what happens when a teacher teaches a lesson is not seen; and thus, without the interviewer, this knowledge would not be revealed (Kvale & Brinkmann, 2009). Interviews prior to lessons and units were used to gather data on TACK planned. Other interviews and observations where the teacher referred to what they had planned also were added to the body of data for TACK planned, but were categorized as: data collected about planning but not collected prior to execution.

Prior to data collection, I created an observation protocol (Appendix C) to collect data during observations and videotaped the teachers as they taught their lessons. The observation protocol consisted of six questions:

1. How did the teacher explain concepts, definitions, and procedures?
2. What are areas of teacher difficulties in explanations?
3. What was the sequence of information presented?
4. What was the time spent on phases of the lesson?
5. What student difficulties arose? If so, how was it shown? What did the teacher do in response?
6. What were actual student levels of prerequisite skills and how did the teacher teach to those levels?

These questions were generated from definitions of each aspect of TACK; specialized content knowledge, knowledge of content and students, and knowledge of content and teaching. The choice for number of observations as the minimum of three days is similar to Pesek and Kirshner's (2000) choice of number of observations.

Decisions for the design of this research, qualitative, multi-case study, participants, and sources of data were based on literature and discussions with my committee members. The basis of these decisions contributes to the quality of this study (Corbin & Strauss, 2008). The next section further discusses the "quality" of this study.

Quality of Research

This is a qualitative study, and as the researcher, I am the key instrument of data collection. This section details how the findings are reliable and valid. "Reliability pertains to the consistency and trustworthiness of research findings… [and] Validity refers in ordinary language to the truth, the correctness, and the strength of the statement" (Kvale & Brinkmann, 2009, p. 245-246). There are many perspectives to evaluating qualitative research. Different perspectives come with different key terms to describe them. Some, like Kvale and Brinkmann (2009) use reliability and validity, while others, like Corbin and Strauss (2008) prefer the phrase "quality of research". Lincoln and Guba (1985) used the terms: credibility, transferability, dependability, and conformability. Creswell (2007) summarized many different perspectives, and I found that the term "quality" best fits this study. In the following paragraphs, quality of research of this study was discussed in general as well as it pertains to key stages of the study; reviewing the literature, gaining permissions for the study, collecting data, data analysis, and reporting the study.

Creswell (2007) provided several suggestions for establishing "good" research; utilize peer review, provide rich thick description, make researcher bias known, ensure member checking, and include triangulation. I asked two individuals to code portions of the data for inter-rater reliability. These two individuals are graduate students within my program of secondary mathematics education. In writing this dissertation for both data collected and analysis, I used rich thick description to provide enough context to show how I arrived at my conclusions. Following this section, I have a section labeled "Researcher Bias", in which I described the experiences that I had which informed my theoretical perspective. Since this is a dissertation, my committee provided peer review and expert opinion. Video and audio recordings served to make sure that transcriptions were accurate and documented things in the classroom pertaining to research that I may have missed when observing.

For inter-rater reliability, Miles and Huberman (1994) suggested that the equation to calculate reliability is the number of agreements divided by the total number of agreements and disagreements. Using this formula, Miles and Huberman (1994) stated that the first time reliability is not better than 70%, second time a few days later with better definitions should be around 80%; and eventually, final check-coding about two-thirds of the way through the study should be about 90% reliability. Portions of interviews (generally 5 minutes) and observations (generally chosen for the multiple codes that showed up within a short amount of time. During the first round, reliability was about 60% for both raters. For the second round, reliability was up to 16/21, roughly 76% for one and 80% for the other. For the final round, reliability was 17/18, or roughly 94% for one, and 100% for the other.

The literature review, impetus, methods for data collection, and methods for data analysis were approved by a committee of four professors; two mathematics educators, one educational researcher, and one mathematician. Following approval from the committee, approval from the school district was obtained, followed by the Institutional Review Board. As stated above, during data collection, I checked often with individual members of my committee. All identifying aspects on physical information like written lesson outlines were removed prior to my possession of them. During transcription, student names were dealt with in one of two ways: numbered for events like S, S1, S2, and so on; or pseudonyms were given to students whose names were repeated a lot and out of context of events. No actual student names were typed into transcriptions. During data analysis all formal levels of analysis were revisited multiple times to ensure nothing was left behind. Also, when coding, as mentioned above, I had two individuals to establish inter-rater reliability.

Data on what teachers planned and executed were collected from many different sources. Unit interviews and pre-observation interviews collected data on what the teachers planned to teach. Audio from the teachers' perspective, and video and audio recording from an observer's perspective of teacher knowledge during a classroom observation. During observations, when possible, I also wrote notes of teacher knowledge I observed and questions to ask teachers about their knowledge shown during the observation. After observations, teachers reported rationale for some of the classroom decisions, giving further details on teacher knowledge during classroom instruction. These multiple sources at multiple times provided triangulation to get a more reliable account of teacher knowledge. In the next section, I present my researcher bias.

Researcher Bias. Besides the need I saw in the literature for further investigation, my professional and personal background was the impetus of my need to know about the differences in pedagogical content knowledge (PCK) in general. I was a high school mathematics teacher for 4.5 years in California and taught high school geometry among other mathematics courses. As semesters went on, I picked up more geometry classes to teach because those were the classes that the other teachers did not want to teach. I come from the perspective that one is never done learning. Teachers should acknowledge the responsibility that their job entails, and improve their practice day-to-day, year-to-year.

My parents were educated in Taiwan and my maternal grandmother, whom I called by her Chinese title is *Popo* (maternal grandmother) was educated in China – one of the first women in her village to be educated, was a high school mathematics curriculum specialist in Taiwan. Growing up, I was greatly influenced by her teaching methodology since I did not speak English fluently until the third grade, and the English as a Second Language (ESL) program did not provide the services for the language I spoke; thus learning was not done in the classroom, but rather at home. My memory of mathematics learning in the early grades was highly conceptual as *Popo* drilled math facts and taught me how to memorize procedures, but also taught me why those procedures are the way they are. I was not exposed to the discovery approach, a type of conceptual teaching, until later on in school, but did not find the two methods contradictory in terms of how much I learned.

Through teaching, I discovered that my students learned more and retained more knowledge when I taught conceptually versus procedurally. The time it took to cover the same material also decreased when I taught conceptually. Due to the lack of studies that are similar to this study, I relied on the conceptions from research related to this study to guide me as I conducted this study.

So far, theoretical perspectives, philosophical assumptions, and participants and sources of data in the research design were based on literature and discussions with my committee members. Since I was the instrument for data collection, it was necessary for me to make my biases transparent. In order to make this study replicable, the data collection procedures are also made transparent. The procedures for data collection were also based on literature and discussions with my committee members.

Procedures for Collection of Data. Once all necessary approvals were met, I e-mailed the teachers with a description of the study and possible dates to set up initial meetings. Below I described procedures in relation to each stage of meetings.

Initial Meeting. This meeting was for the purpose of introducing the teacher to the study, setting up a timeline for completing data collection and getting the necessary forms determined for students. Teachers were informed that this study looks at teacher knowledge. To prepare the teachers for the interview task of making a concept map, a sample concept map (Appendix E) was given and discussed at this meeting.

Interview for Unit Plan and Concept Map. In some cases, this interview was combined with the initial meeting when the teacher was ready. The purpose of this interview was to get the teachers' overall plan for the main concept(s) covered and provide a conceptual background for the consecutive lessons to be observed. Two interview tasks were asked of the teachers; concept map (Appendix D) and unit plan interview (Appendix B). Teachers were reassured that the concept maps could be modified each time they were interviewed for the duration of the study. I provided a copy of Appendix D, the concept map; and paper for the teachers to write on for the unit plan interview. All of these interviews were video and audiotaped.

Interviews for Lesson Plans. The purpose of these interviews was to get a closer look at exactly what the teacher planned to teach in the preselected consecutive days within the unit. These interviews were conducted after the interview for the unit plan and concept map, but before each observation. Teachers were able to modify their concept maps at any time during these interviews. These interviews were also audio and video taped. Interviews for lesson plans also followed the same format as Appendix B.

Observations of Lessons. Teachers chose the dates of a minimum of three observations on the basis that the dates chosen were to best show the development of the concept or skill. All consecutive days were within a five-day week, which allowed for opportunity to observe changes made in planning and execution of lessons. For each teacher, observations of both units were with the same group of students. I used the observation protocol (Appendix C) to collect data during observations and videotaped the teachers as they taught their lessons. Due to the necessity of following the teacher around to video record interactions, minimal note were taken via the Observation Protocol during the observation. For privacy purposes, care was taken to record only student work during teacher-student interactions and not their faces; so when I walked around the room, the video camera was pointed at the floor.

Interview for Reflections of Lessons. These interviews were conducted to provide details to what was observed when the teacher was teaching. Teachers were asked questions about their thoughts while watching video and audio taped evidence to provide context. The purpose of that was to provide the "why" and "how" behind the decisions and judgments made in the classroom. Questions asked in post-execution interviews stemmed from similarities and differences in the planning and executing of the lesson. Teachers were asked to clarify relationships between items planned and executed and why they stayed the same or changed.

Teacher Self-Reflections of Lessons. For the lessons within the unit for which I was not in the classroom to observe, teachers e-mailed me their reflections of what they planned on teaching and how it unfolded in the classroom. These were for the purpose of giving better context for the consecutive lessons. Thus, teachers answered two main questions for self-reflection. First, what and how do you plan on teaching the lesson? Second, what happened as you expected and what changed, and why? Over the course of the unit, answers to these questions gave additional context to what I observe for the preset consecutive lesson.

Interview for Reflections of Units. The purpose of these interviews was to provide a context for the consecutive lessons further than daily self-reflections for days on the unit I did not observed. These interviews attained the teacher's perspective on how the unit was executed in comparison to what was planned. This was also an opportunity for me to ask additional questions from the unit; and was an opportunity for teachers to share any remaining details about the unit that was not shared in previous interviews, observations, or teacher self-reflections. Table 3-2 is a table of the sequence of data collection.

Table 3-2 Sequence of Data Collection

- Initial Meeting
- Interview for Unit Plan and Concept Map
- Reflections for the days not observed in the unit prior to consecutive observations
- Minimum of three consecutive observations:
  - Interview of lesson
  - Observation of lesson
  - Interview after lesson
- Reflections for the days not observed in the unit after consecutive observations but before the end of the unit
- Unit reflection interview

In this section I explicated the quality of this research study. The introduction to this section detailed how quality of this study was upheld in each aspect of this study. The next section discusses analysis of TACK and analysis through the geometry filter.

Analysis of TACK

Teachers' Applied Content Knowledge was based on a subset of Mathematical Knowledge for Teaching from Ball and colleagues described in Chapter 2 section titled "TACK" on page 81. The three domains of Mathematical Knowledge for Teaching I rename TACK are Knowledge of Content and Teaching (KCT), Knowledge of Content and Students (KCS), and Specialized Content Knowledge (SCK). Specialized Content Knowledge (SCK) is the knowledge of concepts, definitions, and procedures for teaching (Ball, Thames, & Phelps, 2008). Knowledge of content and teaching (KCT) is the sequence of information or concepts taught, time it takes to cover each item of the sequence of information, explanations to teach, and questions to teach (Ball, Thames, &

Phelps, 2008). Knowledge of content and students (KCS) is lesson modifications, student difficulties, and student prerequisite skills (Ball, Thames, & Phelps, 2008). From this, the eight areas of teacher knowledge that make up TACK are:

- Knowledge of concepts, definitions, and procedures for teaching
- Sequence of information to be presented
- Timing for phases of the lesson
- Ways to explain concepts, definitions, and procedures
- Ways to ask questions to guide student thinking
- Modifications to lessons based on students' needs.
- Student misconceptions and difficulties and how to address them
- Knowledge of level of prerequisite skills of students

When asking about SCK planning during interviews, the main question asked is: What knowledge of concepts, definitions, or procedures for teaching did the teacher exhibit? The question in connection to the corresponding observation is: What knowledge of concepts, definitions, or procedures for teaching did the teacher exhibit? Similar questions were asked of the other seven areas. Table 3-3 lists the areas of focus for teacher knowledge that are components of TACK separated by the two aspects of the research question; planning and execution.

Table 3-3 Planning and Execution areas of focus of TACK

	Planning	Execution
Specialized Content Knowledge	Knowledge of concepts, definitions, and procedures for teaching	Knowledge of concepts, definitions, and procedures used in teaching
Knowledge of Content and Teaching	Plan for sequence of information to be presented.	Sequence of information presented.
	Planned time for phases of the lesson.	Time spent on phases of the lesson.
	How to explain the concepts, definitions, and procedures.	Explained concepts, definitions, and procedures.
	Planned key questions to guide student thinking.	Questions asked of students to guide their thinking.
Knowledge of Content and Students	Lesson modifications based on perceived student difficulties.	Did those student difficulties still arise? If so, how was it shown? What did the teacher do in response?
	Other anticipated areas of student difficulties and how to help students overcome.	Actual presented areas of student difficulties and how the teacher helped students overcome.
	Anticipated student levels of prerequisite skills and how the teacher plans accordingly.	Actual student levels of prerequisite skills and how the teacher taught accordingly.

This leads to the separation of the research question "What connections are there between the high school geometry TACK shown during the planning of the lesson and the teachers' actual executions of the lesson?" into eight more explicit questions.

- What are the connections between planned and executed specialized content knowledge?
- What are the connections between planned and executed sequence of information to be presented?
- What are the connections between planned and executed time spent on phases of the lesson?

- What are the connections between planned and executed explanations of concepts, definitions and procedures?
- What are the connections between planned and executed key questions to guide student thinking?
- What are the connections between planned and executed lesson modifications?
- What are the connections between planned and observed teachers' acknowledgement of student difficulties and how to overcome them?
- What are the connections between planned and observed teachers' knowledge of student prerequisite skills?

I answered these questions following Creswell's (2007) approach to data analysis and representation for a case study. Answers to these questions within the cases were described in context. I used categories to establish themes or patterns. When interpreting the data, I used direct interpretation and developed naturalistic generalizations. When describing the findings from the cases, I used narratives, tables, and figures.

As stated in the Research Design section, the research design follows that of a case study. However, to get a look at the relationships between the aspects of TACK, I used approaches from grounded theory. According to Creswell (2007), describing open coding categories is the first step to describing data analysis.

For coding using the grounded theory approach, I determined eight *a priori* codes from literature of KCT, KCS, and SCK. These *a priori* codes follow what Kelle (2007) calls "common sense categories". These codes followed directly from literature (Ball, Thames, & Phelps, 2008) and what I described as aspects of TACK at the beginning of this section. Table 3-4 shows *a priori* codes in tabular form. A copy of the *a priori* codes in tabular form can also be found in the Coding Guide in Appendix H on page 242.

Table 3-4 *a priori* codes

	Planning	Execution
Specialized Content Knowledge	SCK-P Knowledge of concepts, definitions, and procedures for teaching	SCK-E Observable knowledge of concepts, definitions, and procedures used in teaching
Knowledge of Content and Teaching	KCTseq-P Plan for sequence of information to be presented.	KCTseq-E Sequence of information presented.
	KCTtime-P Planned time for phases of the lesson.	KCTtime-E Time spent on phases of the lesson.
	KCTexplain-P How to explain the concepts, definitions, and procedures.	KCTexplain-E Explained concepts, definitions, and procedures.
	KCTquestions-P Planned key questions to guide student thinking.	KCTquestions-E Questions asked of students to guide their thinking.
Knowledge of Content and Students	KCSmod-P Lesson modifications based on perceived student difficulties.	KCSmod-E Did those student difficulties still arise? If so, how was it shown? What did the teacher do in response?
	KCSdifficulties-P Other anticipated areas of student difficulties and how to help students overcome.	KCSdifficulties-E Actual presented areas of student difficulties and how the teacher helped students overcome.
	KCSprerequisite-P Anticipated student levels of prerequisite skills and how the teacher plans accordingly.	KCSprerequisite-E Actual student level s of prerequisite skills and how the teacher taught accordingly.

When approaching data analysis through grounded theory, open codes; in the case of this study, *a priori* codes and emergent codes; were described before engaging in axial coding. During axial coding, one code at a time was chosen to focus on the causal condition, context, intervening conditions, strategies, and consequences. Categories were then interrelated to provide a story of how the different aspects of TACK interacted. Conclusions needed the combination of both approaches to data analysis from case study and grounded theory. Themes that came out through data analysis through case study approach were compared to codes and findings from axial coding from grounded theory. This method gave the most cohesive report for the answer to the research question: What connections are there between the high school geometry TACK shown during the planning of the lesson and the teachers' actual executions of the lesson?

In this section I described the data analysis of TACK through both the case study approach and grounded theory approach. Findings from both approaches brought together in the Conclusions chapter were the best method to answer the research question. However, since this study looked at geometry Teachers' Applied Content Knowledge, knowledge of teaching and learning geometry was also an important aspect to consider. The next section discusses how I applied geometry filters as a lens to look through what the teacher planned and what was executed.

Analysis through Geometry Filter

Since this study focused on teachers' knowledge related to high school geometry, there were three pieces of the geometry filter for which to sort the data for geometry content; Breyfogle and Lynch's (2010) geometric understanding with examples of teacher activities, Mistretta's (2000) phases of learning from the van Hiele Model, and

Driscoll's (2007) geometric habits of mind. For the geometry filter, three questions arise. First, what van Hiele level did the teacher's activity support? Table 3-5 shows only the van Hiele levels and description from Table 2-6 on page 32. Van Hiele levels were identified based on what the teacher planned to teach and what actually happened in the classroom. Van Hiele levels identified for what a teacher planned was based on what teachers described they were going to conduct in the classroom; and levels for activities that actually happened in the classroom were based on what the teachers actually did with the students during class. Table 3-5 is an abridged version of Breyfogle and Lynch's (2000) van Hiele Levels.

Table 3-5 Abridged version of Breyfogle and Lynch's (2000) van Hiele Levels

Level	Name	Description
	Visualization	See geometric shapes as a whole; do not focus on their particular attributes.
1	Analysis	Recognize that each shape has different properties; identify the shape by that property.
2	Informal Deduction	See the interrelationships between figures.
3	Formal Deduction	Construct proofs rather than just memorize them; see the possibility of developing a proof in more than one way.
4	Rigor	Learn that geometry needs to be understood in the abstract; see the "construction" of geometric systems.

From the van Hiele levels, Mistretta (2000) proposed phases of learning that supports movement from one van Hiele level to the next. Thus, the second question, what phases of learning did the teacher support? Table 3-6 on the next page displays phases of learning from the van Hiele model proposed by Mistretta (2000). Similar to analysis of the van Hiele levels, phases were determined for what the teacher planned to teach and what actually happened in the classroom. Phases determined for what a teacher planned was based on what teachers described they were going to conduct in the classroom; and phases for what actually happened in the classroom were based on what the teachers actually did with the students during class.

Table 3-6 Phases of Learning from the van Hiele Model (Mistretta, 2000, p. 367)

Phase	Description
Information	Discussions are held where the teacher learns of the students' prior knowledge and experience with the subject matter at hand.
Directed Orientation	The teacher provides activities that allow students to become more acquainted with the material being taught.
Explication	A transition between reliance on the teacher and students' self-reliance is made.
Free Orientation	The teacher is attentive to the inventive ability of the students. Tasks that can be approached in numerous ways are presented to the students.
Integration	The students summarize what was learned during the lesson.

The third piece of the geometry filter was through Driscoll's (2007) geometric habits of mind. Thus, this filter provided insights to the geometric habits of mind that were supported by the teacher; and, of those, what the indicator(s) showed; basic or advanced. Table 3-7 on the next page shows the geometric habits of mind and their indicators.

Table 3-7 Geometric Habits of Mind and Their Indicators (Driscoll, 2007, pp.12-15)

Geometric Habit of Mind	Indicators
Reasoning with Relationships	Basic: Identification of figures presented in a problem and correct enumeration of their properties
	Advanced: Relating multiple figures in a problem through proportional reasoning and reasoning through symmetry
Generalizing Geometric Ideas	Basic: Uses one problem situation to generate another, or when the solver intuits that he or she hasn't found all the solutions
	Advanced: Generate all solutions and make a convincing argument as to why there are no more; or wondering what happens if a problem's context is changed
Investigating Invariants	Basic: Decides to try a transformation of figures in a problem without being prompted, and considers what has changed and what has not changed
	Advanced: Consider extreme cases for what is being asked by a problem
Balancing Exploration and Reflection	Basic: Drawing, playing, and/or exploring with occasional (though maybe not be consistent) stock-taking
	Advanced: approaching a problem by imagining what a final solution would look like, then reasoning backward; or making what Herbst (2006) calls "reasoned conjectures" about solutions with strategies for testing the conjectures

Below, Table 3-8 is a summary of the components of the three pieces of the geometry filter. I used this as a quick reference of all three pieces of the geometry filter on one page.

Table 3-8 Geometry Filter

Van Hiele Level	Phase	Geometric Habits of Mind
Visualization	Information	Reasoning with Relationships (B/A)
Analysis	Directed Orientation	Generalizing Geometric Ideas (B/A)
Informal Deduction	Explication	Investigating Invariants (B/A)
Formal Deduction	Free Orientation	Balancing Exploration and Reflection (B/A)
Rigor	Integration	

This section described the three pieces of the geometry filter; van Hiele Levels, phases of learning from the van Hiele Model, and geometric habits of mind. The geometry filter adds to findings from each TACK analysis, as teaching and learning specific to geometry is highlighted in this lens for analysis. Analysis was an iterative process, but there were two main categories of time when data analysis occurred; during data collection and after the observation or unit. The following section describes the methods to this process in more detail.

Analysis During Data Collection. Between pre-observation interviews and the actual observation of a lesson, characteristics pertaining to what the teacher planned and what the teacher executed were noted in order to ask the teacher about it in the post-observation interview. These characteristics are specific to the subcategories within TACK. Answers to pre-observation questions (Appendix B) were jotted down on either paper or noted in my electronic notepad. During the lesson, when possible, notes were filled out on my electronic notepad as set up in the observation protocol (Appendix C).

This section describes the process in which pre-observation interviews and observations were analyzed in reference to its respective TACK aspect.

For each day of observation, two lists of sequence of information were generated; one of what the teacher planned, and the other of what happened in class. During data collection, these lists were created informally, sometimes without first transcribing, due to limited time between pre-observation interview, observation, and post-observation interview. An "event" within sequence of information was when the teacher moved on to another example, problem, or activity. These statements were then grouped into "events". Often times during pre-observation interviews, the teachers did not mention going over homework or checking homework, but it was something that they did during the lesson. Post-observation interviews revealed that the teachers planned for those activities, but did not feel like it was necessary to tell me, they were more interested in telling me what the lesson was going to be for the day. Thus, when comparing what the teacher planned and what was executed, the sequence of events for executed was compiled first, and what teachers planned to do were noted next to what happened. Anything extra that the teacher planned to do was also noted next to where it should have been in the lesson, or below if it did not fit; and this comparison document guided the post-observation questions.

Teachers were asked how much time they planned to spend on events and phases. In general, phases for both teachers were bellringer/work, checking homework, going over bellringer/work, going over homework, lesson, and summary. Using recordings after the observation before the end of the unit, I complied two lists; planned and executed, demarked by phases. Within each phase were the appropriate events, timing, explanations, questions, misconceptions, modifications, student prerequisite skills, and evidence of specialized content knowledge. This comparison allowed for a deeper look at the data so that I could ask teachers for clarification before the end of data collection. All follow up questions from this process have timestamps of where in the recording the question arose for the teacher to watch to refresh their memory of what happened surrounding the moment in question.

Analysis After Data Collection for Each Observation/Unit. In general, analysis occurred in an organic iterative manner. Of Carney's (1990) three levels of analytical abstraction, at the level of summarizing and packaging data, I transcribed interviews and classroom observations so I could just work with text. Also at this level, initial coding took place, where I highlighted segments of transcriptions and coded for the component of TACK. The second level of Carney's (1990) three levels of analytical abstraction, repackaging and aggregating the data, was when I looked over all the highlighted segments and determined ways to sort within each aspect of TACK; the KCT, KCS, and SCK; that made the most sense for that aspect. Finally, the third level, synthesis of data (Carney, 1990), was when I took what was determined in level two, and created the comparison of the two cases within data collected for each teacher; what was planned and what was executed during the lesson. At each level were different ways in which I displayed data for the next level of analysis, and finally drew conclusions. Before describing what was done to analyze data at each step, I briefly expounded on how I displayed the data in the following paragraphs.

Miles and Huberman (1994) described two main formats for displaying qualitative data; matrices and networks. Depending on what was needed to see connections between planning and execution of a lesson; for some elements of TACK, matrices were used, and others, networks. Matrices have defined rows and columns, while networks have a series of nodes that link between them (Miles & Huberman, 1994). Matrices are helpful in understanding the "flow, location, and connection of events" (Miles & Huberman, 1994, p. 93). Within a network, the nodes are points and the links are the lines that connect them (Miles & Huberman, 1994). Networks are useful when the focus is "on more than a few variables at a time" (Miles & Huberman, 1994, p. 94).

Level 1. At this first level of formally analyzing data, I used a two-column matrix for each data source; one column for the transcription and the other for notes on the transcription. What was in the notes column depended on the source of data. The sources of data being transcripts of concept map interviews, unit interviews, pre/post observation interviews, and observations. Each row of the matrix for interviews contained either a question and answer; or if the answer involved multiple events, like when a teacher goes through what they were going to teach that day describing multiple problems and activities, then a row for each event. For observations, rows separated phases, or when multiple events occurred in a phase, I separated those events into separate rows. These rows were an organizational tactic for data reduction (Miles & Huberman, 1994). For each row of notes, the first sentence gives a summary of what was in the transcription of that row, then explanations of highlighted portions of the transcription. Segments of transcription data that pertained to any of the eight *a priori* codes were highlighted, and the corresponding code was marked in the "notes" column with a short description of the highlighted portion as it pertained to the code. Some highlighted portions have multiple codes, so descriptors distinguished them.

At this level of analysis, anything that could be taken as related to one of the eight *a priori* codes (Appendix A) were highlighted and coded. When the teacher mentioned order of examples, problems, or activities; those statements were coded as sequence of information executed. So during classroom observations, when the teacher stated, " next we will go on to…" that was coded as "KCTseq-E", for knowledge of content and teaching sequence executed. Whenever the teacher mentioned amount of time for a phase in the pre-lesson interview; that was coded as "KCTtime-P", for time for phases planned. Time for phases of the lesson for observations were coded from video and audio recording timestamps. The remaining codes follow the same process. For full codebook including directions on unitizing and coding, refer to Appendix H. Table 3-9 shows the codes.

Table 3-9 TACK with subdomains and listed codes

	Planning	Execution
Specialized Content Knowledge	SCK-P Knowledge of concepts, definitions, and procedures for teaching	SCK-E Knowledge of concepts, definitions, and procedures used in teaching
Knowledge of Content and Teaching	KCTseq-P Plan for sequence of information to be presented.	KCTseq-E Sequence of information presented.
	KCTtime-P Planned time for phases of the lesson.	KCTtime-E Time spent on phases of the lesson.
	KCTexplain-P How to explain the concepts, definitions, and procedures.	KCTexplain-E Explained concepts, definitions, and procedures.

	KCTquestions-P Planned key questions to guide student thinking.	KCTquestions-E Questions asked of students to guide their thinking.
Knowledge of Content and Students	KCSmod-P Lesson modifications based on perceived student difficulties.	KCSmod-E Did those student difficulties still arise? If so, how was it shown? What did the teacher do in response?
	KCSdifficulties-P Other anticipated areas of student difficulties and how to help students overcome.	KCSdifficulties-E Actual presented areas of student difficulties and how the teacher helped students overcome.
	KCSprerequisite-P Anticipated student levels of prerequisite skills and how the teacher plans accordingly.	KCSprerequisite-E Actual student level s of prerequisite skills and how the teacher taught accordingly.

Level 2. This level of analysis enabled me to take what was found in level one, and decide how to display and organize the data to better see connections (Miles & Huberman, 1994). Since I already had matrices from level one, I decided to extend observation matrices by two more columns to the left to compare what the teacher planned with what actually happened in the class; one column for corresponding notes, and the other for the parts of the original transcription. So for each classroom observation, I took the matrix from level one analysis and added all relevant parts of the corresponding pre-observation interview to the two newly created columns. The setup of columns is shown below:

Planned		Executed	
Parts of interview	Notes	Notes	Classroom Transcription

I also looked through transcriptions of interviews and observations where the teacher stated anything relevant to the planning of the classroom observation. For example, when I started with the level one analysis matrix of unit 1, lesson 2, I looked back at all interviews and observations prior to unit 1, lesson 2 for specific references to the planning or execution of unit 1, lesson 2. Those were also added to the planning columns, but they were noted for which source they came from as well. Once this was complete, I added another two columns to the right; notes and transcriptions of corresponding reflection comments pertaining to what happened in class. A similar process was used in looking at other references to the observation after the post-observation interview. So the document containing most of the data for the lesson consisted of three main columns; planned, executed, and reflection. Each of the columns were then divided into two columns, notes and transcription. The setup of the columns is shown below:

Planned		Executed		Reflected	
Transcription	Notes	Notes	Transcription	Transcription	Notes

In some cases, multiple highlighted segments, when combined together, form a complete idea, like an explanation of a concept. These occurred within the multiple codes, creating multiple variables to display. With multiple variables, I knew from Miles and Huberman (1994) that I needed to generate a network containing nodes and links to see relationships. Nodes for different subdomains of TACK have different definitions, because of their different focuses. Below is a list of definitions for nodes specific to the subdomain of TACK. An additional copy of these definitions is available in Appendix G.

Definitions

- Nodes for sequences – examples, problems, definitions, and theorems; i.e. each separate bellwork problem as a node
- Nodes for explanations – when a student asks a question, the teacher explains rather than calling on another student, or asks a question in return, then that is a node for explain. If the same explanation is given for the same question, whether or not asked by the same student; both count within one node. If the same explanation is given to another question, it is a different node. Explanations can also be unsolicited, each topic of explanation counting as a node.
- Nodes for questions – There are two situations where Mrs. Lotus poses questions related to content, to engage in student interaction or in response to student interaction. Each question eliciting a specific response from students is considered a node. Multiple questions eliciting the same specific response is also considered one node. Questions in response to student questions follow the same counting algorithm.
- Nodes for modifications – planned lesson modifications are modifications like the choice of one example over another or a purposeful choice of a problem in order to bring out or avoid certain perceived student difficulties. Each modification counts as one node.
- Nodes for difficulties – each anticipated difficulty that the teacher does not plan on specifically addressing, but acknowledges existence, is considered a node. Further, all difficulties that arise during the execution of the lesson that was not

previously verbalized to me is also considered a node. Same difficulty on multiple students at multiple times still counts as one node.

- Nodes for prerequisite skills – each skill a teacher mentioned that the student should know either in interviews or during class time, counts as a node. Same prerequisite skills mentioned for multiple students at multiple times still counts as one node.

Level 3. Once nodes were established, level three was actually putting the nodes into a network. For this level, notes from level two and one were constantly revisited. One method in which I used to analyze sequence data was with nodes on a two-column relationship graphic like what is shown below in Figure 3-1.

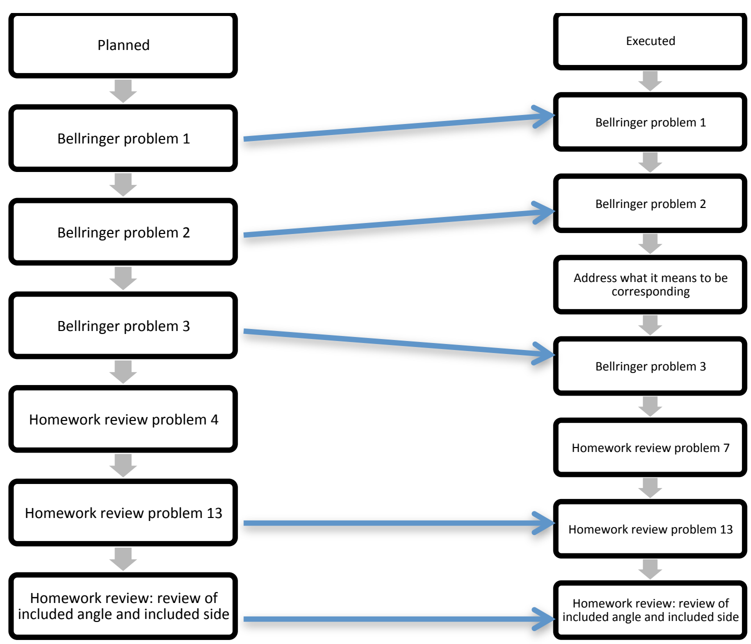


Figure 3-1 Sample node diagram

Nodes like this were similarly used to analyze teacher knowledge of explanations, questions, modifications, anticipated areas of student difficulties, and anticipated student prerequisite skills. This excludes teachers' knowledge of timing. Ideally, this seems like it should work for timing, however, in interviews the teachers repeatedly stated that they do not have specific times for problems, phases, or sometimes even days.

Also using node terminology, matrices of the days of observations were summarized with transcriptions for easier comparison. For analysis levels 1 and 2, I kept transcriptions in the matrices as not to miss any important data. For ease of viewing and conclusion making as not to overload with text (Miles & Huberman, 1994), I left out original transcriptions and kept summary statements only.

Summary of Order of Analysis. First level of analysis involved a lot of coding. It was the initial examination of data to focus in on TACK. With the elements of TACK found in level one, level two was defining and determining how to find connections between planned and executed subdomain of TACK. Level three was putting the matrices and networks together in a way that is useful to draw conclusions. Each level was revisited multiple times so that data would not be missed. From matrices and networks created in levels two and three, conclusions about geometry content could be made as well.

Summary of Theoretical Perspectives and Methodologies

This chapter detailed the theoretical perspectives and methodologies of this study. To set the background for the frameworks and methodologies for this study, I described how I saw this study contribute to scholarship as a theory-making endeavor (Silver & Herbst, 2007). The theoretical perspective of this study is constructivism, where an active participant constructs knowledge when existing knowledge interacts with new experiences (Simon, 1993; Steffe & D'Ambrosio, 1995). Constructivism, as a theory of learning, aligns with why teachers need to teach in a way that supports students to learn relationally. Further, constructivism aligned with many of the teacher attributes of teachers of students that showed achievement. Thus, an important criterion for teachers for this study was that they were constructivist teachers.

To study these teachers in a meaningful way, philosophical assumptions; ontological, epistemological, axiological, rhetorical, and methodological; were outlined. I used direct quotes and paraphrased in the words of the teachers to provide evidence of geometry Teachers' Applied Content Knowledge (TACK). I collaborated with teachers, spending time interviewing and observing, to create an accurate picture of geometry TACK at the time of data collection. In this chapter of the study, I openly discussed the methods used for data collection and analysis, complete with making my biases known. The language used in the following chapters follows that of qualitative research; narratives with occasional use of first-person pronouns. As detailed in the data collection and analysis, I worked with data collected from observations and interviews before drawing conclusions; and this was an iterative process, continuously going back to the data.

There were two main lenses for data analysis; TACK and geometry filter. TACK looked at specialized content knowledge, knowledge of content and students, and knowledge of content and teaching. Although high school geometry was the context for discussing TACK in this study, data analysis through TACK would not provide a comprehensive depiction of geometry TACK. Thus, data analysis through the geometry filter focused on the teacher's knowledge of teaching and learning geometry, and the connections between planned and executed lessons. The next chapter discusses findings from TACK and geometry filter first for the case of each teacher, and then I summarize findings from both teachers.

Chapter 4: Research Findings

Over the course of two months, with a total of twenty school visits, I collected about 40 hours of data in the form of video and audio recordings. I observed Mrs. Orchid, a teacher at the high school, in one class for four consecutive days of one unit and then for three consecutive days of a second unit. I observed Mrs. Lotus, a teacher at the middle school, in one class for three consecutive days of two units. Each observation lasted about ninety minutes. Below is a tabular representation of data collected for each teacher.

Table 4-1 Data Collected

Teacher	Unit	Observation
Mrs. Orchid	1 Pre, Post	1 Pre, Post 2 Pre, Post 3 Pre, Post 4 Pre, Post
	2 Pre	1 Pre, Post 2 Pre, Post 3 Pre, Post
Mrs. Lotus	1 Pre, Post	1 Pre, Post 2 Pre, Post 3 Pre, Post
	2 Pre, Post	1 Pre, Post 2 Pre, Post 3 Pre, Post

Preliminary data analysis occurred while data was still being collected. Data from the pre-observation interview and the corresponding observation were analyzed in an informal manner before the post-observation interview with Mrs. Orchid. Circumstances in scheduling did not allow for Mrs. Orchid to debrief with me within a month after the second unit. I decided that data collected after each day of observation and teacher self-reflection for each day of a unit after my observations sufficed to give an accurate picture of what happened during the unit as a whole.

Since each consecutive lesson in a unit built upon the previous lesson, I analyzed data as I was collecting it. Teachers' Applied Content Knowledge (TACK) and geometry filter, described in chapter 3, were used for analysis during data collection. As mentioned in the previous chapter, there were eight explicit questions regarding Teacher Applied Content Knowledge (TACK) shown during planning and execution of a lesson.

- What are the connections between planned and executed specialized content knowledge?
- What are the connections between planned and executed sequence of information to be presented?
- What are the connections between planned and executed time spent on phases of the lesson?
- What are the connections between planned and executed explanations of concepts, definitions, and procedures?
- What are the connections between planned and executed key questions to guide student thinking?
- What are the connections between planned and executed lesson modifications?
- What are the connections between planned and observed teachers' acknowledgement of student difficulties and how to overcome them?
- What are the connections between planned and observed teachers' knowledge of student prerequisite skills?

In this chapter, these questions and the geometry filter is discussed in the context of each of the units observed.

This chapter is organized in two main sections— Mrs. Orchid and Mrs. Lotus. For each teacher, I describe the context in which they taught before going into detail about the units. Discussions of each unit start with general information about the unit, and then the analysis of the lessons observed within the unit through aspects of TACK and the geometry filter, and findings that surfaced from the analysis.

Mrs. Orchid

Mrs. Orchid taught geometry twice a day, and the class that I observed was the earlier of the two, immediately following the lunch period. Class duration was about 90 minutes. Due to compatibility of time, I generally had an hour to conduct interviews before an observation. At times, when scheduling allowed, we scheduled post-observation interviews on the same day as the observation. When this was not possible, we combined post-observation interviews with pre-observation interviews of the next observation.

Her geometry class was comprised of students from ninth through twelfth grade in high school. There were twenty-four students observed for each of the lessons of both units. Student desks were arranged in groups of four all front facing, but when it was time for group work, students turned desks to face each other. Her general routine for class was bellringer, check homework, go over bell ringer, go over homework, lesson, and summary. Bellringers were problems for students to work on independently at the beginning of class with limited teacher instruction. Their purpose was to review prerequisite skills needed for the lesson or to clarify something done the previous day.

Mrs. Orchid graduated from a mathematics education program where one of the emphases of the program was effective use of technology, and she took it upon herself to incorporate as much useful technology as possible. The technology that I saw in her classroom included her teacher's computer, laptop, document camera, SmartBoard, and graphing calculators. She also had some remaining whiteboard space that was not covered by the SmartBoard, for which she used to write some notes and the date.

This introductory section described Mrs. Orchid's class I observed, time of day, the students, classroom procedures, and technology available in the classroom. The rest of this section is dedicated to Mrs. Orchid reporting findings from Mrs. Orchid's units. First, the context for each unit will be given, then concept maps, what Mrs. Orchid planned for the unit, and then what was observed and planned for each observed lesson is discussed. Teacher's Applied Content Knowledge (TACK) and findings through the geometry filter are discussed within the context of each unit. At the end of each unit is a summary of findings from the unit. At the end of both units is a summary of findings from both of Mrs. Orchid's units and themes from her units.

Unit 1. Mrs. Orchid was eager for me to observe the first unit. The two main reasons for this were that it included an activity for congruent triangles that she had used every year since she started student teaching, and the activity in which students discovered parallelogram properties showcased effective use of technology, which Mrs. Orchid considered an important part of the mathematics classroom. However, this was the first time she had combined congruent triangles and quadrilateral properties in the same unit, using congruent triangles to prove parallelogram properties. So Mrs. Orchid was apprehensive about how the unit would work as a whole. The following sections describe the concept maps and the general plan for the unit before delving into the breakdown of what was planned and executed for lessons observed.

Concept Maps. Since the unit was about using congruent triangles to prove properties of quadrilaterals, she ended up making two separate concept maps, one for congruent triangles and the other for quadrilaterals. She started with describing to me what each category title meant to her and then added nodes in that category. Early on in the interview for this concept map, Mrs. Orchid asked how much she should share about her knowledge, and she stated that there would not be enough time for her to tell me everything. I assured her that she could add to it during any interview time associated with the unit. So Mrs. Orchid explained concepts about congruent triangles that were directly related to the unit. She ended up not adding anything to this concept map throughout the interview, but referred to concepts when discussing parts of the unit. Figure 4-1 is the first of the two concept maps, congruent triangles.

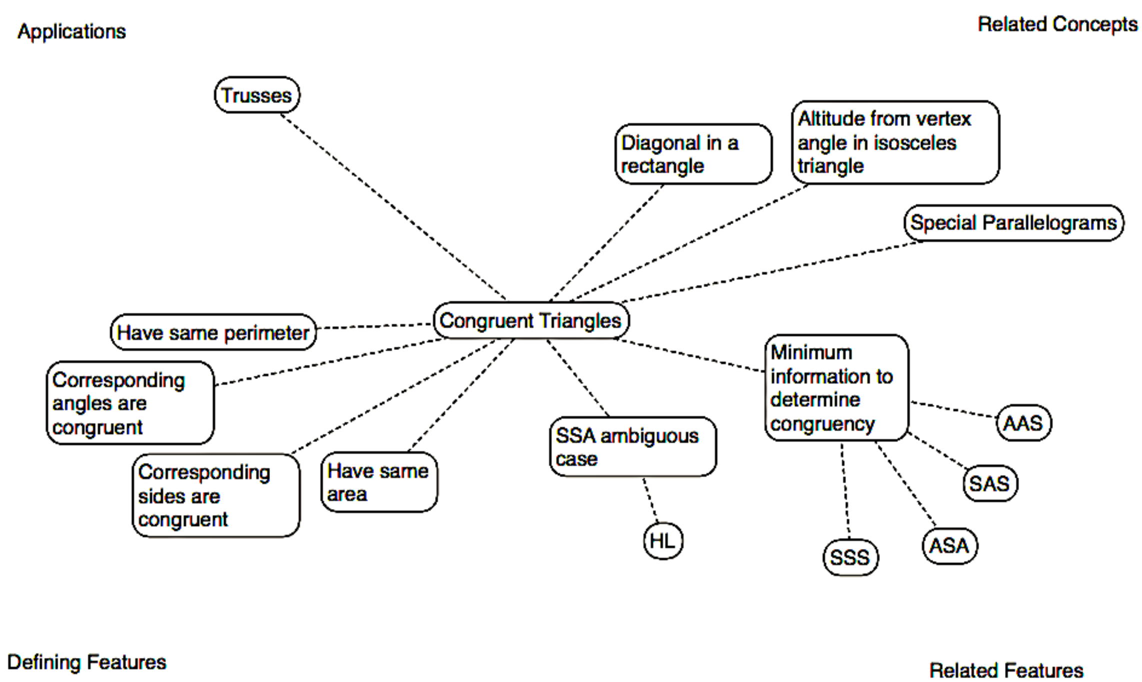


Figure 4-1 Mrs. Orchid's concept map for congruent triangles

Mrs. Orchid started with the defining features of congruent triangles. She stated that congruent triangles have the same perimeter and have the same area and that corresponding sides and angles are congruent. Then she moved on to related features. She started with listing the triangle congruence shortcut theorems, side-side-side (SSS), angle-angle-side (AAS), angle-side-angle (ASA), and side-angle-side (SAS). Then she wrote that the triangle congruence shortcuts were the "minimum information to determine congruency" and connected the nodes. Also a part of related features was the side-side-angle (SSA) ambiguous case which she connected to the hypotenuse-leg theorem. Mrs. Orchid stated that the side-side-angle is the ambiguous case because it only works, statistically, half the time. Related to that is hypotenuse-leg, specific to right triangles. During discussion of related features, Mrs. Orchid constantly referred back to an application from architecture, which involved trusses. A truss is a structure with triangles sharing a side (Ching, 2011). An example of a truss is shown in Figure 4-2.

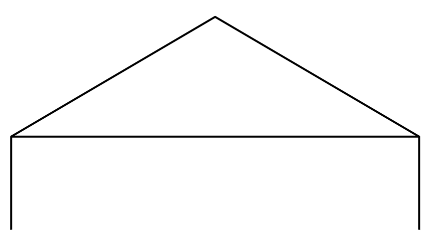


Figure 4-2 An example of a truss for the roof of a house

She stated that this was the only application at the time of the first observation, because she assigned bonus points for students to share other real-world examples of congruent triangles, and the students had not turned in any yet. She planned on adding students' examples to discussions as students brought them. By the end of the unit, no student had brought extra examples.

Under related concepts, Mrs. Orchid started talking about quadrilateral properties and stated that she needed to make another concept map on quadrilaterals on a separate page. On the concept map for triangles, she listed the diagonal in a rectangle and special parallelograms. She stated that the diagonal in a rectangle creates two congruent triangles. Before moving on to creating the concept map for quadrilaterals, she added that the altitude from the vertex of an isosceles triangle creates two congruent triangles.

For the concept map on quadrilaterals, Mrs. Orchid started talking about defining features and then the quadrilateral classifications. She stated that there are a lot of defining features. Mrs. Orchid started defining features by stating that quadrilaterals have four sides. Then she stated that there are alternate definitions: rectangles, rhombi, and squares. Then she stated that the trapezoid and its special case, the isosceles trapezoid, are not like the rectangle, rhombus, or square, in that the latter shapes required two pairs of parallel sides, whereas trapezoids only required one pair of parallel sides. The two related features were that there could be constructions of quadrilaterals based on one or more properties and that there were other alternate definitions of quadrilaterals. A related concept that Mrs. Orchid shared was that isosceles triangle properties were similar to isosceles trapezoid properties. She stated that she used the idea of cutting an isosceles triangle parallel to the base to show students that for an isosceles trapezoid, the base angles and nonparallel sides are congruent. Figure 4-3 is Mrs. Orchid's concept map for quadrilaterals.

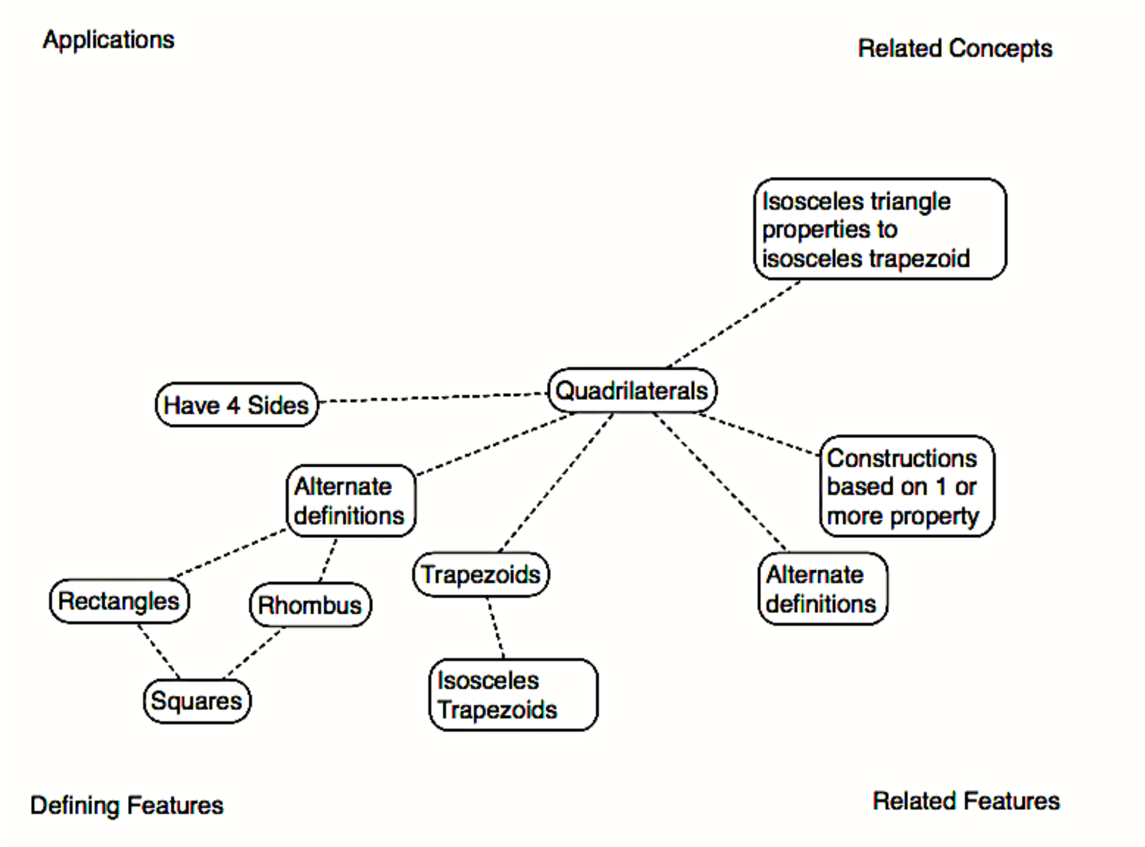


Figure 4-3 Mrs. Orchid's concept map for quadrilaterals

As Mrs. Orchid expanded the concept map for quadrilaterals, she began talking about the connections between using congruent triangles and quadrilaterals. She also discussed the new textbook being used and new content standards, which then led to her talking about how she taught and planned on teaching some of these concepts and how books and past content standards addressed those concepts.

Unit 1 Planned. This unit was titled "Using congruent triangles to prove properties of quadrilaterals." Mrs. Orchid planned to start off the unit on the first day with an introduction to congruent triangles, continue on days two and three with an activity that built triangle congruence theorems, introduce quadrilaterals on day four, and continue to parallelogram properties on day five. She planned to discuss minimum properties or characteristics of parallelograms on day six, special parallelograms with a focus on proofs on day seven, and trapezoids on day eight. She planned to discuss minimum properties or characteristics of parallelograms on day six, special parallelograms with a focus on proofs on day seven, and trapezoids on day eight. The unit would conclude with review and a test on days nine and ten, respectively. Mrs. Orchid stated that, in general, she plans with the knowledge that the number of days for a unit may be extended by a day or two, depending on the needs of her students.

When asked why she sequenced the unit the way she did, she stated that since the point of the unit was to use congruent triangles to prove the properties of quadrilaterals, it would first be necessary to spend time establishing what it means for two triangles to be congruent. Mrs. Orchid stated that, from past experience, she knew it took several days for students to learn this well. She expanded further by stating that the focus was not just on figuring out whether two triangles were congruent, but on proving that they were congruent by using triangle congruence theorems. Mrs. Orchid felt that triangle word problems in the book would solidify the idea that corresponding parts of congruent triangles were congruent and that it was essential to understand this idea before using it to explore and prove properties of quadrilaterals.

Mrs. Orchid noted that even though the concept of congruent triangles was something students should have learned in eighth grade, she expanded their knowledge by focusing on reasoning and proving the congruence of triangles. Since the rest of the unit depended on student knowledge of congruent triangles, corresponding parts, and how to mark triangles as congruent, Mrs. Orchid decided that it would be beneficial to spend a full day on congruent triangles. The prerequisites for this unit that Mrs. Orchid already covered earlier in the semester were basic familiarity with Geometer's Sketchpad, triangle properties, and some basic algebra skills like being able to solve an equation for a variable (*e.g.*, $3x + 7 = 10$). The two examples of triangle properties that Mrs. Orchid gave were (1) the ability to reason that equilateral triangles are also equiangular and (2) that the sum of the interior angles of a triangle is 180°. As a general rule, Mrs. Orchid stated that she covered all prerequisite skills needed for a lesson prior to the lesson. This either occurred right before the lesson, in the form of a bellringer, or was something that was covered earlier in the semester.

According to Mrs. Orchid, the general area of anticipated difficulty for this unit was the idea of sufficiency. From past experience with students, Mrs. Orchid stated that students generally had a difficult time determining what was a sufficient amount of information for a definition. Mrs. Orchid pointed out that students usually added redundant information. In terms of triangle congruency, Mrs. Orchid stated that it was easy for students to determine whether two triangles were congruent. However, in determining congruency of two triangles, understanding the minimum amount of information needed, and why it was the minimum amount was difficult for students. In order to develop the idea of sufficiency, Mrs. Orchid's plan was to coach correct phrasing and ask whether information was useful. The other area of anticipated difficulty within this unit was proofs. In preparation for student areas of difficulty, Mrs. Orchid stated that working in groups and sharing results as a whole class generally helped struggling students. Other areas of anticipated difficulty were activity-specific and will be discussed in the context of the observation days on which the difficulties became apparent.

Observed Lessons: Planned and Executed. In discussions with Mrs. Orchid, we decided that it would be best for me to observe the two-day activity (beginning with day two of the planned unit) on building triangle congruence theorems, as well as the subsequent two days, during which the connections from triangle congruence to quadrilaterals were to be established. The activities of the first two days of observation were designed to build on the idea of identifying congruent parts of a triangle and the triangle congruence shortcut theorems. At the time of the unit interview, Mrs. Orchid had the two-day activity and parallelogram properties (days one, two, and four of the observation) planned. The third day of observation was slated to be an introduction to quadrilaterals, but she expected to plan that lesson closer to when it was going to be taught. This unit is the only unit that I observed for four consecutive days. All other units were only three days each.

Mrs. Orchid called the two-day activity developing triangle congruence shortcuts "Triangle in a Bag." The Triangle in a Bag activity started with the premise that Mrs. Orchid had a triangle hidden in a bag. Students could not see the triangle. They were supposed to recreate that triangle by asking Mrs. Orchid questions. The answers to those questions were supposed to help students recreate the hidden triangle. Mrs. Orchid planned to repeat this process five times, each time with a different triangle. Mrs. Orchid referred to the processes with each triangle as "rounds."

For each round, Mrs. Orchid had prepared a piece of paper that listed the attributes of the triangle: angle measures and side lengths. Also in the bag, were cardstock cutouts of the triangles to the specified dimensions stated on the cardstock. Vertices were labeled on the cutout. She used the cardstock to compare to student drawings. See Figure 4-4 for the first two triangles in the order prepared for presentation on Observation Day 1.

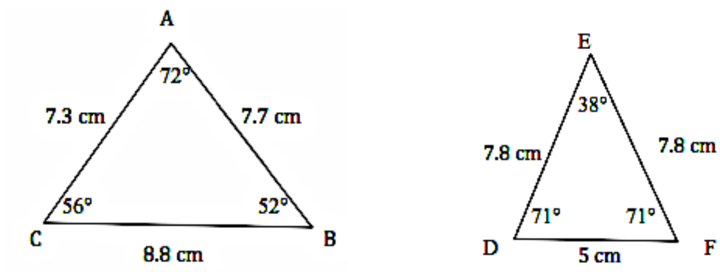
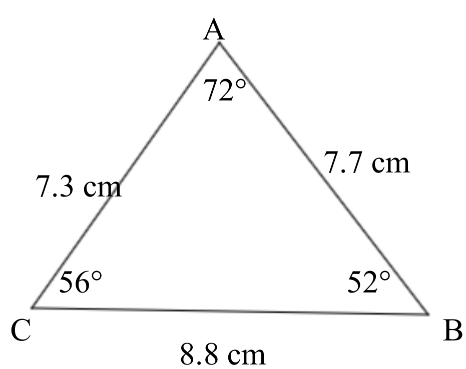


Figure 4-4 First two triangles of Triangle in a Bag activity

The goals for the first day of this activity was to get students to start asking questions and get a feel for what were productive questions and what were not. For example, one student asked if the sum of the interior angles added to one hundred eighty degrees. Anticipated difficulties also were observed, like asking for more information than what was necessary. For example, one group correctly drew triangle CAB, and their questions are listed next to Figure 4-5.



What degrees are each angle of the triangle?
What type of triangle is it?
What is each side length?
What side is the longest side?

Figure 4-5 Triangle and associated student questions.

Mrs. Orchid stated that the question “What type of triangle is it?” was unnecessary. The question “What side is the longest side?” helped students label the vertices.

Additionally for the first day, Mrs. Orchid was hoping that students would be able to start reducing the number of questions to the minimum of three. She planned for students to go through about two rounds. Observations for Triangle in a Bag for this day of class occurred as planned.

For the second day of the Triangle in a Bag Activity, Mrs. Orchid planned to start the day by summarizing what students learned from the two rounds the day before and then continuing with the rounds. After the three remaining rounds, Mrs. Orchid planned on having students work on classwork from the Triangle in a Bag activity. The two questions on the classwork printout were:

1. What must be true about two triangles in order for them to be congruent?
2. What did the activity “Triangle in a bag” have to do with congruent triangles?

For the first question, Mrs. Orchid hoped that students would be able to state in their own words that in order for two triangles to be congruent, all parts have to be equal. For the second question, Mrs. Orchid wanted students to draw the connection that the triangles they drew and the triangle hidden in the bag were supposed to be congruent.

After the first two questions were answered, Mrs. Orchid planned for groups to present different triangle shortcuts and fill out a chart summarizing the shortcuts. Mrs. Orchid postulated that there would not be enough time for students to present all five of the triangle congruence shortcuts (side-side-side, side-angle-side, angle-side-angle, angle-angle-side, and hypotenuse leg for right triangles) if all five shortcuts were present. Due to lack of time, Mrs. Orchid only had three groups present: side-angle-side, side-side-angle, and angle-side-angle.

When groups led the class in filling out the summary boxes, Mrs. Orchid instructed students to highlight the parts of the triangle they were talking about to help other students understand the differences between the shortcuts. Mrs. Orchid knew which groups to call up because as students were getting the correct triangle with the correct three questions, she marked their papers with the triangle congruence shortcut they used without telling them why she did so. For each presentation, Mrs. Orchid reiterated, expanded, and questioned student statements to prompt them for proper vocabulary and development of concepts. Mrs. Orchid asked and explained all that she planned, in addition to eliciting student specific statements.

The third day of observation started off with the bellringer addressing the side-side-angle ambiguous case. Mrs. Orchid planned to start this day with a bellringer for students to work on while Mrs. Orchid checked homework. The planned bellringer posed two questions:

1. How many non-congruent triangles can you create with the following measures?
   Angle A = 37°
   AB = 7cm
   BC = 4.5 cm
2. Can this combination of measures be used to decide if two triangles are congruent? Why or why not?

Mrs. Orchid also planned on providing students with the following Figure 4-6 to use.

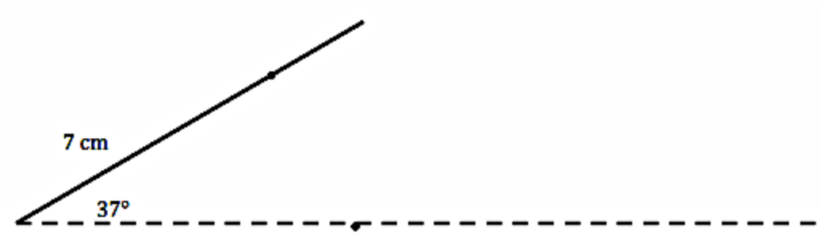


Figure 4-6 Bellringer Observation Day 3

After discussing the bellringer, Mrs. Orchid planned on discussing the reflexive property, that something is congruent to itself. The four problems Mrs. Orchid selected from the book for classwork were selected for content and context variations. Students solved these problems mainly by using the "corresponding parts of congruent triangles are congruent" (CPCTC) relationship. The context for these problems ranged from courier service routes to airplane runway lengths. Mrs. Orchid wanted students to be exposed to something other than construction examples. During the pre-observation interview, Mrs. Orchid talked about how to solve each problem. The last problem asked for students to write a flow chart proof, and Mrs. Orchid stated that she had not covered that with her students and that she wanted her students to write a paragraph proof instead. There was only enough time left in class for students to work on the first problem, the one in the context of couriers. Planned questions and explanations were observed in addition to others Mrs. Orchid stated while interacting with what students said and did.

The fourth day of observation also started with a bellringer and review of the bellringer. The bellringer reviewed relationships of angle measures when two parallel lines are cut by another set of parallel lines not parallel to the first set. This bellringer reminded students of the necessary prerequisite skills for the activity of the day. The activity was for students to discover properties of parallelograms via Geometer's

Sketchpad. Mrs. Orchid introduced the activity with having students attempt to construct a parallelogram with Mrs. Orchid leading them to the correct steps before they went to the computer lab and discovered properties in pairs. After most students were done in the lab, Mrs. Orchid led discussion back in the classroom on summarizing properties of parallelograms. Timing occurred as anticipated, Mrs. Orchid executed all that she planned.

Unit 1 TACK. This section addresses three main ideas: specialized content knowledge, knowledge of content and teaching, and knowledge of content and students. These are the three aspects of Teachers' Applied Content Knowledge (TACK). For each aspect of TACK, I discuss connections between what was planned and what was executed for the unit as a whole. Connections between planned and executed aspects of TACK for each day of Mrs. Orchid's Unit 1 are in Appendix J.

Criteria for determining specialized content knowledge (SCK) are in the list of mathematical tasks of teaching in Appendix H under the "Codes" section of the Coding Guide. However, for convenience, I list it below as well.

- presenting mathematical ideas,
- responding to students' "why" questions,
- finding an example to make a specific mathematical point,
- recognizing what is involved in using a particular representation,
- linking representations to underlying ideas and to other representations,
- connecting a topic being taught to topics from prior or future years,
- explaining mathematical goals and purposes to parents,
- appraising and adapting the mathematical content of textbooks,
- modifying tasks to be either easier or harder,
- evaluating the plausibility of students' claims (often quickly),

- giving or evaluating mathematical explanations,
- choosing and developing useable definitions,
- using mathematical notation and language and critiquing its use,
- asking productive mathematical questions,
- selecting representations for particular purposes,
- inspecting equivalencies (Ball, Thames, & Phelps, 2008, p. 10).

Modifying tasks to be easier or harder, and explaining mathematical goals and purposes to parents were not planned or observed throughout the days that I observed Mrs. Orchid. Interviews were always scheduled, so during interview time Mrs. Orchid did not communicate to parents in my presence. Thus, this occurred outside the realm of data collection for this study.

During observations, Mrs. Orchid stated additional responses to students' "why" questions. For example, during rounds with the Triangle in a Bag activity, Mrs. Orchid generally responded to students' "why" questions with explanations and questions. When Mrs. Orchid responded with questions, she was guiding students to answer their own questions. A group of students had what they thought was enough information to draw a triangle, but when they started drawing, they noticed it did not look like a triangle. They had the measures of two sides and an angle, but when drawn, it only looked like a very slanted "v". One student asked why it did not seem like a triangle. Mrs. Orchid's response was "How could you make that a triangle? What would you have to do?" A response like this was not stated during the interview prior to observation of the lesson. For all other mathematical tasks of teaching, what Mrs. Orchid planned was observed.

For knowledge of content and teaching (KCT) I looked at the sequence of information to be presented within a lesson, timing of phases in the lesson, explanations, and questions. In looking at what was planned and observed, sequences of phases of lessons and timing of phases of the lesson occurred as planned. For each observed day of the unit, additional explanation and questions were observed. For example, during the Triangle in a Bag activity, Mrs. Orchid gave additional explanations for getting students to determine the usefulness of questions to minimize the amount of information needed. Some of these were specific to student questions, like the question of whether the interior angles of a triangle added up to 180°. Below is the conversation that followed after Mrs. Orchid saw the written question "Do the angles add up to 180°?"

> Mrs. Orchid: no, you're not going to make me answer those questions
> {Students in the group laugh}
> Mrs. Orchid: I'm not going to let you ask me that question
> {Students in the group laugh}
> Student 1: well, why not?
> Mrs. Orchid: Why am I not going to let you ask that question?
> Student 1: because its always going to be
> Mrs. Orchid: Yes, no, it's always
> Student 1: 180
> Mrs. Orchid: Cause it's a
> Student 1: Triangle
> Mrs. Orchid: yeah
> {Students in the group laugh}
> Student 1: oh
> Student 2: oh oh
> Mrs. Orchid: oh yeah
> Mrs. Orchid: okay, so what you two are doing look totally different, so you might want to start talking and see, yeah… yeah…

Mrs. Orchid responded to this situation later in class by asking the students if knowing the interior angles of a triangle added up to 180° helped recreate the triangle in the bag.

For knowledge of content and students (KCS), I looked at modifications to the lesson, student misconceptions and or difficulties, and prerequisite knowledge students needed for the lesson. For the first three days of observation, there were no planned or observed modifications to the lesson. On the fourth day of observation, the only modification to the lesson was not allowing this class of students to use trial and error on their own in the computer lab to figure out on Geometer's Sketchpad how to construct a parallelogram. In the pre-observation interview, Mrs. Orchid stated:

> There have been some classes in the past where um I can send them to the computer lab and tell them to figure out how to construct a parallelogram [Me: Um hm] and give them hints walking around and they'll work on it. I'm not too confident with their ability to do that, and the reason is I don't have the time, so we're going to talk about it as a class. [Me: Um hm] And they'll jot down notes on how to do it, [Me: Um hm] and then they'll go to the computer lab and do it.

For all observed days, additional student misconceptions and difficulties were observed, but Mrs. Orchid stated that she had seen those difficulties before. All prerequisite knowledge that Mrs. Orchid stated that was needed for the lessons were observed.

In Mrs. Orchid's first observed unit, all TACK aspects that were shared during interviews before the unit and observations were observed. However, additional explanations, questions, student misconceptions, and student difficulties were observed. Mindful that this section explicated planned and executed TACK, and the focus of this study is on geometry teachers' applied content knowledge, the next section uses the geometry filter as a magnifying glass to examine Mrs. Orchid's knowledge of teaching geometry from planned to executed.

Unit 1 Geometry Filter. This section expands on Mrs. Orchid's teacher knowledge shown in Unit 1 through the geometry filter. The three aspects of the filter are van Hiele levels (Breyfogle & Lynch, 2010), phases of learning based on the van Hiele levels (Mistretta 2000), and geometric habits of mind (Driscoll, 2007). Teachers did not label van Hiele levels (Breyfogle & Lynch, 2010), phases of learning based on the van

Hiele levels (Mistretta 2000), and geometric habits of mind (Driscoll, 2007) in their interviews with me nor did they explicitly state it during observations. I identified each through descriptions in literature compared to classroom observation and interview statements. Tables of each of the geometry filters are provided below in their respective sections. A complete list of the geometry filters is in Appendix I. Expanded explanations for the geometry filters are in Chapter 3. Appendix K provides daily accounts of van Hiele levels (Breyfogle & Lynch, 2010), phases of learning based on the van Hiele levels (Mistretta 2000), and geometric habits of mind (Driscoll, 2007) of observed lessons for this unit. The geometry filter provides the geometry context to TACK as the research question focused on geometry teachers' applied content knowledge of teaching geometry.

Van Hiele Levels. For convenience, Table 4-2 shows the van Hiele Levels modified from Breyfogle and Lynch (2010, p. 234). Expanded explanations of the van Hiele Levels are in Chapter 3.

Table 4-2 Van Hiele Levels abridged from Breyfogle and Lynch (2010, p. 234)

Level	Name	Description
	Visualization	See geometric shapes as a whole; do not focus on their particular attributes.
1	Analysis	Recognize that each shape has different properties; identify the shape by that property.
2	Informal Deduction	See the interrelationships between figures.
3	Formal Deduction	Construct proofs rather than just memorize them; see the possibility of developing a proof in more than one way.
4	Rigor	Learn that geometry needs to be understood in the abstract; see the "construction" of geometric systems.

From interviews, Mrs. Orchid stated that students should practice informal proofs by justifying their answers throughout the geometry course. This belief was based on research and professional experience in mathematics education. In this unit, Mrs. Orchid planned for several opportunities for students to write formal proofs, one of them a question on classwork for the third observed day of the unit. However, due to time, that problem was assigned for homework. For the remaining days, I determined that planned and observed activities were at van Hiele Level 2, informal deduction. For example, in the Triangle in a Bag activity, students saw relationships between the hidden triangle and the triangle being created. Mrs. Orchid led students in discussions where they stated their reasoning and justified why the triangle they constructed was congruent to the triangle in the bag. Throughout the observations in the unit, Mrs. Orchid asked students questions to get them to justify their answers.

Planned and executed informal deduction connected to various aspects of TACK from interviews and observations. During interviews and observations, Mrs. Orchid's planned and observed key questions to guide student thinking, for the most part, were questions challenging students to justify their answers. Some students that were having difficulty with the Triangle in a Bag activity initially had problems recognizing properties of triangles, this being a problem I identified as van Hiele Level 1. Mrs. Orchid facilitated discussion about properties of triangles, before asking students to justify why asking about these properties of triangles were useful to create a congruent triangle; which I categorized as first discussing at van Hiele Level 1, and then Level 2. This showed Mrs. Orchid's knowledge of van Hiele Levels and how to move students through the distinct levels for topics in geometry. Conversations with Mrs. Orchid after classroom observations verified my interpretation of her understanding of van Hiele Levels and moving students through the Levels.

Phases of Learning Based on the van Hiele Model. For convenience, Table 4-3 shows the phases of learning from the van Hiele model. Expanded explanations of the phases of learning based on the van Hiele model are in Chapter 3.

Table 4-3 Phases of Learning from the van Hiele Model (Mistretta, 2000, p. 367)

Phase	Description
Information	Discussions are held where the teacher learns of the students' prior knowledge and experience with the subject matter at hand.
Directed Orientation	The teacher provides activities that allow students to become more acquainted with the material being taught
Explication	A transition between reliance on the teacher and students' self-reliance is made.
Free Orientation	The teacher is attentive to the inventive ability of the students. Tasks that can be approached in numerous ways are presented to the students.
Integration	The students summarize what was learned during the lesson.

From interviews and observations, I identified planned and executed phases of learning based on the van Hiele Model. All identified phases of learning based on the van Hiele Model planned were observed within the four days of observation. For the first three days of observation, I identified four phases planned and observed: directed orientation, explication, free orientation, and integration. Within interview and observation data on the four days of observation, I did not find evidence of the information phase. In discussions with Mrs. Orchid, I learned that an activity typical of the information phase was planned and executed on the first day of the unit, the day before I started observing. On the first day of the unit, Mrs. Orchid assessed student prior knowledge of triangle congruence through discussions and activities where knowledge of triangle congruence, or lack of, would be apparent. Students worked on activities involving triangle congruence on the first three days of observation for this unit, thus the purpose of the activities representative of the information phase for this topic already occurred before I started observing. On the fourth day of observation, Mrs. Orchid asked students what they knew about defining and constructing parallelograms before starting the activity in the computer lab. I viewed this activity as showing the information phase for and how to construct quadrilaterals using Geometer's Sketchpad.

Looking at what Mrs. Orchid planned and executed through the phases of learning illuminated Mrs. Orchid's knowledge of presenting mathematical ideas from the perspective of teaching geometry. Although Mrs. Orchid did not explicitly refer to the phases of learning when describing planned and executed, evidence of the phases emerged when Mrs. Orchid discussed sequence of information to be presented within a lesson, presenting mathematical ideas, and connecting a topic being taught to topics from prior or future years. When Mrs. Orchid discussed the overall plans for the unit, I learned that the information phase would only be present in two days of the unit; triangle congruence and quadrilaterals. All other phases were present in each observed day of the unit.

The Triangle in a Bag activity, application problems, and parallelogram activities allowed students to become more acquainted with the material being taught. Each day, through various times of the day, Mrs. Orchid transitioned between reliance on her and students' self-reliance. Evidence of this was revealed in Mrs. Orchid's directions for what students should do throughout the activities and responses to students' questions. Each activity contained tasks that students could approach in a number ways. At the end of each day, Mrs. Orchid took at least five minutes to have students summarize what was learned during the lesson.

Geometric Habits of Mind. For convenience, Table 4-4 shows Driscoll's (2007) geometric habits of mind. Expanded explanations for the geometric habits of mind are in chapter 3.

Table 4-4 Geometric Habits of Mind and Their Indicators (Driscoll, 2007, pp.12-15)

Geometric Habit of Mind	Indicators
Reasoning with Relationships	Basic: Identification of figures presented in a problem and correct enumeration of their properties
	Advanced: Relating multiple figures in a problem through proportional reasoning and reasoning through symmetry
Generalizing Geometric Ideas	Basic: Uses one problem situation to generate another, or when the solver intuits that he or she hasn't found all the solutions
	Advanced: Generate all solutions and make a convincing argument as to why there are no more; or wondering what happens if a problem's context is changed
Investigating Invariants	Basic: Decides to try a transformation of figures in a problem without being prompted, and considers what has changed and what has not changed
	Advanced: Consider extreme cases for what is being asked by a problem

Balancing Exploration and Reflection	Basic: Drawing, playing, and/or exploring with occasional (though maybe not be consistent) stock-taking Advanced: approaching a problem by imagining what a final solution would look like, then reasoning backward; or making what Herbst (2006) calls "reasoned conjectures" about solutions with strategies for testing the conjectures

Table 4-4 included basic and advanced indicator for each of the geometric habits of mind. For this unit, at different portions of the lesson, both basic and advanced were observed. In the daily observation account of geometric habits of mind, unless otherwise stated, both basic and advanced were planned and executed.

For all four days observed during this unit, I observed evidence of Reasoning with Relationships and Balancing Exploration and Reflection during planned and observed. Mrs. Orchid's plan for the first day of observation only showed basic, but when Mrs. Orchid asked students to justify answers, some students used reasoning through symmetry, which I identified to be advanced. Mrs. Orchid stated that this discussion was not something she had planned for because students had not been exposed to transformations in her class yet. However, she was pleasantly surprised. On the second day of observation, evidence from interviews and observations showed generalizing geometric ideas as both planned and observed. An unplanned discussion about triangle congruence that showed "generalizing geometric ideas" occurred at the end of group work in the computer lab.

> Student: Miss Orchid? Are the triangles inside are they congruent? Because they have the same measurements from the diagonals, this line in this line
> Mrs. Orchid: So what shortcut trick are you using, what theorem are you using
> Student: side side side?
> Mrs. Orchid: Hm, very interesting, you're two days ahead of us
> Student: that's shocking
> Mrs. Orchid: very good

Even though this was not planned, Mrs. Orchid showed the knowledge to answer students' questions. In addition, on the second day of observation, questions that I identified that fostered investigating invariants were not stated during planning but were observed. On the fourth day of observation, activities that I considered that fostered investigating invariants was both planned and observed. However, on the fourth day of observation, I did not see evidence to determine generalizing geometric ideas during planning but I saw evidence when the lesson was observed. Observations of geometric habits of mind that were observed but not planned occurred in the form of additional questions Mrs. Orchid asked of some students who finished ahead of the majority of the class. Once these questions surfaced during group work, Mrs. Orchid brought those discussions into the full class discussions as well.

Even though Mrs. Orchid did not state geometric habits of mind that she was addressing in her lesson, the fact that she was teaching geometry opens the possibility that instruction would adhere to some geometric habits of mind. Evidence for Mrs. Orchid's plans for fostering geometric habits of mind emerged when she discussed presenting mathematical ideas, asking productive mathematical questions, linking representations to underlying ideas and to other representations, connecting topic being taught to topics from prior or future years, sequence of information to be presented, and key questions to guide student thinking. During observations, evidence of Mrs. Orchid fostering geometric habits showed in the TACK aspects in addition to responding to students' "why" questions.

In general, what I identified during planning was observed during the executed lesson. From planned to executed, additional identified phases of learning based on the van Hiele model and geometric habits of mind were observed. For the third day of observation, Mrs. Orchid planned for informal and formal deduction, which I determined as van Hiele levels 2 and 3, but during class, only van Hiele level 2 was observed.

Findings from Mrs. Orchid Unit 1. The general finding for Mrs. Orchid's first unit was that geometry TACK was observed as planned. Through both TACK analysis and the geometry filter; sequence of activities, timing, questions, explanations, areas of student difficulties, modifications, student prerequisite knowledge, van Hiele Levels, phases of learning based on the van Hiele Level, and geometric habits of mind were observed as planned. Additional questions, explanations, student difficulties, phase of learning based on the van Hiele model, and geometric habit of mind beyond what was planned were observed. Additional questions and explanations seemed to be a fairly strong trend for this unit, consistently appearing in each observed lesson. Findings from the second unit add to the understanding of Mrs. Orchid's geometry TACK planned and executed.

Unit 2. Mrs. Orchid was excited to share this unit with me because she created a trigonometry activity that she had not taught before. She had not taught Law of Sines and Law of Cosines in high school geometry until Common Core State Standards, which Mrs. Orchid incorporated the year I observed her. Mrs. Orchid stated that she was eager for me to observe the activity she created but was apprehensive about the fact that she had not taught this unit before. She created this activity through verbal collaboration with other mathematics educators, research, standards, and textbooks.

Concept Map. For the concept map for the second observed unit, Mrs. Orchid recorded her voice in tandem with the construction of the concept map on her SmartBoard, but it was lost due to a computer virus. She waited a couple of days to try to send it once the problem was fixed, but there were still technical issues. Therefore I only received the map without audio. Figure 4-7 is a re-creation of the concept map that Mrs. Orchid sent me. She did not use the categories of defining features, related features, related concepts, and applications like the concept maps she created for the first unit. I still included this concept map to elucidate the concepts pertinent to this unit.

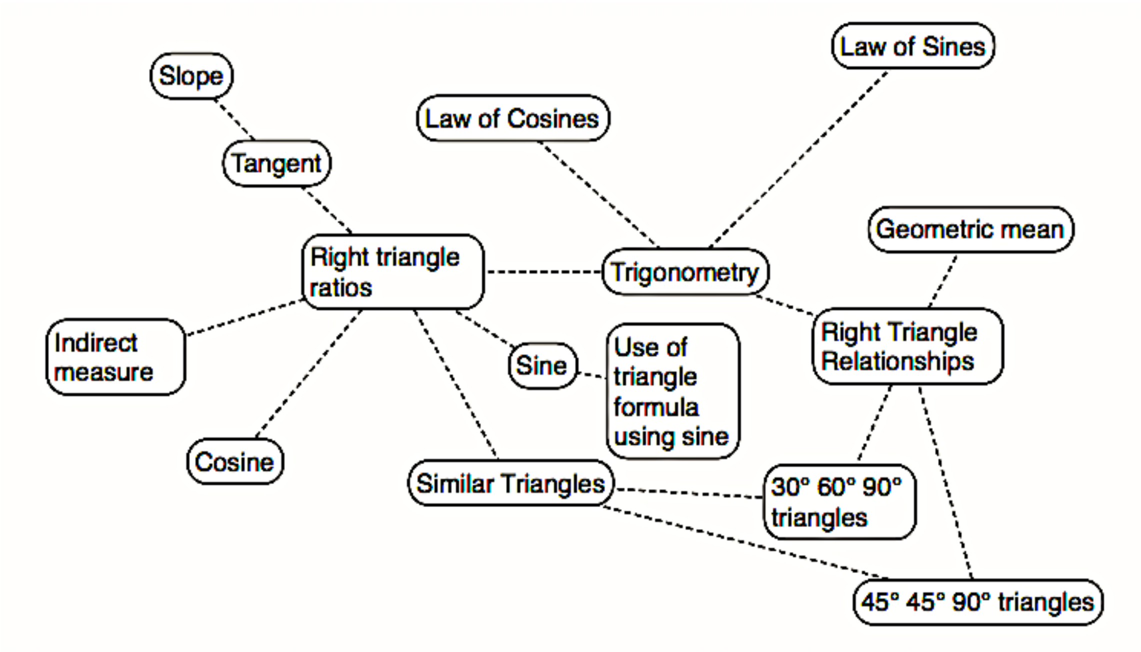


Figure 4-7 Mrs. Orchid Concept Map Unit 2

The focus of this unit was trigonometry. Coming from the main idea of trigonometry as shown on the map are right triangle ratios, right triangle relationships, Law of Sines, and Law of Cosines. These are concepts that Mrs. Orchid planned to teach in the trigonometry unit. Concepts related to right triangle ratios are indirect measure, similar triangles, sine, cosine, and tangent. Topics coming from right triangle relationships are geometric mean, 30° 60° 90° triangles, and 45° 45° 90° triangles. Since this diagram was not in front of Mrs. Orchid and me at the time of this interview, descriptions of the groups of nodes came from Mrs. Orchid's explanation of concepts to be covered in the unit.

Unit 2 Planned. The main focus of this unit was trigonometry. Due to a change in the school holiday schedule, Mrs. Orchid planned for more students to be absent and miss the day of the test for the previous unit. Thus, for the first day of this unit, Mrs. Orchid planned for some students to be taking a test from the previous unit and other students to be working on a worksheet reviewing simplifying radicals with a focus on square roots. For the second day of the unit, Mrs. Orchid planned for 45°-45°-90° triangles and for the third day of the unit, 30° -60° -90° triangles. On the fourth day, identifying opposite and adjacent legs were planned to be the bellringer in introduction to tangent ratio. The fifth day of the unit was sine and cosine ratio. The sixth day was an activity called "A Pigpen for Monica," calculating area of a triangle using the sine ratio. The seventh and eight day of the unit was Law of Sines and Law of Cosines. The ninth day was review, and the tenth day was the unit test. The three days I observed for this unit was the sixth, seventh, and eighth days of the unit.

Observed Lessons: Planned and Executed. When discussing this unit with Mrs. Orchid, she was very excited about teaching and sharing with me the Pigpen for Monica activity. Thus, she wanted me to observe the sixth, seventh, and eighth days of the unit, starting with Pigpen for Monica on the sixth day. Activities on the seventh and eighth days of the unit, Law of Sines and Law of Cosines, related to knowledge from Pigpen for

Monica activity. From here on out, I refer to days of observation, one through three, rather than the days of the unit.

The first observation of this unit started with a bellringer that consisted of four problems where variations on base and height were given to calculate the area of a triangle, which is half the base times the height. She decided to use four different triangles, as each one of them offered different points of discussion. The first point Mrs. Orchid wanted to deal with is the misconception students have that the base is on the bottom, so some of the triangles had the base on top. For the first and second problem, the base was not at the bottom, and the base and height measurements were both labeled and given. Figure 4-8 shows the orientations of the triangles in the bellringer problems Mrs. Orchid planned.

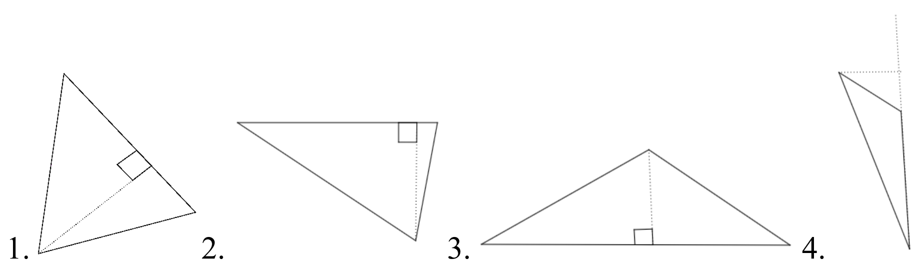


Figure 4-8 Orientations of the triangles in the bellringer problems

For the third problem, the length of the base and the side measurement of the left side were given. Also, the angle in between the two was given. For this problem, the anticipated misconception was that students might confuse what looked congruent versus what is marked congruent. The height for triangle number three looked like it bisected the base, but it really did not. Students were expected to use the tangent ratio in order to get the other side of the base, but Mrs. Orchid expected that some students would use sine ratio, cosine ratio, or the Pythagorean Theorem. The fourth problem had the height outside the triangle, and Mrs. Orchid hoped that students would be able to notice that the triangle was obtuse. Each of these triangles had different talking points in order for Mrs. Orchid to address possible misconceptions that students might have had that were crucial to the understanding of the Pigpen for Monica activity.

Mrs. Orchid's " A Pigpen for Monica" activity posed that Sofia, a girl who is building a pigpen for her pig Monica, has two pieces of fencing of eight feet and six feet long, and the side of a barn of 24 feet long. The question is posed, "What angle between the fences would provide Monica with the most area in her pen? How can Sofia find the area? How can Sofia be sure that she has determined the greatest possible area for Monica?"

The first part of the activity planned for students to draw the shape of the pen and calculate the area given certain angles of attachment of the fences to each other. Mrs. Orchid wanted students to draw the pen before discussing the questions posed by the activity so that they could think about the application of the problem. She wanted students to discuss if they needed all 24 feet of the barn, leading to a discussion about triangle inequality. Mrs. Orchid wanted students to keep their bellringer out so that they could make the connection between number three on the bellringer and how to find the height to calculate the area for the problems in Part 1. She planned on discussing with the students that both eight and six were possible bases and that the area, regardless of the base chosen to calculate, would be the same. Having students calculate the area three times for different angles in Part 1 helped them to see the pattern and gave them a reason to arrive at a formula. Three guiding questions at the end of Part 1 aided in the development of the concept of rewriting the formula for the area of a triangle, ½ base x height, to ½ base x aSin C. Figure 4-9 show what the letters “a” and “C” represent on a triangle, in the rewritten formula for area of a triangle “base x aSin C”.

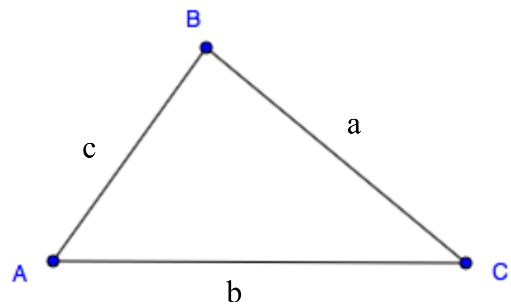


Figure 4-9 Triangle with angles and sides labeled for reference to the formula for area

Part 2 of the activity explored whether or not the formula always works. The first question asks students to sketch the pigpen using 120° and draw the altitude. The second question asks students to determine the height. This is where the students could not draw the altitude inside the triangle, and where Mrs. Orchid hoped that students would draw the connection to number four on their bellringer. In order to calculate the height, since students were using a trigonometry chart that only goes up to 89°, Mrs. Orchid had students use the supplement of 120°, 60° to find the height. Figure 4-10 shows the relationship between the supplements and the height.

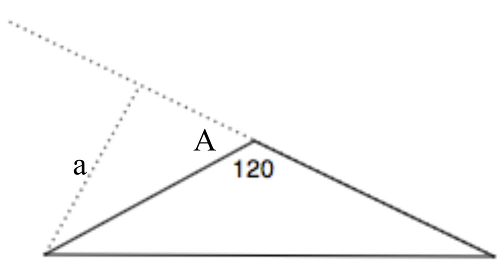


Figure 4-10 Measure of angle A is 60°, the supplement of 120°. Height is side a.

Mrs. Orchid planned for students to make the connection that the sine ratio of supplements are the same, but she was not prepared to delve deeper into understanding why they are the same due to time. Part 3 has four questions:

1. What angle measurement would yield the greatest area for the pigpen? Explain.
2. How can Sofia be sure that it is the greatest area possible? Explain.
3. Sofia discovered that she has only enough hay to cover 22 ft^2 and Monica must have hay covering the entire area of her pigpen. What angle should Sofia use to attach the fences so that she can cover the entire area of Monica's pen with hay?
4. Sofia's dad said that she can use only 1/3 of the barn wall for the pigpen. Does this change your solution? If yes, explain and find the new solution. If no, explain why it would be the same solution.

The question posed at the beginning of the activity finally is posed in question 1 of part 3: "What angle measurement would yield the greatest area for the pigpen? Explain." Question two: "How can Sofia be sure that it is the greatest area possible? Explain." This one is an informal proof for the first question. The third question was created, because after this whole activity, Mrs. Orchid stated that she realized that the answer did not require students to use the sine relationship. She changed the third question so that 90° could not be the answer. Mrs. Orchid stated that due to time, she thinks it is acceptable not to cover number four when doing this activity. She had not taught this lesson before so she was unsure of the pacing.

After the first day of observation, Mrs. Orchid realized that the Pigpen for Monica activity was going to take longer than expected. Thus, she decided to spend an extra day on Pigpen for Monica, the second day of observation, and move on to Law of Sines for the third day. The Law of Sines activity started with Mrs. Orchid discussing with the class the requirements of a triangle in order to use the sine ratio and what they knew they could do if an altitude was drawn. Students were given triangle ABC, and then they were instructed to draw height *k* from vertex B, shown in Figure 4-11.

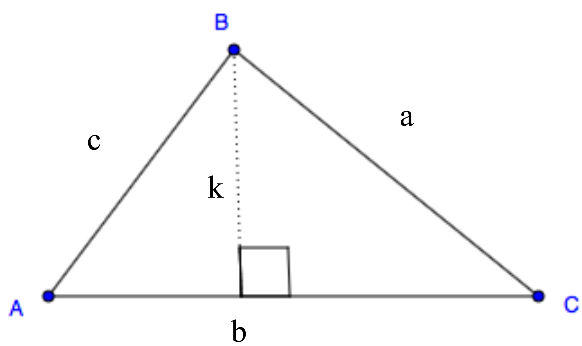


Figure 4-11 Figure of Triangle Labeled for students to set up Law of Sines

Then Mrs. Orchid led students through setting up the sine ratio for sin A and sin C, which is sin A= $\frac{k}{c}$ and sin C= $\frac{k}{a}$. Solving for *k* for each of them yielded *k* = c(sin A) and *k* = a(sin C), which then meant that c(sin A) = a(sin C), or $\frac{\sin C}{c} = \frac{\sin A}{a}$. Similar steps are taken with B. Deriving the Law of Sines this way connected to the Pigpen for Monica activity and helps develop a conceptual understanding of the Law of Sines. For the first and third days of observation, what was planned was observed. Discussions ran longer than expected on the second day, so students finished the Pigpen for Monica activity on the third day of observation.

Students experienced all the activities that were planned for this unit. The major changes to plans were that A Pigpen for Monica took three days instead of one, Law of Sines only took half a day, and review of the unit shared a day with Law of Cosines, so the unit test was given on the day that was originally planned. Throughout each day, homework took longer than anticipated, and student participation was much lower than Mrs. Orchid expected. Mrs. Orchid viewed this class as the well-behaved class, in that they usually did their work and turned in homework. However, for this unit, they were less engaged, and Mrs. Orchid believed it was because the semester was about to end.

Unit 2 TACK. Of the three days of this unit, only the first day was executed as planned during the unit interview. The two other observed days of the unit each took more time to do the activities anticipated. Interviews before the observation each day gave Mrs. Orchid the opportunity to share changes made to the observation day from the initial plan during the unit interview. Compared to what Mrs. Orchid planned for each day of observation right before the observation, the first and third days of observation occurred as planned. This section addresses three main ideas: specialized content knowledge, knowledge of content and teaching, and knowledge of content and students. These are the three aspects of Teachers' Applied Content Knowledge (TACK). For each aspect of TACK, I discuss connections between what was planned and what was executed for the unit as a whole. Connections between planned and executed aspects of TACK for each day of Mrs. Orchid's Unit 2 are in Appendix J.

From each day's pre-lesson interview, observation, and reflection interview, all planned sequences, explanations, and questions were observed. Timing for the first and third observations was observed as planned, but homework review went too long for the second day. Additional explanations, questions were observed. These additional explanation and questions were in response to students' questions and additional areas of difficulty.

All planned student difficulties and prerequisite knowledge were observed throughout the observed days of the unit. Additional student difficulties were also observed. For example, on the second observation day, Mrs. Orchid did not anticipate to go over how to use the calculator for trigonometry until students asked about it. Modifications to the lesson for Observation Day 3 were made based on student behavior and limited time. The Pigpen for Monica activity took longer than expected, so she created a way to help students review what was learned the previous days without taking more time than necessary.

Similar to Mrs. Orchid's first unit, explaining mathematical goals and purposes to parents and modifying tasks to be easier or harder were not planned or observed. Even though Mrs. Orchid modified the activity on the third day, the modification did not change the difficulty of the activity. Since Mrs. Orchid developed her own activities for the days of observation for this unit, she did not discuss appraising and adapting the mathematical content of textbooks. For the remaining mathematical tasks of teaching, what was planned was observed.

For this unit, all of what was planned was observed. Additional student difficulties, explanations, and key questions were observed. This section looked at planned and observed TACK. The next section expounds upon the geometry aspects of planned and observed TACK using the geometry filter.

Unit 2 Geometry Filter. Since this unit was on trigonometry, only relevant aspects of the geometry filter will be discussed. Similar to the discussion for Mrs. Orchid's Unit 1 geometry filter, I determined van Hiele levels (Breyfogle & Lynch, 2010), phases of learning based on the van Hiele levels (Mistretta, 2000), and geometric habits of mind (Driscoll, 2007) through descriptions in literature compared to classroom observation and interview statements.

Throughout the observed lessons, from interviews and observations, Mrs. Orchid asked students to justify their answers in discussions, which I identified as van Hiele Level 2, informal deduction (Breyfogle & Lynch, 2000). Evidence that led me to determine van Hiele Level 2 in relation to evidence for TACK appeared in planned and executed presenting mathematical ideas, responding to students' "why" questions, and questions to guide student thinking. On the first observed day, during the Pigpen for Monica activity, Mrs. Orchid asked students to identify the shape of the pigpen. I determined that identifying the shape of the pigpen showed van Hiele Level 1, analysis. TACK aspects that provided evidence for this van Hiele Level was in planned and executed presenting mathematical ideas and questions to guide student thinking.

As for phases of learning based on the van Hiele Model (Mistretta, 2000), evidence from interviews and observations led me to conclude that all three observed days showed directed orientation, explication, and free orientation. The Pigpen for Monica activity and Law of Sines activity provided students with the opportunity to become more acquainted with the material being taught. Evidence of orientation in relation to TACK aspects was presenting mathematical ideas and sequence of information to be presented. I determined that explication was planned and executed in all three days of observation, and evidence of this from TACK was presenting mathematical ideas, directions for the activities, and key questions to guide student thinking. Most of the problems in A Pigpen for Monica activity were attentive to the inventive ability of students. Evidence of this in relation to TACK aspects appeared when Mrs. Orchid discussed presenting mathematical ideas and sequence of information to be presented. I determined that integration was planned for all three days of observation, but it was only observed on the first and third days of observation. On the second day of observation, Mrs. Orchid ran out of time, and students were not prompted to summarize what they had learned that day. In relation to TACK evidence, Mrs. Orchid did not question for students to summarize, and the planned sequence of summarizing what was learned that day was not observed.

Evidence from interviews and observations led me to determine that each day of observation emphasized a different geometric habit of mind (Driscoll, 2007). On the first day, Mrs. Orchid prompted students to reason with relationships, primarily in getting students to draw the shape of the pigpen. Also during the Pigpen for Monica activity, Mrs. Orchid asked students if they could think of other angle measures where the sines of those angles are equivalent. This showed the geometric habit of mind called generalizing geometric ideas (Driscoll, 2007). The plan for the second observation day during the Law of Sines activity was for students to make reasoned conjectures about what the relationship would be, but this plan did not occur. This exemplified balancing explorations and reflections. However, on the third observation day of this unit, during the Law of Sines activity, the activity was planned and observed. All geometric habits of mind for this unit were connected to TACK in presentation of mathematical ideas, responding to students' "why" questions, sequence of information to be presented, and key questions to guide student thinking.

Findings from Mrs. Orchid Unit 2. The general finding for Mrs. Orchid's second unit was that geometry TACK was observed as planned; with the exception of timing and sequence not occurring as planned for the second day of observation. Through both TACK analysis and the geometry filter for all lessons observed; questions, explanations, areas of student difficulties, modifications, student prerequisite knowledge, van Hiele Levels, phases of learning based on the van Hiele Level, and geometric habits of mind were observed as planned. Additional questions, explanations, and student difficulties, were observed. Additional questions and explanations seemed to be a pretty strong trend for this unit as well, consistently appearing in each observed lesson.

Summary of Unit 1 and Unit 2. For both units, in general, what Mrs. Orchid planned was observed. Additional explanations, questions, student difficulties, and prerequisite knowledge were observed. In reflecting with Mrs. Orchid, these were due to student input. Additional student difficulties were observed with additional explanations and questions from Mrs. Orchid. Otherwise, in general, what students said, did, or asked caused Mrs. Orchid to add explanations and questions. Most lessons did not have modifications. The modifications to the lessons that were made were due to time and student involvement. The modification to a lesson in the first unit was due to student behavior. Part of the decision for modification in the second unit was due to the fact that students did not have the prerequisite knowledge that Mrs. Orchid had assumed in her plans, and thus she was willing to stray from the plans in order to make sure students understood before moving on to the next topic.

Modifications to lessons in both units did not change planned or executed aspects of the geometry filter. Throughout both units, Mrs. Orchid showed evidence that she fostered all geometric habits of mind and used all phases of learning based on the van Hiele Model. All van Hiele Levels planned were observed. Through interactions with students, Mrs. Orchid was observed facilitating additional van Hiele Levels. Consistency in planned and executed, without sacrificing cognitive demand, for pieces of the geometry filter showed Mrs. Orchid's deep knowledge of teaching and learning geometry. Throughout data analysis of both of Mrs. Orchid's units, some themes emerged.

Themes from Unit 1 and Unit 2. Certain patterns specific to Mrs. Orchid appeared through the multiple iterations of combing through data. These patterns were related to themes consistent with both teachers: timing, questioning, familiarity with activities used, areas of apprehension, and sources of knowledge. In six of the seven observed lessons, at least one phase within the lesson exceeded the time Mrs. Orchid's anticipated. In every observed lesson, Mrs. Orchid restated planned key questions at least three times, with many scaffolding questions, to get students to give answers that "showed understanding". From interviews with Mrs. Orchid about timing and teacher-student dialogues in class, she stated that her students needed more wait time in order to verbalize their understanding. From my observations, this showed Mrs. Orchid's persistence to get students to make those connections of new material to what they know, showing relational understanding.

Mrs. Orchid was very familiar with many of the activities used in the first unit. However even in those lessons, there were still areas of apprehension. Even though she used the activity Triangle in a Bag many times before, she had not used a document camera with students presenting before. Further, in the first unit, she had not taught the activities within the first unit together before, and was apprehensive about how the unit would fit together as a whole. Mrs. Orchid was apprehensive about sharing the second unit with me, because she had not taught the content or the activity before; but wanted to share the unit because it contained a lesson she created, A Pigpen for Monica. Throughout all areas of apprehension, her vast sources of knowledge provided her with multiple resources to use in planning and execution of the activities, lessons, and units.

Mrs. Lotus

Similar to Mrs. Orchid, Mrs. Lotus taught the high school geometry course, but she taught the course to eighth graders in a junior high school. She taught geometry twice a day, once at the beginning of the day and the other after lunch. The class that I observed was the one after lunch. The geometry class I observed was also about ninety minutes long. Her available time to meet me was within two hours after the time for the class I observed. Thus, pre-observation interviews were conducted about 20 hours before each observation. I was able to code pre-observation interviews prior to the observation, but I had to take notes during observation and immediately compare pre-observation data to what just happened in the classroom in order to conduct the post-observation interview. Fortunately, I was able to ask in subsequent reflections what I was not able to ask right away.

Throughout all my observations, there were 21 students. Students sat in groups of four at the long sides of two long tables placed together. Her general routine for class was also bellwork, check homework, go over bellwork, go over homework, lesson, and summary. However, bellwork problems were review of skills for students to work on independently at the beginning of class with limited teacher instruction. Usually Mrs. Lotus used bellwork to refresh skills from two lessons prior, but sometimes it was to review a concept that seemed important, but not necessarily tied to the lesson.

Technology that was available in her classroom was an interactive whiteboard, document camera, and teacher computer. Also, through a program at the school, each student had an assigned laptop from the beginning of the semester, so they used it on a daily basis in her classroom. Mrs. Lotus used all available technology on a daily basis, from students adding to notes provided by Mrs. Lotus on their laptops to Mrs. Lotus showing students geometric constructions on a dynamic geometry program on the interactive whiteboard while they construct using the same program on their laptops. Mrs. Lotus stated that technology was a big emphasis at her school.

This introductory section gave the context for Mrs. Lotus's class I observed, time of day, students, classroom procedures, and technology available in the classroom. The remainder of the section focuses on findings from Mrs. Lotus' two observed units. First, the context for each unit will be given, then concept maps, what Mrs. Lotus planned for the unit, and then what was observed and planned for each observed lesson is discussed. Teacher's Applied Content Knowledge (TACK) and findings through the geometry filter are discussed within the context of each unit.

Unit 1. Mrs. Lotus wanted me to observe this unit because of her familiarity with the content and the interesting activities that she typically does with the students for this unit. In the unit interview, Mrs. Lotus stated that she looked for new projects for some of the days within this unit, but upon further reflection, she decided that the original project that she used in years past better addressed what students needed to learn.

Concept Map. During the initial meeting with Mrs. Lotus, she understood the example of the sample concept map I gave her, and she was able to produce a sample one for the subject squares. For the actual study, she attempted a concept map for the first unit. She started with an oval with the word “triangles” in the middle, and from that oval, links were drawn to other ovals, each connecting to the oval “triangles” only. These other ovals contained the words inequalities, special segments, congruence tests, and classifications. She stated that it was too messy and did not want to continue on that sheet of paper. With encouragement to continue, she politely stated that she would think about it and redraw it later. She took the paper and asked for the concept map not to be recorded. During subsequent interviews, when I asked her to add to the concept map, she stated that she had nothing to add.

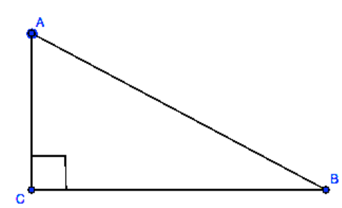
Unit 1 Planned. The main idea of this unit was to introduce students to properties of triangles and triangle congruence. For the first day of the unit, Mrs. Lotus planned to learn students’ prior knowledge, establish vocabulary terms for triangles by sides and angles, and work on proving conjectures related to triangle sum conjecture and interior and exterior angles. On the second and third day of the unit, Mrs. Lotus planned for students to work on an activity that developed conceptual understanding of different points of concurrency on a triangle. Mrs. Lotus stated that this activity might be a two-day lesson. The next concepts before the unit test were triangle inequality and triangle congruence. Triangle inequality was supposed to take a day, and triangle congruence was supposed to take two days. Mrs. Lotus decided that it would be best for me to observe the activity that develops conceptual understanding of different points of concurrency of a triangle and triangle inequality. By the start of the first day of observation, Mrs. Lotus noticed that there were some things from the first day of the unit that needed to be addressed on the second day of the unit. Thus, the first day of observation was going to be dedicated to notes and proofs, and the second and third days of observation were going to be spent doing the activity that developed conceptual understanding of different points of concurrency on a triangle.

Observed Lessons: Planned and Executed. On the first observed day of the unit, Mrs. Lotus reviewed the notes from the day before on definitions and naming parts of triangles, proved some conjectures related to the triangle sum theorem, and started on the Equally Wet problem. The triangle sum theorem states that the sum of the interior angles of a triangle is 180°. The corollaries that Mrs. Lotus asked students to prove were:

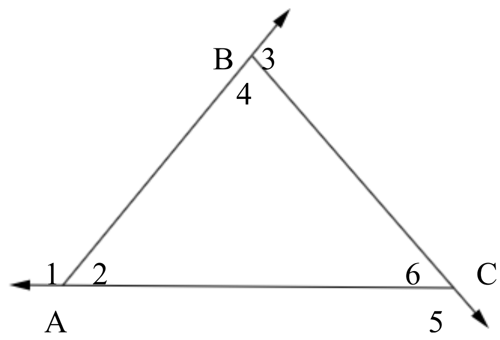
1. The acute angles of a right triangle are complementary

   Given: $\triangle ABC$ with $m\angle C = 90°$

   Prove: $m\angle A + m\angle B = 90°$



2. What is the sum of the measures of the exterior angles of a triangle? Prove your answer.



The Equally Wet problem posed that a girl named Leslie planted three flowers, and needed to place a sprinkler so that all three flowers could get equally watered by being equally spaced from the sprinkler. The three questions that the problem posed were:

1. Determine which arrangement of flowers will make this possible and which are impossible.
2. For the arrangements found for which it will be possible, describe how Leslie can find the correct locations for the sprinkler.
3. Can you generalize this problem?

Mrs. Lotus gave students a few minutes at the end of class to discuss the first two questions from this problem and asked students to spend at most 15 minutes working on the first two problems for homework.

For the next day, Mrs. Lotus started the class by discussing their ideas about problem numbers 1 and 2 from Equally Wet. During student discussion, different attempts at constructing the middle from three points allowed Mrs. Lotus to transition to an activity on Geometer's Sketchpad in which students constructed orthocenters, incenters, and circumcenters. On the third day of observations, students explored the other points of concurrency in a triangle, summarized the different points of concurrency, and practiced with some real world and textbook problems. Each day of the observation was executed as planned.

Unit 1 TACK. Similar to both of Mrs. Orchid's unit Teachers' Applied Content Knowledge (TACK) analysis, codes for specialized content knowledge, knowledge of content and teaching, and knowledge of content and students were determined from literature prior to collecting data for this study. Analysis was conducted in the procedure discussed in chapter 3. For the observed lessons in the unit, timing and sequencing went exactly as planned by Mrs. Lotus. All planned explanations, questions, and student difficulties were observed. Additional student difficulties were observed, and Mrs. Lotus provided corresponding additional explanations and questions that had not been planned. Also, due to students' specific questions related to how they were justifying an answer or writing a proof, Mrs. Lotus showed additional explanations and questions. All specialized content knowledge shown during planning was observed during the execution of the lesson.

For this unit, all of what was planned was observed. Additional student difficulties, explanations, and key questions were observed. The additional explanations and questions were in response to additional student difficulties. This section focused on planned and observed TACK. The next section expands on the geometry aspects of planned and observed TACK using the geometry filter.

Unit 1 Geometry Filter. From what I gathered on interviews and observations and put through the geometry filter; evidence for levels, phases, and habits of mind that I identified in what Mrs. Lotus planned was observed for each observation day. Throughout different phases of instruction in each observed day, Mrs. Lotus asked students to justify their answers, which showed van Hiele Level 2, informal deduction. For example, on the second day of observation, during the discussion of the findings for the Equally Wet problem, when students presented their solutions on the board, Mrs. Lotus asked students why they thought their solution worked in all cases. Evidence from TACK that showed van Hiele Level 2 during observations explanations and key questions to guide student thinking. Mrs. Lotus stated that she planned on having students prove conjectures formally when talking about presenting mathematical ideas and sequence of information to be covered during the day. During the observation, Mrs. Lotus asked students to prove two conjectures formally. Thus, this showed van Hiele Level 3, formal deduction.

Throughout the three observed lessons, the phases of learning based on the van Hiele Levels that I determined were both planned and observed through interviews and observations were directed orientation, explication, free orientation, and integration. From initial interviews with Mrs. Lotus, her philosophy of what phases of learning should be present each day of instruction corresponded with the phases of learning based on the van Hiele Levels. For example, on the first day of observation, problems during notes and discussion for the homework activity provided students with the opportunity to become more familiar with the material. Then on the second and third days of observation for directed orientation, Mrs. Lotus's different construction activities provided students with opportunities to become more acquainted with the material. The phases of learning were still present in each day, even though the activities were different. This showed Mrs. Lotus's specialized content, knowledge in presenting mathematical ideas, and knowledge of content and teaching, sequence of information to be presented.

For each observed day of this unit, I determined that Mrs. Lotus consistently showed fostering three geometric habits of mind: reasoning with relationships, generalizing geometric ideas, and balancing exploration and reflection. Like the activities for the phases of learning, activities that exemplified the geometric habits of mind were different, but those habits of mind were still consistently present in all three days of observation. For example, on the first day of observation, students reasoned with relationships of the sum of interior angles of a triangle, interior and exterior angles of a triangle, exterior angles of a triangle, and possible points equidistant from two and three points. On the second day of observation, students reasoned with relationships of the intersection of the three perpendicular bisectors of a triangle and the intersection of three angle bisectors of a triangle. Similar to phases of learning, this showed knowledge in presenting mathematical ideas and sequence of information to be presented; specialized content knowledge and knowledge of content and teaching respectively. However, in addition, basic and advanced levels of the geometric habits of mind were planned and observed, and evidence between basic and advanced were in Mrs. Lotus's activity directions and key questions to guide student thinking. The phases of learning based on the van Hiele Model and geometric habits of mind stayed consistent for all three days of observation.

Findings from Mrs. Lotus Unit 1. The general finding for Mrs. Lotus's first unit was that geometry TACK was observed as planned. Through both TACK analysis and the geometry filter for all lessons observed; sequence of activities, timing, questions, explanations, areas of student difficulties, modifications, student prerequisite knowledge, van Hiele Levels, phases of learning based on the van Hiele Level, and geometric habits of mind were observed as planned. Additional questions, explanations, and student difficulties, were observed. For every lesson of every observed day of this unit, Mrs. Lotus planned and executed every phase of learning based on the van Hiele Model.

Unit 2 Concept Map and Overview. For the concept map interview of this unit, Mrs. Lotus showed me a list of lessons to be covered for this unit. When I inquired about how this would fit into a concept map, she stated that the list was the concept map. They were in sequence of lessons in the unit. When I asked her to elaborate, she started talking about the unit and how each lesson builds on the topic in the unit.

Unit 2 is titled "Similar Triangles". Day one is on ratios and proportions, day two similarity, day three and four proportions in similar triangles, day five and six indirect measurement, and day seven right triangle proportions. I observed days two, three and four; similarity and proportions in similar triangles. Mrs. Lotus wanted me to observe those days because of the different types of application problems in these sections.

Observed lessons: Planned and executed. The main activity on the first day of observation was the dilations activity. For this activity, Mrs. Lotus gave directions, and then had students work in groups on the activity. Within each group, two students started with triangles of their own creation, and two students started with trapezoids of their own creation. Students were given time to work in groups on constructing the dilations and making conjectures about observations. When Mrs. Lotus noticed that most students were done, she led the class in discussing their findings. Then Mrs. Lotus lectured for a bit about similar polygons by giving a real world example and noted the specificity of language for mathematics and real world. Then Mrs. Lotus led the whole class through 6 problems, starting with one where they had to see if two triangles were similar and progressing in difficulty. For each problem, students were allowed time to work individually and discuss as a group before sharing their answers with the teacher. They also progressed in a way to lead to a theorem about perimeters of proportional or similar figures. Mrs. Lotus showed students this proof to expose them to more proofs and their structure. Then Mrs. Lotus introduced the similarity triangle conjecture by having students compare it to what they knew about similar figures. She then introduced the first test for similarity, all angles congruent.

Lesson two was different from the other lessons in that Mrs. Lotus's computer caught a virus, and so all of her PowerPoints that she had from before were inaccessible. She still knew what she needed to teach and had the activity handout already printed so she was going to still use it. Instead of using saved sketches on GSP, Mrs. Lotus had to construct using compass and ruler under the document camera. Also, due to technical difficulty, she decided not to have bellwork.

Continuing with the dilation activity, Mrs. Lotus led students by giving them instructions for dilating a triangle and then had them measure it for similarity. Each set of directions was a similarity test. The first test was sides without angles, and then the next test focused on two angles and a side. After the discussion summarizing all the ways to determine if two triangles are similar, including the definition, Mrs. Lotus gave a problem where students had to determine if two of the three given triangles were similar. Most of the students said no, and then one student actually did the calculation and found that the triangles were similar. When Mrs. Lotus asked why the students said no, most of them answered that the problem required too much work and that they were too lazy to do it.

After that example, students had to write a proof on how to determine if two triangles were similar. After the proof, three examples were given, and then class ended. What Mrs. Lotus planned was executed despite not having technology. The pre lesson interview was done earlier that morning so Mrs. Lotus was already prepared for the unavailability of technology (SmartBoard with all its functions, computer, and access to school-wide hard drive).

Since certain parts of technology were still unavailable, Mrs. Lotus used other programs to run class as she normally did. So, class started with bellwork and review of bellwork. Instead of notes being on a hard drive for students to access, Mrs. Lotus set up another digital location to where all students could go to retrieve notes for the day. Thus the order of the application problems was presented as follows; pantograph cat, pantograph daisy, rock wall, and river. For each progressive application problem, Mrs. Lotus gave less information for what students needed to use to solve the problem. While students were working on the river problem, Mrs. Lotus walked around and checked homework. Then Mrs. Lotus went over homework, talked about the river problem, and tied in student responses to the next theorem. Then Mrs. Lotus focused on students creating a formal proof of a corollary of the theorem. After going over the proof, Mrs. Lotus showed another theorem and had students prove a corollary to that theorem. After that, there were two straightforward problems before the last problem, which was an application problem, before the end of class. What Mrs. Lotus planned for, happened in class, and student interactions added to what Mrs. Lotus had to say and explain, but none were points of difficulty or misconceptions of content.

Unit 2 TACK. Mrs. Lotus relied heavily on her PowerPoint slides to be used on her SmartBoard. Within this unit, technical difficulties within the school made Mrs. Lotus unable to use her slides at all. Thus, she had to resort to using the document camera like a whiteboard for two lessons. This technical difficulty, however, did not change what was taught and what she planned. She had copies of the slides in a file cabinet just in case something like this happens. Student difficulties that came up within this unit were converting units of measure. This was something that Mrs. Lotus had not planned for, but was able to address. One difficulty that she planned for was students not reading through long word problems. Her planned modification was to not use long word problems on a test. However, she stated that sometimes, standardized tests assess in a way that long word problems are used. She stated that although it is important for students to be able to read through long word problems eventually, due to the fact that her students are eighth graders now, she did not think that their grades should be penalized for their immature behavior at the moment.

Unit 2 Geometry Filter. Similar to Unit 1, from what I gathered from interviews and observations, through the geometry filter, all that Mrs. Lotus planned were observed. The geometry filter highlights certain areas of TACK specific to teaching geometry.

Throughout the three days of observation, in homework review and discussion of problems throughout the lesson, Mrs. Lotus challenged students to informally prove why the answers they came up with were true. This showed van Hiele Level 2, informal deduction. On the first and third days of observation, Mrs. Lotus set up opportunities for students to write formal proofs, van Hiele Level 3. The decision to set up opportunities for students to write formal proofs relates to specialized content knowledge, presentation of mathematical ideas and connecting a topic being taught to topics from prior or future years.

All phases of learning, that I identified, showed evidence in planned and observed throughout the three days of observation. For example, on the second day of observation, Mrs. Lotus learned of students' prior knowledge and experience with the lesson of the day during homework review from the previous day. I decided that this showed the information phase, where the teacher learns of students' prior knowledge and experience with the subject through discussions (Mistretta, 2000). The construction activities showed "directed orientation", where these activities were provided for students to become more familiar with the material being taught (Mistretta, 2000). Discussion of the conjectures from constructions and problems summarizing what was learned showed explication and free orientation phases, because there was a transition of reliance on teacher to student self reliance, and the questions themselves were open ended and were attentive to the inventive abilities of the students (Mistretta, 2000). Also, the two problems at the end of the lesson showed integration phase because students were using what they learned throughout the day to solve those problems (Mistretta, 2000).

Throughout the observed days of the unit, three of the geometric habits of mind were planned and observed; reasoning with relationships, investigating invariants, and balancing exploration and reflection (Driscoll, 2007). From what I observed, during the dilation activity, students were reasoning with relationships of the original figure and the dilated figure. They made observations of the similarities and differences between the original figure and dilated figure. "Students talking with group members to generalize observations of their figures", I identified this to be "generalizing geometric ideas" (Driscoll, 2007).

Findings from Mrs. Lotus Unit 2. Despite technological difficulties and Mrs. Lotus' inability to access PowerPoint slides that she usually used, all planned geometry TACK was observed. Through both TACK analysis and the geometry filter for all lessons observed; sequence of activities, timing, questions, explanations, areas of student difficulties, modifications, student prerequisite knowledge, van Hiele Levels, phases of learning based on the van Hiele Level, and geometric habits of mind were observed as planned. Additional questions and explanations were observed. Based on observations, for each observed day of this unit, van Hiele Model phases of learning that I identified during planning were also observed.

Summary of Unit 1 and Unit 2. For both of Mrs. Lotus' units, geometry TACK planned was observed. For the first unit, additional explanations, questions, and student difficulties, were observed. For the second unit, only additional explanations and questions were observed. Mrs. Lotus credits the lack of student difficulties shown in class to the fact that they are self-motivated. "If they don't understand something, they'll ask each other. If it's something big, they'll usually go home and ask a parent or someone else." The only modification was lack of use of the interactive whiteboard with PowerPoint slide. The activities were executed as planned.

All aspects of the geometry filter were observed as planned. However, in the first unit for geometric habits of mind, Mrs. Lotus fostered reasoning with relationships, generalizing geometric ideas, and balancing exploration and reflection. In the second unit, Mrs. Lotus fostered reasoning with relationships, investigating invariants, and balancing exploration and reflection. Investigating invariants was not in the first unit, and generalizing geometric ideas was not in the second unit. Plans and how Mrs. Lotus executed them showed comprehensive knowledge of teaching and learning geometry. For Mrs. Lotus, all geometry TACK planned was observed. Additional context for geometry TACK for Mrs. Lotus appeared when themes through data analysis emerged.

Themes from Unit 1 and Unit 2. Themes surfaced in relation to timing, questioning, familiarity with activities used, areas of apprehension, and sources of knowledge. For all of Mrs. Lotus' observed lessons, timing occurred as anticipated. There were two lessons where activities took slightly less time than anticipated, and students had more time to work on homework for the next day. In questioning students, Mrs. Lotus usually gets multiple students to answer one question; and if there was a wrong answer, students discussed as a whole class what the correct answer should be. Thus, for big key questions that lead students into discussion, Mrs. Lotus was observed facilitating the discussion.

Mrs. Lotus was very familiar with all the activities for all six observations in both units. Despite her familiarity with these activities, her area of apprehension was when technological difficulty arose, and she was not sure of exact timing for phases of the lesson. Another area of apprehension was having me observe an activity she had not taught before. She thought that it would be interesting for me to observe this activity, but later changed the activity to one that she used before because it matched the standards better. The new activity came from one of Mrs. Lotus' periodical searches of new activities to use with students. To search for new activities, she stated that she used Internet searches, professional development, and collaboration with colleagues. Mrs.

Lotus stated that sometimes, she learns that an old activity can be used in a new way, especially from collaborating with other teachers. Her decision in what to teach comes from state standards, textbook, standardized assessments, professional development, college classes, personal education experiences, and professional experience as a teacher.

Connections between Planned and Executed TACK

For each lesson, and consequently for each unit, TACK components were compared from planned and executed. In general, all four units showed similar findings. What was planned was observed. In the next three sections, I describe findings from both teachers in the context of their category of TACK; knowledge of content and teaching, knowledge of content and students, and specialized content knowledge.

Knowledge of Content and Teaching. Knowledge of content and teaching (KCT) involves the knowledge of sequence of phases within a lesson or unit, timing for phases, explanations for concepts, and key questions to guide student thinking. For both teachers, in general, all KCT planned was executed. For both teachers, additional explanations and questions were observed. Sequences and timing occurred as planned as well, but this was dependent on each teacher's definition of time. Mrs. Lotus gave timing in minute estimates for each phase of a lesson, whereas Mrs. Orchid gave general times like "the rounds will take most of the class time".

Knowledge of Content and Students. Knowledge of content and students is knowledge of content specifically involving students (Ball, Thames, & Phillips, 2008). For this study, knowledge of content and students is demonstrated by the teacher in three ways: modifications to what is being taught based on student difficulties; student misconceptions and how to help students overcome them; and knowledge of student prerequisite skills.

Both Mrs. Orchid and Mrs. Lotus had a lesson within their units where there was a planned and observed modification. Modifications were due to student difficulties and technical difficulty. Standards for student learning was not compromised in their modifications. For both teachers, all anticipated student difficulties were observed. Additional unanticipated areas of student difficulties were student difficulties that both teachers had the knowledge to support student learning through their vast source of knowledge. All anticipated prerequisite skills were observed for students of both teachers. Both Mrs. Orchid and Mrs. Lotus were prepared for the mixed ability in prerequisite knowledge. Prior to the focus activity of the day, they had other notes and activities that addressed the prerequisite skills for those students that lack the skills. Further, they both believed that discussions among students who understand and who may not understand yet, are more meaningful than the teacher telling the student what to do.

Specialized Content Knowledge. To discuss aspects of specialized content knowledge (SCK) specifically, in my analysis of SCK, I looked at planned and executed SCK through mathematical tasks for teaching (Hill, Ball, Schilling, 2008). In presenting mathematical ideas, both teachers presented the way that they planned. All planned responses to students' "why" questions were observed, but with student interaction, additional responses to students' "why" questions were also observed. For each lesson observed, both teachers found examples to make specific mathematical points. For each lesson, each teacher shared her selection of representations for particular purposes.

Generally, these were based on knowledge of content and students from previous experience. In discussing representations, both teachers discussed their links to underlying ideas and other representations. Findings were similar for other mathematical tasks for teaching.

For both teachers, all knowledge shown of mathematical tasks for teaching during planning were shown during the observation. Throughout the units for both teachers, I did not observe "explaining mathematical goals and purposes to parents" and "modifying tasks to be easier or harder". Some mathematical tasks for teaching like "evaluating the plausibility of students' claims" had more evidence in the classroom than during interviews. However, all SCK planned were also observed. Additional evidence for SCK that were not stated as planned was also observed.

Geometry Filter for Mrs. Orchid and Mrs. Lotus

For both teachers, though the geometry filter; levels, phases, and habits of mind I saw during planning was observed during the execution of the lesson. For Mrs. Lotus, the van Hiele Levels, phases of learning, and geometric habits of mind that I identified were all executed exactly as planned. Since Mrs. Orchid's second unit was on trigonometry, applicable aspects of the geometry filter were addressed. Both teachers' first unit focused on informal deduction, van Hiele Level 2, and developing formal deduction, van Hiele Level 3 within some activities. Mrs. Lotus's second unit had formal justification each day observed.

When only looking at observed days of the unit for Mrs. Orchid, I found evidence to support the latter four phases of instruction based on the van Hiele Levels, excluding "information" phase until the fourth observation of the first unit. When looking at both of

Mrs. Orchid's units as a whole, evidence for all phases of learning based on the van Hiele Model were present. Mrs. Lotus's lessons showed all five phases for each observation. Of the four geometric habits of mind, "investigating invariants" was the geometric habit of mind for which I found the least occurrence. Through the geometry filter, very little differences appeared between planned and observed.

Connections of Geometry Filter to TACK

I used the geometry filter to illuminate connections between planned and executed Teachers' Applied Content Knowledge. Similar in format to how I discussed the geometry filter for each unit, the connections between geometry filter and TACK will be discussed by van Hiele Levels, phases of learning based on the van Hiele Model, and geometric habits of mind.

When looking at the van Hiele Level in planning of a lesson, aspects of TACK that were highlighted were: specialized content knowledge, specifically, presenting mathematical ideas and connecting a topic being taught to topics from prior or future years; and knowledge of content and teaching, specifically, sequence of information to be presented, explanations, and key questions to guide student thinking. In determining the van Hiele Level during the observation of a lesson, aspects of TACK that were highlighted in addition to those when looking during the planning of a lesson was specialized content knowledge, specifically, responding to students' "why" questions, finding an example to make a specific mathematical point, and asking productive mathematical questions.

Both teachers' knowledge of how students learn determined their structure of instruction to be very similar to phases of learning based on the van Hiele Model. Since the information phase is where discussions are held where the teacher learns of the students' prior knowledge and experience with the subject matter at hand (Mistretta, 2000, p. 367), connections to TACK were: knowledge of content and teaching, specifically, sequence of information to be presented, timing (how long to allow for and how much time was used in class), and key questions to guide student thinking; and knowledge of content and students, specifically, student prerequisite knowledge. Directed orientation is where the teacher provides activities that allow students to become more acquainted with the material being taught (Mistretta, 2000, p. 367), connections to TACK were: presenting mathematical ideas, an aspect of specialized content knowledge and sequence of topics to present, an aspect of knowledge of content and teaching.

During explication, a transition between reliance on the teacher and students' self-reliance is made (Mistretta, 2000, p. 367). Aspects of TACK illuminated when looking at explication were: presenting mathematical ideas, sequence of topics to be presented, explanations, and questions to guide student thinking. During observations, explication phase was apparent when there was a steep drop in number of questions and explanations teachers gave, and instead, both Mrs. Orchid and Mrs. Lotus prompted students to ask each other questions or gave some points for the students to consider and then immediately walk away. Free orientation is where teachers gave students tasks that can be approached in numerous ways (Mistretta, 2000, p. 367). The connection to aspects of TACK were: specialized content knowledge, specifically, presenting mathematical ideas, connecting a topic being taught to topics from prior or future years, and asking productive mathematical questions; knowledge of content and teaching, specifically, sequence of information to be presented and key questions to guide student thinking; and knowledge of content and students, students' prerequisite knowledge. For integration, where students summarize what was learned during the lesson (Mistretta, 2000, p. 367), connections to TACK were: specialized content knowledge, specifically, presenting mathematical ideas and asking productive mathematical questions; and knowledge of content and teaching, specifically, sequence of information to be presented, explanations, and key questions to guide student thinking. Interesting note, however, for Mrs. Orchid, when there was limited time; integration phase was left out of one observation day.

All four geometric habits of mind (Driscoll, 2007) were fostered between the two units for each teacher. Evidence that helped me determined geometric habits of mind during planning connected to aspects of TACK were: presentation of mathematical ideas, an aspect of specialized content knowledge; sequence of ideas to be presented, explanations, and key questions to guide student thinking, aspects of knowledge of content and teaching; and student prerequisite knowledge, an aspect of knowledge of content and students.

This section was not an exhaustive list of connections, but rather, only the strongest patterns of connections for aspects of the geometry filter to aspects of TACK were discussed. Some other connections may exist, but through my analysis, evidence from the aspects of TACK described above helped determine the aspect of geometry filter each teacher addressed. Some connections were specific to one lesson, like for homework review one lesson, Mrs. Orchid fostered the geometric habit of mind "investigating invariants" to help students clear up a misconception about corresponding parts of congruent triangles.

Comparison of the Cases: Mrs. Lotus and Mrs. Orchid

The context of the two cases, Mrs. Lotus and Mrs. Orchid were different. The class observed of Mrs. Orchid was high school 9th through 11th graders, while Mrs. Lotus's students were 8th graders in an advanced high school geometry course. The number of students was similar, Mrs. Orchid had 24 students, and Mrs. Lotus had 25. Both teachers were defined as exemplary teachers, and looking at similarities can illuminate what both share as exemplary teachers. Looking at differences sheds light to the different contexts in which each of these exemplary teachers work.

Similarities. Both teachers had a variety of knowledge, specifically specialized content knowledge when discussing the overall plans for the unit. They shared knowledge of curriculum, general student misconceptions, philosophy of questioning and explaining things to students, and overall where the concepts they were about to teach fit into the course and future math classes. For both teachers, generally what was planned was observed.

In emergent codes, both teachers gave activity directions before starting an activity. Within both units for each teacher, there was birdwalking, conversation not related to classroom procedures or content, (Hunter, 2004) about the temperature of the room and what students were wearing. Classroom administration were observed for both teachers as well, and within that, both teachers had intercom interruptions at some point during at least one of the units. Once students were working in groups, both teachers walked around, and at times and showed verification - statements made to students from the teacher letting them know that what they are doing is either correct or incorrect without questioning or explaining.

Differences. The main difference between Mrs. Orchid and Mrs. Lotus was the presence of activity directions and classroom administration. Mrs. Orchid had at least three times the number of units-of-analysis that were coded classroom administration and at least twice the number of units-of-analysis that were coded activity directions. Of the units-of-analysis that were coded classroom administration, most were discipline related; asking students to be quiet, put away their phone, work on the assignment, and not sleeping during class. Also, the second most number of units-of-analysis within classroom administration for Mrs. Orchid was related to students using the restroom. Also, Mrs. Orchid repeated similar activity directions at least four times for each activity, while Mrs. Lotus repeated similar activity directions at most, twice.

Even thought both teachers stated that they did not have plans for timing, upon further questioning, both gave rough estimates of times that they thought would be necessary for each phase of the lesson. Mrs. Lotus followed her timing almost perfectly, even though she did not look at the clock in the room very often. She went on to the next phase of the lesson when students seemed like they were ready to move on to the next topic. Mrs. Lotus stated that she based moving on to the next topic on students' facial expressions, correct answers, and ability to explain. Thus, her observed timing followed planned timing almost exactly. However, due to difficulties in the classroom, Mrs. Orchid's timing for the second unit did not go as planned. The same planned sequence, explanations, and questions were eventually addressed; but timing was not exactly observed as planned.

During discussions of students' answers in class, for Mrs. Lotus's observations, students usually answered with the correct answer complete with justification. Thus, many of Mrs. Lotus's remaining follow-up questions were "Did anyone get anything different?"; "Does everyone agree?"; and "Questions?" Student responses to these questions typically were yes, and no to the last two questions. For "Did anyone get anything different?" students that did get something different usually spoke up and were correct. If they were not correct, other students explained what was incorrect, and the class self-corrected. For Mrs. Orchid's class, students often answered with the correct answer as well, but Mrs. Orchid asked several questions to help students explain their answers and justify correctly. If a student was incorrect, other students waited for Mrs. Orchid to tell them whether or not it was correct, and why it was incorrect. Mrs. Orchid facilitated students explaining "why" questions by asking students questions.

Addressing the Research Question from Analysis

The research question was "What connections are there between the high school geometry TACK shown during the planning of the lesson and the teachers' actual executions of the lesson?" To discuss connections between teacher knowledge planned and executed specific to the aspects of TACK, I separated the research question into eight more explicit questions.

- What are the connections between planned and executed specialized content knowledge?
- What are the connections between planned and executed sequence of information to be presented?

- What are the connections between planned and executed time spent on phases of the lesson?
- What are the connections between planned and executed explanations of concepts, definitions and procedures?
- What are the connections between planned and executed key questions to guide student thinking?
- What are the connections between planned and executed lesson modifications?
- What are the connections between planned and observed teachers' acknowledgement of student difficulties and how to overcome them?
- What are the connections between planned and observed teachers' knowledge of student prerequisite skills?

The following are results to each of the eight explicit questions based on the data collected and analysis described in the previous chapter.

- Specialized content knowledge – All planned were observed during execution of the lesson. Additional responses to students' "why" questions and examples to make specific mathematical points were observed.
- Sequence of information to be presented – All planned sequences of information were observed during execution of the lesson. For Mrs. Orchid, in one class, an additional sequence of "calculator procedures" was added to observation that was not planned when more students that anticipated had difficulty using their calculators.
- Time spent on phases of the lesson – Most of what was planned was observed during the execution of the lesson. In the instances where planned timing was not observed, observed phases took longer than planned.
- Explanations of concepts, definitions and procedures – All explanations of concepts, definitions and procedures were executed as planned. However,

additional explanations of concepts, definitions and procedures were observed during the execution of the lesson.

- Key questions to guide student thinking – All key questions to guide student thinking were executed as planned. However, additional questions to guide student thinking were observed during the execution of the lesson.
- Lesson modifications – All planned lesson modifications were observed during the execution of the lesson.
- Teachers' acknowledgement of student difficulties and how to overcome them – All student difficulties and strategies to overcome them that were stated prior to the observed lesson were observed during the execution of the lesson. Additional student difficulties and strategies to overcome them were observed during the execution of the lesson.
- Teachers' knowledge of student prerequisite skills – All student prerequisite skills stated during lesson planning were observed in execution of the lesson.

All of teacher knowledge shown during lesson planning was observed during the execution of the lesson. Through student interactions, additional teacher knowledge was observed during the executed lesson. These were the results from analysis of the data collection before axial coding.

Axial Coding

Axial coding gave a different perspective on individual TACK and emergent codes. Unitizing for units of analysis and coding followed the coding guide in Appendix H. Units of analysis in general adhered to Miles and Huberman's (1994) definition of a unit of analysis; "a sentence or multisentence chunk" (p. 65) where a single code can be easily assigned. Garrett's (2010) definition of topic and unit in relation to units of analysis was more applicable to this study. Topic, in relevant to this study, based off of Garrett's (2010) was defined as a complete idea, aspect of TACK, or topic of conversation upon which the attention of the speaker or speakers is focused. Garrett (2010) defined a unit as "A word, sentence, paragraph or several consecutive sentences or paragraphs focusing on the same topic" (p. 264).

Before starting data collection, there were eight a priori codes; specialized content knowledge, knowledge of content and teaching – sequences, knowledge of content and teaching – time, knowledge of content and teaching – explanations, knowledge of content and teaching – questions, knowledge of content and students – modifications, knowledge of content and students – difficulties, and knowledge of content and students – prerequisites. Emergent codes, codes that showed up as themes in the units of analysis that are not a priori codes were activity directions, birdwalking (Hunter, 2004), classroom administration, curriculum, researcher technical interactions, verification, and other.

Activity directions are discussions of what students are about to do and how to do it. Birdwalking is defined as teachers engaging in discussion with students about something not related to the lesson or any of the above codes (Hunter, 2004). Classroom administration, for this study, is defined as discussions pertaining to taking attendance, intercom interruptions, bells sounding for other classes, students asking to use school facilities outside the classroom, discipline, and other classroom procedures. Curriculum, in the context of this study, is defined as discussions pertaining to opinions and ideas of textbook being used or standards for instruction. "Researcher technical interactions" were discussions regarding administrative aspects of the study like camera angle and scheduling. Verification is defined as statements made to students from the teacher letting them know that what they are doing is either correct or incorrect without questioning or explaining. Since everything is coded, "other" is a code that was used to represent a segment that did not fit anywhere else. For a detailed list with examples of emergent codes, see Appendix H.

Once unitized, themes began to appear. During pre-observation interviews, conversation topics that appeared were regarding administration of the study, activity directions, curriculum, specialized content knowledge, sequencing of activities, timing for phases of lessons, explanations of content, questions to guide student thinking of topic, modifications to lessons, student difficulties, and student prerequisite skills. During observations, activity directions, birdwalking (Hunter, 2004), classroom administration, curriculum, verification, specialized content knowledge, sequencing of activities, timing for phases of lessons, explanations of content, questions to guide student thinking of topic, modifications to lessons, student difficulties, and student prerequisite skills were observed. The following sections describe the relationships between the codes when each a priori code underwent axial coding, starting with specialized content knowledge.

Specialized Content Knowledge. From the defining features of specialized content knowledge (SCK), many aspects of SCK overlap with knowledge of content and teaching, knowledge of content and students, and some emergent codes. For example, an aspect of specialized content knowledge is "responding to students' "why" questions". When students asked Mrs. Orchid and Mrs. Lotus "why" type of questions, both teachers responded with key questions and explanations. Table 4-5 lists SCK and codes related to specific aspects.

Table 4-5 Aspects of SCK and related codes

Aspect of SCK	Related codes
presenting mathematical ideas	Explanations, questions, modifications, difficulties, and prerequisites
responding to students' "why" questions	Explanations, questions, difficulties, and prerequisites
finding an example to make a specific mathematical point	Explanations, questions, and difficulties
recognizing what is involved in using a particular representation	Explanations and questions
connecting a topic being taught to topics from prior or future years	Explanations, questions, modifications, difficulties, and prerequisites
explaining mathematical goals and purposes to parents	not stated in interviews or observed
appraising and adapting the mathematical content of textbooks	Explanations, questions, modifications, difficulties, prerequisites, and curriculum
modifying tasks to be either easier or harder	Explanations, questions, modifications, difficulties, and prerequisites
evaluating the plausibility of students' claims (often quickly)	explanations, questions, and verification
giving or evaluating mathematical explanations	explanations, questions, and verification
choosing and developing useable definitions	Explanations, questions, modifications, difficulties, and prerequisites
using mathematical notation and language and critiquing its use	Explanations, questions, modifications, difficulties, and prerequisites
asking productive mathematical questions	Explanations, questions, difficulties, and prerequisites
selecting representations for particular purposes	Explanations, questions, modifications, difficulties, and prerequisites
inspecting equivalencies	explanations, questions, and verification

Thus from the table, it is apparent that SCK very closely related to KCT in aspects like responding to students' why questions and asking productive mathematical questions.

SCK is very closely related to KCS in aspects like presenting mathematical ideas and selecting representations for particular purposes.

Knowledge of Content and Teaching. During pre-observation interviews, both teachers used the sequence of information to discuss timing, explanations, questions, difficulties, and prerequisite skills. In analyzing data, observed geometry TACK depended on sequence. During observations, teacher's mentioning of sequence was usually accompanied by a comment about time. Although it is unclear if sequence causes change from planned to executed time, or if time causes change from planned to executed sequence; time and sequence are closely related.

During interviews when teachers did not explicitly state time allotment for activity within the lesson, I asked about specific timing for phases. In response to how much time is taken, both teachers talked about timing and then included student prerequisite skills and difficulties, and the questions and explanations necessary for student difficulties and prerequisite skills. Also, because both teachers were getting adjusted to new curriculum in the form of standards and textbooks, both teachers discussed what they did in the past and how much time these phases would have taken and what had changed since then. As stated for sequences, during observations, teachers made note of timing whether it took longer or shorter than expected when the teacher moved on to the next phase of the lesson.

Explanations and questions were nearly inseparable. Both teachers discussed key questions with explanations and vice versa, both in interviews and classroom observations. Also, during observations, when teachers asked a question, and student response seemed confusing, teachers rephrased and explained the student's explanation.

Also during observations, when students asked questions, many times, the teachers responded with a question or series of questions to explain.

Knowledge of Content and Students. For both teachers, most of the lessons observed were lessons that they taught many times before. Both teachers stated that modifications that they decided were necessary were already changed to fit their students the first couple of times they taught the lessons. Thus, modifications were rare. The only modifications that occurred during the observations that I conducted were due to technical difficulties or student difficulties. These modifications changed timing, explanations, and questions.

During interviews, student difficulties were discussed in the context of discussing sequences, explanations, questions and prerequisites. When student difficulties were not explicitly discussed, after prompting from me, teachers discussed student difficulties with prerequisite knowledge, explanations, and questions. During observations, unplanned student difficulties resulted in additional explanations and questions from the teachers. In one case, student difficulties changed timing, and caused the need for lesson modification the next day.

During interviews, prerequisite knowledge needed was discussed in the context of discussing sequences, explanations, questions and student difficulties. When teachers did not discuss prerequisite skills explicitly, when I asked, they responded with what prerequisite skills they thought students needed in addition to student difficulties that arose due to the lack of the prerequisite knowledge.

Summary of Themes from Mrs. Orchid and Mrs. Lotus

Looking through interviews and observations from Mrs. Orchid and Mrs. Lotus, some themes surfaced: reasons for choices of units to share, source of knowledge, use of technology, areas of apprehension, and familiarity with activities. Through interviews and classroom observations, I found that both teachers conducted class with the basis that students need to construct their own knowledge in order to learn, students need to learn conceptually, and therefore, they felt that they needed to teach conceptually. When asked why they conducted class the way they intended and executed, they stated that it was due to a mixture of state standards, textbook, standardized assessments, professional development, college classes, personal education experiences, and professional experience as a teacher.

Both teachers chose lessons where topics and activities would be interesting to watch. Mrs. Lotus's chosen units were topics that she was very familiar with and had interesting activities to share. Mrs. Orchid's first unit had a very interesting activity that she wanted to share, and the second unit was an activity that she created that she wanted to share. Elements within the lessons that both teachers chose for me to observe were engaging activities, open ended tasks, collaboration among students, and expecting students to justify their answers. Prerequisite knowledge required of students for activities were always addressed right before the activity or had been addressed at some point within the semester prior to the activity. Thus, student learning was based on a foundation of prerequisite knowledge that each teacher already built earlier in the semester.

Additional questions asked of students and explanations were observed for every lesson. Both teachers stated that it would be impossible for them to list every student question, difficulty, or misconception prior to the observation of the lesson. However, Mrs. Orchid and Mrs. Lotus had experienced all observed additional student questions, areas of difficulty, and misconceptions previously in other lessons. They stated that they were confident in their knowledge to meet any student difficulty. Both of them also added that if they did not know the answer, then they told the students to look it up, and then they looked it up too, and discuss the problem the next day. This was mainly due to their vast source of knowledge: college coursework, textbook, professional development, teaching experience, and collaboration with colleagues.

Both teachers had interactive whiteboards and they used them to present information on a daily basis; and during the course of this study, both experienced technical difficulties where saved items could not be accessed, especially items saved specifically for the interactive whiteboards. Both teachers still taught the lessons as planned, as the only change in the lesson was that information was printed for students to see instead of having it on the interactive whiteboard where the teachers could write over what was printed. For Mrs. Lotus, instead of having the steps already on slides to present to students, she presented step-by-step constructions using the document camera. Prior to the lesson, Mrs. Lotus stated that the lesson may take a little bit longer because she had to take time to draw out the steps, but after the lesson, she was pleasantly surprised by the minimal change in predetermined time.

Both Mrs. Orchid and Mrs. Lotus discussed how they planned to teach activities that they had not previously taught. Mrs. Lotus ended replacing the new activity with an activity that she taught before because she felt that it fit better with what the students needed to know. However, some common themes arose from how they planned to teach the activity and how they expected for student difficulties and misconceptions. Since they both had not taught the activities before, there were no modifications based on previous experience. Student areas of difficulties and misconceptions were based on previous experiences with other students. For both teachers, areas of difficulties and misconceptions that showed up during the activity that matched with what they anticipated were the difficulties and misconceptions in prerequisite knowledge. However, additional difficulties and misconceptions that were observed mostly were related to the concepts learned during the activity. Figure 4-12 illustrates this point.

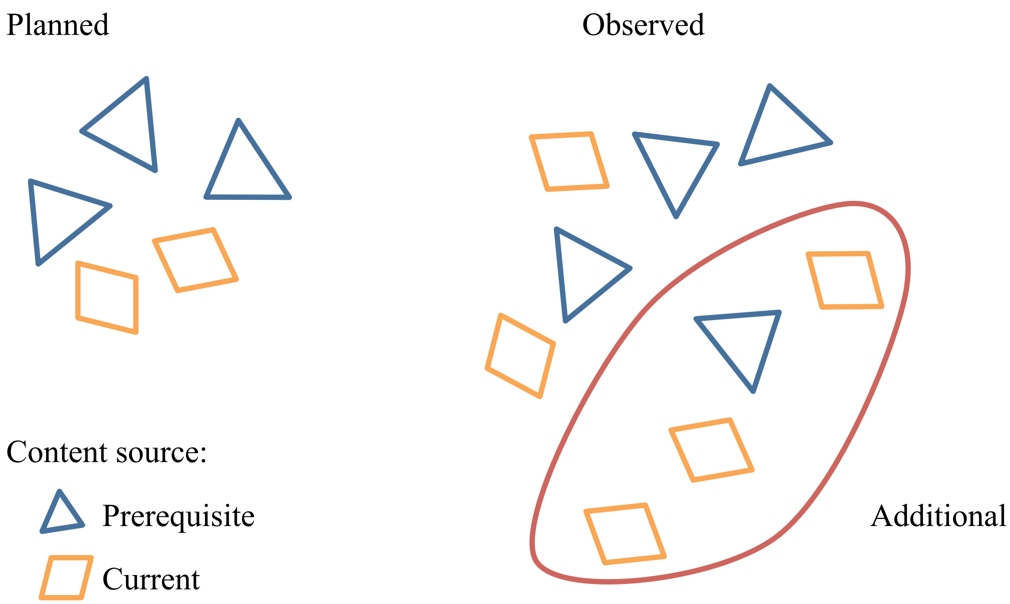


Figure 4-12 Content source of student difficulties and misconceptions

Student difficulties and misconceptions stated during planning were also observed. Of the additional student difficulties and misconceptions that were observed, more were related to the current activity than prerequisite knowledge.

Summary of Research Findings

In the first half of this chapter, I organized data collected by teacher, and then by unit, starting with Mrs. Orchid. For each teacher, I discussed overall data collection with that teacher, overall plans for the unit, concept map(s) for the unit, and planned and executed teacher knowledge for the observed lessons of the unit. Next I discussed analysis of the unit, expounding on TACK, geometry filter, and themes from analysis without axial coding. The second half of this chapter compiled findings in a more succinct fashion, by discussing connections between planned and executed TACK for both teachers, the geometry filter for both teachers, connections of the geometry filter to TACK. After pulling together results from analysis prior to axial coding, I addressed the expanded research question, eight questions each addressing an aspect of TACK. Results from addressing the research question based on analysis without axial coding confirmed the need to employ axial coding to further explore relationships among the codes. Results from axial coding were organized by aspect of TACK, specialized content knowledge, knowledge of content and teaching, and knowledge of content and students. Through discussion of teachers, TACK, geometry filter, and analysis, some themes began to emerge. The last section of this chapter addressed the themes that emerged from analysis.

Chapter 5: Conclusions, Discussions, and Suggestions for Future Research

My interest in high school geometry is based on my experience as a former high school geometry teacher. Coincidentally, a review of the literature revealed that nationally, student achievement in geometry is an area of concern (Clements, 2003; Lappan, 1999; Mullis, Martin, & Foy, 2008; Shaughnessy, 2011); and geometry is one of the areas of mathematics that is necessary for many careers (National Research Panel, 1989; WeUseMath.org, 2011). Moreover, teacher knowledge of geometry has an impact on student achievement (Monk, 1994; Hill, Rowan, & Ball, 2005). Thus, the decision to study geometry teachers' knowledge, specifically geometry Teachers' Applied Content Knowledge (TACK) was due to my interest followed by an examination of literature.

Formed from literature, the research question for this dissertation was: What connections are there between the high school geometry Teachers' Applied Content Knowledge shown during the planning of the lesson and the teachers' actual executions of the lesson? In the previous chapter, I discussed answers to the research question from findings specific to the different aspects of TACK: specialized content knowledge, sequence of information to be presented, timing for phases of the lesson, explanations, key questions to guide student thinking, student areas of difficulty and misconceptions, modifications to the lesson or activities, and student prerequisite skills. For each of these, the exemplary teachers showed, what was planned during observations of lessons. Student engagement during observations often brought about additional instances of these actions than was planned. From axial coding, some aspects of TACK and other teacher knowledge and skills seemed more connected than others. This chapter brings together the findings from Chapter 4 and literature from the first three chapters into: conclusions for this study, discussions related to the conclusions, and suggestions for further research. Before discussing conclusions, I discuss limitations to this study.

Limitations

There are three main limitations to this study; exclusion of students, time allotted to interview teachers, and technological difficulties. Students were not interviewed but their actions were an integral part of this study. From observations, students influenced teachers to give more questions and explanations than what the teachers planned. Certain types of student interactions may have elicited certain types of teacher responses. Additional interviews with students and video cameras stationary at each table would have given more information from students' perspectives. This would have been nice to study, but exceeded the scope of the research question for this study. Student interactions are a possible area for further research.

Due to the timeline being set for a minimum of three consecutive lessons, and what happened one day may impact the next day; there was very limited time to interview the teacher on their reflection of the lesson and the interview prior to the next lesson. Due to time being limited, both teachers explained within the time allotted what they felt was important for me to know about the unit. If I had more time, I think that the data collected from the concept maps would be more robust. In retrospect, for the scope of the study, conversations from interviews yielded the information sought from concept map interviews.

For the first couple of observations, when the video camera was switched from being plugged in to a battery source, video and audio data recording stopped for about 6 seconds. When this occurred, the recorder attached to the teacher was still recording audio; so there was no lapse in audio recording. There were only six of these instances, all of which occurred during transitions between activities when the camera was switched from being stationary to mobile to follow the teacher around. For one observation, the teacher put the audio recorder in her pocket, and the entire recording consisted of rustling sounds. Fortunately for that day of data collection, the video camera recorded class time without breaks. Technological difficulties prevented Mrs. Orchid from e-mailing me the second concept map with audio data of her explaining the concept map. Despite all these limitations, the quality of this study was not compromised. The following section expounds upon conclusions for this this study.

Conclusions

The exemplary teachers in this study showed ownership of depth and breadth of Teachers' Applied Content Knowledge for teaching high school geometry. The purpose of this study was to see the connections between teacher knowledge shown while planning and executing the lesson. The simple response is: teacher knowledge shown during planning was observed during execution of the lessons for the two exemplary teachers I studied. However, due to student engagement in the lesson, additional teacher knowledge was shown during the execution of the lesson. This study elucidated some characteristics of exemplary teachers, namely how they:

- Explained concepts
- Modified lessons

- Addressed student difficulties
- Conducted classroom procedures
- Utilized technology
- Discussed planned and executed lessons

Discussions of these characteristics were based on findings from analysis of TACK, geometry filters, areas of apprehension, sources of teacher knowledge, and changes to lessons from planned to executed.

How did exemplary teachers explain concepts to students? Both Mrs. Orchid and Mrs. Lotus were knowledgeable of the concepts they were going to teach their students. In interviews prior to observations, both teachers discussed how to explain concepts to students along with student areas of difficulties, prerequisite knowledge, curriculum (textbook and standards), and questions to ask students to help explain the concept. Since the teachers had taught the courses for consecutive years, they were able to anticipate student difficulties and plan accordingly. For most lessons, both teachers had planned questions that would facilitate student understanding of concepts. They also had explanations of concepts prepared.

During execution of the lesson, when students asked for an explanation or the teacher felt that it was appropriate to give an explanation, both teachers used questions to elicit student understanding of the concepts or skills rather than responding to students' questions by telling them the answers. When the teachers utilized series of questions, the questions started with accessing knowledge from prior lessons and progressed to current content being taught. This line of questioning showed that teachers were considerate of students' zone of proximal development (Bay-Williams & Herrera, 2007); the area bounded by the student's ability to solve problems independently and the students' ability to solve problems with the help of someone more capable (Vygotsky, 1978). This evidence of teacher knowledge also supports what both teachers stated about their views of how students' learn, connecting new knowledge to existing knowledge; showing that they were constructivist teachers (Steffe & D'Ambrosio, 1995) and in alignment with the Teaching and Learning Principles of *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics [NCTM], 2000). Counting statements of explanations of concepts and questions to explain concepts, both teachers had at least double the number of questions than statements for each lesson. In some lessons the number of questions quadrupled the number of statements.

How did exemplary teachers modify lessons for students? Some of the activities in the lessons that Mrs. Orchid and Mrs. Lotus taught had been used in previous years. For those activities, both teachers had modified their lessons throughout the years. Therefore, at the point of this study, the teachers were prepared for student areas of difficulties by providing appropriate levels of prerequisite knowledge, and carefully planned explanations and questions. Thus, for activities that the teachers taught many times before, modifications were limited to students' behavioral dynamics and technological difficulties. Modifications due to student behavior only changed the level of supervision from the teacher during an activity. For example, instead of coming up with the steps to construct a parallelogram in pairs of students, students discussed the topic as a whole class led by the teacher. Technological difficulties were unforeseen, however both teachers quickly found ways to conduct activities as planned. They showed familiarity with how to teach the content using other types of technology or methods. Modifications did not affect content taught or cognitive demand.

Lessons in one unit for one teacher contained activities that the teacher had not taught before because the content was new to the high school geometry course as a result of new state standards. From how the teacher planned during the unit interview, activities were modified due to student difficulties and students' behavioral dynamics. Despite the modifications, students were still held to the same cognitive demand and content standards; all within the same amount of overall time, days allotted for the unit.

How did exemplary teachers address student difficulties? Similar to the discussion on modifications, through the years of teaching, these two teachers have seen most students' difficulties to the activities that they used before. Thus, they have made the necessary changes to explanations and questions, and made modifications to activities when appropriate. When multiple students showed the same difficulty, teachers readdressed the class with either explanations with statements and/or questions, or a modification to the lesson where the teacher taught procedures for what was needed; in the case of this study, how to use a calculator for a specific function. When I observed that a student was at a lower van Hiele Level than the activity required, I noticed that the teachers questioned at the level the student was on, and gradually brought the student to where he/she needed to be; mindful not to skip levels and work within their zone of proximal development (Bay-Williams & Herrera, 2007). Both teachers stated multiple times during interviews that they have seem all of the difficulties that occurred in class before when they taught the lesson.

Both teachers noted that occasionally, there have been new student difficulties that they had not seen before. With new student difficulties that they have not encountered before, if they knew the solution to the difficulty, they guided students to the correct solution. However, if the solution was not something they understood readily, they asked students to take it home and research the solution for extra credit. If the teachers did not arrive at the solution within a day through research on their own with colleague or literature, they contacted professors at the local university for help. Although none of this was observed during this study, discussions with their colleagues and professors showed this was a reoccurring cycle throughout the years.

How did exemplary teachers conduct classroom procedures? Both teachers knew the sequence of content to cover within an observation day very well. At times, without looking at the clock, they made comments about whether or not the class was on track for timing for the activity. I believe that the teachers' ability to accomplish classroom administrative procedures and conduct student learning was largely due to their respect for time and what they, the teacher, and students should be doing at any given time. For example, while students were working on warm-up activities, the teachers checked homework, took attendance, and redirected off-task behavior; while walking around and evaluating the plausibility of student's claims on the warm-up problems. When students exhibited off-task behavior, like talking about something unrelated or sleeping during an activity; teachers stated activity directions and what students were supposed to do within the activity. Teachers disseminated materials that were needed for the day while giving activity directions or while students worked on something else. From the time students walked into these teachers' classrooms to the time they left the classroom, students always had something to do related to the content. All of class time was utilized for activities planned for the day. Not only were these two teachers comfortable with the content and how to teach it, they were also very familiar with conducting everyday classroom procedures in a way that was supportive of teaching the content.

How did exemplary teachers utilize technology? Technology; in terms of interactive whiteboard, document camera, teacher computer, and student computers; was readily available to both teachers. Both teachers exhibited knowledge of how to use all available technology, not only for themselves, but was able to facilitate student learning using the technology. Teachers provided students opportunities to use technology to aid in learning the content, but also helped students learn how to use technology when needed. Throughout the entire study, use of technology supported learning, and did not detract from learning.

How did exemplary teachers discuss planned and executed lessons? When the teacher discussed what was planned all aspects of TACK were discussed together. Their discussions were guided by the sequence of phases of the lesson, taking time to talk about other aspects of TACK as they related to phases of the lesson. They showed knowledge of how to teach geometry from planning and enacting appropriate Hiele Levels (Breyfogle & Lynch, 2010), phases of learning based on the van Hiele Model (Mistretta, 2000), and geometric habits of mind (Driscoll, 2007) even though they did not mention these by their proper names or titles. A theme that emerged from their discussion of planned lessons were sources of knowledge for content and how to teach the content. Their plans for what to teach and how to teach it were based on many sources of knowledge; teaching experience, coursework, textbook, professional development, and collaboration. The source of knowledge most often referred to was previous experience teaching the activity or topic.

Summary. Many characteristics in the way the exemplary teachers in this study talked about and taught geometry was similar. They often explained concepts with questions, modified lessons when necessary without sacrificing the integrity of the lesson, addressed student difficulties at the appropriate level so students could learn, showed effective use of time management for both teaching content for student learning and classroom administrative tasks, and shared multiple sources of knowledge that were frequently used to continue to improve their skills as a teacher. These characteristics showed depth and breadth of their knowledge of mathematics and how to teach it. Comparison of the teachers' characteristics with literature on teaching and learning geometry showed that they were very knowledgeable in how to teach geometry in ways that helped students to learn with understanding.

Findings compared to literature on teaching and learning geometry

From observations of Mrs. Orchid and Mrs. Lotus, many similarities were observed between their classroom attributes and those shown by teachers of students showing achievement in geometry (Han, 2007; Jones, 2000; Marrades & Gutiérrez, 2000). These attributes were use of technology (Han, 2007; Jones, 2000; Marrades & Gutiérrez, 2000); questioning to get students to justify their answers (Han, 2007; Jones, 2000; Marrades & Gutiérrez, 2000); and employed the process of analyzing, coaching, and re-voicing questions back to students (Martin, et al. 2005).

Similar to Han (2007), Jones (2000), and Marrades and Gutiérrez (2000); Mrs. Orchid and Mrs. Lotus used technology in the classroom, specifically dynamic geometry software (DGS). Both teachers used DGS in at least one of the lessons they shared with me throughout the study. For some activities involving DGS, both Mrs. Orchid and Mrs.

Lotus provided students with figures to use to explore properties instead of creating their own (Marrades & Gutiérrez, 2000). While using DGS, similar to Marrades and Gutiérrez (2000), activities were structured in three phases; create and/or explore, generate and/or verify conjectures, and justify conjectures. For Mrs. Orchid, this was the format for the parallelogram properties activity; and for Mrs. Lotus, this was the points of concurrency in a triangle activity. While using DGS, both teachers encouraged collaboration; Mrs. Orchid's students in pairs, and Mrs. Lotus' students in groups of four.

Throughout group work during activities and student presentations, Mrs. Lotus and Mrs. Orchid questioned students with questions such as "why do you think this is true?" and "can you prove it?" (Han, 2007; Jones, 2000; Marrades & Gutiérrez, 2000). For Mrs. Orchid, throughout the class time, she used these questions to check for understanding. Both teachers chose units with lessons that had engaging activities and open-ended tasks. Students were encouraged to collaborate in groups and justify their answers.

From what I determined regarding the geometry filter, both Mrs. Orchid and Mrs. Lotus showed knowledge of teaching and learning geometry as they planned and enacted appropriate van Hiele Levels (Breyfogle & Lynch, 2010), all phases of learning based on the van Hiele Model (Mistretta, 2000), and fostered geometric habits of mind (Driscoll, 2007). In addition to knowledge of teaching and learning geometry, they both had vast knowledge of teaching and learning mathematics. Through interviews and observations, it was apparent that they had many sources of knowledge. These sources were: teaching experience, coursework, textbook, professional development, and collaboration. When interviewed, both teachers were very secure in their plans except in the area of timing when it involved new curriculum, new textbook material, and changing availability of classroom technology.

A recent paper explicating initial results from developing an instrument to measure mathematical knowledge for teaching high school geometry (MKT-g) looked at MKT-g scores compared to number of years teaching geometry, number of years teaching total, and number of college courses taken in geometry (Herbst & Kosko, 2012). They found that pilot data showed correlations between scores of domains of MKT-g, knowledge of content and teaching, knowledge of content and students, common content knowledge, and specialized content knowledge; and years of experience teaching geometry. Thus, years of teaching experience in geometry seemed to be a good indicator of teachers' knowledge on MKT-g, encompassing all the areas of focus for this study; specialized content knowledge, knowledge of content and teaching, and knowledge of content and students. I would predict from this that both of the teachers I chose for this study, with at least 10 years of teaching geometry in the past 10 years, would do well on MKT-g. Evidence from interviews and classroom observations substantiated this assumption.

Some of these characteristics of these exemplary teachers overlap with Stein and Smith's (2011) five practices for facilitating effective inquiry-oriented classrooms: anticipating how students will approach a task, monitoring students as they work on tasks in class, selecting students' strategies to discuss during class, sequencing order of students' presentations, and connecting ideas in ways so that students can learn. In order to facilitate effective inquiry-oriented classrooms using Stein and Smith's (2011) practices, teachers need the knowledge base to do so. This study showed that the exemplary teachers in this study had the knowledge to facilitate effective inquiry-oriented classrooms.

For every lesson that the teacher has taught before, when asked about misconceptions and planned explanations and questions, teachers referred to reflections from previous experiences teaching the lesson. For example, Mrs. Orchid's Triangle in a Bag activity has been done at least a few times. She stated that when she taught this lesson the first few times, she was learning different unanticipated student responses. Now, she has taught the lesson enough times to where an unanticipated student response is rare. This is in alignment with Stein and Smith's (2011) strategy of anticipation supported by documenting student responses year to year. Some student misconceptions and areas of difficulty were in other areas of math, but both teachers stated how these difficulties lie in a deeper misconception that affects a current topic. Both teachers' plans for sequencing (Stein & Smith, 2011) were heavily based on anticipation (Stein & Smith, 2011).

For both teachers, many responses to students' questions, that were content related, were in the form of a question. Through a series of questions eliciting student responses, eventually, the student answers their own question. For example, a student asked Mrs. Orchid if the order of stating two congruent triangles could be changed. Mrs. Orchid replied "only if what?" The student replied with "if they are both matching", and Mrs. Orchid asked the student to name it differently like she stated. Mrs. Lotus frequently responded to her students in the same manner. This relates into Stein and Smith's (2011) practice of monitoring, which is supported by anticipating student responses beforehand.

Both teachers shared extensive knowledge of anticipated student responses before the execution of the lesson, and they were observed during the execution of the lesson.

Implications for Teaching and Learning

Of the various sources of knowledge that the exemplary teachers in this study shared, and in alignment with Herbst and Kosko's (2012) study; experience teaching geometry is an important source of knowledge for teaching geometry. For the teachers in this study, when there was no prior experience of a new activity, much of geometry TACK knowledge came from experience teaching related content and activities, and other sources. This produces implications for teaching geometry, preparing preservice teachers, and providing professional development for inservice teachers. Aspects of characteristics of exemplary teachers important to implications for teaching and learning are: explaining concepts, asking key questions to guide student learning, modifying lessons, predicting student difficulties, addressing student difficulties, conducting classroom procedures, and utilizing technology.

Teaching Geometry. In order to support relational understanding, teachers teaching geometry need to know explanations of concepts and the potential pitfalls in the explanations related to student prerequisite knowledge and student areas of difficulty or misconceptions. During class, when asked by students to give an explanation, teachers teaching geometry should have a supply of questions to ask the student asking the question or other students. When students give answers or explanations, teachers teaching geometry need to ask students to justify their answers; as ability to justify is not only a good indication of understanding, but it provides students with experience in proving informally (Breyfogle & Lynch, 2010; Driscoll, 2007; Mistretta, 2000).

It is advisable for teachers teaching geometry to have a variety of sources of knowledge including but not limited to teaching experience, coursework, textbook, professional development, and collaboration. Teachers stated that recommendations from multiple sources help in identifying areas of student difficulties or misconceptions, and appropriate modifications to activities without sacrificing cognitive demand (Stein, Engle, Smith, & Hughes, 2008). Once areas of student difficulties or misconceptions are identified, ways to guide students to learn through the difficulties or misconceptions need to be primed prior to execution of the lesson. If the use of technology is necessary in an activity, teachers need to know how to use the technology themselves, be able to help students use it, and facilitate student learning using the technology without taking away from the concepts to be learned (Hughes, 2010). In the event of technological difficulties, teachers need to have back-up plans such as patty paper or compass and straight edge for constructions.

Another aspect of being well prepared involves teachers knowing sequence of phases of the lesson and timing for them very well. This helps with time management when teaching. Both teachers in this study used times when students were working to walk around to students that needed individual help, check students for understanding, or carry out classroom administrative tasks like taking attendance.

Preparing Preservice Teachers. Preservice teachers have limited teaching experience. Preparing them with a good foundation of explanations of concepts, key questions to guide student learning, acceptable modifications to lessons, possible student difficulties and how to address them, effective use of time during class, and use of technology is important. Providing them with meaningful experiences to learn these can include, but is not limited to, exposure to literature in mathematics education that expounds on these, collaboration with experienced exemplary teachers, and guided teaching experience to start their own mental portfolio of geometry TACK based on teaching experience.

Providing Professional Development. Since the previous section explained implications for preparing preservice teachers, this section focuses on providing professional development for inservice teachers. Teachers in this study most often referred to previous teaching experience as a source of knowledge, teachers with different previous teaching experience would benefit from experiences these exemplary teachers have to offer (van Driel & Berry, 2012). Results from this study showed that knowledge of explaining concepts, asking key questions to guide student learning, modifying lessons, addressing student difficulties, conducting classroom procedures, and utilizing technology for activities teachers are familiar with can be shared with other teachers that are less familiar with the activity. In order to continue to improve as teachers, there is a fundamental need to collaborate with other teachers to share ideas and brainstorm solutions for areas of difficulty. There are always areas for improvement and ideas to share.

Summary. Since the most important source of knowledge for geometry TACK for an activity is experience teaching that activity; for those that have not had experience teaching the activity, I think it is important to provide meaningful experience teaching the activity. For both preservice and inservice teachers, I think that a meaningful experience connects geometry TACK for the activity to whatever their knowledge base may be for teaching the activity, and provides support for them to teach the activity to have at least one experience teaching the activity.

Implications for Research

Due to the amount of additional data on Teachers' Applied Content Knowledge (TACK) observed in the executed lesson when compared to the planned lesson, I think it is important to address implications for research on teacher knowledge. Clearly, observations are necessary to see what teachers do in context when studying teacher knowledge. When given the choice between paper and pencil assessment and observing a teacher, due to the student component, observation gives more TACK data. Looking at pre-observation and post-observation interviews, both were equally important when studying TACK planned and TACK executed. Pre-observation interviews gave a good baseline of what the teacher thought of prior to student interactions. Post-interviews where teachers watched parts of their lesson brought to light what teachers "forgot" to tell me about the lesson or did not think was important to share. From interviews with both teachers, there were many remarks made to the importance of student interaction in showing their knowledge that they otherwise did not think to share.

Further Areas of Study

From what I have seen through analysis of data, I saw four immediate areas of study; deeper look at questions and explanations, student component, other teachers at different stages of experience, and other subjects. As seen from both teachers, the students caused increased questions and explanations. Thus this is two pronged: looking more in depth to student involvement in teacher planning and execution of lessons, and types of questions and explanations that were in addition to those planned. A lot of questions and explanations observed during the execution of the lesson not stated prior to the execution of the lesson showed the connecting practice (Stein & Smith, 2011).

Also, it would be interesting to look at teachers at different stages of experience, not necessary number of years teaching, but experience. In my experience with novice teachers, specialized content knowledge based on previous experiences teaching the content is much less than experienced teachers like Mrs. Orchid and Mrs. Lotus. It would be interesting to see what the observable differences are. Also, the TACK connections were observed in relation to high school geometry. My first inclination would be to see other levels of geometry since the geometry filter is already in place. However, other areas of mathematics can be explored in a similar fashion as well, like Algebra and Statistics.

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Appendix

Appendix A

	Planning	Execution
Specialized Content Knowledge	Knowledge of concepts, definitions, and procedures for teaching	Observable knowledge of concepts, definitions, and procedures used in teaching
Knowledge of Content and Teaching	Plan for sequence of information to be presented.	Sequence of information presented.
	Planned time for phases of the lesson.	Time spent on phases of the lesson.
	How to explain the concepts, definitions, and procedures.	Explained concepts, definitions, and procedures.
	Planned key questions to guide student thinking.	Questions asked of students to guide their thinking.
Knowledge of Content and Students	Lesson modifications based on perceived student difficulties.	Did those student difficulties still arise? If so, how was it shown? What did the teacher do in response?
	Other anticipated areas of student difficulties and how to help students overcome.	Actual presented areas of student difficulties and how the teacher helped students overcome.
	Anticipated student levels of prerequisite skills and how the teacher plans accordingly.	Actual student level s of prerequisite skills and how the teacher taught accordingly.

Appendix B

Interview Task (videotaped)

Prior to the interview, the teachers will be asked to write a list of concepts to be covered during the lesson/unit. Teachers will also bring with them any materials that will be used during class, like handouts, pre-determined questions, and homework.

Initial Questions Regarding the List

1. Are these in the order of presentation to the class? If so, why? If not, what would it be, and why did you arrange it that way?
2. How do you plan on teaching this concept? Why?
3. What prior knowledge is needed for this concept? How much of the prior knowledge do your students know? How do you know what they know?
4. Which concepts will be harder than others for students to learn, why? Which concepts will be easier, why? Is the answer the same for all students? Why, why not?
5. How will you know that students understand this concept?
6. What will make you decide to go on to the next concept?

Supplementary Questions if not answered from Initial Questions

1. What are some student misconceptions? How are you prepared for those?
2. What are some analogies or comparisons you have prepared in case students need more explanations about the concept?
3. How do you plan on ending the lesson? Where does the lesson end?
4. What are some key questions you plan on using to guide students' thinking?
5. What are some things that I did not ask about that you would like to share?

Appendix C

Observation protocol

7. How did the teacher explain concepts, definitions, and procedures?

8. What are areas of teacher difficulties in explanations?

9. What was the sequence of information presented?

10. What was the time spent on phases of the lesson?

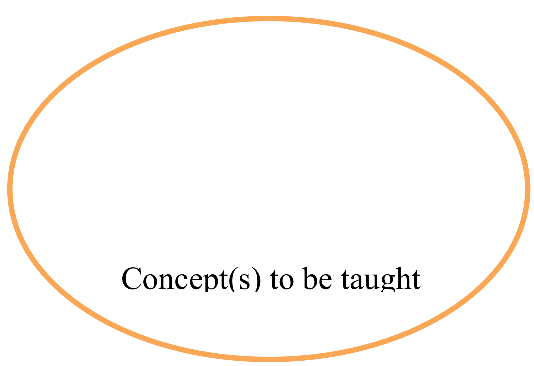
11. What student difficulties arose? If so, how was it shown? What did the teacher do in response?

12. What were actual student levels of prerequisite skills and how did the teacher teach to those levels?

Appendix D

Concept Map

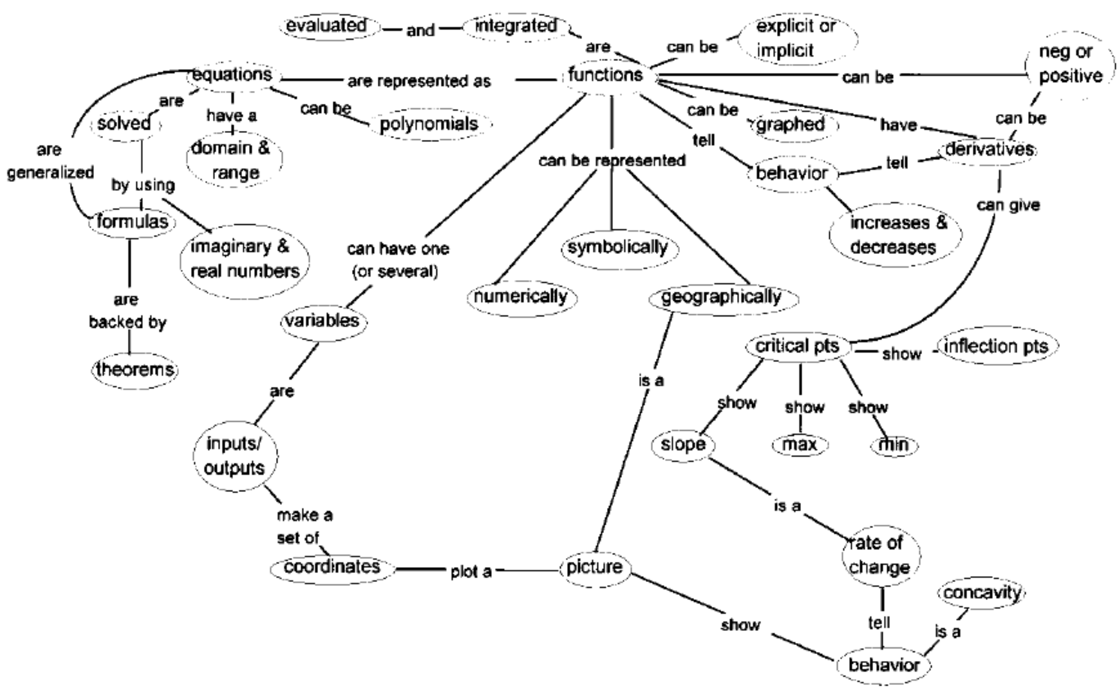
Applications | Related Concepts



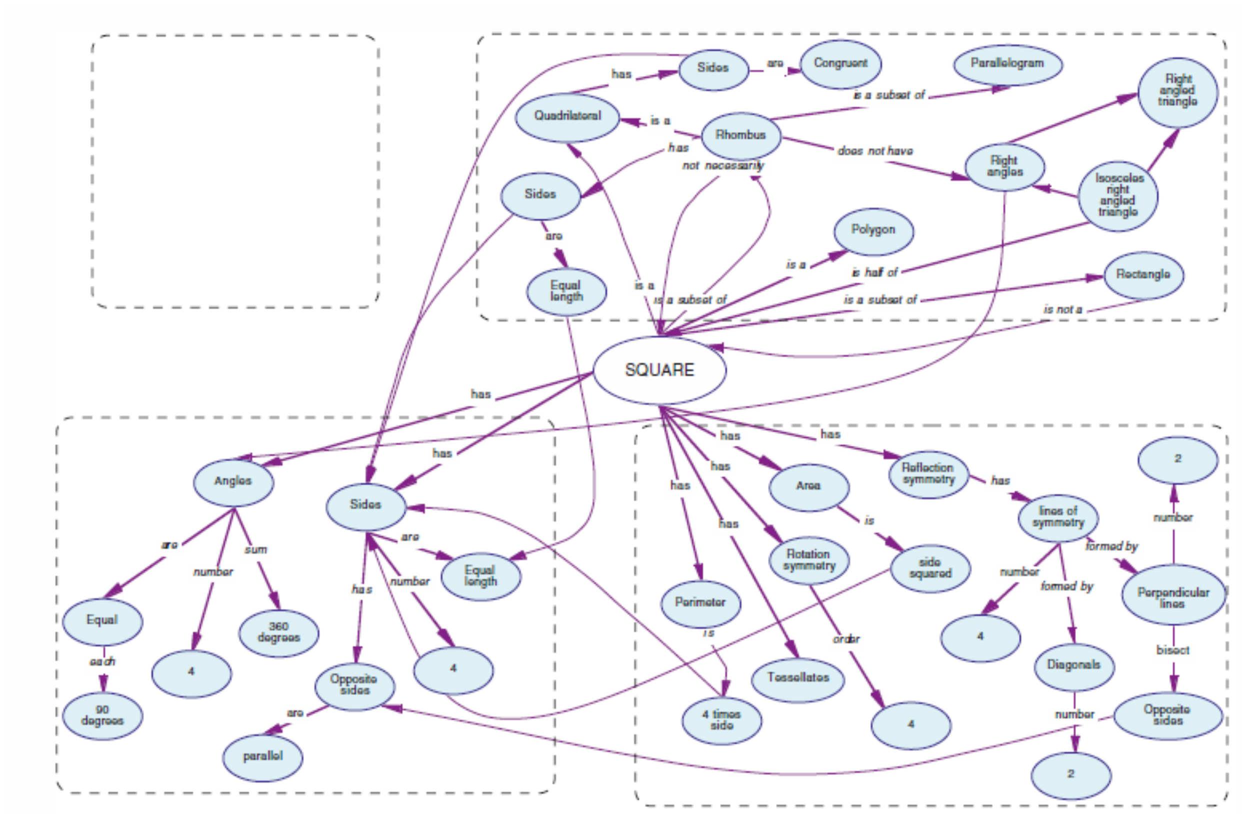
Defining Features | Related Features

Appendix E

Sample Concept Maps



(Williams, 1998, p. 417)



(Lawson & Chinnappan, 2000, p. 211

Appendix F

Meeting	Purpose
Initial Meeting	Overview of study, concept map introduction, set up timeline, forms for conducting study (school based).
Unit Plan and Concept Map Interview	Background information on unit and TACK
Interview for Lesson 1	Collect data for planning
Observation 1	Collect data for enacting
Interview for Reflection of Lesson 1	Triangulate data
Interview for Lesson 2	Collect data for planning
Observation 2	Collect data for enacting
Interview for Reflection of Lesson 2	Triangulate data
Interview for Lesson 3	Collect data for planning
Observation 3	Collect data for enacting
Interview for Reflection of Lesson 3	Triangulate data

Appendix G

Definitions

- Nodes for sequences – examples, problems, definitions, and theorems; i.e. each separate bellwork problem as a node
- Nodes for explain – when a student asks a question, the teacher explains rather than calling on another student, or asks a question in return, then that is a node for explain. If the same explanation is given for the same question, whether or not asked by the same student; both count within one node. If the same explanation is given to another question, it is a different node. Explanations can also be unsolicited, each topic of explanation counting as a node.
- Nodes for questions – There are two situations where Mrs. Lotus poses questions related to content, to engage in student interaction or in response to student interaction. Each question eliciting a specific response from students is considered a node. Multiple questions eliciting the same specific response is also considered one mode. Questions in response to student questions follow the same counting algorithm.
- Nodes for modifications – planned lesson modifications are modifications like the choice of one example over another or a purposeful choice of a problem in order to bring out or avoid certain perceived student difficulties. Each modification counts as one node.
- Nodes for difficulties – each anticipated difficulty that the teacher does not plan on specifically addressing, but acknowledges existence, is considered a node.

Further, all difficulties that arise during the execution of the lesson that was not previously verbalized to me is also considered a node. Same difficulty on multiple students at multiple times still counts as one node.

- Nodes for prerequisite skills – each skill a teacher mentioned that the student should know either in interviews or during class time, counts as a node. Same prerequisite skills mentioned for multiple students at multiple times still counts as one node.
- Phases – overall sections of class where there is an overarching unified goal; namely bellwork, homework review, lesson of the day including notes and activity, and working on homework for the next day.

Appendix H

Coding Guide

This guide sets up the units for analysis for both interview transcripts as well as observation transcripts. Interview transcripts included what the teacher what the teacher wrote while talking, included preprinted material, and had reproductions of what the teacher pointed at while they were talking. For classroom observations, timing for phases of instruction, teacher movement throughout the room, and whom the teacher was addressing was noted.

Unitizing

Definitions used to describe the units of analysis for this study follows those set forth by Garrett (2010).

Term	Definition
Topic	A complete idea, aspect of TACK, or topic of conversation upon which the attention of the speaker or speakers is focused.
Unit	A word, sentence, paragraph or several consecutive sentences or paragraphs focusing on the same topic (p. 264)

Examples:

Separation by question in interview

INTERVIEWER: so what are they supposed to get at by the end of today?

MRS. ORCHID: by the end of the today, the goal is that they will have had enough opportunities to try and figure out which questions [Me: um hm] are most appropriate are most helpful for duplicating the triangle [Me: um hm] um and at least some of the groups will have narrowed it down to three questions that are most helpful. Um so that's that's just the goal for today, just to try to get that [Me: um hm] and we're not putting a name on any of the, you know, sets of three questions or anything. That will happen tomorrow

Separation

INTERVIEWER: great, so you'll know then that students understand in that way when

MRS. ORCHID: if their groups are able to come up with the right triangle questions, then I'll that groups understand, I won't know if individual students necessarily understand, um but hopefully, I don't know, as I'm encouraging them to collaborate together, everybody will be participating hopefully. [Me: okay]

Separation within an answer in interview

INTERVIEWER: Um, so we're starting, um can you explain to me your list of what you're going to cover?

MRS. LOTUS: okay, I sat down and I was just going to look at I guess background. [me: um hm] I've changed books, I guess we changed books [me: um hm] plus we changed from our typical course of study to the common core course of study [me: um hm] and I um ah all that entails. The new book is very different from the more traditional book that I've used forever. [me: okay] so I sat down and I know I'm going to talk about triangles and I think that's what the next month, two to three months, two months [me: um hm] I'll be talking about triangles so I was looking at all right what do I want what do I need to tell them about triangles, what do they need to know and then looking at what do I do what have I done that's in the book and where is it. So I'm, the list is more "okay, now I need to talk about this this this" this is what's in the book [me: um hm] um, the new book so that's my list.

Within Mrs. Lotus' answer to my question on explaining what she planned on covering, she first talked about changing of textbooks and standards required of students. Below shows how I separated this portion of the transcript.

Section 1

INTERVIEWER: Um, so we're starting, um can you explain to me your list of what you're going to cover?

MRS. LOTUS: okay, I sat down and I was just going to look at I guess background. [me: um hm]

Section 2

I've changed books, I guess we changed books [me: um hm] plus we changed from our typical course of study to the common core course of study [me: um hm] and I um ah all that entails. The new book is very different from the more traditional book that I've used forever. [me: okay]

Section 3

so I sat down and I know I'm going to talk about triangles and I think that's what the next month, two to three months, two months [me: um hm] I'll be talking about triangles so I was looking at all right what do I want what do I need to tell them about triangles, what do they need to know and then looking at what do I do what have I done that's in the book and where is it. So I'm, the list is more "okay, now I need to talk about this this this" this is what's in the book [me: um hm] um, the new book so that's my list.

Separation of activity during executed lesson (Separation of Phases)

Mrs. Lotus: … Now its not something you know because its something somebody learned. Okay, any questions before we go on? Okay, so that's where we left yesterday. This is how far we've gotten. So today we're going to pick up with the corollary to a theorem. Now a corollary is another statement that is related to a theorem that we already learned that we can use that theorem to prove.

Mrs. Lotus was reviewing what students had discussed the previous day, and then she went on to the notes for the lesson of the day. Below shows the separation:

(cont.) Review from the previous day

Mrs. Lotus: … Now its not something you know because its something somebody learned. Okay, any questions before we go on? Okay, so that's where we left yesterday. This is how far we've gotten.

Notes for today

So today we're going to pick up with the corollary to a theorem. Now a corollary is another statement that is related to a theorem that we already learned that we can use that theorem to prove…

Separation within an activity (Separation within Phases)

Mrs. Orchid: any other groups need more drawing paper? Before we start?
Student: we do
{teacher hands drawing paper}
Mrs. Orchid: anybody else?
{no one raises their hand}

Mrs. Orchid: Okay, round three begins right now. What is our goal for the number of questions?
Several students: three
Mrs. Orchid: three

Mrs. Orchid had just summarized round two within the Triangle in a Bag Activity. The natural separation is between the summary of round two and the beginning of round three of the activity.

(cont.) Summarized round two

Mrs. Orchid: any other groups need more drawing paper? Before we start?

Student: we do

{teacher hands drawing paper}

Mrs. Orchid: anybody else?

{no one raises their hand}

Beginning of round three for students to work together

Mrs. Orchid: Okay, round three begins right now. What is our goal for the number of questions?

Several students: three

Mrs. Orchid: three …

Separation of topics within classroom transcription

Mrs. Orchid: any other groups need more drawing paper? Before we start?
Student: we do
{teacher hands drawing paper}
Mrs. Orchid: anybody else?
{no one raises their hand}
Mrs. Orchid: Okay, round three begins right now. What is our goal for the number of questions?
Several students: three
Mrs. Orchid: three

Using the same transcript as above, there is separation within an activity by the nature of the activity, but also by what was being said.

(cont.) Summarized round two – students needed materials

Mrs. Orchid: any other groups need more drawing paper? Before we start?

Student: we do

{teacher hands drawing paper}

Mrs. Orchid: anybody else?

{no one raises their hand}

Beginning of round three for students to work together – directions to begin

Mrs. Orchid: Okay, round three begins right now.

Question asked to help start students on round three

Mrs. Orchid: What is our goal for the number of questions?

Several students: three

Mrs. Orchid: three …

Codes

A Priori Codes and their Descriptions

Specialized content knowledge

Knowledge of concepts, definitions, and procedures for teaching. Specifically:

- presenting mathematical ideas,
- responding to students' "why" questions,
- finding an example to make a specific mathematical point,
- recognizing what is involved in using a particular representation,
- linking representations to underlying ideas and to other representations,
- connecting a topic being taught to topics from prior or future years,
- explaining mathematical goals and purposes to parents,
- appraising and adapting the mathematical content of textbooks,
- modifying tasks to be either easier or harder,
- evaluating the plausibility of students' claims (often quickly),
- giving or evaluating mathematical explanations,
- choosing and developing useable definitions,
- using mathematical notation and language and critiquing its use,
- asking productive mathematical questions,
- selecting representations for particular purposes,
- inspecting equivalencies (Ball, Thames, & Phelps, 2008, p. 10).

Knowledge of Content and Teaching

- Sequence of information
- Time for phases of the lesson
- Explanations of concepts, definitions, and procedures
- Key questions to guide students' thinking

Knowledge of Content and Students

- Modifications based on student difficulties

- Anticipated areas of student difficulties and how to help students overcome
- Student levels of prerequisite skills and how the teacher plans accordingly

Below is a table of the *a priori* codes from literature:

	Planning	Execution
Specialized Content Knowledge	SCK-P Knowledge of concepts, definitions, and procedures for teaching	SCK-E Observable knowledge of concepts, definitions, and procedures used in teaching
Knowledge of Content and Teaching	KCTseq-P Plan for sequence of information to be presented.	KCTseq-E Sequence of information presented.
	KCTtime-P Planned time for phases of the lesson.	KCTtime-E Time spent on phases of the lesson.
	KCTexplain-P How to explain the concepts, definitions, and procedures.	KCTexplain-E Explained concepts, definitions, and procedures.
	KCTquestions-P Planned key questions to guide student thinking.	KCTquestions-E Questions asked of students to guide their thinking.
Knowledge of Content and Students	KCSmod-P Lesson modifications based on perceived student difficulties.	KCSmod-E Did those student difficulties still arise? If so, how was it shown? What did the teacher do in response?
	KCSdifficulties-P Other anticipated areas of student difficulties and how to help students overcome.	KCSdifficulties-E Actual presented areas of student difficulties and how the teacher helped students overcome.
	KCSprerequisite-P Anticipated student levels of prerequisite skills and how the teacher plans accordingly.	KCSprerequisite-E Actual student level s of prerequisite skills and how the teacher taught accordingly.

Emergent Codes

Emergent codes, topics and subjects of discussion that showed up in interviews and classroom observations are listed below.

> (AD) Activity directions – discussions of what students are about to do and how to do it. For example, directions for an activity or directions for students to take out homework to be checked.
>
> (B) Birdwalking – teachers engaging in discussion with students about something not related to the lesson or any of the above codes, e.g. asking students how their day was while checking homework and the student talking to the teacher about what their family did that weekend. This was not an apparent attempt at behavior modification (e.g. getting student to sit up if they had their head down).
>
> (CA) Classroom administration – discussions pertaining to taking attendance, intercom interruptions, bells sounding for other classes, students asking to use school facilities outside the classroom, discipline, and other classroom procedures.
>
> (Cu) Curriculum – discussion pertaining to opinions and ideas of textbook being used or standards for instruction.
>
> (RTI) Researcher Technical Interactions – discussion regarding administrative aspects of the study like camera angle and scheduling.
>
> (V) Verification – statements made to students from the teacher letting them know that what they are doing is either correct or incorrect without questioning or explaining. For example: during groupwork, teacher telling students that they are doing a good job and on the right track.
>
> (O) Other – since everything is coded, this is where a segment that does not fit anywhere else is coded. Since the focus is on teacher knowledge, some statements that students make that the teacher did not respond to verbally or nonverbally became categorized as this code.

Appendix I

Van Hiele Levels Abridged from (Breyfogle & Lynch, 2010)

Level	Name	Description
	Visualization	See geometric shapes as a whole; do not focus on their particular attributes.
1	Analysis	Recognize that each shape has different properties; identify the shape by that property.
2	Informal Deduction	See the interrelationships between figures.
3	Formal Deduction	Construct proofs rather than just memorize them; see the possibility of developing a proof in more than one way.
4	Rigor	Learn that geometry needs to be understood in the abstract; see the "construction" of geometric systems.

Phases of Learning from the van Hiele Model (Mistretta, 2000, p. 367)

Phase	Description
Information	Discussions are held where the teacher learns of the students' prior knowledge and experience with the subject matter at hand.
Directed Orientation	The teacher provides activities that allow students to become more acquainted with the material being taught
Explication	A transition between reliance on the teacher and students' self-reliance is made.
Free Orientation	The teacher is attentive to the inventive ability of the students. Tasks that can be approached in numerous ways are presented to the students.

Integration	The students summarize what was learned during the lesson.

Geometric Habits of Mind and Their Indicators (Driscoll, 2007, pp.12-15)

Geometric Habit of Mind	Indicators
Reasoning with Relationships	Basic: Identification of figures presented in a problem and correct enumeration of their properties Advanced: Relating multiple figures in a problem through proportional reasoning and reasoning through symmetry
Generalizing Geometric Ideas	Basic: Uses one problem situation to generate another, or when the solver intuits that he or she hasn't found all the solutions Advanced: Generate all solutions and make a convincing argument as to why there are no more; or wondering what happens if a problem's context is changed
Investigating Invariants	Basic: Decides to try a transformation of figures in a problem without being prompted, and considers what has changed and what has not changed Advanced: Consider extreme cases for what is being asked by a problem

Balancing Exploration and Reflection	Basic: Drawing, playing, and/or exploring with occasional (though maybe not be consistent) stock-taking Advanced: approaching a problem by imagining what a final solution would look like, then reasoning backward; or making what Herbst (2006) calls “reasoned conjectures” about solutions with strategies for testing the conjectures

Appendix J

Mrs. Orchid Unit 1 Observation 1 KCT

	Observed as Planned	Additional Observed
Sequence	Reviewed homework with presentations and answering questions, review test, and the two rounds for Triangle in a Bag.	None
Time	Everything listed in the sequence was covered by the end of class time.	“Homework and test review may have taken a bit longer than expected, but discussion for Triangle in a Bag was not rushed.”
Explanations	Triangle in a Bag • Students made the connection that they are drawing congruent triangles • Used example of constructing trusses • To help determine number of questions, reminded students that if a question yielded more than one answer, then it counted as more than one question.	• Explained some congruency notation practices during homework presentations • Explained how to do a homework problem and several test problems • Explained several mistakes students made on the test • Explanations as to why certain student written questions cannot be answered • Explanations for drawing • Explanations getting students to determine usefulness of questions and to minimize amount of information needed • Explanations for the necessity of labeling vertices
Questions	Triangle in a Bag - Prompted students to ask useful questions by asking “how is this question useful?”	Homework and Test Review: • “Can you elaborate on that?” • “Please explain what he/she said.” • “Why do you think you’re answer is right/correct?” Triangle in a Bag: • Questions to determine usefulness of information asked • Questions to help minimize information asked

Mrs. Orchid Unit 1 Observation 1 KCS

Observed as Planned	Additional Observed

Modifications	None	None
Student Difficulties/ Misconceptions	Student areas of difficulties for Triangle in a Bag: asking useful questions, getting to the minimum number of questions, recognizing the connection to congruent triangles, misplaced vertices, and construction	None
Prerequisite Knowledge	Student prerequisites knowledge for Triangle in a Bag: how to use a ruler and a protractor to measure angles and side lengths, and create angle measures and side lengths of specified measurements	Triangle inequality

Mrs. Orchid Unit 1 Observation 1 SCK

Mathematical Tasks of Teaching	Planned and Executed
Presenting mathematical ideas	"Triangle in a Bag" ideas were observed as planned. Ideas from questions reviewed for homework and test were observed in class.
Responding to students' "why" questions	All planned responses to students' "why" questions were observed for "Triangle in a Bag". Additional responses to students' "why" questions were observed for all phases of the lesson.
Finding an example to make a specific mathematical point	For extra credit, Mrs. Orchid asked students to find examples of congruent triangles other than construction, with the goal in mind to use these as examples. Trusses example was in the textbook.
Selecting representations for particular purposes	Uniform congruence marks minimizes confusion.
Recognizing what is involved in using a particular representation	Mrs. Orchid chose for students to show triangle congruence marks on the diagrams in the way that the book used it or Geometer's Sketchpad to minimize confusion.
Linking representations to underlying ideas and to other representations	Mrs. Orchid emphasized to students that corresponding parts had the same congruence marks, and vice versa.
Connecting a topic being taught to topics from prior or future years	Students learned how to mark corresponding parts of congruent triangles congruent in the lesson prior. This lesson develops the idea of triangle congruence shortcuts. Mrs. Orchid planned on using triangle congruence to prove parallelogram properties.
Explaining mathematical goals and purposes to parents	None observed
Appraising and adapting the mathematical content of textbooks	For this lesson, other than the homework review from the previous day, the textbook was not a part of the lesson or homework for this observation day.
Modifying tasks to be either easier or harder	None observed
Evaluating the plausibility of students' claims (often quickly)	During the observation, she evaluated plausibility of students' claims during Triangle in a Bag rounds and when students answered questions.
Giving or evaluating mathematical explanations	Giving mathematical explanations was shown both in planned and executed phases of the lesson. Evaluating mathematical explanations was observed in all phases of the executed lesson.

Choosing and developing useable definitions	None observed
Using mathematical notation and language and critiquing its use	Throughout the observation, Mrs. Orchid corrected terminology when students were not using the correct term, modeled correct terminology, and emphasized congruency notation practices.
Asking productive mathematical questions	Observed questions, including planned, incited discussion and led to students providing correct answers.
Inspecting equivalencies	For Triangle in a Bag rounds, Mrs. Orchid determined how different questions could be asking for the same information.

Mrs. Orchid Unit 1 Observation 2 KCT

KCT	Observed as Planned	Additional Observed
Sequence	Finish the rounds for Triangle in a Bag, group presentations, and classwork questions	None
Time	The planned sequence was covered by the end of class time.	Group discussion went longer than expected due to new technology, and classwork discussion went shorter than planned
Explanations	Triangle in a Bag • Explanations from rounds for yesterday Classwork • Group presentations on how they were able to create congruent triangles – different groups present different shortcuts • Correct vocabulary will be coached during group presentations • Highlight parts students knew when presenting • Explanations for congruence shortcut theorem specific to theorem	• Explanations as to why the question cannot be answered specific to student questions • Explanations for drawing • Explanations getting students to determine usefulness of questions and to minimize amount of information needed • Explanations for the necessity of labeling vertices • Explained how a set of questions can be different • Re-voicing and elaborating student explanations for triangle congruence shortcut theorems
Questions	Triangle in a Bag • Same questions as the day prior • Classwork questions	• Questions to determine usefulness of information asked • Questions to help minimize information asked

Mrs. Orchid Unit 1 Observation 2 KCS

KCS	Observed as Planned	Additional Observations
Modifications	No modifications	No modifications

Student Difficulties/ Misconceptions	Student areas of difficulties for Triangle in a Bag: asking useful questions, getting to the minimum number of questions, recognizing the connection to congruent triangles, misplaced vertices, construction, triangle JKL, and recognizing included side or angle.	Difficulty for recognizing included side or angle were not observed.
Prerequisite Knowledge	Prerequisite knowledge for Triangle in a Bag: how to use a ruler and a protractor to measure angles and side lengths, and create angle measures and side lengths of specified measurements	Triangle inequality

Mrs. Orchid Unit 1 Observation 2 SCK

Mathematical Tasks of Teaching	Planned and Executed
Presenting mathematical ideas	"Triangle in a Bag" ideas were observed as planned.
Responding to students' "why" questions	During rounds in Triangle in a Bag, Mrs. Orchid responded with explanations and questions.
Finding an example to make a specific mathematical point	Examples specific to situation.
Selecting representations for particular purposes	Mrs. Orchid chose for students organize the triangle congruence shortcuts into a graphic organizer. It helps students see necessary information on specific triangle congruence shortcut easily.
Recognizing what is involved in using a particular representation	Mrs. Orchid created a blank table for students to fill out to organize the triangle congruence shortcuts. The graphic organizer had three columns to organize information for the triangle congruence shortcuts: shortcut name, picture, and written description. In each row was a different triangle congruence shortcut.
Linking representations to underlying ideas and to other representations	For shortcut name, Mrs. Orchid also asked students to write the abbreviation. When discussing the picture and writing the description, Mrs. Orchid introduced the term "included" where pertinent.
Connecting a topic being taught to topics from prior or future years	Students learned how to mark corresponding parts of congruent triangles congruent in the lesson prior. This lesson develops the idea of triangle congruence shortcuts. Mrs. Orchid planned on using triangle congruence to prove parallelogram properties.
Explaining mathematical goals and purposes to parents	None observed
Appraising and adapting the mathematical content of textbooks	None observed
Modifying tasks to be either easier or harder	None observed
Evaluating the plausibility of students' claims (often quickly)	During the observation, she evaluated plausibility of students' claims during Triangle in a Bag rounds, statements during presentations, and in general when students answered questions.

Giving or evaluating mathematical explanations	Giving mathematical explanations was shown both in planned and executed phases of the lesson.
Choosing and developing useable definitions	Definitions for included angle, included side, and triangle congruence shortcuts: ASA, SAS, AAS, SSS, and HL.
Using mathematical notation and language and critiquing its use	Throughout the observation, Mrs. Orchid corrected terminology when students were not using the correct term, modeled correct terminology, and emphasized congruency notation practices.
Asking productive mathematical questions	Observed questions, including planned, incited discussion and led to students providing correct answers.
Inspecting equivalencies	For Triangle in a Bag rounds, Mrs. Orchid determined how different questions could be asking for the same information.

Mrs. Orchid Unit 1 Observation 3 KCT

KCT	Observed as Planned	Additional Observations
Sequence	Sequence was for students to work on the bellringer, homework check, bellringer review, review homework, and then classwork problem 1.	None
Time	No planned timing other than homework will take too long	There was not enough time for classwork problems 2-4.
Explanations	Bellringer • Directions for the activity • Take two measurements and see how many triangles can be created • It's not included in the shortcut theorems because it's called the ambiguous case Homework Review • Prepared to answer any questions students may have • Use different student heights as example for why side-side does not work Classwork • Using CPCTC to see the importance of using congruent triangles • Courier – angle-side-angle • Runway problem – side-angle-side with alternate interior angles forcing the lines to be parallel • Construction – trusses, side-side-side Construction – connecting first floor to second floor, perpendicular	Bellringer • How to draw and measure • Theorems have to work every time Homework Review • How to mark triangles to see which parts are congruent • Re-voiced student explanations to answers to homework problems • CPCTC Classwork • How to mark parts congruent to see whether or not a side or angle is "included" • Re-voiced student explanations

Questions	Bellringer • Are you sure there's not another triangle you can draw? • Does angle B have to be that big? Can we make it smaller? Can we make it larger? What will happen if you do? Homework Review • What is your proof? What evidence do you have? Classwork • How do you know they are congruent triangles? • Do you already know the hypotenuse is congruent? • What other information can we get now about the triangles?	Bellringer • What the problem is asking • Drawing • Whether or not this is a theorem Homework Review • What is your proof? What evidence do you have? • Questions specific to homework question Classwork • How do you know they are congruent triangles? • Do you already know the hypotenuse is congruent? • What other information can we get now about the triangles?

Mrs. Orchid Unit 1 Observation 3 KCS

KCS	Observed as Planned	Additional Observations
Modifications	No modifications	No modifications
Student Difficulties/ Misconceptions	Planned homework misconceptions and misconception of what an included angle and what an included side is.	All planned difficulties/misconceptions were observed except the homework problem where both triangles have two sides and an angle marked congruent, but one has an included angle and the other does not.
Prerequisite Knowledge	Planned prerequisites for Triangle in a Bag: how to use a ruler and a protractor to measure angles and side lengths, and create angle measures and side lengths of specified measurements	All planned prerequisites were observed.

Mrs. Orchid Unit 1 Observation 3 SCK

Mathematical Tasks of Teaching	Planned and Executed
Presenting mathematical ideas	Bellringer, homework review, and classwork ideas were presented as planned. Additional homework problems were presented and classwork ideas 2-4 were not presented during the observation.
Responding to students' "why" questions	Students asked "why" questions during homework review. Mrs. Orchid responded by having other students explain, and coached correct vocabulary or ideas where necessary.

Finding an example to make a specific mathematical point	Examples specific to situation.
Selecting representations for particular purposes	For classwork problems, some problems did not have a pre-drawn figure so students needed to create their own, and problems that had figures were not drawn to scale.
Recognizing what is involved in using a particular representation	Mrs. Orchid chose problems that did not have figures so that students can practice creating their own figures from text. Other figures that were not drawn to scale were used because students needed to rely on information labeled in the figure. These have the potential for student difficulties and misconceptions.
Linking representations to underlying ideas and to other representations	Understanding how to draw figures from text and use pre-drawn figures for classwork problems involved congruence marking from the beginning of the unit. This also links to future problems where students need to draw and interpret figures.
Connecting a topic being taught to topics from prior or future years	From Triangle in a Bag, students learned triangle congruence shortcut theorems. Students used triangle congruence shortcut theorems in classwork problems. Classwork problems helped students see triangle congruence in different contexts. In this unit, students used triangle congruence to prove parallelogram properties.
Explaining mathematical goals and purposes to parents	Not observed
Appraising and adapting the mathematical content of textbooks	Mrs. Orchid used four problems out of the textbook for classwork. However, she changed one of the problems not to require students to create a flow chart proof.
Modifying tasks to be either easier or harder	None observed
Evaluating the plausibility of students' claims (often quickly)	During the observation, she evaluated plausibility of students' claims during homework review, classwork, and when students answered questions.
Giving or evaluating mathematical explanations	Planned mathematical explanations were given during the observation. Additional explanations were also observed. Mrs. Orchid evaluated student mathematical explanations throughout the observation.
Choosing and developing useable definitions	None observed
Using mathematical notation and language and critiquing its use	Throughout the observation, Mrs. Orchid corrected terminology when students were not using the correct term, modeled correct terminology, and emphasized congruency notation practices.
Asking productive mathematical questions	Observed questions, including planned, incited discussion and led to students providing correct answers.
Inspecting equivalencies	Mrs. Orchid determined how different congruence statements could be equivalent.

Mrs. Orchid Unit 1 Observation 4 KCT

KCT	Observed as Planned	Additional Observations
Sequence	Work on the bellringer, go over bellringer, define parallelogram, construct a parallelogram using GSP, properties of parallelogram activity, summarize parallelogram activity, and start on homework.	None
Time	Sequence was completed by the end of class	Teacher was accidentally locked out of room, so transition time was 10 minutes longer than expected.

Explanations	Bellringer • Want them to focus on angle properties inside two sets of intersecting parallel lines Defining "Parallelogram" • Talk about what a good definition of a parallelogram is • Talk about what "all sides are parallel" could mean • If students say "square" or "rectangle", talk about parallelogram classifications Constructing a Parallelogram • List step by step Parallelogram Activity • Compare and measure • Remind students of definitions of parts of a parallelogram Summary • Have students explain answers to questions from activity and fill out graphic organizer	Bellringer: • Directions on the bellringer and where the missing information is for students to solve • Definition of transversal • Alternate interior and alternate exterior angle relationships when parallel lines are cut by a transversal Defining "Parallelogram": • Symbolic notation for parallel • Defined parts of a parallelogram Constructing a Parallelogram: Difference between a sketch or drawing and a construction Parallelogram Activity: • Directions for constructing • Difference between "distance" and "length" • Explanations specific to student questions and answers Summary: How to mark congruence of segments
Questions	Bellringer • What are you supposed to be looking at? Defining "Parallelogram" • What is a parallelogram? • What is a good definition? Constructing a Parallelogram • How would you do this? Parallelogram Activity • Are you sure you are looking at the correct items? Summary Ask students to elaborate on explanations	Bellringer • Questions to get students to use the sum of the measures of the interior angles of a quadrilateral to solve for an angle measure. • Questions that asked students to recognize the relationship of consecutive interior angles. Parallelogram Activity: • What do you call that point in the middle? • What happens when you cut something in the middle? What's the math word for that?

Mrs. Orchid Unit 1 Observation 4 KCS

KCS	Observed as Planned	Additional Observations
Modification	Have students learn how to construct before going to the lab instead of learning to construct by trial and error on their own.	None

Student Difficulties/ Misconceptions	Bellringer • There should be none Defining "Parallelogram" • Thinking that a squares or rectangles are the only parallelograms • Thinking that four parallel lines, or two sets of parallel lines is sufficient information for the definition of a parallelogram • Sufficiency for the definition of a parallelogram Constructing a Parallelogram • Students may not be able to construct a parallelogram in GSP, they may just draw it. Parallelogram Activity • Selecting not enough or too much items on GSP • Not knowing definitions of parts of a parallelogram • No knowing which angles are "opposite" or "consecutive" Summary • Explanations for relationships between two diagonals	Bellringer • Definition of transversal • Alternate interior and alternate exterior angle relationships when parallel lines are cut by a transversal • Sum of the measures of the interior angles of a quadrilateral
Prerequisite Knowledge	Parallel lines, consecutive interior angles theorem, knowledge from bellringer (calculating missing information on angles from given information in a diagram), and a working definition of parallelograms.	Definition of a transversal, alternate interior and alternate exterior angle relationships when parallel lines are cut by a transversal

Mrs. Orchid Unit 1 Observation 4 SCK

Mathematical Tasks of Teaching	Planned and Executed
Presenting mathematical ideas	All planned ideas in bellringer and parallelogram activity were observed. Additional ideas were also observed.

Responding to students' "why" questions	All planned responses to students' "why" questions were observed. Additional responses to students' "why" questions were also observed.
Finding an example to make a specific mathematical point	In asking students to define what a parallelogram is, Mrs. Orchid had two counterexamples ready: all sides parallel – four parallel lines to each other, and square or rectangle – talk about quadrilateral classifications.
Selecting representations for particular purposes	Mrs. Orchid chose for students to use GSP constructed parallelograms to discover properties.
Recognizing what is involved in using a particular representation	Using GSP created parallelograms required students to have access to a computer and knowledge to construct a parallelogram.
Linking representations to underlying ideas and to other representations	Constructing the parallelogram was based on a working definition of "parallelogram" Mrs. Orchid established in class. Students were familiar with GSP. Students will be using GSP in future lessons.
Connecting a topic being taught to topics from prior or future years	Some students many know some properties of parallelograms from previous classes. This lesson was for students to discover properties of parallelograms. By the end of the unit, students used triangle congruence to prove properties of parallelograms.
Explaining mathematical goals and purposes to parents	None observed
Appraising and adapting the mathematical content of textbooks	None observed
Modifying tasks to be either easier or harder	None observed
Evaluating the plausibility of students' claims (often quickly)	Mrs. Orchid evaluated plausibility of students' claims during each phase of the lesson.
Giving or evaluating mathematical explanations	Giving and evaluating mathematical explanations was shown both in planned and executed phases of the lesson.
Choosing and developing useable definitions	Defined parallelogram in order to construct it on GSP.
Using mathematical notation and language and critiquing its use	Throughout the observation, Mrs. Orchid corrected terminology when students were not using the correct term, modeled correct terminology, and emphasized congruency notation practices.
Asking productive mathematical questions	Observed questions, including planned, incited discussion and led to students providing correct answers.
Inspecting equivalencies	Throughout the lesson, Mrs. Orchid determined how different student questions or statements could be equivalent.

Mrs. Orchid Unit 2 Observation 1 KCT

KCT	Planned as Observed	Additional Observations

Sequence	Planned sequence was for students to work on the bellringer, homework check, bellringer review, review homework, and then develop the area formula in the Pigpen activity.	How to use the calculator during homework review
Time	No planned timing other than bellwork and homework will take too long	Bellwork and homework took too long but Mrs. Orchid planned for that.
Explanations	Bellringer • Dividing by two is the same as multiplying by 0.5. • Base of a triangle does not have to be on the bottom. • If it is not marked to be the middle in the diagram, do not assume it is. • How to use the area of a triangle formula • How to find the height of a triangle Homework Review • Prepared to answer any questions students may have • Explanations for why a trigonometry ratio is the appropriate one to use for a set of information. Pigpen for Monica • Explain the major problem for the activity – finding the angle to place the fences so Monica will have the most area in her pigpen. • Notice the pattern to develop a formula. • Explanations to help students verbalize the formula • Prompt students to summarize what they learned	Bellringer • Directions for the activity • Gave example of measuring a person's height, it needs to be perpendicular to the ground • Rounding for an answer that is not an integer. Homework Review • How to use the calculator for trigonometry functions • Answers specific to student questions Pigpen for Monica • Directions for the activity • Triangle inequality • What it means for a pen to be closed
Questions	Bellringer • Are you sure you're using the correct trigonometry ratio? How do you know? • What's the reason for that? • What does this dotted segment (height outside the triangle) mean? Homework Review • Are you sure you're using the correct trigonometry ratio? • Why is that the correct answer? Pigpen for Monica • What will the pigpen look like? • What do you need to find the area? • Was it a good idea to make the angle smaller or bigger? • Where would the height be? Can you draw a segment that would represent the height? Where can we put the height so we can calculate it? • How would you solve for h? • Do you notice a pattern? What repeats?	Bellringer • What units are we working with? • Clarifying questions Homework Review • Why do I not want to spend time going over (specific homework problem numbers) • Questions regarding set up of calculators Pigpen for Monica • What is the problem asking you to do? • Where to put the angle and what kind of angle it would be • Is there another place to put the height? • Identifying triangle parts

Mrs. Orchid Unit 2 Observation 1 KCS

KCS	Observed as Planned	Additional Observations
Modifications	No modifications	No modifications
Student Difficulties/ Misconceptions	Bellringer • Base of a triangle has to be on bottom • If a side looks like it is bisected in the diagram, it is bisected. • Height cannot be calculated outside an obtuse triangle. • Use the wrong trigonometry ratio to find the length of a missing side. Homework Review • Determining the correct trigonometry ratio • Determining in relation to the angle, which side is opposite and which sides is adjacent. Pigpen for Monica • Drawing a height that can be determined with the information given • Verbalizing the formula for area using Sine	Difficulty drawing the shape of the pigpen
Prerequisite Knowledge	Planned prerequisite for bellringer was how to plug numbers into an equation and solve. For all of the activities of the day is the difference between sine, cosine, and tangent ratios.	None

Mrs. Orchid Unit 2 Observation 1 SCK

Mathematical Tasks of Teaching	Planned and Executed
Presenting mathematical ideas	All planned ideas in bellringer, homework review, and area formula for Pigpen for Monica was observed. Additional ideas were also observed.
Responding to students' "why" questions	All planned responses to students' "why" questions were observed.
Finding an example to make a specific mathematical point	No examples given in pre-observation interview. Measuring a person's height needing to be perpendicular to the ground was observed during executed lesson.
Selecting representations for particular purposes	For the bellringer, Mrs. Orchid chose to have the triangles draw out to feature different orientations of triangles.
Recognizing what is involved in using a particular representation	Mrs. Orchid used triangles in different orientations to incite and address misconceptions.
Linking representations to underlying ideas and to other representations	In Pigpen activity, Mrs. Orchid asked students to continue to draw triangles to give a picture to the problem. She stated that being able to draw a useful diagram is necessary to solving word problems.
Connecting a topic being taught to topics from prior or future years	The Pigpen activity took what students knew about area formula for a triangle and improved upon it to use the sine ratio. This linked to Law of Sines.
Explaining mathematical goals and purposes to parents	None observed
Appraising and adapting the mathematical content of textbooks	None observed
Modifying tasks to be either easier or harder	None observed
Evaluating the plausibility of students' claims (often quickly)	Mrs. Orchid evaluated plausibility of students' claims during each phase of the lesson.
Giving or evaluating mathematical explanations	Giving and evaluating mathematical explanations was shown both in planned and executed phases of the lesson.
Choosing and developing useable definitions	Defined parallelogram in order to construct it on GSP.
Using mathematical notation and language and critiquing its use	Throughout the observation, Mrs. Orchid corrected terminology when students were not using the correct term, and modeled correct terminology.
Asking productive mathematical questions	Observed questions, including planned, incited discussion and led to students providing correct answers.
Inspecting equivalencies	Throughout the lesson, Mrs. Orchid determined how different student questions or statements could be equivalent.

Mrs. Orchid Unit 2 Observation 2 KCT

KCT	Observed as Planned	Additional Observations
Sequence	Planned sequence was for students to review homework problems using area formula ½ base x aSinC, left over sine and cosine ratio homework; finish the Pigpen activity, and then start on the Law of Sines activity.	Only homework review and part of the Pigpen activity was observed.
Time	At least finish sequence up to Pigpen activity.	Homework took too long. Did not finish the Pigpen activity and did not start Law of Sines.
Explanations	Pigpen for Monica Part 2 • Directions of what the problem is asking • Determine if the formula developed in part 1 always works • It may appear to students that the formula does not work for obtuse angles • Use the formula for 120° and see if the answer is the same for the supplement. • Notice pattern of values of angles for sine and cosine greater than 90° • Not going to say "Sine of supplementary angles are the same" • Students should then notice that the formula can be used with an obtuse angle, as well as acute. Law of Sines - Every time during the Pigpen activity, when students are solving for the height, and then if they set them up to each other, it ends up being the Law of Sines.	All planned observed in addition to: Homework Review • How to use the calculator for trigonometry functions • Explanation of when to use table and when to use calculator • Answers specific to student questions Pigpen for Monica Part 2 • Directions to draw the correct height • Height needs to be perpendicular to the base
Questions	Pigpen for Monica Part 2 • Does the new area formula always work? Will it work for obtuse angles? • Discuss why using the area formula for 120° yields the same answer as the supplement. • Can you fine other angles that have the same sine ratio? • What's the relationship between 60° and 120°? • What about right angles? Can we use this with 90°? Law of Sines - During the Pigpen activity, when students are solving for the height, and then if they set them up to each other, it ends up being the Law of Sines.	Pigpen for Monica • Why is that the height? • What is the problem with this situation? (obtuse with height outside)

Mrs. Orchid Unit 2 Observation 2 KCS

KCS	Observed as Planned	Additional Observations
Modifications	No modifications	During homework Mrs. Orchid led students to answer homework problems.
Student Difficulties/ Misconceptions	Bellringer • Base of a triangle has to be on bottom • If a side looks like it is bisected in the diagram, it is bisected. • Height cannot be calculated outside an obtuse triangle. • Use the wrong trigonometry ratio to find the length of a missing side. Homework Review • Determining the correct trigonometry ratio • Determining in relation to the angle, which side is opposite and which sides is adjacent. Pigpen for Monica • Drawing a height that can be determined with the information given • Verbalizing the formula for area using Sine	None
Prerequisite Knowledge	Planned prerequisite was for students to be able to use the area formula ½ base x aSinC from the two homework problems.	None

Mrs. Orchid Unit 2 Observation 2 SCK

Mathematical Tasks of Teaching	Planned and Executed
Presenting mathematical ideas	Not all planned ideas were observed.
Responding to students' "why" questions	All planned responses to students' "why" questions were observed. Additional responses to students' "why" questions were also observed.
Finding an example to make a specific mathematical point	No examples given in pre-observation interview. Measuring a person's height needing to be perpendicular to the ground was observed during executed lesson.
Selecting representations for particular purposes	Mrs. Orchid asked students to draw out the triangle for each problem.
Recognizing what is involved in using a particular representation	Mrs. Orchid understood that drawings are not accurate, and asked students to label appropriately.
Linking representations to underlying ideas and to other representations	In Pigpen activity, Mrs. Orchid asked students to continue to draw triangles to give a picture to the problem. She stated that being able to draw a useful diagram is necessary to solving word problems.
Connecting a topic being taught to topics from prior or future years	The Pigpen activity took what students knew about area formula for a triangle and improved upon it to use the sine ratio. This linked to Law of Sines.
Explaining mathematical goals and purposes to parents	None observed
Appraising and adapting the mathematical content of textbooks	None observed
Modifying tasks to be either easier or harder	None observed
Evaluating the plausibility of students' claims (often quickly)	Mrs. Orchid evaluated plausibility of students' claims during each phase of the lesson.
Giving or evaluating mathematical explanations	Giving and evaluating mathematical explanations was shown both in planned and executed phases of the lesson.
Choosing and developing useable definitions	Defined parallelogram in order to construct it on GSP.
Using mathematical notation and language and critiquing its use	Throughout the observation, Mrs. Orchid corrected terminology when students were not using the correct term, and modeled correct terminology.
Asking productive mathematical questions	Observed questions, including planned, incited discussion and led to students providing correct answers.
Inspecting equivalencies	Throughout the lesson, Mrs. Orchid determined how different student questions or statements could be equivalent.

Mrs. Orchid Unit 2 Observation 3 KCT

KCT	Observed as Planned	Additional Observations
Sequence	Planned sequence was for students to watch and follow long instructions of a video for the first two parts of Pigpen activity, discuss homework (third part of Pigpen activity), and then derive Law of Sines following the Law of Sines activity.	None
Time	No planned time for the day, except to go through all of the planned sequence; which is to finish the Law of Sines Activity.	None
Explanations	Pigpen for Monica Part 1 • Directions for the activity • Explain the major problem for the activity – finding the angle to place the fences so Monica will have the most area in her pigpen. • Triangle inequality • What it means for the pen to be closed • Notice the pattern to develop a formula. • Explanations to help students verbalize the formula Pigpen for Monica Part 2 • Directions of what the part 2 is asking • Determine if the formula developed in part 1 always works • Directions to draw the correct height • Height needs to be perpendicular to the base • It may appear to students that the formula does not work for obtuse angles • Use the formula for 120° and see if the answer is the same for the supplement. • Notice pattern of values of angles for sine and cosine greater than 90° • Not going to say "Sine of supplementary angles are the same" • Students should then notice that the formula can be used with an obtuse angle, as well as acute. Pigpen for Monica Part 3 (Homework review) • Examine the trig table and see what is the maximum value for sine. • The supplements match Law of Sines • Every time during the Pigpen activity, when students are solving for the height, and then if they set them up to each other, it ends up being the Law of Sines. • Directions to sketch the triangle to derive Law of Sines	Pigpen for Monica Part 1 • Explanations during each problem where Mrs. Orchid used slightly different numbers for calculations and how it is similar to prior days. • Naming sides and angles of a triangle • Notice the area has not changed using different heights of the same triangle • Answers specific to student questions Pigpen for Monica Part 2 • Explanation that although the relationship between 60 and 120 is that double of 60 is 120, there may be another relationship. • Understanding what the relationship between sine of 60° and 120° is "key". Pigpen for Monica Part 3 • Explanations specific to each problem in this part • "The trig table I gave you is designed to show acute angles of a right triangle only" • Make sure that the calculator is in degrees mode and not radians • Appropriate use of different area formulas Law of Sines • "Oblique is a fancy word for non-right triangles or non-right" • Directions to set up sine ratios for different angles in a way so that students can see that they're equal

Questions	Pigpen for Monica Part 1 • What is the problem asking you to do? • Where to put the angle and what kind of angle it would be • What will the pigpen look like? • What do you need to find the area? • Was it a good idea to make the angle smaller or bigger? • Where would the height be? Can you draw a segment that would represent the height? Where can we put the height so we can calculate it? • Is there another place to put the height? • Identifying triangle parts • How would you solve for h? • Do you notice a pattern? What repeats? Pigpen for Monica Part 2 • Does the new area formula always work? Will it work for obtuse angles? • Why is that the height? • What is the problem with this situation? (obtuse with height outside) • Discuss why using the area formula for 120° yields the same answer as the supplement. • Can you fine other angles that have the same sine ratio? • What's the relationship between 60° and 120°? • What about right angles? Can we use this with 90°? Pigpen for Monica Part 3 (Homework review) • Preplanned questions in the activity • How can you be sure that this is the maximum area? Law of Sines – questions to lead students to see equivalency of Law of Sines	Pigpen for Monica Part 1 • Is every problem you're going to do involve sides of eight and six? • What is the problem with this situation? (obtuse with height outside) Pigpen for Monica Part 2 • Does this only work for 60 and 120? What else would it work for? Pigpen for Monica Part 3 • Questions to lead students to set up the Sine ratios so that the pattern of them being equal can be seen • Why is this possible? Are you sure? • What do you notice?

Mrs. Orchid Unit 2 Observation 3 KCS (no additional observations)

KCS	Observed as Planned
Modifications	Have students watch a video for the first two parts of Pigpen activity. All discussions involved the teacher and the whole class, no group work. Students were seated in rows.
Student Difficulties/ Modifications	Mrs. Orchid planned the lesson in anticipation for any difficulties that may occur; instruction on how where to draw the height that can be used for the problem, using the calculator properly, and questions that highlighted the relationship between sines of supplementary angles.
Prerequisite Knowledge	Prerequisite knowledge for students was how to plug numbers into an equation and solve. Other than that, Mrs. Orchid planned to give students enough explanation in order for them to answer each question in the activities.

Appendix K

Mrs. Orchid Unit 1 Observation 1 Geometry Filter

Geometry Filter	Planned and Executed	Classroom Evidence
Van Hiele Levels	Level 2: Informal Deduction	See relationships between hidden triangle and the triangle bring created. Discussions where students state their reasoning and justify.
Phases of Learning	Directed Orientation	The rounds in the Triangle in a Bag activity allowed students to become more acquainted with the material being taught.
	Explication	With each round of Triangle in a Bag, there was a transition of students being less dependent of Mrs. Orchid.
	Free Orientation	For each of the rounds of Triangle in a Bag, Mrs. Orchid is attentive to the inventive nature of students and the task of creating a congruent triangle can be approached in many different ways.
	Integration	At the end of each round, Mrs. Orchid led students to summarize what was learned in each round.
Geometric Habits of Mind	Reasoning with Relationships	Students were reasoning with what were necessary relationships between two triangles to create duplicates.
	Balancing Exploration and Reflection	Throughout the rounds in the Triangle in a Bag activity, students were balancing exploration and reflection.

Mrs. Orchid Unit 1 Observation 2 Geometry Filter

Geometry Filter	Planned and Executed	Classroom Evidence
Van Hiele Levels	Level 2: Informal Deduction	See relationships between hidden triangle and the triangle bring created. Discussions where students state their reasoning and justify.
Phases of Learning	Directed Orientation	The rounds in the Triangle in a Bag activity allowed students to become more acquainted with the material being taught.
	Explication	With each round of Triangle in a Bag, there was a transition of students being less dependent of Mrs. Orchid.
	Free Orientation	For each of the rounds of Triangle in a Bag, Mrs. Orchid is attentive to the inventive nature of students and the task of creating a congruent triangle can be approached in many different ways.
	Integration	At the end of all the rounds, students summarized their findings through group presentations and a graphic organizer.
Geometric Habits of Mind	Reasoning with Relationships	Students were reasoning with what were necessary relationships between two triangles to create duplicates.
	Balancing Exploration	Throughout the rounds in the Triangle in a Bag activity, students were balancing exploration and reflection.

	and Reflection	
	Executed, not planned: Investigating Invariants	Mrs. Orchid placed the triangle template in different orientations forcing students to tell her how to transform the triangle to fit the one they drew.

Mrs. Orchid Unit 1 Observation 3 Geometry Filter

Geometry Filter	Planned and Executed	Classroom Evidence
Van Hiele Levels	Level 2	Justifying answers through discussion.
	Level 3	Planned but not observed, students writing a formal proof.
Phases of Learning	Directed Orientation	Bellringer and application problems allowed students to become more acquainted with the material being taught.
	Explication	Homework review with student presentations and group work on application problems showed a transition of students being less dependent of Mrs. Orchid.
	Free Orientation	The application problems can be approached in many different ways.
	Integration	At the end of each activity, bellringer review, homework review, and classwork summary, students summarized what was learned.
Geometric Habits of Mind	Reasoning with Relationships	Students were reasoning with what were necessary relationships between two triangles to create duplicates.
	Balancing Exploration and Reflection	Throughout the rounds in the Triangle in a Bag activity, students were balancing exploration and reflection.
	Executed, not planned: Investigating Invariants	Mrs. Orchid placed the triangle template in different orientations forcing students to tell her how to transform the triangle to fit the one they drew.

Mrs. Orchid Unit 1 Observation 4 Geometry Filter

Geometry Filter	Planned and Executed	Classroom Evidence
Van Hiele Levels	Level 2: Informal Deduction	For bellwork review and activity discussions, Mrs. Orchid prompted students to state why they believe the answers they came up with were true.
Phases of Learning	Information	Bellringer, defining parallelogram, and construction of parallelogram, Mrs. Orchid learned of students' prior knowledge through discussions.
	Directed Orientation	Exploration of parallelogram properties, students were becoming more familiar with the material.
	Explication	During the exploration of parallelogram properties, there was a transition of reliance from teacher to student self-reliance.
	Free Orientation	Students could use a variety of methods to test properties of parallelograms.
	Integration	After the exploration, Mrs. Orchid led students in summarizing their findings.
Geometric Habits of Mind	Reasoning with Relationships	Students were reasoning with relationships of angle measures and lengths of sides and diagonals within a parallelogram.
	Balancing Exploration and Reflection	Throughout the activity, students were exploring and reflecting on what they found
	Investigating Invariants	Mrs. Orchid prompted students to move the parallelograms around to see relationships.
	Generalizing Geometric Ideas	This was not planned, but observed. For students that were done sooner than other students, Mrs. Orchid asked them to see if there were other observations to be made.

Mrs. Orchid Unit 2 Lesson 1 Geometry Filter

Geometry Filter	Planned and Executed	Classroom Evidence
Van Hiele Levels	Level 1: Analysis	Identify the shape of the pigpen.
	Level 2: Informal Deduction	Discussions where students state their reasoning and justify.
Phases of Learning	Directed Orientation	Bellwork and homework review were activities that allowed students to become more acquainted with the material that was necessary for the Pigpen activity.
	Explication	Mrs. Orchid set up the questions in the Pigpen activity so that students can transition from reliance on Mrs. Orchid to themselves.
	Free Orientation	Bellwork, homework, and activity problems had multiple ways to get the answer.
	Integration	At the end of the Pigpen activity, Mrs. Orchid prompted students to summarize what they learned.
Geometric Habits of Mind	Reasoning with Relationships	Identify shape of pigpen, triangle.

Mrs. Orchid Unit 2 Lesson 2 Geometry Filter

Geometry Filter	Planned and Executed	Classroom Evidence
Van Hiele Levels	Level 2: Informal Deduction	Discussions where students state their reasoning and justify.
Phases of Learning	Directed Orientation	Homework review allowed students to become more acquainted with the material that was necessary for the Pigpen activity.
	Explication	Mrs. Orchid set up the questions in the Pigpen activity so that students can transition from reliance on Mrs. Orchid to themselves.
	Free Orientation	Homework and Pigpen activity problems had multiple ways to get the answer.
Geometric Habits of Mind	Balancing Explorations and Reflection	Planned for students during the Law of Sines activity where they make reasoned conjectures about what the relationship would be. This was not observed during the lesson.

Mrs. Orchid Unit 2 Lesson 3 Geometry Filter

Geometry Filter	Planned and Executed	Classroom Evidence
Van Hiele Levels	Level 2: Informal Deduction	Discussions where students stated their reasoning and justified.
Phases of Learning	Directed Orientation	Pigpen activity allowed students to be more familiar with using the sine function in an area formula. Law of Sines activity helped students derive the Law of Sines
	Explication	Mrs. Orchid wrote the questions in the Pigpen activity so that students can transition from reliance on Mrs. Orchid to themselves. Students worked on Part 3 of the Pigpen activity on their own for homework the night before.
	Free Orientation	Part 3 of Pigpen Activity problems had multiple ways to get the answer.
	Integration	Review at the end of each part of Pigpen activity as well as Law of Sines activity is where Mrs. Orchid led students to summarize what they learned.
Geometric Habits of Mind	Generalizing Geometric Ideas	Mrs. Orchid asked students if they could think of other pairs of angle measures where the sine of those angle measures are equivalent.

Mrs. Lotus Unit 1 Observation 1 Geometry Filter

Geometry Filter	Planned and Executed	Classroom Evidence
Van Hiele Levels	Level 2	For bellwork review, notes, and homework activity discussion, Mrs. Lotus prompted students to state why they believe the answers they came up with were true.
	Level 3	Students were challenged to prove a corollary
Phases of Learning	Information	Bellwork and review from the previous day, Mrs. Lotus learned of students' prior knowledge.
	Directed Orientation	Problems during notes and discussion for the homework activity provided students with the opportunity to become more familiar with the material.
	Explication	Problems during notes and student discussion of homework problem showed the transition of reliance from teacher to student self-reliance.
	Free Orientation	Students could use a variety of methods prove the corollary and approach the homework problem.
	Integration	After each activity, Mrs. Lotus led students in summarizing their findings.

Geometric Habits of Mind	Reasoning with Relationships	Students were reasoning with relationships of the sum of interior angles of a triangle, interior and exterior angles of a triangle, exterior angles of a triangle, and possible points equidistant from two and three points.
	Generalize Geometric Ideas	Proving about sum of interior angles, relationships between interior and exterior angles, and exterior angles of a triangle challenged students to generalize geometric ideas.
	Balancing Exploration and Reflection	Throughout the lesson, students made reasoned conjectures and tested them.

Mrs. Lotus Unit 1 Observation 2 Geometry Filter

Geometry Filter	Planned and Executed	Classroom Evidence
Van Hiele Levels	Level 2	For bellwork review, homework review, and Equally Wet discussion, Mrs. Lotus prompted students to state why they believe the answers they came up with were true.
Phases of Learning	Information	Bellwork and review of Equally Wet, Mrs. Lotus learned of students' prior knowledge.
	Directed Orientation	Different construction activities provided students opportunities to become more acquainted with the material.
	Explication	Group work during the GSP activity showed the transition of reliance from teacher to student self-reliance.
	Free Orientation	Students could use a variety of methods to create the constructions to form conjectures in the GSP activity.
	Integration	After each GSP construction, Mrs. Lotus led students in summarizing their findings.
Geometric Habits of Mind	Reasoning with Relationships	Students reasoned with relationships of the intersection of the three perpendicular bisectors of a triangle and the intersection of three angle bisectors of a triangle.
	Generalize Geometric Ideas	For each of these intersections, Mrs. Lotus asked students if they found all possible solutions.
	Balancing Exploration and Reflection	For each problem, students explored and then reflected on the activity when Mrs. Lotus led the whole class in discussion

Mrs. Lotus Unit 1 Geometry Filter

Day	Geometry Filter	Planned and Executed
1	Van Hiele Levels	Level 2: Informal Deduction, Level 3: Formal Deduction
	Phases of Learning	Directed Orientation, Explication, Free Orientation, Integration
	Geometric Habits of Mind	Reasoning with Relationships, Generalizing Geometric Ideas, Balancing Exploration and Reflection
2	Van Hiele Levels	Level 2: Informal Deduction
	Phases of Learning	Directed Orientation, Explication, Free Orientation, Integration
	Geometric Habits of Mind	Reasoning with Relationships, Generalizing Geometric Ideas, Balancing Exploration and Reflection
3	Van Hiele Levels	Level 2: Informal Deduction
	Phases of Learning	Directed Orientation, Explication, Free Orientation, Integration
	Geometric Habits of Mind	Reasoning with Relationships, Generalizing Geometric Ideas, Balancing Exploration and Reflection

Mrs. Lotus Unit 2 Observation 1 Geometry Filter

Geometry Filter	Planned and Executed	Classroom Evidence
Van Hiele Levels	Level 2	In bellwork review and textbook homework review, Mrs. Lotus challenged students to informally prove why the answers they came up with were true; asking students to justify with the question "why?"
	Level 3	Construct a proof for a corollary to a theorem
Phases of Learning	Information	Bellwork and review homework from the previous day was where Mrs. Lotus learned of students' prior knowledge and experience with the lesson of the day.
	Directed Orientation	Classwork problems and the dilations activity provided students opportunities to become more acquainted with the material.
	Explication	Group work during the dilation activity and classwork problems showed the transition of reliance from teacher to student self-reliance.
	Free Orientation	Students could use a variety of methods to make conjectures in the dilation activity as well as the classwork problems.
	Integration	After each classwork problem and group work for the dilation activity, Mrs. Lotus led students in summarizing their findings.

Geometric Habits of Mind	Reasoning with Relationships	During the dilation activity, students were reasoning with relationships of the original figure and the dilated figure. They made observations of the similarities and differences between the original figure and dilated figure.
	Generalize Geometric Ideas	Students talking with group members to generalize observations of their figures.
	Balancing Exploration and Reflection	For each problem and the dilation activity, students explored on their own and as a group before summarizing as a whole class and reflecting on how others approached the problem.

Mrs. Lotus Unit 2 Observation 2 Geometry Filter

Geometry Filter	Planned and Executed	Classroom Evidence
Van Hiele Levels	Level 2	In homework review and discussion of problems throughout the lesson, Mrs. Lotus challenged students to informally prove why the answers they came up with were true
Phases of Learning	Information	During homework review was when Mrs. Lotus learned of students' prior knowledge and experience with the lesson of the day.
	Directed Orientation	The construction activities showed "directed orientation", where these activities were provided for students to become more familiar with the material being taught.
	Explication	During discussion of the conjectures from constructions and problems showed a transition of reliance on teacher to student self reliance
		The questions and discussion of the conjectures from constructions were open-ended and was attentive to the inventive abilities of the students.
	Free Orientation	The two problems at the end of the lesson showed integration phase because students were using what they learned throughout the day to solve those problems.
	Integration	
Geometric Habits of Mind	Reasoning with Relationships	During the dilation activity, students were reasoning with relationships of the original figure and the dilated figure. They made observations of the similarities and differences between the original figure and dilated figure.
	Generalize Geometric Ideas	Students talking with group members to generalize observations of their figures.
	Balancing Exploration and Reflection	For each problem and the dilation activity, students explored on their own and as a group before summarizing as a whole class and reflecting on how others approached the problem.

Mrs. Lotus Unit 2 Observation 3 Geometry Filter

Geometry Filter	Planned and Executed	Classroom Evidence
Van Hiele Levels	Level 2	For each answer during bellwork review and in class problems, Mrs. Lotus challenged students to justify their answers.
	Level 3	Mrs. Lotus challenged students to prove two corollaries related to theorems.
Phases of Learning	Information	During bellwork Mrs. Lotus learned of students' prior knowledge and experience with the lesson of the day.
	Directed Orientation	The application problems were activities that allowed students to become more acquainted to the topic of the day.
	Explication	Progression of the application problems and proof of the two corollaries, showed a transition of reliance on teacher to student self reliance
	Free Orientation	The application problems and proofs were open ended, thus being attentive to the inventive abilities of the students
	Integration	At the end of the day, there were two summary problems that required knowledge of material learned throughout the day. Discussion of these two problems helped students summarize what was learned during the lesson, thus showing integration phase.
Geometric Habits of Mind	Reasoning with Relationships	For each application problem, students were reasoning with relationships of the figures. Each application problem asked a different aspect of similarity; determining similarity and determining missing measures if similar.
	Generalize Geometric Ideas	When challenging students to create proofs, Mrs. Lotus asked students if the statements in their proofs worked in all cases or just the case they were trying to make.
	Balancing Exploration and Reflection	Through working on the application problems and proofs, students explored when working individually and discussion as a group. Students reflected also in discussing in small groups and as a whole group.

Appendix K

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