Matrix Algebras over Strongly Non-Singular Rings

by

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Abstract

We consider some existing results regarding rings for which the classes of torsion-free and non-singular right modules coincide. Here, a right R-module M is *non-singular* if xI is nonzero for every nonzero $x \in M$ and every essential right ideal I of R, and a right R-module M is *torsion-free* if $Tor_1^R(M, R/Rr) = 0$ for every $r \in R$. In particular, we consider a ring R for which the classes of torsion-free and non-singular right S-modules coincide for every ring S Morita-equivalent to R. We make use of these results, as well as the existence of a Morita-equivalence between a ring R and the $n \times n$ matrix ring $Mat_n(R)$, to characterize rings whose $n \times n$ matrix ring is a Baer-ring. A ring is *Baer* if every right (or left) annihilator is generated by an idempotent. Semi-hereditary, strongly non-singular, and Utumi rings will play an important role, and we explore these concepts and relevant results as well.

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Chapter 1

Introduction

In this thesis, we consider the relationship between a ring R and $Mat_n(R)$, the $n \times n$ matrix ring over R. In particular, we investigate necessary and sufficient conditions placed on R so that $Mat_n(R)$ is a Baer-ring. A ring is a *Baer-ring* if every right (or left) annihilator ideal is generated by an idempotent. In determining these conditions, we make use of the existence of a Morita-equivalence between R and $Mat_n(R)$ (Proposition 6.2), as well as the fact that $Mat_n(R)$ is isomorphic to the endomorphism ring of any free right R-module with basis $\{x_i\}_{i=1}^n$(Lemma 2.6). Here, two rings are *Morita-equivalent* if their module categories are equivalent, and the *endomorphism ring* $End_R(M)$ of a right R-module M is the set of all R-homomorphisms $f : M \to M$, which is a ring under pointwise addition and composition of functions.

The concepts of torsion-freeness and non-singularity of modules will also come into play. In particular, we consider rings for which the classes of torsion-free and non-singular right S-modules coincide for every ring S Morita-equivalent to R. Albrecht, Dauns, and Fuchs investigate such rings in [1]. A module M over a ring R is *torsion-free in the classical sense* if $xr \neq 0$ for every nonzero $x \in M$ and every regular $r \in R$, where $r \in R$ is *regular* if it is not a left or right zero-divisor. For commutative rings, this is a useful way to define such modules, especially for integral domains since regular elements are precisely the nonzero elements. In the case R is non-commutative, then the set $M_t = \{x \in M \mid ann_r(x)$ contains some regular element of R$\}$, which is usually referred to as the *torsion-submodule* in the commutative setting, is not necessarily a submodule of M. There are other ways in which torsion-freeness can be defined in the non-commutative setting. In [7], Hattori calls a right R-module M *torsion-free* if $Tor_1^R(M, R/Rr) = 0$ for every $r \in R$. This is based on homological properties of modules and coincides with the classical definition in the case R is commutative. In [6], Goodearl defines the singular submodule and non-singularity of modules in the general non-commutative setting, which is closely related to the concept of torsion submodules and torsion-freeness. We look at relevant background information on torsion-freeness and non-singularity in Chapters 4 and 5.

Albrecht, Dauns, and Fuchs found that S is right strongly non-singular and the classes of torsion-free and non-singular S-modules coincide for every ring S Morita equivalent to a ring R if and only if R is right strongly non-singular, right semi-hereditary, and does not contain an infinite set of orthogonal idempotents [1, Theorem 5.1]. A ring is *right strongly non-singular* if its maximal right ring of quotients is a perfect left localization. These rings will be explored in Section 5.2, and semi-hereditary rings will be defined and explored in Chapter 2. We make use of this theorem and take it a step further to show that $Mat_n(R)$ is a right and left Utumi Baer-ring if and only if the classes of torsion-free and non-singular S-modules coincide for every ring S Morita equivalent to a ring R. Note that we remove the condition that every Morita-equivalent ring S need be strongly non-singular. Instead, we assume that our ring R is right Utumi, and from this we also get that $Mat_n(R)$ is both right and left Utumi. We define Utumi rings in Section 5.3.

Unless noted otherwise, commutativity of a ring is not assumed, but all rings are assumed to have a multiplicative identity.

Chapter 2

Semi-hereditary Rings and p.p.-rings

We begin by looking at projective modules. A right R-module P is *projective* if given right R-modules A and B, an epimorphism $\pi : A \to B$, and a homomorphism $\varphi : P \to$ B, then there exists a homomorphism $\psi :$ P $\to$ A such that $\pi\psi = \varphi$. In particular, every free right R-module is projective [9, Theorem 3.1]. We make use of the following well-known characterization of projective modules:

Theorem 2.1. *[9] Let R be a ring. The following are equivalent for a right R-module P:*

(a) P is projective

(b) P is isomorphic to a direct summand of a free right R-module. In other words, there is a free right R-module $F = Q \bigoplus N$, where N is a right R-module and $Q \cong P$.

(c) For any right R-module M and epimorphism $\varphi : M \to P$, $M = \ker(\varphi) \bigoplus N$.

Let Mod_R be the category of all right R-modules for a ring R. A *complex* in Mod_R is a sequence of right R-modules and R-homomorphisms in Mod_R,

$$... \to A_{k+1} \xrightarrow{\alpha_{k+1}} A_k \xrightarrow{\alpha_k} A_{k-1} \to ...$$

such that $\alpha_{k+1}\alpha_k = 0$ for every $k \in \mathbb{Z}$. Observe $\alpha_{k+1}\alpha_k = 0$ implies that $im(\alpha_{k+1}) \subseteq \ker(\alpha_k)$. The sequence is called *exact* if $im(\alpha_{k+1}) = \ker(\alpha_k)$ for every $k \in \mathbb{Z}$. An exact sequence $0 \to A \overset{\alpha}{\to} B \overset{\beta}{\to} C \to 0$ of right R-modules is referred to as a *short exact sequence*. Such an exact sequence is said to *split* if there exists an R-homomorphism $\gamma : C \to B$ such that $\beta\gamma = 1_C$, where 1_C is the identity map on C.

Lemma 2.2. *[9] Let $0 \to A \overset{\alpha}{\to} B \overset{\beta}{\to} C \to 0$ be a sequence of right R-modules. If this sequence is split exact, then $B \cong A \bigoplus C$.*

Proof. If the exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ of right R-modules splits, then there exists an R-homomorphism $\gamma : C \to B$ such that $\beta\gamma \cong 1_C$. Observe that since α is a monomorphism, $im(\alpha) \cong A$. Moreover, if $x \in \ker(\gamma)$, then $\gamma(x) = 0$. However, $\beta(0) = \beta\gamma(x) = x$ since $\beta\gamma = 1_C$. Thus, $x = 0$ and γ is also a monomorphism. Hence, $im(\beta) \cong C$. Therefore, to show that $B \cong A \bigoplus C$, it suffices to show that $B \cong im(\alpha) \bigoplus im(\gamma)$.

Let $b \in B$. Then $\beta(b) \in C$ and $\gamma\beta(b) \in im(\gamma)$. Furthermore, $b - \gamma\beta(b) \in \ker(\beta) = im(\alpha)$ since $\beta(b - \gamma\beta(b)) = \beta(b) - \beta\gamma\beta(b) = \beta(b) - \beta(b) = 0$. Hence, $b = [b - \gamma\beta(b)] + \gamma\beta(b) \in im(\alpha) + im(\gamma)$. Suppose, $x \in im(\alpha) \cap im(\gamma)$. Then, there exists some $a \in A$ such that $\alpha(a) = x$, and there exists some $c \in C$ such that $\gamma(c) = x$. Now, $\alpha(a) \in im(\alpha) = \ker(\beta)$, which implies $\beta(x) = \beta\alpha(a) = 0$. However, it is also the case that $\beta(x) = \beta\gamma(c) = c$. Hence, $c = 0$ and it follows that $x = \gamma(c) = \gamma(0) = 0$. Thus, $im(\alpha) \cap im(\gamma) = 0$. Therefore, $B \cong im(\alpha) \bigoplus im(\gamma) \cong A \bigoplus C$. □

Proposition 2.3. *[9] The following are equivalent for a right R-module P:*

(a) P is projective.

(b) The sequence $0 \to Hom_R(P, A) \xrightarrow{Hom_R(P,\varphi)} Hom_R(P, B) \xrightarrow{Hom_R(P,\psi)} Hom_R(P, C) \to 0$ *is exact whenever* $0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$ *is a an exact sequence of right R-modules.*

Proof. $(a) \Rightarrow (b)$: Suppose P is projective. Observe that the functor $Hom_R(P, \_)$ is left exact [9, Theorem 2.38]. Thus, if $0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$ is exact, then

$$0 \to Hom_R(P, A) \xrightarrow{Hom_R(P,\varphi)} Hom_R(P, B) \xrightarrow{Hom_R(P,\psi)} Hom_R(P, C)$$

is exact. Therefore, it remains to be shown that $Hom_R(P, \psi)$ is an epimorphism. Let $\alpha \in Hom_R(P, C)$. Since P is projective, there exists a homomorphism $\beta : P \to B$ such that $\alpha = \psi\beta$. Hence, $Hom_R(P, \psi)(\beta) = \psi\beta = \alpha$. Therefore, $Hom_R(P, \psi)$ is an epimorphism.

$(b) \Rightarrow (a)$: Let P be a right R-module and assume exactness of $0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$ implies exactness of $0 \to Hom_R(P, A) \xrightarrow{Hom_R(P,\varphi)} Hom_R(P, B) \xrightarrow{Hom_R(P,\psi)} Hom_R(P, C) \to 0$.

This implies $Hom_R(P, \psi)$ is an epimorphism. Thus, if $\alpha \in Hom_R(P, \psi)$, then there exists some $\beta \in Hom_R(P, B)$ such that $Hom_R(P, \psi)(\beta) = \psi\beta = \alpha$. That is, given an epimorphism $\psi : B \to C$ and a homomorphism $\alpha : P \to C$, there exists a homomorphism $\beta : P \to B$ such that $\alpha = \psi\beta$. Therefore, P is projective. □

A ring R is a *right p.p.-ring* if every principal right ideal is projective as a right R-module. A ring R is *right semi-hereditary* if every finitely generated right ideal is projective as a right R-module. For a right R-module M and any subset $S \subseteq M$, define the *right annihilator of* S in R as $ann_r(S) = \{r \in \mathrm{R} \mid xr = 0 \text{ for every } x \in S\}$. The right annihilator of S is a right ideal of R. Similarly, the *left annihilator of* S in R can be defined for a left R-module M as $ann_l(S) = \{r \in \mathrm{R} \mid rx = 0 \text{ for every } x \in S\}$. The left annihilator of S is a left ideal of R. The following proposition shows that right p.p.-rings can be defined in terms of annihilators of elements and idempotents, where an idempotent is an element $e \in R$ such that $e^2 = e$.

Proposition 2.4. *A ring R is a right p.p.-ring if and only if for every $x \in R$ there exists some idempotent $e \in R$ such that $ann_r(x) = eR$.*

Proof. For $x \in R$, consider the function $f_x : R \to xR$ given by $r \mapsto xr$. This is a well-defined epimorphism. Then R is a right p.p.-ring if and only if the principal right ideal xR is projective for for every $x \in R$ if and only if $\ker(f_x)$ is a direct summand of R for every $x \in R$. Observe that for each $x \in R$, $\ker(f_x) = ann_r(x)$. Hence, R is a right p.p.-ring if and only if $ann_r(x)$ is a direct summand of R. Note that every direct summand of R is generated by an idempotent since $R \cong eR \bigoplus (1-e)R$ for any idempotent $e \in R$. Thus, as a direct summand, $ann_r(x) = eR$ for some idempotent $e \in R$. Therefore, R is a right p.p.-ring if and only if for every $x \in R$ there is some idempotent $e \in R$ such that $ann_r(x) = eR$. □

Let $Mat_n(R)$ denote the set of all $n \times n$ matrices with entries in R. Under standard matrix addition and multiplication, $Mat_n(R)$ is a ring. A useful characterization of semi-hereditary rings is that such rings are precisely those for which $Mat_n(R)$ is a right p.p.-ring for every $0 < n < \omega$. To show this, the following two lemmas will be needed:

Lemma 2.5. *[9] A ring R is right semi-hereditary if and only if every finitely generated submodule U of a projective right R-module P is projective.*

Proof. Suppose R is right semi-hereditary and let U be a submodule of a projective right R-module P. By Theorem 2.1, $P \bigoplus N$ is free for some right R-module N. Hence, P is a submodule of a free module, and it follows that any submodule of P is also a submodule of a free module. Thus, without loss of generality, it can be assumed that P is a free right R-module. Moreover, since U is finitely generated, it can be assumed that P is finitely generated with basis $X = \{x_1, x_2, ..., x_n\}$ for some $0 < n < \omega$.

Inductively, it will be shown that U is a finite direct sum of finitely generated right ideals. If $n = 1$, then $P = x_1R \cong R$. Since submodules of the right R-module R are right ideals, U is a finitely generated right ideal. Suppose $n > 1$ and assume U is a finite direct sum of finitely generated right ideals for $k < n$. Let $V = U \cap (x_1R + x_2R + ... + x_{n-1}R)$. Then, V is a finitely generated submodule of a free right R-module with basis $\{x_1, x_2, ..., x_{n-1}\}$. By assumption, V is a finite direct sum of finitely generated right ideals. Note that if $u \in U$, then $u = v + x_n r$ with $v \in V$ and $r \in R$. This expression for u is unique since X is a linearly independent spanning set. Thus, the map $\varphi : U \to R$ defined by $\varphi(u) = \varphi(v + x_n r) = r$ is a well-defined homomorphism.

Now, $im(\varphi)$ is a finitely generated right ideal of R since it is the epimorphic image of the finitely generated right R-module U. Hence, $im(\varphi)$ is projective since R is right semi-hereditary. Consider the short exact sequence $0 \to K \overset{\iota}{\to} U \overset{\varphi}{\to} im(\varphi) \to 0$, where $K = \ker \varphi$ and ι is the inclusion map. This sequence splits since $im(\varphi)$ is projective, and thus $U \cong K \bigoplus im(\varphi)$ by Lemma 2.2. Hence, U is a finite direct sum of finitely generated right ideals since both K and $im(\varphi)$ are finitely generated right ideals. Since R is right semi-hereditary, each of these right ideals is projective. Therefore, U is projective as the direct sum of projective right ideals.

Conversely, suppose that if P is a projective right R-module, then every finitely generated submodule U of P is projective. Let I be a finitely generated right ideal of R. Note that R is a free right R-module and thus projective. Hence, I is a finitely generated submodule of R, and by assumption I is projective. Therefore, R is right semi-hereditary. □

Lemma 2.6. *Let R be a ring, and F a finitely generated free right R-module with basis $\{x_i\}_{i=1}^n$ for $0 < n < \omega$. Then, $Mat_n(R) \cong End_R(F)$.*

Proof. Let $S = End_R(F)$ and take $f \in S$. Then, $f(x_k) \in F$ for each $k = 1, 2, ..., n$. Hence, $f(x_k)$ is of the form $\sum_{i=1}^{n} x_i a_{ik}$, where $a_{ik} \in R$ for every i and every k. Let $A = \{a_{ik}\}$ be the $n \times n$ matrix whose i-kth entry is a_{ik}, and let $\varphi : S \to Mat_n(R)$ be defined by $f \mapsto A$. If $f, g \in S$ are such that $f = g$, then $f(x_k) = g(x_k)$ for every $k = 1, 2, ..., n$. Hence, φ is well-defined. Furthermore, if $f(x_k) = \sum_{i=1}^{n} x_i a_{ik}$ and $g(x_k) = \sum_{i=1}^{n} x_i b_{ik}$ for $k = 1, 2, ..., n$, then $(f + g)(x_k) = f(x_k) + g(x_k) = \sum_{i=1}^{n} x_i(a_{ik} + b_{ik})$. Thus, if $A = \{a_{ik}\}$ and $B = \{b_{ik}\}$ are the $n \times n$ matrices with entries determined by f and g respectively, then $A + B = \{a_{ik} + b_{ik}\}$ is the $n \times n$ matrix with entries determined by $f + g$. Hence, $\varphi(f + g) = A + B = \varphi f + \varphi g$.

To see that φ is a ring homomorphism, it remains to be seen that $\varphi(fg) = \varphi(f)\varphi(g) = AB$. In other words, it needs to be shown that the entries of the matrix AB are determined by $fg(x_j)$ for $j = 1, 2, ..., n$. Observe that if $A = \{a_{ik}\}$ and $B = \{b_{ik}\}$ are $n \times n$ matrices, then under standard matrix multiplication AB is the $n \times n$ matrix whose i-jth entry is $\sum_{k=1}^{n} a_{ik} b_{kj}$. This is indeed the matrix determined by the endomorphism fg since the following holds: $fg(x_j) = f(\sum_{k=1}^{n} x_k b_{kj}) = \sum_{k=1}^{n} f(x_k) b_{kj} = \sum_{k=1}^{n} \sum_{i=1}^{n} x_i a_{ik} b_{kj} = \sum_{i=1}^{n} x_i \sum_{k=1}^{n} a_{ik} b_{kj}$.

Finally, note that if $A = \{a_{ik}\} \in Mat_n(R)$, then $\sum_{i=1}^{n} x_i a_{ik} \in F$ and $\hat{f} : x_j \mapsto \sum_{i=1}^{n} x_i a_{ik}$ is an R-homomorphism from $\{x_i\}_{i=1}^n$ into F. This can be extended to an endomorphism $f \in F$. It readily follows that $\psi : Mat_n(R) \to S$ defined by $\{a_{ik}\} \mapsto f$ is a well-defined ring homomorphism. Moreover, $\varphi\psi(\{a_{ik}\}) = \varphi(f) = \{a_{ik}\}$ and $\psi\varphi(f) = \psi(\{a_{ik}\}) = f$. Thus, φ and ψ are inverses, and therefore φ is an isomorphism between $S = End_R(F)$ and $Mat_n(R)$. □

Theorem 2.7. *[3] A ring R is right semi-hereditary if and only if $Mat_n(R)$ is a right p.p.-ring for every $0 < n < \omega$.*

Proof. Suppose R is right semi-hereditary. For $0 < n < \omega$, let F be a finitely generated free right R-module with basis $\{x_i\}_{i=1}^n$. By Lemma 2.6, $Mat_n(R) \cong End_R(F)$. Therefore, it suffices to show that $S = End_R(F)$ is a right p.p.-ring. Take $s \in S$. Since F is finitely-generated, sF is a finitely generated submodule of F. Free modules are projective, and thus sF is projective by Lemma 2.5. Since sF is an epimorphic image of F, Theorem 2.1 shows that $F \cong \ker s \bigoplus N$ for some right R-module N. Thus, $\ker s = eF$ for some nonzero idempotent $e \in S$. Suppose $r \in ann_r(s) = \{t \in S \mid st(f) = 0 \text{ for every } f \in F\}$. Then, $sr = 0$ and $r \in \ker s = eF \subseteq eS$. On the other hand, suppose $et \in eS$. Since $sef = 0$ for every $f \in F$, $set(f) = 0$ for every $f \in F$. Hence, $et \in ann_r(s)$. Therefore, $ann_r(s) = eS$ and $S = End_R(F) \cong Mat_n(R)$ is a right p.p.-ring.

Suppose $Mat_n(R)$ is a right p.p.-ring for every $0 < n < \omega$. Let I be a finitely generated right ideal of R with generating set $\{a_1, a_2, ..., a_k\}$, and take F to be a free right R-module with basis $\{x_1, x_2, ..., x_k\}$. Note that there exists a submodule K of F which is isomorphic to I. Hence, K is also generated by k elements, say $b_1, b_2, ..., b_k$. Let $S = Mat_k(R) \cong End_R(F)$. For any $f \in F$, there exists $r_1, r_2, ..., r_k \in R$ such that $f = x_1r_1 + x_2r_2 + ... + x_kr_k$. Let $s \in S$ be the well-defined homomorphism defined by $s(f) = s(x_1r_1 + x_2r_2 + ... + x_nr_n) = b_1r_1 + b_2r_2 + ... + b_nr_k$. Note that $im(s) = K$ and thus $s : F \to K$ is an epimorphism.

It will now be shown that $\ker(s) = ann_r(s)F$. Here, as before, $ann_r(s)$ refers to the annihilator in S. If $y = \sum_{i=1}^{n} t_if_i \in ann_r(s)F$, then $st_if_i = 0$ for every $i = 1, 2, ..., n$. Hence, $y \in \ker(s)$. On the other hand, let $f \in \ker(s)$. Now, fR is a submodule of F, and so we can find some $t \in S$ such that $t : F \to fR$ is an epimorphism and $tf = f$. Then, for any $x \in F$, $s[t(x)] = s(fr)$ for some $r \in R$. However, $s(fr) = (sf)r = 0$. Thus, $t \in ann_r(s)$ and

$f = tf \in ann_r(s)F$. Therefore, $\ker(s) = ann_r(s)F$. Moreover, since $Mat_k(R) \cong End_R(F)$ is a right p.p.-ring by assumption, $ann_r(s) = eS$ for some idempotent $e \in S$. Observe that $SF = F$ since $\sum_{i=1}^{n} s_i f_i \in F$ for $s_i \in S$ and $f_i \in F$, and $f = 1_F(f) \in SF$ for any $f \in F$. Hence, $\ker(s) = ann_r(s)F = eSF = eF$. Thus, $\ker(s)$ is a direct summand of F. It then follows from Theorem 2.1 that $I \cong K$ is projective since $s : F \to K$ is a an epimorphism. Therefore, R is a right semi-hereditary ring. □

Two idempotents e and f are called *orthogonal* if $ef = 0$ and $fe = 0$. If R contains only finite sets of orthogonal idempotents, then being a p.p.-ring is right-left-symmetric. Moreover, if R is a right (or left) p.p.-ring not containing an infinite set of orthogonal idempotents, then it satisfies both the ascending and descending chain conditions on annihilators (Theorem 2.11). A ring R satisfies the *ascending chain condition* on annihilators if given any ascending chain $I_0 \subseteq I_1 \subseteq ... \subseteq I_n \subseteq ...$ of annihilators, there exists some $k < \omega$ such that $I_n = I_k$ for every $n \geq k$. Similarly, R satisfies the *descending chain condition* on annihilators if every descending chain of annihilators terminates for some $k < \omega$. Before proving Theorem 2.11, we look at some basic results regarding annihilators and the chain conditions.

Lemma 2.8. *Let S and T be subsets of a ring R such that $S \subseteq T$. Then, $ann_r(T) \subseteq ann_r(S)$ and $ann_l(T) \subseteq ann_l(S)$.*

Proof. For $r \in ann_r(T)$ and $t \in T$, $tr = 0$. Let $s \in S \subseteq T$. Then, $sr = 0$ and hence $r \in ann_r(S)$. Thus, $ann_r(T) \subseteq ann_r(S)$. A similar computation shows the theorem holds for left annihilators. □

Lemma 2.9. *Let U be a subset of a ring R, and let $A = ann_r(U) = \{r \in R \mid ur = 0$ for every $u \in U\}$. Then, $ann_r(ann_l(A)) = A$.*

Proof. Suppose $r \in ann_r(ann_l(A))$, and let $u \in U$. Then, $ua = 0$ for every $a \in A$. Hence, $u \in ann_l(A)$, and thus $ur = 0$. Therefore, $ann_r(ann_l(A)) \subseteq A$. Conversely, suppose $a \in A$. Then, $ba = 0$ for every $b \in ann_l(A)$. Hence, $a \in ann_r(ann_l(A))$. Therefore, $A \subseteq ann_r(ann_l(A))$. □

Lemma 2.10. *R satisfies the ascending chain condition on right annihilators if and only if R satisfies the descending chain condition on left annihilators.*

Proof. Suppose R satisfies the ascending chain condition on right annihilators. Let $ann_l(U_1) \supseteq ann_l(U_2) \supseteq ...$ be a descending chain of left annihilators. Note that if $ann_l(U_i) \supseteq ann_l(U_j)$, then $ann_r(ann_l(U_1)) \subseteq ann_r(ann_l(U_2)) \subseteq ...$ is an ascending chain of right annihilators by Lemma 2.8. By the ascending chain condition on right annihilators, there is some $k < \omega$ such that $ann_r(ann_l(U_n)) = ann_r(ann_l(U_k))$ for every $n \geq k$. Therefore, $ann_l(ann_r(ann_l(U_n))) = ann_l(ann_r(ann_l(U_k)))$ for every $n \geq k$, and by a symmetric version of Lemma 2.9 it follows that $ann_l(U_n) = ann_l(U_n)$ for every $n \geq k$. A similar argument shows that the descending chain condition on left annihilators implies the ascending chain condition for right annihilators. □

Theorem 2.11. *[3] Let R be a right p.p.-ring which does not contain an infinite set of orthogonal idempotents. Then R is also a left p.p.-ring, every right or left annihilator in R is generated by an idempotent, and R satisfies both the ascending and descending chain condition for right annihilators.*

Proof. Let $A = ann_r(U)$ for some subset U of R and consider B $= ann_l(A)$. Suppose B contains nonzero orthogonal idempotents $e_1, ..., e_n$, and let e $= e_1 + ... + e_n$. Note that e is also an idempotent since $e^2 = (e_1 + ... + e_n)(e_1 + ... + e_n) = e_1^2 + ... + e_n^2 + e_1 e_2 + ... + e_{n-1}e_n = e_1 + ... + e_n = e$. Suppose $B = Re$. The claim is that $A = (1-e)R$, and hence A is generated by an idempotent. To see this, first note that $ann_r(B) = ann_r(ann_l(A)) = A$ by Lemma 2.9. Thus, it needs to be shown that $ann_r(B) = (1-e)R$. If $b \in B = Re$, then $b = se$ for some $s \in R$. For all $r \in R$, we obtain $b(1-e)r = se(1-e)r = (se - se^2)r = (se - se)r = 0$. Hence, $(1-e)R \subseteq ann_r(B)$. On the other hand, suppose $r \in ann_r(B)$. Then, $r = r - er + er = (1-e)r + er$. Note that $e \in B = ann_l(A)$, and so $er = 0$ since $r \in ann_r(B) = A$. Thus, $r = (1-e)r \in (1-e)R$, and hence $ann_r(B) \subseteq (1-e)R$. Therefore, if $B = Re$, then A is generated by an idempotent.

If $B \neq Re$, then select $b \in B \backslash Re$, and observe $ba = 0$ for every $a \in A$ since $b \neq re$ for any $r \in R$. Therefore, $B \neq Be$, which implies $B(1-e) \neq 0$. Let $0 \neq y \in B(1-e)$, say $y = s(1-e)$ for some $s \in B$. Since R is a right p.p.-ring, $ann_r(y) = (1-f)R$ for some idempotent $f \in R$. Observe that f is nonzero. For otherwise, $ann_r(y) = R$ and $y = 0$, which is a contradiction. If $0 \neq a \in A$, then $ya = s(1-e)a = sa - sea = 0 - s \cdot 0 = 0$. Thus, $a \in ann_r(y) = (1-f)R$, and so $A \subseteq (1-f)R$. Hence, $fA \subseteq f(1-f)R = 0$ and $f \in ann_l(A) = B$. Observe that $e \in ann_r(y) = (1-f)R$ since $ye = s(1-e)e = 0$, and so $e = (1-f)t$ for some $t \in R$. Thus, $(1-f)e = (1-f)(1-f)t = (1-f)t = e$, and so $fe = f(1-f)t = (f-f^2)t = 0$. Note also that $fe_i = 0$ for $i = 1, ..., n$, since $ye_i = s(1-e)e_i = s(e_i - ee_i) = s(e_i - e_i) = 0$ and hence $e_i \in ann_r(y)$.

Let $e_{n+1} = (1-e)f = f - ef$. Note e_{n+1} is an idempotent since $fe = 0$ and thus $(f-ef)(f-ef) = f - fef - ef + efef = f - 0 - ef + 0 = f - ef$. Consider e_i for some $i = 1, ..., n$. Then, $e_{n+1}e_i = (1-e)fe_i = (1-e) \cdot 0 = 0$, and $e_ie_{n+1} = e_i(1-e)f = (e_i - e_ie)f = (e_i - e_i)f = 0 \cdot f = 0$. Thus, e_{n+1} is orthogonal to $e_1, ..., e_n$. Furthermore, e_{n+1} is nonzero, since otherwise we have $f = ef$. This would imply $f = f^2 = efef = e \cdot 0 \cdot f = 0$, which is a contradiction. Note also that $e_{n+1} \in B$ since both e and f are in B.

Then, $e_1, ..., e_n, e_{n+1}$ are nonzero orthogonal idempotents contained in B. As before, if $e = e_1 + ... + e_{n+1}$ and $B \neq Re$, then there is a nonzero idempotent $e_{n+2} \in B$ orthogonal to $e_1, ..., e_{n+1}$. Since R does not contain any infinite set of orthogonal idempotents, this process must stop for $e_1, ..., e_k$. Thus, for $e = e_1 + ... + e_k$, $B = Re$ and $A = (1-e)R$. Therefore, each right and left annihilator is generated by an idempotent. From a symmetric version of Proposition 2.4, it follows that R is a left p.p.-ring.

Finally, it needs to be shown that R satisfies the ascending and descending chain conditions for right annihilators. Let $C \subseteq D$ be right annihilators. Then, there are idempotents e and f such that $C = eR$ and $D = fR$. Hence, $eR \subseteq fR$, and it follows that $e = fe$. Thus, $g = f - ef$ is a nonzero idempotent. Furthermore, g and e are orthogonal, since $eg = e(f-ef) = ef - e^2f = ef - ef = 0$ and $ge = (f-ef)e = fe - efe = e - e^2 = 0$. Note that

$fR = eR+gR$. For, if $er+gs \in eR+gR$, then $er+gs = er+(f-ef)s = er+fs+efs \in fR$, and conversely, if $fr \in fR$, then $fr = (f+ef-ef)r = efr+(f-ef)r = efr-gr \in eR+gR$.

Let $I_1 \subseteq I_2 \subseteq ...$ be a chain of right annihilators. Then, for $I_1 \subseteq I_2$, there are idempotents e and f such that $I_1 = eR$ and $I_2 = fR$, and there is an idempotent g orthogonal to e such that $I_2 = I_1 + gR$. It then follows that $I_3 = I_1 + gR + hR$ for some idempotent h orthogonal to both e and g. Since R does not contain an infinite set of orthogonal idempotents, this must terminate with some $k < \omega$ so that $I_n = I_k$ for every $n \geq k$. Therefore, R satisfies the ascending chain condition on right annihilators. The descending chain condition on right annihilators follows from Lemma 2.10. $\square$

Chapter 3

Homological Algebra

Before discussing torsion-freeness and non-singularity of modules, we need some basic results in Homological Algebra regarding tensor products, flat modules, and functors.

3.1 Tensor Products

Let A be a right R-module, B a left R-module, and G any Abelian group. A function $f : A \times B \to G$ is called *R-biadditive*, or *R-bilinear*, if the following conditions are satisfied:

(i) For each $a, a' \in A$ and $b \in B$, $f(a + a', b) = f(a, b) + f(a', b)$,

(ii) For each $a \in A$ and $b, b' \in B$, $f(a, b + b') = f(a, b) + f(a, b')$,

(iii) For each $a \in A$, $b \in B$, and $r \in R$, $f(ar, b) = f(a, rb)$.

Note that in general $f(a + a', b + b') \neq f(a, b) + f(a', b')$. The *tensor product* of A and B, denoted $A \bigotimes_R B$, is an Abelian group and an R-biadditive function $h : A \times B \to A \bigotimes_R B$ having the universal property that whenever G is an Abelian group and $g : A \times B \to G$ is R-biadditive, there is a unique map $f : A \bigotimes_R B \to G$ such that $g = fh$.

Proposition 3.1. *[9] Let R be a ring. Given a right R-module A and a left R-module B, the tensor product $A \bigotimes_R B$ exists.*

Proof. Let F be a free Abelian group with basis $A \times B$, and let U be a subgroup of F generated by all elements of the form $(a + a', b) - (a, b) - (a', b)$, $(a, b + b') - (a, b) - (a, b')$, or $(ar, b) - (a, rb)$, where $a, a' \in A$, $b, b' \in B$, and $r \in R$. Define $A \bigotimes_R B$ to be F/U, and denote $(a, b) + U \in F/U$ as $a \otimes b$. In addition, let $h : A \times B \to A \bigotimes_R B$ be defined by

$(a, b) \mapsto a \otimes b$. Observe that h is a well-defined R-biadditive map. For if $a, a' \in A$ and $b \in B$, then $h(a + a', b) = (a + a', b) + U = (a + a', b) - [(a + a', b) - (a, b) - (a', b)] + U = [(a, b) + U] + [(a', b) + U] = h(a, b) + h(a', b)$. Similarly, $h(a, b + b') = h(a, b) + h(a, b')$ for $b, b' \in B$, and $h(ar, b) = (ar, b) + U = (ar, b) - [(ar, b) - (a, rb)] + U = (a, rb) + U = h(a, rb)$ for $r \in R$.

Let G be any Abelian group and $g : A \times B \to G$ any R-biadditive map. For F/U to be a tensor product, it needs to be shown that there is a function $\varphi : A \bigotimes_R B = F/U \to G$ such that $g = \varphi h$. Define $\hat{f} : A \times B \to G$ by $(a, b) \mapsto g(a, b)$. Each element of F is of the form $\sum_{A \times B} (a, b)n_{(a,b)}$, where $n_{(a,b)} = 0$ for all but finitely many $(a, b) \in A \times B$. Let f be defined by $\sum_{A \times B} (a, b)n_{(a,b)} \mapsto \sum_{A \times B} \hat{f}[(a, b)]n_{(a,b)}$. This is clearly well-defined since $\hat{f}$ is well-defined. Moreover, $f[(a, b)] = \hat{f}[(a, b)]$ for $(a, b) \in A \times B$, and thus f extends $\hat{f}$ to a function on F. Note that if k is another extension of $\hat{f}$, then k must equal f since they are equal on the generating set $A \times B$. Hence, f is a unique extension. Also observe that f is a homomorphism since, given $x, y \in F$, $f(x + y) = f(\sum_{A \times B} (a, b)n_{(a,b)} + \sum_{A \times B} (a', b')m_{(a,b)})$ $= \sum_{A \times B} \hat{f}[(a, b)]n_{(a,b)} + \sum_{A \times B} \hat{f}[(a', b')]m_{(a,b)} = f(x) + f(y)$.

It readily follows from g being R-biadditive that the homomorphism $f : F \to G$ which we have just constructed is also R-biadditive. To see this, observe that if $a, a' \in A$ and $b \in B$, then $f[(a + a', b)] - f[(a, b)] - f[(a', b)] = g[(a + a', b)] - g[(a, b)] - g[(a', b)] = 0$. The other two conditions are satisfied with similar computation. Thus, we have that $f(U) = 0$. Define $\varphi : F/U = A \bigotimes_R B \to G$ by $\varphi(x + U) = f(x)$. If $x + U = x' + U$, then $x - x' \in U$ and hence $f(x - x') \in f(U) = 0$. Thus, $f(x) = f(x')$ and φ is well-defined. Furthermore, $\varphi h(a, b) = \varphi[a \otimes b] = \varphi[(a, b) + U] = f[(a, b)] = g[(a, b)]$. Therefore $A \bigotimes_R B = F/U$ is a tensor product. □

Proposition 3.2. *Let R be a ring, A a right R-module, and B a left R-module. Then, the tensor product $A \bigotimes_R B$ is unique up to isomorphism.*

Proof. It has already been shown that $A \bigotimes_R B$ exists. Suppose H and H' are both tensor products, and let $h : A \times B \to H$ and $h' : A \times B \to H'$ be the respective R-biadditive functions having the universal property. Then, there exists a function $f : H \to H'$ such that $h' = fh$ and a function $f' : H' \to H$ such that $h = f'h'$. Hence, $h = f'fh$ and $h' = ff'h'$. That is, $f'f \cong 1_H$ and $ff' \cong 1_{H'}$. Therefore, $f : H \to H'$ is an isomorphism. □

Each element of $A \bigotimes_R B$ is a finite sum of the form $\sum_{i=1}^{n}(a_i \otimes b_i)$. The elements $a \otimes b$ that generate $A \bigotimes_R B$ are referred to as *tensors*. Given $a, a' \in A$, $b, b' \in B$, and $r \in R$, the following properties hold for tensors:

(i) $(a + a') \otimes b = a \otimes b + a' \otimes b$,

(ii) $a \otimes (b + b') = a \otimes b + a \otimes b'$,

(iii) $ar \otimes b = a \otimes rb$.

These properties can be proved in a method similar to that used in the proof of Proposition 3.1 to show that $h : A \times B \to A \bigotimes_R B$ defined by $(a, b) \mapsto a \otimes b$ is R-biadditive.

Proposition 3.3. *[9] Let R be a ring, $A, A' \in Mod_R$, and $B, B' \in {}_RMod$. If $f : A \to A'$ and $g : B \to B'$ are R-homomorphisms, then there is an induced map $f \otimes g : A \bigotimes_R B \to A' \bigotimes_R B'$ such that $(f \otimes g)(a \otimes b) = f(a) \otimes g(b)$.*

Proof. Let $h : A \times B \to A \bigotimes_R B$ and $h' : A' \times B' \to A' \bigotimes_R B'$ be the respective R-biadditive maps with the universal tensor property. Define $\varphi : A \times B \to A' \times B'$ by $\varphi(a, b) = (f(a), g(b))$. It then follows that $h'\varphi : A \times B \to A' \bigotimes_R B'$ is R-biadditive. For if $a, a' \in A$ and $b \in B$, then $h'\varphi(a + a', b) = h'(f(a + a'), g(b)) = h'[f(a) + f(a'), g(b)] = h'[f(a), g(b)] + h'[f(a'), g(b)] = h'\varphi(a, b) + h'\varphi(a', b)$. Similarly, $h'\varphi(a, b + b') = h'\varphi(a, b) + h'\varphi(a, b')$ and $h'\varphi(ar, b) = h'\varphi(a, rb)$ for $b' \in B$ and $r \in R$. By the universal property of the R-biadditive map h, there exists a map $\hat{\varphi} : A \bigotimes_R B \to A' \bigotimes_R B'$ such that $h'\varphi = \hat{\varphi}h$. Hence, $\hat{\varphi}(a \otimes b) = \hat{\varphi}h(a, b) = h'\varphi(a, b) = h'[f(a), g(b)] = f(a) \otimes g(b)$. Therefore, $f \otimes g = \hat{\varphi}$ is an induced map satisfying $(f \otimes g)(a \otimes b) = f(a) \otimes g(b)$. □

The following lemmas will be needed in a later section:

Lemma 3.4. *[5] Let R be a ring, A a right R-module, and B a left R-module. If $a \otimes b$ is a tensor in $A \bigotimes_R B$, then $a \otimes b = 0$ if and only if there exists $a_1, a_2, ..., a_k \in A$ and $r_1, r_2, ..., r_k \in R$ such that $a = a_1 r_1 + a_2 r_2 + ... + a_k r_k$ and $r_j b = 0$ for $j = 1, 2, ..., k$.*

Lemma 3.5. *For a left R-module M, there is an R-module isomorphism $\varphi : R \bigotimes_R M \to M$ given by $\varphi(r \otimes m) = rm$. Here, R is viewed as a right R-module. Similarly, $N \bigotimes_R R \cong N$ for a right R-module N.*

Proof. First, observe that $R \times M \xrightarrow{\psi} M$ given by $\psi((r, m)) = rm$ is R-biadditive. Thus, we can define an R-module homomorphism $R \bigotimes_R M \xrightarrow{\varphi} M$ that sends each $r \otimes m \in R \bigotimes_R M$ to rm. In other words, $\varphi(r \otimes m) = \psi(r, m)$. Note that for every $s \in R$, $\varphi(s(r \otimes m)) = \varphi(sr \otimes m) = (sr)m = s(rm) = s\varphi(r \otimes m)$.

Let $\alpha : M \to R \bigotimes_R M$ be defined by $\alpha(m) = 1 \otimes m$. Clearly α is a well-defined R-module homomorphism since $\alpha(m + n) = 1 \otimes (m + n) = 1 \otimes m + 1 \otimes n = \alpha(m) + \alpha(n)$, and $\alpha(rm) = 1 \otimes rm = 1r \otimes m = 1 \otimes m$. It follows that $\alpha\varphi(r \otimes m) = \alpha(rm) = 1 \otimes rm = 1r \otimes m = r \otimes m$, and $\varphi\alpha(m) = \varphi(1 \otimes m) = 1m = m$. Thus, φ is a bijection and hence an R-module isomorphism. □

Lemma 3.6. *[9]If $A \xrightarrow{i} B \xrightarrow{p} C \to 0$ is an exact sequence of left R-modules, then for any right R-module M, $M \bigotimes_R A \xrightarrow{1 \otimes i} M \bigotimes_R B \xrightarrow{1 \otimes p} M \bigotimes_R C \to 0$ is an exact sequence.*

Proof. For $M \bigotimes_R A \xrightarrow{1 \otimes i} M \bigotimes_R B \xrightarrow{1 \otimes p} M \bigotimes_R C \to 0$ to be exact, it needs to be shown that $im(1 \otimes i) = \ker(1 \otimes p)$ and $1 \otimes p$ is surjective. Since $im(i) = \ker(p)$ and hence $pia = 0$ for every $a \in A$, it readily follows that $im(1 \otimes i) \subseteq \ker(1 \otimes p)$. For if $\sum(m_j \otimes a_j) \in M \bigotimes_R A$, then $(1 \otimes p)(1 \otimes i)[\sum(m_j \otimes a_j)] = (1 \otimes p)[\sum(1 \otimes i)(m_j \otimes a_j)] = (1 \otimes p)[\sum(m_j \otimes ia_j)] = \sum(1 \otimes p)(m_j \otimes ia_j) = \sum(m_j \otimes pia_j) = \sum(m_j \otimes 0) = 0$. To see that $im(1 \otimes i) = \ker(1 \otimes p)$, first note that since $im(1 \otimes i)$ is contained in the kernel of $1 \otimes p$, there is a uniqe homomorphism $\varphi : M \bigotimes_R B / im(1 \otimes i) \to M \bigotimes_R C$ such that $\varphi[(m \otimes b) + im(1 \otimes i)] = (1 \otimes p)(m \otimes b) = m \otimes pb$ [8, Ch. IV, Theorem 1.7].

It can be shown that φ is an isomorphism, and from this it will follow that $im(1 \otimes i) = \ker(1 \otimes p)$. Note that since the sequence $A \xrightarrow{i} B \xrightarrow{p} C \to 0$ is exact and hence p is surjective, for every $c \in C$ there exists an element $b \in B$ such that $pb = c$. Let the function $f : M \times C \to M \bigotimes_R B/im(1 \otimes i)$ be defined by $(m, c) \mapsto p \otimes b$. If there is another element $b_0 \in B$ such that $pb_0 = c$, then $p(b - b_0) = pb - pb0 = c - c = 0$. Hence, $b - b_0 \in \ker(p) = im(i)$. Thus, there is an $a \in A$ such that $ia = b - b_0$, and it then follows that $m \otimes b - m \otimes b_0 = m \otimes (b - b_0) = m \otimes ia \in im(1 \otimes i)$. Hence, $(m \otimes b - m \otimes b_0) + im(1 \otimes i) = 0$, and therefore f is well-defined. Furthermore, it is easily seen that f is an R-biadditive function. Thus, if $h : (m, c) \mapsto m \otimes c$ is the biadditive function of the tensor product, then there is a homomorphism $\psi : M \bigotimes_R C \to M \bigotimes_R B/im(1 \otimes i)$ such that $\psi h = f$. In other words, $\psi(m \otimes c) = (m \otimes b) + im(1 \otimes i)$.

Observe that $\psi\varphi[(m \otimes b) + im(1 \otimes i)] = \psi(m \otimes pb) = \psi(m \otimes c) = (m \otimes b) + im(1 \otimes i)$ and $\varphi\psi(m \otimes c) = \varphi[(m \otimes b) + im(1 \otimes i)] = m \otimes pb = m \otimes c$. Thus, φ is an isomorphism with inverse ψ. Now, let $\pi : M \bigotimes_R B \to M \bigotimes_R B/im(1 \otimes i)$ be the canonical epimorphism given by $m \otimes b \mapsto m \otimes b + im(1 \otimes i)$. Then, $\varphi\pi(m \otimes b) = \varphi[(m \otimes b) + im(1 \otimes i)] = m \otimes pb = (1 \otimes p)(m \otimes b)$. Hence, $\varphi\pi = 1 \otimes p$. Therefore, since φ is an isomorphism, $\ker(1 \otimes p) = \ker(\varphi\pi) = \ker(\pi) = im(1 + i)$.

Finally, it needs to be shown that $1 \otimes p$ is surjective. Let $\sum(m_j \otimes c_j) \in M \bigotimes_R C$. Since p is surjective, for each j, there exists an element $b_j \in B$ such that $pb_j = c_j$. Thus, $(1 \otimes p)[\sum(m_j \otimes b_j)] = \sum(1 \otimes p)(m_j \otimes b_j) = \sum(m_j \otimes pb_j) = \sum(m_j \otimes c_j)$. Therefore, $1 \otimes p$ is surjective and the sequence $M \bigotimes_R A \xrightarrow{1 \otimes i} M \bigotimes_R B \xrightarrow{1 \otimes p} M \bigotimes_R C \to 0$ is exact. □

A right R-module M is *flat* if $0 \to M \bigotimes_R A \xrightarrow{1_M \otimes \varphi} M \bigotimes_R B \xrightarrow{1_M \otimes \psi} M \bigotimes_R C \to 0$ is an exact sequence of Abelian groups whenever $0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$ is an exact sequence of left R-modules.

Proposition 3.7. *[9] Let R be a ring and let $\{M_i\}_{i \in I}$ be a collection of right R-modules for some index set I. Then, the direct sum $\bigoplus_I M_i$ is flat if and only if M_i is flat for every $i \in I$. Moreover, R is flat as a right R-module, and any projective right R-module P is flat.*

Proof. First note that if $0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$ is an exact sequence of left R-modules, then $M \bigotimes_R A \xrightarrow{1_M \otimes \varphi} M \bigotimes_R B \xrightarrow{1_M \otimes \psi} M \bigotimes_R C \to 0$ is exact by Lemma 3.6. Thus, M is flat if and only if $1_M \otimes \varphi$ is a monomorphism whenever φ is a monomorphism.

Suppose A and B are left R-modules and let $\varphi : A \to B$ be a monomorphism. For $\bigoplus_I M_i$ to be flat, it needs to be shown that $1 \otimes \varphi : (\bigoplus_I M_i) \bigotimes_R A \to (\bigoplus_I M_i) \bigotimes_R B$ is a monomorphism. By [9, Theorem 2.65], there exist isomorphisms $f : (\bigoplus_I M_i) \bigotimes_R A \to (\bigoplus_I M_i \bigotimes_R A)$ and $g : (\bigoplus_I M_i) \bigotimes_R B \to (\bigoplus_I M_i \bigotimes_R B)$ defined by $f : (x_i) \otimes a \mapsto (x_i \otimes a)$ and $g : (x_i) \otimes b \mapsto (x_i \otimes b)$. Furthermore, since $1_{M_i} \otimes \varphi$ is a homomorphism for each $i \in I$, there is a homomorphism $\psi : \bigoplus_I (M_j \bigotimes_R A) \to \bigoplus_I (M_j \bigotimes_R B)$ such that $(x_i \otimes a) \mapsto (x_i \otimes \varphi(a))$. Observe that ψ is a monomorphism if and only if $1_{M_i} \otimes \varphi$ is a monomorphism for each $i \in I$. It then follows that $\psi f = g(1 \otimes \varphi)$ since $\psi f[(x_i) \otimes a] = \psi(x_i \otimes a) = x_i \otimes \varphi(a) = g[(x_i) \otimes \varphi(a)] = g(1 \otimes \varphi)[(x_i \otimes a)]$. Therefore, $\bigoplus_I M_i$ is flat if and only if $1 \otimes \varphi$ is a monomorphism if and only if ψ is a monomorphism if and only if $1_{M_i} \otimes \varphi$ is a monomorphism for each i if and only if M_i is flat for each i.

To see that R is flat as a right R-module, note that Lemma 3.5 gives isomorphisms $f : A \to R \bigotimes_R A$ and $g : B \to R \bigotimes_R B$ defined by $f(a) = 1_R \otimes a$ and $g(b) = 1_R \otimes b$. Observe that $(1_R \otimes \varphi) f(a) = (1_R \otimes \varphi)(1_R \otimes a) = 1_R \otimes \varphi(a) = g\varphi(a)$. Hence, $(1_R \otimes \varphi) = g\varphi f^{-1}$, which is a monomorphism. Therefore, R is flat as a right R-module.

Let P be a projective right R-module. Then there is a free right R-module F and an R-module N such that $F = P \bigoplus N$. As a free module, F is a direct sum of copies of R, which is flat. Hence, F is also flat. Therefore, P is flat as a direct summand of F. ☐

3.2 Bimodules and the Hom and Tensor Functors

Let A be a right R-module. Consider the functor $T_A : {}_R Mod \to Ab$ defined by $T_A(B) = A \bigotimes_R B$ with induced map $T_A(\varphi) = 1_A \otimes \varphi : A \bigotimes_R B \to A \bigotimes_R B'$, where Ab is the category of all Abelian groups and $\varphi \in Hom_R(B, B')$ for left R-modules B and B'. Observe that $T_A(\varphi)(a \otimes b) = a \otimes \varphi(b)$. T_A is sometimes denoted $T_A(\_) = A \bigotimes_R \_$. Similarly, the functor

$T_B(A) = A \bigotimes_R B$ with induced map $\psi \otimes 1_B$ can be defined for a left R-module B and $\psi \in Hom_R(A, A')$. We also consider the functor $Hom_R(A, \_) : Mod_R \to Ab$ with induced map $f_* : Hom_R(A, B) \to Hom_R(A, C)$ defined by $f_*(h) = fh$, where $f : B \to C$ is a homomorphism for right R-modules B and C.

Let R and S be rings and let M be an Abelian group which has both a left R-module structure and a right S-module structure. Then, M is an (R, S)-*bimodule* if $(rx)s = r(xs)$ for every $r \in R$, $s \in S$, and $x \in M$. This is sometimes denoted ${}_RM_S$. In particular, if A is a right R-module and $E = End_R(A)$, then M is an (E, R)-bimodule. Note that for $x \in M$ and $\alpha \in E$, scalar multiplication αx is defined as $\alpha(x)$.

Proposition 3.8. *Let R and S be rings. Suppose M is an (R, S)-bimodule and N is a right S-module. Then, $Hom_S(M_S, N_S)$ is a right R-module and $Hom_S(N_S, M_S)$ is a left R-module.*

Proof. First, observe that $Hom_S(M_S, N_S)$ is an Abelian group. For if $f, g \in Hom_S(M_S, N_S)$, then $f(xr) = f(x)r$ and $g(xr) = g(x)r$ for every $r \in R$. Hence, $f + g \in Hom_S(M_S, N_S)$ since $(f+g)(xr) = f(xr)+g(xr) = f(x)r+g(x)r = (f+g)(x)r$. Moreover, if $h \in Hom_S(M_S, N_S)$, then $[f + (g + h)](x) = f(x) + (g + h)(x) = f(x) + g(x) + h(x) = (f + g)(x) + h(x) = [(f + g) + h](x)$. Hence, $Hom_S(M_S, N_S)$ is associative. Furthermore, the map $\alpha : a \mapsto 0$ acts as the zero element. Finally, note that if $f \in Hom_S(M_S, N_S)$, then $g : M \to N$ defined by $g(x) = -f(x)$ is such that $(f + g)(x) = f(x) + g(x) = f(x) - f(x) = 0$. Hence, every element of $Hom_S(M_S, N_S)$ has an inverse. Therefore, $Hom_S(M_S, N_S)$ is an Abelian group.

Now, let $\varphi \in Hom_S(M_S, N_S)$, $r, r' \in R$, and $x \in M$. Define the right R-module structure on $Hom_S(M_S, N_S)$ by $(\varphi r)(x) = \varphi(rx)$. Then, $(\varphi + \psi)(r)(x) = (\varphi r + \psi r)(x) = (\varphi r)(x) + (\psi r)(x) = \varphi(rx) + \psi(rx) = (\varphi + \psi)(rx)$ for $\psi \in Hom_S(M_S, N_S)$. Moreover, $[\varphi(r + r')](x) = \varphi[(r + r')x] = \varphi[rx + r'x] = \varphi(rx) + \varphi(r'x) = (\varphi r)(x) + (\varphi r')(x)$ for $r' \in R$. Finally, observe that $[\varphi(rr')](x) = \varphi[(rr')(x)] = \varphi[r(r'x)] = (\varphi r)(r'x)$. Therefore, $Hom_S(M_S, N_S)$ satisfies the conditions of a right R-module. Similarly, $Hom_S(N_S, M_S)$ is a left R-module with $(r\pi)(x) = r\pi(x)$ for any $\pi \in Hom_S(N_S, M_S)$. □

Proposition 3.9. *[9] Let R be a subring of S. Suppose M is an (R,S)-bimodule and A is a right R-module. Then, $A \bigotimes_R M$ is a right S-module. In particular, S is an (R,S)-bimodule and hence $A \bigotimes_R S$ is a right S-module.*

Proof. Let $y = \sum_{i=1}^{n}(a_i \otimes x_i) \in A \bigotimes_R M$ and let $s \in S$. Define the right S-module structure on $A \bigotimes_R M$ by $(\sum_{i=1}^{n}(a_i \otimes x_i))s = \sum_{i=1}^{n}(a_i \otimes x_i s)$. To see that this does define a right S-module, consider the well-defined map $\mu_s : M \to M$ defined by $\mu_s(x) = xs$. By the bimodule structure of M, $r\mu_s(x) = r(xs) = (rx)s = \mu_s(rx)$ for $r \in R$. Hence, $\mu_s \in Hom_R(M,M)$. Consider the functor $T_A(\_) = A \bigotimes_S \_$. By Proposition 3.3, there is a well-defined homomorphism $T_A(\mu_s) = 1_A \otimes \mu_s : A \bigotimes_R M \to A \bigotimes_R M$ such that $(1_A \otimes \mu_s)(a \otimes x) = a \otimes \mu_s(x) = a \otimes xs$. If the element ys is defined by $ys = (1_A \otimes \mu_s)(y) = (1_A \otimes \mu_s)(\sum_{i=1}^{n}(a_i \otimes x_i)) = \sum_{i=1}^{n}(1_A \otimes \mu_s)(a_i \otimes x_i) = \sum_{i=1}^{n}(a_i \otimes x_i s)$, then the S-module structure is well-defined since $(1_A \otimes \mu_s)$ is a well-defined homomorphism and $\sum_{i=1}^{n}(a_i \otimes x_i s) \in A \bigotimes_R M$. The remaining right S-module conditions follow readily. Moreover, it is easy to see that S satisfies the conditions of an (R,S)-bimodule. Therefore, given any right R-module A, $A \bigotimes_R S$ is a right S-module. □

Proposition 3.10. *Let $R \leq S$ be rings and let M be an (R,S)-bimodule. Then, the following hold:*

(a) The functor $T_M(\_) = \_ \bigotimes_R M : Mod_R \to Ab$ is actually a functor $Mod_R \to Mod_S$.

(b) The functor $Hom_S(M, \_) : Mod_S \to Ab$ is actually a functor $Mod_S \to Mod_R$.

Proof. (a): It has already been shown in Proposition 3.9 that $T_M(A) = A \bigotimes_R M$ is a right S-module for any right R-module A. It needs to be shown that if $\psi \in Hom_R(A, A')$ for $A' \in Mod_R$, then $T_M(\psi) = \psi \otimes 1_M \in Hom_S(A \bigotimes_R M, A' \bigotimes_R M)$. In other words, it needs to be shown that $\psi \otimes 1_M$ is an S-homomorphism. Let $s \in S$. Then, $(\psi \otimes 1_M)(a \otimes x)s =$

$(\psi(a) \otimes x)s = \psi(a) \otimes xs = (\psi \otimes 1_M)(a \otimes xs) = (\psi \otimes 1_M)[(a \otimes x)s]$. Thus, $T_M(\psi)$ is a morphism in Mod_S, and therefore $T_M(\_)$ is a functor with values in Mod_S.

(b): Given any right S-module N, $Hom_S(M, N)$ is a right R-module by Proposition 3.8. It needs to be shown that if $f : N \to N'$ is a homomorphism for $N, N' \in Mod_S$, then the induced map $f_* = Hom_R(M, f) : Hom_S(M, N) \to Hom_S(M, N')$ defined by $f_*(\varphi) = f\varphi$ is an R-homomorphism. Note that if $\varphi, \psi \in Hom_S(M, N)$, then $f(\varphi+\psi) = f\varphi+f\psi$. Hence, f_* is a homomorphism since $f_*(\varphi+\psi) = f(\varphi+\psi) = f\varphi+f\psi = f_*\varphi+f_*\psi$. Let $r \in R$. Observe that $(\varphi r)(x) = \varphi(rx)$ by Proposition 3.8. Moreover, since M has a left R-module structure and $f\varphi$ is an element of the right R-module $Hom_S(M, N')$, Proposition 3.8 also shows that $[f\varphi(x)]r = f[\varphi r](x) = f\varphi(rx)$ for $x \in M$. Thus, $[f_*(\varphi(x))]r = [f\varphi(x)]r = f\varphi(rx) = f_*[\varphi(rx)] = f_*[(\varphi r)(x)]$. Hence, f_* is an R-homomorphism, and therefore $Hom_S(M, \_)$ is a functor with values in Mod_R. □

The following lemmas will be used later to show $Mod_R \cong Mod_{Mat_n}(R)$. The proofs are omitted and can be found in *Rings and Categories of Modules* by Frank Anderson and Kent Fuller.

Lemma 3.11. *[2, Proposition 20.10] Let R and S be rings, M a right R-module, N a right S-module, and P an (S, R)-bimodule. If M is finitely generated and projective, then $\mu : N \bigotimes_S Hom_R(M, P) \to Hom_R(M, N \bigotimes_S P)$ defined by $\mu(y \otimes f)(x) = y \otimes f(x)$ is a natural isomorphism. Here, $x \in M$, $y \in N$, and $f \in Hom_R(M, P)$.*

Lemma 3.12. *[2, Proposition 20.11] Let R and S be rings, M a right R-module, N a left S-module, and P an (S, R)-bimodule. If M is finitely generated and projective, then $\nu : Hom_R(P, M) \bigotimes_S N \to Hom_R(Hom_S(N, P), M)$ defined by $\nu(f \otimes y)(g) = fg(y)$ is a natural isomorphism. Here, $f \in Hom_R(P, M)$, $g \in Hom_S(N, P)$, and $y \in N$.*

3.3 The Tor Functor

Consider the exact sequence $P = \cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} A \to 0$ of right R-modules, where P_j is projective for every j. Such an exact sequence is called a *projective resolution* of the right R-module A. Note that a projective resolution can be formed for any projective right R-module A since every right R-module is the epimorphic image of a projective right R-module. Define the *deleted projective resolution*, denoted P_A, by removing the morphism ϵ and the right R-module A. Note that the projective resolution is an exact sequence, and hence $im(d_{i+1}) = \ker(d_i)$. Therefore, $d_i d_{i+1} = 0$ for every $i \in \mathbb{Z}^+$, and thus the projective resolution P and the deleted projective resolution P_A are both complexes. However, P_A is not necessarily exact since $im(d_1) = \ker(\epsilon)$, which may not equal the kernel of the morphism $P_0 \to 0$. Now, we can form the *induced complex* TP_A, which is defined as $\cdots \to T(P_2) \xrightarrow{T(d_2)} T(P_1) \xrightarrow{T(d_1)} T(P_0) \to 0$.

For $n \in \mathbb{Z}$, the n^{th} *homology* is $H_n(C) = Z_n(C)/B_n(C)$, where C is a complex, $Z_n(C) = \ker(d_n)$, and $B_n(C) = im(d_{n+1})$. Hence, $H_n(C) = \ker(d_n)/im(d_{n+1})$. If we consider the deleted projective resolution P_A as defined above, then $\cdots \to P_2 \bigotimes_R B \xrightarrow{d_2 \otimes 1_B} P_1 \bigotimes_R B \xrightarrow{d_1 \otimes 1_B} P_0 \bigotimes B \to 0$ is the induced complex $T_B P_A$ of the functor $T_B(\_) = \_ \bigotimes_R B$. Define the *Tor functor* to be $Tor_n^R(A, B) = H_n(T_B P_A) = \ker(d_n \otimes 1_B)/im(d_{n+1} \otimes 1_B)$. Note that $Tor_n^R(A, B)$ does not depend on the choice of projective resolution [9]. The functor $Tor_n^R(A, \_)$ is referred to as the *left derived functor* of $A \bigotimes_R B$. The following two well-known propositions will be useful later:

Proposition 3.13. *[9] If $M \in Mod_R$ and $0 \to A \to B \to C \to 0$ is an exact sequence of left R-modules, then the induced sequence* $... \to Tor_{n+1}^R(M, C) \to Tor_n^R(M, A) \to Tor_n^R(M, B) \to Tor_n^R(M, C) \to ... \to Tor_1^R(M, C) \to M \bigotimes_R A \to M \bigotimes_R B \to M \bigotimes_R C \to 0$ *is exact.*

Proposition 3.14. *[9] A right R-module M is flat if and only if $Tor_n^R(M, X) = 0$ for every left R-module X and every $n \geq 1$.*

Chapter 4

Torsion-free Rings and Modules

In 1960, Hattori used the homological properties of classical torsion-free modules over integral domains to give a more general definition of torsion-freeness. He defines a right R-module M to be *torsion-free* if $Tor_1^R(M, R/Rr) = 0$ for every $r \in R$, and he defines a left R-module N to be *torsion-free* if $Tor_1^R(R/sR, N) = 0$ for every $s \in R$ [7]. The following equivalent definition of torsion-freeness is also given by Hattori in [7, Proposition 1]:

Proposition 4.1. *[7] The following are equivalent for a right R-module M.*

(a) M is torsion-free

(b) For each $x \in M$ and $r \in R$, $xr = 0$ implies the existence of $x_1, x_2, ..., x_k \in M$ and $r_1, r_2, ...r_k \in R$ such that $x = \sum_{j=1}^{k} x_j r_j$ and $r_j r = 0$ for every $j = 1, 2, ..., k$.

Proof. Consider the exact sequence $0 \to Rr \xrightarrow{\iota} R \xrightarrow{\pi} R/Rr \to 0$ of left R-modules, where ι is the inclusion map and π is the epimorphism $r \mapsto r + Rr$. This induces a long exact sequence $X = ... \to Tor_1^R(M, R/Rr) \xrightarrow{f} M \bigotimes_R Rr \xrightarrow{1_M \otimes \iota} M \bigotimes_R R \cong M \xrightarrow{1_M \otimes \pi} M \bigotimes_R R/Rr \to 0$ [9, Corollary 6.30]. Observe that condition (b) is equivalent to $1_M \otimes \iota$ being a monomorphism. For if $1_M \otimes \iota : x \otimes r \mapsto xr$ is a monomorphism, then $xr = 0$ implies $x \otimes r = 0$. Hence, there exists $x_1, x_2, ..., x_k \in M$ and $r_1, r_2, ..., r_k \in R$ such that $x = x_1r_1 + x_2r_2 + ... + x_kr_k$ and $r_jr = 0$ for $j = 1, 2, ..., k$ by Lemma 3.4. On the other hand, if $xr = 0$ implies $x = x_1r_1 + x_2r_2 + ... + x_kr_k$ and $r_jr = 0$, then $x \otimes r = x_1r_1 + x_2r_2 + ... + x_kr_k \otimes r = x_1 \otimes r_1r + x_2 \otimes r_2r + ... + x_k \otimes r_kr = 0$. Hence, $\ker(1_M \otimes \iota) = 0$ and $1_M \otimes \iota$ is a monomorphism.

To complete the proof, it needs to be shown that M is torsion-free if and only if $1_M \otimes \iota$ is a monomorphism. If M is torsion-free, then $Tor_1^R(M, R/Rr) = 0$. Thus, $0 \to M \bigotimes_R Rr \xrightarrow{1_M \otimes \iota}$

$M \bigotimes_R R \cong M \xrightarrow{1_M \otimes \pi} M \bigotimes_R R/Rr \to 0$ is exact and so $1_M \otimes \iota$ is a monomorphism. Conversely, if $1_M \otimes \iota$ is a monomorphism, then $im(f) = \ker(1_M \otimes \iota) = 0$ in the induced sequence X. However, f is a monomorphism. Hence, $0 = im(f) \cong Tor_1^R(M, R/Rr)$. □

A ring R is *torsion-free* if every finitely generated right (or left) ideal is torsion-free as a right (or left) R-module. Hattori shows in [7] that a ring R is torsion-free if and only if every principal left ideal of R is flat. To see this, observe that if $0 \to J \xrightarrow{i} R \xrightarrow{p} R/J \to 0$ is an exact sequence of right R-modules with J finitely generated, then $0 \to J \bigotimes_R Rr \xrightarrow{i \otimes 1_{Rr}} R \bigotimes_R Rr \xrightarrow{p \otimes 1_{Rr}} R/J \bigotimes_R Rr \to 0$ is an exact sequence whenever Rr is flat. This is the case if and only if $Tor_1^R(R/J, Rr) = 0$. Hattori gives a natural isomorphism in [7, Proposition 7] showing that $Tor_1^R(R/J, Rr) \cong Tor_1^R(J, R/Rr)$. Hence, $Tor_1^R(J, R/Rr) = 0$ if and only if Rr is flat for every $r \in R$. That is, every finitely generated right ideal is torsion-free if and only if every principal left ideal is flat.

In 2004, John Dauns and Lazlo Fuchs provided the following useful characterization of torsion-free rings:

Theorem 4.2. *[4]The following are equivalent for a ring R:*

(a) R is torsion-free.

(b) For every $s, r \in R$, $sr = 0$ if and only if $s \in s \cdot ann_l(r)$. In other words, $sr = 0$ if and only if $s = su$ and $ur = 0$ for some $u \in R$.

Proof. (a) ⇒ (b): Suppose R is a torsion-free ring. For $s \in R$, sR is torsion-free as a right R-module. By Proposition 4.1, if $a \in sR$ and $r \in R$ with $ar = 0$, then there exists $u \in R$ so that $a = su$ and $ur = 0$. Hence, if $sr = 0$, we have $s = su$ and $ur = 0$ for some $u \in R$, since $s = s \cdot 1 \in sR$. Conversely, if there is some $u \in R$ such that $s = su$ and $ur = 0$, then $sr = (su)r = s(ur) = s \cdot 0 = 0$. Therefore, $sr = 0$ if and only if $s = su$ and $ur = 0$ for some $u \in R$.

(b) ⇒ (a): Assume that $sr = 0$ for every $s, r \in R$ if and only if $s = su$ and $ur = 0$ for some $u \in R$. Let Rr be a finitely generated left ideal of R. Assume that the sequence

$0 \to J \to R \to R/J \to 0$ is exact with J finitely generated. Then, R is a torsion-free ring if $0 \to J \bigotimes_R Rr \xrightarrow{\varphi} R \bigotimes_R Rr \xrightarrow{\psi} R/J \bigotimes_R Rr \to 0$ is exact. By Lemma 3.6, it follows that $J \bigotimes_R Rr \xrightarrow{\varphi} R \bigotimes_R Rr \xrightarrow{\psi} R/J \bigotimes_R Rr \to 0$ is exact. In order for the entire sequence to be exact, it needs to be shown that φ is a monomorphism.

Note that $R \bigotimes_R Rr \cong Rr$ by Lemma 3.5. Consider $j \otimes sr \in J \bigotimes_R Rr$. Since $j \otimes sr = js \otimes r$ and $js \in J$, tensors in $J \bigotimes_R Rr$ can be written as $k \otimes r$ for some $k \in J$. Thus, it needs to be shown that $J \bigotimes_R Rr \xrightarrow{\varphi} Rr$ given by $\varphi(k \otimes r) = kr$ is a monomorphism. Let $k \otimes r \in \ker \varphi$. Then $\varphi(k \otimes r) = kr = 0$. By assumption, there exists some $u \in R$ such that $k = ku$ and $ur = 0$. Then, $k \otimes r = ku \otimes r = k \otimes ur = k \otimes 0 = 0$. Thus, $\ker \varphi = 0$ and φ is a monomoprhism. Therefore, $0 \to J \bigotimes_R Rr \xrightarrow{\varphi} R \bigotimes_R Rr \xrightarrow{\psi} R/J \bigotimes_R Rr \to 0$ is an exact sequence, and hence R is a torsion-free ring. □

Proposition 4.3. *[7, Proposition 7] A ring R is torsion-free if and only if every submodule of a torsion-free right R-module is torsion-free.*

Proof. Suppose R is torsion-free and let N be a submodule of a torsion-free right R-module M. Consider the exact sequence $0 \to N \xrightarrow{\iota} M \xrightarrow{\pi} M/N \to 0$, where ι is the inclusion map and π is the canonical epimorphism. As noted above, if R is torsion-free, then the principal left ideal Rr is flat for every $r \in R$. Hence, $0 \to N \bigotimes_R Rr \to M \bigotimes_R Rr \to M/N \bigotimes_R Rr \to 0$ is exact and so $Tor_1^R(M/N, Rr) \cong 0$. Observe that $Tor_1^R(M, R/Rr) \cong 0$ since M is torsion-free. If we consider the long exact sequence derived from the functor $Tor_n^R(\_, R/Rr)$, then $0 \cong Tor_1^R(M/N, Rr) \cong Tor_2^R(M/N, R/Rr) \to Tor_1^R(N, R/Rr) \to Tor_1^R(M, R/Rr) \cong 0$ is exact. Therefore, $Tor_1^R(N, R/Rr) = 0$ and N is torsion-free. On the other hand, if every submodule of a torsion-free right R-module is torsion-free, then every finitely generated right ideal of R is torsion-free since R itself is torsion-free as a right R-module. □

Theorem 4.4. *[4]A ring R is a right p.p.-ring if and only if R is torsion-free and, for each* $x \in R$, $ann_r(x)$ *is finitely generated.*

Proof. Suppose R is a right p.p.-ring. Then, for each $r \in R$, $ann_r(r) = eR$ for some idempotent $e \in R$. Let $s \in R$ be such that $rs = 0$. Then, $s \in ann_r(r)$, and hence $s = es'$ for some $s' \in R$. It follows that $es = e^2s' = es' = s$. Furthermore, $e = e^2 \in eR = ann_r(r)$ and hence $re = 0$. Note also that if $s = es$ and $re = 0$, then $s \in eR = ann_r(r)$ and hence $rs = 0$. Thus, $rs = 0$ if and only if $s = es$ and $re = 0$. Therefore, R is a torsion-free ring by a symmetric version of Theorem 4.2. Moreover, since R is a right p.p.-ring, $ann_r(r)$ is generated by an idempotent and thus finitely generated.

Conversely, suppose R is a torsion-free ring and the right annihilator of every element of R is finitely generated. Let $s \in R$ and let $\{s_1, ..., s_n\}$ be the finite set of generators for $ann_r(s)$. Note that each $s_i \in ann_r(s)$, and so $ss_i = 0$ for each $i = 1, ..., n$. Let $S = \bigoplus^{n} R$ be the direct sum of n copies of R, and consider S as a left R-module. Let $s' = (s_1, ...s_n) \in S$. Note that S is a torsion-free left R-module since it is the direct sum of copies of R, which is torsion-free as a left R-module. Thus, the submodule Rs' of S is torsion-free by Proposition 4.3. Hence, Proposition 4.1 gives some $u \in R$ such that $s' = us'$ and $su = 0$, and thus $u \in ann_r(s)$. Note that $s_i = us_i$ for each $i = 1, ..., n$. This implies that $s_i \in uR$ for each i, and so $\{s_1, ..., s_n\} \subseteq uR$. It follows that $ann_r(s) = s_1R + ... + s_nR \subseteq uR$. Suppose $x \in uR$. Then, $x = ut$ for some $t \in R$. Thus, $sx = sut = 0 \cdot t = 0$, and so $x \in ann_r(s)$. Therefore, $ann_r(s) = uR$.

Now, since R is a torsion-free ring, uR is torsion-free as a finitely generated right ideal of R. By a symmetric version of Theorem 4.2, since $su = 0$, there exists an $e \in uR = ann_r(s)$ such that $u = eu$ and $se = 0$. Let $x \in uR$. Then, $x = ut = eut \in eR$ for some $t \in R$. Hence, $uR \subseteq eR$. On the other hand, suppose $y \in eR$. Then, for some $v \in R$, $y = ev$ and $sy = sev = 0 \cdot v = 0$. Thus, $y \in ann_r(s)$ and $eR \subseteq ann_r(s) = uR$. Hence, $ann_r(s) = uR = eR$ and $e = ur$ for some $r \in R$. It then follows that e is an idempotent since $e^2 = eux = ux = e$. Therefore, $ann_r(s)$ is generated by an idempotent and so R is a right p.p.-ring. □

Lemma 4.5. *If R is a right p.p.-ring and $e \in R$ is a nonzero idempotent, then $eR = ann_r(x)$ for some $x \in R$. In particular, $eR = ann_r(1 - e)$.*

Proof. If $er \in eR$, then $(1-e)er = (e-e^2)r = (e-e)r = 0$. Hence, $er \in ann_r(1-e)$ and $eR \subseteq ann_r(1-e)$. On the other hand, if $s \in ann_r(1-e)$, then $(1-e)s = 0$. Hence, $s - es = 0$, and so $s = es \in eR$. Therefore, $eR = ann_r(1-e)$. □

Proposition 4.6. *[1] If R is a right and left p.p.-ring which does not contain an infinite set of orthogonal idempotents and M is a torsion-free right R-module, then $ann_r(x)$ is generated by an idempotent for every $x \in M$.*

Proof. Let R be a right and left p.p.-ring which does not contain an infinite set of orthogonal idempotents. Take M to be a torsion-free right R-module and let $A = ann_r(x)$ for some nonzero $x \in M$. Suppose $r_0 \in R$ is such that $xr_0 = 0$. Note that the cyclic submodule xR is torsion-free since R is a right p.p.-ring. Moreover, $ann_l(r_0) = Re_0$ for some idempotent $e_0 \in R$ since R is a left p.p.-ring. By Proposition 4.1, there exists $xs_1, xs_2, ..., xs_n \in xR$ and $t_1e_0, t_2e_0..., t_ne_0 \in Re_0 = ann_l(r_0)$ such that $x = xs_1t_1e_0 + xs_2t_2e_0 + ... + xs_nt_ne_0$. Hence, $xe_0 = xs_1t_1e_0^2 + xs_2t_2e_0^2 + ... + xs_nt_ne_0^2 = x$. Thus, $0 = x - xe_0 = x(1-e_0)$. Therefore, if $(1-e_0)r \in (1-e_0)R$, then $x(1-e_0)r = 0$ and $(1-e_0)R \subseteq A$.

Now, if there exists some $r_1 \in A \backslash (1-e_0)R$, then $r_1 \neq (1-e_0)r_1$ and hence $e_0r_1 \neq 0$. However, $xe_0r_1 = xr_1 = 0$. Since R is a left p.p.-ring, $ann_l(e_0r_1) = R(1-f)$ for some idempotent $1-f$. Note that as before it follows from Proposition 4.1 that $x = x(1-f)$ since $xe_0r_1 = 0$. Furthermore, $1 - e_0 \in ann_l(e_0r_1) = R(1-f)$ since $(1-e_0)e_0r_1 = e_0r_1 - e_0r_1 = 0$. Hence, there is some $r \in R$ such that $(1-e_0)f = r(1-f)f = r(f-f) = 0$. Thus, $e_0f = f$. Let $e_1 = (1-f)e_0 = e_0 - fe_0$. Then, $e_1^2 = (e_0 - fe_0)(e_0 - fe_0) = e_0 - e_0fe_0 - fe_0 + fe_0fe_0 = e_0 - fe_0 - fe_0 + fe_0 = e_0 - fe_0 = e_1$. Thus, e_1 is an idempotent. Moreover, e_1 is nonzero, since otherwise $e_0 = fe_0$ and hence $e_0 = 0$.

Now, $e_1e_0 = (1-f)e_0e_0 = (1-f)e_0 = e_1$, and Lemma 4.5 shows that $(1-e_0)R = ann_r(e_0)$ and $(1-e_1)R = ann_r(e_1)$. Thus, if $r \in ann_r(e_0)$, then $e_1r = e_1e_0r = 0$. Hence, $r \in ann_r(e_1) = (1-e_1)R$, and so $(1-e_0)R \subseteq (1-e_1)R$. Moreover, $e_1e_0r_1 = e_1r_1 = (1-f)e_0r_1 = 0$ since $1-f \in ann_r(e_0r_1)$. Thus, $e_0r_1 \in ann_r(e_1) = (1-e_1)R$. However, e_0r_1 is nonzero and hence $e_0r_1 \notin ann_r(e_0) = (1-e_0)R$. Thus, $(1-e_0)R \subset (1-e_1)R$ is a proper inclusion. By supposing there is some $r_2 \in A\backslash(1-e_1)R$ and repeating these steps, and then supposing there is some $r_3 \in A\backslash(1-e_2)R$ and so on, we can construct an ascending chain $(1-e_0)R \subset (1-e_1)R \subset (1-e_2)R \subset ....$ However, this chain must terminate at some point since R only contains finite sets of orthogonal idempotents. Therefore, there is some idempotent $e \in R$ such that $A = (1-e)R$. □

Proposition 4.7. *[1] If R is a right and left p.p.-ring not containing an infinite set of orthogonal idempotents, then a cyclic submodule of a torsion-free right R-module is projective.*

Proof. Let M be a torsion-free right R-module, and take N to be a cyclic submodule of M. Then, N is of the form xR for some $x \in N \leq M$. By Proposition 4.6, $ann_r(x) = eR$ for some idempotent $e \in R$. If $f : R \to xR$ is the epimorphism defined by $r \mapsto xr$, then $xR \cong R/\ker(f) = R/ann_r(x)$ by the First Isomorphism Theorem. It then follows that $xR \cong R/ann_r(x) \cong [eR \bigoplus (1-eR)]/ann_r(x) \cong [eR \bigoplus (1-e)R]/eR \cong (1-e)R$. Therefore, N is a principal right ideal of R, and thus projective, since R is a right p.p.-ring. □

A ring R is a *Baer-ring* if $ann_r(A)$ is generated by an idempotent for every subset A of R. Note that if R is Baer, then $ann_r(ann_l(A)) = eR$ for some idempotent $e \in R$. Hence, $ann_l(A) = ann_l(ann_r(ann_l(A))) = ann_l(eR) = R(1-e)$ by Lemma 4.5. Thus, $ann_r(A)$ is generated by an idempotent if and only if $ann_l(A)$ is generated by an idempotent. Therefore, the property that R is a Baer ring is right-left-symmetric. The following theorem from Dauns and Fuchs [4] gives conditions for which a ring R is Baer:

Theorem 4.8. *[4] If R is a torsion-free ring and right annihilators of elements are finitely generated and satisfy the ascending chain condition, then R is a Baer-ring.*

Proof. It follows from *Theorem* 4.4 that R is a right p.p.-ring since $ann_r(x)$ is finitely generated for every $x \in R$. Thus, for each $x \in R$, there is some idempotent $e \in R$ such that $ann_r(x) = eR$. Suppose R contains an infinite set E of orthogonal idempotents. Consider two idempotents e_1 and e_2 in E, and let $e_1 r \in e_1 R$. Note that since e_1 and e_2 are orthogonal idempotents, $e_1r = (e_1 + 0)r = (e_1^2 + e_2e_1)r = (e_1 + e_2)e_1r \in (e_1 + e2)R$. Therefore, $e_1R \subseteq (e_1 + e_2)R$. Inductively, we can construct an ascending chain of principal ideals generated by idempotents. For if $e_1, ..., e_n, e_{n+1}$ are orthogonal idempotents in the infinite set and $(e_1 + ... + e_n)r \in (e_1 + ... + e_n)R$, then $(e_1 + e_2 + ... + e_n)r = (e_1^2 + e_2^2 ... + e_n^2 + 0)r = [(e_1^2 + e_1e_2 + ...e_1e_n) + (e_2e_1 + e_2^2 + ... + e_2e_n) + ... + (e_ne_1 + ... + e_n^2) + (e_{n+1}e_1 + ... + e_{n+1}e_n)]r = (e_1 + ... + e_{n+1})(e_1 + ... + e_n)r \in (e_1 + ... + e_{n+1})R$.

Hence, $e_1R \subseteq (e_1 + e_2)R \subseteq ... \subseteq (e_1 + ... + e_n)R \subseteq (e_1 + ... + e_{n+1})R \subseteq ...$ is an ascending chain of principal ideals generated by idempotents. Furthermore, this will be an infinite chain since there are an infinite number of idempotents in E. Note that by Lemma 4.5, for each $n \in \mathbb{Z}^+$, $(e_1 + ... + e_n)R = ann_r(x)$ for some $x \in R$. Thus, an infinite ascending chain of right annihilators has been constructed, contradicting the ascending chain condition on right annihilators. Therefore, R does not contain an infinite set of orthogonal idempotents. Since R is a right p.p.-ring which does not contain an infinite set of orthogonal idempotents, by Theorem 2.11 every right annihilator in R is generated by an idempotent. Therefore, R is a Baer-ring. □

Chapter 5

Non-singularity

5.1 Essential Submodules and the Singular Submodule

Let R be a ring and consider a submodule A of a right R-module M. If $A \cap B$ is nonzero for every nonzero submodule B of M, then A is said to be an *essential* submodule of M. This is denoted $A \leq^e M$. In other words, $A \leq^e M$ if and only if $B = 0$ whenever $B \leq M$ is such that $A \cap B = 0$. A monomorphism $\alpha : A \to B$ is called *essential* if $im(A) \leq^e B$.

Proposition 5.1. *[2, Corollary 5.13] A monomorphism $\alpha : A \to B$ is essential if and only if, for every right R-module C and every $\beta \in Hom_R(B, C)$, β is a monomorphism whenever $\beta\alpha$ is a monomorphism.*

The *singular submodule of* M is defined as $\mathbf{Z}(M) = \{x \in M \mid xI = 0$ for some essential right ideal I of R$\}$. Equivalently, $\mathbf{Z}(M) = \{x \in M \mid ann_r(x) \leq^e R\}$. For if $I \leq^e R$ and $x \in M$ is such that $xI = 0$, then for any nonzero right ideal J of R, there is an element $a \in I \cap J$. Since $a \in I$, $xa = 0$. Hence, $a \in ann_r(x) \cap J$ and so $ann_r(x) \leq^e R$. On the other hand, note that $ann_r(x)$ is a right ideal of R such that $x \cdot ann_r(x) = 0$. A right R-module M is called *singular* if $\mathbf{Z}(M) = M$ and *non-singular* if $\mathbf{Z}(M) = 0$. If R is viewed a right R-module, then the *right singular ideal of* R is $\mathbf{Z}_r(R) = \mathbf{Z}(R_R)$. The ring R is *right non-singular* if it is non-singular as a right R-module.

Proposition 5.2. *[6] A right R-module A is non-singular if and only if $Hom_R(C, A) = 0$ for every singular right R-module C.*

Proof. Suppose A is a non-singular right R-module and C is a singular right R-module. Let $f \in Hom_R(C, A)$. If it can be shown that $f(\mathbf{Z}(C)) \leq \mathbf{Z}(A)$, then the proof follows readily since $f(C) = f(\mathbf{Z}(C))$ and $\mathbf{Z}(A) = 0$. Suppose $x \in \mathbf{Z}(C)$. Then, $ann_r(x) \leq^e R$. Hence, if I is any nonzero right ideal of R, then there exists some $y \in I$ such that $xy = 0$. Then, $f(x)y = f(xy) = f(0) = 0$ and $y \in ann_r(f(x)) \cap I$. Thus, $ann_r(f(x)) \leq^e R$ and so $f(x) \in \mathbf{Z}(A)$. Therefore, $f(\mathbf{Z}(C)) \leq \mathbf{Z}(A)$.

Conversely, suppose A is a right R-module and $Hom_R(C, A) = 0$ for every singular right R-module C. Then, $Hom_R(\mathbf{Z}(A), A) = 0$ since the singular submodule $\mathbf{Z}(A)$ is singular. Hence, the inclusion map $\iota : \mathbf{Z}(A) \to A$ given by $\iota(x) = x$ is a zero map. Thus, $\mathbf{Z}(A) = \iota(\mathbf{Z}(A)) = 0$. Therefore, A is a non-singular right R-module. □

Proposition 5.3. *[6] The following are equivalent for a right R-module C:*

(a) C is singular.

(b) There exists an exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ such that f is essential.

Proof. $(a) \Rightarrow (b)$: Suppose C is a right R-module. Let $0 \to A \xrightarrow{\iota} B \xrightarrow{g} C \to 0$ be an exact sequence of right R-modules such that B is free and ι is the inclusion map. Let $\{x_\alpha\}_{\alpha \in K}$ be a basis for B for some index K. Then, for each $\alpha \in K$, $g(x_\alpha) \in C = \mathbf{Z}(C)$. Hence, there exists an essential right ideal I_α of R such that $g(x_\alpha I_\alpha) = g(x_\alpha)I_\alpha = 0$. Thus, for each $\alpha \in K$ and each $i_\alpha \in I_\alpha$, $x_\alpha i_\alpha \in \ker g = A$. That is, $x_\alpha I_\alpha \leq A$ for each $\alpha \in K$, and it follows that $\bigoplus_K x_\alpha I_\alpha \leq A$. If $x_\alpha J$ is a nonzero right ideal of $x_\alpha R$, then J is a nonzero right ideal of R, and there is a nonzero element $y \in I_\alpha \cap J$. Then it readily follows that $x_\alpha y \in x_\alpha I_\alpha \cap x_\alpha J$ is nonzero. Hence, $x_\alpha I_\alpha \leq^e x_\alpha R$ for each $\alpha \in K$. Thus, $\bigoplus_K x_\alpha I_\alpha \leq^e \bigoplus_K x_\alpha R = B$. Therefore, A is also essential in B since $\bigoplus_K x_\alpha I_\alpha \leq A$. It then follows from the exactness of the sequence that $im(A) \cong A \leq^e B$.

$(b) \Rightarrow (a)$: Assume $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is an exact sequence of right R-modules such that $im(A) \leq^e B$. For each $b \in B$, define $h_b : R \to B$ by $h_b(r) = br$, and let $I_b = \{r \in R \mid br \in im(A)\}$. Note that I_b is a nonzero right ideal of R. Suppose I_b is not essential in R. Then there is a nonzero right ideal J of R such that $I_b \cap J = 0$. Moreover, if $s \in \ker(h_b)$, then $h_b(s) = bs = 0 \in im(A)$ and it follows that $\ker(h_b) \subseteq I_b$. Hence,

$\ker(h_b) \cap J = 0$. Thus, $h_b|_J$ is a monomorphism. This implies that $h_b(J)$ must be a nonzero right ideal of B since J is a nonzero right ideal of R. Thus, $h_b(J) \cap im(A) \neq 0$ by the assumption that $im(A) \leq^e B$. Then for some nonzero $j \in J$, $bj = h_b(j) \in im(A)$. Hence, $j \in I_b \cap J$, which is a contradiction. Therefore, I_b is an essential right ideal of R. Note that for every $b \in B$, if $bi \in bI_b$, then $bi \in im(A)$. Then by exactness of the sequence, $bI_b \subseteq im(A) = \ker g$. Hence, $g(b)I_b = g(bI_b) = 0$, which implies $g(b) \in \mathbf{Z}(C)$. Since this is the case for every $b \in B$, $g(B) \subseteq \mathbf{Z}(C)$. Furthermore, since the sequence is exact, $C = g(B) \subseteq \mathbf{Z}(C)$. Therefore, $C = \mathbf{Z}(C)$. □

Proposition 5.4. *If R is a right p.p.-ring, then R is a right non-singular ring.*

Proof. Let R be a right p.p.-ring and take any $x \in R$. Suppose $ann_r(x) \leq^e R$. Since R is a right p.p.-ring, $ann_r(x) = eR$ for some idempotent $e \in R$. Observe that $R = eR \bigoplus (1-e)R$. Hence, $ann_r(x) \cap (1-e)R = 0$. However, this implies that $(1-e)R = 0$ since $ann_r(x) \leq^e R$. Hence, $1 - e = 0$, and so $ann_r(x) = 1R = R$. Thus, $xr = 0$ for every $r \in R$, which implies $x = 0$. Therefore, R is right non-singular. □

5.2 The Maximal Ring of Quotients and Right Strongly Non-singular Rings

The maximal ring of quotients and strongly non-singular rings will play an important role in determining which rings satisfy the condition that the classes of torsion-free and non-singular modules coincides. We explore these concepts in this section. If R is a subring of a ring Q, then Q is a *classical right ring of quotients* of R if every regular element of R is a unit in Q and every element of Q is of the form rs^{-1}, where $r, s \in R$ with s regular [8]. For a ring which is not necessarily commutative, such a Q may not exist. Thus, we consider a more general way to define the right ring of quotients which guarantees its existence for any ring R.

Let A be a submodule of a right R-module B. If $Hom_R(M/A, B) = 0$ for every right R-module M satisfying $A \leq M \leq B$, then B is a *rational extension* of A. This is denoted $A \leq^r B$.

Lemma 5.5. *[6] Let B be a non-singular right R-module and take any submodule A of B. Then, $A \leq^r B$ if and only if $A \leq^e B$.*

Proof. Suppose $A \leq^r B$ and let $M \leq B$ be such that $M \cap A = 0$. Now, $M \bigoplus A$ is a right R-module satisfying $A \leq M \bigoplus A \leq B$. Hence, $Hom_R([M \bigoplus A]/A, B) = 0$. Consider $f : (M \bigoplus A)/A \to M$ defined by $(m+a)+A \mapsto m$ for $m \in M$ and $a \in A$. If $m, m_0 \in M$ and $a, a_0 \in A$ are such that $(m + a) + A = (m_0) + a_0) + A$, then $(m - m_0) + (a - a_0) \in A$. Hence, $m-m_0 \in A$. However, $M \cap A = 0$ and so $m-m_0 = 0$. Thus, f is well-defined. Moreover, f is an isomorphism. For if $m \in M$, then $f[(m+a)+A] = m$ for any $a \in A$, and $f[(m+a)+A] = 0$ implies that $(m + a) + A = m + A = 0$. Observe that $f \in Hom_R([M \bigoplus A]/A, B) = 0$ since $M \leq B$. Thus, $M = im(f) = 0$ and therefore $A \leq^e B$. Note that this implication does not require B to be right non-singular.

On the other hand, suppose $A \leq^e B$ and take M to be a right R-module such that $A \leq M \leq B$. Then, any nonzero submodule N of B is such that $A \cap N \neq 0$. Hence, any nonzero submodule K of M is such that $A \cap K \neq 0$ since any such submodule is also a submodule of B. Thus, $A \leq^e M$. Consider the exact sequence $0 \to A \xrightarrow{\iota} M \xrightarrow{\pi} M/A \to 0$, where ι is the inclusion map and π is the canonical epimorphism. Observe that $im(\iota) = A \leq^e M$. Hence, M/A is singular by Proposition 5.3. It then follows from Proposition 5.2 that $Hom_R(M/A, B) = 0$ since B is nonsingular. Therefore, B is a rational extension of A. □

A right R-module E is called *injective* if, given any two right R-modules A and B, a monomorphism $\alpha : A \to B$, and a homomorphism $\varphi : A \to E$, there exists a homomorphism $\psi : B \to E$ such that $\varphi = \psi\alpha$. If E is injective and $M_R \leq^e E_R$, then E is called an *injective* *hull* of M. Every right R-module M has an injective hull, which is unique up to isomorphism [6, Theorems 1.10, 1.11].

Let R be a subring of a ring Q. If $R_R \leq^r Q_R$, then Q is a *right ring of quotients of* R. Observe that R is a right ring of quotients of itself since $R_R \leq^r R_R$. Similarly, if ${}_RR \leq^r {}_RQ$, then Q is a *left ring of quotients of* R. Let Q be a right ring of quotients of R such that given any other right ring of quotients P of R, the inclusion map $\mu : R \to Q$ extends to a monomorphism $\nu : P \to Q$. Here, Q is called a *maximal right ring of quotients of* R. This is denoted Q^r when there is no confusion as to which ring the maximal quotient ring applies, and $Q^r(R)$ otherwise. The *maximal left ring of quotients* Q^l is similarly defined. In general, $Q^r \neq Q^l$.

Theorem 5.6. *[6] For any ring R, the maximal right ring of quotients $Q^r(R)$ exists. In particular, if E is the injective hull of R_R and $T = End_R(E)$, then $Q = \cap\{\ker\delta \mid \delta \in T$ and $\delta R = 0\}$ is a maximal right ring of quotients.*

Proof. If E is the injective hull of R, then $\tau x = \tau(x)$ defines a left T-module structure on E for $\tau \in T$ and $x \in E$. Let $T_0 = End_T(E)$ and define $\omega(x) = x\omega$ for $\omega \in T_0$ and $x \in E$. Consider the homomorphisms $\psi : T \to E$ and $\varphi : T_0 \to E$ defined by $\psi\tau = \tau 1$ and $\varphi\omega = 1\omega$. It is easily seen that ψ is an epimorphism and φ is a monomorphism. Let $x \in E$ and consider the homomorphism $\sigma : R \to xR$ defined by $\sigma(r) = xr$. Since R is a subring of E, σ can be extended to a homomorphism $\tau : E \to E$. Thus, $\tau(1) = \sigma(1) = x$ and so $\psi(\tau) = \tau(1) = x$. Therefore, ψ is an epimorphism. Now, suppose $\omega \in \ker\varphi$. Then $1\omega = \varphi(\omega) = 0$. If $x \in E$, then $\tau 1 = x$ for some $\tau \in T$ since ψ is an epimorphism. Hence, $\omega(x) = x\omega = (\tau 1)\omega = \tau(1\omega) = \tau(0) = 0$. Therefore, $\omega = 0$ and φ is a monomorphism.

If $\delta \in T$ is such that $\delta R = 0$, then $\delta(1\omega) = (\delta 1)\omega = 0$ for every $\omega \in T_0$. Hence, $1\omega \in Q$. Therefore, φ can actually be defined as a map $T_0 \to Q$. It readily follows that φ maps onto Q and hence $\varphi : T_0 \to Q$ is an isomorphism. To see this, let $x \in Q$ and consider $\nu : E \to E$ defined by $(\tau 1)\nu = \tau x$. This can be defined for every $\tau \in T$ since

φ is a well-defined epimorphism onto E. Thus, if $1_E \in T$ is the identity map on E, then $\varphi(\nu) = 1\nu = [1_E(1)]\nu = 1_E(x) = x$. Therefore, φ is onto.

We now define multiplication on Q. For $x, y \in Q$, let $x \cdot y = \varphi[(\varphi^{-1}x)(\varphi^{-1}y)] = 1(\varphi^{-1}x)(\varphi^{-1}y)$. Clearly $x \cdot y \in Q$ and it is easily seen to be associative. Since φ is an isomorphism, if $r \in R$, then there exists some $\omega \in T_0$ such that $\varphi(\omega) = 1\omega = r$. Thus, if $x \in Q$, then $x \cdot r = 1(\varphi^{-1}x)(\varphi^{-1}r) = (\varphi\varphi^{-1}x)(\omega) = x\omega = (x1)\omega = x(1\omega) = xr$. It follows from [6, Theorem 2.26] that this multiplication defines a unique ring structure on Q which is consistent with the R-module structure..

To see that Q is a right ring of quotients, suppose $R \leq M \leq Q$ for some right R-module M and let $\alpha \in Hom_R(M/R, Q)$. Consider the epimorphism $\pi : M \to M/R$ given by $x \mapsto x + R$, and define $\gamma = \alpha\pi : M \to Q$. Observe that $\gamma R = 0$ since $\gamma(r) = \alpha\pi(r) = \alpha(r + R) = 0$ for any $r \in R$. Moreover, γ can be extended to a map $\beta \in T$ such that $\beta R = 0$. Since Q is the intersection of the kernels of all homomorphisms $\delta \in T$ satisfying $\delta R = 0$, $M \subseteq Q \subseteq \ker\beta$. Thus, $\gamma M = \beta M = 0$ and so $\alpha(x + R) = \gamma(x) = 0$ for any $x \in M$. Therefore, $R \leq^r Q$ and Q is a right ring of Quotients.

To see that Q^r is a maximal right ring of quotients, let P be another right ring of quotients. Then $R_R \leq^r P_R$ by definition, and hence $R_R \leq^e P_R$ by Lemma 5.5. If $\iota : R \to P$ and $\mu : R \to E$ are the inclusion maps, then by injectivity of E, there exists a homomorphism $\nu : P \to E$ such that $\nu\iota = \mu$. Observe that $R \cap \ker\nu = \ker\mu = 0$. This implies $\ker\nu = 0$ since R is essential in P and $\ker\nu$ is a submodule of P. Therefore, the inclusion map $\mu : R \to E$ can be extended to a monomorphism $\nu : P \to E$. Moreover, [6, Theorem 2.26] shows that νP is contained in Q, and hence the inclusion map $R \to Q$ can be extended to a monomorphism $\nu : P \to Q$. Finally, note that since $R \leq \nu P \leq Q$ and $R_R \leq^r Q_R$, $Hom_R(\nu P/R, Q) = 0$. Hence, given $x \in P$, the homomorphism $\sigma : \nu P/R \to Q$ defined by $\sigma(\nu y + R) = \nu(xy) - (\nu x)(\nu y)$ is the zero map. Therefore, ν is a ring homomorphism and Q is a maximal right ring of quotients of R. $\square$

Goodearl shows in [6, Corollary 2.31] that Q^r is injective as a right R-module. Therefore, $Q^r(R)$ is an injective hull of R since $R_R \leq^e Q^r_R$ by Lemma 5.5. Moreover, since the injective hull is unique up to isomorphism, we can refer to $Q^r(R)$ as the injective hull of R. The following results about maximal quotient rings will be needed later. The proofs are omitted.

Proposition 5.7. *[1, Proposition 2.2] For a right non-singular ring R, R is a left p.p.-ring such that $Q^r(R)$ is torsion-free as a right R-module if and only if all non-singular right R-modules are torsion-free.*

Theorem 5.8. *[10, Ch. XII, Proposition 7.2] If R is a right non-singular ring and M is a finitely generated non-singular right R-module, then there exists a monomorphism $\varphi : M \to \oplus_n Q^r$ for some $n < \omega$. In other words, M is isomorphic to a submodule of a free Q^r-module.*

For a ring R, its maximal right ring of quotients Q^r is a *perfect left localization of R* if Q^r is flat as a right R-module and the multiplication map $\varphi : Q^r \bigotimes_R Q^r \to Q^r$, defined by $\varphi(a \otimes b) = ab$, is an isomorphism. If R is a right non-singular ring for which Q^r is a perfect left localization, then R is called *right strongly non-singular.* Goodearl provides the following useful characterization of right strongly non-singular rings:

Theorem 5.9. *[6, Theorem 5.17] Let R be a right non-singular ring. Then, R is right strongly non-singular if and only if every finitely generated non-singular right R-module is isomorphic to a finitely generated submodule of a free right R-module.*

Corollary 5.10. *[6, Theorem 5.18] Let R be a right non-singular ring. Then, R is right semi-hereditary, right strongly non-singular if and only if every finitely generated non-singular right R-module is projective.*

Proof. For a right non-singular ring R, suppose R is right semi-hereditary, right strongly non-singular. Let M be a finitely generated non-singular right R-module. By Theorem 5.9, M is isomorphic to a finitely generated submodule of a free right R-module F. Therefore, since R is right semi-hereditary, M is projective by Lemma 2.5.

Conversely, assume every finitely generated non-singular right R-module is projective. Since R is right non-singular, every finitely generated right ideal of R is non-singular. Hence, every finitely generated right ideal is projective and R is right semi-hereditary. Furthermore, every finitely generated non-singular right R-module is a direct summand, and hence a submodule, of a free right R-module. Therefore, R is right strongly non-singular by Theorem 5.9. □

5.3 Coincidence of Classes of Torsion-free and Non-singular Modules

We know turn our attention to rings for which the classes of torsion-free and non-singular right R-modules coincide, which is investigated in [1] by Albrecht, Dauns, and Fuchs. A few definitions are needed before stating their theorems in full. A ring is *right semi-simple* if it can be written as a direct sum of modules which have no proper nonzero submodules, and a ring is *right Artinian* if it satisfies the descending chain condition on right ideals. Assume *semi-simple Artinain* to mean *right semi-simple, right Artinian.* The following results from Stenström consider rings with semi-simple right maximal ring of quotients.

Proposition 5.11. *[10, Ch. XI, Proposition 5.4] Let R be a ring whose maximal right ring of quotients is semi-simple. Then, $Q^r = Q^l$ if and only if Q^r is flat as a right R-module.*

Theorem 5.12. *[10, Ch. XII, Corollaries 2.6,2.8] Let R be a ring and suppose $Q^r(R)$ is semi-simple. Then:*

(a) Q^r is a perfect right localization of R. In other words, if R is left non-singular, then it is left strongly non-singular.

(b) If M is any non-singular right R-module, then $M \bigotimes_R Q^r$ is the injective hull of M.

A ring R is *von Neumann regular* if, given any $r \in R$, there exists some $s \in R$ such that $r = rsr$. These rings are of interest because R is von Neumann regular if and only if every right R-module is flat [9, Theorem 4.9]. The following lemmas will be needed in the next chapter.

Lemma 5.13. *[9] If R is a semi-simple Artinian ring, then R is von Neumann regular.*

Proof. The Wedderburn-Artin Theorem states that R is semi-simple Artinian if and only if it is isomorphic to a finite direct product of matrix rings over division rings. For any division ring D, $Mat_n(D) \cong End_D(\bigoplus^n D)$ is von Neumann regular [9]. Therefore, R is von Neumann regular since direct products of regular rings are regular. □

Lemma 5.14. *[10] A ring R is right non-singular if and only if Q^r is von Neumann regular.*

Proof. Stenström shows in [10, Ch. XII] that if R is right non-singular, then $Q^r \cong End_R(E)$, where $E \cong Q^r$ is the injective hull of R. In [10, Ch. V, Proposition 6.1], it is shown that such rings are regular.

Conversely, assume Q^r is von Neumann regular. Let I be an essential right ideal of R and take $x \in R$ to be nonzero. Suppose $xI = 0$. Since Q^r is regular, there exists some $q \in Q$ such that $xqx = x$. Hence, qxR is a nonzero right ideal of R, and so $I \cap qxR \neq 0$. Thus, $0 \neq qxr \in I$ for some nonzero $r \in R$. However, $xr = xqxr \in xI = 0$. This implies $qxr = 0$, which is a contradiction. Therefore, $xI \neq 0$ and R is right non-singular. □

Let R be a ring and M a right R-module. A submodule U of M is ***S**-closed* if M/U is non-singular. The following lemma shows that annihilators of elements are **S**-closed for non-singular rings.

Lemma 5.15. *If R is a right non-singular ring, then for any $x \in R$, $ann_r(x)$ is **S**-closed.*

Proof. Let R be right non-singular. It needs to be shown that $R/ann_r(x)$ is non-singular for any $x \in R$. That is, for $x \in R$, $\mathbf{Z}(R/ann_r(x)) = \{r + ann_r(x) \mid (r + ann_r(x))I = 0$ for some $I \leq^e R\} = 0$. Let $0 \neq r + ann_r(x) \in R/ann_r(x)$ and I be a nonzero essential right ideal of R such that $(r + ann_r(x))I = 0$. Then, for any $a \in I$, $ra + ann_r(x) = 0$. Hence, $ra \in ann_r(x)$ and $xra = 0$ for every $a \in I$. In other words, $(xr)I = 0$. If $xr \neq 0$, then there is a contradiction since $I \leq^e R$ and $\mathbf{Z}(R) = 0$. Thus, $xr = 0$ and $r \in ann_r(x)$. Therefore, $r + ann_r(x) = 0$, and it follows readily that $\mathbf{Z}(R/ann_r(x)) = 0$. □

If R is a right non-singular ring and every $\mathbf{S}$-closed right ideal of R is a right annihilator, then R is referred to as a *right Utumi ring*. Similarly, R is a *left Utumi ring* if R is left non-singular and every $\mathbf{S}$-closed left ideal of R is a left annihilator. The following result from Goodearl characterizes non-singular rings which are both right and left Utumi.

Theorem 5.16. *[6, Theorem 2.38] If R is a right and left non-singular ring, then $Q^r = Q^l$ if and only if every R is both right and left Utumi.*

For a ring R, if every direct sum of nonzero right ideals of R contains only finitely many direct summands, then R is said to have *finite right Goldie-dimension*. Denote the Goldie-dimension of R as G-dim R_R. If a ring R with finite right Goldie-dimension also satisfies the ascending chain condition on right annihilators, then R is a *right Goldie-ring*. The maximal right quotient ring Q^r is a *semi-perfect left localization of* R if Q^r_R is torsion-free and the multiplication map $Q^r \bigotimes_R Q^r \to Q^r$ is an isomorphism. The following is a useful characterization of rings with finite right Goldie dimension:

Theorem 5.17. *[10, Ch. XII, Theorem 2.5] If R is a right non-singular ring, then Q^r is semi-simple if and only if R has finite right Goldie dimension.*

We are now ready to state two key results form Albrecht, Fuchs, and Dauns, which consider rings for which the classes of torsion-free and non-singular modules coincide. These will be needed in the next chapter to prove the main theorem of this thesis. The proof of Theorem 5.18 is omitted.

Theorem 5.18. *[1, Theorem 3.7] The following are equivalent for a ring R:*

(a) R is a right Goldie right p.p.-ring and Q^r is a semi-perfect left localization of R.

(b) R is a right Utumi p.p.-ring which does not contain an infinite set of orthogonal idempotents.

(c) R is a right non-singular ring which does not contain an infinite set of orthogonal idempotents, and every finitely generated non-singular right R-module is torsion-free.

(d) A right R-module M is torsion-free if and only if M is non-singular.

Furthermore, if R satisfies any of the equivalent conditions, then R is a Baer-ring and Q^r is semi-simple Artinian.

Theorem 5.19. *[1] The following are equivalent for a ring R:*

*(a) R is a right and left non-singular ring which does not contain an infinite set of orthogonal idempotents, and every **S**-closed left or right ideal is generated by an idempotent.*

(b) R is a right or left p.p.-ring, and $Q^r = Q^l$ is semi-simple Artinian.

(c) R is a right strongly non-singular right p.p.-ring which does not contain an infinite set of orthogonal idempotents.

(d) R is right strongly non-singular, and a right R-module is torsion-free if and only if it is non-singular.

(e) For a right R-module M, the following are equivalent:

(i) M is torsion-free

(ii) M is non-singular

(iii) If $E(M)$ is the injective hull of M, then $E(M)$ is flat.

Proof. $(a) \Rightarrow (b)$: Assume R is right and left non-singular, contains no infinite set of orthogonal idempotents, and every **S**-closed right or left ideal is generated by an idempotent. Let I be an **S**-closed right ideal of R. Then, $I = eR$ for some idempotent $e \in R$. As shown in the proof of Lemma 4.5, $eR = ann_r(1-e)$. Thus, $I = eR$ is the right annihilator of $1-e$. Note that a symmetric argument shows that if J is an **S**-closed left ideal of R, then $J = Rf$ is a left annihilator of $1-f$ for some idempotent $f \in R$. Hence, R is both a right and left Utumi ring. By Lemma 5.15, since R is a right non-singular ring, $ann_r(x)$ is **S**-closed for every $x \in R$. This implies that $ann_r(x)$ is generated by an idempotent for every $x \in R$. Therefore, R is a right p.p-ring. A symmetric argument shows that R is also a left p.p.-ring since condition (a) applies to both right and left ideals. Note that R satisfies condition (b) of Theorem 5.18 since it is a right Utumi p.p.-ring which does not contain an infinite set of orthogonal idempotents. Hence, Q^r is semi-simple Artinian by Theorem 5.18. Furthermore, since every right and left $\mathbf{S}$-closed ideal is an annihilator, R is right and left Utumi. Therefore, $Q^r = Q^l$ by Theorem 5.16.

$(b) \Rightarrow (c)$: Suppose R is a right p.p.-ring and $Q^r = Q^l$ is semi-simple Artinian. Since R is a right p.p.-ring, it is also a right non-singular ring. Hence, R has finite right Goldie dimension by Theorem 5.17. Suppose R contains an infinite set of orthogonal idempotents. Consider two orthogonal idempotents e and f, and let $x \in eR \cap fR$. Then, $x = er = fs$ for some r, $s \in r$. This implies that $x = 0$ since $er = e^2 r = efs = 0$. Thus, $eR \cap fR = 0$ for any two orthogonal idempotents e and f in the infinite set, and $eR \bigoplus fR$ is direct. Hence, R contains an infinite direct sum of nonzero right submodules, which contradicts R having finite right Goldie dimension. Therefore, R does not contain an infinite set of orthogonal idempotents.

By Theorem 5.12, since R is semi-simple Artinian, R is a left strongly non-singular ring. Hence, the multiplication map $\varphi : Q^r \bigotimes_R Q^r \rightarrow Q^r$, defined by $\varphi(q \otimes p) = qp$, is an isomorphism. Note that this also implies that Q^r is flat as a left R-module. However, in order for R to be right strongly non-singular, it needs to be shown that Q^r is flat as a right R-module. By Proposition 5.11, Q^r is indeed flat as a right R-module since $Q^r = Q^l$ is assumed to be semi-simple Artinian. Therefore, R is a right strongly non-singular ring which does not contain an infinite set of orthogonal idempotents. Note that Theorem 2.11 shows that R is also a left p.p.-ring. Thus, if we had instead assumed that R is a left p.p.-ring, then a symmetric argument could be used to show that R is also a right p.p.-ring, and the latter part of the proof would remain the same.

$(c) \Rightarrow (d)$: Assume R is a right strongly non-singular right p.p.-ring which does not contain an infinite set of orthogonal idempotents. Then, Q^r is flat as a right R-module, which follows from R being right strongly non-singular. Since flat R-modules are torsion-free, this implies that Q^r is torsion-free. By Theorem 2.11, since R is a right p.p.-ring and does not contain an infinite set of orthogonal idempotents, R is also a left p.p.-ring. Hence, every non-singular right R-module is torsion-free by Proposition 5.7. Thus, R satisfies condition (c) of Theorem 5.18, which implies that a right R-module M is torsion-free if and only if M is non-singular.

$(d) \Rightarrow (e)$: Suppose R is right strongly non-singular, and a right R-module is torsion-free if and only if it is non-singular. Then, conditions (i) and (ii) of (e) are clearly equivalent, and it suffices to show that a right R-module is non-singular if and only if its injective hull is flat. Suppose M is a non-singular right R-module. Note that R satisfies condition (d) of Theorem 5.18, and hence Q^r is semi-simple Artinian. By Theorem 5.12, $M \bigotimes_R Q^r$ is an injective hull of M. Thus, if $E(M)$ denotes the injective hull of M, then $E(M) \cong M \bigotimes_R Q^r$, since an injective hull of a right R-module is unique up to isomorphism. This implies that $E(M)$ is a right Q^r-module, since $M \bigotimes_R Q^r$ is a right Q^r-module by Proposition 3.9. Furthermore, since Q^r is semi-simple Artinian, every Q^r-module is projective. Hence, $E(M)$ is projective and thus isomorphic to a direct summand of a free Q^r-module F. Note that Q^r is flat as a right R-module since R is right strongly non-singular. Thus, Proposition 3.7 shows that any free Q^r-module is flat since such modules can be written as $\bigoplus_{i \in I} M_i$ for some index set I, where M_i is isomorphic to Q^r for every $i \in I$. This implies that $E(M)$ is flat by Proposition 3.7 since it is a direct summand of the flat right R-module $F = \bigoplus_{i \in I} M_i$.

On the other hand, assume that the injective hull $E(M)$ of some right R-module M is flat. Noting again that R satisfies condition (d) of Theorem 5.18, it follows that R is a right p.p.-ring. Thus, R is a torsion-free ring by Theorem 4.4. Since flat R-modules are torsion-free, $E(M)$ is torsion-free as a right R-module. Hence, M is a submodule of a torsion-free right R-module. Thus, M is a torsion free right R-module by Proposition 4.3. Therefore, M is non-singular since a right R-module is torsion-free if and only if it is non-singular by assumption.

$(e) \Rightarrow (a)$: For a right R-module M, assume that M is torsion-free if and only if M is non-singular if and only if the injective hull $E(M)$ is flat. By Theorem 5.18, R is a right p.p.-ring which does not contain an infinite set of orthogonal idempotents. It then follows from Proposition 5.4 that R is a right non-singular ring. Hence, R is also a left p.p.-ring by Proposition 5.7, since every non-singular right R-module is torsion-free, and a symmetric argument for Proposition 5.4 shows that R is left non-singular.

The injective hull $E(R)$ is flat as a right R-module since R is assumed to be right non-singular. Hence, Q^r is flat as a right R-module, since Q^r is the injective hull of R. We've already shown that R satisfies the equivalent conditions of Theorem 5.18, which implies that Q^r is a semi-simple Artinian ring. Thus, it follows from Proposition 5.11 that $Q^r = Q^l$. Since R is both right and left non-singular, every **S**-closed right ideal of R is a right annihilator and every **S**-closed left ideal of R is a left annihilator by Theorem 5.16. Furthermore, note that Theorem 5.18 shows that R is a Baer-ring. Hence, every annihilator is generated by an idempotent. Therefore, every **S**-closed right ideal and every **S**-closed left ideal is generated by an idempotent. $\square$

Chapter 6

Morita Equivalence

Before proving the main theorem, we discuss Morita equivalences. In particular, we show that there is a Morita equivalence between R and $Mat_n(R)$ for any $0 < n < \omega$. This is then used to show that the classes of torsion-free and non-singular $Mat_n(R)$-modules coincide for certain conditions placed on R.

Let R and S be rings. The categories Mod_R and Mod_S are *equivalent* (or *isomorphic*) if there are functors $F : Mod_R \to Mod_S$ and $G : Mod_S \to Mod_R$ such that $FG \cong 1_{Mod_S}$ and $GF \cong 1_{Mod_R}$. Note that these are natural isomorphisms. In other words, if $\eta : GF \to 1_{Mod_R}$ denotes the natural isomorphism, then for each $M, N \in Mod_R$, there exist isomorphisms $\eta_M : GF(M) \to M$ and $\eta_N : GF(N) \to N$ such that $\beta\eta_M = \eta_N GF(\beta)$ whenever $\beta \in Hom_R(M, N)$. Here, $GF(\beta)$ denotes the induced homomorphism. The functors F and G are referred to as an *equivalence* of Mod_R and Mod_S. If such an equivalence exists, then R and S are said to be *Morita-equivalent*. In [10, Ch. IV, Corollary 10.2], Stenström shows that R and S are Morita-equivalent if and only if there are bimodules ${}_SP_R$ and ${}_RQ_S$ such that $P \bigotimes_R Q \cong S$ and $Q \bigotimes_S P \cong R$. A property P is referred to as *Morita-invariant* if for every ring R satisfying P, every ring S Morita-equivalent to R also satisfies P.

A *generator* of Mod_R is a right R-module P satisfying the condition that every right R-module M is a quotient of $\bigoplus_I P$. Note that R and any free right R-module are generators of Mod_R. A *progenerator* of Mod_R is a generator which is finitely generated and projective.

Lemma 6.1. *[2] Let R be a ring, P a progenerator of Mod_R, and $S = End_R(P)$. Then, there is an equivalence $F : Mod_R \to Mod_S$ given by $F(M) = Hom_R(P, M)$ with inverse $G : Mod_S \to Mod_R$ given by $G(N) = N \bigotimes_S P$.*

Proof. As a projective generator of Mod_R, P is a right R-module. P also has a left S-module structure with $(f * g)(x) = f(g(x))$ for $f, g \in S$ and $x \in P$, where multiplication in the endomorphism ring is defined as composition of functions. It then readily follows that P is an (S, R)-bimodule since $f(xr) = f(x)r$ for any $f \in S$ and $r \in R$. Thus, $F = Hom_S(P, \_)$ is a functor $Mod_S \to Mod_R$ and $G = \_ \bigotimes_R P$ is a functor $Mod_R \to Mod_S$ by Proposition 3.10.

It needs to be shown that $GF \cong 1_{Mod_R}$ and $FG \cong 1_{Mod_S}$ are natural isomorphisms. Since P is a progenerator of Mod_R, it is finitely generated and projective as a right R-module. Thus, it follows from Lemma 3.12 that if M is any right R-module, then $GF(M) = G(Hom_R(P, M)) = Hom_R(P, M) \bigotimes_S P \cong Hom_R(Hom_S(P, P), M) \cong Hom_R(End_S(P), M) \cong Hom_R(R, M) \cong M$. Similarly, given any right S-module N, $FG(N) = F(N \bigotimes_S P) = Hom_R(P, N \bigotimes_S P) \cong N \bigotimes_S Hom_R(P, P) = N \bigotimes_S S \cong N$ by Lemma 3.11. Therefore, F is an equivalence with inverse G. □

Proposition 6.2. *Let R be a ring. For every $0 < n < \omega$, R is Morita-equivalent to $Mat_n(R)$.*

Proof. Let P be a finitely generated free right R-module with basis $\{x_i\}_{i=1}^n$ for $0 < n < \omega$. Then, P is a progenerator of Mod_R and $Mat_n(R) \cong End_R(P)$ by Lemma 2.6. Therefore, the equivalence of Lemma 6.1 is a Morita-equivalence between R and $Mat_n(R)$. □

Lemma 6.3. *[10, Ch. X, Proposition 3.2] If R and S are Morita-equivalent, then the maximal ring of quotients, $Q^r(R)$ and $Q^r(S)$, are also Morita equivalent.*

Proposition 6.4. *Let R and S be Morita-equivalent rings with equivalence $F : Mod_R \to Mod_S$ and $G : Mod_S \to Mod_R$.*

(i) If U is an essential submodule of a right R-module M, then $F(U)$ is an essential submodule of the right S-module $F(M)$.

(ii) If M is a non-singular right R-module, then $F(M)$ is a non-singular right S-module.

In other words, essentiality and non-singularity are Morita-invariant properties.

Proof. (i): Let $U \leq^e M$. Then, the inclusion map $\iota : U \to M$ is an essential monomorphism. Consider the induced homomorphism $F(\iota) : F(U) \to F(M)$. Note that since ι is a monomorphism, $F(\iota)$ is a monomorphism [2, Proposition 21.2]. Let W be any right S-module and take $\beta \in Hom_S(F(M), W)$ to be such that $\beta F(\iota) : F(U) \to W$ is a monomorphism. There is a natural isomorphism $\Phi_{U,W} : Hom_S(F(U), W) \to Hom_R(U, G(W))$ defined by $\gamma \mapsto G(\gamma)\eta_U^{-1}$, where η_U denotes the isomorphism $GF(U) \to U$ [2, 21.1]. Hence, $\Phi_{U,W}(\beta F(\iota))$ is a monomorphism. Moreover, $\Phi_{U,W}(\beta F(\iota)) = G(hF(\iota))\eta_U^{-1} = G(h)GF(\iota)\eta_U^{-1} = G(h)\eta_M^{-1}\eta_M GF(\iota)\eta_U^{-1} = \Phi_{M,W}(\beta)\iota\eta_U\eta_U^{-1} = \Phi_{M,W}(\beta)\iota$. Thus, $\Phi_{M,W}(\beta)\iota$ is a monomorphism and it follows from Proposition 5.1 that $\Phi_{M,W}(\beta)$ is a monomorphism since ι is essential. Furthermore, $\Phi_{M,W}(\beta)$ is a monomorphism if and only if β is a monomorphism [2, Lemma 21.3]. Hence, $F(\iota)$ is an essential monomoprhism by Proposition 5.1. Therefore, $F(U) \cong im(F(\iota)) \leq^e F(M)$.

(ii): Let M be a non-singular right R-module. It needs to be shown that $F(M)$ is a non-singular right S-module and in view of Proposition 5.2 it suffices to show that $Hom_S(C, F(M)) = 0$ for any singular right S-module C. By Proposition 5.3, there is an exact sequence $0 \to A \xrightarrow{f} F \to C \to 0$ of right S-modules such that $f(A) \leq^e F$ and F is free. Then, $G(f(A)) \leq^e G(B)$ by (i). Hence, $0 \to G(A) \xrightarrow{G(f)} G(B) \to G(C) \to 0$ is an exact sequence of right R-modules such that $G(f(A)) \leq^e G(B)$. Thus, $G(C)$ is a singular right R-module by Proposition 5.3. Since $G(C)$ is singular and M is non-singular, $Hom_R(G(C), M) = 0$ by Proposition 5.2. Therefore, $Hom_S(C, F(M)) \cong Hom_R(G(C), M) = 0$. Observe that in this proof, it is also shown that singularity is Morita-invariant since we show that $G(C)$ is singular for an arbitrary singular module C. □

We now prove the main theorem of this thesis.

Theorem 6.5. *The following are equivalent for a ring R:*

(a) R is a right strongly non-singular, right semi-hereditary, right Utumi ring not containing an infinite set of orthogonal idempotents.

(b) Whenever S is Morita-equivalent to R, then the classes of torsion-free right S-modules and non-singular right S-modules coincide.

(c) For every $0 < n < \omega$, $Mat_n(R)$ is a right and left Utumi Baer-ring not containing an infinite set of orthogonal idempotents.

Moreover, if R is such a ring, then the corresponding left conditions are also satisfied.

Proof. $(a) \Rightarrow (b)$: Assume R is a right strongly non-singular, right semi-hereditary, right Utumi ring not containing an infinite set of orthogonal idempotents. Let R and S be Morita equivalent, and let $F : Mod_R \to Mod_S$ and $G : Mod_S \to Mod_R$ be an equivalence. Also, take N to be a finitely generated non-singular right R-module. Since R is right strongly non-singular, N is isomorphic to finitely generated submodule V of a free right R-module by Theorem 5.9. Furthermore, since R is right semi-hereditary and free R-modules are projective, $V \cong N$ is projective by Lemma 2.5. Thus, since projective modules are torsion-free, it follows that finitely generated non-singular right R-modules are torsion-free. Therefore, R satisfies condition (c) of Theorem 5.18, which implies that the maximal ring of quotients $Q^r(R)$ is a semi-simple Artinian ring. Note that $Q^r(R)$ and $Q^r(S)$ are Morita-equivalent by Lemma 6.3. Hence, $Q^r(S)$ is also semi-simple Artinian, since semi-simpleness and Artinian are properties preserved under a Morita-equivalence [2]. Furthermore, $Q^r(S)$ is a regular ring by Lemma 5.13. Therefore, Lemma 5.14 shows that S is right non-singular.

Let M be a finitely generated non-singular right S-module. Then, $G(M)$ is a finitely generated non-singular right R-module since non-singularity and being finitely generated are both Morita-invariant properties [2]. Thus, since R is a right strongly non-singular ring, $G(M)$ is isomorphic to a finitely generated submodule of a free right R-module P by Theorem 5.9. Note that as a free right R-module, P is projective, which is also a Morita-invariant property [2]. Hence, $F(P)$ is a projective right S-module. Furthermore, since $G(M)$ is isomorphic to a finitely generated submodule of P, $FG(M) \cong M$ is isomorphic to a finitely generated submodule U of $F(P)$. Now, $F(P)$ is projective and hence a submodule of a free right S-module, which implies $U \cong M$ is a submodule of a free right S-module. Therefore, M is isomorphic to a finitely generated submodule of a free right S-module, and S is right strongly non-singular by Theorem 5.9.

It has been shown that S is a right non-singular ring with a semi-simple Artinian maximal right ring of quotients. Thus, S has finite right Goldie dimension by Theorem 5.17. Hence, S cannot contain an infinite set of orthogonal idempotents. Moreover, S is a right p.p.-ring since R is right semi-hereditary. For if P is a principal right ideal of S, then $G(P)$ is a finitely generated right ideal of the right semi-hereditary ring R, which implies that $G(P)$ is projective. Hence, $FG(P) \cong P$ is projective, which again follows from projectivity being Morita-invariant. Then, S is a right strongly non-singular right p.p.-ring which does not contain an infinite set of orthogonal idempotents. Therefore, a right S-module is torsion-free if and only if it is non-singular by Theorem 5.19.

$(b) \Rightarrow (a)$: Assume that the classes of torsion-free and non-singular S-modules coincide for every ring S Morita-equivalent to R. Thus, since $Mat_n(R)$ is Morita-equivalent to R for every $0 < n < \omega$, the classes of torsion-free right $Mat_n(R)$-modules and non-singular right $Mat_n(R)$-modules coincide for every $0 < n < \omega$. Hence, $Mat_n(R)$ is a right Utumi p.p.-ring which does not contain an infinite set of orthogonal idempotents by Theorem 5.18. Thus, R is right semi-hereditary by Theorem 2.7. In particular, since these conditions are satisfied for every $0 < n < \omega$, they are satisfied for $n = 1$. Hence, $R \cong Mat_1(R)$ is a right semi-hereditary right Utumi ring not containing an infinite set of orthogonal idempotents.

It needs to be shown that R is right strongly non-singular. Let M be a finitely generated non-singular right R-module. By Corollary 5.10, R is right strongly non-singular if M is projective. Let $0 \to U \to F = \bigoplus^{n} R \to M \to 0$ be an exact sequence of right R-modules. Since F is a finitely generated free right R-module, it is a progenerator of Mod_R. Hence, $0 \to Hom_R(F, U) \to Hom_R(F, F) = End_R(F) \to Hom_R(F, M) \to 0$ is exact by Proposition 2.3. Moreover, if $S = End_R(F) \cong Mat_n(R)$, then $F : Mod_R \to Mod_S$ given by $F(M) = Hom_R(F, M)$ and $G : Mod_S \to Mod_R$ given by $G(N) = N \bigotimes_S F$ is an equivalence by Lemma 6.1. Thus, $Hom_R(F, M)$ is a non-singular right S-module by Proposition 6.4 (ii). Furthermore, since S is Morita-equivalent to R, the S-module $Hom_R(F, M)$ is torsion-free by assumption. Note that since the sequence is exact, $Hom_R(F, M) \cong S/Hom_R(F, U)$. Thus, $Hom_R(F, M)$ is cyclic as an S-module since $Hom_R(F, U)$ is a right ideal of the right S-module S. Note also that S is a left p.p.-ring by Theorem 2.11 since S is a right p.p.-ring which does not contain an infinite set of orthogonal idempotents. Thus, the cyclic torsion-free right S-module $Hom_R(F, M)$ is projective by Proposition 4.7. Therefore, $M \cong GF(M) = G(Hom_R(F, M))$ is a projective right R-module and R is right strongly non-singular.

$(a) \Rightarrow (c)$: Assume R is right strongly non-singular, right semi-hereditary, right Utumi, and does not contain an infinite set of orthogonal idempotents. It has been shown that any ring S Morita-equivalent to such a ring is right strongly non-singular and the classes of torsion-free and non-singular right S-modules coincide. Thus, $Mat_n(R)$ is right strongly non-singular and a right $Mat_n(R)$-module is torsion-free if and only if it is non-singular, which follows from $Mat_n(R)$ being Morita-equivalent to R for any $0 < n < \omega$. By Theorem 5.19, $Mat_n(R)$ is a right strongly non-singular right p.p.-ring which does not contain an infinite set of orthogonal idempotents. It then follows from Theorem 2.11 that $Mat_n(R)$ satisfies the ascending chain condition on right annihilators. Furthermore, Theorem 4.4 shows that since $Mat_n(R)$ is a right p.p.-ring, $Mat_n(R)$ is a torsion-free ring such that right annihilators of elements are finitely generated. Hence, $Mat_n(R)$ is a Baer-ring by Theorem 4.8. Moreover, Theorem 5.19 shows that every **S**-closed one-sided ideal of $Mat_n(R)$ is generated by an idempotent. Thus, every right ideal of $Mat_n(R)$ is a right annihilator and every left ideal of $Mat_n(R)$ is a left annihilator. Hence, $Mat_n(R)$ is a right and left Utumi ring.

$(c) \Rightarrow (a)$: Suppose $Mat_n(R)$ is a right and left Utumi Baer-ring for every $0 < n < \omega$ and does not contain an infinite set of orthogonal idempotents. Then, $Mat_n(R)$ is a right p.p.-ring, and so R is right semi-hereditary by Theorem 2.7. Furthermore, since $Mat_n(R)$ satisfies these conditions for every $0 < n < \omega$, $R \cong Mat_1(R)$ is a right and left Utumi Baer-ring not containing an infinite set of orthogonal idempotents. Thus, every **S**-closed one-sided ideal of R is an annihilator and hence generated by an idempotent. Therefore, since R is a right and left p.p.-ring and hence right and left non-singular, R is right strongly non-singular by Theorem 5.19. □

Corollary 6.6. *The following are equivalent for a ring R which does not contain an infinite set of orthogonal idempotents:*

(a) R is a right and left Utumi, right semi-hereditary ring.

(b) For every $0 < n < \omega$, $Mat_n(R)$ is a Baer-ring, and $Q^r(R)$ is torsion-free as a right R-module.

Proof. $(a) \Rightarrow (b)$: Suppose R is right and left Utumi and right semi-hereditary. Then, R is a right p.p.-ring and hence right non-singular. Moreover, R is a left p.p.-ring by Theorem 2.11, which implies that R is also a left non-singular ring. Since R is both right and left Utumi, $Q^r(R) = Q^l(R)$ by Theorem 5.16. Furthermore, since R is a right Utumi right p.p.-ring which does not contain an infinite set of orthogonal idempotents, $Q^r(R) = Q^l(R)$ is semi-simple Artinian and torsion-free by Theorem 5.18. Therefore, R is right strongly non-singular by Theorem 5.19.

Since R is a right strongly non-singular, right semi-hereditary, right Utumi ring not containing an infinite set of orthogonal idempotents, the classes of torsion-free and non-singular right $Mat_n(R)$-modules coincide by Theorem 6.5. Moreover, the proof of Theorem 6.5 shows that $Mat_n(R)$ is right strongly non-singular. Thus, $Mat_n(R)$ is a right strongly non-singular, right p.p.-ring not contain an infinite set of orthogonal idempotents by Theorem 5.19. It then follows from Theorem 2.11 that $Mat_n(R)$ satisfies the ascending chain condition on right annihilators. Since $Mat_n(R)$ is a right p.p.-ring, Theorem 4.4 shows that $Mat_n(R)$ is a torsion-free ring such that right annihilators of elements are finitely generated. Hence, $Mat_n(R)$ is a Baer-ring by Theorem 4.8.

$(b) \Rightarrow (a)$: Assume $Mat_n(R)$ is a Baer-ring for every $0 < n < \omega$, and $Q^r(R)$ is torsion-free as a right R-module. Since $Mat_n(R)$ is a Baer-ring, it is both a right and left p.p.-ring.

Hence, R is both right and left semi-hereditary by Theorem 2.7. It then readily follows that R is right and left non-singular. Note also that $R \cong Mat_1(R)$ is a Baer-ring since $Mat_n(R)$ is Baer for every $0 < n < \omega$. Let I be a proper **S**-closed right ideal of R. Then, R/I is non-singular as a right R-module. Furthermore, R/I is cyclic and thus finitely generated. Hence, R/I is isomorphic to a submodule of a free Q^r-module by Theorem 5.8. Since Q^r is assumed to be torsion-free as a right R-module, it follows from Proposition 4.6 that I is generated by an idempotent $e \in R$. Hence, $I = ann_r(1-e)$ by Lemma 4.5 and R is right Utumi. Observe that the argument works for **S**-closed left ideals as well, and so R is also left Utumi. □

We conclude by considering two examples, the first of which illustrates why the condition of being right semi-hereditary is necessary in the main theorem. Let $R = \mathbb{Z}[x]$. As an integral domain, R is a strongly non-singular p.p.-ring not containing an infinite set of orthogonal idempotents [1, Corollary 3.10]. By Theorem 5.19, the classes of torsion-free and non-singular right R-modules coincide, and by Theorem 5.18 R is right Utumi. However, R is not semi-hereditary since the ideal $x\mathbb{Z}[x] + 2\mathbb{Z}[x]$ of $\mathbb{Z}[x]$ is not projective. As seen in the proof of Theorem 2.7, this implies $S = Mat_2(R)$ is not a right or left p.p.-ring, and hence not a Baer ring. Therefore, the main theorem does not hold if R is not assumed to be right semi-hereditary. Moreover, this example shows that the classes of torsion-free and non-singular S-modules do not necessarily coincide, even if this holds for R and there is a Morita-equivalence between R and S.

Finally, we consider an example from [3] which details a ring with finite right Goldie-dimension but infinite left Goldie-dimension. In the context of this thesis, this example provides a right Utumi Baer-ring which is not left Utumi. Let $K = F(y)$ for some field F and consider the endomorphism f of K determined by $y \mapsto y^2$. The ring we consider is $R = K[x]$ with coefficients written on the right and multiplication defined according to $kx = xf(k)$ for $k \in K$. Observe that $yx = xy^2$. It can be shown that $Rx \cap Rxy = 0$, and hence $Rxy \bigoplus Rxyx \bigoplus Rxyx^2 \bigoplus ... \bigoplus Rxyx^k \bigoplus ...$ is an infinite direct sum of left ideals of

R. Thus, R has infinite left Goldie-dimension. On the other hand, every right ideal of R is a principal ideal [3], and thus R is right Noetherian. Hence, R is a right Goldie-ring. It then follows from Theorem 5.18 that R is a right Utumi Baer ring and Q^r is semi-simple Artinian. However, R having infinite left Goldie-dimension but finite right Goldie-dimension implies that $Q^r \neq Q^l$ [1, Proposition 4.1]. Therefore, Theorem 5.16 shows that R cannot be left Utumi.

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