Inverse Eigenvalue Problem for Euclidean Distance Matrices

by

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Abstract

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This paper examines the inverse eigenvalue problem (IEP) for the particular class of Euclidean distance matrices ($\mathbb{EDM}$). Studying the necessary and sufficient conditions for a matrix to be an element of $\mathbb{EDM}$ gives us a better understanding as to the necessary conditions placed on the numbers to be the eigenvalues of some Euclidean distance matrix. Using this necessary conditions, Hayden was able to solve the IEP order n using a Hadamard matrix of the same order. After 10 years, an error in his construction of Hayden's solution order $n+1$ was noted and corrected accordingly however the result was not a solution to the IEP of order $n+1$.

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List of Symbols

$\mathbb{EDM}$	set of Euclidean distance matrices
$\mathbf{x}$	bold letter x indicates vector
$\mathbb{R}^n$	real vector space with dimension n
$\mathbb{R}^{n\times n}$	set of all real valued matrices of order n
$\sigma(D)$	set of eigenvalues of matrix D
$\\| \bullet \\|$	norm of a vector
$\\| \bullet \\|_2$	Euclidean norm of a vector
M^T	transpose of matrix M
$\mathbb{SYM}_0^+$	set of all symmetric non-negative matrices with 0 diagonal
$\{\mathbf{x}\}^\perp$	orthogonal subspace of vector $\mathbf{x}$
IEP	inverse eigenvalue problem

Chapter 1

What is an inverse eigenvalue problem?

An inverse eigenvalue problem deals with the reconstruction of a matrix from given spectral data (including complete or partial information of eigenvalues or eigenvectors). Often, reconstructing a matrix with some particular structure (such as being a circulant matrix or a tridiagonal matrix) given spectral data is the aim. Also, given spectral data, does a matrix with some particular structure even exist? Considerable time has been spent on finding the necessary and sufficient conditions in order for a given inverse eigenvalue problem to have a solution. An efficient algorithm for finding such a solution to an inverse eigenvalue problem is normally quite difficult.

1.1 Examples of solved inverse eigenvalue problems

For some classes of matrices, the inverse eigenvalue problem has been solved. For instance, the inverse eigenvalue problem for a circulant matrix $C \in \mathbb{R}^{n\times n}$ has been solved, along with symmetric tridiagonal matrix $T \in \mathbb{R}^{n\times n}$.

1.1.1 Circulant matrices

A circulant matrix $C \in \mathbb{R}^{n\times n}$ is of the form $\begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} \\ \alpha_{n-1} & \alpha_0 & \alpha_1 & \cdots & \alpha_{n-2} \\ \alpha_{n-2} & \alpha_{n-1} & \alpha_0 & \cdots & \alpha_{n-3} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \alpha_1 & \alpha_2 & \dots & \alpha_{n-1} & \alpha_0 \end{bmatrix}$.

Each row of a circulant matrix is the previous row cycled horizontally one place to the right.

Define a basic circulant $P = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}$, where $\alpha_{n-1} = 1$, and $\alpha_i = 0$ for $i \neq n-1$.

Notice that the minimal polynomial for P is $q(t) = t^n - 1$. So $\sigma(P)$ is the set of distinct n^{th} roots of unity, say $\{\gamma^0, \gamma^1, \dots, \gamma^{n-1}\}$ where $\gamma = e^{\frac{2\pi i}{n}}$. Let $\{\lambda_0, \lambda_1, \dots, \lambda_{n-1}\}$ be the eigenvalues of circulant matrix C^*, and therefore the roots of characteristic polynomial of C^*. By interpolation, there is a polynomial $Q(x)$ of degree $n-1$ that maps each γ_i to λ_i. So, $C^* = Q(P)$ and hence $\sigma(Q(P)) = \sigma(C^*) = \{\lambda_0, \dots, \lambda_{n-1}\}$.

1.1.2 Real symmetric tridiagonal matrices

Real symmetric tridiagonal matrices are of the form $\begin{bmatrix} \alpha_1 & \beta_1 & 0 & \cdots & 0 \\ \beta_1 & \alpha_2 & \beta_2 & \ddots & \vdots \\ 0 & \beta_2 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \alpha_{n-1} & \beta_{n-1} \\ 0 & \dots & 0 & \beta_{n-1} & \alpha_n \end{bmatrix}$.

Given real numbers $\lambda_1 < \mu_1 < \lambda_2 < \cdots < \lambda_{n-1} < \mu_{n-1} < \lambda_n$, there exists a unique tridiagonal (or Jacobi matrix), $T = \begin{bmatrix} \alpha_1 & \beta_1 & 0 & \cdots & 0 \\ \beta_1 & \alpha_2 & \beta_2 & \ddots & \vdots \\ 0 & \beta_2 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \alpha_{n-1} & \beta_{n-1} \\ 0 & \dots & 0 & \beta_{n-1} & \alpha_n \end{bmatrix}$ with $\beta_i > 0$ such that $\sigma(T) = \{\lambda_1, \dots, \lambda_n\}$ and $\sigma(T_{1,1}) = \{\mu_1, \dots, \mu_{n-1}\}$, where $T_{1,1} \in \mathbb{R}^{(n-1)\times(n-1)}$ denotes the matrix $T \in \mathbb{R}^{n\times n}$ with column 1 and row 1 removed [1].

Chapter 2

Definition and existence of an Euclidean distance matrix

Distance matrices are a class of matrices subject to the condition that each entry represents the distance pairwise between points in a finite dimension k. Storing distances among points in matrix form is relatively efficient. There are obviously necessary conditions for an $n \times n$ matrix M to possibly be a distance matrix; however the initial obvious conditions are not sufficient to classify as a distance matrix. Euclidean distance matrices are a sub-class of distance matrices in that the distance is measured using the Euclidean metric.

2.1 $\mathbb{EDM}$ definition

An Euclidean space is a finite-dimensional real vector space $\mathbb{R}^k$ with inner product defined by $\langle \mathbf{x}, \mathbf{y} \rangle = \langle (x_1, x_2, \ldots, x_k), (y_1, y_2, \ldots y_k) \rangle = x_1 y_1 + x_2 y_2 + \cdots + x_k y_k$. The inner product induces a distance or metric $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|_2 = \sqrt{\Sigma_{i=1}^{k} (\mathbf{x}_i - \mathbf{y}_i)^2}$. An element of the set of Euclidean distance matries, denoted $\mathbb{EDM}$, is derived by a complete list of the squared distances between pairs of points from a list of k $\{\mathbf{x}_i, i = 1 \ldots k\}$. The entries of the distance matrix are defined by $d_{ij} = \|\mathbf{x}_i - \mathbf{x}_j\|_2^2 = \langle \mathbf{x}_i - \mathbf{x}_j, \mathbf{x}_i - \mathbf{x}_j \rangle$, which represents the distance (squared) between the i^{th} and j^{th} point. Note that the number of points in the list N, does not have to equal the dimension of the space k.

The absolute distance squared between points $\mathbf{x}_i$ and $\mathbf{x}_j$ for $i, j = 1, \ldots, N$ must satisfy the metric space properties:

1. $d_{ij} \geq 0$, $i \neq j$
2. $d_{ij} = 0$ if and only if $\mathbf{x}_i = \mathbf{x}_j$

3. $d_{ij} = d_{ji}$

4. $d_{ij} \leq d_{ik} + d_{kj}$, $i \neq j \neq k$

A distance matrix D has N^2 entries but since it is symmetric and has diagonal 0, D contains only $N(N-1)/2$ pieces of information. Note that, $\mathbb{EDM}$, the set of Euclidean squared-distance matrices, is a subset of $\mathbb{R}_+^{N \times N}$.

2.2 Characterization of $\mathbb{EDM}$

Although the entries in any distance matrix D must satisfy the metric properties given above, that is not a sufficient condition for the matrix D to be in $\mathbb{EDM}$. It wasn't until 1982, that J. C. Gower gave necessary and sufficient conditions for a matrix D to be in $\mathbb{EDM}$. His result, according to many mathematicians, is rather late given its significance. Gower's results are explained in Theorem 2.1. Following the proof, is an example as to why matrix a $A \in \mathbb{R}^{n \times n}$, with entries only satisfying the metric properties may not be and element of $\mathbb{EDM}$.

Theorem 2.1. *Let $\mathbf{s} \in \mathbb{R}^n$, with $\mathbf{s}^T\mathbf{e} = 1$, $\mathbf{e}$ the all 1's vector. D is a distance matrix if and only if*

1. *D is symmetric,*

2. *$diag(D) = 0$ and*

3. *$(I - \mathbf{e}\mathbf{s}^T)D(I - \mathbf{s}\mathbf{e}^T)$ is negative semidefinite with rank bounded above by k.*

Proof. ($\Rightarrow$) Let $D = [d_{ij}]$ be a distance matrix defined by the points in the list $\{\mathbf{x}_\ell \in \mathbb{R}^n$, $\ell = 1, \ldots, N\}$. Define $X := [\mathbf{x}_1, \ldots, \mathbf{x}_N] \in \mathbb{R}^{n \times N}$ where each vector in the list is a column of the matrix X. Each entry in D, $d_{ij} = \|\mathbf{x}_i - \mathbf{x}_j\|^2 = \|\mathbf{x}_i\|^2 + \|\mathbf{x}_j\|^2 - 2\mathbf{x}_i^T\mathbf{x}_j$. Therefore $D = X\mathbf{e}^T + \mathbf{e}X^T - 2P^TP$ where $P_{k \times n} = [\mathbf{x}_1, \ldots, \mathbf{x}_n]$, $\mathbf{x}_i \in \mathbb{R}^k$ and $X = [\|\mathbf{x}_1\|^2, \ldots, \|\mathbf{x}_n\|^2]^T$. Let $\mathbf{u} \in \mathbb{R}^n$ then $\mathbf{u}^T(I - \mathbf{e}\mathbf{s}^T)D(I - \mathbf{s}\mathbf{e}^T)\mathbf{u}$=$((I - \mathbf{s}\mathbf{e}^T)\mathbf{u})^T D(I - \mathbf{s}\mathbf{e}^T)\mathbf{u}$. Let $\mathbf{v} = (I - \mathbf{s}\mathbf{e}^T)\mathbf{u}$ then $\mathbf{v} = (I - \mathbf{s}\mathbf{e}^T)\mathbf{u} \in \{\mathbf{e}\}^\perp$ since $\mathbf{e}^T(I - \mathbf{s}\mathbf{e}^T)\mathbf{u} = (\mathbf{e}^T - ((\mathbf{e}^T\mathbf{s})\mathbf{e}^T)\mathbf{u} = (\mathbf{e}^T - \mathbf{e}^T)\mathbf{u} = \mathbf{0}^T \cdot \mathbf{u} = 0$.

$$\begin{aligned}
\mathbf{v}^T D\mathbf{v} &= \mathbf{v}^T(X\mathbf{e}^T + \mathbf{e}X^T - 2P^TP)\mathbf{v} \\
&= \mathbf{v}^T X\mathbf{e}^T\mathbf{v} + \mathbf{v}^T\mathbf{e}X^T\mathbf{v} - 2\mathbf{v}^TP^TP\mathbf{v} \\
&= -2(P\mathbf{v})^T(P\mathbf{v}) \\
&= -2\|P\mathbf{v}\|^2 \le 0
\end{aligned}$$

($\Leftarrow$) Let $D = D^T$, diag(D)=0 and $(I-\mathbf{e}\mathbf{s}^T)D(I-\mathbf{s}\mathbf{e}^T)$ be negative semidefinite. Therefore $-\frac{1}{2}(I-\mathbf{e}\mathbf{s}^T)D(I-\mathbf{s}\mathbf{e}^T)$ is positive semidefinite. Since rank$\{(I-\mathbf{e}\mathbf{s}^T)D(I-\mathbf{s}\mathbf{e}^T)\} \le k$, $\exists\ Q_{k\times n} = [\mathbf{y}_1, ..., \mathbf{y}_n]$, $\mathbf{y}_i \in \mathbb{R}^k$ such that $Q^TQ = -\frac{1}{2}(I-\mathbf{e}\mathbf{s}^T)D(I-\mathbf{s}\mathbf{e}^T)$. Then, $-2Q^TQ = (I-\mathbf{e}\mathbf{s}^T)D(I-\mathbf{s}\mathbf{e}^T) = D - \mathbf{e}\mathbf{s}^TD - D\mathbf{s}\mathbf{e}^T + \mathbf{e}\mathbf{s}^TD\mathbf{s}\mathbf{e}^T$. Let $\mathbf{g} = D\mathbf{s} - \frac{1}{2}\mathbf{s}^TD\mathbf{s}\mathbf{e}$. Then $-2Q^TQ = D - \mathbf{g}\mathbf{e}^T - \mathbf{e}\mathbf{g}^T$. After rearranging, $D = \mathbf{g}\mathbf{e}^T + \mathbf{e}\mathbf{g}^T - 2Q^TQ$. So, $0 = d_{ii} = g_i + g_i - 2\mathbf{y}_i^T\mathbf{y}_i$, and therefore $g_i = \mathbf{y}_i^T\mathbf{y}_i = \|\mathbf{y}_i\|^2$. Finally, $d_{ij} = \|\mathbf{y}_i\|^2 + \|\mathbf{y}_j\|^2 - 2\mathbf{y}_i^T\mathbf{y}_j = \|\mathbf{y}_i - \mathbf{y}_j\|^2$.

$\square$

Observation 2.2. *If* $\mathbf{s}^T\mathbf{e} = 1$ *then* $(I - \mathbf{s}\mathbf{e}^T)\mathbf{x} \in \{\mathbf{e}\}^\perp$ *for all* $\mathbf{x}$ *since* $\mathbf{e}^T(I-\mathbf{s}\mathbf{e}^T)\mathbf{x} = (\mathbf{e}^T - \mathbf{e}^T)\mathbf{x} = 0$. *Therefore, condition 3 of Theorem 2.1 can be restated as* D *is negative semidefinite on the subspace* $\{\mathbf{e}\}^\perp$ *for* $D \in \mathbb{EDM}$.

Observation 2.3. *By the pure definition and metric properties, if* $D \in \mathbb{EDM}$, D *must be symmetric, positive real valued with* 0 *diagonal to have a chance of being in* $\mathbb{EDM}$. *From now on, denote the set of* $n \times n$ *square symmetric zero diagonal non-negative matrices as* $\mathbb{SYM}_0^+$. *Therefore Theorem 2.1 could be restated more simply: Let* $D \in \mathbb{SYM}_0^+$. *Then* $D \in \mathbb{EDM}$ *iff* D *is negative semidefinite on* $\{\mathbf{e}\}^\perp$.

Together, conditions *1*, *2*, and *3* of Theorem 2.1 are the necessary and sufficient conditions for a matrix $D \in \mathbb{EDM}$. The matrix $A \in \mathbb{SYM}_0^+$ solely satisfying the metric properties does not guarantee $A \in \mathbb{EDM}$.

Example 1: Define

$$A = \begin{bmatrix} 0 & 1 & 1 & 4 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 4 & 1 & 1 & 0 \end{bmatrix}.$$

Upon inspection, the entries in A satisfy each metric property 1-4 and conditions *1* and *2* of Theorem 2.1 therefore A is in $\mathbb{SYM}_0^+$. In checking the last condition, let $\mathbf{s} = \begin{bmatrix} 1 & -1 & 0 & 1 \end{bmatrix}^T$, with $\mathbf{s}^T\mathbf{e} = 1$. Compute $\hat{A} = (I - \mathbf{e}\mathbf{s}^T)A(I - \mathbf{s}\mathbf{e}^T)$

$$\begin{aligned} \hat{A} &= \begin{bmatrix} 0 & 1 & 0 & -1 \\ -1 & 2 & 0 & -1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 4 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 4 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & -1 & -1 \\ 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 0 & 1 & 2 \\ 0 & 0 & 2 & 0 \\ 1 & 2 & 2 & 1 \\ 2 & 0 & 1 & -2 \end{bmatrix}. \end{aligned}$$

Condition *3* of Theorem 2.1 states $\hat{A} = (I - \mathbf{e}\mathbf{s}^T)A(I - \mathbf{s}\mathbf{e}^T)$ must be negative semidefinite in order for $A \in \mathbb{EDM}$. By the definition of negative semidefinite, and our computation, the inequality $\mathbf{x}^T(I - \mathbf{e}\mathbf{s}^T)A(I - \mathbf{s}\mathbf{e}^T)\mathbf{x} = \mathbf{x}^T\hat{A}\mathbf{x} \le 0$ must hold for any $\mathbf{x} \in \mathbb{R}^4$. But $\mathbf{x}^T\hat{A}\mathbf{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -2 & 0 & 1 & 2 \\ 0 & 0 & 2 & 0 \\ 1 & 2 & 2 & 1 \\ 2 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = 2$. Therefore the matrix A cannot be an Euclidean distance matrix.

2.3 Eigenvalue properties of $\mathbb{EDM}$

Before attempting to solve an inverse eigenvalue problem for $D \in \mathbb{EDM}$, as with any class of matrices, it is important to have an understanding of the way the eigenvalues of D behave, relationships with their eigenvectors and other spectral properties of $\mathbb{EDM}$ as a set. Each potential distance matrix $M \in \mathbb{SYM}_0^+$ is symmetric and real-valued. Therefore $\sigma(M) \in \mathbb{R}$ since $\bar{\lambda}\langle \mathbf{x}, \mathbf{x}\rangle = \langle \lambda\mathbf{x}, \mathbf{x}\rangle = \langle M\mathbf{x}, \mathbf{x}\rangle = \langle \mathbf{x}, M^T\mathbf{x}\rangle = \langle \mathbf{x}, M\mathbf{x}\rangle = \langle \mathbf{x}, \lambda\mathbf{x}\rangle = \lambda\langle \mathbf{x}, \mathbf{x}\rangle$ for any $\lambda \in \sigma(M)$. Moreover, if $M \in \mathbb{SYM}_0^+$ with $\sigma(M) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, then $trace(M) = 0$; meaning that $\sum_{i=1}^{n} \lambda_i = 0$. These properties of eigenvalues of any matrix in $\mathbb{SYM}_0^+$ gives us an obvious starting point for the purpose of reconstructing an Euclidean distance matrix from spectral data. Next it will be useful to state and prove the Courant-Fisher Theorem for the real symmetric case.

Theorem 2.4 (The Courant-Fischer Theorem)**.** *Let A be an $n \times n$ real symmetric matrix with ordered eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \leq \lambda_n$ and L a subspace of $\mathbb{R}^n$. Define $F(L) = F_A(L) = max\{\mathbf{x}^T A\mathbf{x} | \mathbf{x} \in L, \|\mathbf{x}\|_2 = 1\}$, and define $f(L) = f_A(L) = min\{\mathbf{x}^T A\mathbf{x} | \mathbf{x} \in L, \|\mathbf{x}\|_2 = 1\}$. Then, $\lambda_k = min\{F(L) | dim(L) = k\} = max\{f(L) | dim(L) = n - k + 1\}$.*

Proof. Let A be an $n \times n$ real symmetric matrix with ordered eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \leq \lambda_n$ and L a subspace of $\mathbb{R}^n$. Define $F(L) = F_A(L) = max\{\mathbf{x}^T A\mathbf{x} | x \in L, \|\mathbf{x}\|_2 = 1\}$, and define $f(L) = f_A(L) = min\{\mathbf{x}^T A\mathbf{x} | x \in L, \|\mathbf{x}\|_2 = 1\}$. Then $f(L) \leq min\{f(L) | dim(L) = k\} \leq max\{f(L) | dim(L) = k\} \leq max\{f(L) | dim(L) = n - k + 1\} = \lambda_k$. Similarly, $F(L) \geq max\{F(L) | dim(L) = n - k + 1\} \geq min\{F(L) | dim(L) = n - k + 1\} \geq min\{F(L) | dim(L) = k\} = \lambda_k$. □

Definition 2.5. *A symmetric matrix B is said to be almost negative semidefinite if $\mathbf{x}^T B\mathbf{x} \leq 0$ for all $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{x}^T\mathbf{e} = 0$, ($\mathbf{x} \in \{\mathbf{e}\}^\perp$).*

The following theorem was adapted from J. Ferland and J.P. Crouzeix's theorem on properties of almost positive semidefinite matrices and gives some useful conditions for the analogous almost negative semidefinite matrices. This, in turn, gives alternate necessary and sufficient conditions for a matrix A to be in $\mathbb{EDM}$ involving spectral data, which is useful for the study of inverse eigenvalue problems, [3].

Theorem 2.6. *Let $D \in \mathbb{SYM}_0^+$ and $D \neq 0$. $D \in \mathbb{EDM}$ if and only if*

1. *D has exactly one positive eigenvalue,*
2. *$\exists\ \mathbf{w} \in \mathbb{R}^n$ such that $D\mathbf{w} = \mathbf{e}$ and $\mathbf{w}^T\mathbf{e} \geq 0$.*

Proof. $(\Rightarrow)(1)$ Let $D \in \mathbb{EDM}$. Suppose $L \subseteq \mathbb{R}^n$. Define $F(L) = \max\{\mathbf{x}^T D\mathbf{x} | \ \mathbf{x} \in L, \|\mathbf{x}\|_2 = 1\}$. By the Courant-Fisher Theorem, $\lambda_k = \min\{F(L) | \dim(L) = k\}$. Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the ordered eigenvalues of D. $\lambda_{n-1} = \min\{F(\{\mathbf{e}\}^\perp) | \dim(\{\mathbf{e}\}^\perp) = n-1\} \leq F(\{\mathbf{e}\}^\perp)$. By Theorem 2.1, $F(\{\mathbf{e}\}^\perp) \leq 0$ because $\left(I - \mathbf{s}\mathbf{e}^T\right)\mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in \{\mathbf{e}\}^\perp$. So, $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1} \leq 0$. If $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = 0$, then $\lambda_n = 0$ since $Trace(D) = \sum_{i=1}^{n} \lambda_i = 0$. If D had all zero eigenvalues, D would be orthonormally similar to the 0 matrix which is a contradiction. Therefore there are some non-zero eigenvalues of D and $\lambda_n > 0$.

$(\Rightarrow)$ (2) By (1) D is not negative semidefinite. Suppose $\mathbf{e} \notin \text{range}(D)$, therefore there does not exist $\mathbf{w} \in \mathbb{R}^n$ such that $D\mathbf{w} = \mathbf{e}$. Recall, $\mathbf{e}$ can be written as $\mathbf{e} = \mathbf{x} + \mathbf{y}$ where $\mathbf{x} \in \ker(D)$ and $\mathbf{y} \in \text{range}(D)$. Note that $D = D^T$, so range(D)=range(D^T). Since $\mathbf{e} \notin \text{range}(D)$, $\mathbf{x} \neq 0$. $\mathbf{x}^T\mathbf{e} = \mathbf{x}^T(\mathbf{x}+\mathbf{y}) = \mathbf{x}^T\mathbf{x} + \mathbf{x}^T\mathbf{y} = \mathbf{x}^T\mathbf{x}$ since $\ker(D) \perp \text{range}(D)$, so $\mathbf{x}^T\mathbf{y} = 0$. Let $\mathbf{v} \in \mathbb{R}^n$. Define $\mathbf{h} = \left(I - \frac{1}{\mathbf{x}^T\mathbf{e}}\mathbf{x}\mathbf{e}^T\right)\mathbf{v}$.

$$\begin{aligned}
\mathbf{h}^T D\mathbf{h} &= \left(\left(I - \frac{1}{\mathbf{x}^T\mathbf{e}}\mathbf{x}\mathbf{e}^T\right)\mathbf{v}\right)^T D\left(\left(I - \frac{1}{\mathbf{x}^T\mathbf{e}}\mathbf{x}\mathbf{e}^T\right)\mathbf{v}\right) \\
&= \mathbf{v}^T D\mathbf{v} - \frac{1}{\mathbf{x}^T\mathbf{e}}\mathbf{v}^T D\mathbf{x}\mathbf{e}^T\mathbf{v} - \frac{1}{\mathbf{x}^T\mathbf{e}}\mathbf{v}^T\mathbf{e}\mathbf{x}^T D\mathbf{v} + \frac{1}{(\mathbf{x}^T\mathbf{e})^2}\mathbf{v}^T\mathbf{e}\mathbf{x}^T D\mathbf{x}\mathbf{e}^T\mathbf{v} \\
&= \mathbf{v}^T D\mathbf{v}.
\end{aligned}$$

Note that

$$\begin{aligned}
\mathbf{h}^T\mathbf{e} &= \mathbf{v}^T\left(I - \frac{1}{\mathbf{x}^T\mathbf{e}}\mathbf{e}\mathbf{x}^T\right)\mathbf{e} \\
&= \mathbf{v}^T\mathbf{e} - \mathbf{v}^T\left(\frac{1}{\mathbf{x}^T\mathbf{e}}\mathbf{e}\mathbf{x}^T\mathbf{e}\right) \\
&= \mathbf{v}^T\mathbf{e} - \mathbf{v}^T\mathbf{e} \\
&= 0.
\end{aligned}$$

So by Theorem 2.1 with $\mathbf{s} = \left(\frac{1}{\mathbf{x}^T\mathbf{e}}\right)\mathbf{x}$, $\mathbf{h}^T D\mathbf{h} = \mathbf{v}^T D\mathbf{v} \le 0$. D cannot be negative semidefinite so $\mathbf{e} \in \text{range}(D)$. Therefore there exists $\mathbf{w} \in \mathbb{R}^n$ such that $D\mathbf{w} = \mathbf{e}$.

Now suppose $\mathbf{w}^T\mathbf{e} \ne 0$. We are going to show $\mathbf{w}^T\mathbf{e} > 0$. Let $\mathbf{u}$ be an eigenvector corresponding to the positive eigenvalue of D. Let $\mathbf{g} = \left(I - (\frac{1}{\mathbf{w}^T\mathbf{e}})\mathbf{w}\mathbf{e}^T\right)\mathbf{u} = \left(\mathbf{u} - (\frac{1}{\mathbf{w}^T\mathbf{e}})\mathbf{w}\mathbf{e}^T\mathbf{u}\right)$. Re arrange to obtain, $\mathbf{u} = \mathbf{g} + (\frac{1}{\mathbf{w}^T\mathbf{e}})\mathbf{w}\mathbf{e}^T\mathbf{u}$, then compute

$$\begin{aligned}
\mathbf{u}^T D\mathbf{u} &= \left(\mathbf{g} + \frac{1}{\mathbf{w}^T\mathbf{e}}\mathbf{w}\mathbf{e}^T\mathbf{u}\right)^T D\left(\mathbf{g} + \frac{1}{\mathbf{w}^T\mathbf{e}}\mathbf{w}\mathbf{e}^T\mathbf{u}\right) \\
&= \mathbf{g}^T D\mathbf{g} + \mathbf{g}^T D\frac{1}{\mathbf{w}^T\mathbf{e}}\mathbf{w}\mathbf{e}^T\mathbf{u} + \mathbf{u}^T\frac{1}{\mathbf{e}^T\mathbf{w}}\mathbf{e}\mathbf{w}^T D\mathbf{g} + \mathbf{u}^T\frac{1}{(\mathbf{e}^T\mathbf{w})^2}\mathbf{e}\mathbf{w}^T D\mathbf{w}\mathbf{e}^T\mathbf{u}.
\end{aligned}$$

Using $D\mathbf{w} = \mathbf{e}$ to simplify,

$$\mathbf{u}^T D\mathbf{u} = \mathbf{g}^T D\mathbf{g} + \frac{\mathbf{g}^T\mathbf{e}\mathbf{e}^T\mathbf{u}}{\mathbf{w}^T\mathbf{e}} + \frac{\mathbf{u}^T\mathbf{e}\mathbf{e}^T\mathbf{g}}{\mathbf{e}^T\mathbf{w}} + \frac{\mathbf{u}^T\mathbf{e}\mathbf{w}^T\mathbf{e}\mathbf{e}^T\mathbf{u}}{\mathbf{e}^T\mathbf{w}\mathbf{w}^T\mathbf{e}}.$$

Since $\mathbf{g}^T\mathbf{e} = \left(\left(I - \frac{1}{\mathbf{w}^T\mathbf{e}}\mathbf{w}\mathbf{e}^T\right)\mathbf{u}\right)^T\mathbf{e} = 0$,

We obtain

$$
\begin{aligned}
\mathbf{u}^T D\mathbf{u} &= \mathbf{g}^T D\mathbf{g} + \frac{(\mathbf{u}^T\mathbf{e})(\mathbf{u}^T\mathbf{e})^T}{(\mathbf{w}^T\mathbf{e})(\mathbf{w}^T\mathbf{e})^T}(\mathbf{w}^T\mathbf{e}) \\
&= \mathbf{g}^T D\mathbf{g} + \left(\frac{\mathbf{u}^T\mathbf{e}}{\mathbf{w}^T\mathbf{e}}\right)^2 \mathbf{w}^T\mathbf{e}.
\end{aligned}
$$

Since $\mathbf{u}$ is an eigenvector corresponding to the positive eigenvalue of D, $\mathbf{u}^T D\mathbf{u} > 0$. Noting that $\mathbf{g} \in \{\mathbf{e}\}^\perp$ and $\mathbf{g}^T D\mathbf{g} \leq 0$, by Theorem 2.1. Therefore $\mathbf{w}^T\mathbf{e} > 0$.

($\Leftarrow$) Let λ be the positive eigenvalue of D with orthonormal eigenvector $\mathbf{u} \neq \mathbf{0}$. i.e. $D\mathbf{u} = \lambda\mathbf{u}$. Since the remaining eigenvalues of D are non-positive, D can be written as $D = \lambda\mathbf{u}\mathbf{u}^T - C^T C$. By assumption, there exists $\mathbf{w} \in \mathbb{R}^n$ satisfying $D\mathbf{w} = \mathbf{e}$ and $\mathbf{w}^T\mathbf{e} \geq 0$.

$D\mathbf{w} = (\lambda\mathbf{u}\mathbf{u}^T - C^T C)\mathbf{w} = \lambda\mathbf{u}^T\mathbf{w}\mathbf{u} - C^T C\mathbf{w} = \mathbf{e}$ (*)

and,

$\mathbf{w}^T D\mathbf{w} = \lambda\mathbf{w}^T\mathbf{u}^T\mathbf{w}\mathbf{u} - \mathbf{w}^T C^T C\mathbf{w} = \mathbf{w}^T\mathbf{e} \geq 0.$ (**)

If $\mathbf{u}^T\mathbf{w} = 0$, then $-\|C\mathbf{w}\|^2 = -\mathbf{w}^T C^T C\mathbf{w} = \mathbf{w}^T\mathbf{e} \geq 0$, by (**). This implies $C\mathbf{w} = \mathbf{0}$. Then (*) leads to the contradiction $\mathbf{e} = \mathbf{0}$. We conclude $\mathbf{u}^T\mathbf{w} \neq 0$, and $C\mathbf{w} \neq 0$. Since $\lambda > 0$ and $\mathbf{u}^T\mathbf{w} \neq 0$, (*) may be rearranged to give $\mathbf{u} = \frac{1}{\lambda\mathbf{u}^T\mathbf{w}}(\mathbf{e} + C^T C\mathbf{w})$.

$$
\begin{aligned}
D &= \lambda\mathbf{u}\mathbf{u}^T - C^T C \\
&= \lambda\left(\frac{1}{\lambda\mathbf{u}^T\mathbf{w}}(\mathbf{e} + C^T C\mathbf{w})\right)\left(\frac{1}{\lambda\mathbf{u}^T\mathbf{w}}(\mathbf{e} + C^T C\mathbf{w})\right)^T - C^T C \\
&= \frac{1}{\lambda(\mathbf{u}^T\mathbf{w})^2}(\mathbf{e}\mathbf{e}^T + \mathbf{e}\mathbf{w}^T C^T C + C^T C\mathbf{w}\mathbf{e}^T + C^T C\mathbf{w}\mathbf{w}^T C^T C) - C^T C.
\end{aligned}
$$

Let $\mathbf{h} \in \mathbb{R}^n$ satisfy $\mathbf{h}^T\mathbf{e} = 0$. Then

$$\mathbf{h}^T D\mathbf{h} = \frac{1}{\lambda(\mathbf{u}^T\mathbf{w})^2}(\mathbf{h}^T C^T C\mathbf{w}\mathbf{w}^T C^T C\mathbf{h}) - \mathbf{h}^T C^T C\mathbf{h}.$$

By (**), $\lambda(\mathbf{u}^T\mathbf{w})^2 \geq \|C\mathbf{w}\|^2$. Therefore,

$$\begin{aligned}
\mathbf{h}^T D\mathbf{h} &\leq \frac{1}{\|C\mathbf{w}\|^2}(\mathbf{h}^{\mathbf{T}} C^T C\mathbf{w})^2 - \|C\mathbf{h}\|^2 \\
&= \frac{1}{(C\mathbf{w})^T(C\mathbf{w})}[(C\mathbf{h})^T(C\mathbf{w})]^2 - (C\mathbf{h})^T(C\mathbf{h}) \\
&\leq 0,
\end{aligned}$$

since by Cauchy-Schwarz we have $[(C\mathbf{h})^T(C\mathbf{w})]^2 \leq (C\mathbf{w})(C\mathbf{w})^T(C\mathbf{h})(C\mathbf{h})$. Hence D is negative semidefinite on the subspace $\{\mathbf{e}\}^\perp$. So by Theorem 2.1, $D \in \mathbb{EDM}$. □

Observation 2.7. *An important fact to note is if there exists two vectors* $\mathbf{w}_1, \mathbf{w}_2 \in \mathbb{R}^n$ *such that* $D\mathbf{w}_1 = D\mathbf{w}_2 = \mathbf{e}$*, then* $\mathbf{w}_1 - \mathbf{w}_2 = \mathbf{z} \in ker(D)$*. So,* $\mathbf{e}^T\mathbf{w}_1 - \mathbf{e}^T\mathbf{w}_2 = \mathbf{e}^T\mathbf{z}$*. However* $\mathbf{e}^T\mathbf{z} = \mathbf{w}_1^T D\mathbf{z} = 0$*, therefore the conclusion is* $\mathbf{w}_1^T\mathbf{e} = \mathbf{w}_2^T\mathbf{e}$*.*

A special simple case of Theorem 2.6, which will be used later, is when D has eigenvector $\mathbf{e}$ associated with its only positive eigenvalue.

Corollary 2.8. *Let* $D \in \mathbb{SYM}_0^+$ *and have only one positive eigenvalue* λ_1*, with eigenvector* $\mathbf{e}$*. Then* D *is a distance matrix.*

Proof. Suppose $D \in \mathbb{SYM}_0^+$ and has only one positive eigenvalue λ_1, with eigenvector $\frac{1}{\sqrt{n}}\mathbf{e}$. All other eigenvalues of D are nonpositive, $D = \lambda_1 \frac{1}{n}\mathbf{e}\mathbf{e}^T - C^T C$. If $\mathbf{x}^T\mathbf{e} = 0$,

$$\begin{aligned}
\mathbf{x}^T D\mathbf{x} &= \lambda_1 \frac{1}{n}\mathbf{x}^T\mathbf{e}\mathbf{e}\mathbf{x}^T - \mathbf{x}^T C^T C\mathbf{x}^T \\
&= -\mathbf{x}^T C^T C\mathbf{x}^T = -\|C\mathbf{x}\|^2 \\
&\leq 0.
\end{aligned}$$

Therefore by Theorem 2.1 since matrix D is symmetric with $diag(D) = 0$ and is negative semidefinite on $\{\mathbf{e}\}^\perp$, $D \in \mathbb{EDM}$. $\square$

By Theorem 2.6, D must have exactly one positive eigenvalue; call it λ_1. So necessarily $\lambda_1 = -\lambda_2 - \cdots - \lambda_n$ since $\sum_{i=1}^{n} \lambda_i = 0$.

Chapter 3

Connections between Hadamard and Euclidean distance matrices

So far, we have characterized $\mathbb{EDM}$ as a set of matrices all of which are symmetric and have zero diagonal from the geometry of distance and metric properties. By Theorem 2.1, any D in $\mathbb{EDM}$ must be negative semidefinite on the subspace $\{\mathbf{e}\}^{\perp}$. We also found from Theorem 2.6, D in $\mathbb{EDM}$ must have exactly one positive eigenvalue. The all 1's vector $\mathbf{e}$ must be of the form $D\mathbf{w}$ for some $\mathbf{w}$ with $\mathbf{w}^T\mathbf{e} \geq 0$. And the sum of the eigenvalues of D is zero. In this section we discuss how to construct a distance matrix with specified eigenvalues from a known Hadamard matrix.

3.1 Hadamard matrices

First some preliminary definitions and background.

Definition 3.1. *A matrix $H \in \mathbb{R}^{n\times n}$ is a Hadamard matrix if each entry is ± 1 and $H^T H = nI$ meaning the rows, and consequently columns, are pairwise orthogonal and H is non-singular.*

Historically, Hadamard matrices were known as far back as 1867 and it is believed there is a connection between Hadamard matrices and tessellations. Hadamard matrices are only known to exist if $n = 1, 2$ or $n \equiv 0 \mod 4$ and it is unknown if this condition is sufficient [2]. In order to show connection between Hadamard matrices and $\mathbb{EDM}$, it is very useful to point out shared properties for a set real numbers in common with the eigenvalues for $D \in \mathbb{EDM}$.

Lemma 3.2. *Let $\lambda_1 \geq 0 \geq \lambda_2 \geq \ldots \geq \lambda_n$ where $\sum_{i=1}^{n} \lambda_i = 0$, then $(n-1)|\lambda_n| \geq \lambda_1$ and $(n-1)|\lambda_2| \leq \lambda_1$.*

Proof. By the given assumption, $\lambda_1 = |\lambda_2| + |\lambda_3| + \ldots + |\lambda_n|$ and $|\lambda_2| \leq |\lambda_3| \leq \ldots \leq |\lambda_n|$. Therefore $(n-1)|\lambda_2| \leq |\lambda_2| \leq |\lambda_3| \leq \ldots \leq |\lambda_n| \leq (n-1)|\lambda_n|$, and by replacing with λ_1, $(n-1) \leq \lambda_1 \leq (n-1)|\lambda_n|$. $\square$

3.2 Solution to the IEP based on a given Hadamard matrix

The next theorem published by Thomas Hayden, Robert Reams, and James Wells in 1999 [7], shows the closest and perhaps strongest connection distance matrices have with Hadamard matrices and solves the inverse eigenvalue problem for $\mathbb{EDM}$ in a finite number of cases.

Theorem 3.3. *Let n be such that there exists a Hadamard matrix of order n. Let $\lambda_1 \geq 0 \geq \lambda_2 \geq \ldots \geq \lambda_n$ and $\displaystyle\sum_{i=1}^{n} \lambda_i = 0$, then there exists a distance matrix D with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$.*

Proof. Let $\mathbf{e} \in \mathbb{R}^n$, and $H \in \mathbb{R}^{n\times n}$ be a Hadamard matrix of order n. Let $U = \frac{1}{\sqrt{n}}H$ so that U is an orthonormal matrix. Let $\Lambda = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Therefore $D = U^T\Lambda U$ is symmetric and has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. H has one row of all ones, assume it is row 1 of H.

$$D\mathbf{e} = U^T\Lambda U\mathbf{e} = \tfrac{1}{n}H^T\Lambda H\mathbf{e} = \tfrac{1}{n}H^T\begin{bmatrix} n\lambda_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \tfrac{1}{n}\begin{bmatrix} n\lambda_1 \\ n\lambda_1 \\ \vdots \\ n\lambda_1 \end{bmatrix} = \lambda_1\mathbf{e}.$$ Therefore $\mathbf{e}$ is an eigenvector of D corresponding to eigenvalue λ_1. It can be computed $d_{ii} = \sum_{i=1}^{n}\frac{1}{n}\lambda_i = 0$ since $\sum_{i=1}^{n}\lambda_i = 0$. By Corollary 2.8, $D \in \mathbb{EDM}$. $\square$

For the purpose of constructing $D \in \mathbb{EDM}$ of any finite order given particular spectral properties, this is sufficient but not necessary condition. For example, there exists distance matrices of order 3 so they cannot be Hadamard. Furthermore, it is not even known if there exists a Hadamard matrix for every $n \geq 4$ which is a multiple of 4 [2].

3.2.1 An example constructing a distance matrix using a Hadamard matrix of the same order

Let $n = 4$, assign $\lambda_1 = 6, \lambda_2 = -1, \lambda_3 = -2, \lambda_4 = -3$,

so that

$$\Lambda = \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}.$$

Let Hadamard matrix,

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix}, \text{then } U = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

So $U^T \Lambda U = \begin{bmatrix} 0 & 2 & 2.5 & 1.5 \\ 2 & 0 & 1.5 & 2.5 \\ 2.5 & 1.5 & 0 & 2 \\ 1.5 & 2.5 & 2 & 0 \end{bmatrix}$ is a distance matrix by Theorem 3.3; call it D. Also, by Corollary 2.8, since $D\mathbf{e} = 6\mathbf{e}$; D is in $\mathbb{EDM}$.

3.3 Solving the inverse eigenvalue problem for other orders

The next step in solving the inverse eigenvalue problem of Euclidean distance matrices for all orders of n, has been focused especially on the $n+1$ case. Starting with a matrix $D \in \mathbb{EDM}$ of order n constructed from given spectral data, is there any way to border the matrix D to get a matrix $\hat{D}$ of size $n+1$ with $\hat{D} \in \mathbb{EDM}$?

In order to begin looking at the inverse eigenvalue problem for the $n+1$ case we need a few preliminary theorems due to Fiedler [3].

Theorem 3.4. *Let A be a symmetric $m \times m$ matrix with eigenvalues $\alpha_1, \ldots, \alpha_m$ and let $\mathbf{u}$ be a unit eigenvector corresponding to α_1. Let B be a symmetric $n \times n$ matrix with eigenvalues $\beta_1, \ldots, \beta_n$ and let $\mathbf{v}$ be a unit eigenvector corresponding to β_1. Then for any p, the matrix*

$$C = \begin{bmatrix} A & p\mathbf{u}\mathbf{v}^T \\ p\mathbf{v}\mathbf{u}^T & B \end{bmatrix}$$

has eigenvalues $\alpha_2, \ldots, \alpha_m, \beta_2, \ldots, \beta_m, \gamma_1, \gamma_2$ where γ_1 and γ_2 are eigenvalues of the matrix

$$C^* = \begin{bmatrix} \alpha_1 & p \\ p & \beta_1 \end{bmatrix}.$$

Proof. Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ be an orthogonal set of eigenvectors of A,

$$A\mathbf{u}_i = \alpha_i \mathbf{u}_i, \qquad i = 2, \ldots, m.$$

By direct verification $\begin{bmatrix} \mathbf{u}_i \\ 0 \end{bmatrix}$ are eigenvectors of C with the corresponding $i = 2, \ldots, m$.

Similarly, for B, let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be an orthogonal set of eigenvectors of B,

$$B\mathbf{v}_i = \beta_i \mathbf{v}_i, \qquad i = 2, \ldots, n.$$

By direct verification $\begin{bmatrix} \mathbf{v}_i \\ 0 \end{bmatrix}$ are eigenvectors of C with the corresponding $i = 2, \ldots, n$.

Let γ_1 and γ_2 be eigenvalues of C^* with corresponding eigenvectors $(g, h)^T$ and $(j, k)^T$.

$$\begin{bmatrix} \alpha_1 & p \\ p & \beta_1 \end{bmatrix} \begin{bmatrix} g \\ h \end{bmatrix} = \gamma_1 \begin{bmatrix} g \\ h \end{bmatrix}, \text{ gives us the equations } \begin{cases} \alpha_1 g + ph = \gamma_1 g \\ pg + \beta_1 h = \gamma_1 h \end{cases}.$$

$$\begin{bmatrix} \alpha_1 & p \\ p & \beta_1 \end{bmatrix} \begin{bmatrix} j \\ k \end{bmatrix} = \gamma_2 \begin{bmatrix} j \\ k \end{bmatrix}, \text{ gives us the equations } \begin{cases} \alpha_1 j + pk = \gamma_2 j \\ pg + \beta_1 h = \gamma_2 k \end{cases}.$$

Then,

$$\begin{bmatrix} A & p\mathbf{u}\mathbf{v}^T \\ p\mathbf{v}\mathbf{u}^T & B \end{bmatrix} \begin{bmatrix} g\mathbf{u} \\ h\mathbf{v} \end{bmatrix} = \begin{bmatrix} gA\mathbf{u} + hp\mathbf{u} \\ gp\mathbf{v} + hB\mathbf{v} \end{bmatrix} = \begin{bmatrix} (g\alpha_1 + hp)\mathbf{u} \\ (gp + h\beta_1)\mathbf{v} \end{bmatrix} = \gamma_1 \begin{bmatrix} g\mathbf{u} \\ h\mathbf{v} \end{bmatrix}$$

and,

$$\begin{bmatrix} A & p\mathbf{u}\mathbf{v}^T \\ p\mathbf{v}\mathbf{u}^T & B \end{bmatrix} \begin{bmatrix} j\mathbf{u} \\ k\mathbf{v} \end{bmatrix} = \begin{bmatrix} jA\mathbf{u} + kp\mathbf{u} \\ jp\mathbf{v} + kB\mathbf{v} \end{bmatrix} = \begin{bmatrix} (j\alpha_1 + kp)\mathbf{u} \\ (jp + k\beta_1)\mathbf{v} \end{bmatrix} = \gamma_2 \begin{bmatrix} j\mathbf{u} \\ k\mathbf{v} \end{bmatrix}.$$

Therefore γ_1 and γ_2 are eigenvalues of C with corresponding eigenvectors $(g,h)^T$ and $(j,k)^T$. □

Theorem 3.5. *Let $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_k$ be the eigenvalues of the symmetric non-negative matrix $A \in \mathbb{R}^{k \times k}$ and $\beta_1 \geq \beta_2 \geq \ldots \geq \beta_l$ be the eigenvalues of the symmetric non-negative matrix $B \in \mathbb{R}^{l \times l}$, where $\alpha_1 \geq \beta_1$. Also, $A\mathbf{u} = \alpha_1 \mathbf{u}$, $B\mathbf{v} = \beta_1 \mathbf{v}$ and $\mathbf{u}$ and $\mathbf{v}$ are corresponding unit Perron vectors. Then with $p = \sqrt{\sigma(\alpha_1 - \beta_1 + \sigma)}$ the matrix*

$$\begin{bmatrix} A & p\mathbf{u}\mathbf{v}^T \\ p\mathbf{v}\mathbf{u}^T & B \end{bmatrix}$$

has eigenvalues $\alpha_1 + \sigma, \beta_1 - \sigma, \alpha_2, \ldots, \alpha_k, \beta_2, \ldots, \beta_l$ for any $\sigma \geq 0$.

Proof. By Theorem 3.4, $\begin{bmatrix} A & p\mathbf{u}\mathbf{v}^T \\ p\mathbf{v}\mathbf{u}^T & B \end{bmatrix}$ has eigenvalues $\alpha_2, \ldots, \alpha_m, \beta_2, \ldots, \beta_m$. If $\sigma \geq 0$ choose $p = \sqrt{\sigma(\alpha_1 - \beta_1 + \sigma)}$. Set $det\left(\begin{bmatrix} \alpha_1 & p \\ p & \beta_1 \end{bmatrix} - \lambda I\right) = 0$.

$$\begin{aligned} det\left(\begin{bmatrix} \alpha_1 & p \\ p & \beta_1 \end{bmatrix} - \lambda I\right) &= (\alpha_1 - \lambda)(\beta_1 - \lambda) - p^2 \\ &= \lambda^2 - (\alpha_1 + \beta_1)\lambda + (\alpha_1\beta_1 - p^2) = 0 \end{aligned}$$

By the quadratic formula,

$$\begin{aligned} \lambda &= \frac{\alpha_1 + \beta_1 \pm \sqrt{(\alpha_1 + \beta_1)^2 - 4(\alpha_1\beta_1 - p^2)}}{2} \\ &= \frac{1}{2}\left[\alpha_1 + \beta_1 \pm \sqrt{\alpha_1^2 - 2\alpha_1\beta_1 + \beta_1^2 + 4p^2}\right]. \end{aligned}$$

After plugging in p and distributing, and grouping α_1, β_1,

$$\begin{aligned} \lambda &= \frac{1}{2}\left[\alpha_1 + \beta_1 \pm \sqrt{(\alpha_1 - \beta_1)^2 + 4\sigma\alpha_1 - 4\sigma\beta_1 + 4\sigma^2}\right] \\ &= \frac{1}{2}\left[\alpha_1 + \beta_1 \pm \sqrt{(\alpha_1 - \beta_1)^2 + 4\sigma(\alpha_1 - \beta_1) + 4\sigma^2}\right] \\ &= \frac{1}{2}\left[\alpha_1 + \beta_1 \pm \sqrt{(2\sigma + (\alpha_1 - \beta_1))^2}\right] \\ &= \frac{1}{2}[\alpha_1 + \beta_1 \pm (2\sigma + \alpha_1 - \beta_1)] \end{aligned}$$

Therefore $\lambda_1 = \frac{1}{2}[\alpha_1+\beta_1+2\sigma+\alpha_1-\beta_1] = \alpha_1+\sigma_1$, and $\lambda_2 = \frac{1}{2}[\alpha_1+\beta_1-2\sigma-\alpha_1+\beta_1] = \beta_1-\sigma$. By Theorem 3.4, $\alpha_1 + \sigma$ and $\beta_1 - \sigma$ are eigenvalues of the matrix $\begin{bmatrix} A & p\mathbf{u}\mathbf{v}^T \\ p\mathbf{v}\mathbf{u}^T & B \end{bmatrix}$.

$\square$

3.3.1 An error attempting to solve the inverse eigenvalue problem for $\mathbb{EDM}$ for the $(n+1)\times(n+1)$ case

In 1999, T.L. Hayden et. al. claimed to solve the inverse eigenvalue problem for distance matrices $(n+1)\times(n+1)$ using a distance matrix of size n constructed from a Hadamard matrix and specified eigenvalules as in Theorem 3.3. [7].

The theorem states:

Let n be such that there exists a Hadamard matrix of order n. Let $\lambda_1 \geq 0 \geq \lambda_2 \geq \ldots \geq \lambda_{n+1}$ and $\sum_{i=1}^{n+1} \lambda_i = 0$, then there is an $(n+1)\times(n+1)$ distance matrix $\hat{D}$ with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{n+1}$.

Let $n = 4$ and $\lambda_1 = 6.1623$, $\lambda_2 = -0.1623$, $\lambda_3 = -1$, $\lambda_4 = -2$, $\lambda_5 = -3$. Following the construction in the proof in the original paper, we begin with a distance matrix constructed from a Hadamard matrix (same matrix as in our previous example)

$$D = \begin{bmatrix} 0 & 2 & 2.5 & 1.5 \\ 2 & 0 & 1.5 & 2.5 \\ 2.5 & 1.5 & 0 & 2 \\ 1.5 & 2.5 & 2 & 0 \end{bmatrix},$$

with eigenvalues {6.1623+(-0.1623), -1, -2, -3} and Perron vector $\mathbf{e}$. Following Theorem 3.5, define $\hat{\mathrm{D}}$= $\begin{bmatrix} D & p\mathbf{u} \\ p\mathbf{u}^T & 0 \end{bmatrix}$, letting $\mathbf{u}$=$\frac{1}{2}\mathbf{e}$ and $p = \sqrt{-6.1623(-.1623)} \approx 1.0001$. By computing the block matrix

$$\hat{D} \approx \left[\begin{array}{cccc|c} 0 & 2 & 2.5 & 1.5 & .5 \\ 2 & 0 & 1.5 & 2.5 & .5 \\ 2.5 & 1.5 & 0 & 2 & .5 \\ 1.5 & 2.5 & 2 & 0 & .5 \\ \hline .5 & .5 & .5 & .5 & 0 \end{array}\right]$$

with eigenvalues {6.1623, -.1623, -1, -2, -3}. Define $\hat{\mathbf{w}} = [.5, .5, .5, .5, -4]^T$, $D\hat{\mathbf{w}} = \hat{\mathbf{e}}$, but by inspection $\hat{\mathbf{w}}^T\hat{\mathbf{e}} < 0$ and by Theorem 2.6, $\hat{\mathrm{D}}$ cannot be a distance matrix.

Over 10 years after the original paper was published, Jaklic and Modic were able to note the error, salvage some of the proof but yield different results [11].

3.4 Current state of the $n + 1$ case

Building on his solution for solving IEP of $\mathbb{EDM}$'s of order such that a Hadamard matrix exists, Hayden in 1999, considered the $n + 1$ case. Despite a major error in his solution, the paper was published and the inverse eigenvalue problem was supposedly solved for the $n + 1$ case. Jaklic and Modic in 2013 noted the error in the theorem of Hayden's $n+1$ followed the direction of the proof and added a small but significant condition [11]. However, as a result, this did not solve an IEP for $\mathbb{EDM}$; but suffices to give a general construction algorithm for a distance matrix, but not from prescribed spectral data.

Theorem 3.6. *Let $D \in \mathbb{EDM}$ and $D \neq 0$ with Perron eigenvalue r and the corresponding normalized Perron eigenvector $\mathbf{u}$. Let $D\mathbf{w}$=$\mathbf{e}$, $\mathbf{w^T e} \geq 0$ and $p > 0$. Then the matrix $\hat{D}$= $\begin{bmatrix} D & p\mathbf{u} \\ p\mathbf{u}^T & 0 \end{bmatrix} \in \mathbb{EDM}$ iff $p \in [\alpha^-, \alpha^+]$, where $\alpha^\pm := \frac{r}{\mathbf{u^T e} \mp \sqrt{r \mathbf{e^T w}}}$, noting that the denominator can be zero if $\mathbf{u} = \frac{1}{\sqrt{n}\mathbf{e}}$. In this case, take $\alpha^+ = \infty$.*

Proof. By Theorem 2.6 matrix $\hat{D} \in \mathbb{EDM}$ if (1) it has exactly one positive eigenvalue and (2) there exists a vector $\hat{\mathbf{w}} \in \mathbb{R}^{n+1}$, such that $\hat{D}\mathbf{w} = \mathbf{e}$ and $\hat{\mathbf{w}}^T\mathbf{e} \geq 0$.

1. *The matrix $\hat{D}$ has exactly one positive eigenvalue:*

By Theorem 3.4, $\hat{D}$ has $n-1$ non-positive eigenvalues of the original distance matrix D and the eigenvalues of the matrix

$$\begin{bmatrix} r & p \\ p & 0 \end{bmatrix}.$$

This can be seen by looking at the roots of characteristic polynomial $p(x) = x^2 - rx - p^2$ which are $\frac{1}{2}(r \pm \sqrt{r^2+4p^2})$. Because $\sqrt{r^2+4p^2} > r$ and $\hat{D}$ has $n-1$ non-positive eigenvalues, then $\hat{D}$ has exactly one positive eigenvalue.

2. *We now must find values of p such that $\mathbf{e}^T\hat{\mathbf{w}} \geq 0$.*

Define

$$\hat{\mathbf{w}} = \begin{bmatrix} \mathbf{w} - (\frac{\mathbf{u}^T\mathbf{e}}{r} - \frac{1}{p})\mathbf{u} \\ \frac{1}{p}(\mathbf{u}^T\mathbf{e} - \frac{r}{p}) \end{bmatrix}$$

so that $\hat{D}\hat{\mathbf{w}} = \mathbf{e}$.

$$\mathbf{e}^T\hat{\mathbf{w}} = \mathbf{e}^T\mathbf{w} - \frac{1}{r}\mathbf{e}^T\mathbf{u}^T\mathbf{e}\mathbf{u} + \frac{1}{p}\mathbf{e}^T\mathbf{u} + \frac{1}{p}\mathbf{u}^T\mathbf{e} - \frac{r}{p^2} \tag{3.1}$$

$$= \mathbf{e}^T\mathbf{w} - \frac{1}{r}(\mathbf{u}^T\mathbf{e})^2 + \frac{2}{p}\mathbf{u}^T\mathbf{e} - \frac{r}{p^2} \tag{3.2}$$

Case 1: Suppose $\mathbf{u} = \frac{1}{\sqrt{n}}\mathbf{e}$. Then $\mathbf{w} = \frac{1}{r}\mathbf{e}$ since $D\mathbf{u} = r\mathbf{u}$ means $D\frac{1}{\sqrt{n}}\mathbf{e} = r\frac{1}{\sqrt{n}}\mathbf{e}$; and therefore $D\frac{1}{r}\mathbf{e} = \mathbf{e}$. So $\mathbf{w} = \frac{1}{r}\mathbf{e}$ because $D\mathbf{w} = \mathbf{e}$. After substituting those values into (2) and reducing, $\mathbf{e}^T\hat{\mathbf{w}} = \frac{1}{p^2}(2p\sqrt{n} - r)$. So $\mathbf{e}^T\hat{\mathbf{w}} \geq 0$ if and only if $2p\sqrt{n} - r \geq 0$, meaning $p \geq \frac{r}{2\sqrt{n}}$. Using what we defined vectors $\mathbf{u}$ and $\mathbf{w}$ to be, plugging in and reducing,

$$\alpha^{\pm} = \frac{r}{\mathbf{u}^T\mathbf{e} \mp \sqrt{r\mathbf{e^T w}}} = \frac{r}{\sqrt{n} \mp \sqrt{n}}.$$

Therefore $\mathbf{e}^T\hat{\mathbf{w}} \geq 0$ if and only if $p \in [\alpha^-, \alpha^+] = [\frac{r}{2\sqrt{n}}, \infty]$.

Case 2: Suppose $\mathbf{u} \neq \frac{1}{\sqrt{n}}\mathbf{e}$. Refer to equation (3.2), rearrange and set equal to zero.

$$\mathbf{e}^T\mathbf{w} - \frac{1}{r}(\mathbf{u}^T\mathbf{e})^2 + \frac{2}{p}\mathbf{u}^T\mathbf{e} - \frac{r}{p^2} = 0,$$

$$\left(-\frac{1}{r}\right)(\mathbf{u}^T\mathbf{e})^2 + \left(\frac{2}{p}\right)\mathbf{u}^T\mathbf{e} + \left(\mathbf{e}^T\mathbf{w} - \frac{r}{p^2}\right) = 0.$$

By the quadratic formula the roots of the polynomial equation
$f(x) = -\frac{1}{r}x^2 + \frac{2}{p}x + \left(\mathbf{e}^T\mathbf{w} - \frac{r}{p^2}\right)$ are $\frac{r}{p} \pm \sqrt{r(\mathbf{e}^T\mathbf{w})}$.

Therefore, $\mathbf{e}^T\hat{\mathbf{w}} = \left(\mathbf{u}^T\mathbf{e} - \frac{r}{p} + \sqrt{r(\mathbf{e}^T\mathbf{w})}\right)\left(\mathbf{u}^T\mathbf{e} - \frac{r}{p} - \sqrt{r(\mathbf{e}^T\mathbf{w})}\right) = 0$, noting that $\left(\mathbf{u}^T\mathbf{e} - \frac{r}{p} + \sqrt{r(\mathbf{e}^T\mathbf{w})}\right) \geq \left(\mathbf{u}^T\mathbf{e} - \frac{r}{p} - \sqrt{r(\mathbf{e}^T\mathbf{w})}\right)$. Setting each term equal to 0 and solving for p, we achvieve,

$$p = \frac{r}{\mathbf{u}^T\mathbf{e} + \sqrt{r(\mathbf{e}^T\mathbf{w})}} \text{ or } p = \frac{r}{\mathbf{u}^T\mathbf{e} - \sqrt{r(\mathbf{e}^T\mathbf{w})}}.$$

Observe that $[\alpha^-, \alpha^+]$ is well defined since for any $\mathbf{w} \in \mathbb{R}^n$ satisfying $D\mathbf{w} = \mathbf{e}$, $\mathbf{e}^T\mathbf{w}$ is unique. Refer to observation 2.7. So if $p \in [\alpha^-, \alpha^+]$, where $\alpha^{\pm} = \frac{r}{\mathbf{u}^T\mathbf{e} \mp \sqrt{r\mathbf{e}^T\mathbf{w}}}$ given any $\mathbf{u} \in \mathbb{R}^n$, then $\mathbf{e}^T\hat{\mathbf{w}} \geq 0$.

In conclusion, $\hat{\mathrm{D}} = \begin{bmatrix} D & p\mathbf{u} \\ p\mathbf{u}^T & 0 \end{bmatrix} \in \mathbb{EDM}$ where $D \in \mathbb{EDM}$ of any size, iff $p \in [\alpha^-, \alpha^+]$, where $\alpha^{\pm} = \frac{r}{\mathbf{u^T e} \mp \sqrt{r\mathbf{e^T w}}}$. □

Looking back at Example 3.3.1, with $\mathbf{u} = \frac{1}{2}\mathbf{e}$, $\frac{1}{6}\mathbf{w} = \mathbf{e}$ and $r = 6$. By inspection, the interval $[\alpha^-, \alpha^+] = [\frac{3}{2}, \infty]$. Note $p \approx 1.0001 \notin [\alpha^-, \alpha^+]$. Therefore, by Theorem 3.6, $\hat{\mathrm{D}} = \begin{bmatrix} D & p\mathbf{u} \\ p\mathbf{u}^T & 0 \end{bmatrix}$ can not be a distance matrix.

By including the condition $p \in [\alpha^+, \alpha^-]$, Theorem 3.6, guarantees that $\hat{\mathrm{D}}$ is a distance matrix but does not necessarily solve the inverse eigenvalue problem. The added condition for $p \in [\alpha^-, \alpha^+]$ and the definition of α means p is defined in terms of r which, recall, is an eigenvalue of our original distance matrix D.

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