Linear Topological Spaces

by

William C. McGuffey

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Approved by

Michel Smith, Chair, Professor of Mathematics and Statistics
Ulrich Albrecht, Professor of Mathematics and Statistics
Piotr Minc, Professor of Mathematics and Statistics

Abstract

In this thesis several topics from Topology, Linear Algebra, and Real Analysis are combined in the study of linear topological spaces. We begin with a brief look at linear spaces before moving on to study some basic properties of the structure of linear topological spaces including the localization of a topological basis. Then we turn our attention to linear spaces with a metric topology. In particular, we consider problems involving normed linear spaces, bounded linear transformations, and Hilbert spaces.

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Chapter 1

Introduction

This thesis is a compilation of solutions to problems assigned under the direction of Professor Michel Smith. The material in this thesis was developed through a one-on-one Moore method directed study. I was given notes on linear topological spaces which included definitions and theorems to be proved. During weekly meetings I presented my solutions to various problems and we discussed approaches to solving problems. A large portion of the notes came from *Linear Topological Spaces* by John L. Kelley and Isaac Namioka, but other material came about when we were side-tracked during our weekly discussions. All proofs in this thesis can be assumed to be my own unless otherwise stated. The style of the thesis reflects the way in which it was developed. It could be used as a framework for an introductory course on linear topological spaces, and it reads like a set of class notes with solutions included. A course on linear topological spaces could be useful for upper-level undergraduate students or new graduate students because it integrates many areas of mathematics including Topology, Linear Algebra, and Real Analysis.

1.1 Notation and Terminology

Throughout this thesis I try to be as consistent as possible with notation. I use uppercase English letters such as X, Y, and Z to denote *linear spaces*, *topological spaces*, or *linear topological spaces*. The elements of these spaces are denoted by lowercase letters such as x, y, and z. The elements of a linear space are called *vectors*, but we refer to the elements of a linear topological space as *points*. An arbitrary set of vectors or points will be denoted by uppercase English letters such as A, B, E, F, and M, while the symbols U, V, and W will usually be reserved for *open* subsets of a linear topological space.

A linear space consists of a set X together with two operations, addition $+$ and scalar multiplication $\cdot$, and is denoted by $(X, +, \cdot)$. However, the operations are usually omitted, and the space is simply denoted by X. Moreover, a topological space consists of a set X together with a collection $\mathcal{T}$ of subsets of X and is denoted by $(X, \mathcal{T})$. Again, the topology is omitted, and we simply denote the space by X.

Associated with a linear space is an underlying *field* of (real or complex) numbers, denoted by $\mathbb{F}$. The elements of the field are called *scalars* and are denoted by lowercase Greek letters such as λ and μ. The real part of a complex number λ will be denoted by $\Re e\lambda$, and the imaginary part will be denoted by $\Im m\lambda$. However, occassionally the symbols m or M may be used to denote a scalar whenever it is used to demonstrate a type of *boundedness.* When integers are used as an *index* they will be denoted by i, j, k, m, n, or N. As usual, the complex numbers are denoted by $\mathbb{C}$, the real numbers by $\mathbb{R}$, the rationals by $\mathbb{Q}$, the integers by $\mathbb{Z}$, and the natural numbers by $\mathbb{N}$.

A *linear transformation* between linear spaces will usually be denoted by the uppercase letter T. When the domain and range of a linear transformation are linear topological spaces, it may be referred to as a *linear map.*

In general, a *function* may be denoted by lowercase letters f, g, or h, or occassionally by uppercase letters F and G. A function f can also be denoted by the arrow $\rightarrow$; for example, $x \rightarrow f(x)$. If A is a set and f is a function, then $f^{-1}(A)$ denotes the set consisting of points x in the domain whose image $f(x)$ is an element of A, and $f(A)$ denotes the set consisting of each point that is the image of an element of A.

A *collection* of sets will be denoted by uppercase script letters such as $\mathcal{A}$, $\mathcal{B}$, $\mathcal{G}$, and $\mathcal{T}$. However, the script letter $\mathcal{H}$ will be used to denote a Hilbert space. A countable collection of sets is denoted by $\{A_i\}_{i=1}^{\infty}$, where A_i is a set for each positive integer i. In general, a collection of sets is denoted by $\{A_\alpha : \alpha \in \Gamma\}$, where A_α is a set for each *index* α in the index set Γ, to indicate that the collection is not necessarily countable.

If A and B are sets of points, then $A \times B$ denotes the *Cartesian product* of A and B, which is the set of all ordered pairs (a, b) where a is an element of A and b is an element of B. If $\{A_\alpha : \alpha \in \Gamma\}$ is a collection of sets, then the Cartesian product of all sets in the collection is denoted by $\prod_{\alpha \in \Gamma} A_\alpha$ and elements of this product are denoted by $(x_\alpha)_{\alpha \in \Gamma}$ or simply (x_α). If the index set is countable, we denote the product by $\prod_{n=1}^{\infty} A_n$ and write sequences as (x_n).

If a sequence (x_n) *converges* to a point x, we will denote this by $x_n \to x$.

If A is a subset of a topological space X, then $\mathrm{Int}(A)$ denotes the *interior* of the set A and $\overline{A}$ denotes the *closure* of A. The *complement* of A in X is denoted by $X \setminus A$.

The vertical bars $|\cdot|$ have many uses throughout this thesis, but each use is standard.

- If A is a set, then $|A|$ is used to denote the *cardinality* of A.
- If λ is a scalar, then $|\lambda|$ denotes the *magnitude* or *absolute value* of λ.
- If T is a bounded linear transformation, then $|T|$ is a *norm.*

The vertical bars $\|\cdot\|$ are denote the *norm* on a normed linear space. The angled brackets $\langle \cdot, \cdot \rangle$ are used to denote an inner product.

1.2 Preliminary Material

Throughout this thesis I assume some prior knowledge of both linear spaces and topological spaces. I will often use some basic theorems and concepts from Topology without stating them explicitly in the body of the thesis. Among these theorems are those that would be seen in an undergraduate course. Each of the following theorems can be found in Munkres' *Topology* [3].

Theorem. *Let f be a function from a topological space X to a topological space Y. Then the following are equivalent:*

(1) f is continuous.

(2) $f(\overline{A}) \subset \overline{f(A)}$ *for every* $A \subset X$.

(3) $f^{-1}(B)$ *is closed in* X *for every closed set* $B \subset Y$.

(4) For each $x \in X$ *and each neighborhood* V *of* $f(x)$ *in* Y *there is a neighborhood* U *of* x *in* X *such that* $f(U) \subset V$.

Theorem. *A product of compact spaces is compact.*

Theorem. *If* $A \subset X$ *and* $B \subset Y$, *then* $\overline{A} \times \overline{B} = \overline{A \times B}$ *in the product space* $X \times Y$.

Theorem. *Let* A *be a subset of the topological space* X. *Then* $x \in \overline{A}$ *if and only if every open neighborhood* U *of* x *intersects* A.

Theorem. *Let* X *be a topological space; let* $A \subset X$. *If there is a sequence of points of* A *converging to* x, *then* $x \in \bar{A}$*; the converse holds if* X *is metric (or first-countable, in general).*

Theorem. *Let* $f : X \to Y$. *If the function* f *is continuous, then for every convergent sequence* $x_n \to x$ *in* X, *the sequence* $(f(x_n))$ *converges to* $f(x)$. *The converse holds if* X *is metric (or first-countable, in general).*

Theorem. *Let* X *be a metrizable space. Then, the following are equivalent:*

(1) X *is compact.*

(2) X *is limit point compact.*

(3) X *is sequentially compact.*

Theorem. *A topological space* X *is regular if and only if given a point* x *of* X *and an open neighborhood* U *of* x, *there is an open neighborhood* V *of* x *such that* $\overline{V} \subset U$.

Theorem. *If* X *is a compact metric space, then* X *is complete.*

Chapter 2

Linear Spaces

In this chapter we discuss the elementary structure of linear spaces, linear subspaces, and linear transformations between spaces. We also prove the existence of a Hamel base for any linear space and the uniqueness of its cardinality. We start with some elementary definitions and terminology for linear spaces which will be used throughout this thesis. We assume some prior knowledge about the structure of linear spaces, including the axioms of the underlying field and the operations of addition and scalar multiplication. Throughout this chapter assume that X is a linear space over a scalar field $\mathbb{F}$ unless otherwise stated.

2.1 Elementary Properties

Definition. A subset E of a linear space X is said to be *linearly independent* provided for each finite subset $\{x_1, \ldots, x_n\}$ of E,

$$\lambda_1 x_1 + \cdots + \lambda_n x_n = 0 \qquad \text{if and only if} \qquad \lambda_1 = \cdots = \lambda_n = 0$$

Definition. Given any subset E of a linear space X, the *span* of E, denoted by $\mathrm{span}(E)$, is the set of all finite linear combinations of vectors in E, that is,

$$\mathrm{span}(E) = \{\lambda_1 x_1 + \cdots + \lambda_n x_n : n \in \mathbb{N}, x_i \in E, \lambda_i \in \mathbb{F}, 1 \leq i \leq n\}$$

Definition. A subset B of a linear space X is called a *Hamel base* if and only if for each non-zero vector $x \in X$ there is a unique set $\{b_1, \ldots, b_n\}$ of vectors in B and a unique set $\{\lambda_1, \ldots, \lambda_n\}$ of scalars so that $x = \lambda_1 b_1 + \cdots + \lambda_n b_n$.

Definition. Given two subsets A and B of a linear space X, the algebraic sum $A+B$ denotes the set consisting of all sums $a+b$, where $a \in A$ and $b \in B$. In particular, if $x \in X$, then the *x-translate* of the set A is defined by

$$x + A = \{x\} + A = \{x + a : a \in A\}$$

Given a scalar $\lambda \in \mathbb{F}$ and a subset A of X, λA denotes the set consisting of all scalar multiples λa with $a \in A$.

Definition. A *linear transformation* T from a linear space X to a linear space Y (over the same scalar field $\mathbb{F}$) is a function satisfying

1. $T(x+y) = T(x) + T(y)$, for all $x, y \in X$, and
2. $T(\lambda x) = \lambda T(x)$, for all $x \in X$ and for all $\lambda \in \mathbb{F}$.

Observation. Suppose that T is a linear transformation from the linear space X to the linear space Y.

(a) If $A, B \subset X$, then $T(A+B) = T(A) + T(B)$.

Proof. If $y \in T(A+B)$, then there is a vector $x \in A+B$ such that $y = T(x)$. Because $x \in A+B$, we can write $x = a+b$, where $a \in A$ and $b \in B$. Consequently,

$$y = T(x) = T(a+b) = T(a) + T(b) \in T(A) + T(B)$$

On the other hand, if $y \in T(A) + T(B)$, then we can write $y = y_1 + y_2$, where $y_1 \in T(A)$ and $y_2 \in T(B)$. So, there exist $a \in A$ and $b \in B$ such that $y_1 = T(a)$ and $y_2 = T(b)$. Hence,

$$y = y_1 + y_2 = T(a) + T(b) = T(a+b) \in T(A+B)$$

Therefore, $T(A+B) = T(A) + T(B)$. $\square$

(b) If $A \subset X$ and $\lambda \in \mathbb{F}$, then $T(\lambda A) = \lambda T(A)$.

Proof. If $y \in T(\lambda A)$, then there is a point $x \in \lambda A$ such that $y = T(x)$. Because $x \in \lambda A$, we can write $x = \lambda a$, where $a \in A$. As a result,

$$y = T(x) = T(\lambda a) = \lambda T(a)$$

On the other hand, if $y \in \lambda T(A)$, then we can write $y = \lambda y_1$, where $y_1 \in \lambda T(A)$. So, there exists $a \in A$ such that $y_1 = \lambda T(a)$. Therefore,

$$y = \lambda y_1 = \lambda T(a) = T(\lambda a) \in T(\lambda A)$$

Thus, $T(\lambda A) = \lambda T(A)$. $\square$

Definition. Given a linear transformation T from a linear space X to a linear space Y, the *nullspace* of T, denoted by $\text{null}(T)$, is the set of all elements of X that are mapped by T to the zero vector 0_Y of Y, that is,

$$\text{null}(T) = T^{-1}(0_Y) = \{x \in X : T(x) = 0_Y\}$$

The following observation demonstrates that the pre-image of an element in a linear space under a linear transformation is a translate of the nullspace of the linear transformation. We will use this fact in the theorem that follows.

Observation. Let T be a surjective linear transformation from a linear space X to a linear space Y. If $T(x) = y$, then $T^{-1}(y) = x + T^{-1}(0)$.

Proof. For each $y \in Y$ there exists a vector $x \in X$ such that $T(x) = y$ since T is surjective. For any element u of $x + T^{-1}(0)$, we can write $u = x + v$, where $T(v) = 0$, if and only if

$$T(u) = T(x + v) = T(x) + T(v) = T(x) + 0 = T(x) = y$$

In other words, u is in $x + T^{-1}(0)$ if and only if $T(u) = y$. Hence, $x + T^{-1}(0) = T^{-1}(y)$. $\square$

Theorem 2.1 (Induced mapping theorem)**.** *Suppose that X, Y, and Z are linear spaces and that $T : X \to Y$ and $S : Y \to Z$ are linear transformations such that the nullspace of T contains the nullspace of S. Moreover, suppose that S is surjective. Then there exists a unique linear transformation $U : Z \to Y$ such that $T = U \circ S$. Furthermore, U is one-to-one if and only if* $\text{null}(T) = \text{null}(S)$.

Proof. Given a vector z_0 in Z there is a vector x_0 of X such that $S(x_0) = z_0$ because S is a surjection. Then, $S^{-1}(z_0)$ is the x_0-translate of $S^{-1}(0_Z)$. Now, because the nullspace of T contains the nullspace of S, it follows that

$$S^{-1}(z_0) = x_0 + S^{-1}(0_Z) \subset x_0 + T^{-1}(0_Y)$$

As a result, for any $x \in S^{-1}(z_0)$, we can write $x = x_0 + v$ for some $v \in T^{-1}(0_Y)$. Then,

$$T(x) = T(x_0 + v) = T(x_0) + T(v) = T(x_0) + 0 = T(x_0)$$

Therefore, $T(x) = T(x_0)$ for every $x \in S^{-1}(z_0)$. Let $y_0 = T(x_0)$ and define $U(z_0) = y_0$. Then, for all $x \in S^{-1}(z_0)$, we have

$$(U \circ S)(x) = U(S(x)) = U(z_0) = y_0 = T(x)$$

That is, $T = U \circ S$ on $S^{-1}(z_0)$. Because this can be done for each $z \in Z$, it follows that $T = U \circ S$ on $\bigcup_{z \in Z} S^{-1}(z) = X$. Furthermore, $T^{-1} = (U \circ S)^{-1} = S^{-1} \circ U^{-1}$. Thus, U is one-to-one if and only if $U^{-1}(0_Y) = \{0_Z\}$ if and only if

$$T^{-1}(0_Y) = (S^{-1} \circ U^{-1})(0_Y) = S^{-1}(U^{-1}(0_Y)) = S^{-1}(0_Z)$$

Consequently, U is one-to-one if and only if $\text{null}(T) = \text{null}(S)$. $\square$

2.2 Base and Dimension

In this section, we establish the existence of a Hamel base for any linear space. Every non-trivial linear space contains a non-zero vector. The set consisting only of this non-zero vector is linearly independent. According to Theorem 2.3, this linearly independent set lies in a maximal linearly independent set. By Theorem 2.4, this maximal linearly independent set is a Hamel base for the linear space. Consequently, these theorems demonstrate that every linear space has a Hamel base. Then, we show that any two Hamel bases for a given linear space must have the same cardinality.

Theorem 2.2 (Zorn's Lemma)**.** *Suppose that S is a set and $\mathcal{G}$ is a collection of subsets of S so that if $\mathcal{G}'$ is a monotonic subcollection of $\mathcal{G}$ then there is an element of $\mathcal{G}$ that contains every element of $\mathcal{G}'$. Then there is an element of $\mathcal{G}$ which is a subset of no other element of $\mathcal{G}$. (There is a maximal element of $\mathcal{G}$.)*

Theorem 2.3. *Every linearly independent subset of a linear space X lies in a maximal linearly independent subset of X.*

Proof. We will apply Zorn's lemma to prove this theorem. Suppose that $E \subset X$ is a linearly independent set. The collection $\mathcal{G}$ of all linearly independent subsets of X which contain E is non-empty because $E \in \mathcal{G}$. Suppose that $\mathcal{G}' \subset \mathcal{G}$ is a monotonic subcollection, that is, for any $B_1, B_2 \in \mathcal{G}'$, either $B_1 \subset B_2$ or $B_2 \subset B_1$. Let B_0 denote the union of all the sets in the collection $\mathcal{G}'$. From the definition of B_0 it is clear that B_0 contains each element of $\mathcal{G}'$. So, we just need to show that $B_0 \in \mathcal{G}$, that is, B_0 is linearly independent and contains E.

Clearly, B_0 contains E because it is defined as the union of sets each of which contains E. Now, we just need to show that B_0 is linearly independent. Assume that B_0 is not linearly independent. Then, there is a finite subset $\{b_1, \dots, b_n\}$ of B_0 and scalars $\lambda_1, \dots, \lambda_n \in \mathbb{F}$, not all zero, such that

$$\lambda_1 b_1 + \cdots + \lambda_n b_n = 0$$

Since B_0 is the union of all sets in the collection $\mathcal{G}'$ and since $b_1, \ldots, b_n \in B_0$, there must be finitely many (at most n) sets $B_1, \ldots, B_k \in \mathcal{G}'$ such that $b_1, \ldots, b_n \in \bigcup_{j=1}^{k} B_j$. Because $\mathcal{G}'$ is a monotonic collection we can assume, without loss of generality, that $B_1 \subset B_2 \subset \cdots \subset B_k$. So, the union of these sets is B_k and we have $b_1, \ldots, b_n \in B_k$. However, this contradicts the assumption that B_k is linearly independent because $\lambda_1 b_1 + \cdots + \lambda_n b_n = 0$, where $\lambda_1, \ldots, \lambda_n$ are not all zero. So, B_0 must be linearly independent and is therefore an element of the collection $\mathcal{G}$. By Zorn's lemma, there is an element of $\mathcal{G}$ which is a subset of no other elements of $\mathcal{G}$. That is, E is contained in a maximal linearly independent set. $\square$

Theorem 2.4. *The subset B of a linear space X is a Hamel base if and only if B is a maximally linearly independent subset of X.*

Proof. Suppose that B is a Hamel base for the linear space X. Given a vector $x \in X$, there is a unique representation of x as a linear combination $x = \lambda_1 b_1 + \cdots + \lambda_n b_n$, where $b_1, \ldots, b_n \in B$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$. As a result,

$$0 = x - x = (\lambda_1 - \lambda_1) b_1 + \cdots + (\lambda_n - \lambda_n) b_n = 0 \cdot b_1 + \cdots + 0 \cdot b_n$$

In other words, given a finite subset $\{b_1, \ldots, b_n\}$ of B, the only representation of the zero vector as a linear combination of $b_1, \ldots, b_n$ is with coefficients all equal to zero. Therefore, B must be linearly independent. Assume that B is not maximal with respect to linear independence. Then B is properly contained in a maximal linearly independent set B' by the previous theorem. If $b \in B' \setminus B$, then b is an element of X that has no representation as a linear combination of the elements of B, contradicting the assumption that B is a Hamel base. Thus, B must be a maximal linearly independent set.

Conversely, suppose that B is a maxial linearly independent subset of X. Given $x \in X$, we have the unique representation $x = 1 \cdot x$ because B is linearly independent. Now, suppose that $x \notin B$. Let $B' = B \cup \{x\}$. Since B is a maximal linearly independent set, it follows that B' is linearly dependent. Then, there is a finite subset $\{b_1, \ldots, b_n, x\}$ of B' and scalars $\lambda_1, \ldots, \lambda_n, \lambda \in \mathbb{F}$, not all zero, such that

$$0 = \lambda x + \lambda_1 b_1 + \cdots + \lambda_n b_n$$

In particular, $\lambda \neq 0$ for otherwise $0 = \lambda_1 b_1 + \cdots + \lambda_n b_n$, with $\lambda_1, \ldots, \lambda_n$ not all zero, would contradict linear independence of B. Therefore, we have

$$\begin{aligned} -\lambda x &= \lambda_1 b_1 + \cdots + \lambda_n b_n \\ x &= \left(-\tfrac{\lambda_1}{\lambda}\right) b_1 + \cdots + \left(-\tfrac{\lambda_n}{\lambda}\right) b_n \end{aligned}$$

So, we have a representation of x as a linear combination of elements of B. Assume that we have two such representations:

$$x = \lambda_1 b_1 + \cdots + \lambda_n b_n \qquad \text{and} \qquad x = \mu_1 b_1 + \cdots + \mu_n b_n$$

Then, we can write the zero vector as

$$0 = x - x = (\lambda_1 - \mu_1) b_1 + \cdots + (\lambda_n - \mu_n) b_n$$

Since B is linearly independent, it follows that $\lambda_i = \mu_i$, for each $i \in \{1, \ldots, n\}$. Therefore, the scalars in the representation of x are unique. Hence, B is a Hamel base for X. $\square$

Our next goal is show that any two Hamel bases for a given linear space must have the same cardinality. We take our first step toward this result with the following lemma, allows us to consider the case where the linear space has a finite Hamel base and the case where the Hamel base is infinite separately.

Lemma 2.1. *If X is a linear space with a finite Hamel base, then all bases for X are finite.*

Proof. Suppose that $A = \{a_1, \ldots, a_n\}$ is a finite Hamel base for X, and suppose that B is another Hamel base for X. Then, each vector in A can be expressed as a finite linear combination of vectors in B. Because A is finite, we only need finitely many vectors, say $\{b_1, \ldots, b_m\}$, in B to represent any element of A as a linear combination. Let B' denote this finite subset of B. For each $i \in \{1, \ldots, n\}$, there exist scalars $\lambda_{i,1}, \ldots, \lambda_{i,m} \in \mathbb{F}$ such that $a_i = \lambda_{i,1}b_1 + \cdots + \lambda_{i,m}b_m$. Moreover, B' is linearly independent as a result of being a subset of B. Assume that $B' \subsetneq B$, and choose $b \in B \setminus B'$. Because A is a Hamel base, we can write $b = \mu_1 a_1 + \cdots + \mu_n a_n$. Now, we have

$$b = \mu_1(\lambda_{1,1}b_1 + \cdots + \lambda_{1,m}b_m) + \cdots + \mu_n(\lambda_{n,1}b_1 + \cdots + \lambda_{n,m}b_m)$$

After collecting like terms and relabeling the scalars, we have

$$b = \lambda_1 b_1 + \cdots \lambda_m b_m$$

Therefore,

$$0 = -1 \cdot b + \lambda_1 b_1 + \cdots + \lambda_m b_m$$

contradicting the linear independence of B'. Thus, $B' = B$ and B is therefore finite. $\square$

The proofs of the next lemma and the following theorem are not my own. I was exposed to these proofs in an Abstract Algebra course with Dr. Ulrich Albrecht. Using this lemma we will be able to establish a process of replacing the elements of one Hamel base with elements of another Hamel base in order to show that the two bases have the same cardinality.

Lemma 2.2 (Steinitz exchange lemma)**.** *Let $B = \{b_1, \ldots, b_n\}$ be a finite Hamel base for a linear space X. If $x = \lambda_1 b_1 + \cdots + \lambda_n b_n$ and $\lambda_i \neq 0$, then $B_x = \{b_1, \ldots, b_{i-1}, x, b_{i+1}, \ldots, b_n\}$ is also a Hamel base for X.*

Proof. Because $x = \lambda_1 b_1 + \cdots + \lambda_n b_n$ and $\lambda_i \neq 0$, we can solve for b_i to get

$$b_i = \frac{1}{\lambda_i}x - \left(\sum_{j \neq i} \frac{\lambda_j}{\lambda_i} b_j\right)$$

Given a vector y in X with representation $y = \mu_1 b_1 + \cdots + \mu_n b_n$, we can substitute b_i in the expression for y, which gives

$$y = \mu_1 b_1 + \cdots + \mu_{i-1} b_{i-1} + \mu_i \left(\frac{1}{\lambda_i} x - \sum_{j \neq i} \frac{\lambda_j}{\lambda_i} b_j \right) + \mu_{i+1} b_{i+1} + \cdots + \mu_n b_n$$

After distributing μ_i and collecting like terms, we have the following representation of y as a linear combination of the vectors in $B_x = \{b_1, \ldots, b_{i-1}, x, b_{i+1}, \ldots, b_n\}$.

$$y = \sum_{1 \leq j < i} \left(\mu_j - \frac{\mu_i \lambda_j}{\lambda_i} \right) b_j + \frac{\mu_i}{\lambda_i} x + \sum_{i < j \leq n} \left(\mu_j - \frac{\mu_i \lambda_j}{\lambda_i} \right) b_j$$

As a result, the vector y is in the span of B_x and therefore B_x spans all of X. Now, to show that B_x is linearly independent, suppose we have

$$0 = \gamma_1 b_1 + \cdots + \gamma_{i-1} b_{i-1} + \gamma_i x + \gamma_{i+1} + b_{i+1} + \cdots + a_n b_n$$

Substituting the representation $x = \lambda_1 b_1 + \cdots + \lambda_n b_n$ for the vector x, we have

$$0 = \gamma_1 b_1 + \cdots + \gamma_i (\lambda_1 b_1 + \cdots + \lambda_n b_n) + \cdots + \gamma_n b_n$$
$$0 = \sum_{1 \leq j < i} (\gamma_j + \gamma_i \lambda_j) b_j + \gamma_i \lambda_i b_i + \sum_{i < j \leq n} (\gamma_j + \gamma_i \lambda_j) b_j$$

Since B is linearly independent, this expression of zero in terms of the vectors of B implies $\gamma_i \lambda_i = 0$, which implies that $\gamma_i = 0$ because $\lambda_i \neq 0$ by assumption. So, the last equation becomes

$$0 = \gamma_1 b_1 + \cdots + \gamma_{i-1} b_{i-1} + \gamma_{i+1} b_{i+1} + \cdots + \gamma_n b_n$$

Again, since B is linearly independent, we have $\gamma_j = 0$ for all $j \in \{1, \ldots, n\}$. Therefore, B_x is linearly independent. As a result, B_x is a Hamel base for X. □

Theorem 2.5. *If A and B are Hamel bases for X, then $|A| = |B|$.*

Proof. According to Lemma 2.1, we can prove this first for finite Hamel bases and then for infinite Hamel bases separately. Suppose that $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_n\}$ are finite Hamel bases for X, and suppose that $m \neq n$. Without loss of generality, assume that $m < n$. Since A and B are linearly independent, every vector in each set is non-zero. Since B is a Hamel base, each vector of A is in the span of B. In particular,

$$a_1 = \lambda_{1,1}b_1 + \lambda_{1,2}b_2 + \cdots + \lambda_{1,n}b_n$$

for some scalars $\lambda_{1,1}, \ldots, \lambda_{1,n} \in \mathbb{F}$. Because $a_1 \neq 0$, we can re-order the finitely many elements of B so that $\lambda_{1,1} \neq 0$. By the Steinitz exchange lemma, the set $B_1 = \{a_1, b_2, \ldots, b_n\}$ is also a Hamel base for X. So, a_2 is in the span of B_1, and we have

$$a_2 = \lambda_{2,1}a_1 + \lambda_{2,2}b_2 + \cdots + \lambda_{2,n}b_n$$

where $\lambda_{2,1}, \ldots, \lambda_{2,n} \in \mathbb{F}$. Because A is linearly independent, $\lambda_{2,1} = 0$. Without loss of generality, assume that $\lambda_{2,2} \neq 0$ because $a_2 \neq 0$ and we can re-order the elements of B_1 accordingly. By the Steinitz exchange lemma, the set $B_2 = \{a_1, a_2, b_3, \ldots, b_n\}$ is a Hamel base for X.

Continuing this process, at the mth step we find that

$$B_m = \{a_1, \ldots, a_m, b_{m+1}, \ldots, b_n\} = A \cup \{b_{m+1}, \ldots, b_n\}$$

is a Hamel base for X. By linear independence of B_m, the vectors $b_{m+1}, \ldots, b_n$ cannot be represented as linear combinations of the vectors in A. This contradicts the assumption that A is a Hamel base for X because $b_{m+1}, \ldots, b_n$ are not in the span of A. Therefore, $m \geq n$. By a symmetric argument in A and B, we obtain $m \leq n$. Hence, $m = n$ and $|A| = |B|$.

Now, suppose that A is a Hamel base for X with cardinality $|A| \geq \aleph_0$, and let B be another Hamel base for X. Given a vector $a \in A$, let B_a denote the finite subset of B that is needed to express a as a linear combination with non-zero scalar coefficients. Since B is a Hamel base, B_a is uniquely determined by a. Define

$$B' = \bigcup_{a \in A} B_a$$

Since A spans X, so does B'. Assume that $B' \subsetneq B$, and choose $b \in B \setminus B'$ so that b can be expressed as

$$b = \lambda_1 b_1 + \cdots + \lambda_n b_n$$

for some $b_1, \ldots, b_n \in B'$ and for some $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$. Then,

$$0 = (-1)b + \lambda_1 b_1 + \cdots + \lambda_n b_n$$

gives a non-trivial representation of the zero vector as a linear combination of elements of the Hamel base B, contradicting linear independence of B. Therefore, $B' = B$. As a result,

$$|B| = \left| \bigcup_{a \in A} B_a \right| \leq \aleph_0 |A| = |A|$$

By a symmetric argument in A and B, we also have

$$|A| \leq \aleph_0 |B| = |B|$$

Therefore, $|A| = |B|$. $\square$

Definition. The *dimension* of a linear space X is the cardinality of a Hamel base for X. If X has a finite Hamel base, then X is said to be *finite-dimensional*. Otherwise, X is *infinite-dimensional*.

2.3 Linear Subspaces

A linear subspace is essentially a subset of a linear space that is also a linear space in its own right. The intersection of a collection of linear subspaces is a linear subspace. The span of any subset of a linear space is the smallest linear subspace that contains the subset. Moreover, the algebraic sum of two linear subspaces is the span of their union and is therefore a linear subspace. Each problem in this section comes from a set of exercises in Royden's *Real Analysis* [4].

Definition. A non-empty subset E of a linear space X is a *linear subspace* if and only if $\lambda x + \mu y \in E$ whenever $x, y \in E$ and $\lambda, \mu \in \mathbb{F}$.

Observation. Let E be a non-empty subset of a linear space X. Then E is a linear subspace if and only if $E + E = E$ and $\lambda E = E$, for each non-zero scalar $\lambda \in \mathbb{F}$.

Proof. Suppose that E is a linear subspace. If $x \in E$, then $x = \frac{1}{2}x + \frac{1}{2}x$ is an element of $E + E$ because E is closed under scalar multiplication. On the other hand, given $x \in E + E$ we can write $x = x_1 + x_2$ for some $x_1, x_2 \in E$. Since E is a linear subspace, it is closed under addition and consequently $x_1 + x_2 = x \in E$. Hence, $E = E + E$. Now, given $x \in E$, we can write $x = \lambda(\lambda^{-1}x)$, for some non-zero scalar λ, where $\lambda^{-1}x$ is in E because E is a linear subspace. On the other hand, if $x \in E$ and λ is any scalar, then $\lambda x \in E$ since E is closed under scalar multiplication. Hence, $\lambda E = E$.

Conversely, suppose that $E + E = E$ and $\lambda E = E$, for each non-zero scalar λ. Given vectors $x, y \in E$ and non-zero scalars $\lambda, \mu \in \mathbb{F}$, λx and μy are elements of E because $\lambda E = E$ and $\mu E = E$ by assumption. If either λ or μ is zero, then λx and μy are still elements of E because $0 \in E$. Moreover, because $E + E = E$, we have $\lambda x + \mu y \in E$. Therefore, E is a linear subspace. $\square$

Lemma 2.3. *The intersection of a collection of linear subspaces is a linear subspace.*

Proof. Let $\{E_\alpha : \alpha \in \Gamma\}$ be a collection of linear subspaces of a linear space X. Let E denote the intersection $\bigcap_{\alpha\in\Gamma} E_\alpha$. Then, E is non-empty because $0 \in E_\alpha$, for each $\alpha \in \Gamma$. For any x and y in E, we have $x, y \in E_\alpha$, for each $\alpha \in \Gamma$. Since each E_α is a linear subspace, we have $\lambda x + \mu y \in E_\alpha$, for any scalars $\lambda, \mu \in \mathbb{F}$ and for each $\alpha \in \Gamma$. Hence, $\lambda x + \mu y \in E$. Therefore, the intersection E is a linear subspace of X. $\square$

Theorem 2.6. *Given a subset E of a linear space X, there exists a smallest linear subspace containing E. This linear subspace is the span of E.*

Proof. Let E be a subset of a linear space X. Let $\mathcal{A}$ be the collection of all linear subspaces of X containing E. Then, $\mathcal{A}$ is non-empty because $X \in \mathcal{A}$. By Lemma 2.3, the intersection A of all elements in the collection $\mathcal{A}$ is a linear subspace of X. This is clearly the smallest linear subspace containing E; if B is another linear subpace containing E, then $B \in \mathcal{A}$ so that B contains the intersection A of elements of $\mathcal{A}$.

Now, we want to show that the intersection A of all linear subspaces containing E is the span of E. If $x \in A$, then x belongs to every linear subspace that contains E. So, we must show that $\mathrm{span}(E)$ is a linear subspace (it clearly contains E). First, the span of E is non-empty whenever E is non-empty. Now, if x and y are vectors in the span of E, then there exist vectors $x_1, \ldots, x_n \in E$ and $y_1, \ldots y_m \in E$ together with scalars $\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_m \in \mathbb{F}$ such that

$$x = \sum_{i=1}^{n} \lambda_i x_i \qquad \text{and} \qquad y = \sum_{i=1}^{m} \mu_i y_i$$

So, for any scalars $\lambda, \mu \in \mathbb{F}$, we have

$$\lambda x + \mu y = \lambda \sum_{i=1}^{n} \lambda_i x_i + \mu \sum_{i=1}^{m} \mu_i y_i = \sum_{i=1}^{n} \lambda\lambda_i x_i + \sum_{i=1}^{m} \mu\mu_i y_i$$

Because $\lambda x + \mu y$ is a finite linear combination of elements of E, it follows that $\alpha x + \mu y$ is in the span of E. Therefore, the span of E is a linear subspace of X that contains E, from which it follows that $A \subset \mathrm{span}(E)$.

Finally, we need to show that the span of E is contained in the intersection A so that it is the smallest linear subspace containing E. Given a vector x in the span of E, there exist $x_1, \ldots, x_n \in E$ and scalars $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$ such that $x = \lambda_1 x_1 + \cdots + \lambda_n x_n$. Because A is a linear subspace of X containing E, it follows that A contains the finite linear combination $\lambda_1 x_1 + \cdots + \lambda_n x_n$ of elements of E. Hence, $\operatorname{span}(E) \subset A$. Thus, $\operatorname{span}(E) = A$ and the span of E is the smallest linear subspace of X that contains E. $\square$

Corollary 2.1. *If each of A and B is a linear subspace of a linear space X, then so is $A+B$. Moreover, $A + B$ is the span of $A \cup B$.*

Proof. Suppose that A and B are linear subspaces of a linear space X. According to Theorem 2.6, we only need to show that $A + B$ is the span of $A \cup B$. If $x \in A + B$, then $x = a + b$, for some $a \in A$ and $b \in B$. As a result, x is a finite linear combination of elements of $A \cup B$. So, x is in the span of $A \cup B$. Hence, $A + B \subset \operatorname{span}(A \cup B)$. On the other hand, if $x \in \operatorname{span}(A \cup B)$, then x can be expressed as a finite linear combination of elements of $A \cup B$. So, there are scalars $\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_m \in \mathbb{F}$ such that

$$x = \sum_{i=1}^{n} \lambda_i a_i + \sum_{i=1}^{m} \mu_i b_i$$

where $a_1, \ldots, a_n \in A$ and $b_1, \ldots, b_m \in B$. Because A is a linear subspace, it follows that $a = \lambda_1 a_1 + \cdots + \lambda_n a_n \in A$. Similarly, $b = \mu_1 b_1 + \cdots + \mu_m b_m \in B$ because B is a linear subspace. Thus, $x = a + b$ is an element of $A + B$. Therefore, $A + B = \operatorname{span}(A \cup B)$. By the previous theorem, $A + B$ is the smallest linear subspace containing $A \cup B$. $\square$

Chapter 3

Linear Topological Spaces

A linear space X over a field $\mathbb{F}$ has two algebraic operations: addition and scalar multiplication. In order to call something a linear topological space we have to describe the relationship between the topology and the linear structure. Because continuous functions are the main object of study in topology, it is natural to define a linear topological space to be a linear space together with a topology that makes the operations of addition and scalar multiplication continuous functions. In this chapter we consider some basic properties of linear topological spaces including the effects of the linear operators on sets, the localization of a topological basis, and other computations involving linear operators and sets.

Definition. A linear space X over a field $\mathbb{F}$ is a *linear topological space* provided there exists a topology $\mathcal{T}_X$ for X and a topology $\mathcal{T}_{\mathbb{F}}$ for $\mathbb{F}$ such that addition $(x, y) \to x + y$ from $X \times X$ into X and scalar multiplication $(\lambda, x) \to \lambda x$ from $\mathbb{F} \times X$ into X are continuous functions according to their respective product topologies.

3.1 Linear Operations That Are Homeomorphisms

By the definition of a linear topological space, the operations of addition and scalar multiplication are continuous. The following lemma will be used to show that *translation* (addition by a fixed point) and multiplication by a fixed non-zero scalar are also continuous functions. As a result, we will see that translation by a point and multiplication by a fixed non-zero scalar are homeomorphisms and therefore preserve topological properties.

Lemma 3.1. *Let X, Y, and Z be topological spaces and let $F : X \times Y \to Z$ be a continuous function. For any point x_0 of X the function f defined by restricting the domain of F to the subspace $\{x_0\} \times X$ is continuous.*

Proof. Let V be an open subset of Z. Because F is continuous, the pre-image $F^{-1}(V)$ is an open subset of $X \times Y$. By definition, $F^{-1}(V)$ consists of all points (x, y) in $X \times Y$ such that $F(x, y) \in V$. Fix a point $x_0 \in X$ and let f denote the function obtained by restricting the domain of F to the subspace $\{x_0\} \times X$. Then, $f^{-1}(V)$ is the set of all points (x_0, y) in $X \times Y$ such that $f(x_0, y) \in V$. From the definition of $F^{-1}(V)$ and $f^{-1}(V)$, it is clear that $f^{-1}(V)$ is the intersection of the open subset $F^{-1}(V)$ of $X \times Y$ and the subset $\{x_0\} \times Y$ of $X \times Y$. As a resul, $f^{-1}(V)$ is an open subset of $\{x_0\} \times Y$ in the subspace topology. Hence, f is continuous. $\square$

Theorem 3.1. *Given a point x_0 of a linear topological space X, the translation map defined by $x \to x_0 + x$ is a homeomorphism.*

Proof. Let f denote the translation map from X into X. First, f maps X onto itself since for each $x \in X$, we have

$$x = 0 + x = (x_0 - x_0) + x = x_0 + (x - x_0) = f(x - x_0)$$

where $x - x_0 \in X$ so that $x \in f(X)$. Now, we show that f is one-to-one. If $f(x_1) = f(x_2)$, then $x_0 + x_1 = x_0 + x_2$. Adding $-x_0$ to each side of this equation gives $x_1 = x_2$. Thus, f is a bijection. Now, f is obtained by restricting the domain $X \times X$ of the continuous addition operation to the slice $\{x_0\} \times X$. By Lemma 3.1, f is continuous. Notice that $f^{-1} : X \to X$ is defined by $f^{-1}(x) = -x_0 + x$. Consequently, f^{-1} is also continuous by the lemma. Therefore, f is a homeomorphism. $\square$

Corollary 3.1. *Given a point x of X and a subset E of X which satisfies a topological property P, the translate $x + E$ also satisfies the property P.*

Proof. Fix $x_0 \in X$ and let f denote translation by x_0 as a function from X to X, which is a homeomorphism by Theorem 3.1. Given a subset E of X,

$$x_0 + E = \{x_0 + x : x \in E\} = \{f(x) : x \in E\} = f(E)$$

In other words, the translate $x_0 + E$ is the homeomorphic image of the set E. $\square$

Corollary 3.2. *If A is any subset of X and U is an open subset of X, then $A + U$ is open.*

Proof. If A is any subset of X and U is an open subset of X, then

$$A + U = \{a + u : a \in A, u \in U\} = \bigcup_{a \in A} (a + U)$$

By the previous corollary, $a + U$ is open, for each $a \in A$, so that $A + U$ is a union of open sets. Thus, $A + U$ is open. $\square$

We have a similar result for multiplication by a fixed non-zero scalar. Clearly, multiplication by zero is not a homeomorphism for any non-trivial linear topological space X because it does not map X onto X. It is also not one-to-one because a linear map is one-to-one if and only if its nullspace is trivial. More importantly, the inverse of multiplication by zero does not exist because the inverse function of multiplication by a given scalar is multiplication by its reciprocal. None of these is a problem for multiplication by non-zero scalars.

Theorem 3.2. *Given a non-zero scalar $\lambda \in \mathbb{F}$, the map $x \to \lambda x$ is a homeomorphism.*

Proof. Fix a non-zero scalar λ and let g denote multiplication by λ as a function from X into X. First, g maps X onto X since for each $x \in X$, we have

$$x = 1x = \left(\lambda\lambda^{-1}\right) x = \lambda \left(\lambda^{-1} x\right) = g\left(\lambda^{-1} x\right)$$

where $\lambda^{-1} x \in X$ so that $x \in g(X)$. Now, we show that g is one-to-one. If $g(x_1) = g(x_2)$, then $\lambda x_1 = \lambda x_2$. Multiplying each side by λ^{-1} gives $x_1 = x_2$. Thus, g is a bijection. Notice that $g^{-1} : X \to X$ is defined by $g(x) = \lambda^{-1} x$. By Lemma 3.1, g and g^{-1} are both continuous. Thus, g is a homeomorphism. $\square$

Corollary 3.3. *If $\lambda \in \mathbb{F}$ is a non-zero scalar and $E \subset X$ satisfies a topological property P, then λE also satisfies P. In particular, a non-zero scalar multiple of an open open neighborhood of zero is an open neighborhood of zero.*

Proof. Fix a non-zero scalar λ and let g be the homeomorphism from the previous theorem. Given a subset E of X,

$$\lambda E = \{\lambda x : x \in E\} = \{g(x) : x \in E\} = g(E)$$

That is, the set λE is the homeomorphic image of the set E. $\square$

Now that we have established that translation by a fixed point is a homeomorphism, we can prove the following theorem. If a linear transformation between linear topological spaces is continuous at one point, then we can guarantee continuity at any other point by using the open set definition of continuity together with the fact that translation by a point is a homeomorphism. This theorem will be useful to us in later chapters because it allows us to check for continuity of a linear map by simply checking for continuity at the zero element of the domain space.

Theorem 3.3. *A linear map between linear topological spaces is continuous if and only if it is continuous at one of its points.*

Proof. Suppose that each of X and Y is a linear topological space and that $T : X \to Y$ is a linear transformation. If T is continuous, then it is continuous at each point of X. Conversely, suppose that T is continuous at a point x_0 of X. Let $x \in X$ be an arbirary point and let V be an open neighborhood of $T(x)$ in Y. Then,

$$T(x_0) - T(x) + V = T(x_0 - x) + V$$

is an open neighborhood of $T(x_0)$. Since T is continuous at x_0, there is an open subset U of X containing x_0 such that $T(U) \subset T(x_0 - x) + V$. Since U is a neighborhood of x_0, it follows that $(x - x_0) + U$ is an open neighborhood of x, and we have

$$T[(x - x_0) + U] = T(x - x_0) + T(U) \subset T(x - x_0) + [T(x_0 - x) + V]$$

Now, because T is linear, it follows that

$$T[(x - x_0) + U] \subset T(x) - T(x_0) + T(x_0) - T(x) + V = V$$

Therefore, $(x - x_0) + U$ is an open neighborhood of x in X such that T maps all of its points into the open neighborhood V of $T(x)$ in Y. Hence, T is continuous at x. Because x was chosen arbitrarily, T is continuous at each point of X. □

3.2 Homogeneity

One useful property of linear topological spaces is homogeneity. In homogeneous spaces, all points play the same role with respect to the topology. For example, if one point of a homogeneous space is a limit point, then each point of the space must be a limit point. Homogeneity of a linear topological space depends only on the property that translation is a homeomorphism. For instance, translation in two-dimensional Euclidean space $\mathbb{R}^2$ corresponds to changing the location of the origin, which results in a plane that is topologically equivalent to the original plane. Because this property depends on only one operation, we will see that homogeneity is a property of topological groups in general.

Definition. A topological space X is said to be *homogeneous* if and only if for each pair x and y of points in X there is a homeomorphism h of X onto itself such that $h(x) = y$.

Theorem 3.4. *A linear topological space is homogeneous.*

Proof. Let X be a linear topological space. Given two points x_0 and y_0 in X, let h denote the translation map $x \to (y_0 - x_0) + x$ as a function from X to X. Then, h is a homeomorphism of X onto itself by Theorem 3.1, and we have

$$h(x_0) = (y_0 - x_0) + x_0 = y_0 + (-x_0 + x_0) = y_0 + 0 = y_0$$

Because this homeomorphism can be defined for any pair of points of X, it follows that the linear topological space X is homogeneous. $\square$

A *topological group* is a group $(G, *)$ such that the maps $(x, y) \to x * y$ and $x \to x^{-1}$ are continuous. A linear space together with the addition operation is an Abelian group. Notice that the proof that a linear topological space is homogeneous depends only on the translation map being a homeomorphism. Suppose that in the definition of a linear topological space a non-communtative operation is used instead of $+$. For instance, let $(G, *)$ be a topological group that is not necessarily Abelian. Given $x_0, y_0 \in G$, we can define $h(x) = (y_0 * x_0^{-1}) * x$ or $h(x) = x * (x_0^{-1} * y_0)$. In either case, h is a homeomorphism and $h(x_0) = y_0$. So, topological groups in general are homogeneous.

3.3 Local Topological Basis

Because translation by a point is a homeomorphism, the topology for a linear topological space is determined by a local topological basis at zero. A basis for the topology on the linear space is given by the collection consisting of all translates of elements of a local topological basis at zero. In this section we consider some properties of the elements of a local topological basis at zero. These properties are true for open neighborhoods of zero, so I will occassionally use the term "open neighborhood of zero" in place of "element of a local topological basis at zero." Throughout this section, assume that X is a linear topological space with topology $\mathcal{T}$, and assume that $\mathcal{B}_0$ is a local topological basis at the zero point of X.

Theorem 3.5. *The collection $\mathcal{B} = \{x + U : x \in X, U \in \mathcal{B}_0\}$ is a basis for X.*

Proof. Given a point x in X and an element U of the local topological basis $\mathcal{B}_0$, we have $x \in x + U$. Therefore, every element of X lies in some element of $\mathcal{B}$. Fix $x_0 \in X$. Suppose that $x_1 + U_1$ and $x_2 + U_2$ are two elements of $\mathcal{B}$ such that x_0 lies in their intersection. Then, the translates $(x_1 - x_0) + U_1$ and $(x_2 - x_0) + U_2$ are open neighborhoods of zero. So, there is an element U_0 of $\mathcal{B}_0$ such that $0 \in U_0$ and U_0 lies in the intersection of $(x_1 - x_0) + U_1$ and

$(x_2 - x_0) + U_2$. As a result, $x_0 \in x_0 + U_0$ and $x_0 + U_0$ is contained in the intersection of $x_1 + U_1$ and $x_2 + U_2$. Thus, the collection $\mathcal{B}$ is a basis for the topology of X. $\square$

Corollary 3.4. *In the previous theorem, we can replace $\mathcal{B}_0$ with a local topological basis at any other point of X.*

Proof. Given a point $x \in X$ and a local topological basis $\mathcal{B}_x$ at x, the collection

$$\mathcal{B}_0 = \{-x + U : U \in \mathcal{B}_x\}$$

is a local topological basis at zero. Then, the collection

$$\mathcal{B} = \{(y - x) + U : y \in X, U \in \mathcal{B}_x\}$$

is a topological basis for X as a result of Theorem 3.5. $\square$

Theorem 3.6. *If U is an open neighborhood of zero, then there is an open neighborhood V of zero such that $V + V \subset U$.*

Proof. Let U be an open set containing zero and define $A = \{(x, y) \in U \times U : x + y \in U\}$. Notice that A is the intersection of the open set $U \times U$ and the pre-image of U under the continuous addition operation. By continuity of the addition operation, A is an open neighborhood of $(0, 0)$ in $X \times X$. So, there is a basis element (in the product topology) $V_1 \times V_2$ that contains $(0, 0)$ and lies in A. In particular, V_1 and V_2 are open neighborhoods of zero in X. So, there is an element V of the local topological basis $\mathcal{B}_0$ at zero such that $V \subset V_1 \cap V_2$. Then $V \times V \subset V_1 \times V_2 \subset A$, which implies that $V + V \subset U$. $\square$

This theorem can be used for arguments that require estimating the closeness of two points even in a non-metric linear topological space. The set V in the theorem essentially plays the role of the $\epsilon/2$-ball in a metric space. Using an inductive argument, the theorem can be generalized to guarantee the existence of an open neighborhood V of zero satisfying $V_1 + V_2 + \cdots + V_n \subset U$, where n is any natural number and $V_i = V$, for all $i \in \{1, 2, \ldots, n\}$.

The following definitions provide some useful and insightful terminology for describing sets. These are especially useful in describing the open neighborhoods of zero. These definitions are given by Aliprantis and Border [1].

Definition. Let X be a linear space. The *line segment* joining two elements x and y of X is the set $\{\lambda x + (1-\lambda)y : 0 \leq \lambda \leq 1\}$. A subset E of a linear space X is said to be:

- *convex* if it contains the line segment joining any two of its points.
- *absorbing* (or *radial*) if for any $x \in X$, some multiple of E includes the line segment joining x and zero, that is, if there exists a scalar $\lambda_0 \in \mathbb{F}$ so that $\lambda x \in E$, for all $\lambda \in \mathbb{F}$ satisfying $|\lambda| \leq |\lambda_0|$.
- *balanced* (or *circled*) if for each $x \in E$, the line segment joining x and $-x$ is contained in E, that is, $\lambda x \in E$ for any $x \in E$ and for any $\lambda \in \mathbb{F}$ satisfying $|\lambda| \leq 1$.
- *symmetric* if $-x \in E$ whenever $x \in E$, that is, $-E = E$.
- *star-shaped about zero* if it includes the line segment joining each of its points with zero, that is, if $\lambda x \in E$ for any $x \in E$ and for all λ satisfying $0 \leq \lambda \leq 1$.

The underlying tool used in the proof of Theorem 3.6 is continuity of the addition operation on a linear space. On the other hand, the following theorem is a consequence of continuity of the scalar multiplication operation. The proof of this theorem is adapted from Aliprantis and Border [1]. I came across the proof while reading their discussion of the terminology given in the previous definition. While reading through various sources I tried to avoid reading proofs, but this proof was given informally before the statement of the thereom, and I did not realize what they were proving until it was too late.

Theorem 3.7. *Every open neighborhood of zero contains a balanced neighborhood of zero.*

Proof. Let U be an open neighborhood of zero. By continuity of scalar multiplication, there is a neighborhood $A \times V$ of $(0,0)$ in $\mathbb{F} \times X$ such that $\lambda V \subset U$, for all $\lambda \in A$. In particular, there is a number $\epsilon > 0$ such that $\lambda \in A$ whenever $|\lambda| < \epsilon$. Fix $\lambda_0 \in \mathbb{F}$ with $0 < |\lambda_0| < \epsilon$. Since $\lambda_0 \neq 0$, multiplication by λ_0 is a homeomorphism. Hence, $\lambda_0 V$ is an open neighborhood of zero and $\lambda V \subset U$, for all $\lambda \in \mathbb{F}$ satisfying $|\lambda| \leq |\lambda_0|$. Define

$$W = \bigcup \{\lambda V : |\lambda| \leq |\lambda_0|\}$$

Then, W is contained in U because it is a union of subsets of U. If $x \in W$, then there exists a scalar $\mu \in \mathbb{F}$ with $|\mu| \leq |\lambda_0|$ and a point $v \in V$ such that $x = \mu v$. Therefore, if $\lambda \in \mathbb{F}$ satisfies $|\lambda| \leq 1$, then $|\lambda\mu| \leq |\mu| \leq |\lambda_0|$ and

$$\lambda x = \lambda(\mu v) = (\lambda\mu)v \in (\lambda\mu)V \subset W$$

where $(\lambda\mu)V \subset W$ because $|\lambda\mu| \leq |\lambda_0|$. So, $\lambda W \subset W$, for each scalar λ satisfying $|\lambda| \leq \lambda_0$; that is, W is a balanced neighborhood of zero that lies in U. $\square$

Theorem 3.8. *Every open neighborhood of zero is an absorbing set.*

Proof. Assume that U is a neighborhood of zero which is not radial. In particular, for each positive integer n, there exists a scalar $\lambda_n \in \mathbb{F}$ such that $|\lambda_n| < \frac{1}{n}$ and $\lambda_n x \notin U$. In $\mathbb{F} \times X$, the sequence (λ_n, x) clearly converges to the point $(0, x)$. By continuity of scalar multiplication, the sequence $(\lambda_n x)$ must converge to $0 \cdot x = 0$. Since $(\lambda_n x)$ is a sequence of points in the closed set $X \setminus U$, the sequential limit point 0 must belong to $X \setminus U$; that is, $0 \in X \setminus U$, contradicting the assumption that U is an open neighborhood of zero. As a result, there must be a scalar λ_0 such that $\lambda x \in U$, for all λ satisfying $|\lambda| \leq |\lambda_0|$. $\square$

Corollary 3.5. *If U is an open neighborhood of zero and x is a point of X, then there is a scalar $\lambda \in \mathbb{F}$ such that $x \in \lambda U$.*

Proof. By Theorem 3.8, U is an absorbing set. So, there is a nonzero scalar $\mu \in \mathbb{F}$ such that $\mu x \in U$. Let $\lambda = \mu^{-1}$. Then, $\lambda(\mu x) \in \lambda U$ and $\lambda(\mu x) = \lambda\lambda^{-1}x = x$. Hence, $x \in \lambda U$. $\square$

This corollary does not hold in general if the point x is replaced by any subset E of X. For instance, a line in two-dimensional Euclidean space $\mathbb{R}^2$ cannot be contained in a scalar multiple of a neighborhood of zero whenever that neighborhood is not all of $\mathbb{R}^2$. It seems that this property is related to boundedness. So, we are able to define the concept of a bounded set even in linear topological spaces on which there is no metric.

Definition. A subset E of a linear topological space X is said to be *bounded* if for each neighborhood U of zero there is a scalar $\lambda \in \mathbb{F}$ such that $E \subset \lambda U$.

3.4 More Properties of Linear Topological Spaces

In this section we continue our discussion of properties of linear topological spaces. In particular, we apply some of the properties of the local topological basis at zero. Assume throughout this section that X is a linear topological space with topology $\mathcal{T}$, and let $\mathcal{B}_0$ denote a local topological basis at zero.

Theorem 3.9. *If the topology $\mathcal{T}$ for a linear topological space X satisfies the T_1 axiom, then $\{0\} = \bigcap\{U : U \in \mathcal{B}_0\}$.*

Proof. Clearly, zero is contained in each element of $\mathcal{B}_0$ so that $\{0\} \subset \bigcap\{U : U \in \mathcal{B}_0\}$. Because the topology of X satisfies the T_1 axiom, given a non-zero point x in X, there is an open neighborhood V of 0 such that $x \notin V$. So, there is a local basis element U of $\mathcal{B}_0$ that contains zero and lies in V. Consequently, $x \notin U$ because $x \notin V$ and $U \subset V$. In other words, for each non-zero $x \in X$, there is an element of $\mathcal{B}_0$ that does not contain x. So, $x \notin \bigcap\{U : U \in \mathcal{B}_0\}$. Therefore, $\bigcap\{U : U \in \mathcal{B}_0\} \subset \{0\}$. Hence, $\{0\} = \bigcap\{U : U \in \mathcal{B}_0\}$. □

Theorem 3.10. *If M is a subset of X, then $\overline{M} = \bigcap\{M + U : U \in \mathcal{B}_0\}$.*

Proof. Suppose that $x \in \overline{M}$, and let U be an element of the local topological basis $\mathcal{B}_0$. Then, $x - U$ is an open neighborhood of x. So, $x - U$ contains a point y of M distinct from x. This implies that $y - x \in -U$ and therefore $x - y \in U$. Hence, $x = y + (x - y) \in M + U$. So, for each element U in $\mathcal{B}_0$, $x \in M + U$. Consequently, $\overline{M} \subset \bigcap\{M + U : U \in \mathcal{B}_0\}$.

On the other hand, suppose that $x \in \bigcap\{M + U : U \in \mathcal{B}_0\}$, and let V be an open neighborhood of x. If $x \in M$, then $x = x + 0 \in M + V$, as desired. Now, suppose that $x \notin M$. Because V is open, there is an element U of the local topological basis $\mathcal{B}_0$ such that $x \in x - U$ and $x - U \subset V$. Because $x \in \bigcap\{M + U : U \in \mathcal{B}_0\}$, there is a point y of M distinct from x such that $x = y + (x - y)$, where $x - y \in U$. Therefore, $y = x - (x - y) \in x - U \subset V$. Hence, every open neighborhood of x contains a point of M distinct from x. In other words, $x \in \overline{M}$ and $\bigcap\{M + U : U \in \mathcal{B}_0\} \subset \overline{M}$. Thus, $\overline{M} = \bigcap\{M + U : U \in \mathcal{B}_0\}$. □

Theorem 3.11. *The closure of a linear subspace is a linear subspace.*

Proof. Suppose that M is a linear subspace of a linear topological space X. According to Theorem 3.10, we have $\overline{M} = \bigcap\{M + U : U \in \mathcal{B}_0\}$. First, $\overline{M}$ is non-empty because M is non-empty. Given an arbitrary open set U in $\mathcal{B}_0$, we can choose an open set V in $\mathcal{B}_0$ such that $V + V \subset U$. For any $x, y \in \overline{M}$, there exist $a, b \in M$ and $u, v \in V$ such that $x = a + u$ and $y = b + v$. Therefore,

$$x + y = (a + u) + (b + v) = (a + b) + (u + v)$$

where $a + b \in M$ because M is a linear subspace and $u + v \in V + V \subset U$. Since $x + y \in M + U$ for any element U of $\mathcal{B}_0$, it follows that $x + y \in \overline{M}$. So, $\overline{M}$ is closed under addition. For $\lambda = 0$ and for any $x \in \overline{M}$, $\lambda x = 0 \in \overline{M}$. Now, assume $\lambda \neq 0$. Given an open set U in $\mathcal{B}_0$, the set $\lambda^{-1}U$ is an open neighborhood of zero and it therefore contains some element V of the local topological basis $\mathcal{B}_0$. For any $x \in \overline{M}$, there exists $a \in M$ and $v \in V$ so that

$$\lambda x = \lambda(a + v) = \lambda a + \lambda v$$

where $\lambda a \in M$ because M is a linear subspace and $\lambda v = \lambda\lambda^{-1}u = u$, for some $u \in U$, because $V \subset \lambda^{-1}U$. Since $\lambda x \in M + U$ for any element U of $\mathcal{B}_0$, it follows that $\lambda x \in \overline{M}$. Therefore, $\overline{M}$ is closed under scalar multiplication. Thus, $\overline{M}$ is a linear subspace. □

Theorem 3.12. *A linear topological space is regular.*

Proof. Suppose that X is a linear topological space. Let A be a closed subset of X and let x be a point that is not in A. Since A is closed, $X \setminus A$ is open and $-x + X \setminus A$ is a neighborhood of zero. So, there is an open neighborhood U of zero such that $U + U \subset -x + X \setminus A$. The set $V = U \cap (-U)$ is a balanced ($V = -V$) neighborhood of zero satisfying $V + V \subset -x + X \setminus A$. Then, $x + V$ is an open neighborhood of x and $A + V$ is an open set containing A. Assume that there is a point y in the intersection $(x + V) \cap (A + V)$. Then, there exists $v_1 \in V$ such that $y = x + v_1$, and there exists $v_2 \in V$ such that $y = a + v_2$, for some $a \in A$. As a result, $a = x + v_1 - v_2 \in x + V - V$. Because V is balanced, it follows that $V - V = V + V \subset -x + X \setminus A$ and therefore $x + V - V \subset X \setminus A$. Therefore, $a \in x + V - V \subset X \setminus A$, contradicting the assumption that $a \in A$. Thus, $x + V$ and $A + V$ are disjoint open sets containing x and A, respectively, and X is a regular space. □

Theorem 3.13. *Let $\{X_i\}_{i=1}^{\infty}$ be a collection of linear topological spaces, and let X denote the Cartesian product $\prod_{i=1}^{\infty} X_i$. The space X in the product topology, together with addition and scalar multiplication defined coordinate-wise, is a linear topological space.*

Proof. Let f denote coordinate-wise addition as a function from $X \times X$ into X. Given points $(x_i)_{i=1}^{\infty}$ and $(y_i)_{i=1}^{\infty}$ of X, let U be an arbitrary open subset of X containing $(x_i + y_i)_{i=1}^{\infty}$. Then, there is a basis element (of the product topology)

$$\prod_{i=1}^{\infty} U_i = U_1 \times U_2 \times \cdots U_n \times X_{n+1} \times X_{n+2} \times \cdots$$

where n is some fixed integer and $U_i = X_i$ for $i > n$, such that $(x_i + y_i)_{i=1}^{\infty} \subset \prod_{i=1}^{\infty} U_i \subset U$. For each $i \in \{1, \ldots, n\}$, the addition function $f_i : X_i \times X_i \to X_i$, defined by $f_i(x_i, y_i) = x_i + y_i$, is continuous because X_i is a linear topological space. So, there is an open subset $V_i \times W_i$ of $X_i \times X_i$ containing (x_i, y_i) such that $V_i + W_i \subset U_i$. For $i > n$, let $V_i = X_i$ and $W_i = X_i$.

Then,

$$V = \prod_{i=1}^{\infty} V_i \qquad \text{and} \qquad W = \prod_{i=1}^{\infty} W_i$$

are open subsets of X so that $V \times W$ is an open subset of $X \times X$. Moreover, for any $((x_i')_{i=1}^{\infty}, (y_i')_{i=1}^{\infty}) \in V \times W$, we have

$$f((x_i')_{i=1}^{\infty}, (y_i')_{i=1}^{\infty}) = (x_i' + y_i')_{i=1}^{\infty} \in \prod_{i=1}^{\infty}(V_i + W_i) \subset \prod_{i=1}^{\infty} U_i \subset U$$

Therefore, coordinate-wise addition on X is a continuous function.

Now, let g denote coordinate-wise scalar multiplication as a function from $\mathbb{F} \times X$ into X. Fix a scalar $\lambda_0 \in \mathbb{F}$ and let $(x_i)_{i=1}^{\infty}$ be a point of X. Given an open subset U of X containing $(\lambda_0 x_i)_{i=1}^{\infty}$, there is a basis element

$$\prod_{i=1}^{\infty} U_i = U_1 \times U_2 \times \cdots \times U_n \times X_{n+1} \times X_{n+2} \times \cdots$$

containing $(\lambda_0 x_i)_{i=1}^{\infty}$ such that $\prod_{i=1}^{\infty} U_i \subset U$. For each $i \in \{1, \ldots, n\}$, the scalar multiplication $g_i : \mathbb{F} \times X_i \to X_i$, defined by $g_i(\lambda, x_i) = \lambda x_i$, is continuous because X_i is a linear topological space. For each $i \in \{1, \ldots, n\}$, there exists an open subset V_i of $\mathbb{F}$ containing λ_0 and an open subset W_i of X_i containing $(x_i)_{i=1}^{\infty}$ such that $\lambda x_i \in U_i$, for each $\lambda \in V_i$ and for each $x_i \in W_i$. Let $V = \bigcap_{i=1}^{n} V_i$. Then, V is an open subset of $\mathbb{F}$ containing λ_0 such that $\lambda W_i \subset U_i$, for all $\lambda \in V$. For $i > n$, let $W_i = X_i$ so that $\prod_{i=1}^{\infty} V_i$ is an open set in X. As a result, $V \times W$ is an open subset of $\mathbb{F} \times X$ such that, for any $(\lambda, (x_i')_{i=1}^{\infty}) \in V \times W$,

$$g(\lambda, (x_i')_{i=1}^{\infty}) = (\lambda x_i')_{i=1}^{\infty} \in \prod_{i=1}^{\infty} U_i \subset U$$

Therefore, coordinate-wise scalar multiplication on X is continuous. Because both addition and scalar multiplication are continuous according to their respective product topologies, it follows that X is a linear topological space. $\square$

3.5 Elementary Computations

The theorems in this section rely heavily on the fact that the addition operation on a linear space X is a continuous function from $X \times X$ into X. In particular, we consider the relationship between the addition operation and the interiors and closures of sets. We will also find conditions to determine if an algebraic sum of sets is closed or compact. Throughout this section we will assume that X is a linear topological space with subsets A and B. We will use F to denote the addition operation on X in order to more easily express images and pre-images of sets under the addition operation.

Theorem 3.14. *If each of A and B is a subset of a linear topological space X, then*

$$\mathrm{Int}(A+B) = \mathrm{Int}(A) + \mathrm{Int}(B).$$

Proof. If U and V are open subsets of X satisfying $U \subset A$ and $V \subset B$, then $U + V$ is an open subset of X satisfying $U + V \subset A + B$. Consequently, we have

$$\mathrm{Int}(A) + \mathrm{Int}(B) \subset \mathrm{Int}(A+B)$$

On the other hand, let $F : X \times X \to X$ denote the addition operation on X. Suppose that $x \in \mathrm{Int}(A+B) = \mathrm{Int}(F(A \times B))$. Then, there is an open subset W of X such that $x \in W \subset F(A \times B)$. Because F is continuous, $F^{-1}(W)$ is an open neighborhood of $F^{-1}(x)$. We can write $F^{-1}(x) = (a, b)$, for some $a \in A$ and for some $b \in B$. So, there is an open subset $U \times V$ of $X \times X$ such that $(a, b) \in U \times V$ and $U \times V \subset F^{-1}(W)$. As a result, U is an open subset of X satisfying $a \in U$ and $U \subset A$, which implies that $a \in \mathrm{Int}(A)$. Similarly, $b \in \mathrm{Int}(B)$. Therefore, $F^{-1}(x) \in \mathrm{Int}(A) \times \mathrm{Int}(B)$. As a result, we have

$$x = F(a, b) \in F(\mathrm{Int}(A) \times \mathrm{Int}(B)) = \mathrm{Int}(A) + \mathrm{Int}(B)$$

Hence, $\mathrm{Int}(A+B) \subset \mathrm{Int}(A) + \mathrm{Int}(B)$ so that we have both inclusions. $\square$

Lemma 3.2. *If A and B are subsets of a linear topological space X, then*

$$\overline{A} + \overline{B} \subset \overline{A+B}.$$

Proof. Let F denote the addition on X as a function from $X \times X$ into X. Because F is continuous, the image of the closure of a set is contained in the closure of the image of the set. Notice that $\overline{A} \times \overline{B} = \overline{A \times B}$ in the product space $X \times X$. As a result, we have

$$\overline{A} + \overline{B} = F(\overline{A} \times \overline{B}) = F(\overline{A \times B}) \subset \overline{F(A \times B)} = \overline{A+B}$$

Hence, $\overline{A} + \overline{B} \subset \overline{A+B}$. $\square$

Example 3.1. The other inclusion in Lemma 3.2 does not necessarily hold. For instance, let

$$A = \{(x,y) \in \mathbb{R}^2 : x > 0, y \geq 1/x\} \qquad \text{and} \qquad B = \{(x,y) \in \mathbb{R}^2 : x < 0, y \geq -1/x\}.$$

Both A and B are closed so that $\overline{A} = A$ and $\overline{B} = B$. The algebraic sum is given by $A + B = \{(x,y) \in \mathbb{R}^2 : y > 0\}$, the closure of which is the set $\overline{A+B} = \{(x,y) \in \mathbb{R}^2 : y \geq 0\}$. Because $A+B$ is not closed, we have $\overline{A+B} \not\subset A + B = \overline{A} + \overline{B}$.

Theorem 3.15. *Let A be a subset of a linear topological space X. For any point x of X,*

$$\overline{x+A} = x + \overline{A}.$$

Proof. The inclusion $x + \overline{A} \subset \overline{x+A}$ results from Lemma 3.2. For the other inclusion, suppose that $p \in \overline{x+A}$. Write $p = x + y$ so that $y = -x + p$. Let U be an open neighborhood of y. Then, $x + U$ is an open neighborhood of $x + y = p$. Since $p \in \overline{x+A}$, it follows that $x + U$ intersects $x + A$. As a result, U must intersect A. Therefore, every open neighborhood of y intersects A. In other words, $y \in \overline{A}$ so that $p = x + y \in x + \overline{A}$. Hence, $\overline{x+A} \subset x + \overline{A}$. $\square$

The previous theorem may be generalized a little bit. Given any subset A of a linear topological space X and a finite subset $F = \{x_1, \ldots, x_n\}$ of X, the set $F + A$ is the finite union of the translates $x_i + A$, for $i \in \{1, \ldots, n\}$. Because the closure of a finite union of sets is equivalent to the union of the closures of the sets, Theorem 3.15 implies that the closure of $F + A$ is equal to the union of the translates $x_i + \overline{A}$. Therefore, $\overline{F + A} = F + \overline{A}$. This establishes the following corollary.

Corollary 3.6. *If A is any subset of a linear topological space X and F is a finite subset of X, then $\overline{F + A} = F + \overline{A}$.*

Theorem 3.16. *If each of A and B are compact subsets of X, then $A + B$ is compact.*

Proof. Because A and B are compact subsets of X, their product $A \times B$ is compact. Let $F : X \times X \to X$ be the addition operation defined on X. Then,

$$F(A \times B) = \{a + b : a \in A, b \in B\} = A + B$$

So, $A+B$ is the continuous image of the compact set $A \times B$. Therefore, $A+B$ is compact. □

The proof of the previous theorem uses two basic ideas. One is that the product of compact sets is compact, and the other is that the image of a compact set under a continuous map is compact. Given two sets A and B satisfying a topological property that is preserved under products and continuous maps, the sum $A + B$ will also satisfy that property.

Recall that the sum of two sets is open whenever at least one of the sets is open. The next example demonstrates that another assumption is necessary to guarantee that the sum of two sets is closed whenever at least one is closed.

Example 3.2. If A and B are closed subsets of a linear topological space X, then $A + B$ is not necessarily closed. For instance, let $X = \mathbb{R}^2$ and consider the subsets

$$A = \{(x, y) : x > 0, y \geq 1/x\} \qquad \text{and} \qquad B = \{(x, y) : x <, y \geq -1/x\}$$

Each of A and B is closed, but the sum $A + B = \{(x, y) : y > 0\}$ is not closed.

Although the sum $A + B$ is not necessarily closed whenever both A and B are closed, it is true that $A + B$ is closed if we include the additional assumption that one of the two sets is compact. In order to show this, we will first prove another result that can be useful.

Theorem 3.17. *Suppose that A is a closed subset of X and B is a compact subset of X such that A and B are disjoint. Then there is an open neighborhood V of zero such that A and $B + V$ are disjoint.*

Proof. Let $x \in B$. Because A and B are disjoint, it follows that x belongs to the open set $X \setminus A$. So, $-x + X \setminus A$ is an open neighborhood of zero. By Theorem 3.6, there is a neighborhood W_x of zero satisfying $W_x + W_x \subset -x + X \setminus A$. For each $x \in B$, $x + W_x$ is an open neighborhood of x, and therefore the collection $\{x + W_x : x \in B\}$ is an open covering of B. Because B is compact, there exist finitely many points $x_1, \ldots, x_n \in B$ such that $B \subset \bigcup_{i=1}^{n} (x_i + W_{x_i})$. Let $V = \bigcap_{i=1}^{n} W_{x_i}$, which is an open set because it is a finite intersection of open sets. Given $x \in B$, $x \in x_i + W_{x_i}$, for some $i \in \{1, \ldots, n\}$. In particular, for some $i \in \{1, \ldots, n\}$, we have

$$x + V \subset x + W_{x_i} \subset (x_i + W_{x_i}) + W_{x_i} = x_i + (W_{x_i} + W_{x_i})$$

By construction, $W_{x_i} + W_{x_i} \subset -x_i + X \setminus A$ so that $x_i + (W_{x_i} + W_{x_i}) \subset X \setminus A$, and we have $x + V \subset X \setminus A$. In other words, $A \cap (x + V) = \emptyset$, for all $x \in B$. Thus, $A \cap (B + V) = \emptyset$. $\square$

Theorem 3.18. *If A is a closed subset of X and B is a compact subset of X, then the algebraic sum $A + B$ is closed.*

Proof. If $X = A + B$, then the statement holds because X is closed. Suppose that $A + B \subsetneq X$ and let $x \in X \setminus (A + B)$. Then, $A \cap (x - B) = \emptyset$ because otherwise a point z in this intersection would satisfy $z = a$, for some $a \in A$, and it would satisfy $z = x - b$, for some $b \in B$, so that x could be expressed as $x = a + b$, contrary to assumption. Notice that $-B$ is compact because B is compact and multiplication by a non-zero scalar is a homeomorphism. By Theorem 3.17, there is an open neighborhood V of 0 such that $A \cap (x + V - B) = \emptyset$, that is, $(A + B) \cap (x + V) = \emptyset$. Since $x + V$ is an open neighborhood of x contained in $X \setminus (A + B)$, it follows that $A + B$ is closed. □

3.6 Linear Functionals

Given a linear space X over a scalar field $\mathbb{F}$, a *linear functional* is a linear map from X into the scalar field $\mathbb{F}$. Let $\mathcal{F}(X)$ denote the collection of linear functionals on X. Define addition and scalar multiplication on $\mathcal{F}(X)$ as follows: if $f, g \in \mathcal{F}(X)$, then $f + g$ is the linear functional on X such that $(f + g)(x) = f(x) + g(x)$, for each $x \in X$; if $f \in \mathcal{F}(X)$ and λ is any scalar, then λf is the element of $\mathcal{F}(X)$ such that $(\lambda f)(x) = \lambda f(x)$, for each $x \in X$. The collection $\mathcal{F}(X)$ together with addition and scalar multiplication defined in this way is a linear space.

Theorem 3.19. *Suppose that X is a linear topological space, and let $f \in \mathcal{F}(X)$ be a linear functional which is not identically zero. The following are equivalent:*

(1) f is continuous

(2) the null space of f is closed

(3) the null space of f is not dense in X

(4) f is bounded on a neighborhood of zero

Proof.

- If f is continuous, then the nullspace of f is closed.

  Suppose that f is continuous, and let N denote the nullspace of f. Then, $X \setminus N$ is non-empty because f is not identically zero. Given a point $x_0 \in X \setminus N$, we have $f(x_0) \neq 0$. Without loss of generality, assume that $f(x_0) > 0$, and let $U = (0, \infty)$.

Since f is continuous, $f^{-1}(U)$ is an open set containing x_0. Now, for each $x \in f^{-1}(U)$, we have $f(x) \in (0, \infty)$ so that $f(x) \neq 0$. Therefore, $f^{-1}(U)$ is an open set containing x_0 such that $f^{-1}(U) \subset X \setminus N$. Hence, $X \setminus N$ is open and the nullspace N is closed.

- If the nullspace of f is closed, then the null space of f is not dense in X.

  Assume that the nullspace N of f is dense in X, that is, $\overline{N} = X$. By assumption, N is closed so that $N = \overline{N}$. However, this implies that $N = \overline{N} = X$, which contradicts the assumption that f is not identically zero.

- If the nullspace of f is not dense in X, then f is bounded on a neighborhood of zero.

  Suppose that the nullspace N of f is not dense in X. Then, there is a point x and a neighborhood U of zero such that $x + U$ does not intersect N. Because U is a neighborhood of zero, it contains a balanced neighborhood V of zero, that is, $\lambda V \subset U$, for all scalars λ with $|\lambda| \leq 1$. Assume f is not bounded on V. Then, there exists $v \in V$ such that $|f(v)| \geq |f(x)|$. Choose $\lambda \in \mathbb{F}$ with $|\lambda| \leq 1$ such that $|\lambda| \cdot |f(v)| = |f(x)|$. Then,

  $$|f(x)| = |\lambda||f(v)| = |\lambda f(v)| = |f(\lambda v)|$$

  Because $|\lambda| \leq 1$, we have $\lambda V \subset U$ so that there is an element u of U with $\lambda v = u$ and hence $|f(x)| = |f(u)|$. Without loss of generality, assume that $f(u) = -f(x)$ because we can take $\pm\lambda$ as needed. Then,

  $$0 = f(x) + f(u) = f(x + u)$$

  As a result, $x + u$ is an element of the nullspace N of f, contrary to the assumption that $x + U$ does not intersect N. Thus, f must be bounded on V.

- If f is bounded on a neighborhood of zero, then f is continuous.

  Suppose that f is bounded on an open set U containing zero. Then, there is a positive number M such that $|f(x)| < M$, for all $x \in U$. Given $\epsilon > 0$, let $\lambda = \frac{\epsilon}{M}$ and define $V = \lambda U$. For each $x \in V$, there exists $u \in U$ such that $x = \lambda u$. As a result, $|f(x)| = |f(\lambda u)| = \lambda |f(u)|$. Because $u \in U$, it follows that $|f(u)| < M$ so that $|f(x)| < \lambda M = \epsilon$. Therefore, f is continuous at zero. Since f is linear, it follows that f is continuous at each of its points.

$\square$

Chapter 4

Metric Linear Topological Spaces

In the remaining chapters we will study linear topological spaces for which the topology is induced by a metric. I will assume some familiarity with metric spaces throughout the rest of the thesis. Of particular interest will be spaces with a *translation-invariant* metric d, meaning that $d(x, y) = d(x + z, y + z)$, for any points x, y, and z in X. Given a point x of a metric space X, the ϵ-ball centered at x will be denoted by $B_\epsilon(x)$.

4.1 Totally Bounded Sets

A subset of a metric space is said to be totally bounded if for each $\epsilon > 0$ the set can be covered by finitely many ϵ-balls. The following definition extends this concept to linear topological spaces which are not necessarily metric spaces. In this section we show that the two definitions are equivalent for metric linear topological spaces.

Definition. A subset B of a linear topological space X is *totally bounded* if for each neighborhood U of zero there exists a finite set F such that $B \subset F + U$.

It is clear that the set F in the definition of a totally bounded set may be assumed to be a subset of B. It is also clear that a subset of a totally bounded set is totally bounded. The following theorems demonstrate other properties of totally bounded sets.

Theorem 4.1. *The closure of a totally bounded set is totally bounded.*

Proof. Suppose that B is a totally bounded subset of a linear topological space X, and let U be an open neighborhood of zero. Because linear topological spaces are regular, there is a neighborhood V of zero such that $\overline{V} \subset U$. By assumption, there is a finite set F such that

$B \subset F+V$. Then, $\overline{B} \subset \overline{F+V}$. Since F is finite, we have $\overline{F+V} = F+\overline{V}$ by Corollary 3.6. As a result, $\overline{B} \subset F+\overline{V} \subset F+U$. Thus, $\overline{B}$ is a totally bounded set. □

Theorem 4.2. *The image of a totally bounded set under a continuous linear map is totally bounded.*

Proof. Suppose that B is a totally bounded subset of X and $T : X \to Y$ is a continuous linear map. Let V be a neighborhood of zero in Y. For any $y \in T(B)$, there exists $x \in B$ such that $T(x) = y$. Since $y+V$ is a neighborhood of y and T is continuous, there is an open neighborhood U of x in X such that $T(U) \subset y+V$. Because U is a neighborhood of x, it follows that $-x+U$ is a neighborhood of zero. Since B is totally bounded, there is a finite set F such that $B \subset F+(-x+U)$. Consequently, we have $T(B) \subset T(-x+F+U)$ so that $T(B) \subset -T(x)+T(F)+T(U)$ by linearity of T. Because $T(U) \subset y+V$ and $y = T(x)$, it follows that $T(B) \subset T(F)+V$. Moreover, $T(F)$ is finite since F is finite. Therefore, $T(B)$ is totally bounded. □

Theorem 4.3. *A subset of a product is totally bounded if and only if each of its projections is totally bounded.*

Proof. Let $\{X_\alpha : \alpha \in \Gamma\}$ be a collection of linear topological spaces. Let π_β denote the projection map of $\prod_{\alpha\in\Gamma} X_\alpha$ onto X_β. Suppose that B is a totally bounded subset of the product space $\prod_{\alpha\in\Gamma} X_\alpha$. Fix $\beta \in \Gamma$, and let U_β be an arbitrary neighborhood of zero in X_β. Then, $\prod_{\alpha\in\Gamma} U_\alpha$, where $U_\alpha = X_\alpha$ for $\alpha \neq \beta$, is an open neighborhood of zero in $\prod_{\alpha\in\Gamma} X_\alpha$. By assumption, there is a finite set F such that $B \subset F + \prod_{\alpha\in\Gamma} U_\alpha$. Then, since the projection map π_β is linear,

$$\pi_\beta(B) \subset \pi_\beta\left(F + \prod_{\alpha\in\Gamma} U_\alpha\right) = \pi_\beta(F) + U_\beta$$

where $\pi_\beta(F)$ is finite. Therefore, $\pi_\beta(B)$ is totally bounded.

Conversely, suppose that $\pi_\alpha(B)$ is totally bounded, for each $\alpha \in \Gamma$. Without loss of generality, we can show that B is contained in a finite translate of a (topological) basis element containing zero. Let $\prod_{\alpha\in\Gamma} U_\alpha$ be a neighborhood of zero, where $U_\alpha \neq X_\alpha$ for only finitely many indices, say $\alpha_1, \ldots, \alpha_n$, in Γ. For each $\alpha \in \Gamma$, U_α is a neighborhood of zero in X_α. By assumption, there exists a finite set F_α such that $\pi_\alpha(B) \subset F_\alpha + U_\alpha$. For each $\alpha \notin \{\alpha_1, \ldots, \alpha_n\}$, we can take $F_\alpha = \{0\}$ since $U_\alpha = X_\alpha$. Let F be the subset of $\prod_{\alpha\in\Gamma} X_\alpha$ defined by $\prod_{\alpha\in\Gamma} F_\alpha$, that is,

$$F = \big\{(x_\alpha) : x_{\alpha_j} \in F_{\alpha_j} \text{ for } j = 1, \ldots, n; x_\alpha = 0 \text{ otherwise}\big\}$$

Then, F is finite since each F_{α_j} is finite. For each $\alpha \in \Gamma$, we have $\pi_\alpha(B) \subset F_\alpha + U_\alpha$, and therefore

$$B \subset \prod_{\alpha\in\Gamma}(F_\alpha + U_\alpha) = \prod_{\alpha\in\Gamma} F_\alpha + \prod_{\alpha\in\Gamma} U_\alpha = F + U$$

Thus, B is totally bounded. □

Theorem 4.4. *A scalar multiple of a totally bounded set is totally bounded.*

Proof. Suppose that B is a totally bounded subset of X. Clearly, $0 \cdot B = \{0\}$ is totally bounded. Let λ be a non-zero scalar. If U is any neighborhood of zero, then $\lambda^{-1}U$ is also a neighborhood of zero. Since B is totally bounded, there is a finite set $F = \{x_1, \ldots, x_n\}$ such that $B \subset F + \lambda^{-1}U$. As a result, $\lambda B \subset \lambda\left(F + \lambda^{-1}U\right) = \lambda F + U$, where $\lambda F = \{\lambda x_1, \ldots, \lambda x_n\}$ is finite. So, λB is totally bounded. □

If the metric for a linear topological space is translation-invariant, then the ϵ-ball centered at a point is equivalent to the translate of the ϵ-ball centered at zero.

Lemma 4.1. *Suppose that X is a metric linear topological space with translation-invariant metric d. If $x \in X$ and $\epsilon > 0$, then $B_\epsilon(x) = x + B_\epsilon(0)$.*

Proof. Suppose that $x \in x_0 + B_\epsilon(0)$, that is, $x = x_0 + y$, for some $y \in B_\epsilon(0)$. Because the metric d is translation-invariant and $d(0, y) < \epsilon$, we have $d(x_0, x_0 + y) < \epsilon$. Since $x = x_0 + y$, it follows that $d(x_0, x) < \epsilon$ so that $x \in B_\epsilon(x_0)$. On the other hand, suppose that

$x \in B_\epsilon(x_0)$, that is, $d(x, x_0) < \epsilon$. Because the metric is translation-invariant, $d(x - x_0, 0) < \epsilon$. Consequently, $x - x_0 \in B_\epsilon(0)$ implies that $x \in x_0 + B_\epsilon(0)$. □

Theorem 4.5. *Let X be a metric linear topological space with translation-invariant metric d. A subset B of X is totally bounded if and only if for each $\epsilon > 0$ there is a finite covering of B by open ϵ-balls.*

Proof. Suppose that B is a totally bounded subset of X. Given $\epsilon > 0$, the ϵ-ball $B_\epsilon(0)$ is an open neighborhood of zero. By assumption, there exists a finite set $F = \{x_1, \ldots, x_n\}$ such that $B \subset F + B_\epsilon(0)$. Then, $F + B_\epsilon(0) = \bigcup_{i=1}^{n}(x_i + B_\epsilon(0))$. Because the metric is translation-invariant, we have $x_i + B_\epsilon(0) = B_\epsilon(x_i)$, for each $i \in \{1, \ldots, n\}$, by Lemma 4.1. Therefore, $B \subset F + B_\epsilon(0) \subset \bigcup_{i=1}^{n} B_\epsilon(x_i)$.

Now, suppose that for each $\epsilon > 0$, there is a finite covering of B by ϵ-balls. Let U be an open neighborhood of zero in X. Then, there exists $\epsilon > 0$ such that $B_\epsilon(0) \subset U$. By assumption, there exists a finite set $F = \{x_1, \ldots, x_n\}$ such that $B \subset \bigcup_{i=1}^{n} B_\epsilon(x_i)$. Therefore,

$$B \subset \bigcup_{i=1}^{n} B_\epsilon(x_i) = \bigcup_{i=1}^{n}(x_i + B_\epsilon(0)) \subset \bigcup_{i=1}^{n}(x_i + U) = \bigcup_{i=1}^{n} x_i + U = F + U$$

Hence, B is totally bounded. □

4.2 Completeness

A sequence (x_n) in a metric space (X, d) is said to be a Cauchy sequence if for each $\epsilon > 0$ there is an integer N such that $d(x_m, x_n) < \epsilon$, for all $m, n \geq N$. However, we can extend this definition to linear topological spaces which are not necessarily metric spaces. In this section we show that the two definitions are equivalent for linear topological spaces with a translation-invariant metric.

Definition. Let X be a linear topological space. A sequence (x_n) in X is said to be a *Cauchy* sequence if and only if for each neighborhood U of zero, there exists an integer N such that $x_m - x_n \in U$, for all $m, n \geq N$.

Theorem 4.6. *Suppose that X is a metric linear topological space with translation-invariant metric d. A sequence (x_n) in X is Cauchy if and only if for each $\epsilon > 0$ there exists an integer N such that $d(x_m, x_n) < \epsilon$, for all $m, n \geq N$.*

Proof. Suppose that (x_n) is a Cauchy sequence. Given $\epsilon > 0$, the ϵ-ball $B_\epsilon(0)$ is a neighborhood of zero. By assumption, there exists an integer N such that $x_m - x_n \in B_\epsilon(0)$, for $m, n \geq N$. Therefore, $x_m \in x_n + B_\epsilon(0) = B_\epsilon(x_n)$, that is $d(x_m, x_n) < \epsilon$, for all $m, n \geq N$. For the converse, let U be any open neighborhood of zero. Then, there is an $\epsilon > 0$ such that $B_\epsilon(0) \subset U$. By assumption, there exists an N such that $d(x_m, x_n) < \epsilon$, for all $m, n \geq N$. Because the metric is translation-invariant, this implies that $d(0, x_m - x_n) < \epsilon$, for all $m, n \geq N$. Hence, $x_m - x_n \in B_\epsilon(0) \subset U$, for all $m, n \geq N$. $\square$

Definition. A metric space X is *complete* if and only if every Cauchy sequence of X converges to a point of X.

Chapter 5

Normed Linear Spaces

A normed linear space has a natural topology, which is the metric topology induced by the norm metric. First we will show that the operators on a normed linear space are continuous with respect to the norm metric, proving that any normed linear space is a linear topological space. Because we are now working in a metric space, we will be able to utilize the sequence definition of continuity and the sequence lemma.

5.1 Normed Linear Spaces

A norm on a linear space can be thought of as a function that gives the length of a vector in the space. The norm on a linear space leads to a natural metric for which the distance between two points is the length of the vector joining them.

Definition. A *norm* on a linear space X is a function $\|\cdot\| : X \to [0, \infty)$ satisfying, for all points $x, y \in X$ and for all scalars $\lambda \in \mathbb{F}$,

(a) (*positive definiteness*) $\|x\| = 0$ if and only if $x = 0$

(b) (*homogeneity*) $\|\lambda x\| = |\lambda| \|x\|$

(c) (*triangle inequality*) $\|x + y\| \leq \|x\| + \|y\|$

Theorem 5.1. *The function $d(x, y) = \|x - y\|$ is a metric for X.*

Proof. Clearly, $d(x, y) = \|x - y\| \geq 0$ by definition of the norm. By positive definiteness of the norm, $\|x-y\| = 0$ if and only if $x-y = 0$ if and only if $x = y$. Moreover, by homogeneity of the norm,

$$\|x - y\| = \|-(y - x)\| = |-1| \cdot \|y - x\| = \|y - x\|$$

So, the function d is symmetric. Lastly, the triangle inequality is satisfied since

$$\|x - z\| = \|x - y + y - z\| \leq \|x - y\| + \|y - z\|$$

Therefore, d is a metric for X. $\square$

The *norm metric*, defined by $d(x, y) = \|x-y\|$, on a normed linear space X is *translation-invariant* because $\|(x+z) - (y+z)\| = \|(x-y) + (z-z)\| = \|x-y\|$, for all x, y, and z in X. Moreover, the norm metric is *homogeneous*, that is, $d(\lambda x, \lambda y) = |\lambda| d(x, y)$, for all x and y in X and for any scalar λ. This follows directly from homogeneity of the norm since $\|\lambda x - \lambda y\| = \|\lambda(x-y)\| = |\lambda|\|x-y\|$.

In the following theorem, we need to show that the addition on X is a continuous function from $X \times X$ into X and that scalar mutliplication on X is a continuous function from $\mathbb{F} \times X$ into X. Because the topology of X is induced by the metric d, we can give both $X \times X$ and $\mathbb{F} \times X$ a metric topology which is equivalent to the product topology. As a result, a function on these metric spaces is continuous if and only if the function preserves sequential limit points.

Theorem 5.2. *The normed linear space X together with the topology induced by the norm metric is a linear topological space.*

Proof. Let (x_n) and (y_n) be convergent sequences in X with $x_n \to x$ and $y_n \to y$. Given $\epsilon > 0$, choose an integer N such that $\|x_n - x\| < \frac{\epsilon}{2}$ and $\|y_n - y\| < \frac{\epsilon}{2}$, for all $n \geq N$. By the triangle inequality, for all $n \geq N$, we have

$$\|(x_n + y_n) - (x + y)\| \leq \|x_n - x\| + \|y_n - y\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Hence, $x_n + y_n \to x + y$. Therefore, the addition operation on X is continuous.

Now, let (x_n) be a convergent sequence in X as before, and let (λ_n) be a sequence of scalars converging to a scalar λ. Because the sequence (λ_n) of scalars converges, it is bounded. Let M be a positive number such that $|\lambda_n| \leq M$, for all n. Given $\epsilon > 0$, choose an integer N such that $\|x_n - x\| < \frac{\epsilon}{2M}$ and $|\lambda_n - \lambda| < \frac{\epsilon}{2\|x\|}$, for all $n \geq N$. Then, we have

$$\begin{aligned}
\|\lambda_n x_n - \lambda x\| &= \|\lambda_n x_n - \lambda_n x + \lambda_n x - \lambda x\| \\
&\leq \|\lambda_n (x_n - x)\| + \|(\lambda_n - \lambda)x\| \\
&= |\lambda_n| \cdot \|x_n - x\| + |\lambda_n - \lambda| \cdot \|x\| \\
&< |\lambda_n| \left(\tfrac{\epsilon}{2M}\right) + \left(\tfrac{\epsilon}{2\|x\|}\right) \|x\| \leq \epsilon
\end{aligned}$$

Hence, $\lambda_n x_n \to \lambda x$. Therefore, scalar multiplication on X is a continuous function. Because both operations are continuous according to the respective product topologies, it follows that X is a linear topological space. $\square$

Lemma 5.1. *For all x and y in a normed linear space X,*

$$|\|x\| - \|y\|| \leq \|x - y\|$$

Proof. By the triangle inequality, we have $\|x\| = \|x - y + y\| \leq \|x - y\| + \|y\|$. Subtracting $\|y\|$ from each side of the inequality results in the inequality $\|x\| - \|y\| \leq \|x - y\|$. Similarly, $\|y\| - \|x\| \leq \|y - x\| = \|x - y\|$. Therefore, because $\|y\| - \|x\| = -(\|x\| - \|y\|)$, we have the desired inequality. $\square$

Lemma 5.2. *If X is a normed linear space, then the norm $\|\cdot\|$ defined on X is uniformly continuous (and therefore continuous).*

Proof. Given $\epsilon > 0$, let x and y be points of X such that $\|x - y\| < \epsilon$. Then, we have $|\|x\| - \|y\|| \leq \|x - y\| < \epsilon$ by Lemma 5.1. Therefore, the norm is uniformly continuous. $\square$

Because the norm $\|\cdot\|$ defined on a normed linear space is uniformly continuous, the norm preserves sequential limit points. In other words, if (x_n) is a sequence in X such that $x_n \to x$, then $\|x_n\| \to \|x\|$. The next theorem demonstrates that the operation of addition on a normed linear space is also uniformly continuous.

Theorem 5.3. *The addition operation on a normed linear space X is a uniformly continuous function from $X \times X$ into X.*

Proof. Let ρ denote the metric defined on $X \times X$ given by

$$\rho((x_1, y_1), (x_2, y_2)) = \|x_1 - x_2\| + \|y_1 - y_2\|$$

The topology generated by this metric is equvialent to the product topology on $X \times X$. Given $\epsilon > 0$ and points (x_1, y_1) and (x_2, y_2) in $X \times X$ satisfying $\rho((x_1, y_1), (x_2, y_2)) < \epsilon$, we have

$$\|(x_1 + y_1) - (x_2 + y_2)\| \leq \|x_1 - x_2\| + \|y_1 - y_2\| < \epsilon$$

Therefore, the addition on X is a uniformly continuous function from $X \times X$ into X. □

5.2 Bounded Linear Transformations

Linear transformations are essential in the study of linear spaces, and continuous functions are essential in topology. In the study of linear topological spaces, we are concerned mostly with continuous linear transformations. So, it will be useful to determine conditions under which linear transformations are continuous. We will see that a linear transformation is continuous if and only if it is bounded. All linear transformations in a finite-dimensional space are bounded and therefore continuous.

Let $\mathrm{Lin}(X)$ denote the collection of all linear transformations from X into itself. For any linear transformation T in $\mathrm{Lin}(X)$, define

$$|T| = \sup_{x \neq 0} \frac{\|T(x)\|}{\|x\|}$$

Notice that $|T|$ can also be expressed as $\sup\limits_{\|x\|=1} \|T(x)\|$ or as $\sup\limits_{\|x\|\leq 1} \|T(x)\|$.

Example 5.1. The set $X = \mathcal{C}^\infty([0,1])$ of smooth (infinitely differentiable) real-valued functions defined on the unit interval $[0,1]$ together with the maximum norm

$$\|f\|_\infty = \max\{f(x) : 0 \leq x \leq 1\}$$

is a normed linear space. Let D denote the derivative operator, that is, $D(f) = f'$. Consider the sequence (e^{nx}) of functions in X. Then,

$$\frac{\|D(e^{nx})\|}{\|e^{nx}\|} = \frac{\|ne^{nx}\|}{\|e^{nx}\|} = \frac{|n|\|e^{nx}\|}{\|e^{nx}\|} = |n| \to \infty$$

As a result, $|D|$ is unbounded, that is, $|D| = \infty$.

Definition. A linear transformation such that $|T| < \infty$ is said to be *bounded*.

Let $\mathrm{Lin_B}(X)$ denote the set of all bounded linear transformations on the linear topological space X. Define addition and scalar multiplication on $\mathrm{Lin_B}(X)$ as follows: If T_1 and T_2 are bounded linear transformations on X, then $T_1 + T_2$ is the element of $\mathrm{Lin_B}(X)$ so that $(T_1 + T_2)(x) = T_1(x) + T_2(x)$, for all $x \in X$; if $T \in \mathrm{Lin_B}(X)$ and $\lambda \in \mathbb{F}$ is any scalar, then λT is the element of $\mathrm{Lin_B}(X)$ so that $(\lambda T)(x) = \lambda T(x)$, for all $x \in X$.

Theorem 5.4. *The function* $|\cdot|$ *is a norm on* $\mathrm{Lin_B}(X)$.

Proof. Given $T \in \mathrm{Lin_B}(X)$, we have $|T| = \sup_{\|x\|=1} \|T(x)\| \geq 0$ because the supremum over a set of non-negative numbers is non-negative. Moreover, $|T| = 0$ if and only if $\|T(x)\| = 0$, for all x satisfying $\|x\| = 1$, if and only if $T(x) = 0$, for all $x \in X$, i.e., T is the zero map. Now, for any scalar λ,

$$|\lambda T| = \sup_{\|x\|=1} \|(\lambda T)(x)\| = \sup_{\|x\|=1} \|\lambda \cdot T(x)\| = \sup_{\|x\|=1} |\lambda|\|T(x)\| = |\lambda| \sup_{\|x\|=1} \|T(x)\| = |\lambda| \cdot |T|$$

Lastly, for any $T_1, T_2 \in \mathrm{Lin_B}(X)$, we have

$$\begin{aligned}|T_1 + T_2| = \sup_{\|x\|=1} \|(T_1 + T_2)(x)\| &= \sup_{\|x\|=1} \|T_1(x) + T_2(x)\| \\ &\leq \sup_{\|x\|=1} (\|T_1(x)\| + \|T_2(x)\|) \\ &\leq \sup_{\|x\|=1} \|T_1(x)\| + \sup_{\|x\|=1} \|T_2(x)\| = |T_1| + |T_2|\end{aligned}$$

Therefore, $(\mathrm{Lin_B}(X), |\cdot|)$ is a normed linear space. □

Since $(\mathrm{Lin_B}(X), |\cdot|)$ is a normed linear space, it follows that the $|T_1 - T_2|$ defines the norm metric on $\mathrm{Lin_B}(X)$. According to Theorem 5.2, the linear space $\mathrm{Lin_B}(X)$ together with the norm metric forms a linear topological space. This establishes the following corollary.

Corollary 5.1. *The set* $\mathrm{Lin_B}(X)$ *of bounded linear transformations on a normed linear space is a linear topological space.*

Theorem 5.5. *A linear transformation* T *is continuous if and only if* $T \in \mathrm{Lin_B}(X)$.

Proof. Suppose that T is a continuous linear map on X. By continuity of T at 0, there exists $\delta > 0$ such that $\|T(x) - T(0)\| < 1$ whenever $\|x\| < \delta$. Because T is linear, $T(0) = 0$ so that $\|T(x)\| < 1$ whenever $\|x\| < \delta$. Let $\lambda = \frac{\delta}{2}$. For any point x in X satisfying $\|x\| = 1$, we have $\|\lambda x\| = \lambda\|x\| = \lambda < \delta$. By continuity of T at zero, $\|T(\lambda x)\| < 1$. Because T is linear, $\|T(\lambda x)\| = \|\lambda T(x)\| = \lambda\|T(x)\|$. As a result, $\lambda\|T(x)\| < 1$ so that $\|T(x)\| < \lambda^{-1}$, for all $x \in X$ satisfying $\|x\| = 1$. Therefore, $|T| \leq \lambda^{-1}$ so that T is bounded, that is, $T \in \mathrm{Lin_B}(X)$.

Conversely suppose that $T \in \mathrm{Lin_B}(X)$. Because $|T| < \infty$, there is a number $M > 0$ satisfying $\|T(x)\| \leq M\|x\|$, for each point x of X. Given $\epsilon > 0$, let $\delta = \frac{\epsilon}{M}$. For any $x \in X$ that satisfies $\|x\| < \delta$, we have $M\|x\| < \epsilon$. Since T is linear, we have $T(0) = 0$. By assumption, $\|T(x)\| \leq M\|x\|$; therefore, $\|x - 0\| < \delta$ implies that $\|T(x) - T(0)\| < \epsilon$. So, T is continuous at zero and therefore continuous at every point of X. □

As the next theorem states, all linear transformations on a finite-dimensional space are bounded. As a result, the previous theorem implies that all linear transformation on a finite-dimensional space are continuous.

Theorem 5.6. *If X is finite-dimensional, then* $\mathrm{Lin}(X) = \mathrm{Lin}_{\mathrm{B}}(X)$.

Chapter 6

Hilbert Spaces

6.1 Inner Product Spaces

Normed linear spaces have nice properties, but they have even better properties when the norm is induced by an inner product. In spaces with a norm given by an inner product, we are able to use the Cauchy-Schwarz inequality and the Parallelogram Law. After showing that an inner product space is a linear topological space, we turn our attention to problems set in a Hilbert space.

Definition. Let X be a linear space over $\mathbb{R}$. A *inner product* on X is a function $\langle \cdot, \cdot \rangle$ from $X \times X$ into $\mathbb{R}$ that satisfies, for all $x, y, z \in X$ and for all $\lambda, \mu \in \mathbb{R}$,

(a) $\langle x, x \rangle \geq 0$, and $\langle x, x \rangle = 0$ if and only if $x = 0$.

(b) $\langle x, y \rangle = \langle y, x \rangle$

(c) $\langle \lambda x + \mu y, z \rangle = \lambda \langle x, z \rangle + \mu \langle y, z \rangle$

In this case, X is said to be a *real inner product space.* We can also define a *complex inner product space* by allowing the scalars λ and μ in the definition to be complex numbers and by changing part (b) of the definition to

(d) $\langle x, y \rangle = \overline{\langle y, x \rangle}$

where $\overline{z}$ denotes the complex conjugate of the complex number z.

If X is an inner product space, the *inner product norm* on X is defined by $\|x\| = \sqrt{\langle x, x \rangle}$. Before we show that this does in fact define a norm on X, we will state a well-known theorem that holds in inner product spaces. The Cauchy-Schwarz inequality will be used to show that the inner product norm is actually a norm.

Theorem (Cauchy-Schwarz inequality)**.** *Let X be an inner product space with inner product $\langle \cdot, \cdot \rangle$. For all $x, y \in X$,*

$$|\langle x, y\rangle|^2 \leq \langle x, x\rangle \cdot \langle y, y\rangle$$

Moreover, equality holds if and only if x and y are linearly dependent.

We will omit the proof of the Cauchy-Scwarz inequality. Notice that the inequality can be expressed in terms of the inner product norm. By taking the square root of each side we obtain the inequality $|\langle x, y\rangle| \leq \|x\| \cdot \|y\|$. Another useful property of inner product spaces is the Parallelogram Law. This property will be important in a problem in the next section.

Theorem 6.1 (Parallelogram Law)**.** *Let X be an inner product space. For all $x, y \in X$,*

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

Proof. By the properties of the inner product norm, we have

$$\begin{aligned} \|x + y\|^2 = \langle x + y, x + y\rangle &= \langle x, x + y\rangle + \langle y, x + y\rangle \\ &= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle \\ &= \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle \end{aligned}$$

Similarly, $\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y\rangle$. Adding together these two equations gives the Parallelogram Law. □

Theorem 6.2. *Let X be an inner product space. Then, $\|x\| = \sqrt{\langle x, x\rangle}$ is a norm on X.*

Proof. Clearly, $\|x\| \geq 0$ for all $x \in X$. Moreover, $\|x\| = 0$ if and only if $\langle x, x\rangle = 0$ if and only if $x = 0$. Now, for any scalar λ,

$$\|\lambda x\| = \sqrt{\langle \lambda x, \lambda, x\rangle} = \sqrt{\lambda^2\langle x, x\rangle} = |\lambda|\sqrt{\langle x, x\rangle} = |\lambda| \cdot \|x\|$$

Lastly, for all x and y in X, we have

$$\|x+y\|^2 = \langle x+y, x+y\rangle = \|x\|^2 + \|y\|^2 + \langle x, y\rangle + \langle y, x\rangle$$

Now, if X is a real inner product space, then $\langle x, y\rangle = \langle y, x\rangle$. But, if X is a complex inner product space, then $\langle x, y\rangle = \overline{\langle y, x\rangle}$. In either case, $\langle x, y\rangle + \langle y, x\rangle \leq 2|\langle x, y\rangle|$. Now, by the Cauchy-Scwarz inequality, we have $|\langle x, y\rangle| \leq \|x\|\|y\|$. Thus,

$$\|x+y\|^2 \leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| = (\|x\| + \|y\|)^2$$

Taking the square root of both sides gives the triangle inequality. Thus, $\|x\| = \sqrt{\langle x, x\rangle}$ defines a norm on X, called the *inner product norm*. $\square$

Because $\|x\| = \sqrt{\langle x, x\rangle}$ defines a norm on X, the metric induced by the inner product norm is given by $\|x-y\| = \sqrt{\langle x-y, x-y\rangle}$ according to Theorem 5.1. Moreover, with this metric the inner product space X is a linear topological space by Theorem 5.2.

Corollary 6.1. *An inner product space X together with the topology induced by the inner product norm metric is a linear topological space.*

Theorem 6.3. *Let X be an inner product space with inner product $\langle \cdot, \cdot\rangle$. The inner product is a continuous function from $X \times X$ into $\mathbb{R}$.*

Proof. Suppose that (x_n) and (y_n) are sequences in X with $x_n \to x$ and $y_n \to y$. Then,

$$\begin{aligned}
|\langle x_n, y_n\rangle - \langle x, y\rangle| &= |\langle x_n, y_n\rangle - \langle x_n, y\rangle + \langle x_n, y\rangle - \langle x, y\rangle| \\
&= |\langle x_n, y_n - y\rangle + \langle x_n - x, y\rangle| \\
&\leq |\langle x_n, y_n\rangle| + |\langle x_n - x, y\rangle| \\
&\leq \|x_n\| \cdot \|y_n - y\| + \|x_n - x\| \cdot \|y\|
\end{aligned}$$

Because (x_n) converges there is a positive number M such that $\|x_n\| \leq M$, for all n. Given $\epsilon > 0$, choose $N \in \mathbb{N}$ such that $\|x_n - x\| < \frac{\epsilon}{2\|y\|}$ and $\|y_n - y\| < \frac{\epsilon}{2M}$, for all $n \geq N$. Then, for $n \geq N$,

$$|\langle x_n, y_n \rangle - \langle x, y \rangle| \leq \|x_n\| \cdot \|y_n - y\| + \|x_n - x\| \cdot \|y\| < \epsilon$$

Hence, $\langle x_n, y_n \rangle \to \langle x, y \rangle$. Thus, the inner product is continuous. $\square$

6.2 Hilbert Spaces

This final section includes some problems concerning projections and orthogonality in Hilbert spaces, including the uniqueness of the projection onto a closed and convex set. We will also see that a Hilbert space can be written as the direct sum of a closed linear subspace and its orthogonal complement. Lastly, we consider the relationship between a set, the closure of its span, the orthogonal complement, and the double orthogonal complement.

Definition. A *Hilbert space* $\mathcal{H}$ is an inner product space that is complete with respect to the inner product norm.

Theorem 6.4. *Suppose that F is a closed and convex subset of a Hilbert space $\mathcal{H}$. If $x \in \mathcal{H}$, then there is a unique element $P(x)$ in F satisfying*

$$\|x - P(x)\| = \inf\{\|x - z\| : z \in F\}$$

The mapping P is called the projection *of $\mathcal{H}$ on F.*

Proof. Fix $x_0 \in \mathcal{H}$, and let $\delta = \inf\{\|x_0 - x\| : x \in F\}$. If $x_0 \in F$, then $\|x_0 - x_0\| = 0$ and therefore $\delta = 0$. Moreover, x_0 is unique because $\|x_0 - x\| = 0$ if and only if $x_0 = x$. So, we can now assume that $x_0 \notin F$. For each $n \in \mathbb{N}$, we can choose $x_n \in F$ satisfying $\delta \leq \|x_0 - x_n\| < \delta + \frac{1}{n}$; otherwise, $\delta + \frac{1}{n}$ would be a lower bound of $\{\|x_0 - x\| : x \in F\}$ that is greater than the greatest lower bound δ. Without loss of generality, we can assume that $x_0 = 0$ because translation by $-x_0$ is a homeomorphism and the metric is translation-invariant so that our assumptions are not afffected. Then, $\delta \le \|x_n\| < \delta + \frac{1}{n}$ for each $n \in \mathbb{N}$, which implies that $\|x_n\| \to \delta$. Because $\mathcal{H}$ is a Hilbert space, we have the Parallelogram Law, so that for any $m, n \in \mathbb{N}$,

$$\begin{aligned} &\|x_m - x_n\|^2 = 2\|x_m\|^2 + 2\|x_n\|^2 - \|x_m + x_n\|^2 \\ \text{or} \quad &\|x_m - x_n\|^2 = 2\|x_m\|^2 + 2\|x_n\|^2 - 4\left\|\tfrac{x_m+x_n}{2}\right\|^2 \end{aligned}$$

Now, because F is convex and (x_n) is a sequence in F, it follows that $\frac{x_m+x_n}{2}$ is an element of F, for every $m, n \in \mathbb{N}$. By definition of δ, we have

$$\left\|\frac{x_m + x_n}{2}\right\| \ge \delta \qquad \text{which implies} \qquad \left\|\frac{x_m + x_n}{2}\right\|^2 \ge \delta^2.$$

So, if we replace $\left\|\frac{x_m+x_n}{2}\right\|^2$ by δ^2 in the Parallelogram Law, we will be subtracting a smaller quantity on the right-hand side so that we get the inequality

$$\|x_m - x_n\|^2 \le 2\|x_m\|^2 + 2\|x_n\|^2 - 4\delta^2$$

Because $\|x_m\| \to \delta$ and $\|x_n\| \to \delta$, the right-hand side of this inequality goes to zero. Therefore, $\|x_m - x_n\| \to 0$, and (x_n) is a Cauchy sequence. Because $\mathcal{H}$ is complete and F is closed, x_n converges to a point x^* of F satisfying $\|x^*\| = \delta$. For uniqueness, suppose there is another point y^* of F such that $\|y^*\| = \delta$. Then, by the Parallelogram Law,

$$\begin{aligned} \|x^* - y^*\|^2 &= 2\|x^*\|^2 + 2\|y^*\|^2 - 4\left\|\frac{x_m + x_n}{2}\right\|^2 \\ &= 2\delta^2 + 2\delta^2 - 4\left\|\frac{x_m + x_n}{2}\right\|^2 \\ &\le 4\delta^2 - 4\delta^2 = 0 \end{aligned}$$

Thus, $x^* = y^*$ and the projection $P(x_0) = x^*$ is unique. $\square$

Definition. Let A be a subset of a Hilbert space $\mathcal{H}$. A point x in $\mathcal{H}$ is *orthogonal* to A, written $x \perp A$, if $\langle x, a\rangle = 0$ for every $a \in A$. The *orthogonal complement* of A in $\mathcal{H}$, denoted by $A^{\perp}$, is the set $\{x : x \perp A\}$.

Theorem 6.5. *Suppose that M is a closed linear subspace of a Hilbert space $\mathcal{H}$. Then, $\langle x - P(x), y\rangle = 0$, for each $x \in \mathcal{H}$ and each $y \in M$. Moreover, $P(x)$ is the unique point of M such that $x - P(x) \in M^{\perp}$.*

Proof. Let x be a point of $\mathcal{H}$, and let p denote the projection of x onto the closed, convex set M, that is, p is the unique point of M that is closer to x than any other point of M. Given a point $y \in M$ and a scalar λ, we have $\lambda y \in M$ and $p - \lambda y \in M$ because M is a linear subspace of $\mathcal{H}$. As a result,

$$\|x - (p - \lambda y)\|^2 \geq \|x - p\|^2 \tag{6.1}$$

By properties of the inner product norm, we have

$$\begin{aligned}
\|x - (p - \lambda y)\|^2 &= \langle (x - p) + \lambda y, (x - p) + \lambda y\rangle \\
&= \|x - p\|^2 + \langle \lambda y, x - p\rangle + \langle x - p, \lambda y\rangle + \|\lambda y\|^2 \\
&\leq \|x - p\|^2 + 2\Re e(\lambda\langle x - p, y\rangle) + |\lambda|^2\|y\|^2
\end{aligned}$$

Assuming that the underlying scalar field is $\mathbb{R}$ and combining the previous inequality with inequality (6.1) gives

$$\|x - p\|^2 + 2\lambda\langle x - p, y\rangle + |\lambda|^2\|y\|^2 \geq \|x - p\|^2$$

$$2\lambda\langle x - p, y\rangle + |\lambda|^2\|y\|^2 \geq 0$$

Consequently,

$$2\lambda\langle x - p, y\rangle \geq -|\lambda|^2\|y\|^2 \tag{6.2}$$

If $\lambda > 0$, then we can divide by λ without affecting inequality (6.2) to get

$$2\langle x - p, y\rangle \geq -\lambda\|y\|^2$$

As $\lambda \to 0^+$, we have $\lambda\|y\|^2 \to 0$ so that $\langle x - p, y\rangle \geq 0$. Now, assuming that $\lambda < 0$, we can divide both sides of inequality (6.2) by λ, but the inequality will be reversed:

$$2\langle x - p, y\rangle \leq -\lambda\|y\|^2$$

As $\lambda \to 0^-$, we have $\lambda\|y\|^2 \to 0$ so that $\langle x - p, y\rangle \leq 0$. Therefore, $\langle x - p, y\rangle = 0$. A similar argument can be used if the scalar field is the complex numbers. Because y was chosen arbitrarily, it follows that $x - p \in M^\perp$. Now, suppose that q is another point of M such that $x - q \in M^\perp$. For any $y \in M$, we have

$$\langle p - q, y\rangle = \langle p - x + x - q, y\rangle = \langle x - q, y\rangle - \langle x - p, y\rangle = 0$$

In particular, $p - q \in M$ because M is a linear subspace. So,

$$\|p - q\|^2 = \langle p - q, p - q\rangle = 0$$

Hence, $p - q = 0$ so that $p = q$. $\square$

Definition. A linear space X is a *direct sum* of linear subspaces A and B if and only if each element of X can be expressed uniquely as a sum of an element of A and an element of B.

Theorem 6.6. *If M is a closed linear subspace of a Hilbert space $\mathcal{H}$, then $\mathcal{H} = M \oplus M^\perp$.*

Proof. Given $x \in \mathcal{H}$, let p denote the projection of x onto the closed, convex set M. Write $x = p + (x - p)$. By the previous theorem, p is the unique point of M such that $x - p \in M^\perp$. Therefore, $p + (x - p)$ is a unique expression for x as a sum of an element of M and an element of $M^\perp$. $\square$

Theorem 6.7. *$A^\perp$ is a closed linear subspace of $\mathcal{H}$.*

Proof. First, the orthogonal complement $A^\perp$ is non-empty because $0 \in A^\perp$ since $\langle 0, a\rangle = 0$ for each $a \in A$. Furthermore, for any points x and y in $A^\perp$ and for any scalars λ and μ, we have

$$\langle \lambda x + \mu y, a\rangle = \lambda\langle x, a\rangle + \mu\langle y, a\rangle = \lambda \cdot 0 + \mu \cdot 0 = 0$$

Hence, $\lambda x + \mu y \in A^\perp$ so that $A^\perp$ is a linear subspace of $\mathcal{H}$.

Now, let (x_n) be a sequence in $A^\perp$ which converges to a point x in $\mathcal{H}$. Fix a point a in A. Given a point a of A, we have $\langle x_n, a\rangle \to \langle x, a\rangle$ by continuity of the inner product. Because $x_n \in A^\perp$, for each n, we have $\langle x_n, a\rangle = 0$. Therefore, by continuity of the inner product,

$$\langle x, a\rangle = \lim_{n\to\infty} \langle x_n, a\rangle = 0.$$

Hence, the limit x of the sequence (x_n) belongs to $A^\perp$ so that $A^\perp$ is closed. Thus the orthogonal complement of any subset of $\mathcal{H}$ is a closed linear subspace of $\mathcal{H}$. □

Observation. If $A \subset B$, then $A^\perp \supset B^\perp$.

Proof. Suppose that $A \subset B$, and let $x \in B^\perp$. Then, $\langle x, b\rangle = 0$, for all $b \in B$. Because $A \subset B$, it follows that $\langle x, a\rangle = 0$, for all $a \in A$. Hence, $B^\perp \subset A^\perp$. □

Theorem 6.8. *The orthogonal complement $A^\perp$ of A is identical with the orthogonal complement of the closure of the span of A.*

Proof. Let $\tilde{A}$ denote the closure of the span of A. Because $A \subset \mathrm{span}(A) \subset \tilde{A}$, we have $\tilde{A}^\perp \subset A^\perp$ by the previous observation. To get the other inclusion, let $x \in A^\perp$. Given $y \in \tilde{A}$, there is a sequence (y_n) of points in the span of A such that $y_n \to y$. For each n, we can express y_n as a finite linear combination

$$y_n = \sum_{i=1}^{k_n} \lambda_i^{(n)} a_i^{(n)} \qquad \text{where } \lambda_i^{(n)} \in \mathbb{F} \text{ and } a_i^{(n)} \in A$$

For each $n \in \mathbb{N}$, we have

$$\langle x, y_n \rangle = \left\langle x, \sum_{i=1}^{k_n} \lambda_i^{(n)} a_i^{(n)} \right\rangle = \sum_{i=1}^{k_n} \langle x, \lambda_i^{(n)} a_i^{(n)} \rangle = \sum_{i=1}^{k_n} \lambda_i^{(n)} \langle x, a_i^{(n)} \rangle = 0$$

where $\left\langle x, a_i^{(n)} \right\rangle = 0$, for all $i \in \{1, \ldots, k_n\}$, $n \in \mathbb{N}$, because $x \in A^{\perp}$. By continuity of the inner product, we have

$$\langle x, y \rangle = \lim_{n \to \infty} \langle x, y_n \rangle = 0$$

So, $\langle x, y \rangle = 0$, for all $y \in \tilde{A}$. Hence, $x \in \tilde{A}^{\perp}$. Therefore, $A^{\perp} = \tilde{A}^{\perp}$. $\square$

Theorem 6.9. *$A^{\perp\perp}$ is the closure of the span of A.*

Proof. Let $\tilde{A}$ denote the closure of the span of A. If $x \in \tilde{A}$, then there is a sequence (x_n) of points in the span of A such that $x_n \to x$. For each n, x_n can be expressed as a finite linear combination of elements of A:

$$x_n = \sum_{i=1}^{k_n} \lambda_i^{(n)} a_i^{(n)}$$

For any $y \in A^{\perp}$ and for fixed n, we have

$$\langle x_n, y \rangle = \left\langle \sum_{i=1}^{k_n} \lambda_i^{(n)} a_i^{(n)}, y \right\rangle = \sum_{i=1}^{k_n} \lambda_i^{(n)} \left\langle a_i^{(n)}, y \right\rangle = 0$$

By continuity of the inner product, we have $\langle x, y \rangle = \lim_{n \to \infty} \langle x_n, y \rangle = 0$. Since $\langle x, y \rangle = 0$ for any $y \in A^{\perp}$, it follows that $x \in A^{\perp\perp}$. Hence, $\tilde{A} \subset A^{\perp\perp}$. On the other hand, suppose that $x \in A^{\perp\perp}$. Because $\tilde{A}$ is a closed linear subspace of $\mathcal{H}$, we have $\mathcal{H} = \tilde{A} \oplus \tilde{A}^{\perp}$. So, we can write, $x = p + y$, where p is the unique projection of x onto $\tilde{A}$ and $y \in \tilde{A}^{\perp} = A^{\perp}$. Consequently, $\langle x, y \rangle = 0$ because $x \in A^{\perp\perp}$. Now, $p \in \tilde{A} \subset A^{\perp\perp}$ so that $\langle p, y \rangle = 0$. Therefore,

$$\|y\|^2 = \langle y, y \rangle = \langle x - p, y \rangle = \langle x, y \rangle - \langle p, y \rangle = 0$$

Thus, $y = 0$ from which it follows that $x = p \in \tilde{A}$. Hence $\tilde{A} = A^{\perp\perp}$. $\square$

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