SECURITY AND SECURE-DOMINATING SETS IN GRAPHS

by

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Abstract

All graphs in this paper are finite and simple. Let $G = (V, E)$ be a graph, $x \in V$ and $S \subseteq V$. Following convention, we let $N(x) := \{u : ux \in E\}$ and $N[x] := \{x\} \cup N(x)$. For $S \subseteq V$, $N(S) := \cup_{x \in S} N(x)$, and $N[S] = N(S) \cup S$. Security in graphs was first defined by Brigham, Dutton, and Hedetniemi in 2007 (BDH), [1]. Given a graph $G = (V, E)$ an *attack* on $S = \{s_1, s_2, ..., s_k\} \subseteq V$ is defined to be a collection of pairwise disjoint sets $A = \{A_1, A_2, ..., A_k\}$ for which $A_i \subseteq N[s_i] - S, 1 \leq i \leq k$. A *defense* of S is a collection of pairwise disjoint sets $D = \{D_1, D_2, ..., D_k\}$ such that $D_i \subseteq N[s_i] \cap S, 1 \leq i \leq k$. An attack A is *defendable* if there is a defense D such that$|D_i| \geq |A_i|$ for $1 \leq i \leq k$. We say that D *defends against* A, in this case. S is *secure* if there is a defense of S against every attack on S.

In Chapter 2 we give an efficient algorithm for finding a defense against an attack, when one exists. In Chapter 3 we give some fundamental results about secure-dominating sets in graphs, and determine the minimum order of a secure-dominating set in paths, cycles, and complete multipartite graphs. In Chapter 4 we give efficient tests for the security of S when $G[S]$, the subgraph of G induced by S, satisfies certain conditions.

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This is in loving memory of my father, Charles A. Jones.

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Chapter 1

Introduction

All graphs in this paper are finite and simple. Let $G = (V, E)$ be a graph, $x \in V$ and $S \subseteq V$. Following convention, we let $N(x) := \{u : ux \in E\}$ and $N[x] := \{x\} \cup N(x)$. For $S \subseteq V$, $N(S) := \cup_{x \in S} N(x)$, and $N[S] = N(S) \cup S$. The graph G may be indicated by the subscript, N_G, if this is not clear by the context.

Security in graphs was first defined by Brigham, Dutton, and Hedetniemi in 2007, [1]. Given a graph $G = (V, E)$ an *attack* on $S = \{s_1, s_2, ..., s_k\} \subseteq V$ is defined to be a collection of pairwise disjoint sets $A = \{A_1, A_2, ..., A_k\}$ for which $A_i \subseteq N[s_i] - S, 1 \leq i \leq k$. A *defense* of S is a collection of pairwise disjoint sets $D = \{D_1, D_2, ..., D_k\}$ such that $D_i \subseteq N[s_i] \cap S, 1 \leq i \leq k$. An attack A is *defendable* if there is a defense D such that $|D_i| \geq |A_i|$ for $1 \leq i \leq k$. We say that D *defends against* A, in this case. Thus each vertex in $N[S] - S$ can attack only one of its neighbors in S, and each vertex in S can defend itself or one of its neighbors in S. The set S is defined to be *secure* if for every attack A on S there exists a defense D of S which defends against A. The following theorem is from the 2007 paper Security in Graphs, by Brigham, Dutton, and Hedetniemi, [1].

Theorem 1.0.1. *(Brigham, Dutton, and Hedetniemi 2007; see also [5])- A set $S \subseteq V(G)$ is secure in G if and only if for any $X \subseteq S$, $|N[X] \cap S| \geq |N(X) - S|$.*

They also define the security number of a graph G as $s(G) := min\{|S| : S \text{ is a secure set in } G\}$. Some results and unsolved questions about the security of a set S in a graph can be found in [1], [2], and [6]. In the subsequent chapters we will continue to delve into the topic of security. In particular, we will, in Chapter 2, be examining how to algorithmically find a sufficient defense D of S, if one is possible, given an attack A. In Chapter 3, we will examine the relationship between ideas found in the fields of graph security and domination theory. The contents of Chapter 3 are largely the same as the contents of [6]. In Chapter 4, we will look at efficient tests for security of sets $S \subseteq V(G)$ when $G[S]$, the subgraph of G induced by S, satisfies certain conditions.

Chapter 2

Hall's Theorem and Security

In this chapter we consider cases where the set $S \subseteq G$ is secure. We will consider two versions of a therorem that was first provided by Pillip Hall in 1935, and determine how the appliciation of the idea of Hall's Theorem along with specific algorithms can be used to successfully find a defense D to a given attack A on the set S, if there exist such a D.

2.1 Definitions

Definition 2.0.2. *Given a graph $G = (V, E)$, a matching M in G is a set of pairwise non-adjacent edges, meaning no two edges in M share a common vertex.*

Definition 2.0.3. *A vertex v is matched by a matching M, or saturated by M, if it is an endpoint of an edge contained in the matching M. Otherwise the vertex v is unmatched or unsaturated by M.*

Definition 2.0.4. *A bipartite graph $G = (V, E)$ is a graph in which the vertex set V is partitioned into two disjoint subsets X and Y where every edge $e \in E$ has exactly one vertex in X and one in Y.*

Definition 2.0.5. *A maximal matching is a matching M in a graph G with the property that M is not properly contained in another matching in G.*

Definition 2.0.6. *A maximum matching in a graph is a matching in the graph such that no other matching in the graph has more edges than M.*

Definition 2.0.7. *The set $S \subseteq V(G)$ is secure in G if every attack on S is defendable.*

Definition 2.0.8. *The set $S \subseteq V(G)$ is $ultra-secure$ in G if there exists a single defense which defends against every attack on S.*

Definition 2.0.9. *The security number of a graph G is $s(G) := min\{|S| : S$ is a secure set in $G\}$.*

Definition 2.0.10. *A system of distinct representatives (SDR) for a sequence of (not necessarily distinct) sets $S_1, S_2, ..., S_m$ is a sequence of distinct elements $x_1, x_2, ..., x_m$ such that $x_i \in S_i$ for all $i = 1, 2, ..., m$.*

Theorem 2.0.11. *(Hall's Theorem for Bipartite Graphs)- A bipartite graph G with bipartition X,Y (with X finite) has a matching which saturates X iff $|N(A)| \geq |A|$ for all subsets A of X.*

Theorem 2.0.12. *(Hall's Theorem for an SDR, 1935, [3])-There exists a system of distinct representatives for a family of sets $\mathbb{S} := \{S_1, S_2, ..., S_m\}$ iff for every $J \subseteq \{1, ..., m\}$, $|\cup_{j \in J} S_j| \geq |J|$.*

The second of the two theorems just above is the original theorem of Phillip Hall [3] and Theorem 2.0.11 is an "equivalent" version that emerged, we are not sure exactly when. For the purposes if this chapter it will be useful to understand the connection between the two forms, because such an understanding will show how to form systems of distinct representatives by finding maximum matchings in bipartite graphs. We can then exploit an essential insight in the paper by Isaak, Johnson, and Petrie [5] to see that finding a defense against an attack, if there is one, can be achieved by finding a system of distinct representatives. Therefore, finding a defense against an attack (if there is one!) can be achieved by finding a maximum matching in a bipartite graph, and there are excellent and efficient algorithms for that task.

On the connection between the two forms of Hall's Theorem, first note that the "only if" statements are both quite clear. If a bipartite graph with bipartition X, Y has a matching M which saturates X, then for any $A \subseteq X$, $N(A)$ contains $\{v \in Y | uv \in M$ for some $u \in A\}$, a set of size $|A|$ because M is a matching which saturates $A \subseteq X$, and so $|N(A)| \geq |A|$. If

$S_1, ..., S_m$ are sets with a system of distinct representatives $x_1, ..., x_m$, and $J \subseteq \{1, ..., m\}$, then $\cup_{j \in J} S_j$ contains $\{x_j | j \in J\}$, a set of size $|J|$ because the x_j are distinct, and therefore

$$|\cup_{j \in J} S_j| \geq |J|.$$

Suppose the "if" assertion, i.e., the "backwards" implication, of Theorem 2.0.11 holds. We will show that the backwards implication of Theorem 2.0.12 holds. This connection is particularly important for the purpose of this chapter.

Suppose that $S_1, ..., S_m$ are sets, and for every $J \subseteq \{1, ..., m\}$, $|\cup_{j \in J} S_j| \geq |J|$. Make a bipartite graph G with bipartition

$$X = \{1, ..., m\}$$

$$Y = \cup_{i=1}^{m} S_i.$$

Let $x \in X$ and $y \in Y$ be adjacent in G if and only if $y \in S_x$.

Now, if $A \subseteq X$ then,

$$|N(A)| = |\cup_{x \in A} S_x| \geq |A|,$$

by the assumption above, with A replacing J. Therefore, by the assumption that Theorem 2.0.11 holds, there is a matching M in G which saturates X. For $x = 1, ..., m$ let $y_x \in Y$ be such that $xy_x \in M$. Then becuase M is a matching, $y_1, ..., y_m$ consitute a system of distinct representatives of $S_1, ..., S_m$. Thus Theorem 2.0.12 holds.

Now suppose that Theorem 2.0.12 holds. Suppose that G is a bipartite graph with bipartition X, Y, with $|X| < \infty$, and suppose that for all $A \subseteq X$, $|N(A)| \geq |A|$. Now take the view that X is a finite index set, and consider the indexed collection $[N(x); x \in X]$. Because, for any $A \subseteq X$,

$$|\cup_{x \in A} N(x)| = |N(A)| \geq |A|,$$

by assumption, we see that the condition in Theorem 2.0.12 holds, with X replacing $\{1, ..., m\}$ and the sets $N(x), x \in X$, replacing the $S_j, j \in \{1, ..., m\}$. (The role of the generic subset $J \subseteq \{1, ..., m\}$ is played by $A \subseteq X$.) Therefore, because Theorem 2.0.12 is assumed to be true, there is an indexed collection $[y(x); x \in X]$ such that for all $x \in X$, $y(x) \in N(x)$ and if $x, x' \in X$, $x \neq x'$ implies $y(x) \neq y(x')$. (In other words, the $y(x), x \in X$, are distinct.) Then $M = \{xy(x) | x \in X\}$ is a matching in G which saturates X. Thus the "if" assertion in Theorem 2.0.11 holds. Therefore, Theorems 2.0.11 and 2.0.12 are "equivalent."

The next result is a useful generalization of the original version of Hall's Theorem which replaces a "system of distinct representatives" with a "system of pairwise disjoint subsets" with prescribed cardinalities. Perhaps first noticed by Rado [6] and then, independently, by Halmos and Vaughan [3], the result, which we will sometimes refer to as the HRHV theorem, is an easy consequence of Hall's Theorem; on the other hand , Hall's Theorem in its original form, Theorem 2.0.12, is clearly the special case $k_i = 1, i = 1, ..., n$, of the HRHV theorem.

The proof of Theorem 2.0.13 will be given in detail, since it shows how to reduce the problem of finding pairwise disjoint subset representatives of given sets to finding a system of distinct representatives of another list of sets, and that is exactly what we need in order to algorithmize the contruction of defenses against attacks.

Theorem 2.0.13. *(HRHV, [4],[7])- Suppose $A_1, ..., A_n$ are sets, and $k_1, ..., k_n$ are non-negative integers. Then $\exists$ $B_1, ..., B_n$, pairwise disjoint, such that for $i = 1, ..., n$, $B_i \subseteq A_i$ and $|B_i| = k_i$, if and only if for each $\emptyset \neq J \subseteq \{1, ..., n\}$, $|\cup_{j \in J} A_j| \geq \sum_{j \in J} k_i$.*

Proof. $\Rightarrow$: Suppose that $B_i \subseteq A_i$, $|B_i| = k_i$ $i = 1, ..., n$, and $B_1, ..., B_n$ are pairwise disjoint. Suppose that $\emptyset \neq J \subseteq \{1, ..., n\}$. Then

$$|\cup_{j \in J} A_j| \geq |\cup_{j \in J} B_j| = \textstyle\sum_{j \in J} |B_j| \qquad \text{(because the } B_j \text{ are pairwise disjoint)}$$

$$= \textstyle\sum_{j \in J} k_j$$

$\Leftarrow$: (From Hall's Theorem)- Make a new list of sets, repeating each A_i, k_i times :

$$A_1, ..., A_1,\ k_1 \text{ times}$$

$$A_2, ..., A_2,\ k_2 \text{ times}$$

$$\vdots$$

$$\vdots$$

$$A_n, ..., A_n,\ k_n \text{ times}$$

Assume that for all $\emptyset \neq J \subseteq \{1, ..., n\}$, $|\cup_{j\in J} A_j| \geq \sum_{j\in J} k_j$. By Hall's Theorem, there is a system of distinct representatives of the sets in this list if, for each selection of sets from the list, or array, the number of elements in the union of those sets is at least the number of selections made.

Take a sub array and let $J = \{j \in \{1, ..., n\}|$ at least one A_j has been selected$\}$. Then the cardinality of the union of the selected sets is equal to $|\cup_{j\in J} A_j| \geq \sum_{j\in J} k_j \geq$ number of places in the sub array. We conclude by Hall's Theorem that this list admits a system of distinct representatives.

Collect the k_1 representatives of A_1 together into a set $B_1 \subseteq A_1$, $|B_1| = k_1$, collect the k_2 representatives of A_2 into B_2, etc.

$B_1, ..., B_n$ are pairwise disjoint sets satisfying $B_j \subseteq A_j$, $|B_j| = k_j, j = 1, ..., n$. □

Given finite sets $A_1, ..., A_n$ and non-negative integers $k_1, ..., k_n$ to algorithmically find $B_1, ..., B_n$ satisfying the conclusion of Theorem 2.0.13, if such B_i exist, we make a bipartite graph, as in Figure 2.1.

A matching in this graph which saturates the left hand side will determine $B_1, ..., B_n$. Such a matching will be a maximum matching in the graph, and if such a matching exists, then every maximum matching will saturate the left hand side of the graph.

There are at least two good algorithms for finding a maximum matching in a bipartite graph. We describe the simplest, the augmenting path algorithm, which can be used to find maximum matchings in any graph, not only bipartite graphs, in Appendix A.

Now we are ready to decribe an algorithm for finding a defense, when there is one, aganist an attack on a set S of vertices in a graph G. Suppose $S = \{s_1, ..., s_m\}$, and suppose that s_i is being attacked by a_i attackers from $N(s_i)\backslash S, i = 1, ..., m$. A defense against this attack would consist of a list of pairwise disjoint sets $D_1, ..., D_m$

satisfying:

(i) $D_i \subseteq N[s_i] \cap S, i = 1, ..., m$, and

(ii) $|D_i| \geq a_i$

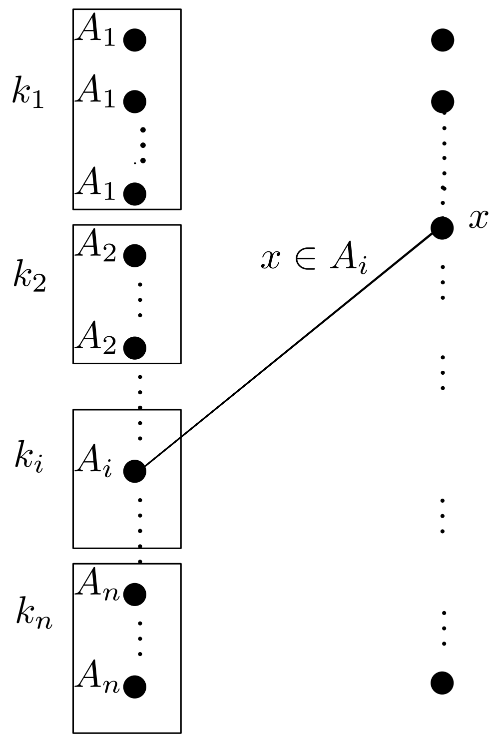

Figure 2.1: Note that the RHS is the $\cup_{i=1}^{n} A_i$

Each D_i is the set of vertices to be assigned to defend s_i, whence (i) and (ii). The D_i must be pairwise disjoint because no vertex in S is allowed to defend more than one vertex in S.

Clearly, If we can find pairwise disjoint $D_1, ..., D_m$ satisfying (i) and (ii) then we can find such D_i satisfying (ii) with equality. So now we are in the situation dealt with in the HRHV theorem, with the a_i here playing the roles of the k_i there and the sets $N[s_i] \cap S$

playing the roles of the A_i, and with the sought for disjoint subset representatives denoted $D_1, ..., D_m$ rather than $B_1, ..., B_n$.

Now, by the thread of thought that runs from the proof of the HRHV theorem back through the original (SDR) form of Hall's Theorem to the "bipartite graph" equivalent theorem, we see that we need not pause over the question of whether or not $D_1, ..., D_m$ can be found. We can simply set out to find $D_1, ..., D_m$ by finding a maximum matching in a certain bipartite graph; if a maximum matching in that bipartite graph saturates one side of the bipartition, then we can read off the D_i from the matching. Otherwise, there is no defense against the attack.

We form a bipartite graph as follows. Let $U_1, ..., U_t$, $t = \sum_{i=1}^{m} a_i$, be a list of sets obtained by repeating each set $N[s_i] \cap S$ a_i times, $i = 1, ..., m$. Let one side of the bipartition be $X = \{u_1, ..., u_t\}$. The u_j are meant to *represent*, or *correspond to*, the U_j. Let the other side be S, but let's call it Y, for the moment. Now we declare that $u_j \in X$ and $y \in Y$ are adjacent in this bipartite graph if and only if $y \in U_j$.

There is a defense against the attack if and only if $D_1, ..., D_m$ (pairwise disjoint, $D_j \subseteq N[s_j] \cap S$, and $|D_j| = a_j, j = 1, ..., m$) exist, and the D_j exist if and only if there is a matching in this bipartite graph which saturates X; finally, a matching in the bipartite graph can have no more than $|X|$ edges, and that many only if it saturates X. So the way is clear: run the algorithm, let it find a maximum matching, and if that maximum matching does not saturate X, then all is lost. There is no defense against this attack. Otherwise, if X is saturated, then we can form the D_j using the matching as described in the proof of the HRHV theorem.

Example: Let G be the graph in Figure 2.2, with $S = \{s_1, ..., s_{11}\} \subseteq V(G)$ under attack, as indicated. (S is, in fact, secure in G, but we do not need to know that to find a defense against this attack.) By inspection, the numbers $a_1, ..., a_{11}$ of attackers of $s_1, ..., s_{11}$, repectively, are $a_1 = 1 = a_2 = a_3 = a_4$, $a_5 = 0$, $a_6 = 2$, $a_7 = 1$, $a_8 = 0$, $a_9 = 2$, $a_{10} = 1$, $a_{11} = 0$. We form a bipartite graph as prescribed previously, with this convention: when

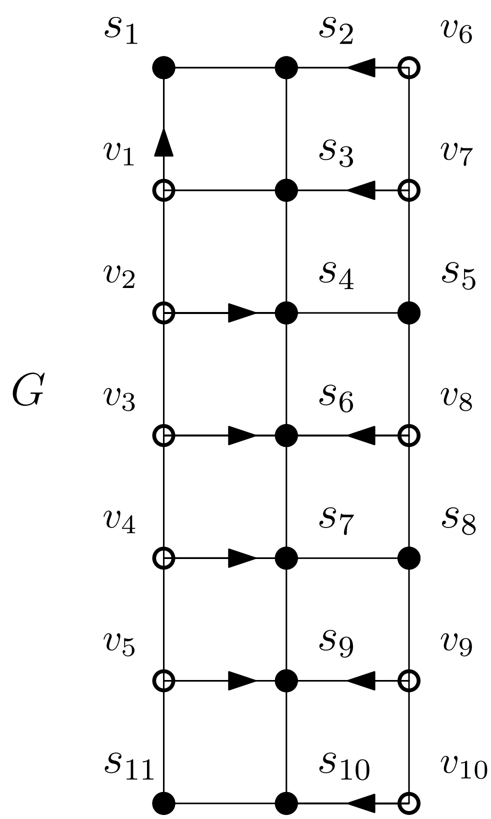

Figure 2.2: $G = P_7 \Box P_3$

$a_i = 1$, the vertex on the left side associated with $N[s_i] \cap S$ will be u_i, and when $a_i = 2$, the two vertices associated with $N[s_i] \cap S$ will be u_{i1} and u_{i2} This graph appears in Figure 2.3.

To find a defense against this attack, we first find a maximal matching in the graph in Figure 2.3, which happens to be not maximum, and proceed from there to a maximum matching, using the augmenting path algorithm. In Appendix B we perform this process on Figure 2.3 resulting in a saturation of the vertex set of left-hand side the graph, and; therefore, producing a maximum matching in the graph. The successful defense is given in Appendix B.

An Open Problem

What if there is no successful defense against an attack? In that case, we might wish to make the best of a bad situation by finding a defense which maximizes the number of *survivors*, elements of S that have at least as many defenders as attackers.

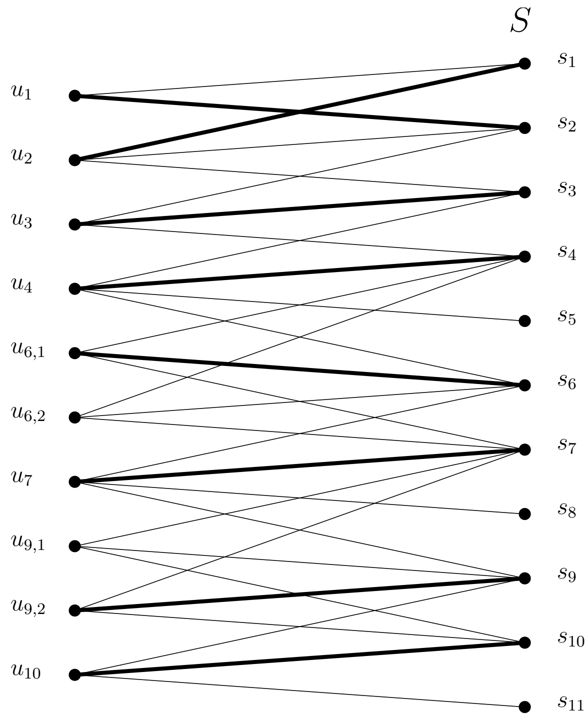

Figure 2.3: The bipartite graph produced from G, with a maximal matching chosen greedily.

More generally, the elements of S may be assigned positive weights beforehand, indicating the importance of their survival to the S tribe, and we would then wish to find a defense which maximizes the sum of the weights of the survivors. Simply maximizing the number of survivors corresponds to the weighted problem with all assigned weights the same.

We can solve such a problem, given an attack on S, by forming a bipartite graph as above and comparing all possible defenses corresponding to all possible maximum matchings in that graph. But that is not efficient. The sun may go nova before we are done and the attack will have happened.

Chapter 3

Secure-dominating sets In Graphs

3.1 Definitions

Definition 3.0.14. *In a graph G, a set $S \subseteq V(G)$ is a dominating set in G if every vertex in G not in S has a neighbor in S.*

Definition 3.0.15. *The domination number of G is $\gamma(G) := min\{|S| : \{S$ is a dominating set of $G\}$*

Definition 3.0.16. *A $secure-dominating$ set in G is a set $S \subseteq V(G)$ that is both a secure set in G and also a dominating set in G.*

Definition 3.0.17. *The $secure - domination$ $number$ of G is $\gamma_s(G) := min\{|S| : \{S$ is a secure-dominating set in $G\}$.*

Definition 3.0.18. *A connected $secure - dominating$ set in a connected graph G is a secure-dominating set $S \subseteq V(G)$ which induces a connected subgraph of G.*

Definition 3.0.19. *The connected $secure - domination$ $number$ of a connected graph G is $\gamma_s^c(G) := min\{|S| : \{S$ is a connected secure-dominating set of $G\}$.*

Theorem 3.0.20. *(Brigham, Dutton, and Hedetniemi 2007; see also [5])- A set $S \subseteq V(G)$ is secure in G if and only if for any $X \subseteq S$, $|N[X] \cap S| \geq |N(X) - S|$.*

Notation: The $n - cube$, Q_n, is the simple graph whose vertices are the n-tuples with entries in $\{0, 1\}$ and whose edges are the pairs of n-tuples that differ in exactly one position. The path and cycle on n vertices will be denoted P_n and C_n, respectively. These are standard notations. For other standard notations (K_n, $K_{m,n}$, etc) see almost any graph theory text;

for instance, *Introduction to Graph Theory* by Douglas West. As in that text, we will let $\Box$ denote the Cartesian product.

Proposition 3.0.21. *For any G, $\gamma_s(G) \geq \frac{|V(G)|}{2}$.*

Proof. Suppose that $X \subseteq S$ is a secure-dominating set in G, and let $n = |V(G)|$. Let $X = S$ in Theorem 3.0.20:

$$\begin{aligned} |S| \geq |N(S) - S| &= |V(G) - S| \qquad \text{(because } S \text{ is dominating)} \\ &= |V(G)| - |S| \\ &\Rightarrow 2\,|S| \geq n \\ &\Rightarrow |S| \geq \tfrac{n}{2} \end{aligned}$$

$\Box$

Corollary 3.0.22. *Suppose that $S \subseteq V(G)$, S is dominating in G, each $v \in S$ is adjacent to at most one vertex in $V(G) \backslash S$, and $|S| = \lceil \frac{|V(G)|}{2} \rceil$. Then S is a secure-dominating set in G, and $\gamma_s(G) = \lceil \frac{|V(G)|}{2} \rceil$.*

Proof. In view of Proposition 3.0.21, it suffices to show that S is secure. In fact, S is ultra-secure: let each vertex of S defend itself. This defense will defend against any attack. $\Box$

Corollary 3.0.23. *For $n \geq 2$, $\gamma_s(Q_n) = \gamma_s^c(Q_n) = 2^{n-1}$.*

Proof. The claim clearly holds for $n = 2$. For $n > 2$, $Q_n = Q_{n-1} \Box Q_2 = Q_{n-1} \Box K_2$, a graph formed by joining two copies of Q_{n-1} by a (perfect) matching. Letting S be the vertex set of either copy of Q_{n-1}, the conclusion follows from Corollary 3.0.22, and the additional observation that Q_{n-1} is connected. $\Box$

Corollary 3.0.24. *If G is the Petersen graph then $\gamma_s(G) = \gamma_s^c(G) = 5$.*

Proof. Let S be the vertex set of one of the obvious 5-cycles in the usual drawing of the Petersen graph. $\Box$

Proposition 3.0.25. $\gamma_s(K_{m,n}) = \gamma_s^c(K_{m,n}) = \lceil \frac{m+n}{2} \rceil$.

Proof. Let $K_{m,n}$ have bipartition A and B with $|A| = |m|, |B| = |n|$. We may as well suppose that $m \leq n$. Since, by Proposition 3.0.21, $\gamma_s(K_{m,n}) \geq \lceil \frac{m+n}{2} \rceil$, to prove the claim of this proposition it suffices to find $S \subseteq A \cup B$, secure and dominating, with $|S| = \lceil \frac{m+n}{2} \rceil$. The main idea is to form S by taking about half of the vertices of A, and about half of the vertices of B.

Cases:

(1) If m and n are both even, take $\frac{m}{2}$ vertices of A and $\frac{n}{2}$ vertices of B for S.

(2) If m is even and n is odd, take $\frac{m}{2}$ vertices of A and $\lceil \frac{n}{2} \rceil$ vertices of B for S.

(3) If m is odd and n is even, take $\lceil \frac{m}{2} \rceil$ vertices of A and $\frac{n}{2}$ vertices of B for S.

(4) If m and n are odd, take $\lceil \frac{m}{2} \rceil$ vertices of A and $\lfloor \frac{n}{2} \rfloor$ vertices of B for S.

In every case, S is dominating and connected. We verify that S is secure by applying Theorem 3.0.20. Let $S_1 = S \cap A$, $S_2 = S \cap B$. Suppose that $\emptyset \neq X \subseteq S$ and let $X_1 = X \cap A$, $X_2 = X \cap B$. If both X_1 and X_2 are non-empty then $N(X) - S = (A \cup B) \backslash S$ so,

$$|N(X) - S| = \lfloor \tfrac{m+n}{2} \rfloor, \text{ while}$$
$$|N[X] \cap S| = |S| = \lceil \tfrac{m+n}{2} \rceil \geq |N(X) - S|.$$

If $X = X_1$ then $|N(X) - S| = |B - S_2|$, while $|N[X] \cap S| = |X| + |S_2|$. Since $|S_2| \geq |B - S_2| - 1$ in every case, and $X = X_1 \neq \emptyset$, it follows that $|N[X] \cap S| \geq |N(X) - S|$. If $X = X_2$, the conclusion $|N[X] \cap S| \geq |N(X) \backslash S|$ is achieved similarly.

$\square$

Proposition 3.0.26. *Suppose that $r \geq 3$ and that $n_1, ..., n_r$ are all positive integers. Then* $\gamma_s(K_{n_1,...,n_r}) = \gamma_s^c(K_{n_1,...,n_r}) = \lceil \frac{n_1+...+n_r}{2} \rceil$

Proof. The proof will be on the model of the proof of Proposition 3.0.25. It suffices to exhibit a connected secure-dominating set $S \subseteq V(G)$, $G = K_{n_1,\ldots,n_r}$ with $|S| = \lceil \frac{n}{2} \rceil$, where $n = n_1 + \ldots + n_r$. It will be straightforward to present such a set; showing that it is secure will use Theorem 3.0.20. It will be obvious that it is connected and dominating.

Let $V_1, \ldots, V_r$ be the parts of G, with $|V_i| = n_i$, $i = 1, \ldots, r$. Let $I := \{i \in \{1, \ldots, r\} : \{n_i$ is odd$\}$ and $J = \{1, \ldots, r\} \backslash I$. Form S as follows; For $j \in J$, put $\frac{n_j}{2}$ vertices of V_j in S; for $\lceil \frac{|I|}{2} \rceil$ values of $i \in I$ for which n_i is smallest, put $\lceil \frac{n_i}{2} \rceil$ vertices of V_i in S, and for the other values of $i \in I$, put $\lfloor \frac{n_i}{2} \rfloor$ vertices of V_i in S. Then $|S| = \lceil \frac{n}{2} \rceil$, and S is clearly connected and dominating.

Let $S_i = S \cap V_i$, $i = 1, \ldots, r$. Suppose $\emptyset \neq X \subseteq S$. Let $X_i = X \cap S_i$, $i = 1, \ldots, r$. Let $U := \{i \in \{1, \ldots, r\} : X_i = \emptyset\}$. If

$$\begin{aligned} r - |U| &\geq 2 \text{ then} \\ |N[X] \cap S| &= |S| \\ &= \lceil \tfrac{n}{2} \rceil \geq \lfloor \tfrac{n}{2} \rfloor \\ &= |N(X) \backslash S| \,. \end{aligned}$$

(Noting that $N(X) = N[X] = V(G)$, in this case.)

Otherwise, $X_i = \emptyset$ except for one value of i, call it j. Then $N(X) = V(G) \backslash V_j$, $N[X] = (V(G) \backslash V_j) \cup X_j$. Therefore,

$$|N[X] \cap S| = |(S \backslash S_j) \cup X_j| = \sum_{i \neq j} |S_i| + |X_j| \geq \sum_{i \neq j} |V_i \backslash S_i| = |N(X) \backslash S|;$$

the last inequality holds because for each $j \in \{1, \ldots, r\}$, $\sum_{i \neq j} |S_i| \geq \sum_{i \neq j} |V_i \backslash S_i| - 1$, and $|X_j| \geq 1$. $\square$

Proposition 3.0.27. *(a) If $n \geq 1$, $\gamma_s(P_n) = \lceil \frac{n}{2} \rceil$; for $n \leq 5$, $\gamma_s^c(P_n) = \gamma_s(P_n)$, and for $n \geq 6$, $\gamma_s^c(P_n) = n - 2$.*

(b) If $n \geq 3$, $\gamma_s(C_n) = \lceil \frac{n}{2} \rceil$ unless $n \equiv 2$ (mod 4), in which case $\gamma_s(C_n) = \frac{n}{2} + 1$; if $3 \leq n \leq 6$, $\gamma_s^c(C_n) = \gamma_s(C_n)$, and if $n > 6$, $\gamma_s^c(C_n) = n - 2$.

Proof. If a set of vertices on either a path or a cycle contains a vertex with two neighbors, neither of which are in the set, then the set is not secure. Conversely, if a set of vertices on either a path or a cycle induces a union of subpaths of lengths ≥ 1, or a union of such subpaths together with one or both of the end vertices, in the case of a path, then that set is secure. Finding such a set which is also dominating, with $\lceil \frac{n}{2} \rceil$ vertices, in either P_n or C_n, except in the case of C_n when $n \equiv 2 \pmod 4$, is an easy exercise; we illustrate by indicating a minimum secure-dominating set in P_6.

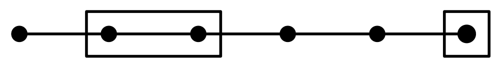

Figure 3.1: Example of a minimum secure-dominating set in P_6.

In the case of C_n, $n = 6, 10, 14, \ldots$, since $\frac{n}{2}$ is odd, and a minimum secure-dominating set in C_n must induce a union of paths of lengths ≥ 1, with at most two vertices between two successive subpaths, you just cannot find such a set with $\frac{n}{2}$ vertices, but you can find such a set with $\frac{n}{2} + 1$ vertices. We leave the verification to the reader. The claims about γ_s^c are obvious. $\square$

Proposition 3.0.28. *If G and H are finite simple graphs, then $\gamma_s(G \square H) \leq \gamma_s(G) \cdot |V(H)|$.*

Proof. Let $S \subseteq V(G)$ be a secure-dominating set in G such that $|S| = \gamma_s(G)$, and consider the set $S \times V(H) \subseteq V(G \square H)$. Obviously $S \times V(H)$ is dominating in $G \square H$, so if it is secure in $G \square H$, then

$$\gamma_s(G \square H) \leq |S \times V(H)| = \gamma_s(G)\,|V(H)|.$$

Consider an attack in $G \square H$ on $S \times V(H)$. A vertex $(u, w) \in S \times V(H)$ has, for neighbors outside of $S \times V(H)$, only vertices (v, w), $v \in V(G) \backslash S$. Consequently, we can think of this attack as consisting of $|V(H)|$ possibly different attacks on S in G, one for each $w \in V(H)$. For each of these attacks we can defend S with a defense $D(w) = [D_u(w); u \in S]$; taking the "union" of these defenses, by letting (u, w), $u \in S$, $w \in V(H)$, be defended by $D_u(w) \times \{w\}$, gives a defense of $S \times V(H)$ against the attack. $\square$

Corollary 3.0.29. *If $\gamma_s(G) = \frac{|V(G)|}{2}$, then for every finite simple H, $\gamma_s(G \square H) = \frac{|V(G \square H)|}{2}$.*

So, in the cases of P_{2k}, K_{2k}, and C_{4k}, the secure-domination number of their Cartesian products with any other finite simple graph is as small as possible, in accord with Proposition 3.0.21. We conjecture that for odd m and n,

$$\gamma_s(P_m \square P_n) = \lceil \tfrac{mn}{2} \rceil, \text{ and even that for odd } m, n \geq 3, \ \gamma_s^c(P_m \square P_n) = \lceil \tfrac{mn}{2} \rceil,$$

but we are not ready to prove this, as yet. And the connected secure domination numbers of the other Cartesian products among the paths, cycles and complete graphs of odd order (or of order $\equiv 2 \pmod 4$, for the cycles) we leave as open questions.

Chapter 4

Tests For the Security of S When $G[S]$ Satisfies Certain Conditions

In this Chapter we will find efficient tests for security of sets $S \subseteq V(G)$ when $G[S]$, the subgraph of G induced by S, is the complement of a forest. The theorem by Brigham, Dutton, and Hedetniemi in 2007 (BDH), [1], will assist us in this endeavor, and help us to provide some interesting results for testing for security of sets S.

We restate Theorem BDH for the reader's convenience.

Theorem 4.0.30. *(Brigham, Dutton, and Hedetniemi 2007; see also [5])- A set $S \subseteq V(G)$ is secure in G if and only if for any $X \subseteq S$, $|N[X] \cap S| \geq |N(X) - S|$.*

The straightforward test for security in G of a set $S \subseteq V(G)$ given by Theorem BDH potentially requires the performance of $2^{|S|} - 1$ tasks, one task for each non-empty subset X of S: count the vertices in $N[X] \cap S$ and count the vertices in $N(X) - S$, and compare the counts. Each task is easy to perform if G is described by an adjacency matrix A, form $B = A + I$, where I is the $V(G) \times V(G)$ identity matrix. Given $X \subseteq S \subseteq V(G)$, to count $N[X] \cap S$, count the columns of B indexed by elements of S which have a 1 in at least on row indexed by an element of X, and to count $N(X) - S$, count the columns of B indexed by $V(G) - S$ which have at least one 1 in a row indexed by an element of X.

4.1 Definitions

Definition 4.0.31. *A tree is a simple graph G that satisfies any of the following equivalent conditions:*

(i) G is connected and has no cycles.

(ii) G has no cycles, and a simple cycle is formed if any new edge is added to G.

(iii) G is connected, but is not connected if any single edge is removed from G.

(iv) Any two vertices in G are connected in G by a unique simple path.

Definition 4.0.32. *The degree in G of a vertex v of a graph G is the number of edges of G which touch v; the degree of v is denoted $deg(v)$ or $d_G(v)$.*

Definition 4.0.33. *A $forest$ is a disjoint union of trees.*

Definition 4.0.34. *A $leaf$ in a graph is a vertex of degree 1.*

Definition 4.0.35. *A clique in a simple graph $G = (V, E)$ is a complete subgraph of G, a subgraph in which any two distinct vertices are adjacent.*

Definition 4.0.36. *The maximum degree of a graph G is the largest vertex degree of G, denoted $\triangle(G)$.*

Proposition 4.0.37. *Suppose that $X \subseteq S \subseteq V(G)$ and that*

(i) $|S| \geq |N(S) - S|$, and

(ii) X is dominating in G[S].

Then $|N[X] \cap S| \geq |N(X) - S|$.

Proof. Let $H = G[S]$. Then $N[X] \cap S = N_H[X] = |S|$, since X is dominating in H. Therefore,

$$|N[X] \cap S| = |S| \geq |N(S) - S| \geq |N(X) - S|.$$

$\square$

Corollary 4.0.38. *Suppose that $\emptyset \neq S \subseteq V(G)$. Then testing S for security in G requires inspecting, at most, the neighbor sets of S itself and the non-empty subsets of S which are not dominating in G[S].*

Corollary 4.0.39. *Suppose that $\emptyset \neq S \subseteq V(G)$, and a is a non-negative integer, $a < |S|$, such that all sets $X \subseteq S$ satisfying $|X| > a$ are dominating in $G[S]$. Then testing S for security in G requires inspecting, at most, $1 + \sum_{k=1}^{a} \binom{|S|}{k}$ subsets of S, namely, S itself and the non-empty subsets of S of cardinality $\leq a$.*

Corollary 4.0.40. *Suppose that $\emptyset \neq S \subseteq V(G)$. Then testing S for security in G requires inspecting, at most, $1+\sum_{k=1}^{\triangle(\overline{G[S]})} \binom{|S|}{k}$ subsets of S, namely, S itself and the non-empty subsets of S of cardinality $\leq \triangle(\overline{G[S]})$.*

Proof. Let $H = G[S]$ and $a = \triangle(\overline{H})$. By the preceding corollary, it suffices to show that any $X \subseteq S$ satisfying $|X| > a$ is dominating in H. If $v \in V(H) = S$ is not in X and is not adjacent to any $x \in X$, then $d_{\overline{H}}(v) \geq |X| > a = \triangle(\overline{H})$, which is impossible. Therefore, any $X \subseteq S$ satisfying $|X| > a$ is dominating in $H = G[S]$. □

For instances: If $S \subseteq V(G)$ and $G[S]$ is a matching plus isolates, then the last corollary says that the security of S can be checked by verifying the BDH inequality for $|S|+1$ subsets of S, S itself and its singleton subsets. And if $\overline{G(S)}$ is a disjoint union of paths, not all single edges or isolates, then $\triangle(\overline{G(S)}) = 2$ and the security of S in G can be verified (or falsified) by checking the BDH inequality for no more than $\binom{|S|}{2} + |S| + 1$ subsets of S.

But in these two cases, even though the number of tasks we may have to perform to certify or disqualify S as a secure set in G is polynomial in $|S|$, it still may be much more than we have to do. In the case where $\overline{G[S]}$ is a matching plus isolates, any vertex in S not saturated by the matching is dominating in $G[S]$, and so will not have to be checked for satisfaction of the BDH inequality (assuming that S itself has already passed the BDH test, $|S| \geq |N(S) - S|$). Similarly, when $\overline{G[S]}$ is a disjoint union of paths, inculding paths of length 0, those isolates in $\overline{G[S]}$ will be dominating in G(S), and so will numerous doubleton subsets of S.

Proposition 4.0.41. *Suppose that H is a finite simple graph, $u, v \in V(H), u \neq v$, and either* *(i)* $uv \in E(\overline{H})$ *and* $N_{\overline{H}}(u) \cap N_{\overline{H}}(v) = \emptyset$*, or*

(ii) $dist_{\overline{H}}(u, v) > 2$*.*

Then $\{u, v\}$ *is dominating in* H*.*

Proof. In case (i), since no $w \in V(\overline{H})\backslash\{u, v\}$ is adjacent to both u and v in $\overline{H}$, then each such w must be adjacent to at least one of u and v in H. In case (ii), $dist_{\overline{H}}(u, v) > 2$ again implies that $N_{\overline{H}}(u) \cap N_{\overline{H}}(v) = \emptyset$, and the conclusion that $N_H[\{u, v\}] = V(H)$ follows as in the first case. □

Theorem 4.0.42. *Suppose that* $S \subseteq V(G)$*, and that* $\overline{G[S]}$ *is a forest. Then the security of* S *in* G *is implied by the truth of the BDH inequality* $|N[X] \cap S| \geq |N[X] - S|$ *for*

(1) $X = S$*;*

(2) $X = \{v\}$ *for all* $v \in S$ *which is not isolated in* $\overline{G[S]}$*; and*

(3) $X = N_{\overline{H}}(v)$ *for all* $v \in S$ *which is not isolated in* $\overline{G[S]}$*.*

Proof. Let $H = G[S]$. Since $\overline{H}$ is a forest, $\overline{H}$ is triangle free. Therefore, by Propositioin 4.0.41, if $u, v \in V(H) = S$ are adjacent in $\overline{H}$, or at distance > 2 from each other in $\overline{H}$ (note: if u and v lie in different components of $\overline{H}$ then $dist_{\overline{H}}(u, v) = \infty > 2$), then $\{u, v\}$ is dominating in H. Also, if v is isolated in $\overline{H}$ then v is dominating in H. Therefore, by Proposition 4.0.37, if $X = S$ satisfies the BDH inequality $|N[S] \cap S| = |S| \geq |N(S) - S|$ (where $N = N_G$), then the only non-empty sets $X \subseteq S$ that might possibly fail to satisfy the BDH inequality $|N[X] \cap S| \geq |N(X) - S|$ are either singleton sets v where v is not isolated in $\overline{H}$ or sets $X \subseteq S$ such that $|X| \geq 2$ and each pair of distinct vertices in X are at distance 2 from each other in $\overline{H}$. Let us call these latter subsets X *distance* 2 *simplices* in $\overline{H}$.

Suppose that $X \subseteq N_{\overline{H}}(v)$ for some $v \in S$ and $|X| \geq 2$. For $x, y \in X, x \neq y, x$ cannot be adjacent in $\overline{H}$ to y, nor to any other neighbor of y then v, because $\overline{H}$ contains no cycles. Therefore, in H, x is adjacent to every vertex of S that y is not adjacent to , and to y itself, and the same holds with the roles of x and y reversed. Thus

$$S - \{v\} = N_H[\{x, y\}] \subseteq N_H[X] \subseteq N_H[N_{\overline{H}}(v)] \subseteq S - \{v\}.$$

Consequently, if $Y = N_{\overline{H}}(v)$ satisfies the BDH inequality,

$$|N_H[Y]| = |N_G[Y] \cap S| \geq |N_G(Y) - S|, \text{ then}$$
$$|S| - 1 = |S - v| = |N_H[X]| = |N_G[X] \cap S| = |N_H[Y]| \geq |N_G(Y) - S| \geq |N_G(X) - S|.$$

Therefore, if $Y = N_{\overline{H}}(v)$ satisfies the BDH inequality for every $v \in S$ which is not isolated in $\overline{H}$, then every $X \subseteq N_{\overline{H}}(v)$ for some $v \in S$ such that $|X| \geq 2$ will satisfy the BDH inequality. Consequently, we can finish the proof by showing that each distance two simplex in $\overline{H}$ is a subset of $N_{\overline{H}}(v)$ for some $v \in S$.

Suppose that X is a distance 2 simplex in $\overline{H}$. If $x, y \in X, x \neq y$, then $dist_{\overline{H}}(x, y) = 2$ implies that x and y have a common neighbor in $\overline{H}$, say v. Now, x and y can have no other common neighbor than v in $\overline{H}$, because $\overline{H}$ contains no cycle. If $z \in X \backslash \{x, y\}$, then x and z have a unique common neighbor in $\overline{H}$, say u, and y and z have a unique common neighbor in $\overline{H}$ say w. If u, v, w are distinct then $\overline{H}$ contains a 6-cycle , and if two of them are equal but not equal to the third, then $\overline{H}$ contains a 4-cycle. Since $\overline{H}$ contains no cycle, $u = v = w$.

Then the common neighbor in $\overline{H}, v$, of x and y is also the common neighbor in $\overline{H}$ of x and any other vertex in X. Consequently, $X \subseteq N_{\overline{H}}(v)$. □

So, if $S \subseteq V(G)$ and $\overline{G[S]}$ is a forest, the number of sets $X \subseteq S$ to check for satisfaction of the BDH inequality, to verify the security of S, is no greater than $2|S| + 1$, considerably smaller than $2^{|S|} - 1$. This is not surprising, since $G[S]$ must be a very edge-dense graph to have such a sparse complement, but it is still nice to have this reduction in complexity, and to know exactly which sets $X \subseteq S$ to check.

We owe this simplification of the security check, as well as Corollaries 4.0.38, 4.0.39, and 4.0.40, to Proposition 4.0.37. In every case, this proposition is telling us to check the inequality

$$|S| \geq |N(S) - S|$$

first; if it does not hold then S is not secure, and if it does hold, then we need to check the inequalities $|N[X] \cap S| \geq |N(X) - S|$ only for $\emptyset \neq X \subseteq S$ which are not dominating in $G[S]$, i.e. for which

$$|N[X] \cap S| = \left|N_{G[S]}[X]\right| < |S|.$$

If S passes its test with strict inequality –passing with honors!– then we can go further.

Proposition 4.0.43. *Suppose that $S \subseteq V(G)$ and $|S| = |N(S) - S| + a, a > 0$. Then every $X \subseteq S$ such that $|N[X] \cap S| = \left|N_{G[S]}(X)\right| \geq |S| - a$ satisfies $|N[X] \cap S| \geq |N(X) - S|$.*

Proof. Supposing that $X \subseteq S$ and $|N[X] \cap S| \geq |S| - a$, we have

$$|N[X] \cap S| \geq |S| - a = |N(S) - S| + a - a = |N(S) - S| \geq |N(X) - S|.$$

□

Corollary 4.0.44. *If $S \subseteq V(G)$, $|S| > |N(S) - S|$ and $\overline{G[S]}$ is a forest, then S is secure in G if and only if*

$$|N[v] \cap S| \geq |N(v) - S| \textit{ for all } v \in S$$

such that v is not isolated in $\overline{G[S]}$.

Proof. The "only if" claim is part of Theorem BDH. Suppose that $|N[v] \cap S| \geq |N(v) - S|$ for each $v \in S$ which is not isolated in $\overline{G[S]}$. By Theorem 4.0.42, we can conclude that S is secure in G if $|N[X] \cap S| \geq |N(X) - S|$ for every X which is the open neighborhood in $\overline{G[S]}$ of a single vertex of S. But, by the proof of Theorem 4.0.42, for every such X, if $|X| \geq 2$ then $N[X] \cap S = N_{G[S]}[X] = S \backslash \{v\}$, where v is the vertex with open neighborhood X in $\overline{G[S]}$, whence

$$|N[X] \cap S| = |S| - 1 \geq |S| - a, \text{ where } a = |S| - |N(S) - S| \geq 1,$$

whence the inequality $|N[X] \cap S| \geq |N(X) - S|$ follows from Proposition 4.0.43; and if $|X| = 1$ then X is itself a singleton set whose only element is a vertex which is not isolated in $\overline{G[S]}$, so the described inequality holds by hypothesis. □

Corollary 4.0.44 can be sharpened further. If $|S| > |N(S) - S|$ and $\overline{G[S]}$ is a forest, then to verify the security of S the inequality $|N[v]| \cap S \geq |N(v) - S|$ need be checked only for all $v \in S$ whose degree in $\overline{G[S]}$ exceeds $|S| - |N(S) - S|$.

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Appendices

Appendix A

Augmenting Path Algorithm

Definitions

Definition A.0.45. *Given a matching M in a graph G, an $M-augmenting$ path is a path P in G such that the endpoints of P are $M-unsaturated$, and the edges of P are alternately in M and not in M.*

Definition A.0.46. *$Depth-first$ search (DFS) is an algorithm for traversing or searching tree or graph data structures. One starts at the root (selecting some arbitrary vertex as the root in the case of a graph) and explores as far as possible along each branch before backtracking.*

Definition A.0.47. *$Breadth - first$ search (BFS) is a strategy for searching in a graph when search is limited to essentially two operations: (a) visit and inspect a vertex of a graph; (b) gain access to visit the vertices that neighbor the currently visited vertex. The BFS begins at a root vertex and inspects all the neighboring vertices. Then for each of those neighbor vertices in turn, it inspects their neighbor vertices which were unvisited, and so on.*

Theorem A.0.48. *(Berge's Theorem, [8])- A matching M is maximum in a graph G if and only if there exists no $M - augmenting$ path in G.*

The "only if" assertion of Berge's Theorem is the easier of its two implications to prove, but the proof is the basis of the augmenting path algorithm, so we give it here. Suppose M is a matching in G, and P is an M-augmenting path in G with end vertices u and v. Since neither u nor v is incident to an edge of M, and the edges of P alternate between being in M, P must be of odd length, with one more edge not in M than in M. See figure A.1.

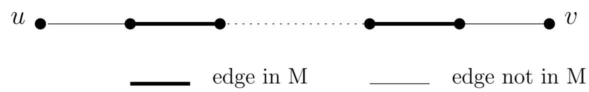

Figure A.1: An M-alternating path of length ≥ 5

Because M is a matching, no vertex of P is incident to an edge of M which is not an edge of P. Therefore,

$$M' = (M \backslash (M \cap E(P))) \cup (E(P) \backslash M),$$

the set of edges obtained by exchanging the edges of M which are on P for the other edges of P is a matching in G with one more edge than M:

$$|M'| = |M| + 1.$$

In the augmenting path algorithm one starts with a maximal matching M, in G, chosen *greedily* – choose any edge to start with, then look for an edge not adjacent to the first, and so on. When no edge not adjacent to those already chosen can be found, you have a maximal matching.

If every vertex of G is saturated by M, then M, is a maximum matching. Otherwise, pick an unsaturated vertex u and do a depth-first search for an M_1-augmenting path with u as one end-vertex. If one is found, use it to produce M_2, a matching in G satisfying $|M_2| = |M_1|+1$, and resume the algorithm with M_2 replacing M_1 and with the supply of unsaturated vertices reduced by 2. If no M_1-augmenting path with u at one end is found, mark u as finished and repeat the search with another M_1-unsaturated vertex. If no M_1-augmenting path is found, then M_1 is a maximum matching.

Since any unsaturated vertex marked as finished never need be started from again, and since any vertex saturated by M_1 will be saturated by M_2, should there be an M_2, it is easy to see that the augmenting path algorithm reaches a conclusion in no more than a constant times $|V(G)| + |E(G)|$ operations.

If G is bipartite, as is the case in our problems, the search for maximum matchings is somewhat simplified. One may as well focus on unsaturated vertices on the smaller side of the bipartition as starting end vertices for an M-augmenting path, where M is the current maximal matching. If there is no unsaturated vertex on the smaller side, then M is a maximum matching.

Appendix B

Running the Augmenting Path Algorithm on Figure 2.3

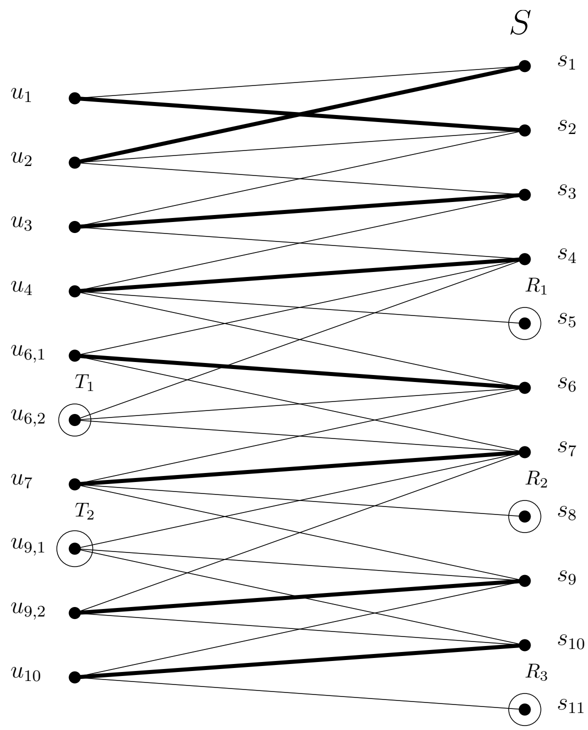

Figure B.1: Selection of unmatched vertices T_i and R_i

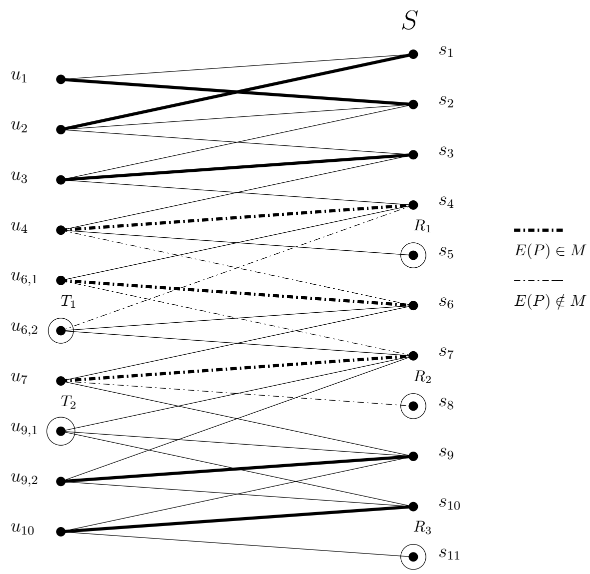

Figure B.2: Arbitrarily selecting T_1 to begin the first iteration of the augmenting-path algorithm by locating an M-augmenting path from T_1 to R_2.

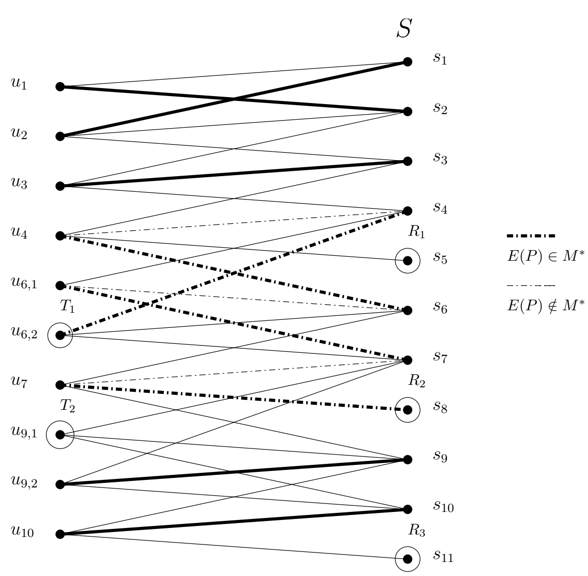

Figure B.3:

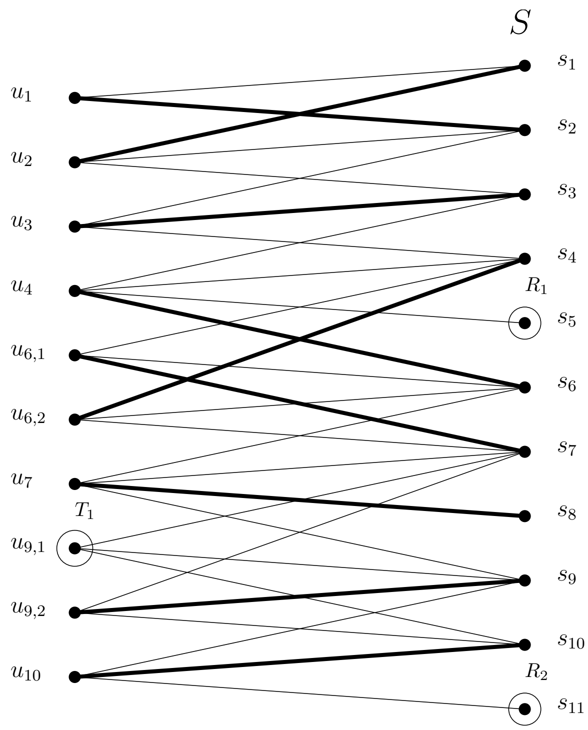

Figure B.4: Selection of the new unmatched vertices T_i and R_i after the first iteration of the augmenting-path algorithm.

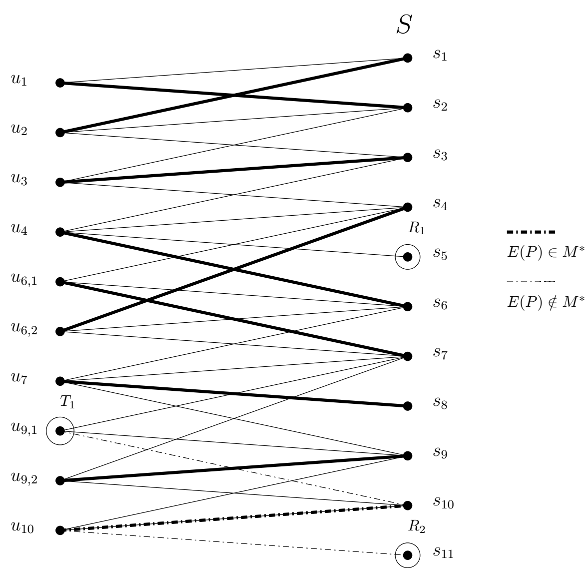

Figure B.5: Arbitrarily selecting the new T_1 to begin the second iteration of the augmenting-path algorithm by locating an M-augmenting path from the new T_1 to the new R_2.

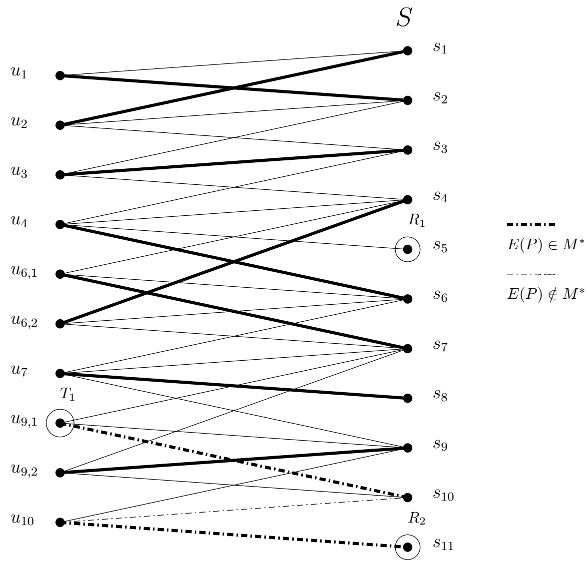

Figure B.6: The bipartite graph produced from G.

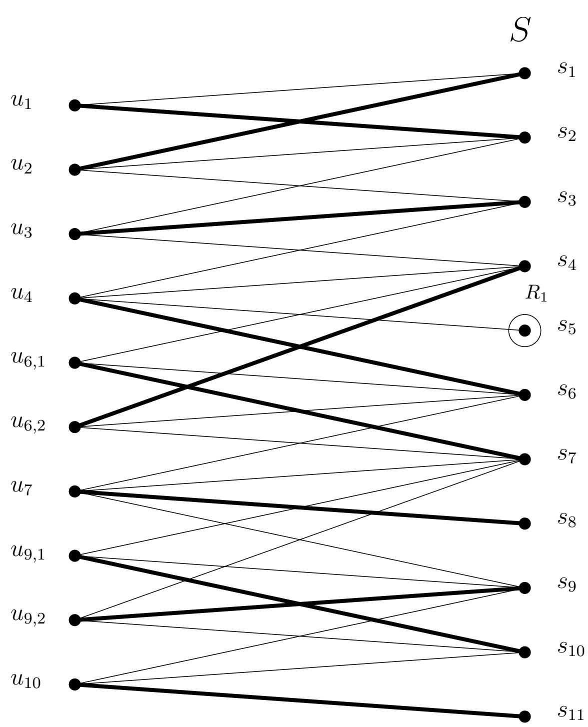

Figure B.7:

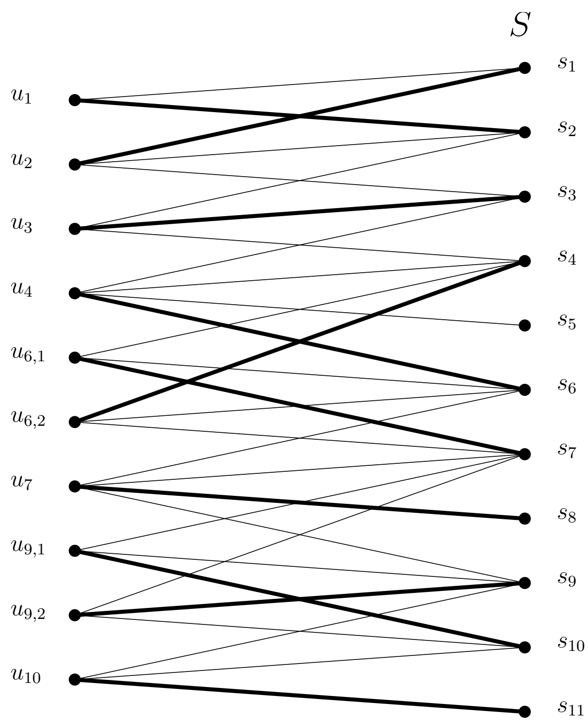

Figure B.8: We have now produced a maximum matching saturating the LHS.