Some Geometry Of Symmetrized Tensor Spaces

by

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Abstract

Let G be a subgroup of the symmetric group S_n for $n \in \mathbb{N}$ and let V be an inner product space. Orthogonality properties of the set of standard (decomposable) symmetrized tensors in $V^{\otimes n}$ corresponding to G have been studied for more than two decades [WG91, HT92, Hol95, DP99, BPR03, Hol04, TS12]. The determination of such properties would be facilitated by an understanding of the more general geometric properties of this set. Building on fairly isolated insights throughout the relevant literature, we obtain some new results, and in doing so we begin to weave a coherent narrative for the future exploration of this geometry.

The space $V^{\otimes n}$ is an orthogonal direct sum of orbital subspaces, so it suffices to study the sets of standard symmetrized tensors in these subspaces. In order to do so, we investigate for each irreducible character χ of G and each subgroup H of G the set Σ of standard vectors in what we call the coset space $\mathcal{C}_H^\chi$. This coset space is itself a vector space and a $\mathbb{C}G$-module, and for each (H, χ)-pairing it corresponds to one and only one orbital subspace of $V^{\otimes n}$. Hence the coset space serves as a proxy tool for our inquiries into the geometry of a given orbital subspace.

The structure of $\mathcal{C}_H^\chi$ as a vector space is easily understood, but for the sake of parallelism with orbital subspaces, we need also to understand its structure as a $\mathbb{C}G$-module. The first result furnishes an isomorphism that renders this structure in the well-known terms of the associated group algebra. The isomorphism may prove useful in the future, as groundwork for a more module-theoretic approach to these matters.

In order to generate meaningful conjectures, we then devote considerable effort to the construction of bases for our coset spaces. As it turns out, finding such a basis is itself a far from trivial problem. After developing sufficient machinery, we obtain bases for coset spaces associated with the groups S_3, S_4, and A_4, allowing the (H, χ)-pairings to vary. We then compute the Gram matrices for the basis vectors in each coset space. The first conjectures arise from these computations. In several cases we find that, possibly after dividing the entries of the Gram matrix by 2, we obtain the Cartan matrix for a crystallographic root system of type A_2 or $A_2 \times A_2$. Thus in these cases a basis for the coset space is also a base for a root system. By reference to the correspondence between coset spaces and orbital subspaces, we conclude that certain symmetrized tensor spaces possess the geometry of crystallographic root systems. The section culminates with a theorem stating that, if a coset space gives rise to a root system, then that root system has irreducible components of type A_1 or A_2, which are simply laced.

In the next section, we examine in depth the geometric properties of $\mathcal{C}_H^\chi$ when the group is dihedral, of order a power of 2. It has already been proven that the standard symmetrized tensors in $V^{\otimes n}$ for this choice of G have an orthogonal basis for every (H, χ)-pairing in the corresponding coset space. We now deploy more explicitly geometric methods to obtain the same conclusion. The result is a more intuitive proof, the methodology of which will hopefully shed light on the orthogonality properties of other sets of standard symmetrized tensors. Ultimately, one goal of future work is to provide necessary and sufficient conditions for a finite group to give rise to an orthogonal basis of symmetrized tensors.

Finally, we show how this more geometric approach confirms a result in the literature. Given an m-dimensional inner product space V, the orbital subspace V_γ^χ of $V^{\otimes n}$ is determined by the irreducible character χ of S_m and the element $\gamma \in \Gamma_{n,m}$, where we may view $\Gamma_{n,m}$ as the set of functions $\alpha : \{1, ..., n\} \to \{1, ..., m\}$. In [TS12], the authors prove that the standard symmetrized tensors in a certain orbital subspace of $V^{\otimes n}$ form a root system of type A_{m-1}. Again we take a more conceptual avenue to the same result. It is a fitting way to close: The proof demonstrates how, by bringing the power of representation and character theory to bear, we can accomplish what was formerly done through combinatorial means.

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Chapter 1

Preliminaries

1.1 Basics of Representation Theory

Let G be a finite group, let K be a field, and let V be a finite-dimensional vector space over K. Denote by $\mathrm{GL}(V)$ the group of invertible linear transformations of V onto itself. We define a **linear K-representation** of G in V to be a group homomorphism $\rho : G \to \mathrm{GL}(V)$. When the field K is understood to be $\mathbb{C}$, as it is throughout this work, we refer to ρ simply as a representation. A group G usually has multiple representations, and we may think of the images of these representations informally as different "snapshots" of the group. By studying the "snapshots," we indirectly obtain information about the group at hand.

Now, given a representation ρ, and an element $g \in G$, we have that $\rho(g) \in \mathrm{GL}(V)$, so it follows that $\rho(g) : V \to V$ can be viewed as a matrix $[\rho(g)]_B$ relative to some ordered basis B. Recall that the **trace** of a matrix is the sum of its diagonal elements. Although we may change the basis for V to some other ordered basis, the trace of $\rho(g)$ remains unchanged relative to this new basis. It makes sense, then, to speak of *the* trace of $\rho(g)$. Hence the following notion is well-defined: The **character afforded by** ρ is the function $\chi : G \to K$ given by $\chi(g) = Tr[\rho(g)]_B$, where B is some ordered basis for V.

Let W be a subspace of V such that $\rho(g)(W) \subseteq W$ for all $g \in G$. Then we obtain a new map $G \to \mathrm{GL}(W)$, called the **sub-representation** of ρ afforded by W. If ρ has no proper, non-trivial sub-representations, then we say that ρ is irreducible, and the character it affords is an **irreducible character**. The set of irreducible characters of a group G, denoted $\mathrm{Irr}(G)$, will play a central role in the ensuing text. Before proceeding, though, we must learn to see representations of G as equivalent to what we call KG-modules.

Given a finite group G and a field K, KG is the vector space whose elements are linear combinations of the form $\sum_{a\in G}\alpha_a a$, where $\alpha_a \in K$. Its basis is the set G, and it gets its additive structure in a natural way. To give KG multiplicative structure, we define multiplication by

$$\left(\sum_{a\in G}\alpha_a a\right)\left(\sum_{b\in G}\beta_b b\right) = \sum_{a,b\in G}(\alpha_a\beta_b)ab.$$

The element $1e$ serves as the multiplicative identity of KG, where e is the identity element of the group G. Now, a K-**algebra** is a ring A with identity that is also a vector space over K such that $\alpha(ab) = (\alpha a)b = a(\alpha b)$ for all $\alpha \in K$ and all $a, b \in A$. With the above structure, KG is a K-algebra, and since its basis is G, we refer to KG as a **group algebra**.

Recall that, for an arbitrary ring R, an R-module is an abelian group M equipped with a map $R \times M \to M$ such that for all $r, s \in R$ and $m, n \in M$,

(i) $r(m+n) = rm + rn$

(ii) $(r+s)m = rm + sm$

(iii) $r(sm) = (rs)m$

(iv) 1m=m

Let V be a vector space over the ring KG, and again let $\rho : G \to \mathrm{GL}(V)$ be a representation of G. Then V becomes a KG-module by defining $gv = \rho(g)(v)$ $(g \in G, v \in V)$ and extending linearly to KG. In this way, each representation gives rise to a KG-module.

Now let V be a KG-module. Using the embedding $K \to KG$ given by $\alpha \mapsto \alpha e$, we view V as a (finite-dimensional) vector space over K. Define $\rho : G \to \mathrm{GL}(V)$ by $\rho(g)(v) = gv$. Then the properties of a KG-module can be used to verify that ρ is a representation of G. Hence each KG-module gives rise to a representation of G, so the two objects are equivalent via this one-to-one correspondence. In this instance, we say that ρ is the **representation of G afforded by V**. If χ is the character afforded by ρ, then we may also say that χ is afforded by V.

Just as a representation has subrepresentations, so too does a KG-module have submodules. If a KG-module V has no non-zero, proper submodules, we call V **simple**. Simple KG-modules correspond to irreducible representations of G. Like the latter, these simple KG-modules afford **irreducible characters**. We will pass freely between representations of G and KG-modules throughout, depending on contextual convenience. As mentioned, we will always take $K = \mathbb{C}$. We now present, for the reader's convenience, a list of results we will need.

1.2 Selected Results

Definition 1.1. For a character χ of G, the positive integer $\chi(e)$ is the **degree** of χ. We say χ is **linear** if $\chi(e) = 1$.

Theorem 1.2. ***([Isa94, p.16])*** *A group G is abelian if and only if every irreducible character of G is linear.*

Proposition 1.3. *Let χ be a linear character of G. Then χ is irreducible.*

Proposition 1.4. ***([Isa94, p.20])*** *Let χ be a character of G. Then for all $g \in G$,*

(i) $|\chi(g)| \leq \chi(e)$, *and*

(ii) $\chi(g^{-1}) = \overline{\chi(g)}$.

Proposition 1.5. ***([Isa94, p.16])*** *The cardinality of the set $Irr(G)$ equals the number of conjugacy classes of G.*

Theorem 1.6. ***([Isa94, p.19]) Generalized Orthogonality Relation.*** *Let $\chi, \psi \in Irr(G)$. Then the following holds for every $h \in G$:*

$$\frac{1}{|G|}\sum_{g\in G}\chi(gh)\psi(g^{-1}) = \delta_{\chi\psi}\frac{\chi(h)}{\chi(e)},$$

where $\delta_{\chi\psi}$ is the Kronecker delta.

The set of all class functions $G \to \mathbb{C}$ is a vector space over $\mathbb{C}$. Since characters are constant on conjugacy classes, characters are class functions, and we have the following definition.

Definition 1.7. Let χ, ψ be characters of G. Then

$$(\chi, \psi) = (\chi, \psi)_G = \frac{1}{|G|}\sum_{g\in G}\chi(g)\overline{\psi(g)}$$

is the **inner product of χ and ψ**.

Proposition 1.8. ***([Isa94, p.21])*** *Let χ be a character of G. Then χ is irreducible iff $(\chi, \chi) = 1$.*

Definition 1.9. Let χ be a character of G. The **kernel** of χ is the subgroup of G defined by $\ker \chi = \{g \in G \mid \chi(g) = \chi(e)\}$.

Theorem 1.10. ***[Isa94, p.81])*** *Let $N \lhd G$ and let $\chi \in Irr(G)$ with $(\chi, 1)_N \neq 0$. Then $N \subseteq ker\chi$.*

Proposition 1.11. ***[Isa94, p.24])*** *Let $N \lhd G$.*

(i) *If χ is a character of G and $N \subseteq ker\chi$, then χ is constant on cosets of N in G, and the function $\hat{\chi}$ on G/N defined by $\hat{\chi}(gN) = \chi(g)$ is a character of G/N.*

(ii) *If $\hat{\chi}$ is a character of G/N, then the function χ defined by $\chi(g) = \hat{\chi}(gN)$ is a character of G.*

(iii) *In both (i) and (ii), $\chi \in Irr(G)$ iff $\hat{\chi} \in Irr(G/N)$.*

Theorem 1.12. *Let $H \leqslant G$ and let $\chi \in Irr(G)$. Then the restriction of χ to H, denoted χ_H, is a character of H.*

Definition 1.13. **([Isa94, p.17])** Let $\chi_i \in \mathrm{Irr}(G)$ for each i. If $\chi = \sum_{i=1}^{k} n_i\chi_i$ is a character, then those χ_i with $n_i > 0$ are called the irreducible **constituents** of χ.

Definition 1.14. **([Isa94, p.62])** Let $H \leqslant G$ and let λ be a class function of H. Then λ^G, the **induced class function** on G, is given by

$$\lambda^G(g) = \frac{1}{|H|}\sum_{x\in G}\lambda^\circ(xgx^{-1}),$$

where λ° is defined by $\lambda^\circ(h) = \lambda(h)$ if $h \in H$ and $\lambda^\circ(y) = 0$ if $y \notin H$.

Proposition 1.15. ***([Isa94, p.63])*** *Let $H \leqslant G$ and suppose λ is a character of H. Then λ^G is a character of G. We say that λ^G is an **induced character** of G.*

Definition 1.16. Let $H \leqslant G$, let λ be a character of H, and let $g \in G$. The **conjugate character of** λ, denoted ${}^g\lambda$, is the character of ${}^gH$ defined by ${}^g\lambda({}^gh) = \lambda(h)$ for all $h \in H$. Here, ${}^gh = ghg^{-1}$, and ${}^gH = \{{}^gh \mid h \in H\}$.

Proposition 1.17. ***[CR62, p.323])*** *Let H be a subgroup of G, let M be a $\mathbb{C}G$-module, and let L be a submodule of M_H. Define L^G as the module corresponding to the induced character λ^G, where L affords λ. If $M = \dot{\sum}_{a\in A} aL$, where A is a set of representatives for the left cosets of H in G, then $M \cong L^G$.*

Definition 1.18. [Hun80, **p.135**] Elements $e_1, \dots, e_n$ in a ring R are **orthogonal central idempotents** if $e_i e_j = \delta_{ij}$, $e_i \in Z(R)$ for each i, and $e_i^2 = e_i$ for each i.

Theorem 1.19. *[**Isa94**] The elements e_χ, $\chi \in Irr(G)$, are orthogonal central idempotents in the ring $\mathbb{C}G$, where $e_\chi = \dfrac{\chi(e)}{|G|} \displaystyle\sum_{g \in G} \chi(g^{-1})g$.*

Proposition 1.20. *[**JL93**] If χ is an irreducible character of G, and V is any $\mathbb{C}G$-module, then $e_\chi \cdot V$ is equal to the sum of those $\mathbb{C}G$-submodules of V that have character χ.*

Proposition 1.21. *[**Suz82, (3.8),p.23**] Let G be a group and let H and K be subgroups of G. For each $x \in G$, the double coset HxK is a union of right cosets of H and the cardinality of the set of these cosets is $|K : K \cap H^x|$.*

Theorem 1.22. ***Frobenius Reciprocity**. Let H be a subgroup of G, let χ be a character of G, and let λ be a character of H. Then $(\lambda^G, \chi) = (\lambda, \chi_H)$.*

Theorem 1.23. ***Mackey's Subgroup Theorem**. Let X and Y be subgroups of a group G. If L is a $\mathbb{C}X$-module, then*

$$(L^G)_Y \cong \bigoplus_{a \in A} (({}^aL)_{{}^aX \cap Y})^Y,$$

where A is a set of representatives for the (Y, X)-double cosets in G.

We will need to refer to the character table of the dihedral group

$$D_{2n} = \langle a, b \,|\, a^n = e, b^2 = e, bab = a^{-1} \rangle$$

[Ser77, pp. 37–38]:

	a^k	ba^k	
ψ_0	1	1	
ψ_1	1	-1	
ψ_2	$(-1)^k$	$(-1)^k$	$(n$ even$)$
ψ_3	$(-1)^k$	$(-1)^{k+1}$	$(n$ even$)$
χ_j	$2\cos(2\pi kj/n)$	0	$(1 \leq j < n/2)$.

Here, the entries in the left column are the distinct irreducible characters of D_{2n}, and the entries in the top row are representatives for the conjugacy classes of D_{2n}.

Irreducible characters of the symmetric group can also be viewed from a combinatorial perspective. We will draw upon this view in the final chapter. For now, we state the pertinent results. The following can be found in [Hog07, pp.17-18]. Throughout, we let n denote a natural number.

Definition 1.24. A tuple $\alpha = [\alpha_1, ..., \alpha_h]$ of non-negative integers is a **(proper) partition** of n, written $\alpha \vdash n$, provided

- $\alpha_i \geq \alpha_{i+1}$ for all $1 \leq i < h$
- $\sum_{i=1}^{h} \alpha_i = n$.

Definition 1.25. The conjugate partition of a partition $\alpha \vdash n$ is the partition $\alpha' \vdash n$ with i-th component α'_i equal to the number of indices j for which $\alpha_j \geq i$.

Definition 1.26. Given two partitions $\alpha = [\alpha_1, ..., \alpha_h]$ and $\beta = [\beta_1, ..., \beta_k]$ of n, α **majorizes** β if $\sum_{i=1}^{j} \alpha_i \geq \sum_{i=1}^{j} \beta_i$ for each $1 \leq j \leq h$.

Definition 1.27. The **Young subgroup** of the symmetric group S_n corresponding to a partition $\alpha = [\alpha_1, \ldots, \alpha_h]$ of n is the internal direct product $S_\alpha = S_{A_1} \times \cdots \times S_{A_h}$, where S_{A_i} is the subgroup of S_n consisting of those permutations that fix every integer *not* in the set

$$A_i = \{1 \le k \le n \mid \sum_{j=1}^{i-1} \alpha_j < k \le \sum_{j=1}^{i} \alpha_j\},$$

where an empty sum is interpreted as zero.

Proposition 1.28. *Each partition $\alpha \vdash n$ uniquely determines an irreducible character χ_α of S_n. The map $\alpha \mapsto \chi_\alpha$ defines a bijection from the set of partitions of n to the set $Irr(S_n)$ of irreducible characters of S_n.*

Proposition 1.29. *If α and β are partitions of n, then the irreducible character χ_α is a constituent of the induced character $(1_{S_\beta})^{S_n}$ if and only if α majorizes β.*

Definition 1.30. The alternating character of the symmetric group S_n is the character ϵ_n given by

$$\epsilon_n(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is even,} \\ -1 & \text{if } \sigma \text{ is odd.} \end{cases}$$

Proposition 1.31. *If α is a partition of n, then $\chi_{\alpha'} = \varepsilon_n \chi_\alpha$, where ε_n is the alternating character of the group S_n.*

1.3 Root Systems

The following definitions and results about root systems can be found in [Hum72, pp.42-44, 47, 52, 55]. Let E be a finite-dimensional vector space over $\mathbb{R}$ endowed with a positive definite symmetric bilinear form (α, β).

Definition 1.32. A **reflection** in E is an invertible linear transformation leaving point-wise fixed some hyperplane (subspace of co-dimension one) and sending any vector orthogonal to that hyperplane into its negative.

Definition 1.33. The reflection σ_α determined by a nonzero vector α is given by

$$\sigma_\alpha(\beta) = \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha.$$

We abbreviate $\frac{2(\beta,\alpha)}{(\alpha,\alpha)}$ by $\langle \beta, \alpha \rangle$.

Definition 1.34. A subset Φ of the Euclidean space E is called a (crystallographic) **root system** in E if the following axioms are satisfied:

(R1) Φ is finite, spans E, and does not contain 0

(R2) If $\alpha \in \Phi$, the only multiples of α in Φ are $\pm\alpha$.

(R3) If $\alpha \in \Phi$, $\sigma_\alpha(\Phi) = \Phi$

(R4) If $\alpha, \beta \in \Phi$, then $\langle \beta, \alpha \rangle \in \mathbb{Z}$

We call the elements of Φ **roots**.

Definition 1.35. The **rank** of the root system Φ is the dimension of E.

Proposition 1.36. *We have* $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 4cos\theta$*, where* θ *is the angle between* α *and* β*.*

Definition 1.37. A subset Δ of Φ is called a **base** if

- (B1) Δ is a basis for E as a vector space
- (B2) Each root $\beta \in \Phi$ can be written as $\beta = \sum k_\alpha \alpha$ $(\alpha \in \Delta)$ with integral coefficients k_α all nonnegative or nonpositive.

Definition 1.38. The root system Φ is **irreducible** if it cannot be partitioned into the union of two proper subsets such that each root in one set is orthogonal to each root in the other.

Definition 1.39. Fix an ordering $(\alpha_1, ..., \alpha_l)$ of the roots in a base Δ of the root system Φ. The matrix $(\langle \alpha_i, \alpha_j \rangle)$ is then called the **Cartan matrix** of Φ. The entries of the Cartan matrix are called **Cartan integers**.

1.4 Coset Spaces and Symmetrized Tensor Spaces

Here we provide a brief account of the construction of coset spaces. We then present some fundamental results, and discuss the nature of the correspondence between coset spaces and orbital subspaces of symmetrized tensor spaces.

Let G be a finite group and let H be a subgroup of G. Denote by G/H the set of (left) cosets of H in G. We denote by $\mathbb{C}(G/H)$ the complex vector space having G/H as basis. Let $\mathrm{Irr}(G)$ denote the set of irreducible characters of G, and let $\chi \in \mathrm{Irr}(G)$. We define a form B_H^χ on $\mathbb{C}(G/H)$ by

$$B_H^\chi(aH, bH) = \frac{\chi(e)}{|H|} \sum_{h \in H} \chi(b^{-1}ah),$$

extending linearly in the first component and antilinearly in the second. By [Hol04, p.2], B_H^χ is a well-defined Hermitian form, and it is G-invariant: For all $g, a, b \in G$, we have that $B_H^\chi(gaH, gbH) = B_H^\chi(aH, bH)$.

Now put $\mathcal{C}_H^\chi := \mathbb{C}(G/H)/\ker B_H^\chi$, where $\ker B_H^\chi := \{x \in \mathbb{C}(G/H) : B_H^\chi(x, y) = 0$ for all $y \in \mathbb{C}(G/H)\}$. We call $\mathcal{C}_H^\chi$ a **coset space**. The set of vectors $\Sigma = \Sigma_H^\chi = \{\overline{aH} \in \mathcal{C}_H^\chi \mid a \in G\}$

are called the **standard vectors** in the coset space. The coset space and its standard vectors will play a pivotal role in the sequel. If $\mathcal{C}_H^\chi$ has a basis consisting of pairwise orthogonal standard vectors, we say that $\mathcal{C}_H^\chi$ has an **o-basis**. We will call G an **o-basis group** if for every $H \leqslant G$ and $\chi \in \mathrm{Irr}(G)$ the vector space $\mathcal{C}_H^\chi$ has a basis that is orthogonal relative to $\overline{B_H^\chi}$ (defined below) consisting entirely of standard vectors.

The form B_H^χ induces a well-defined form $\overline{B_H^\chi}$ on $\mathcal{C}_H^\chi$ given by $\overline{B_H^\chi}(\bar{x}, \bar{y}) = B_H^\chi(x, y)$ ($x, y \in \mathbb{C}(G/H)$, where $\bar{x}$ denotes the coset $x+$ kerB_H^χ. From the natural left action of G on $\mathbb{C}(G/H)$ we get a well-defined action of G on $\mathcal{C}_H^\chi$, and $\overline{B_H^\chi}$ inherits G-invariance. It is a theorem in [Hol04, p.2] that, since B_H^χ is positive semidefinite, the form $\overline{B_H^\chi}$ is positive definite. Thus $\mathcal{C}_H^\chi$ is an inner product space. Given vectors $v_1, \ldots, v_n \in \mathcal{C}_H^\chi$, the matrix of inner products given by $(\overline{B_H^\chi}(v_i, v_j))_{i,j}$ is the **Gram matrix** of the vectors $v_1, ..., v_n$. Also from [Hol04, p.2], we have the following formula for the dimension of the complex vector space $\mathcal{C}_H^\chi$:

$$\dim_{\mathbb{C}} \mathcal{C}_H^\chi = \chi(e)(\chi, 1)_H = \frac{\chi(e)}{|H|} \sum_{h \in H} \chi(h).$$

Before discussing the correspondence between coset spaces and orbital subspaces, we note the following proposition, which we use in a later chapter. Let $N \lhd G$, let $\chi \in \mathrm{Irr}(G)$, and assume that $N \subseteq \ker\chi$. Put $\widehat{G} := G/N$ and denote by $\hat{a}$ the image of $a \in G$ under the canonical epimorphism $G \to \widehat{G}$. The function $\widehat{\chi} : \widehat{G} \to \mathbb{C}$ given by $\widehat{\chi}(\hat{a}) = \chi(a)$ is a well-defined irreducible character of $\widehat{G}$ [Isa94, p.24]. Let $H \leqslant G$.

Proposition 1.40. *[**Hol04**, **p.4**] The map $\varphi : \mathcal{C}_H^\chi \to \mathcal{C}_{\hat{H}}^{\hat{\chi}}$ given by $\varphi(\overline{aH}) = \overline{\hat{a}\widehat{H}}$ is a well-defined linear isometry. In particular, $\mathcal{C}_H^\chi$ has an o-basis if and only if $\mathcal{C}_{\hat{H}}^{\hat{\chi}}$ has an o-basis.*

Let us now explore the notion of a symmetrized tensor space. Fix positive integers n and m and set $\Gamma_{n,m} = \{\gamma \in \mathbb{Z}^n \,|\, 1 \leq \gamma_i \leq m\}$. Fix a subgroup G of the symmetric group S_n. A right action of G on the set $\Gamma_{n,m}$ is given by $\gamma\sigma = (\gamma_{\sigma(1)}, \ldots, \gamma_{\sigma(n)})$ ($\gamma \in \Gamma_{n,m}, \sigma \in G$). The **stabilizer** of $\gamma \in \Gamma_{n,m}$ is the set $G_\gamma = \{\sigma \in G \,|\, \gamma\sigma = \gamma\}$.

Let V be an inner product space of dimension m and let $\{e_i \,|\, 1 \leq i \leq m\}$ be an orthonormal basis for V. The inner product on V induces an inner product on $V^{\otimes n}$ (the nth tensor power of V) and, with respect to this inner product, the set $\{e_\gamma \,|\, \gamma \in \Gamma_{n,m}\}$ is an orthonormal basis for $V^{\otimes n}$, where $e_\gamma = e_{\gamma_1} \otimes \cdots \otimes e_{\gamma_n}$.

The space $V^{\otimes n}$ is a (left) $\mathbb{C}G$-module with action given by $\sigma e_\gamma = e_{\gamma\sigma^{-1}}$ $(\sigma \in G, \gamma \in \Gamma_{n,m})$, extended linearly. The inner product on $V^{\otimes n}$ is G-invariant, which is to say $(\sigma v, \sigma w) = (v, w)$ for all $\sigma \in G$ and all $v, w \in V^{\otimes n}$.

Let $\chi \in \mathrm{Irr}(G)$. The **symmetrizer** corresponding to χ is

$$s^\chi = \frac{\chi(e)}{|G|} \sum_{\sigma \in G} \chi(\sigma^{-1})\sigma \in \mathbb{C}G,$$

where e denotes the identity element of G. This element s^χ is the central idempotent of $\mathbb{C}G$ corresponding to χ [CR62, 33.8].

Let $\gamma \in \Gamma_{n,m}$. The **standard (decomposable) symmetrized tensor** corresponding to χ and γ is $e^\chi_\gamma = s^\chi e_\gamma$. The **orbital subspace** of $V^{\otimes n}$ corresponding to χ and γ, denoted V^χ_γ, is the span of the set $\Psi = \Psi^\chi_\gamma = \{e^\chi_{\gamma\sigma} \,|\, \sigma \in G\}$. The space $V^{\otimes n}$ is an orthogonal direct sum of orbital subspaces.

To understand the nature of the correspondence between coset spaces and orbital subspaces, we rely on the following definition, theorem, and corollary.

Definition 1.41. Let V and V' be inner product spaces, let $\varphi : V \to V'$ be a linear map, and let r be a positive real number. Then φ is a **similarity transformation of ratio r** if $\|\varphi(v)\| = r\|v\|$ for all $v \in V$. In this case, we write $V \sim V'$.

A similarity transformation of ratio 1 is commonly known as an isometry. All similarity transformations, however, preserve angles as well as relative lengths (i.e., $\|\varphi(v)\|/\|\varphi(w)\| = \|v\|/\|w\|$).

Theorem 1.42. *([HH13]) For $\gamma \in \Gamma_{n,m}$, we have $V_\gamma^\chi \sim \mathcal{C}_{G_\gamma}^\chi$.*

Proof. Let $\gamma \in \Gamma_{n,m}$ and put $H = G_\gamma$. The map $G/H \to V_\gamma^\chi$, given by $\sigma H \mapsto e_{\gamma\sigma^{-1}}^\chi$, is well-defined and it induces a surjective linear map $\varphi : \mathbb{C}(G/H) \to V_\gamma^\chi$. For $\sigma, \tau \in G$, we have

$$\begin{aligned}(\varphi(\sigma H), \varphi(\tau H)) = (e_{\gamma\sigma^{-1}}^\chi, e_{\gamma\tau^{-1}}^\chi) &= \frac{\chi(e)}{|G|} \sum_{\mu \in H} \chi(\tau^{-1}\sigma\mu) \\ &= rB_H^\chi(\sigma H, \tau H),\end{aligned} \tag{1.1}$$

where $r = |G : H|^{-1}$ and where the second equality is from [Fre73, p. 339] (with $\bar{\chi}$ in place of χ). Using linearity we get $(\varphi(x), \varphi(y)) = rB_H^\chi(x, y)$ for all $x, y \in \mathbb{C}(G/H)$, and it follows that the induced map $\bar{\varphi} : \mathcal{C}_H^\chi \to V_\gamma^\chi$ given by $\bar{\varphi}(\bar{x}) = \varphi(x)$ is a well-defined bijective similarity transformation of ratio r. □

According to Theorem 1.42 and its proof, every orbital subspace can be identified with a coset space in such a way that the standard symmetrized tensors in the orbital subspace identify, in an angle-preserving and relative length-preserving manner, with the standard vectors in the coset space. The following result says that, conversely, every coset space can be similarly identified with an orbital subspace. The statement requires some explanation: Let $G = \{g_1, \dots, g_n\}$ be a finite group. The **Cayley embedding** of G in the symmetric group $S_{|G|}$ is the monomorphism $\varphi : G \to S_{|G|}$ given by $\varphi(g) = \lambda_g$, with $\lambda_g : G \to G$ defined by $\lambda_g(a) = ga$. Here, we regard λ_g as an element of $S_{|G|}$ by using the identification $\{1, \dots, n\} \leftrightarrow G$, $i \leftrightarrow g_i$. Using this same identification, we write γ_{g_i} to mean γ_i for $\gamma \in \Gamma_{|G|,m}$.

Corollary 1.43. ([HH13]) Assume that $m = \dim V \geq 2$. Let G be a finite group, let $\chi \in \mathrm{Irr}(G)$, and let $H \leqslant G$. Identifying G as a subgroup of $S_{|G|}$ via the Cayley embedding, we have $\mathcal{C}_H^{\chi} \sim V_{\gamma}^{\chi}$, where $\gamma \in \Gamma_{|G|,m}$ is defined by putting γ_g equal to 1 or 2 according as $g \in H$ or $g \notin H$.

Proof. We have $H = G_{\gamma}$, so the claim follows from Theorem 1.42. $\square$

We will reference this correspondence frequently in the fourth and fifth chapters. Now, with preliminaries behind us, we present new work.

Chapter 2

The Module Structure of the Coset Space

The coset space is the fundamental object enabling us to explore the geometry of symmetrized tensor spaces. Accordingly, we should familiarize ourselves with its structure as a $\mathbb{C}G$-module. Although we do not rely heavily on the module-theoretic properties of the coset space throughout, we might glean future insight from its decomposition as a direct sum of simple $\mathbb{C}G$-modules. Let G be a finite group, let $H \leqslant G$, and let $\chi \in \mathrm{Irr}(G)$.

Theorem 2.1. $\mathcal{C}_H^\chi \cong \bigoplus_{j=1}^{(\chi,1)_H} \mathcal{L}_\chi$, *where* $\mathcal{L}_\chi$ *is the simple* $\mathbb{C}G$*-module affording* χ.

Proof. First, we claim that $\mathbb{C}(G/H) \cong (\mathbb{C}_H)^G$. Define $\lambda : \mathbb{C}G \times \mathbb{C}_H \longrightarrow \mathbb{C}(G/H)$ by $\lambda(r,s) = rsH$. Letting $r_1 = \sum_{a\in G} \alpha_a a \in \mathbb{C}G$, $r_2 = \sum_{a\in G} \beta_a a \in \mathbb{C}G$, and $s \in \mathbb{C}_H$, we have that

$$\lambda(r_1 + r_2, s) = \sum_a (\alpha_a + \beta_a) saH = \sum_a \alpha_a saH + \sum_a \beta_a saH = \lambda(r_1, s) + \lambda(r_2, s).$$

Another straightforward argument shows that λ is linear in the second slot as well. Now let $r \in \mathbb{C}G$, $h \in \mathbb{C}H$, and let $s \in \mathbb{C}_H$. Then

$$\lambda(rh, s) = (rh)sH = s(rh)H = sr(hH) = srH = rsH = r(hs)H = \lambda(r, hs),$$

where we have used the fact that $s \in Z(\mathbb{C}G)$ and the fact that H acts trivially on $\mathbb{C}_H$. Thus λ is a middle-linear map, and so there exists a unique homomorphism $\overline{\lambda} : \mathbb{C}G \otimes_{\mathbb{C}H} \mathbb{C}_H \longrightarrow \mathbb{C}(G/H)$ such that $\overline{\lambda}(r \otimes s) = \lambda(r,s)$. In fact, $\overline{\lambda}$ is a $\mathbb{C}G$-homomorphism: Let $a \in G$. Then

$$\overline{\lambda}(a(r \otimes s)) = \overline{\lambda}(ar \otimes s) = (ar)sH = a(rs)H = a\overline{\lambda}(r \otimes s).$$

Now define $\mu : \mathbb{C}(G/H) \longrightarrow \mathbb{C}G \otimes_{\mathbb{C}H} \mathbb{C}_H$ by $\mu(aH) = a \otimes 1$. Let $aH, bH \in G/H$, and suppose $aH = bH$. Then $a^{-1}b \in H$, so $a^{-1}b = h^{-1}$ for some $h \in H$. Thus

$$\mu(aH) = a \otimes 1 = bh \otimes 1 = b \otimes h \cdot 1 = b \otimes 1 = \mu(bH),$$

so μ is well-defined. We claim μ is the inverse map of $\overline{\lambda}$. For $a \in G$, and $r = \sum_{g \in G} \alpha_g gH \in \mathbb{C}(G/H)$, extending linearly gives

$$\mu(ar) = \mu\left(\sum_{g \in G} \alpha_g agH\right) = \left(\sum_{g \in G} \alpha_g ag\right) \otimes 1 = a\left(\sum_{g \in G} \alpha_g g\right) \otimes 1 = a\mu(r),$$

so μ is a $\mathbb{C}G$-homomorphism. Letting r be as above, we then have that

$$(\overline{\lambda} \circ \mu)(r) = \overline{\lambda}(\mu(r)) = \overline{\lambda}\left(\left(\sum_g \alpha_g g\right) \otimes 1\right) = \left(\sum_g \alpha_g g\right) H = \sum_g \alpha_g gH = r.$$

Now let $m \otimes n \in \mathbb{C}G \otimes_{\mathbb{C}H} \mathbb{C}_H$, where $m \in G$ and $n \in \mathbb{C}_H$. Then

$$(\mu \circ \overline{\lambda})(m \otimes n) = \mu(\overline{\lambda}(m \otimes n)) = \mu(mnH) = mn \otimes 1 = m \otimes n.$$

Therefore, $\mathbb{C}(G/H) \cong \mathbb{C}G \otimes_{\mathbb{C}H} \mathbb{C}_H = (\mathbb{C}_H)^G$, establishing the claim.

By virtue of the above isomorphism, $\mathbb{C}(G/H)$ affords the character $(1_H)^G$. Thus the number of times the simple $\mathbb{C}G$-module $\mathcal{L}_\chi$ appears in the direct sum decomposition of $\mathbb{C}(G/H)$ is $((1_H)^G, \chi)$. By Frobenius Reciprocity, we have $((1_H)^G, \chi) = (1_H, \chi|_H) = (\chi, 1)_H$. Thus $\bigoplus_{j=1}^{(\chi,1)_H} \mathcal{L}_\chi$ is the direct sum of the copies of $\mathcal{L}_\chi$ appearing in the decomposition of $\mathbb{C}(G/H)$. Therefore, 1.20 gives $\bigoplus_{j=1}^{(\chi,1)_H} \mathcal{L}_\chi = e_\chi \cdot \mathbb{C}(G/H)$.

We now prove the theorem. Define $\pi_\chi : \mathbb{C}(G/H) \longrightarrow e_\chi \cdot \mathbb{C}(G/H)$ in the obvious way. Since

π_χ is an epimorphism, we need only show that $\ker \pi_\chi = \ker B_H^\chi$. Let $s = \sum_{a\in G} \alpha_a aH \in \mathbb{C}(G/H)$, and let X be a set of left coset representatives of H in G. Then

$$\begin{aligned}
s \in \ker \pi_\chi &\iff \frac{\chi(e)}{|G|}\sum_{\sigma\in G}\chi(\sigma^{-1})\sigma\left(\sum_{a\in G}\alpha_a aH\right) = 0 \\
&\iff \frac{1}{|G|}\sum_{a\in G}\sum_{\sigma\in G}\alpha_a\chi(e)\chi(\sigma^{-1})\sigma aH = 0 \\
&\iff \frac{1}{|G|}\sum_{x\in X}\sum_{a\in G}\sum_{h\in H}\alpha_a\chi(e)\chi(ah^{-1}x^{-1})xha^{-1}aH = 0 \\
&\iff \frac{1}{|G|}\sum_{x\in X}\sum_{a\in G}\sum_{h\in H}(\ \alpha_a\chi(e)\chi(ahx^{-1})\ )xH = 0 \\
&\iff \sum_{a\in G}\sum_{h\in H}\alpha_a\chi(e)\chi(ahx^{-1}) = 0 \quad \forall x \in X \\
&\iff \frac{1}{|H|}\sum_{a\in G}\sum_{h\in H}\alpha_a\chi(e)\chi(x^{-1}ah) = 0 \quad \forall x \in X \\
&\iff \sum_{a\in G}\alpha_a\frac{\chi(e)}{|H|}\sum_{h\in H}\chi(x^{-1}ah) = 0 \quad \forall x \in X \\
&\iff \sum_{a\in G}\alpha_a B_H^\chi(aH, xH) = 0 \quad \forall x \in X \\
&\iff B_H^\chi\left(\sum_a \alpha_a aH, xH\right) = 0 \quad \forall x \in X, \\
&\iff s \in \ker B_H^\chi
\end{aligned}$$

The proof is complete. $\square$

Chapter 3

Construction of Bases for Coset Spaces and Root System Geometry

We fuel our conjectures by first computing the Gram matrices for $\mathcal{C}_H^\chi$ in the case where G is a symmetric group of small order, using various choices for H and χ.

Example 3.1. Let $G = S_3$, let $H = \{e\}$, and let $\chi \in \mathrm{Irr}(G)$ of degree 2. We have that $\dim \mathcal{C}_H^\chi = 4$, and we obtain the basis $\{\overline{H}, \overline{(12)H}, \overline{(13)H}, \overline{(123)H}\}$ by trial and error. After dividing all entries of the resulting Gram matrix by 2, we get

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

This is the Cartan matrix for the rank 4 root system $A_2 \times A_2$.

The construction of bases for coset spaces by trial and error is time-consuming. To expedite the next example, we use the following.

Proposition 3.2. *Let $H \leqslant K \leqslant G$, and let $\{k_iH\}_{i=1}^n$ be a complete set of distinct left cosets of H in K. Let $\chi \in Irr(G)$ and let $a, b \in G$. Then* $B_K^\chi(aK, bK) = \dfrac{|H|}{|K|} \displaystyle\sum_i B_H^\chi(aH, bk_iH)$.

Proof. Let $\chi \in \mathrm{Irr}(G)$, and let $a, b \in G$. For each $k \in K$, we have that $k = k_ih$ for some i and some $h \in H$, and this expression for k is unique, so

$$
\begin{aligned}
B_K^\chi(aK, bK) &= \frac{\chi(e)}{|K|} \sum_{k \in K} \chi(a^{-1}bk) \\
&= \frac{\chi(e)}{|K|} \sum_i \sum_{h \in H} \chi(a^{-1}bk_i h) \\
&= \frac{\chi(e)}{|K|} \sum_i \left(\frac{|H|}{\chi(e)} \cdot \frac{\chi(e)}{|H|} \sum_{h \in H} \chi(a^{-1}bk_i h) \right) \\
&= \frac{|H|}{|K|} \sum_i B_H^\chi(aH, bk_i H).
\end{aligned}
$$

□

Now we vary only the subgroup, but in doing so, we obtain a result that we later prove holds generally for a symmetric group of arbitrarily large degree, given a specific subgroup and a specific degree 2 irreducible character.

Example 3.3. Let $G = S_3$, let $K = \langle (12) \rangle$, and let χ be as in the first example. Then $\dim \mathcal{C}_K^\chi = 2$, and using 3.2, we quickly obtain a basis for $\mathcal{C}_K^\chi$. The Gram matrix corresponds to the Cartan matrix for the root system A_2:

$$
\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}
$$

Already, with two cases, a pattern has begun to emerge.

As we construct bases for coset spaces of higher dimension, we require more results in order to make calculations manageable. The next proposition is helpful in this regard.

Proposition 3.4. *Let $n = \dim \mathcal{C}_H^\chi$, let $a_1, ..., a_n \in G$, and let $\alpha_1, ...\alpha_n \in \mathbb{C}$. The following are equivalent:*

(i) $\mathcal{B} = \{\overline{a_i H}\}_{i=1}^n$ *is a basis for* $\mathcal{C}_H^\chi$

(ii) *If* $B_H^\chi\left(\sum_{j=1}^{n}\alpha_j a_j H, a_i H\right) = 0$ *for all* $1 \leq i \leq n$, *then* $\alpha_1 = \cdots = \alpha_n = 0$.

Proof. Assume (i) holds, and suppose that $B_H^\chi\left(\sum_{j=1}^{n}\alpha_j a_j H, a_i H\right) = 0$ for all $1 \leq i \leq n$. Fix $y \in G$. Since $\mathcal{B}$ is a basis for $\mathcal{C}_H^\chi$, we have that $\overline{yH} = \beta_1\overline{a_1 H} + \cdots + \beta_n\overline{a_n H}$ for some $\beta_i \in \mathbb{C}$, with $1 \leq i \leq n$. Then

$$yH + \ker B_H^\chi = (\ \beta_1 a_1 H + \cdots + \beta_n a_n H\ ) + \ker B_H^\chi$$

$$\Longrightarrow yH - (\ \beta_1 a_1 H + \cdots + \beta_n a_n H\ ) \in \ker B_H^\chi$$

$$\Longrightarrow yH - (\ \beta_1 a_1 H + \cdots + \beta_n a_n H\ ) = k,$$

for some $k \in \ker B_H^\chi$

$$\Longrightarrow yH = (\ \beta_1 a_1 H + \cdots + \beta_n a_n H\ ) + k.$$

Now, we claim that $B_H^\chi\left(\sum_{i=1}^{n}\alpha_i a_i H, yH\right) = 0$, and this will be true if and only if

$$B_H^\chi(\alpha_1 a_1 H + \cdots + \alpha_n a_n H, \beta_1 a_1 H + \cdots + \beta_n a_n H + k) = 0$$

if and only if

$$\sum_{i=1}^{n}\sum_{l=1}^{n}\alpha_i\overline{\beta_l}B_H^\chi(a_i H, a_l H) + \sum_{i=1}^{n}\alpha_i B_H^\chi(a_i H, k) = 0$$

if and only if

$$(1) \qquad \sum_{i=1}^{n}\sum_{l=1}^{n}\alpha_i\overline{\beta_l}B_H^\chi(a_i H, a_l H) = 0.$$

By assumption, we have the following system of equations:

$$(1) \qquad \alpha_1 B_H^\chi(a_1H, a_1H) + \cdots + \alpha_i B_H^\chi(a_iH, a_1H) + \cdots + \alpha_n B_H^\chi(a_nH, a_1H) = 0$$

$$\cdot$$

$$\cdot$$

$$\cdot$$

$$(j) \qquad \alpha_1 B_H^\chi(a_1H, a_jH) + \cdots + \alpha_i B_H^\chi(a_iH, a_jH) + \cdots + \alpha_n B_H^\chi(a_nH, a_jH) = 0$$

$$\cdot$$

$$\cdot$$

$$\cdot$$

$$(n) \qquad \alpha_1 B_H^\chi(a_1H, a_nH) + \cdots + \alpha_i B_H^\chi(a_iH, a_nH) + \cdots + \alpha_n B_H^\chi(a_nH, a_nH) = 0$$

Now, multiply (1) by $\overline{\beta_1}$, (2) by $\overline{\beta_2}$, and continue in this fashion until finally multiplying (n) by $\overline{\beta_n}$. Note that each row remains equal to zero. Summing both sides of the newly obtained system, we have (1) in the chain of double implications, which establishes the claim.

Since $y \in G$ was arbitrary, we have $B_H^\chi\left(\sum_{i=1}^n \alpha_i a_i H, yH\right) = 0$ for all $y \in G$. Hence $\left(\sum_{i=1}^n \alpha_i a_i H\right) \in \ker B_H^\chi$, implying that $\sum_{i=1}^n (\alpha_i a_i H + \ker B_H^\chi) = \ker B_H^\chi$, which implies that $\sum_{i=1}^n \alpha_i \overline{a_i H} = 0$. Since $\mathcal{B}$ is a basis for $\mathcal{C}_H^\chi$, we must have $\alpha_1 = \cdots = \alpha_n = 0$.

Assume (ii) holds. Since the fact that $B_H^\chi\left(\sum_{j=1}^n \alpha_j a_j H, a_i H\right) = 0$ for all i is enough to imply that all the α_i are zero, it certainly follows that if $B_H^\chi\left(\sum_{i=1}^n \alpha_i a_i H, yH\right) = 0$ for all $y \in G$, then all the α_i are zero. By an argument above, this is tantamount to saying that

$\alpha_1\overline{a_1H} + \cdots + \alpha_n\overline{a_nH} = 0$ implies that all the α_i are zero. Hence $\{\overline{a_iH}\}_{i=1}^n$ is a linearly independent set of n vectors, which proves the set constitutes a basis for $\mathcal{C}_H^\chi$.

$\square$

Example 3.5. Let $G = S_4$, let $H = \{e\}$, and let χ be of degree 2. Then $\dim \mathcal{C}_H^\chi = 4$, and using 3.4 we have that the Gram matrix corresponds to the root system $A_2 \times A_2$. This result is in line with our previous findings.

We would like to calculate the Gram matrix for a coset space of higher dimension in order to see what patterns emerge. Since S_4 has two degree 3 irreducible characters, spaces with dimension as high as 9 are available for $H = \{e\}$. In constructing a basis for such a space, however, the number of candidates for basis vectors is daunting. A more manageable task is to find a basis for a 9-dimensional space associated with the subgroup A_4. Even for this, we require two more results. Yet a third result will then be needed to draw a conclusion for S_4. We present those results now.

Proposition 3.6. *Let $H \leqslant K \leqslant G$. Let $\chi \in Irr(G)$, and let $a_1, ..., a_n \in G$. If $\{\overline{a_iK}\}_{i=1}^n$ is linearly independent in $\mathcal{C}_K^\chi$, then $\{\overline{a_iH}\}_{i=1}^n$ is linearly independent in $\mathcal{C}_H^\chi$.*

Proof. Assume that $\{\overline{a_iK}\}_{i=1}^n$ is linearly independent in $\mathcal{C}_K^\chi$, and suppose $\alpha_1\overline{a_1H} + \cdots + \alpha_n\overline{a_nH} = 0$, where $\alpha_i \in \mathbb{C}$ for all i. Then

$$\alpha_1 B_H^\chi(a_1H, yH) + \cdots + \alpha_n B_H^\chi(a_nH, yH) = 0 \qquad \forall y \in G.$$

Let $\{k_jH\}_{j=1}^m$ be a set of distinct left cosets of H in K, and let $x \in G$. Then we have

$$\alpha_1 B_H^\chi(a_1H, xk_1H) + \cdots + \alpha_n B_H^\chi(a_nH, xk_1H) = 0$$

$$\alpha_1 B_H^\chi(a_1H, xk_2H) + \cdots + \alpha_n B_H^\chi(a_nH, xk_2H) = 0$$

$$\cdot$$

$$\cdot$$

$$\cdot$$

$$\alpha_1 B_H^\chi(a_1H, xk_mH) + \cdots + \alpha_n B_H^\chi(a_nH, xk_mH) = 0.$$

Thus by summing the above system of equations, we have

$$\alpha_1 \sum_j B_H^\chi(a_1H, xk_jH) + \cdots + \alpha_n \sum_j B_H^\chi(a_nH, xk_jH) = 0,$$

implying by 3.2 that

$$\alpha_1 \frac{|K|}{|H|} B_K^\chi(a_1K, xK) + \cdots + \alpha_n \frac{|K|}{|H|} B_K^\chi(a_nK, xK) = 0.$$

Factoring $\frac{|K|}{|H|}$ from the left-hand side, we now have that

$$\alpha_1 B_K^\chi(a_1K, xK) + \cdots + \alpha_n B_K^\chi(a_nK, xK) = 0,$$

which in turn implies that

$$B_K^\chi(\alpha_1 a_1K + \cdots + \alpha_n a_nK, xK) = 0 \qquad \forall x \in G.$$

Hence

$$\alpha_1 \overline{a_1K} + \cdots + \alpha_n \overline{a_nK} = 0.$$

By assumption, the vectors $\{\overline{a_iK}\}_{i=1}^n$ are linearly independent, so $\alpha_1 = \cdots = \alpha_n = 0$. We conclude that $\{\overline{a_iH}\}_{i=1}^n$ is linearly independent in $\mathcal{C}_H^\chi$. $\square$

Theorem 3.7. *Let $H \leqslant K \leqslant G$, let $\chi \in Irr(G)$, and assume χ vanishes off K. Let $\{\overline{k_jH}\}_{j=1}^t$ ($k_j \in K$) be linearly independent in $\mathcal{C}_H^\chi$, and let $\{g_iK\}_{i=1}^s$ be a complete set of distinct left cosets of K in G. Then $\{\overline{g_ik_jH} \mid 1 \leq i \leq s,\ 1 \leq j \leq t\}$ is linearly independent in $\mathcal{C}_H^\chi$.*

Proof. First, assume aH and bH are left cosets of H in K such that $aH \subseteq cK$ and $bH \subseteq dK$, where cK and dK are distinct left cosets of K in G. Let $h \in H$. We claim that $a^{-1}bh \notin K$. Suppose otherwise. Since $a \in cK$ we get $aK = cK$. Similarly, $bK = dK$. Thus $aK \neq bK$. Then $a^{-1}bh \in K$ would give $a^{-1}b \in Kh^{-1} = K$, a contradiction. Thus $a^{-1}bh \notin K$.

Now, since $\{\overline{k_jH}\}_{j=1}^t$ is linearly independent in $\mathcal{C}_H^\chi$, so is the set of translates $\mathcal{T}_i := \{\overline{g_ik_1H}, \ldots, \overline{g_ik_tH}\}$ for each $1 \leq i \leq s$. For each i, $\langle \mathcal{T}_i \rangle$ is a subspace of $\mathcal{C}_H^\chi$.

Let $1 \leq m, n \leq s$ with $m \neq n$, and let $1 \leq p, q \leq t$. We have that $k_pH \subseteq K$ and $k_qH \subseteq K$, so it follows that $g_mk_pH \subseteq g_mK$ and $g_nk_qH \subseteq g_nK$. Since $g_mK \neq g_nK$, we get that $\overline{B_H^\chi}(\overline{g_mk_pH}, \overline{g_nk_qH}) = \dfrac{\chi(e)}{|H|} \displaystyle\sum_{h \in H} \chi((g_mk_p)^{-1}(g_nk_q)h) = 0$ by the first claim of the proof and the fact that χ vanishes off K.

Now put $\mathcal{T} = \cup_i \mathcal{T}_i$. From the above it follows that, for all $1 \leq j \leq s$, $\langle \mathcal{T}_j \rangle$ and $\displaystyle\sum_{i \neq j} \langle \mathcal{T}_i \rangle$ are orthogonal, whence $\langle \mathcal{T} \rangle = \displaystyle\dot{\sum_i} \langle \mathcal{T}_i \rangle$, and thus $\mathcal{T} = \{\overline{g_ik_jH} \mid 1 \leq i \leq s,\ 1 \leq j \leq t\}$ is a linearly independent set. $\square$

Theorem 3.8. *Let $H \leqslant K \leqslant G$, and let $\chi \in Irr(G)$, with $\psi = \chi|_K \in Irr(K)$. Then $\mathcal{C}_H^\psi$ and $\mathcal{C}_H^\chi$ are isometric.*

Proof. Let $\{\overline{a_iH}\}_{i=1}^n$ be a basis for $\mathcal{C}_H^\psi$. Since ψ and χ agree on H, the definition of coset space dimension gives that

$$\dim_{\mathbb{C}} \mathcal{C}_H^\psi = \frac{\psi(e)}{|H|}\sum_{h\in H}\psi(h) = \frac{\chi(e)}{|H|}\sum_{h\in H}\chi(h) = \dim_{\mathbb{C}} \mathcal{C}_H^\chi.$$

For each $1 \leq i \leq n$, $a_iH \in K/H \subseteq G/H$, so $\overline{a_iH} \in \mathcal{C}_H^\chi$. Thus there exists a unique and well-defined linear map $\varphi : \mathcal{C}_H^\psi \longrightarrow \mathcal{C}_H^\chi$ such that $\varphi(\overline{a_iH}) = \overline{a_iH}$ for each i. We now show that the inner products on the two spaces agree. Let $\overline{aH}, \overline{bH} \in \mathcal{C}_H^\psi$. Then

$$\overline{B_H^\psi}(\overline{aH},\overline{bH}) = \frac{\psi(e)}{|H|}\sum_{h\in H}\psi(b^{-1}ah) = \frac{\chi(e)}{|H|}\sum_{h\in H}\chi(b^{-1}ah) = \overline{B_H^\chi}(\overline{aH},\overline{bH}) = \overline{B_H^\chi}(\varphi(\overline{aH}),\varphi(\overline{bH})),$$

so by linearity of φ and the inner product, the formula holds when replacing standard vectors with arbitrary vectors. Now letting $v \in \ker\varphi$, we have that $\varphi(v) = 0$, so that $\overline{B_H^\chi}(\varphi(v),\varphi(v)) = 0$. Hence $\overline{B_H^\psi}(v,v) = 0$, implying that $v = 0$. Thus φ is injective, which proves that the two spaces are isometric. □

Example 3.9. Let $G = A_4$, let $H = \{e\}$, and let χ be the degree 3 irreducible character of A_4. Then $\dim \mathcal{C}_H^\chi = 9$. We outline the construction of a basis for $\mathcal{C}_H^\chi$. First put $K = \langle(12)(34)\rangle \leqslant A_4$, so that $\dim \mathcal{C}_K^\chi = 3$. Using 3.4, we obtain the basis $\{\overline{K}, \overline{(12)(34)K}, \overline{(13)(24)K}\}$ for $\mathcal{C}_K^\chi$. Now, by 3.6, replace K with H to get the linearly independent set $\{\overline{H}, \overline{(12)(34)H}, \overline{(13)(24)H}\}$ in $\mathcal{C}_H^\chi$. We have that χ vanishes off $V = \{(1),(12)(34),(13)(24),(14)(23)\}$. Using a set of left cosets of V in A_4, 3.7 gives for $\mathcal{C}_H^\chi$ the linearly independent set

$$\{\overline{H}, \overline{(12)(34)H}, \overline{(13)(24)H}, \overline{(123)H}, \overline{(134)H}, \overline{(243)H}, \overline{(132)H}, \overline{(234)H}, \overline{(124)H}\}.$$

These are 9 vectors, so they constitute a basis for $\mathcal{C}_H^\chi$. We find, however, that the Gram matrix corresponds to no Cartan matrix for a crystallographic root system.

Example 3.10. We have that $\{e\} \leqslant A_4 \leqslant S_4$, and also that the restriction of one of the degree 3 irreducible characters of S_4 is the degree 3 irreducible character of A_4. By 3.8, it follows that the basis acquired in the last example is also a basis for $\mathcal{C}_H^\chi$, where $G = S_4$, $H = \{e\}$, and where χ is the character formerly restricted. Hence we have found a coset space associated with a symmetric group such that the geometry of this coset space is not that of a crystallographic root system. The entries of the Gram matrix for the basis vectors in $\mathcal{C}_H^\chi$ cannot be adjusted to become the entries for a Cartan matrix. This raises the question, however, of precisely which of the Cartan integers we can obtain from the inner product on a coset space. The answer, as we will see shortly, points the way towards a more general connection between coset spaces and root systems. The next results make clear this connection.

Proposition 3.11. *Let $\mathcal{C}_H^\chi$ be a coset space and let $\overline{aH}, \overline{bH} \in \Sigma$. Then $|\overline{B_H^\chi}(\overline{aH}, \overline{bH})| \leq \dim_{\mathbb{C}} \mathcal{C}_H^\chi$.*

Proof. Let $\overline{aH}, \overline{bH} \in \Sigma$. By the Cauchy-Schwarz Inequality, we have that $|\overline{B_H^\chi}(\overline{aH}, \overline{bH})| \leq \|\overline{aH}\|\|\overline{bH}\|$. Note that equality occurs if and only if one vector is a scalar multiple of the other [Axl97]. Thus we infer that

$$\begin{aligned}
|\overline{B_H^\chi}(\overline{aH}, \overline{bH})| &\leq \sqrt{\overline{B_H^\chi}(\overline{aH}, \overline{aH})} \cdot \sqrt{\overline{B_H^\chi}(\overline{bH}, \overline{bH})} \\
&= \sqrt{\frac{\chi(e)}{|H|} \sum_{h \in H} \chi(h)} \cdot \sqrt{\frac{\chi(e)}{|H|} \sum_{h \in H} \chi(h)} \\
&= \sqrt{\dim_{\mathbb{C}} \mathcal{C}_H^\chi} \cdot \sqrt{\dim_{\mathbb{C}} \mathcal{C}_H^\chi} \\
&= \dim_{\mathbb{C}} \mathcal{C}_H^\chi.
\end{aligned}$$

□

Lemma 3.12. *Let $H \leqslant G$ and let $\chi \in Irr(G)$. Assume that $\mathcal{C}_H^\chi$ has a basis consisting of standard vectors. If this basis forms a base for a crystallographic root system, then for all distinct basis vectors $\overline{aH}$ and $\overline{bH}$, either $B_H^\chi(\overline{aH}, \overline{bH}) = 0$ or $|B_H^\chi(\overline{aH}, \overline{bH})| = \frac{1}{2} \dim_{\mathbb{C}} \mathcal{C}_H^\chi$.*

Proof. Assume that $\mathcal{C}_H^\chi$ has a basis consisting of standard vectors, and that this basis forms a base for a crystallographic root system. Let $\overline{aH}$ and $\overline{bH}$ be distinct basis vectors for $\mathcal{C}_H^\chi$. By 3.11, we have the strict inequality

$$|B_H^\chi(\overline{aH}, \overline{bH})| < \dim_{\mathbb{C}} \mathcal{C}_H^\chi.$$

Since the basis vectors for $\mathcal{C}_H^\chi$ form a base for a root system, we must have that $\langle \overline{aH}, \overline{bH} \rangle \in \mathbb{Z}$ for all $\overline{aH}, \overline{bH}$ in the basis. Thus, with $(\overline{aH}, \overline{bH}) := \overline{B_H^\chi}(\overline{aH}, \overline{bH})$, we have

$$\frac{2(\overline{aH}, \overline{bH})}{(\overline{bH}, \overline{bH})} \in \mathbb{Z}.$$

By 3.11, this gives

$$\frac{2(\overline{aH}, \overline{bH})}{\dim_{\mathbb{C}} \mathcal{C}_H^\chi} \in \mathbb{Z}.$$

Now put $|(\overline{aH}, \overline{bH})| = c$ and $\dim_{\mathbb{C}} \mathcal{C}_H^\chi = d$, so that $\dfrac{2c}{d} = k$ for some $k \in \mathbb{Z}^+ \cup \{0\}$. If $k = 0$, then $B_H^\chi(\overline{aH}, \overline{bH}) = c = 0$, so suppose $k > 0$.

By the inequality in 3.11, $c < d$, so we have $k < 2$, which forces $k = 1$. Therefore,

$$|B_H^\chi(\overline{aH}, \overline{bH})| = c = \frac{1}{2}d = \frac{1}{2}\dim_{\mathbb{C}} \mathcal{C}_H^\chi.$$

This completes the proof. $\square$

Theorem 3.13. *Let $H \leqslant G$ and let $\chi \in Irr(G)$. Assume that $\mathcal{C}_H^\chi$ has a basis consisting of standard vectors. If this basis forms a base for a crystallographic root system Φ, each irreducible component of which has rank at most two, then $\Phi = \prod_{i=1}^n R_i$, where $R_i = A_1$ or $R_i = A_2$.*

Proof. Let $\{\alpha_1, ..., \alpha_n\}$ be a basis for $\mathcal{C}_H^\chi$ consisting of standard vectors. By renumbering the α_i if necesssary, we may assume that the matrix $(\langle \alpha_i, \alpha_j \rangle)$ is block diagonal with each block the Cartan matrix of an irreducible root system. The assumption implies that each block is of size either 1×1 or 2×2. A block of size 1×1 corresponds to an irreducible component of type A_1. A block of size 2×2 corresponds to a rank two irreducible root system. If $\alpha = \alpha_i$ and $\beta = \alpha_{i+1}$ correspond to such a block, then $\langle \alpha, \beta \rangle \neq 0$, so the previous lemma implies that $|\langle \alpha, \beta \rangle| = 1$. In this case, we have $4\cos^2\theta = \langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 1$, where θ is the angle between α and β. Hence $\theta = 2\pi/3$, and so the block corresponds to an irreducible root system of type A_2. This proves the theorem. $\square$

We now see that coset spaces give rise to a very restricted class of root systems with irreducible components of rank 1 or 2. When we obtain such a root system, either we get the trivial geometry of A_1, or we get the hexagonal geometry of A_2. It is notable that these root systems are the only simply laced rank 1 and 2 root systems: They are the only rank 1 and 2 root systems all of whose vectors have equal length. We conjecture that the standard vectors in a coset space form a root system isomorphic to a given irreducible root system if and only if the irreducible root system is simply laced. In our concluding remarks, we will mention the progress that has been made on this front.

Chapter 4

The Geometry of Coset Spaces For Certain Dihedral Groups

It was proven in [HT92] that each orbital subspace of a symmetrized tensor space for a dihedral group has an orthogonal basis consisting of standard symmetrized tensors if and only if the dihedral group has order a power of 2. Here we present a more conceptual proof of this result, couched in the terminology of the coset space and its standard vectors. We hope that reliance on the natural geometry of the dihedral group, acting as a group of symmetries in the plane, will lead to further insight into the nature of o-basis groups. Before presenting the new proof, we will require some rather lengthy preliminaries.

Proposition 4.1. *Let V be a real inner product space and let C be a cyclic group of order n that acts on V in such a way that $v \mapsto cv$ is a linear map $V \to V$ for each $c \in C$ and $(cv, cw) = (v, w)$ for each $c \in C$ and $v, w \in V$. Fix $v \in V$ and let $Cv = \{cv : c \in C\}$ be the orbit of v under the action of C. Then Cv is the set of vertices of a regular m-gon for some divisor m of n.*

Proof. We have $C = \langle a \rangle$ for some $a \in C$. The stabilizer C_v of v is a subgroup of C. Let $|C_v| = p$, and set $m := n/p = |C : C_v| = |Cv|$. Now, since C is cyclic, so is C_v, and we have that $C_v = \langle a^m \rangle$. Thus C_v has m distinct cosets, and $\{a^0, a^1, ..., a^{m-1}\}$ is a set of representatives for these cosets. Hence $Cv = \{a^0 v, a^1 v, ..., a^{m-1} v\}$.

Now put $v_i = a^i v$, where $0 \leq i \leq m - 1$. Let e_i be the directed line segment joining v_i and v_{i+1}. Then $e_i = v_{i+1} - v_i$, so

$$\|e_i\|^2 = \|v_{i+1} - v_i\|^2$$

$$= \langle v_{i+1} - v_i, v_{i+1} - v_i \rangle$$

$$= \langle v_{i+1}, v_{i+1} \rangle - \langle v_i, v_{i+1} \rangle - \langle v_{i+1}, v_i \rangle + \langle v_i, v_i \rangle$$

$$= \langle a^{i+1} v, a^{i+1} v \rangle - \langle a^i v, a^{i+1} v \rangle - \langle a^{i+1} v, a^i v \rangle + \langle a^i v, a^i v \rangle$$

$$= \langle v, v \rangle - \langle v, av \rangle - \langle av, v \rangle + \langle v, v \rangle$$

$$= 2\langle v, v \rangle - 2\langle av, v \rangle.$$

Since i was arbitrary, we must have that $\|e_i\|^2 = \|e_j\|^2$ for all $0 \leq i, j \leq m-1$, and hence that $\|e_i\| = \|e_j\|$ for all $0 \leq i, j \leq m-1$.

Now let θ_i be the angle between e_i and e_{i+1}. Since $e_{i+1} = v_{i+1} - v_i$ and $e_i = v_i - v_{i-1}$, we have

$$\langle e_i, e_{i+1} \rangle = \langle v_i - v_{i-1}, v_{i+1} - v_i \rangle$$

$$= \langle v_i, v_{i+1} \rangle - \langle v_{i-1}, v_{i+1} \rangle - \langle v_i, v_i \rangle + \langle v_{i-1}, v_i \rangle$$

$$= \langle a^i v, a^{i+1} v \rangle - \langle a^{i-1} v, a^{i+1} v \rangle - \langle a^i v, a^i v \rangle + \langle a^{i-1} v, a^i v \rangle$$

$$= \langle v, av \rangle - \langle v, a^2 v \rangle - \langle v, v \rangle + \langle v, av \rangle$$

$$= 2\langle v, av \rangle - \langle v a^2 v \rangle - \langle v, v \rangle$$

Thus the inner product is independent of index. Since $\|e_i\| = \|e_j\|$ for all $0 \leq i, j \leq m-1$, it follows that

$$\theta_i = \cos^{-1} \frac{\langle e_i, e_{i+1} \rangle}{\|e_i\| \|e_{i+1}\|}$$

is also independent of index, whence $\theta_i = \theta_j$ for all $0 \leq i, j \leq m-1$. This proves the proposition.

□

Let D_{2n} be the dihedral group of order $2n$. Fix $1 \leq j < n/2$ and let C_n be the cyclic subgroup of all rotations in D_{2n}. The map $\rho : C_n \longrightarrow GL_2(\mathbb{C})$ given by

$$\rho(a^k) = \begin{bmatrix} \cos(2\pi jk/n) & -\sin(2\pi jk/n) \\ \sin(2\pi jk/n) & \cos(2\pi jk/n) \end{bmatrix}$$

is a well-defined homomorphism and hence a representation of C_n. We use this representation to view $\mathbb{C}^2$ as a $\mathbb{C}C_n$-module.

Fix $H \leqslant D_{2n}$ and put $\eta = 2(|H \cap C_n|/|H|)^{1/2}$. For a positive integer m, define

$$B_m = \left\{ \left[\eta \cos \frac{2\pi k}{m}, \eta \sin \frac{2\pi k}{m} \right]^T \mid k \in \mathbb{Z} \right\}$$

viewed as a subset of the inner product space $\mathbb{C}^2$. Then B_m is a regular m-gon. According to 4.1, the action of C_n on $v_0 := (\eta, 0) \in \mathbb{C}^2$ also produces a regular n-gon. Indeed, using the representation given above, we have that

$$C_n v_0 = \left\{ \left[\eta \cos \frac{2\pi jl}{n}, \eta \sin \frac{2\pi jl}{n} \right]^T \mid l \in \mathbb{Z} \right\}$$

Put $n' = n/\gcd(n, j)$.

Lemma 4.2. $B_{n'} = C_n v_0$

Proof. Let $v \in B_{n'}$. Then $v = \left[\eta \cos \frac{2\pi k}{n'}, \eta \sin \frac{2\pi k}{n'} \right]^T$ for some $k \in \mathbb{Z}$. Define $j = j/\gcd(n, j)$. Then $\gcd(n', j') = 1$, so there exists $x, y \in \mathbb{Z}$ such that $xn' + yj' = 1$. We have that

$$\frac{2\pi k}{n'} = \frac{2\pi k \cdot 1}{n'} = \frac{2\pi k(xn' + yj')}{n'} = 2\pi k(x + y\frac{j'}{n'}) = 2\pi k(x + y\frac{j}{n}) = 2\pi kx + \frac{2\pi jky}{n} = \frac{2\pi jky}{n}$$

Since $ky \in \mathbb{Z}$, $v \in C_n v_0$.

Now let $v \in C_n v_0$. Then $v = \left[\eta \cos \frac{2\pi jl}{n}, \eta \sin \frac{2\pi jl}{m}\right]^T$ for some $l \in \mathbb{Z}$. Divide j and n by $\gcd(n, j)$ to get

$$\frac{2\pi jl}{n} = \frac{2\pi j'l}{n'}$$

Since $j'l \in \mathbb{Z}$, we have $v \in B_{n'}$. $\square$

Now put $\chi = \chi_j$ and $E = \{\overline{cH} \mid c \in C_n\} \subseteq \mathcal{C}_H^{\chi}$.

Theorem 4.3. *If $(\chi, 1)_H \neq 0$, then there exists an isometric embedding $B_{n'} \longrightarrow E$.*

Proof. Assume that $(\chi, 1)_H \neq 0$. Define $f : B_{n'} \longrightarrow E$ by $f(cv_0) = \overline{cH}$. We claim that f is well-defined and injective. First, let $x, y \in B_{n'}$ with $x = y$. Then $x = \left[\eta \cos \frac{2\pi p}{n'}, \eta \sin \frac{2\pi p}{n'}\right]^T$ and $y = \left[\eta \cos \frac{2\pi q}{n'}, \eta \sin \frac{2\pi q}{n'}\right]^T$, where $p, q \in \mathbb{Z}$ and $p \equiv q \mod n'$. Let $g = \gcd(n, j)$. Then $p \cdot g \equiv q \cdot g \mod n' \cdot g$, implying that $p \cdot g \equiv q \cdot g \mod n$. Thus

$$\left[\eta \cos \frac{2\pi jpg}{n}, \eta \sin \frac{2\pi jpg}{n}\right]^T = \left[\eta \cos \frac{2\pi jqg}{n}, \eta \sin \frac{2\pi jqg}{n}\right]^T.$$

This establishes a well-defined correspondence between $B_{n'}$ and $C_n v_0$. Since $C_n v_0$ is an orbit, we conclude that f is well-defined.

Now let $\alpha c v_0 \in \mathbb{C}B_{n'}$, and suppose $f(\alpha c v_0) = 0$. Then

$$\begin{aligned}
& \overline{\alpha c H} = 0 \\
\implies\ & \alpha c H + \ker B_H^{\chi} = \ker B_H^{\chi} \\
\implies\ & \alpha c H \in \ker B_H^{\chi} \\
\implies\ & \alpha B_H^{\chi}(cH, dH) = 0 \quad \text{for all } d \in C_n \\
\implies\ & \alpha B_H^{\chi}(cH, H) = 0 \\
\implies\ & \alpha \frac{\chi(e)}{|H|} \sum_{h \in H} \chi(ch) = 0 \\
\implies\ & \alpha \frac{\chi(e)}{|H|} \left[\sum_{h \in H \cap C_n} \chi(ch) + \sum_{h \in H \cap {C_n}^c} \chi(ch) \right] = 0 \\
\implies\ & \alpha \frac{\chi(e)}{|H|} \sum_{h \in H \cap C_n} \chi(ch) = 0 \\
\implies\ & \alpha \cdot \frac{\chi(e)}{|H|} \cdot |H \cap C_n| \cdot \chi(e) = 0 \\
\implies\ & 4\alpha \cdot \frac{|H \cap C_n|}{|H|} = 0 \\
\implies\ & \alpha = 0
\end{aligned}$$

where we have used the fact that χ vanishes on reflections, and the fact that $C_n \subseteq \ker \chi$ [Isa94]. Hence f is injective.

We note that $f(cx) = cf(x)$ for each $c \in C_n$ and $x \in B_{n'}$. Using this, and the fact that both forms are C_n-invariant, we claim that it is enough to check that $(cv_0, v_0) = \overline{B_H^{\chi}}(\overline{cH}, \overline{H})$ in order to establish the isometry. Let $x, y \in B_{n'}$. Then $x = \alpha a v_0$ and $y = \beta b v_0$ for some

$\alpha, \beta \in \mathbb{C}$ and $a, b \in C_n$. Provided the above equality of inner products holds,

$$\begin{aligned}
(x, y) = (\alpha a v_0, \beta b v_0) &= \alpha\overline{\beta}(a v_0, b v_0) \\
&= \alpha\overline{\beta}(b^{-1} a v_0, v_0) \\
&= \alpha\overline{\beta}\,\overline{B_H^{\chi}}(\overline{b^{-1} a H}, \overline{H}) \\
&= \alpha\overline{\beta}\,\overline{B_H^{\chi}}(f(b^{-1} a v_0), f(v_0)) \\
&= \alpha\overline{\beta}\,\overline{B_H^{\chi}}(b^{-1} f(a v_0), f(v_0)) \\
&= \alpha\overline{\beta}\,\overline{B_H^{\chi}}(f(a v_0), b f(v_0)) \\
&= \alpha\overline{\beta}\,\overline{B_H^{\chi}}(f(a v_0), f(b v_0)) \\
&= \overline{B_H^{\chi}}(\alpha f(a v_0), \beta f(b v_0)) \\
&= \overline{B_H^{\chi}}(f(\alpha a v_0), f(\beta b v_0)) \\
&= \overline{B_H^{\chi}}(f(x), f(y)).
\end{aligned}$$

Now let $c \in C_n$, noting that $c = a^k$ for some $k \in \{0, ..., n-1\}$. We have that

$$\begin{aligned}
(cv_0, v_0) &= \left(\left[\eta \cos \frac{2\pi jk}{n}, \eta \sin \frac{2\pi jk}{n} \right], [\eta, 0]^T \right) \\
&= \eta^2 \cos \frac{2\pi jk}{n} \\
&= \frac{4|H \cap C_n|}{|H|} \cdot \cos \frac{2\pi jk}{n} \\
&= \frac{\chi(e)}{|H|} \cdot |H \cap C_n| \cdot 2 \cos \frac{2\pi jk}{n} \\
&= \frac{\chi(e)}{|H|} \cdot |H \cap C_n| \cdot \chi(c) \\
&= \frac{\chi(e)}{|H|} \sum_{h \in H \cap C_n} \chi(ch) \\
&= \frac{\chi(e)}{|H|} \sum_{h \in H \cap C_n} \chi(ch) + \frac{\chi(e)}{|H|} \sum_{h \in H \cap {C_n}^c} \chi(ch) \\
&= \frac{\chi(e)}{|H|} \sum_{h \in H} \chi(ch) \\
&= \overline{B_H^\chi}(\overline{cH}, \overline{H})
\end{aligned}$$

Here we have again used that $C_n \subseteq \ker \chi$, and that χ vanishes off C_n. That f is an isometry now follows.

Lemma 4.4. *Let G be a dihedral group, and let χ be an irreducible character of D_{2n} of degree 2. If $\mathcal{C}_{\{e\}}^\chi$ has an o-basis, then n is even.*

Proof. Since $\dim_{\mathbb{C}} \mathcal{C}_{\{e\}}^\chi = 4$, we have a basis $\{\ \overline{b^{l_1} a^{i_1}\{e\}},\ \overline{b^{l_2} a^{i_2}\{e\}},\ \overline{b^{l_3} a^{i_3}\{e\}},\ \overline{b^{l_4} a^{i_4}\{e\}}\ \}$ for the space, where $0 \leq l_j \leq 1$ and $0 \leq i_j \leq n$. The powers of b cannot all be distinct, so we may assume that $l_1 = l = l_2$. Then we have two basis vectors of the form $\overline{b^l a^{i_1}\{e\}}$ and $\overline{b^l a^{i_2}\{e\}}$, where we may assume $i_1 > i_2$. Left multiplying each of these two basis vectors by $a^{-i_2} b^l$, we obtain $\overline{a^{i_1 - i_2}\{e\}}$ and $\overline{\{e\}}$. Put $k = i_1 - i_2$. Then $\overline{\{e\}}$ and $\overline{a^k\{e\}}$ are a pair of orthogonal elements of E. By the isometry in 4.3, we have a pair of orthogonal elements in $B_{n'}$ as well.

Thus there exists in G a rotation through an angle of $\pi/2$, implying that $4|n$ and hence that n is even. $\square$

We are now ready to prove the main theorem.

Theorem 4.5. *The following are equivalent:*

(i) *D_{2n} has order a power of 2,*

(ii) *for each $H \leqslant D_{2n}$, $\chi \in Irr(D_{2n})$, $\mathcal{C}_H^\chi$ has an o-basis.*

(i $\Rightarrow$ ii) Assume G is a dihedral group with order a power of 2. Fix $H \leqslant G$ and $\chi \in Irr(G)$. If $(\chi, 1)_H = 0$, then $\dim \mathcal{C}_H^\chi = 0$, so the conclusion follows. Suppose $(\chi, 1)_H \neq 0$. If χ has degree 1, then

$$\dim_{\mathbb{C}} \mathcal{C}_H^\chi = \frac{\chi(e)}{|H|} \sum_{h \in H} \chi(h) = \frac{1}{|H|} \sum_{h \in H} \chi(h) \leq \frac{1}{|H|} \sum_{h \in H} \chi(e) = \frac{1}{|H|} \cdot |H| = 1$$

so again the conclusion follows.

Now put $\chi = \chi_j$ for some $1 \leq j < n/2$ and assume $(\chi, 1)_H \neq 0$. Since n' is a power of 2 and $n' > 2$, we have that $4|n'$. Hence $B_{n'}$ has a pair of orthogonal elements, and it follows from 4.3 that E does as well. Assume that $H \nsubseteq C_n$. Put $A = H \cap C_n = \{a^{i_1}, ..., a^{i_r}\}$, where $1 \leq i_j \leq n-1$, and put $B = H \cap {C_n}^c = \{a^{i_1}b, ..., a^{i_r}b\}$, where b is the generating reflection for the dihedral group. Noting that $|A| = |B|$, and that $H = A \sqcup B$, we have that

$$\dim_{\mathbb{C}} \mathcal{C}_H^\chi = \frac{\chi(e)}{|H|} \sum_{h \in H} \chi(h) = \frac{\chi(e)}{|H|} \sum_{h \in A} \chi(h) = \frac{\chi(e)}{|H|} \cdot \frac{|H|}{2} \cdot \chi(e) = 2,$$

again using that $C_n \subseteq \ker \chi$. Hence the two orthogonal elements of E form an o-basis for $\mathcal{C}_H^\chi$.

Assume now that $H \subseteq C_n$. Let $\overline{cH}$ and $\overline{dH}$ be the pair of orthogonal elements guaranteed by 4.3. Then $\{\overline{cH}, \overline{dH}\}$ is linearly independent in $\mathcal{C}_H^\chi$. A complete set of distinct cosets of C_n in G is $\{C_n, bC_n\}$, where b is the generating reflection. By 3.7, $\{\overline{cH}, \overline{dH}, \overline{bcH}, \overline{bdH}\}$ is linearly independent in $\mathcal{C}_H^\chi$. Since $\overline{cH}$ and $\overline{dH}$ are orthogonal, the G-invariance of $\overline{B_H^\chi}$ gives us that $\overline{bcH}$ and $\overline{bdH}$ are orthogonal as well. Consider $\overline{cH}$ and $\overline{bcH}$. We have that

$$\overline{B_H^\chi}(\overline{cH}, \overline{bcH}) = \frac{\chi(e)}{|H|} \sum_{h \in H} \chi(c(bc)^{-1}h) = \frac{\chi(e)}{|H|} \sum_{h \in H} \chi(cc^{-1}b^{-1}h) = \frac{\chi(e)}{|H|} \sum_{h \in H} \chi(bh) = 0,$$

since b is a reflection. Now consider $\overline{cH}$ and $\overline{bdH}$. We have

$$\overline{B_H^\chi}(\overline{cH}, \overline{bdH}) = \frac{\chi(e)}{|H|} \sum_{h \in H} \chi(c(bd)^{-1}h) = \frac{\chi(e)}{|H|} \sum_{h \in H} \chi(cd^{-1}bh) = \frac{\chi(e)}{|H|} \sum_{h \in H} \chi(cd^{-1}h^{-1}b) = 0,$$

since $cd^{-1}h^{-1}$ is a rotation and hence $cd^{-1}h^{-1}b$ is a reflection. Similar arguments show that $\overline{dH}$ is orthogonal to both $\overline{bcH}$ and $\overline{bdH}$, whence the set of all four vectors are pairwise orthogonal. Finally, since $H \subseteq \ker \chi$, we have

$$\dim_{\mathbb{C}} \mathcal{C}_H^\chi = \frac{\chi(e)}{|H|} \sum_{h \in H} \chi(h) = \frac{\chi(e)}{|H|} \cdot |H| \cdot \chi(e) = 4$$

Therefore, $\mathcal{C}_H^\chi$ has an o-basis.

(ii $\Rightarrow$ i) Let $G = D_{2n}$. Assume $\mathcal{C}_H^\chi$ has an o-basis for each $H \leqslant G$ and $\chi \in Irr(G)$. Let $\chi = \chi_1$ and let $H = \{e\}$. Then $n = n'$. Hence for each $\overline{cH} \in E$, there exists $c \in C_n$ such that $f(cv_0) = \overline{cH}$, so that f in Theorem 4.3 is surjective and thus bijective. Thus there exists a pair of orthogonal elements in E and hence in B_n, implying by Lemma 4.4 that n is even. Thus, if $n = 3$, then $\mathcal{C}_H^\chi$ does not have an o-basis for each (H, χ)-pair. If $n = 4$, then $|G| = 8 = 2^3$.

We now proceed by induction on n. Fix $n > 4$. Since n is even, we have that $a^{n/2} \in C_n$.

Let $\pi : G \longrightarrow G/\langle a^{n/2}\rangle$ be the canonical epimorphism, and put $\widehat{G} = G/\langle a^{n/2}\rangle$. Since the quotient of a dihedral group is dihedral, we have that $\widehat{G}$ is a dihedral group of order n. Now fix $\widehat{H} \leqslant \widehat{G}$ and $\widehat{\chi} \in \mathrm{Irr}(\widehat{G})$. Then $\widehat{H} = \pi(H)$ for some $H \leqslant G$ and $\widehat{\chi} = \chi \circ \pi$ for some $\chi \in \mathrm{Irr}(G)$. We have that $\langle a^{n/2}\rangle \subseteq \ker \chi$, so it follows by 1.40 that $\mathcal{C}_H^{\chi}$ and $\mathcal{C}_{\widehat{H}}^{\widehat{\chi}}$ are isometric. By assumption, $\mathcal{C}_H^{\chi}$ has an o-basis for each (H, χ)-pair. Thus $\mathcal{C}_{\widehat{H}}^{\widehat{\chi}}$ has an o-basis for each $(\widehat{H}, \widehat{\chi})$-pair. By the induction hypothesis, $|\widehat{G}| = n$ is a power of 2, whence $|G| = 2|\widehat{G}| = 2n$ is a power of 2 as well. The proof is complete.

$\square$

Chapter 5

Root Systems For A Special Coset Space

We now extend our results on root systems to include a special case. Earlier, we looked at symmetric groups and their associated coset spaces, and we drew conclusions about the type of rank 1 and 2 crystallographic root systems embedded in the geometry of these spaces. If we consider symmetric groups S_m with m arbitrarily large, it turns out that we need not restrict ourselves to rank 1 and 2. By fixing a particular subgroup and irreducible character of S_m, we always obtain the root system A_{m-1}. This result agrees with a conclusion found in [TS12, Theorem 14]. In that paper, however, the authors take a combinatorial approach, drawing upon substantial graph-theoretic preliminaries. Our proof is much more geometrical.

Put $G = S_m$. The inner product space $\mathbb{C}^m = \{a = (a_1, a_2, \ldots, a_m) \mid a_i \in \mathbb{C}\}$ is a $\mathbb{C}G$-module with action given by $\sigma a = (a_{\sigma^{-1}(1)}, a_{\sigma^{-1}(2)}, \ldots, a_{\sigma^{1}(m)})$. This $\mathbb{C}G$-module is an internal direct sum $\mathbb{C}^m = T \dot{+} V$, where $T = \mathbb{C}(1, 1, \ldots, 1)$ and $V = \{a \in \mathbb{C}^m \mid \sum_i a_i = 0\}$.

The $\mathbb{C}G$-module T affords the trivial character 1. Let ψ and ν denote the characters of G afforded by $\mathbb{C}^m$ and V, respectively. By the preceding paragraph, we have $\psi = \nu + 1$.

Denote by $\mathbb{C}_\varepsilon$ the vector space $\mathbb{C}$ on which G acts according to the formula $\sigma x = \varepsilon(\sigma)x$, where ε is the sign character of G. Then $\mathbb{C}_\varepsilon$ is a $\mathbb{C}G$-module affording the character ε.

The $\mathbb{C}G$-module $V_\varepsilon = \mathbb{C}_\varepsilon \otimes V$ affords the character $\chi = \varepsilon\nu$. We identify the vector space V_ε with V using the map $x \otimes v \mapsto xv$. With this identification we define the action of G on V_ε by the formula $\sigma a = \varepsilon(\sigma)(a_{\sigma^{-1}(1)}, a_{\sigma^{-1}(2)}, \ldots, a_{\sigma^{1}(m)})$.

Given a partition α of m, recall that there exists a bijective correspondence $\alpha \mapsto \chi_\alpha$ from the set of partitions of m to the set $\mathrm{Irr}(S_m)$ of irreducible characters of S_m.

For each $1 \leq i \leq m$, let $e_i \in \mathbb{C}^m$ be the m-tuple with jth entry δ_{ij}. The $\mathbb{R}$-span of the set $\Phi = \kappa\{e_i - e_j \,|\, i \neq j\} \subset V_\varepsilon$ is a root system of type A_{m-1} for $\kappa \in \mathbb{R}$. Put $a_\kappa = \kappa(1, -1, 0, \ldots, 0) \in \Phi$.

Fix $H \leqslant G$ and $\chi \in \mathrm{Irr}(G)$. Again, let Σ be the standard vectors of $\mathcal{C}_H^\chi$.

Theorem 5.1. *Assume that $H = \langle(12)\rangle$, and $\chi = \chi_{[2,1^{m-2}]}$. There exists a $\mathbb{C}G$-isomorphism $\bar{\varphi} : \mathcal{C}_H^\chi \to V_\varepsilon$ satisfying the following:*

(i) *$\bar{\varphi}$ is an isometry,*

(ii) *$\bar{\varphi}(\overline{\sigma H}) = \sigma a_\kappa$ for all $\sigma \in G$,*

(iii) $\Phi = \begin{cases} \bar{\varphi}(\Sigma \dot{\cup} -\Sigma) & \textit{if } m = 3, \\ \bar{\varphi}(\Sigma) & \textit{if } m \geq 4. \end{cases}$

The proof will require several lemmas. After inducing the map $\bar{\varphi}$ in the statement of the theorem, we will give the proof of the theorem.

Consider $v_i \in \mathbb{C}^{m-1}$, the $(m-1)$-tuple with j-th entry δ_{ij}. We identify v_i with $e_i \in \mathbb{C}^m$, the m-tuple with the same entries but with a fixed m-th entry of zero. Put $v = v_1 + \cdots + v_{m-1}$. Putting $L = \mathbb{C}v$, we have that L is a submodule of $\mathbb{C}^m|_H$, where we define H as the subgroup of S_m isomorphic to S_{m-1} obtained by fixing m. Throughout, we will identify H and S_{m-1} for the sake of simplicity.

Lemma 5.2. $\mathbb{C}^m \cong L^{S_m}$

Proof. Using 5.2, we must show that $\mathbb{C}^m = \dot{\sum_{a \in A}} aL$, where A is a set of representatives for the left cosets of H in G. The $\mathbb{C}G$-isomorphism then follows.

We claim that a complete set of left coset representatives for H in G is $A = \{(1,m),(2,m),...,(m-1,m),(m,m) = e\}$. Suppose $(i,m)H = (j,m)H$ for some $i \neq j$, with $1 \leq i,j < m$. Note that $m \notin \sigma$ for any $\sigma \in H$, and also that $(i,j) \in H$. If $(j,m)(i,m)H = H$, then $(j,m)(i,m)(i,j) \in H$, in which case $(i,m) \in H$, a contradiction. Since $|G : H| = m$, A is a set of distinct coset representatives of H in G.

We have that $aL = \mathbb{C}av$ for each $a \in A$. Label the elements of A as $a_1 = (1,m)$, $a_2 = (2,m),...,a_m = (m,m)$. We must show that $a_iL \bigcap(\sum_{j \neq i} a_jL) = \{0\}$ for each $1 \leq i \leq m$. To show this, it suffices to prove that the statement holds for $i = 1$, since the other cases are handled similarly.

Let $w \in a_1L \bigcap(\sum_{i>1} a_iL)$. Since $w \in a_1L$,

$$(1) \quad w = \alpha_1 a_1 v = \alpha_1 v_2 + \alpha_1 v_3 + \cdots + \alpha_1 v_m$$

for some $\alpha_1 \in \mathbb{C}$. On the other hand, $w \in \sum_{i>1} a_iL$, so $w = \alpha_2 a_2 v + \alpha_3 a_3 v + \cdots + \alpha_m a_m v$ where $\alpha_i \in \mathbb{C}$ for $2 \leq i \leq m$. Now, v_1 occurs as a summand in the vectors a_2v through a_mv, so the combined coefficient of v_1 as a summand of w is $\alpha_2 + \cdots + \alpha_m$. Equating this coefficient of v_1 with its coefficient in (1), we have

$$(2) \quad \alpha_2 + \cdots + \alpha_m = 0.$$

For each v_i with $i > 1$, v_i fails to occur as a summand in $\alpha_i a_i v$. Hence the combined coefficient of v_i in the expression of $w \in \sum_{i>1} a_iL$ is $\alpha_2 + \cdots + \alpha_{i-1} + \alpha_{i+1} + \cdots + \alpha_m$. Equating this coefficient with the coefficient of v_i in (1) yields

$$(3) \quad \alpha_2 + \cdots + \alpha_{i-1} + \alpha_{i+1} + \cdots + \alpha_m = \alpha_1.$$

Now, taking (2) with each equation of the form (3) corresponding to an $i > 1$, we have a system of m equations on whose left side each α_i occurs $m-1$ times. Summing both sides of this system, we have $(m-1)\sum_{i>1} \alpha_i = (m-1)\alpha_1$, so that $\sum_{i>1} \alpha_i = \alpha_1$. But by (2), this left-hand sum is zero, implying that $\alpha_1 = 0$, and hence that $w = 0$.

Thus $a_i L \bigcap (\sum_{j \neq i} a_j L) = \{0\}$, as claimed. Since $\sum_i a_i L$ is the sum of m submodules, we must have $\mathbb{C}^m = \dot{\sum_i} a_i L$. The isomorphism follows.

□

We continue to view S_{m-1} as a subgroup of S_m.

Lemma 5.3. $\psi = (1_{S_{m-1}})^{S_m}$

Proof. The module $L = \mathbb{C}v$ is one-dimensional, and clearly stable under the action of S_{m-1}, so it affords $1_{S_{m-1}}$. Since $\mathbb{C}^m$ affords ψ, the above isomorphism gives the result. □

Lemma 5.4. *The character ν is irreducible.*

Proof. We have that ψ and ν are the characters of $G = S_m$ afforded by $\mathbb{C}^m$ and V, respectively, and that $\psi = \nu + 1$. Now ν is irreducible if and only if $(\nu, \nu) = (\psi - 1, \psi - 1) = 1$. Put $\lambda = 1_H$, where again $H = S_{m-1}$, so that $\psi = \lambda^G$. Then by using Frobenius Reciprocity twice, we have

$$\begin{aligned}(\lambda^G - 1_G,\ \lambda^G - 1_G) &= (\lambda^G,\ \lambda^G) - 2(\lambda^G,\ 1_G) + (1_G,\ 1_G)\\ &= (\lambda,\ (\lambda^G)_H) - 2(\lambda,\ (1_G)_H) + 1\\ &= (\lambda,\ (\lambda^G)_H) - 2(\lambda,\ \lambda) + 1\\ &= (\lambda,\ (\lambda^G)_H) - 1\end{aligned}$$

It remains to determine that the value of $(\lambda,\ (\lambda^G)_H) = 2$. By Mackey's Subgroup Theorem,

$$(\lambda^G)_H \cong \bigoplus_{a\in A}(\ ({}^a\lambda)_{{}^aH\cap H}\ )^H,$$

where A is a set of representatives for the (H, H)-double cosets in G. Let $A = \{e, (m-1, m)\}$. Then HeH and $H(m-1, m)H$ are (H, H)-double cosets of H in G, and since $(m-1, m) \in H(m-1, m)H$ but $(m-1, m) \notin H$, these cosets are distinct.

Let $K \leqslant H = S_{m-1}$, with $K \cong S_{m-2}$, and now identify K with the subgroup of S_{m-1} fixing $m-1$. With $a = (m-1, m)$, we claim that ${}^aH \cap H = K$. Let let $k \in K$. Then

$$k = ek = (m-1, m)(m-1, m)k = (m-1, m)k(m-1, m) \in {}^aH,$$

and we have ${}^aH \cap H \supseteq K$. Now let $aha^{-1} \in {}^aH \cap H$. Since ${}^aH \cap H \subseteq H$, we have $m \notin aha^{-1}$. Assume $m-1 \in aha^{-1}$. Since conjugation applies a to each entry of h, this implies that $m \in H$, a contradiction. Hence ${}^aH \cap H \subseteq K$, and the claim follows.

Since $K \cong S_{m-2}$, we have $|{}^aH \cap H| = |K| = (m-2)!$, and thus $|H : {}^aH \cap H| = m-1$. Using 1.21, and incorporating the case $a = e$, we now have that

$$|S_m| = m! = (m-1)! + (m-1)!(m-1) = |H| + |H||H :\ {}^aH \cap H| = |H| + |H(m-1, m)H|.$$

By reasons of order, we must in fact have a complete set of (H, H)-double cosets in G. Thus Mackey's Subgroup Theorem gives

$$(\lambda^G)_H = (\lambda_H)^H + ({}^a\lambda_K)^H = \lambda + ({}^a\lambda_K)^H.$$

Then

$$(\lambda, (\lambda^G)_H) = (\lambda, \lambda + ({}^a\lambda_K)^H) = (\lambda, \lambda) + (\lambda, ({}^a\lambda_K)^H).$$

By Frobenius Reciprocity, we have

$$(\lambda, ({}^a\lambda_K)^H) = (\lambda_K, {}^a\lambda_K)$$

Since λ_K is trivial, the definition of the conjugate character yields that ${}^a\lambda_K({}^ak) = 1$ for all ${}^ak \in {}^aK$, so that ${}^a\lambda_K$ is trivial as well. Thus $(\lambda_K, {}^a\lambda_K) = 1$, and we conclude that

$$\begin{aligned}
(\nu, \nu) &= (\psi - 1, \psi - 1) \\
&= (\lambda^G - 1_G, \lambda^G - 1_G) \\
&= (\lambda,\ (\lambda^G)_H) - 1 \\
&= (\lambda, \lambda) + (\lambda, ({}^a\lambda_K)^H) - 1 \\
&= (\lambda, \lambda) + (\lambda_K, {}^a\lambda_K) - 1 \\
&= 2 - 1 = 1,
\end{aligned}$$

whence ν is irreducible. $\square$

Lemma 5.5. *The $\mathbb{C}G$-module V_ε affords the character $\chi = \chi_{[2,1^{m-2}]}$.*

Proof. We have that $\nu + 1 = \psi = (1_{S_{[m-1,1]}})^{S_m}$, after identifying the partition $[m-1, 1]$ of m with the partition $[m-1]$ of $m-1$. The only partitions of m that majorize $[m-1, m]$ must have $m-1$ or m as the first entry, giving $[m-1, 1]$ and $[m]$. Thus by 1.29, we have that $(1_{S_{[m-1,1]}})^{S_m} = c\chi_{[m-1,1]} + d\chi_{[m]}$ for some positive integers c and d. Since $\psi = (1_{S_{[m-1,1]}})^{S_m}$, and $1_{S_m} = \chi_{[m]}$, we must have $\nu = \chi_{[m-1,1]}$.

Now, the conjugate partition of $\alpha = [m-1, 1]$ is the partition α' with i-th component α'_i equal to the number of indices j for which $\alpha_j \geq i$. Hence $\alpha' = [2, 1^{m-2}]$. By 1.31, we have that $\chi_{[2,1^{m-2}]} = \varepsilon_m \chi_{[m-1,1]}$, where ε_m is the sign character of S_m. Since $\mathbb{C}_\varepsilon$ affords ε_m and V affords $\chi_{[m-1,1]}$, we infer that $\chi_{[2,1^{m-2}]}$ is the character afforded by $V_\varepsilon = \mathbb{C}_\varepsilon \otimes V$. $\square$

Since H fixes a_κ, we get a well-defined linear map $\varphi : \mathbb{C}(G/H) \to V_\varepsilon$ satisfying $\varphi(\sigma H) = \sigma a_\kappa$ $(\sigma \in G)$. This map is a $\mathbb{C}G$-homomorphism. For the next result, we prove that φ preserves the bilinear forms on $\mathbb{C}(G/H)$ and V_ε. To this end, we put $\kappa = \sqrt{\chi(e)/|H|} = \sqrt{(m-1)/2}$ and we use the character $\chi = \chi_{[2,1^{m-2}]}$ of degree $m-1$ from 5.5. This character can be defined by

$$\chi(\sigma) = \begin{cases} |\mathrm{Fix}(\sigma)| - 1 & \text{if } \sigma \text{ is even,} \\ 1 - |\mathrm{Fix}(\sigma)| & \text{if } \sigma \text{ is odd.} \end{cases}$$

where $\mathrm{Fix}(\sigma)$ denotes the subset of elements of $\{1, ..., m\}$ fixed by σ. For $1 \leq i \leq m$, we use the notation $i \in \sigma,\ i \notin \sigma$ to indicate that σ moves or fixes the integer i, respectively.

Lemma 5.6. *The map $\varphi : \mathbb{C}(G/H) \to V_\varepsilon$ preserves the bilinear forms.*

Proof. Due to the G-invariance of B_H^χ, it suffices to consider products of elements of the form $\sigma H,\ H$. Throughout, k denotes the number of fixed points of σ, and we calculate the value of χ on an element of G according to the formula above. We proceed by considering cases.

Case 1: Suppose $1, 2 \notin \sigma$.

$B_H^\chi(\sigma H, H) = \frac{m-1}{2}[\chi(\sigma) + \chi(\sigma(1,2))] = \frac{m-1}{2}[(k-1) + (3-k)] = \frac{m-1}{2} \cdot 2 = m - 1 = \kappa^2\langle(1, -1, 0, \dots, 0), (1, -1, 0, \dots, 0)\rangle = \langle\varphi(\sigma H), \varphi(H)\rangle$

Case 2: Suppose $1 \in \sigma,\ 2 \notin \sigma$. (The reverse case is similar)

We have $\sigma(1,2) = (1, \dots) \cdots (1, 2) = (1, 2, \dots) \cdots (\dots)$.

$B_H^\chi(\sigma H, H) = \frac{m-1}{2}[(k-1)+(2-k)] = \frac{m-1}{2} = \kappa^2\langle(1,-1,0,\ldots,0),(0,-1,\ldots,1,\ldots,0)\rangle =$
$\langle\varphi(\sigma H),\varphi(H)\rangle$

Case 3: Suppose $1,2 \in \sigma$, with 1 and 2 in different cycles.

We have $\sigma(1,2) = (1,\ldots)(2,\ldots)\cdots(1,2) = (1,\ldots,2,\ldots)\cdots(\ldots)$.

$B_H^\chi(\sigma H, H) = \frac{m-1}{2}[(k-1)+(1-k)] = 0 = \kappa^2\langle(1,-1,0,\ldots,0),(0,0,\ldots,-1,\ldots,1,\ldots,0)\rangle =$
$\langle\varphi(\sigma H),\varphi(H)\rangle$

Case 4: Suppose $1,2 \in \sigma$, with $1,2$ in same cycle, $(1,2)$ not a cycle.

Subcase i: 1,2 nonadjacent

We have $\sigma(1,2) = (1,\ldots,2,\ldots)\cdots(1,2) = (1,\ldots)(2,\ldots)\cdots(\ldots)$.

$B_H^\chi(\sigma H, H) = \frac{m-1}{2}[(k-1)+(1-k)] = 0 = \kappa^2\langle(1,-1,0,\ldots,0),(0,0,\ldots,-1,\ldots,1,\ldots,0)\rangle =$
$\langle\varphi(\sigma H),\varphi(H)\rangle$

Subcase ii: 1,2 adjacent

We have $\sigma(1,2) = (1,2,\ldots)\cdots(1,2) = (1,\ldots)\cdots(\ldots)$.

$B_H^\chi(\sigma H, H) = \frac{m-1}{2}[(k-1)+(-k)] = -(\frac{m-1}{2}) = \kappa^2\langle(1,-1,0,\ldots,0),(0,1,\ldots,-1,\ldots,0)\rangle =$
$\langle\varphi(\sigma H),\varphi(H)\rangle$

Case 5: Suppose $1,2 \notin \sigma$, and $(1,2)$ a cycle.

We have $\sigma(1,2) = (1,2)(\ldots)\cdots(\ldots)(1,2) = (\ldots)\cdots(\ldots)$.

$B_H^\chi(\sigma H, H) = \frac{m-1}{2}[(k-1)+(-k-1)] = -(m-1) = \kappa^2\langle(1,-1,0,\ldots,0),(-1,1,0,\ldots,0)\rangle =$
$\langle\varphi(\sigma H),\varphi(H)\rangle$

Thus φ preserves the bilinear map. $\square$

The last step in getting the induced map $\bar{\varphi}$ is to realize the kernel of B_H^χ in terms of the orthogonal idempotent $e_\chi = \dfrac{\chi(e)}{|G|}\displaystyle\sum_{\sigma\in G}\chi(\sigma^{-1})\sigma$ associated with G and χ.

Lemma 5.7. *We have ker* $B_H^\chi = (1-e_\chi)\cdot\mathbb{R}(G/H)$.

Proof. Let $\displaystyle\sum_a \alpha_a aH \in \ker B_H^\chi$. It is enough to prove that $e_\chi \cdot \displaystyle\sum_a \alpha_a aH = 0$. We have

$$\begin{aligned}
e_\chi \cdot \sum_a \alpha_a aH &= \frac{\chi(e)}{|G|}\sum_\sigma \chi(\sigma^{-1})\sigma \sum_a \alpha_a aH \\
&= \frac{\chi(e)}{|G|}\sum_\sigma\sum_a \alpha_a\chi(\sigma^{-1})\sigma aH \\
&= \frac{\chi(e)}{|G|}\sum_g\sum_a \alpha_a\chi(ag^{-1})gH \\
&= \frac{\chi(e)}{|G|}\sum_b\sum_a\sum_h \alpha_a\chi(ah^{-1}b^{-1})bH \\
&= \frac{\chi(e)}{|G|}\sum_b\sum_a\sum_h \alpha_a\chi(hb^{-1}a)bH \\
&= \frac{\chi(e)}{|G|}\sum_b\sum_a \alpha_a\sum_h \chi(b^{-1}ah)bH \\
&= \frac{\chi(e)}{|G|}\sum_b\sum_a \alpha_a(\ \sum_h \chi(b^{-1}ah)\ )bH \\
&= \frac{\chi(e)}{|G|}\sum_b\sum_a \alpha_a(\ B_H^\chi(aH, bH)\ )bH \\
&= \frac{\chi(e)}{|G|}\sum_b B_H^\chi(\ \sum_a \alpha_a aH, bH\ )bH \\
&= 0
\end{aligned}$$

Now let $r = \displaystyle\sum_{\psi\neq\chi} e_\psi \sum_a \alpha_a aH \in (1-e_\chi)\cdot\mathbb{R}(G/H)$. It suffices to show that $\dfrac{\psi(e)}{|G|}\displaystyle\sum_\sigma \psi(\sigma^{-1})\sigma\sum_a \alpha_a aH \in$ $\ker B_H^\chi$ for a single summand $\psi \neq \chi$, for then $r \in B_H^\chi$ as well. Then for all b in G, we have

$$
\begin{aligned}
B_H^\chi\left(\frac{\psi(e)}{|G|}\sum_\sigma \psi(\sigma^{-1})\sigma\sum_a \alpha_a aH, bH\right) &= \frac{\psi(e)}{|G|}\sum_\sigma\sum_a \alpha_a\psi(\sigma^{-1})B_H^\chi(\sigma aH, bH)\\
&= \frac{\psi(e)}{|G|}\sum_\sigma\sum_a \alpha_a\psi(\sigma^{-1})\sum_h \chi(b^{-1}\sigma ah)\\
&= \frac{\psi(e)}{1}\sum_a \alpha_a\sum_h \frac{1}{|G|}\sum_\sigma \chi(b^{-1}\sigma ah)\psi(\sigma^{-1})\\
&= \frac{\psi(e)}{1}\sum_a \alpha_a\sum_h \frac{1}{|G|}\sum_\sigma \chi(\sigma ahb^{-1})\psi(\sigma^{-1})\\
&= 0
\end{aligned}
$$

by the Generalized Orthogonality Relation.

□

We now prove Theorem 5.1.

Proof. First, we claim that $\ker B_H^\chi \subseteq \ker\varphi$. Using Lemma 5.7, and the fact that φ is a $\mathbb{C}G$-homomorphism, we have that $\varphi((1-e_\chi)\cdot\sum_{a\in G}\alpha_a aH) = (1-e_\chi)\varphi(\sum_{a\in G}\alpha_a aH)$. Now $\varphi(\sum_{a\in G}\alpha_a aH) \in V_\varepsilon$, and since V_ε affords χ by Lemma 5.5, e_χ acts as the identity map on V_ε. Thus

$$
\begin{aligned}
(1-e_\chi)\varphi(\sum_{a\in G}\alpha_a aH) &= \varphi(\sum_{a\in G}\alpha_a aH) - e_\chi\varphi(\sum_{a\in G}\alpha_a aH)\\
&= \varphi(\sum_{a\in G}\alpha_a aH) - \varphi(\sum_{a\in G}\alpha_a aH)\\
&= 0,
\end{aligned}
$$

and the claim follows.

We now have an induced $\mathbb{C}G$-isomorphism $\bar\varphi : \mathcal{C}_H^\chi \longrightarrow V_\varepsilon$ with $\bar\varphi(\overline{\sigma H}) = \varphi(\sigma H) = \sigma a_\kappa$, proving (ii).

Since $\overline{B_H^\chi}$ is an inner product, and since $\overline{B_H^\chi}(\bar{v}, \bar{w}) = B_H^\chi(v, w) = \langle \varphi(v), \varphi(w) \rangle$ for all $\bar{v}, \bar{w} \in \mathcal{C}_H^\chi$, the induced map preserves the inner product. Therefore $\bar{\varphi}$ is an isometry, yielding (i).

To establish (iii), first let $m = 3$. Since $(132)H = (23)H$ and $(123)H = (13)H$, we have that $\Sigma = \{\overline{(132)H}, \overline{(123)H}, \overline{H}\}$, after choosing representatives for the left cosets of H in G. Hence $\bar{\varphi}(\Sigma) = \{\kappa(1, -1, 0), \kappa(0, 1, -1), \kappa(1, 0, -1)\}$. To map onto the remaining roots $(-1, 1, 0), (0, -1, 1)$, and $(-1, 0, 1)$, we must include $-\Sigma = \{\overline{-(132)H}, \overline{-(123)H}, \overline{-H}\}$ in the domain of $\bar{\varphi}$. After doing so, we get $\bar{\varphi}(\Sigma \dot{\cup} - \Sigma) = \Phi$.

Now let $m \geq 4$. We have that $\bar{\varphi}(\overline{H}) = \sigma a_\kappa = \varepsilon(e) a_\kappa = \kappa(1, -1, 0, ..., 0) \in \Phi$. Since $m \geq 4$, S_m contains the element $\sigma = (12)(34)$, and so $\bar{\varphi}(\overline{\sigma H}) = \sigma a_\kappa = \varepsilon(\sigma) a_\kappa = -a_\kappa = \kappa(-1, 1, 0, \ldots, 0)$. By [Hum72, p.53, Lemma C], we have that S_m acts transitively on the set of those roots having a fixed length, so that the orbit of a_κ under S_m is precisely Φ. That is, $\bar{\varphi}(\Sigma) = \Phi$. The proof is complete. $\square$

Chapter 6

Conclusions

Originally, the coset space was developed as a tool for deciding whether an orbital subspace of a symmetrized tensor space had a basis consisting of pairwise orthogonal standard symmetrized tensors (an o-basis). As we have seen, when G is a dihedral group of order a power of 2, these orbital subspaces V_γ^χ have such a basis for each $\gamma \in \Gamma$ and $\chi \in \mathrm{Irr}(G)$. Dihedral groups are distinctive in the sense that all of their irreducible characters have degree at most 2. Hence for any (H, χ)-pairing, we have that

$$\dim_{\mathbb{C}} \mathcal{C}_H^\chi = \frac{\chi(e)}{|H|} \sum_{h \in H} \chi(h)\overline{1(h)} \leq \frac{\chi(e)}{|H|} \sum_{h \in H} \chi(e) = \frac{\chi(e)}{|H|} \cdot |H|\chi(e) = \chi^2(e) \leq 4.$$

Their relatively small dimension makes the coset spaces for the dihedral group fairly easy to investigate. With the symmetric group, however, the maximum degree of its irreducible characters increases with the cardinality of the group. Consequently the maximum dimensions of coset spaces for these groups grow as well. The approach of constructing bases and computing Gram matrices becomes unwieldy for S_n when n is high, a fact which became apparent with the concluding example of S_4 in Chapter 3.

Still, for the fixed (H, χ)-pairing in Chapter 5, we were able to obtain a geometric description of $\mathcal{C}_H^\chi$ for S_n with n arbitrarily large. In this case, the standard vectors $\Sigma \subseteq \mathcal{C}_H^\chi$ formed a root system of type A_{n-1}. Considering this alongside our prior results, it is notable that, for each of the root systems we obtained, all of its irreducible components were of type A_m for some $m \in \mathbb{N}$. Thus all of the root systems we obtained have roots of equal length: They are "simply laced." Since we arrived at our results, work has been done [HH13] indicating that the following is a theorem:

There exists an orbital subspace such that the standard symmetrized tensors in the subspace form a root system isomorphic to a given irreducible root system if and only if the irreducible root system is simply laced.

We saw a glimmer of this already with our concluding result of Chapter 3, which placed tight constraints on the rank 1 and 2 root systems which could be realized with standard vectors in a coset space. As we established in 3.12, if $\mathcal{C}_H^\chi$ has a basis of standard vectors that forms a base for a root system, then the value of $\overline{B_H^\chi}(\overline{aH}, \overline{bH})$ is either 0 or $\pm\frac{1}{2}\dim_{\mathbb{C}} \mathcal{C}_H^\chi$ for distinct basis vectors $\overline{aH}, \overline{bH}$ in the coset space. Naturally, we revisit the question that arose in the context of research on o-basis groups: Given a finite group G, what conditions on the group insure that for every $H \leqslant G$ and $\chi \in \mathrm{Irr}(G)$, the vector space $\mathcal{C}_H^\chi$ has a basis that is orthogonal relative to $\overline{B_H^\chi}$ consisting entirely of standard vectors? This is a question about the structure of a finite group, and about its irreducible characters. Having posed the question, the formula for the inner product of two vectors in a coset space is itself very suggestive. We have

$$\overline{B_H^\chi}(\overline{aH}, \overline{bH}) = \frac{\chi(e)}{|H|} \sum_{h \in H} \chi(b^{-1}ah).$$

If we could determine the conditions under which an irreducible character vanishes on a subset of G, this could shed light on the values we obtain from the summation on the right. Specifically, note that the right-hand side of the equation above is a sum of the elements of a set of the form $\chi(cH) = \{\chi(ch) \mid h \in H\}$, where $cH \in G/H$. Thus, if we knew when irreducible characters vanish on entire cosets, we could know one way in which this summation becomes zero.

Much literature on zeros of irreducible characters already exists. For instance, we have the following theorem in [Nav01]:

Theorem 6.1. *Let G be a finite group and let $N \lhd G$. Let $\chi \in \mathrm{Irr}(G)$. Then χ_N is not irreducible if and only if χ vanishes on some coset Nx of N in G.*

To illustrate an easy consequence of this theorem, let $N \lhd G$ and let $\chi \in \mathrm{Irr}(G)$. Suppose χ_N is not irreducible. By the theorem, χ vanishes on some coset of N in G. Call this coset xN. Then $\overline{xN}$ and $\overline{N}$ are a pair of orthogonal vectors in $\mathcal{C}_N^\chi$. Extension of results such as this theorem might prove very fruitful in generating more orthogonal cosets, and hence in generating an entire o-basis for a coset space.

We find other results of interest scattered throughout the literature. In [DRB07], the authors prove that if G is a finite solvable group which has an irreducible character χ which vanishes on exactly one conjugacy class, then G has a homomorphic image which is a nontrivial 2-transitive permutation group. Now, Holmes proved in [Hol95] that any 2-transitive subgroup of S_n ($n \geq 3$) is not o-basis. In [Hol04] he proved that the class of o-basis groups is closed under taking homomorphic images. The above results, taken in concert, imply that if G is a finite solvable group which has an irreducible character χ which vanishes on exactly one conjugacy class, then G cannot be o-basis.

Another possible line of inquiry lies in the related notions of a Camina group and the vanishing-off subgroup. Given a finite group G and the set of nonlinear irreducible characters $\mathrm{nl}(G) \subseteq \mathrm{Irr}(G)$, we define the vanishing-off subgroup $V(G) = \langle g \in G \mid \text{ there exists } \chi \in \mathrm{nl}(G) \text{ such that } \chi(g) \neq 0\rangle$ [Isa94]. It turns out that Camina groups can be defined by the condition that $V(G) = G'$ [Lew09]. When restricting attention to Camina groups of prime power order, we find in [DS96] that such groups have nilpotency class at most 3. This suggests a possible connection with results in [Erv07], wherein the author conjectures that having nilpotency class no more than 3 may be a necessary condition for a nilpotent group to be o-basis.

In working towards a proof that a finite group is o-basis, we will always take cognizance of the geometric proof methods we have employed throughout this work. These methods hold promise in furnishing intuitive ways to explore future conjectures in this area.

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