The Stone-Čech Compactification and the Number of Pairwise Nonhomeomorphic Subcontinua of $\beta[0,\infty) \setminus [0,\infty)$

by

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Abstract

An introduction to the Stone-Čech compactification βX of a T_1 completely regular topological space X is given. The method of invariantly embedding linear orders into ultrapowers is used to find $2^{\mathfrak{c}}$ pairwise nonhomeomorphic continua in $\beta\mathbb{R}$, under the assumption that the Continuum Hypothesis fails.

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Chapter 1

Introduction

A continuum is a connected compact Hausdorff space. The Stone-Čech remainder $\mathbb{H}^* = \beta\mathbb{H} \setminus \mathbb{H}$ of the half-line $\mathbb{H} = [0, \infty)$ is a continuum, and every free ultrafilter u on ω generates a subcontinuum $\mathbb{I}_u$ of $\mathbb{H}^*$. The spaces $\mathbb{I}_u$ resemble the interval $\mathbb{I} = [0, 1]$ in several ways, but unlike intervals of reals the "intervals" $\mathbb{I}_u$ can vary considerably. Assuming $\neg$CH, A. Dow [1] demonstrated a family of $2^{\mathfrak{c}}$ free ultrafilters on ω such that the corresponding $\mathbb{I}_u$'s are mutually nonhomeomorphic. This proves the following.

Theorem 1.1 ($\neg$ CH)**.** *There exist $2^{\mathfrak{c}}$ pairwise nonhomeomorphic subcontinua of $\mathbb{H}^*$.*

Prior to this result, only a finite number of subcontinua of $\mathbb{H}^*$ were known to exist in a given model of ZFC. Note that $2^{\mathfrak{c}}$ is the maximum possible number because $\mathbb{H}^*$ has a basis of size $\mathfrak{c}$. The main result of [1] is achieved by first noting that each $\mathbb{I}_u$ is closely related to the linearly ordered ultrapower $\mathbb{R}^\omega/u$. The following theorem says that in order to find $2^{\mathfrak{c}}$ pairwise nonhomeomorphic $\mathbb{I}_u$'s, it suffices to find $2^{\mathfrak{c}}$ pairwise nonisomorphic completions $\overline{\mathbb{R}^\omega/u}$ of ultrapowers $\mathbb{R}^\omega/u$.

Theorem 1.2. *If u and v are free ultrafilters on ω and $\mathbb{I}_u \simeq \mathbb{I}_v$, then $\overline{\mathbb{R}^\omega/u} \approx \overline{\mathbb{R}^\omega/v}$.*

Finding $2^{\mathfrak{c}}$ nonisomorphic completions $\overline{\mathbb{R}^\omega/u}$ is no trivial matter. It was first established in [9] that all ultrapowers $\mathbb{R}^\omega/u$ are isomorphic under CH. Prior to [3], we only knew of the existence of $\mathfrak{c}$ nonisomorphic ultrapowers in certain models of ZFC+$\neg$CH. The authors of [3] show that there are actually $2^{\mathfrak{c}}$ nonisomorphic ultrapowers whenever CH fails. A. Dow was able to modify some of their arguments to prove the following.

Theorem 1.3 ($\neg$ CH)**.** *There exists a family $\{D_\alpha : \alpha < 2^{\mathfrak{c}}\}$ of free ultrafilters on ω such that $\overline{\mathbb{R}^\omega/D_\alpha} \not\approx \overline{\mathbb{R}^\omega/D_\beta}$ for any $\alpha < \beta < 2^{\mathfrak{c}}$.*

The goal of this paper is to develop the tools needed for proving the theorems stated above. In the next chapter, we will state some relevant definitions and theorems from introductory topology and set theory. The Stone-Čech compactification is the subject of Chapter 3, wherein we prove existence and uniqueness results and look at some examples. In Chapter 4 we introduce the ultrapowers $\mathbb{R}^\omega/u$ and prove the aforementioned CH result. In Chapter 5 we examine the spaces $\mathbb{I}_u$ and prove Theorem 1.2. As indicated above, Theorems 1.2 and 1.3 yield a proof of Theorem 1.1. We give a slightly different proof of Theorem 1.1 in Chapter 6 while presenting a series of results from [3]. Finally, we prove Theorem 1.3.

Chapter 2

Some Topology and Set Theory

2.1 Topology

We refer the reader to *Topology* by J. Munkres [6] for the basics.

Theorem 2.1. *Every closed subset of a compact space is compact.* □

Theorem 2.2. *Every compact subspace of a Hausdorff space is closed.* □

Let X and Y be topological spaces, and let $f : X \to Y$ be a function. For $A \in \mathcal{P}(X)$ let $f[A] = \{f(x) : x \in A\}$ and for $B \in \mathcal{P}(Y)$ let $f^{-1}[B] = f^{-1}B = \{x \in X : f(x) \in B\}$.

Theorem 2.3. *If f is continuous and X is compact then $f[X]$ is compact.* □

Theorem 2.4. *Suppose X is compact, Y is Hausdorff, and f is continuous. Then f is closed, and if $f[X]$ contains a dense subset of Y then $f[X] = Y$.* □

Theorem 2.5. *f is continuous iff $f[cl_X A] \subseteq cl_Y f[A]$ for each $A \in \mathcal{P}(X)$.* □

Theorem 2.6. *f is continuous and closed iff $f[cl_X A] = cl_Y f[A]$ for each $A \in \mathcal{P}(X)$.* □

Theorem 2.7. *Suppose Y is Hausdorff, $D \subseteq X$ is dense in X, and $f, g : X \to Y$ are continuous. If $f \restriction D = g \restriction D$ then $f = g$.* □

A collection $\mathcal{C}$ of subsets of X is said to have the *finite intersection property* if every finite subcollection of $\mathcal{C}$ has nonempty intersection.

Theorem 2.8. *X is compact iff $\bigcap \mathcal{C} \neq \varnothing$ whenever $\mathcal{C}$ is a collection of closed subsets of X with the finite intersection property.* □

Theorem 2.9 (Tychonoff's Theorem). *A product of compact spaces is compact (in the product topology).* □

A *compactification* of X is a compact Hausdorff space containing a dense copy of X. Every locally compact Hausdorff space has a *one point compactification*, defined to be the set $\alpha X = X \cup \{\infty\}$ with the following topology: $U \subseteq \alpha X$ is open if

(i) U is open in X, or

(ii) $\infty \in U$ and $\alpha X \setminus U$ is compact.

Theorem 2.10. *If A is a connected subspace of X then $cl_X A$ is connected.* □

A *continuum* is a connected compact Hausdorff space.

Theorem 2.11. *The intersection of a family of continua with the finite intersection property is a continuum. That is, if $\mathcal{C}$ is a collection of compact connected subspaces of X with the finite intersection property, then $\bigcap \mathcal{C}$ is compact and connected as a subspace of X.* □

Theorem 2.12. *Every metrizable space is normal.* □

Theorem 2.13. *Every compact Hausdorff space is normal.* □

Theorem 2.14. *Every F_σ subspace of a normal space is normal.*

Proof. Suppose L is a countable union of closed sets in a normal space X, and A and B are disjoint relatively closed subsets of L. Write A and B as countable unions of closed subsets of X; $A = \bigcup_{i \in \omega} A_i$ and $B = \bigcup_{i \in \omega} B_i$.

Fix $i \in \omega$. Note that $A = E \cap L$ for some closed $E \subseteq X$ with $E \cap B_i = \varnothing$. Similarly, $B = F \cap L$ for some closed $F \subseteq X$ with $F \cap A_i = \varnothing$. By normality of X, there are disjoint open sets U' and V' with $E \subseteq U'$ and $B_i \subseteq V'$. Similarly, there are disjoint open sets U'' and V'' with $A_i \subseteq U''$ and $B \subseteq V''$. Let $U_i = U' \cap U''$ and $V_i = V' \cap V''$. Then U_i and V_i are disjoint open sets containing A_i and B_i, repsectively, and $\overline{U_i} \cap B = \varnothing = A \cap \overline{V_i}$.

Let $U = \bigcup_{n \in \omega} U_n \setminus \bigcup_{i<n} \overline{V_i}$ and $V = \bigcup_{n \in \omega} V_n \setminus \bigcup_{i<n} \overline{U_i}$. Then $U \cap L$ and $V \cap L$ are disjoint open subsets of L containing A and B, respectively. □

Let $C(X)$ be the ring of continuous real-valued functions on X and let $C^*(X)$ be the subring of $C(X)$ consisting of the bounded members of $C(X)$. Let $A, B, Z \subseteq X$.

- A is C^*-*embedded* in X if every function in $C^*(A)$ extends to a function in $C^*(X)$.

- A and B are *completely separated* in X if there exists $f \in C(X)$ such that $f[A] = \{0\}$ and $f[B] = \{1\}$.

- X is *completely regular* if closed sets and singletons are completely separated in X. That is, for each closed $A \subseteq X$ and $p \in X \setminus A$ there exists $f \in C(X)$ with $f[A] = \{0\}$ and $f(p) = 1$. If X is completely regular and T_1, then distinct singletons are completely separated in X.

- Z is a *zero set* of X if there exists $f \in C(X)$ with $Z = f^{-1}\{0\}$. If $f \in C(X)$ then we let $Z(f) = f^{-1}\{0\}$ denote the zero set of f. Let $\mathcal{Z}(X)$ be the collection of all zero sets of X.

Theorem 2.15. *If X is metric then $\mathcal{Z}(X)$ equals the collection of closed subsets of X.* $\square$

Theorem 2.16. *Two sets are completely separated iff they are contained in disjoint zero sets.*

Proof. Let $A, B \subseteq X$. Suppose A and B are completely separated. Let $f \in C(X)$ such that $A \subseteq f^{-1}(\{0\})$ and $B \subseteq f^{-1}(\{1\})$. Then $A \subseteq Z(f)$, $B \subseteq Z(f-1)$, and $Z(f) \cap Z(f-1) = \varnothing$. Conversely, suppose A and B are contained in disjoint zero sets. Let $f_1, f_2 \in C(X)$ s.t. $Z(f_1) \supseteq A$, $Z(f_2) \supseteq B$, and $Z(f_1) \cap Z(f_2) = \varnothing$. Define $f = \frac{|f_1|}{|f_1|+|f_2|}$. Then f is continuous, $f^{-1}(\{0\}) = Z(f_1) \supseteq A$, and $f^{-1}(\{1\}) = Z(f_2) \supseteq B$. $\square$

Theorem 2.17 (Urysohn's Lemma)**.** *Disjoint closed subsets of a normal space are completely separated.* $\square$

Note that Urysohn's Lemma implies every T_1 normal space is completely regular.

Theorem 2.18 (Tietze's Extension Theorem)**.** *Closed subsets of normal spaces are C^*-embedded.* $\square$

A *linear order* is a pair $(L, <)$, where L is a set and $<$ is a binary relation on L such that for all $a, b, c \in L$:

(i) either $a = b$, $a < b$, or $b < a$ (comparability, antisymmetry),

(ii) $a \not< a$ (irreflexivity), and

(iii) If $a < b$ and $b < c$, then $a < c$ (transitivity).

We may refer to a linear order simply by its underlying set when no confusion will arise. Suppose L is a linear order. L is *dense* if for all $l_1 < l_2 \in L$ there exists $l_3 \in L$ with $l_1 < l_3 < l_2$. L is *complete* if every subset of L has a least upper bound. If L is a dense linear order, $(M, \prec)$ is a *completion* of L if

(i) M is complete,

(ii) $L \subseteq M$ and $\prec$ extends the ordering $<$ on L, and

(iii) L is dense in M, i.e., for all $m_1 \prec m_2 \in M$ there exists $l \in L$ with $m_1 \prec l \prec m_2$.

Theorem 2.19. *Every dense linear order has a unique completion (up to isomorphism).* □

Theorem 2.20. *If L is dense and compact in the order topology, then L is complete.* □

Theorem 2.21 (Intermediate Value Theorem)**.** *Suppose L and L' are linear ordered topological spaces, L is complete, and $f : L \to L'$ is continuous. If $a, b \in L$ and r is a point of L' lying between $f(a)$ and $f(b)$, then there exists a point c of L lying between a and b such that $f(c) = r$.* □

Theorem 2.22. *Suppose L and L' are complete linearly ordered topological spaces and $h : L \to L'$ is a homeomorphism. Then h is either order preserving or order reversing.* □

Suppose L and L' are linear orders and $f : L \to L'$. f maps L *cofinally* if for all $l' \in L'$ there exists $l \in L$ such that $l' < f(l)$. f maps L *coinitially* if for all $l' \in L'$ there exists $l \in L$ such that $f(l) < l'$. The *cofinality* of L, denoted $\mathrm{cf}(L)$, is the least ordinal α such that there is a map $f : \alpha \to L$ cofinally into L. The *coinitiality* of L, denoted $\mathrm{coi}(L)$, is the least ordinal α such that there is a map $f : \alpha \to L$ coinitially into L.

2.2 Set Theory

We refer the reader to *Set Theory* by T. Jech [5] for the basics.

Theorem 2.23. *Suppose κ is regular. If $\lambda < \kappa$ and $\{X_\xi : \xi < \lambda\}$ is a collection with $|X_\xi| < \kappa$ for each $\xi < \lambda$, then $\left|\bigcup_{\xi<\lambda} X_\xi\right| < \kappa$.* □

Theorem 2.24. *For every infinite cardinal κ there exists an increasing sequence $\{\alpha_\xi : \xi < \mathrm{cf}(\kappa)\}$ such that $\kappa = \sup_{\xi<\mathrm{cf}(\kappa)} \alpha_\xi$ and $|\alpha_\xi| < \kappa$ for each $\xi < \mathrm{cf}(\kappa)$.* □

Theorem 2.25. *If κ is a limit cardinal, then $2^\kappa = (2^\kappa)^{\mathrm{cf}(\kappa)}$.* □

Theorem 2.26. *If λ is an infinite cardinal, and $\langle \kappa_i : i < \lambda \rangle$ is a nondecreasing sequence of nonzero cardinals, then $\prod_{i<\lambda} \kappa_i = (\sup_{i<\lambda} \kappa_i)^\lambda$.* □

Theorem 2.27. *Suppose κ is singular. There exists a set $\{\kappa_i : i < \mathrm{cf}(\kappa)\}$ of regular cardinals, each $\kappa_i > \omega_1$, such that*

$$\sup_{i<\mathrm{cf}(\kappa)} \kappa_i = \kappa \quad \textit{and} \quad \prod_{i<\mathrm{cf}(\kappa)} 2^{\kappa_i} = 2^\kappa.$$

Proof. As a limit cardinal, κ is the sup of $\mathrm{cf}(\kappa)$ regular cardinals κ_i. We may assume each $\kappa_i > \omega_1$. By Theorems 2.25 and 2.26,

$$\prod_{i<\mathrm{cf}(\kappa)} 2^{\kappa_i} = (\sup_{i<\mathrm{cf}(\kappa)} 2^{\kappa_i})^{\mathrm{cf}(\kappa)} = (2^\kappa)^{\mathrm{cf}(\kappa)} = 2^\kappa.$$

□

Let κ be a regular uncountable cardinal. A set $C \subseteq \kappa$ is a *closed unbounded* subset of κ if C is unbounded in κ and C contains all limit ordinals less than κ. A set $S \subseteq \kappa$ is *stationary* if $S \cap C = \varnothing$ for every closed unbounded subset C of κ.

Theorem 2.28. *The intersection of fewer than κ closed unbounded subsets of κ is closed unbounded.* □

Theorem 2.29 (Pressing Down Lemma). *If κ is regular uncountable, S is a stationary subset of κ, and $f : S \to \kappa$ such that $f(\gamma) < \gamma$ for all $\gamma \in S$, then there is a stationary set $S' \subseteq S$ and $\alpha < \kappa$ such that $f(\gamma) = \alpha$ for all $\gamma \in S'$.* $\square$

Theorem 2.30. *Suppose κ is regular uncountable and $\lambda < \kappa$ is regular. Then $S = \{\alpha < \kappa : cf(\alpha) = \lambda\}$ is stationary in κ, and may be partitioned into κ pairwise disjoint stationary sets.*

Proof. If C is closed unbounded in κ, then the λ-th element of C has cofinality λ, thus $S \cap C \neq \varnothing$. So S is stationary in κ. For each $\alpha \in S$, let $(\alpha_\xi)_{\xi<\lambda}$ be an increasing sequence in κ with $\sup_{\xi<\lambda}\alpha_\xi = \alpha$. For each $\eta < \kappa$ and $\xi < \lambda$ let

$$S_{\eta,\xi} = \{\alpha \in S : \eta \leq \alpha_\xi\}.$$

Claim: There exists $\xi < \lambda$ such that $S_{\eta,\xi}$ is stationary in κ for all $\eta < \kappa$. Well, otherwise for all $\xi < \lambda$ there exists $\eta_\xi < \kappa$ and a closed unbounded C_ξ such that $C_\xi \cap S_{\eta_\xi,\xi} = \varnothing$, so that each element $\alpha \in C_\xi \cap S$ has $\alpha_\xi < \eta_\xi$. Then $C = \bigcap_{\xi<\lambda} C_\xi$ is closed unbounded and $\alpha = \sup_{\xi<\lambda}\alpha_\xi \leq \sup_{\xi<\lambda}\eta_\xi < \kappa$ for each $\alpha \in C \cap S$. But $C \cap S$ is stationary in κ; in particular it is unbounded in κ. Contradiction.

Let $\xi < \lambda$ be given by the claim and define $f(\alpha) = \alpha_\xi$ for each $\alpha \in S$. Then for each $\eta < \kappa$, $f \upharpoonright S_\eta$ is a regressive function on the stationary set S_η. For each $\eta < \kappa$ the Pressing Down Lemma implies there exists a stationary $S'_\eta \subseteq S_\eta$ and $\eta \leq \gamma_\eta$ with $f(\alpha) = \gamma_\eta$ for all $\alpha \in S'_\eta$. Then $\gamma_\eta \neq \gamma_{\eta'}$ implies $S'_\eta \cap S'_{\eta'} \neq \varnothing$. In particular, $\left|\{S'_\eta : \eta < \kappa\}\right| = |\{\gamma_n : \eta < \kappa\}|$. Since the γ_η are unbounded in κ and κ is regular, this set has cardinality κ. While it may not be true that $S = \bigcup_{\eta<\kappa} S'_\eta$, we could simply add the deficit $S \setminus \bigcup_{\eta<\kappa} S'_\eta$ t one of the S'_η. $\square$

Theorem 2.31 (Δ-system Lemma). *If $\mathcal{C}$ is an uncountable collection of finite sets then there exists an uncountable $\mathcal{S} \subseteq \mathcal{C}$ and a set r such that $A \cap B = r$ for any $A \neq B \in \mathcal{S}$.* $\square$

Theorem 2.32 (Ramsey's Theorem). *If f is an n-place function on ω with finite range then there is an infinite $W \subseteq \omega$ such that f is constant on all increasing n-tuples in W^n.* $\square$

Chapter 3

The Stone-Čech Compactification

In this chapter we will show every T_1 completely regular space X has a compactification βX which is unique with respect to certain properties. Note that it is necessary that X be T_1 completely regular for X to have *any* compactification. One of the properties of βX is that every continuous map $f : X \to Y$ from X into a compact Hausdorff space Y has a unique continuous extension $\beta f : \beta X \to Y$. We will prove some useful results concerning the the extensions βf, and then we will examine the spaces $\beta\mathbb{H}$, $\beta\omega$, $\beta\omega_1$, and βL.

3.1 Filters and Normal Bases

Suppose $\mathcal{A}$ is a collection of sets that is closed under finite intersections. An $\mathcal{A}$-*filter* is a nonempty subcollection $\mathcal{D}$ of $\mathcal{A}$ such that

(i) $\varnothing \notin \mathcal{D}$,

(ii) if $A, B \in \mathcal{D}$ then $A \cap B \in \mathcal{D}$, and

(iii) if $A \in \mathcal{D}$ and $A \subseteq B \in \mathcal{A}$ then $B \in \mathcal{D}$.

We will omit reference to $\mathcal{A}$ when no confusion will arise. By properties (i) and (ii) filters have the finite intersection property. On the other hand, given any subcollection $\mathcal{E} \subseteq \mathcal{A}$ with the finite intersection property, let

$$(\mathcal{E}) = \{A \in \mathcal{A} : A \text{ is a superset of a finite intersection of members of } \mathcal{E}\}.$$

Then $(\mathcal{E})$ is a filter containing $\mathcal{E}$, called the *filter generated by* $\mathcal{E}$.

A filter is called an *ultrafilter* if no other filter properly includes it.

Theorem 3.1 (Ultrafilter Lemma)**.** *Every filter may be extended to an ultrafilter.*

Proof. Suppose $\mathcal{D}$ is a filter. Consider the set $\mathbb{P}$ of all filters containing $\mathcal{D}$, partially ordered by inclusion. The union of a chain of filters containing D is a filter containing $\mathcal{D}$, so every chain in $\mathbb{P}$ has an upper bound. By Zorn's Lemma $\mathbb{P}$ has a maximal element p, an ultrafilter containing $\mathcal{D}$ which no other filter properly includes. $\square$

Theorem 3.2. *p is an ultrafilter iff p is a filter and every set in $\mathcal{A}$ which intersects each member of p is in p.*

Proof. Suppose p is an ultrafilter and $A \in \mathcal{A}$ intersects every element of p. Then $p \cup \{A\}$ has the finite intersection property, so we may consider $(p \cup \{A\})$. By maximality of p, $(p \cup \{A\}) \subseteq p$, so $A \in p$. Conversely, suppose p is a filter and every set in $\mathcal{A}$ which intersects each member of p is in p. If $A \in \mathcal{A} \setminus p$ then A does not intersect every member of p so A cannot be added to p to generate a larger filter. $\square$

Theorem 3.3. *If p is an ultrafilter, $A_1, A_2 \in \mathcal{A}$, and $A_1 \cup A_2 \in p$, then $A_1 \in p$ or $A_2 \in p$.*

Proof. Suppose neither is in p. Then by the previous theorem there exist $A', A'' \in p$ with $A_1 \cap A' = \varnothing$ and $A_2 \cap A'' = \varnothing$. Then $A' \cap A'' \in p$ and $(A_1 \cup A_2) \cap (A' \cap A'') = \varnothing$, so $A_1 \cup A_2 \notin p$. $\square$

A filter is *principal* if it consists of all members of $\mathcal{A}$ which contain a particular element of X. That is, a principal filter is a filter of the form $\{A \in \mathcal{A} : x \in A\}$. A principal filter may be an ultrafilter, depending on $\mathcal{A}$. An ultrafilter that is not principal is said to be *free*.

Suppose X is a set. We refer to a $\mathcal{P}(X)$-filter ($\mathcal{P}(X)$-ultrafilter) as simply a *filter on X* (*ultrafilter on X*).

Theorem 3.4. *u is an ultrafilter on X iff u is a filter on X and for each $A \subseteq X$ exactly one of A and $X \setminus A$ belongs to u.*

Proof. ($\Rightarrow$): Suppose u is an ultrafilter on X. Since $A \cup X \setminus A = X \in u$, by the previous theorem we have $A \in u$ or $X \setminus A \in u$. Both cannot be in u because $A \cap X \setminus A = \varnothing$. ($\Leftarrow$): If $A \in \mathcal{P}(X) \setminus u$ then $X \setminus A \in u$ so A cannot be added to u to generate a larger filter. $\square$

The filter on ω consisting of the cofinite subsets of ω is the called the *cofinite filter.*

Theorem 3.5. *An ultrafilter on ω is free iff it contains the cofinite filter.*

Proof. Suppose u is a free ultrafilter on ω. Let $A \subseteq \omega$ such that $\omega \setminus A$ finite. For a contradiction suppose $A \notin u$. Then $\omega \setminus A \in u$. For each $n \in \omega \setminus A$ there exists $A_n \in u$ such that $n \notin A_n$. Then $(\omega \setminus A) \cap \bigcap_{n \in \omega \setminus A} A_n = \varnothing$, contradicting the finite intersection property of u. Conversely, suppose u is an ultrafilter containing the cofinite filter. If $n \in \omega$, then $\omega \setminus \{n\} \in u$, so that $\{n\} \notin u$. $\square$

Suppose X is a T_1 topological space and $\mathcal{L}(X)$ is a *closed lattice base* for X. That is, $\mathcal{L}(X)$ is a collection of closed subsets of X that is closed under finite unions and finite intersections, such that every closed subset of X is an intersection of members of $\mathcal{L}(X)$. $\mathcal{L}(X)$ is a *normal base* for X if, additionally,

(i) for any closed subset A and $x \in X \setminus A$ there exists a member of $\mathcal{L}(X)$ containing x missing A, and

(ii) disjoint members of $\mathcal{L}(X)$ are contained in disjoint complements of members of $\mathcal{L}(X)$.

Theorem 3.6.

(1) If X is completely regular then $\mathcal{Z}(X)$ is a normal base for X.

(2) If X is normal then the collection of closed subsets of X is a normal base for X.

(3) If X is compact Hausdorff then any closed lattice base is a normal base for X.

Proof of (1). $\mathcal{Z}(X)$ is a lattice since $Z(f_1) \cap Z(f_2) = Z(|f_1| + |f_2|)$ and $Z(f_1) \cup Z(f_2) = Z(f_1 \cdot f_2)$. Now let A be closed in X. Under the assumption X is completely regular, for each $x \in X \setminus A$ let $f_x : X \to \mathbb{R}$ be a continuous function with $f(x) = 1$ and $f[A] = \{0\}$. Then $A = \bigcap_{x \in X \setminus A} f_x^{-1}\{0\}$. Thus $\mathcal{Z}(X)$ is a closed lattice base for X. Next we establish the normal base properties. (i): A and $\{x\}$ are completely separated by zero sets Z_1 and Z_2. The set Z_2 is as desired. (ii) Suppose $Z(f_1), Z(f_2) \in \mathcal{Z}(X)$ are disjoint. By Theorem

2.16, $Z(f_1)$ and $Z(f_2)$ are completely separated. That is, there exists $f \in C(X)$ such that $f^{-1}\{0\} = Z(f_1)$ and $f^{-1}\{1\} = Z(f_2)$. Then $Z(f_1) \subseteq f^{-1}(-\infty, \frac{1}{2})$ and $Z(f_2) \subseteq f^{-1}(\frac{1}{2}, \infty)$. As $[\frac{1}{2}, \infty)$ and $(-\infty, \frac{1}{2}]$ are zero sets in $\mathbb{R}$, $f^{-1}[\frac{1}{2}, \infty)$ and $f^{-1}(-\infty, \frac{1}{2}]$ are zero sets in X. We have $Z(f_1) \subseteq f^{-1}(-\infty, \frac{1}{2}) = X \setminus f^{-1}[\frac{1}{2}, \infty)$ and $Z(f_2) \subseteq f^{-1}(\frac{1}{2}, \infty) = X \setminus f^{-1}(-\infty, \frac{1}{2}]$, so that $Z(f_1)$ and $Z(f_2)$ are contained in disjoint complements of zero sets. □

Proof of (2). Trivial. □

Proof of (3). Let $\mathcal{L}(X)$ be a closed lattice base for X. (i): Suppose A closed in X and $x \in X \setminus A$. Since $\{x\}$ is closed in X, $\{x\} = \bigcap\{L \in \mathcal{L}(X) : x \in L\}$. By compactness of X there exists $L \in \mathcal{L}(X)$ such that $L \cap A = \varnothing$. (ii): Suppose A, B are disjoint members of $\mathcal{L}(X)$. By normality of X there are disjoint open sets U and V such that $A \subseteq U$ and $B \subseteq V$. Since X is compact and $\mathcal{L}(X)$ is a closed base for X, there are $L_1, L_2 \in \mathcal{L}(X)$ such that $X \setminus U \subseteq L_1$, $X \setminus V \subseteq L_2$, $A \cap L_1 = \varnothing$, and $B \cap L_2 = \varnothing$. Then $X \setminus L_1$ and $X \setminus L_2$ are disjoint, $A \subseteq X \setminus L_1$, and $B \subseteq X \setminus L_2$. □

Note that if X is metric, the bases in (1) and (2) are identical.

3.2 Construction of βX

Suppose X is a T_1 topological space and $\mathcal{L}(X)$ is a normal base for X. Let $\beta X(\mathcal{L})$ be the set of all $\mathcal{L}(X)$-ultrafilters. For each $L \in \mathcal{L}(X)$ let

$$F(L) = F_{\mathcal{L}(X)}(L) = \{p \in \beta X(\mathcal{L}) : L \in p\}\,.$$

Theorem 3.7. *$F(L_1) \cap F(L_2) = F(L_1 \cap L_2)$ for any $L_1, L_2 \in \mathcal{L}(X)$.*

Proof. ($\subseteq$) holds by filter property (ii). ($\supseteq$) holds by filter property (iii). □

Theorem 3.8. *$F(L_1) \cup F(L_2) = F(L_1 \cup L_2)$ for any $L_1, L_2 \in \mathcal{L}(X)$.*

Proof. ($\subseteq$) holds by filter property (iii). ($\supseteq$) holds by Theorem 3.3. □

Let $\tau_{\mathcal{L}(X)} = \{X \setminus L : L \in \mathcal{L}(X)\}$, and for each $O \in \tau_{\mathcal{L}(X)}$ let

$$B(O) = B_{\mathcal{L}(X)}(O) = \{u \in \beta X(\mathcal{L}) : (\exists L \in u)(L \subseteq O)\}.$$

Theorem 3.9. *$\{B(O) : O \in \tau_{\mathcal{L}(X)}\}$ is a basis for a topology on $\beta X(\mathcal{L})$ for which $\{F(L) : L \in \mathcal{L}(X)\}$ is a closed lattice base.*

Proof. It is easy to see that $B(X) = \beta X$ and $B(U) \cap B(V) = B(U \cap V)$ for any $U, V \in \tau_{\mathcal{L}(X)}$. So $\{B(O) : O \in \tau_{\mathcal{L}(X)}\}$ is a basis for a topology. $\{F(L) : L \in \mathcal{L}(X)\}$ is a closed lattice base by the previous two theorems, together with the fact that $B(O) = \beta X(\mathcal{L}) \setminus F(X \setminus O)$ for any $O \in \tau_{\mathcal{L}(X)}$. $\square$

Theorem 3.10. *$\beta X(\mathcal{L})$ is compact Hausdorff.*

Proof. Suppose $p_1 \neq p_2 \in \beta X(\mathcal{L})$. Then there exist $L_1 \in p_1$ and $L_2 \in p_2$ with $L_1 \cap L_2 = \varnothing$. There exist L_1' and L_2' in $\mathcal{L}(X)$ such that $L_1 \subseteq X \setminus L_1'$, $L_2 \subseteq X \setminus L_2'$, and $X \setminus L_1' \cap X \setminus L_2 = \varnothing$. It follows that $p_1 \in B(X \setminus L_1')$, $p_2 \in B(X \setminus L_2')$, and $B(X \setminus L_1') \cap B(X \setminus L_2') = \varnothing$. This proves $\beta X(\mathcal{L})$ is Hausdorff.

To prove $\beta X(\mathcal{L})$ is compact, it suffices to show that any collection of basic closed subsets of $\beta X(\mathcal{L})$ with the finite intersection property has a nonempty intersection. Let $\{F(L_i) : i \in I\}$ be a collection of basic closed subsets of $\beta X(\mathcal{L})$ with the finite intersection property. Then $\{L_i : i \in I\}$ also has the finite intersection property. Generate an $\mathcal{L}(X)$-filter containing all of the sets L_i, and then extend it to an ultrafilter p in $\beta X(\mathcal{L})$. Then $p \in \bigcap\{F(L_i) : i \in I\}$. $\square$

Define $e : X \to \beta X(\mathcal{L})$ by $x \mapsto \{L \in \mathcal{L}(X) : x \in L\}$. That is, for $x \in X$ let $e(x)$ be the principal $\mathcal{L}(X)$-filter on x. Normal base property (i) guarantees $e(x)$ is an ultrafilter.

Theorem 3.11. *$cl_{\beta X(\mathcal{L})} e[L] = F(L)$ for any $L \in \mathcal{L}(X)$.*

Proof. Let $L \in \mathcal{L}$. Since $e[L] \subseteq F(L)$ and $F(L)$ is closed in $\beta X(\mathcal{L})$, we have $\mathrm{cl}_{\beta X(\mathcal{L})} e[L] \subseteq F(L)$. To show $F(L) \subseteq \mathrm{cl}_{\beta X(\mathcal{L})} e[L]$ it suffices to show $F(L)$ is contained in every basic closed set containing $e[L]$. To that end, suppose $L_0 \in \mathcal{L}(X)$ with $e[L] \subseteq F(L_0)$. Then $L_0 \in e(l)$ for all $l \in L$, so $L \subseteq L_0$. By the superset property of ultrafilters, we have $F(L) \subseteq F(L_0)$. $\square$

Theorem 3.12. *e is a dense embedding.*

Proof. First we show e is injective. Let $x_1 \neq x_2 \in X$. Then $\{x_1\}$ is closed in X and $x_2 \in X \setminus \{x_1\}$. By normal base property (i) there exists $L \in \mathcal{L}(X)$ with $x_2 \in L$, and $L \cap \{x_1\} = \varnothing$. That is, $x_2 \in L$ and $x_1 \notin L$. So $L \in e(x_2)$ but $L \notin e(x_1)$. Thus $e(x_1) \neq e(x_2)$, proving e is injective. e is continuous since $e^{-1}[F(L)] = \{x \in X : L \in e(x)\} = \{x \in X : x \in L\} = L$. The inverse of e defined on $e[X]$ is continuous since $e[L] = F(L) \cap e[X]$. Finally $\mathrm{cl}_{\beta X(\mathcal{L})} e[X] = F(X) = \beta X(\mathcal{L})$ by Theorem 3.11, so that $e[X]$ is dense in $\beta X(\mathcal{L})$. $\square$

We will frequently identify X with its copy $e[X]$ in $\beta X(\mathcal{L})$, viewing points of X as principal $\mathcal{L}(X)$-ultrafilters. For instance, under this identification Theorem 3.11 says $\mathrm{cl}_{\beta X(\mathcal{L})} L = F(L)$ for any $L \in \mathcal{L}(X)$. Theorems 3.10 and 3.12 yield the following.

Theorem 3.13. *$\beta X(\mathcal{L})$ is a compactification of X.* $\square$

Corollary 3.14. *If X is compact Hausdorff, then $X = \beta X(\mathcal{L})$.*

Proof. Suppose X is compact Hausdorff. Since $\beta X(\mathcal{L})$ is Hausdorff, X is closed in $\beta X(\mathcal{L})$. Since X is also dense in $\beta X(\mathcal{L})$, we have $X = \beta X(\mathcal{L})$ $\square$

Theorem 3.15. *$cl_{\beta X(\mathcal{L})}(L_1 \cap L_2) = cl_{\beta X(\mathcal{L})} L_1 \cap cl_{\beta X(\mathcal{L})} L_2$ for any $L_1, L_2 \in \mathcal{L}(X)$.*

Proof. By Theorems 3.7 and 3.11 , $cl_{\beta X(\mathcal{L})} L_1 \cap cl_{\beta X(\mathcal{L})} L_2 = F(L_1) \cap F(L_2) = F(L_1 \cap L_2) = cl_{\beta X(\mathcal{L})}(L_1 \cap L_2)$. $\square$

For the remainder of this paper we will assume X is a T_1 completely regular space, and we will write simply βX for $\beta X(\mathcal{Z})$.

3.3 Properties and Uniqueness

Suppose γX is a compactification of X and consider the following statements.

(1) $\mathrm{cl}_{\gamma X}(Z_1) \cap \mathrm{cl}_{\gamma X}(Z_2) = \mathrm{cl}_{\gamma X}(Z_1 \cap Z_2)$ for any $Z_1, Z_2 \in \mathcal{Z}(X)$.

(2) Disjoint members of $\mathcal{Z}(X)$ have disjoint closures in γX.

(3) Every continuous function from X into a compact Hausdorff space has a unique continuous extension to γX.

(4) X is C^*-embedded in γX.

(5) If αX is any compactification of X, then there is a unique continuous surjection $f : \gamma X \to \alpha X$ which is the identity on X.

Theorem 3.16. *(1) $\Rightarrow$ (2), (3) $\Rightarrow$ (4), (3) $\Rightarrow$ (5), and (4) $\Rightarrow$ (2).*

Proof. (1) $\Rightarrow$ (2): Trivial. (3) $\Rightarrow$ (4): Given $f \in C^*(X)$ let $Y = \mathrm{cl}_{\mathbb{R}} f[X]$ and apply the assumption. (3) $\Rightarrow$ (5): Let f be the continuous extension of the inclusion $i : X \hookrightarrow \alpha X$. By Theorem 2.4 f is a surjection. (4) $\Rightarrow$ (2): Suppose $Z_1, Z_2 \in \mathcal{Z}(X)$ are disjoint. By Theorem 2.16 there exists $f \in C^*(X)$ such that $f(Z_1) = \{0\}$ and $f(Z_2) = \{1\}$. Then $\beta f(cl_{\gamma X}(Z_1)) \subseteq \overline{f(Z_1)} = \{0\}$. Similarly, $\beta f(cl_{\gamma X}(Z_2)) \subseteq \{1\}$. So $cl_{\gamma X}(Z_1) \cap cl_{\gamma X}(Z_2) = \varnothing$. $\square$

Say that ξX *satisfies (n)* iff ξX is a compactification of X and statement (n) is true when all instances of γ are replaced by ξ.

Theorem 3.17. *βX satisfies (1), (2).*

Proof. Theorems 3.15 and 3.16. $\square$

Theorem 3.18. *βX satisfies (3).*

Proof. Suppose f is a continuous function from X into a compact Hausdorff space Y. For each $p \in \beta X$, the collection $\{\mathrm{cl}_Y f[Z] : Z \in p\}$ has the finite intersection property since p does. By compactness of Y we have $\bigcap_{Z \in p} \mathrm{cl}_Y f[Z] \neq \varnothing$ for each $p \in \beta X$.

Define $\beta f : \beta X \to Y$ by

$$\beta f(p) \in \bigcap_{Z \in p} \mathrm{cl}_Y f[Z],$$

choosing $\beta f(p) = f(p)$ when $p \in X$ so that βf extends f. To prove βf is continuous, we show (*) $\beta f[\mathrm{cl}_{\beta X} A] \subseteq \mathrm{cl}_Y f[A]$ for any $A \in \mathcal{P}(X)$. To that end let $A \in \mathcal{P}(X)$. To prove (*) it suffices to show $\beta f[\mathrm{cl}_{\beta X} A] \subseteq Z$ for any $Z \in \mathcal{Z}(Y)$ with $f[A] \subseteq Z$ (Y is compact Hausdorff $\Rightarrow$ Y is T_1 completely regular $\Rightarrow$ $\mathcal{Z}(Y)$ is a closed base for Y). Well, Let $Z \in \mathcal{Z}(Y)$ with $f[A] \subseteq Z$. Then $A \subseteq f^{-1}(Z)$ and $f^{-1}(Z)$ is a zero set in X by continuity of f. Suppose $y \in \beta f[\mathrm{cl}_{\beta X} A]$. Then $y \in \beta f[\mathrm{cl}_{\beta X} f^{-1}(Z)]$. There exists $p \in \mathrm{cl}_{\beta X} f^{-1}(Z)$ such that $\beta f(p) = y$. We have $f^{-1}Z \in p$, so by definition of βf

$$y = \beta f(p) \in \mathrm{cl}_Y f[f^{-1}(Z)] \subseteq \mathrm{cl}_Y Z = Z.$$

Uniqueness now follows from Theorem 2.7, and implies each intersection $\bigcap_{Z \in p} \mathrm{cl}_Y f[Z]$ is actually a singleton. □

Corollary 3.19. *βX satisfies* (4), (5). □

Theorem 3.20. *If γX satisfies* (5) *and ξX satisfies* (2)*, then $\gamma X \simeq \xi X$.*

Proof. Let $h{:}\gamma X \to \xi X$ be the continuous function which is the identity on X. By Theorem 2.4 we just need to show that h is injective to prove h is a homeomorphism. To that end, let $p \neq q \in \gamma X$. There exists $f \in C(\gamma X)$ s.t. $f(p) = 0$ and $f(q) = 1$. Let $Z_1 = \{x \in X : f(x) \leq \frac{1}{3}\}$ and $Z_2 = \{x \in X : f(x) \geq \frac{2}{3}\}$. Suppose V is an open set γX containing p. Then $V \cap f^{-1}(-\infty, \frac{1}{3})$ is a nonempty open set in γX (it contains p). Since X is dense in γX, there exists $x \in V \cap f^{-1}(-\infty, \frac{1}{3}) \cap X$. Then $x \in V \cap Z_1$. Thus every open set in γX containing p contains a point of Z_1. That is, $p \in \mathrm{cl}_{\gamma X} Z_1$. Similarly, $q \in \mathrm{cl}_{\gamma X} Z_2$. By continuity of h we have $h(p) \in h(\mathrm{cl}_{\gamma X} Z_1) \subseteq \mathrm{cl}_{\xi X} h(Z_1) = \mathrm{cl}_{\xi X} Z_1$. Similarly, $h(q) \in \mathrm{cl}_{\xi X} Z_2$. By hypothesis $\mathrm{cl}_{\xi X} Z_1 \cap \mathrm{cl}_{\xi X} Z_2 = \varnothing$, thus $h(p) \neq h(q)$. □

Theorem 3.21. *If γX satisfies any of* (1)-(5) *then $\gamma X \simeq \beta X$.*

Proof. Suppose γX satisfies one of (1)-(5). By Theorem 3.16, γX satisfies (2) or (5). βX satisfies (2) and (5), so by the previous theorem $\gamma X \simeq \beta X$. $\square$

Thus βX is the unique (up to homeomorphism) compactification of X with any (all) of the properties (1)-(5). We call it the *Stone-Čech compactification* of X. The function βf defined in Theorem 3.18 is called the *Stone-Čech extension* of f.

3.4 Additional Properties

Theorem 3.22.

(i) If X is normal and $\mathcal{L}(X)$ is the collection of closed subsets of X, then $\beta X \simeq \beta X(\mathcal{L})$.

(ii) If X is compact Hausdorff and $\mathcal{C}(X)$ is a closed lattice base for X, then $X \simeq \beta X(\mathcal{C})$.

Proof. Suppose X is normal. By Theorem 3.6 and Urysohn's Lemma $\beta X(\mathcal{L})$ is a compactification of X satisfying (2). This proves (i). Suppose X is compact Hausdorff. Then $\mathcal{C}(X)$ is a normal base for X by Theorem 3.6, so the conclusion of (ii) follows from Corollary 3.14. $\square$

Theorem 3.23. *If A is closed in X then $\beta A \simeq cl_{\beta X} A$.*

Proof. Clearly $\mathrm{cl}_{\beta X} A$ is a compactification of A. By Tietze's Extension Theorem and property (4) of βX, every function in $C^*(A)$ has a continuous extension to $\mathrm{cl}_{\beta X} A$. $\square$

Theorem 3.24. *If $f : X \to Y$ is a homeomorphism, γX is a compactification of X, and $h : \gamma X \to \beta Y$ continuously extends f, then h is a homeomorphism.*

Proof. It suffices to show h is injective. Well, $\beta(f^{-1}) \circ h$ is the identity on X. So in fact $\beta(f^{-1}) \circ h$ is the identity on γX, thus h is injective. $\square$

If γX is a compactification of X, then $\gamma X \setminus X$ is called the *remainder* of γX. We will sometimes write X^* for the *Stone-Čech remainder* $\beta X \setminus X$.

Theorem 3.25. *If γX is a compactification of X, then $\beta X \setminus X$ maps continuously onto $\gamma X \setminus X$ (via the Stone-Čech extension of the identity on X).*

Proof. Let $\beta i : \beta X \to \gamma X$ be the Stone-Čech extension of the identity on X. It suffices to show there is no $p \in \beta X \setminus X$ such that $\beta i(p) \in X$. For a contradiction, suppose there is such a p. Then $p \neq \beta i(p)$. Separate p and $\beta i(p)$ with disjoint open sets $U_p, U_{\beta i(p)} \subseteq \beta X$. There exists an open $V \subseteq \gamma X$ such that $V \cap X = U_{\beta i(p)} \cap X$. By continuity of βi there exists an open $W \subseteq \beta X$ containing p, mapping into V. There exists $x \in U_p \cap W \cap X$. But $x \notin U_{\beta i(p)} \cap X = V \cap X$, so $\beta i(x) = x \notin V$. Contradiction. □

Theorem 3.26. *X is locally compact Hausdorff iff X is open in βX.*

Proof. Suppose X is locally compact Hausdorff. Then X has a one-point compactification. The one-point remainder is the continuous image of X^*, so X^* is closed in βX. Conversely, open subspaces of (locally) compact spaces are locally compact. □

Theorem 3.27. *Suppose X is a T_2 locally compact countable union of compact spaces. If A is F_σ in $\beta X \setminus X$, then $cl_{\beta X \setminus X} A \simeq \beta A$.*

Proof. $\beta X \setminus X$ is closed in βX by the previous theorem. So $\mathrm{cl}_{\beta X \setminus X} A$ is a compactification of A. Now we show A is C^*-embedded in $\mathrm{cl}_{\beta X \setminus X} A$. Let $f \in C^*(A)$. The assumption that X is a countably union of compact spaces implies X is F_σ in βX. Since $\beta X \setminus X$ is closed in βX, $X \cup A$ is F_σ in βX and A is closed in $X \cup A$. By Theorem 2.14, $X \cup A$ is normal. Apply Tietze's Extension Theorem to extend f to $\hat{f} \in C^*(X \cup A)$. Since X is dense in $X \cup A$, $\beta(\hat{f} \restriction X)$ must extend $\hat{f}$. $\beta(\hat{f} \restriction X) \restriction \mathrm{cl}_{\beta X \setminus X} A$ is the desired extension of f. □

Theorem 3.28. *If $f : X \to Y$ is surjective, $f^{-1}\{y\}$ is compact for each $y \in Y$, and $\beta f : \beta X \to \beta Y$ is the Stone-Čech extension of f, then $\beta f[X^*] = Y^*$.*

Proof. ($\subseteq$): Suppose $p \in X^*$. Let $y \in Y$. Then $f^{-1}\{y\} \notin p$. There exists $A \in p$ with $A \cap f^{-1}\{y\} = \varnothing$. Then $f[A] \cap \{y\} = \varnothing$. So $\mathrm{cl}_{\beta Y} f[A] \cap \{y\} = \varnothing$. So $\beta f(p) \neq y$. ($\supseteq$): βf is surjective because f is surjective. □

Theorem 3.29. *If $f : X \to Y$ is a closed mapping into a normal space Y and $\beta f : \beta X \to \beta Y$ is the Stone-Čech extension of f, then $\beta f^{-1}\{q\} = \bigcap_{B \in q} cl_{\beta X} f^{-1}(B)$ for any $q \in \beta Y$.*

Proof. Let $q \in \beta Y$. For each $p \in \beta X$ we have

$$\begin{aligned} p \in \mathrm{cl}_{\beta X} f^{-1}(B) \text{ for all } B \in q &\Leftrightarrow f^{-1}(B) \in p \text{ for all } B \in q \\ &\Leftrightarrow f^{-1}(B) \cap A \neq \varnothing \text{ for all } B \in q \text{ and } A \in p \\ &\Leftrightarrow B \cap f(A) \neq \varnothing \text{ for all } B \in q \text{ and } A \in p \\ &\Leftrightarrow f(A) \in q \text{ for all } A \in p \\ &\Leftrightarrow \beta f(p) = q \;\Leftrightarrow\; x \in \beta f^{-1}(\{q\}). \end{aligned}$$

□

3.5 Examples

3.5.1 $\beta\omega$

Let ω be the countably infinite discrete space. Then $\beta\omega$ is just the set of all ultrafilters on ω with a clopen base consisting of the sets $B(A) = F(A)$, $A \subseteq \omega$. The embedded copy of ω consists of the principal ultrafilters $\{A \subseteq \omega : n \in A\}$, $n \in \omega$, while ω^* consists of the free ultrafilters.

Lemma 3.30. *The product space* $2^{\mathfrak{c}}$ *is separable.*

Proof. Let $\mathcal{B}$ be a countable basis for the product space 2^ω. Let D be the union of all sets

$$\left\{ f \in 2^{2^\omega} : (\forall i \in \{1,...,n\})((f \restriction U_i \equiv 0 \vee f \restriction U_i \equiv 1) \wedge (\ f \restriction (2^\omega \setminus U_1 \cup ... \cup U_n) \equiv 0)) \right\}$$

over the finite subsets $\{U_1, ..., U_n\}$ of $\mathcal{B}$. Note that $|D| = \omega$. We will now show that D is dense in $2^{\mathfrak{c}}$. Let U be a nonempty basic open set in 2^{2^ω}. Write $U = \bigcap_{i=1}^n \pi_{\alpha_i}^{-1}\{p_i\}$, where each $p_i \in \{0,1\}$ and the $\alpha_i \in 2^\omega$ are distinct. Since 2^ω is Hausdorff, we may separate the points $\alpha_1, ..., \alpha_n$ with finitely many pairwise disjoint basic open sets $U_1, ..., U_n \in \mathcal{B}$. Let $f \in \mathcal{D}$ with $f \restriction U_i \equiv p_i$ for each i. Then $f \in U \cap D$. □

Theorem 3.31. $|\beta\omega| = 2^{\mathfrak{c}}$.

Proof. Clearly $|\beta\omega| \leq 2^{\mathfrak{c}}$ (note that in general every separable Hausdorff space has cardinality $\leq 2^{\mathfrak{c}}$). By the previous lemma there is a map $f : \omega \to 2^{\mathfrak{c}}$ from ω onto a dense subset of the compact Hausdorff product space $2^{\mathfrak{c}}$. Then βf witnesses $|\beta\omega| \geq 2^{\mathfrak{c}}$. $\square$

3.5.2 $\beta\mathbb{R}$ and $\beta\mathbb{H}$

Let $\mathbb{H} = [0, \infty)$.

Theorem 3.32. $\mathbb{H}^* = \bigcap_{n\in\omega} cl_{\beta\mathbb{H}}[n, \infty)$.

Proof. If $p \in \beta\mathbb{H} \setminus \bigcap_{n\in\omega} \mathrm{cl}_{\beta\mathbb{H}}[n, \infty)$ then there exists $n \in \omega$ and $A \in p$ such that $A \cap [n, \infty) = \varnothing$. Then A is compact and we have $\bigcap p \neq \varnothing$ so that $p \in \mathbb{H}$. This proves $\mathbb{H}^* \subseteq \bigcap_{n\in\omega} \mathrm{cl}_{\beta\mathbb{H}}[n, \infty)$. On the other hand, if $p \in \mathbb{H}$ then there exists $n \in \omega$ with $p < n$ and we have $p \notin \mathrm{cl}_{\beta\mathbb{H}}[n, \infty)$. $\square$

Theorem 3.33. $\mathbb{H}^*$ *is a continuum.*

Proof. For each $n \in \omega$, $\mathrm{cl}_{\beta\mathbb{H}}[n, \infty)$ is compact and connected by Theorem 2.10. So $\mathbb{H}^*$ is the intersection of a nested collection of continua. By Theorem 2.11, $\mathbb{H}^*$ is a continuum. $\square$

Using similar arguments to those in the preceding proofs (let $(-\infty, n] \cup [n, \infty)$ play the role of $[n, \infty)$),

$$\mathbb{R}^* = \bigcap_{n\in\omega} \mathrm{cl}_{\beta\mathbb{R}}(-\infty, n] \cup \bigcap_{n\in\omega} \mathrm{cl}_{\beta\mathbb{R}}[n, \infty).$$

Thus $\mathbb{R}^*$ is compact but not connected, and has two disjoint copies of $\mathbb{H}^*$ as its connected components.

3.5.3 $\beta\omega_1$ and βL

Let ω_1 be the first uncountable ordinal with the order topology. Let $L = \omega_1 \times [0, 1)$ with the lexicographic order topology (the Long Line). Each topological space is locally compact Hausdorff, and therefore has a one-point compactification αX.

Theorem 3.34. *Every continuous real-valued function on ω_1 is eventually constant.*

Proof. Let $f \in C(\omega_1)$. Let S be the stationary set $\{\delta < \omega_1 : \delta \text{ is a limit ordinal}\}$. For each $n \in \mathbb{N}$ we define $\alpha_n < \omega_1$ such that f varies by less than $\frac{1}{n}$ on (α_n, ω_1). Fix $n \in \mathbb{N}$. Using the continuity of f, for each $\delta \in S$ let $g_n(\delta) < \delta$ such that f varies by less than $\frac{1}{n}$ on $(g_n(\delta), \delta]$. By the Pressing Down Lemma, there exists a stationary set $S_n \subseteq S$ and $\alpha_n < \omega_1$ such that $g_n(\delta) = \alpha_n$ for all $\delta \in S_n$. So f varies by less than $\frac{1}{n}$ on $(\alpha_n, \delta]$ for all $\delta \in S_n$. Since S_n is unbounded, f varies by less than $\frac{1}{n}$ on (α_n, ω_1). Since ω_1 is regular, $\sup_{n\in\mathbb{N}}\alpha_n < \omega_1$. Clearly f must be constant on the final segment $(\sup_{n\in\mathbb{N}}\alpha_n, \omega_1)$ of ω_1. $\square$

Theorem 3.34 implies ω_1 is C^*-embedded in its one-point compactification $\alpha\omega_1 = \omega_1 + 1$.

Corollary 3.35. $\beta\omega_1 = \alpha\omega_1$. $\square$

Another way of proving $\beta\omega_1 = \alpha\omega_1$ is to simply show that the remainder of $\beta\omega_1$ cannot have more than one point (use the fact that any two closed unbounded subsets of ω_1 have nonempty intersection).

Theorem 3.36. *Every continuous real-valued function on L is eventually constant.*

Proof. Let $f \in C(L)$. Note that any subspace of L of the form $\{\langle\alpha, x_\alpha\rangle : \alpha < \omega_1\}$ is homeomorphic to ω_1 in the order topology. By Theorem 3.34, f is eventually constant on any set of the form $\{\langle\alpha, x_\alpha\rangle : \alpha < \omega_1\}$. In particular, for each $q \in \mathbb{Q} \cap [0,1)$ there exists $\gamma_q < \omega_1$ and $r_q \in \mathbb{R}$ such that $f(\langle\alpha, q\rangle) = r_q$ for all $\alpha > \gamma_q$. Letting $\gamma = \sup_{q\in\mathbb{Q}\cap[0,1)} \gamma_q < \omega_1$, we have $f(\langle\alpha, q\rangle) = r_q$ for each $\alpha > \gamma$ and $q \in \mathbb{Q} \cap [0,1)$. Since $\mathbb{Q} \cap [0,1)$ is dense in $[0,1)$, we have $f(\{\alpha\} \times [0,1)) = f(\{\beta\} \times [0,1))$ for all $\alpha, \beta > \gamma$. Suppose for a contradiction that f is *not* constant on $\bigcup_{\alpha>\gamma}\{\alpha\} \times [0,1)$. Since $\{\alpha\} \times [0,1)$ is connected and f is continuous, $f(\{\alpha\} \times [0,1))$ is connected in $\mathbb{R}$. So there exists a nonempty interval $(a,b) \subseteq \mathbb{R}$ such that $(a,b) \subseteq f(\{\alpha\} \times [0,1))$ for all $\alpha > \gamma$. As $|(a,b)| \geq \omega_1$, there exists $\{r_\alpha : \gamma < \alpha < \omega_1\} \subseteq (a,b)$ such that $r_\alpha \neq r_\beta$ for any $\gamma < \alpha < \beta < \omega_1$. For each $\alpha > \gamma$ let $\langle\alpha, x_\alpha\rangle \in \{\alpha\} \times [0,1)$ such that $f(\langle\alpha, x_\alpha\rangle) = r_\alpha$. Then f is not eventually constant on $\{\langle\alpha, x_\alpha\rangle : \alpha > \gamma\}$, a contradiction. $\square$

Corollary 3.37. $\beta L = \alpha L$. $\square$

Chapter 4

Ultrapowers

4.1 Definition

Suppose X is a set and u is an ultrafilter on ω. Define a relation on X^ω by

$$f \sim g \Leftrightarrow \{n \in \omega : f(n) = g(n)\} \in u.$$

It is easily checked that $\sim$ is an equivalence relation: symmetry is obvious, reflexivity follows from the fact that $X \in u$, and transitivity follows from the fact that u is closed under finite intersections and supersets. The *ultrapower* X^ω/u is the set of corresponding equivalence classes f/u. If X is a linearly ordered set then we may define a relation on X^ω/u by

$$f/u < g/u \Leftrightarrow \{n \in \omega : f(n) < g(n)\} \in u.$$

Theorem 4.1. *X^ω/u is linearly ordered by $<$.*

Proof. Irreflexivity: $\{n \in \omega : f(n) < f(n)\} = \varnothing \notin u$ so $f/u \not< f/u$. Antisymmetry:

$$f/u < g/u \Rightarrow \{n \in \omega : f((n) < g(n)\} \in u \Rightarrow \{n \in \omega : f(n) \geq g(n)\} \notin u \Rightarrow f/u \ngtr g/u.$$

Transitivity: Suppose $f/u < g/u$ and $g/u < h/u$. Then $\{n \in \omega : f(n) < h(n)\} \supseteq \{n \in \omega : f(n) < g(n)\} \cap \{n \in \omega : g(n) < h(n)\} \in u$, so $f/u < h/u$. Comparability: If $f/u \neq g/u$ then $\{n \in \omega : f(n) = g(n)\} \notin u$. So the complement $\{n \in \omega : f(n) < g(n)\} \cup \{n \in \omega : f(n) > g(n)\}$ is in u. By Theorem 3.3, $\{n \in \omega : f(n) < g(n)\} \in u$ or $\{n \in \omega : f(n) > g(n)\} \in u$, so that $f/u < g/u$ or $g/u > f/u$. $\square$

We will be primarily interested in the case $X = \mathbb{R}$. We may view $\mathbb{R}$ as a linearly ordered subset of $\mathbb{R}^\omega/u$ by identifying $c \in \mathbb{R}$ with the equivalence class f/u, where f is given by $f(n) = c$ for all $n \in \omega$. Under this identification, $\mathbb{R}^\omega/u = \mathbb{R}$ when u is a principal ultrafilter. If u is free then $\mathbb{R}^\omega/u$ properly contains $\mathbb{R}$, and is sometimes called *hyper-real*. In Chapter 6 we will assume $\neg$CH and find $2^\mathfrak{c}$ free ultrafilters $u \in \omega^*$ such that the corresponding $\mathbb{R}^\omega/u$ are pairwise nonisomorphic.

4.2 CH

A dense linear order L is *countably saturated* if for any countable subsets A and B with $A < B$, there exists $v \in L$ such that $A < v < B$.

Theorem 4.2. *$\mathbb{R}^\omega/u$ is countably saturated for any $u \in \omega^*$.*

Proof. Let A and B be countable subsets of $\mathbb{R}^\omega/u$. Let $(a^j/u)_{j\in\omega}$ be an increasing cofinal sequence in A and $(b^k/u)_{k\in\omega}$ a decreasing coinitial sequence in B. Choose a representative $(a^0_i)_{i\in\omega}$ from a^0/u. Assuming a representative has been chosen from a^j/u, select a representative from a^{j+1}/u such that $a^j_i \le a^{j+1}_i$ for all $i \in \omega$. Recursively select representatives from each member of $(b^k/u)_{n\in\omega}$ in a similar manner so that $b^{k+1}_i \le b^k_i$ for all $k, i \in \omega$.

For each $i \in \omega$ let $M_i = \{m \le i : a^m_i < b^m_i\}$. Claim: For each $i \in \omega$ there exists $x_i \in \mathbb{R}$ such that $a^m_i < x_i < b^m_i$ for each $m \in M_i$. Fix $i \in \omega$. Assume $M_i \ne \varnothing$. Let $\rho = \max(M_i)$. Then $a^m_r \le a^\rho_i < b^\rho_i \le b^m_i$ for all $m \in M_i$, by choice of representatives. Let $x_i \in (a^\rho_i, b^\rho_i)$.

Define $(x_i)_{i\in\omega} \in \mathbb{R}^\omega$ as indicated above, setting $x_i = 0$ if $M_i = \varnothing$. Now suppose $j, k \in \omega$. Let $m = \max(j, k)$. Let $E \in u$ such that $a^m_i < b^m_i$ for all $i \in E$. For each $i \in E$ with $i \ge m$ we have $m \in M_i$. Thus $a^m_i < x_i < b^m_i$ for all $i \in E \cap \{n \in \omega : n \ge m\} \in u$. So $a^j/u \le a^m/u < x/u < b^m/u \le b^k/u$. $\square$

Theorem 4.3. *All countably saturated linear orders of cardinality ω_1 are isomorphic.*

Proof. Suppose L and L' are countably saturated linear orders of cardinality ω_1. Enumerate $L = \{l_\xi : \xi < \omega_1\}$ and $L' = \{l'_\xi : \xi < \omega_1\}$. We construct a bijection $\varphi : \omega_1 \to \omega_1$ inductively so that the induced mapping $l_\alpha \mapsto l'_{\varphi(\alpha)}$ is an order preserving isomorphism from L onto L'. Let $\varphi_0 = \{(0,0)\}$. Suppose $\gamma < \omega_1$, and for each $\delta < \gamma$ a bijection φ_δ has been defined between subsets of ω_1 so that

(i) $\delta \subseteq \mathrm{dom}(\varphi_\delta) \cap \mathrm{ran}(\varphi_\delta)$, $|\varphi_\delta| \leq \omega$, $\varphi_i \subseteq \varphi_j$ for all $i < j \leq \delta$, and

(ii) the induced mapping $l_\alpha \mapsto l'_{\varphi_\delta(\alpha)}$ is an order preserving isomorphism from $\{l_\alpha : \alpha \in \mathrm{dom}(\varphi_\delta)\}$ onto $\{l'_\beta : \beta \in \mathrm{ran}(\varphi_\delta)\}$.

Define φ_γ as follows. Let $\psi = \bigcup_{\delta<\gamma} \varphi_\delta$. Let $\xi_1 < \omega_1$ be the least ordinal not in $\mathrm{dom}(\psi)$. Let

$$A = \{\alpha \in \mathrm{dom}(\psi) : l_\alpha < l_{\xi_1}\} \text{ and } B = \{\alpha \in \mathrm{dom}(\psi) : l_{\xi_1} < l_\alpha\}.$$

There exists $\xi_2 < \omega_1$ such that $\{l'_{\psi(\alpha)} : \alpha \in A\} < l'_{\xi_2} < \{l'_{\psi(\alpha)} : \alpha \in B\}$. Extend ψ by defining $\psi(\xi_1) = \xi_2$. Now let $\xi_3 < \omega_1$ be the least ordinal not in $\mathrm{ran}(\psi)$. Let

$$A' = \{\beta \in \mathrm{ran}(\psi) : l'_\beta < l'_{\xi_3}\} \text{ and } B' = \{\beta \in \mathrm{ran}(\psi) : l'_{\xi_3} < l'_\beta\}.$$

There exists $\xi_4 < \omega_1$ such that $\{l_{\psi^{-1}(\beta)} : \beta \in A'\} < l_{\xi_4} < \{l_{\psi^{-1}(\beta)} : \beta \in B'\}$. Extend ψ again by defining $\psi(\xi_4) = \xi_3$. Let $\varphi_\gamma = \psi$. $\square$

Since $|\mathbb{R}^\omega/u| = \mathfrak{c}$, we have the following.

Corollary 4.4 (CH). *$\mathbb{R}^\omega/u \simeq \mathbb{R}^\omega/v$ for all $u, v \in \omega^*$.* $\square$

Chapter 5

The Continua $\mathbb{I}_u$ $(u \in \omega^*)$

5.1 Definition

Let $\mathbb{I} = [0,1]$, $\mathbb{M} = \omega \times \mathbb{I}$, and $\mathbb{I}_n = \{n\} \times \mathbb{I}$. For each $u \in \omega^*$ let

$$\mathbb{I}_u = \bigcap_{A \in u} \mathrm{cl}_{\beta\mathbb{M}} \bigcup_{n \in A} \mathbb{I}_n.$$

Theorem 5.1. $\mathbb{I}_u$ *is a continuum.*

Proof. Clearly $\mathbb{I}_u$ is compact Hausdorff. Now we show $\mathbb{I}_u$ is connected. Suppose not. Then there exist two disjoint closed subsets H and K of $\mathbb{M}$ such that $\mathbb{I}_u \subseteq \mathrm{cl}_{\beta\mathbb{M}}H \cup \mathrm{cl}_{\beta\mathbb{M}}K$ and $\mathrm{cl}_{\beta\mathbb{M}}H \cap \mathbb{I}_u \neq \varnothing \neq \mathrm{cl}_{\beta\mathbb{M}}K \cap \mathbb{I}_u$. There exist $A, B \in u$ such that $H \cap \mathbb{I}_n \neq \varnothing$ for all $n \in A$ and $K \cap \mathbb{I}_n \neq \varnothing$ for all $n \in B$. There exists $C \in u$ such that $\mathbb{I}_n \subseteq (H \cup K)$ for all $n \in C$. There exists $n \in A \cap B \cap C$. Then H and K disconnect $\mathbb{I}_n$, a contradiction. $\square$

$\mathbb{I}_u$ is a subcontinuum of $\mathbb{H}^*$ by the following.

Theorem 5.2. $\mathbb{I}_u \simeq \bigcap_{A \in u} cl_{\beta\mathbb{H}} \bigcup_{n \in A} [n, n+1] \subseteq \mathbb{H}^*$.

Proof. Without loss of generality assume the set E of even numbers in ω is in u. The map $\sigma : E \times \mathbb{I} \to \bigcup_{n \in E} [n, n+1]$ defined by $\sigma(\langle n, x \rangle) = n + x$ is a homeomorphism. By Theorems 3.23 and 3.24 there exists a homeomorphism $\hat{\sigma} : \mathrm{cl}_{\beta\mathbb{M}}(E \times \mathbb{I}) \to \mathrm{cl}_{\beta\mathbb{H}} \bigcup_{n \in E} [n, n+1]$ such that $\hat{\sigma} \restriction (E \times \mathbb{I}) = \sigma$. We have

$$\mathbb{I}_u = \bigcap_{A \in u} \mathrm{cl}_{\beta\mathbb{M}} \bigcup_{n \in A} \mathbb{I}_n = \bigcap_{A \in u} \mathrm{cl}_{\beta\mathbb{M}} \bigcup_{n \in A \cap E} \mathbb{I}_n \simeq \hat{\sigma} \bigcap_{A \in u} \mathrm{cl}_{\beta\mathbb{M}} \bigcup_{n \in A \cap E} \mathbb{I}_n$$

$$= \bigcap_{A\in u} \mathrm{cl}_{\beta\mathbb{H}}\sigma \bigcup_{n\in A\cap E} \mathbb{I}_n = \bigcap_{A\in u} \mathrm{cl}_{\beta\mathbb{H}} \bigcup_{n\in A\cap E} [n, n+1] = \bigcap_{A\in u} \mathrm{cl}_{\beta\mathbb{H}} \bigcup_{n\in A} [n, n+1].$$

Since u is free it contains the filter of cofinite sets. So $\bigcap_{A\in u} \mathrm{cl}_{\beta\mathbb{H}} \bigcup_{n\in A}[n, n+1] \subseteq \mathbb{H}^*$ by Theorem 3.32. $\square$

Thus $\mathbb{I}_u$ is a subcontinuum of $\mathbb{H}^*$. We now give one more representation of $\mathbb{I}_u$. Let $\pi : \mathbb{M} \to \omega$ be defined by $\langle n, x\rangle \mapsto n$. Then π is continuous, closed, and surjective. Let $\beta\pi : \beta\mathbb{M} \to \beta\omega$ continuously extend π. By Theorem 3.29,

$$\mathbb{I}_u = \bigcap_{A\in u} \mathrm{cl}_{\beta\mathbb{M}} \bigcup_{n\in A} \mathbb{I}_n = \bigcap_{A\in u} \mathrm{cl}_{\beta\mathbb{M}}\pi^{-1}(A) = \beta\pi^{-1}(u).$$

In general it can be shown that $\beta f^{-1}\{p\}$ is a continuum whenever $\beta f : \beta X \to \beta Y$ is the continuous extension of a closed continuous surjection $f : X \to Y$ whose singleton preimages are continua.

5.2 Decompositions and the Proof of Theorem 1.2

In this section we will identify $\mathbb{R}$ with the interval $(0, 1)$. For each $x/u \in \mathbb{R}^\omega/u$ choose a representative $(x_n) \in x/u$ and let

$$x_u = \{L \in \mathcal{L}(\mathbb{M}) : (\exists A \in u)(L \supseteq \{\langle n, x_n\rangle : n \in A\})\}.$$

Let $P_u = \{x_u : x/u \in \mathbb{R}^\omega/u\}$.

Theorem 5.3.

(i) $x_u \in \mathbb{I}_u$ *and*

(ii) $x_u = y_u$ *iff* $\{n \in \omega : x_n = y_n\} \in u$.

Proof. It is easily checked that x_u is a filter. To show it is an ultrafilter, suppose $L \in \mathcal{L}(\mathbb{M})$ intersects every set in x_u. Let $A = \{n \in \omega : \langle n, x_n\rangle \in L\}$. If $\omega \setminus A \in u$ then $\{\langle n, x_n\rangle : n \in \omega \setminus A\} \in x_u$, but $L \cap \{\langle n, x_n\rangle : n \in \omega \setminus A\} = \varnothing$. Therefore we must have $A \in u$.

Since $L \supseteq \{\langle n, x_n\rangle : n \in A\}$, we have $L \in x_u$. Thus x_u is an ultrafilter. Clearly it contains all the sets $\bigcup_{n\in A} \mathbb{I}_n$, $A \in u$, so that $x_u \in \mathbb{I}_u$. Now we prove (ii). Let $A = \{n \in \omega : x_n = y_n\}$. Suppose $A \in u$. Let $L \in x_u$. There exists $B \in u$ such that $L \supseteq \{\langle n, x_n\rangle : n \in B\}$. Then $A \cap B \in u$ and $L \supseteq \{\langle n, x_n\rangle : n \in A \cap B\} = \{\langle n, y_n\rangle : n \in A \cap B\}$, so $L \in y_u$. This proves $x_u \subseteq y_u$ and the reverse inclusion follows similarly. Conversely, if $x_u = y_u$ and $\omega \setminus A \in u$ then $\varnothing = \{\langle n, x_n\rangle : n \in \omega \setminus A\} \cap \{\langle n, y_n\rangle : n \in \omega \setminus A\} \in x_u$, a contradiction. $\square$

Thus P_u is a copy of the set $\mathbb{R}^\omega/u$ inside of $\mathbb{I}_u$. Note that not every element in $\mathbb{I}_u$ is of the type x_u. Consider, for example, the collection

$$\left\{\bigcup_{n\in A} \mathbb{I}_n : A \in u\right\} \cup \{\mathbb{M} \setminus G : G \subseteq \mathbb{M} \text{ is open and } \mu(G \cap \mathbb{I}_n) < 1/n \text{ for each } n \in \omega\}.$$

Each set in u is infinite, so sets of the first type contain $\mathbb{I}_n$ for arbitrarily high $n \in \omega$. For sufficiently large n, no finite number of the sets G can cover $\mathbb{I}_n$, so the collection has the finite intersection property. The ultrafilter generated by this collection can contain no set of the form $\{\langle n, x_n\rangle : n \in \omega\}$ since we can cover $\{\langle n, x_n\rangle : n \in \omega\}$ with one of the sets G.

Theorem 5.4. *P_u is dense in $\mathbb{I}_u$.*

Proof. Let $B(O) \cap \mathbb{I}_u$ be a nonempty basic open subset of $\mathbb{I}_u$. Then $A = \{n \in \omega : O \cap \mathbb{I}_n \neq \varnothing\} \in u$. For each $n \in A$ choose $\langle n, x_n\rangle \in O \cap \mathbb{I}_n$ with $x_n \neq 0, 1$. Then $x_u \in B(O) \cap P_u$. $\square$

For each $a_u < b_u \in \mathbb{R}^\omega/u$, define an *interval* in $\mathbb{I}_u$ by

$$[a_u, b_u] = \bigcap_{A\in u} \mathrm{cl}_{\beta\mathbb{M}} \bigcup_{n\in A} \{n\} \times [a_n, b_n]$$

It is easily checked that $[a_u, b_u]$ is well-defined. Each interval is a continuum by the same argument we used to prove $\mathbb{I}_u$ is a continuum.

Define a relation on $\mathbb{I}_u$ by $x \sim y$ iff every interval containing x contains y.

Theorem 5.5. *$\sim$ is an equivalence relation.*

Proof. Clearly $\sim$ is reflexive and transitive, so we just need to prove $\sim$ is symmetric. Suppose $x \nsim y$. Then there is an interval $[a_u, b_u]$ containing x but not y. There exists $L \in y$ such that $L \cap \bigcup_{n\in\omega}\{n\} \times [a_n, b_n] = \varnothing$. For each $n \in \omega$ let $c_n = \sup\{c \in L \cap \mathbb{I}_n : c < a_n\}$ and $d_n = \inf\{d \in L \cap \mathbb{I}_n : b_n < d\}$. Then $y \in [0_u, c_u] \cup [d_u, 1_u]$. Without loss of generality, assume $y \in [0_u, c_u]$. Since $[0_u, c_u] \cap [a_u, b_u] = \varnothing$, we have $x \notin [0_u, c_u]$. Thus $y \nsim x$. □

For each $x \in \mathbb{I}_u$ define $L_x = x/\sim$, the equivalence class of x with respect to $\sim$. The L_x are called a *layers* of $\mathbb{I}_u$.

Theorem 5.6. *Each layer is a continuum.*

Proof. Notice $L_x = \bigcap\{[a_u, b_u] : x \in [a_u, b_u]\}$. Apply Theorem 2.11. □

Theorem 5.7. $L_{x_u} = \{x_u\}$ *for all* $x_u \in P_u$.

Proof. Clearly $x_u \in L_{x_u}$. Now suppose $y \in \mathbb{I}_u \setminus \{x_u\}$. There exists $A \in y$ and $B \in u$ such that $A \cap \{\langle n, x_n\rangle : n \in B\} = \varnothing$. For each $n \in B$ we may define a subinterval $\{n\} \times [a_n, b_n]$ of $\mathbb{I}_n$ containing $\langle n, x_n\rangle$, missing $A \cap \mathbb{I}_n$. Then $x_u \in [a_u, b_u]$ but $y \notin [a_u, b_u]$. So $y \notin L_{x_u}$. □

A partial ordering on the intervals in $\mathbb{I}_u$ is given by $[a_u, b_u] < [c_u, d_u]$ iff $b_u < c_u$. Define a relation $\prec$ on the set of layers of $\mathbb{I}_u$ by $L_x \prec L_y$ if there are intervals I_1 and I_2 with $x \in I_1$, $y \in I_2$, and $I_1 < I_2$. For each $x \in \mathbb{I}_u$ let $[0_u, L_x) = \bigcup_{L_y \prec L_x} L_y$ and $[0_u, L_x] = [0_u, L_x) \cup L_x$.

Theorem 5.8.

(i) $\prec$ *linearly orders* $\mathbb{I}_u/\sim$,

(ii) the canonical epimorphism $\pi : \mathbb{I}_u \hookrightarrow (\mathbb{I}_u/\sim, \prec)$ *is continuous if* $(\mathbb{I}_u/\sim, \prec)$ *has the order topology induced by* $\prec$, *and*

(iii) $(\mathbb{I}_u/\sim, \prec)$ *contains a dense copy of* $\mathbb{R}^\omega/u$.

Proof. (i) $\prec$ is irreflexive: This follows from irreflexivity of the partial ordering $<$ of intervals. $\prec$ is a total ordering: Suppose $L_x \neq L_y$. By Theorem 5.5, x and y are contained in disjoint intervals. One interval must be less than the other, so $L_y \prec L_x$ or $L_x \prec L_y$. $\prec$ is transitive:

Suppose $L_x \prec L_y$ and $L_y \prec L_z$. There exist intervals $I_1 < I_2$ and $I_3 < I_4$ such that $x \in I_1$, $y \in I_2 \cap I_3$, and $z \in I_4$. Then $I_2 \cap I_3$ is an interval and $I_1 < I_2 \cap I_3 < I_4$, so that $L_x \prec L_z$. $\prec$ is antisymmetric: Suppose for a contradiction that $L_x \prec L_y$ and $L_y \prec L_x$. Then there exist $I_1, I_2 \in x$ and $I_3, I_4 \in y$ with $I_1 < I_3$ and $I_4 < I_2$. But $I_1 \cap I_2$ and $I_3 \cap I_4$ are nonempty disjoint intervals with $I_1 \cap I_2 < I_3 \cap I_4$ and $I_3 \cap I_4 < I_1 \cap I_2$, a contradiction.

(ii) We prove that π is continuous by showing $[0_u, L_x]$ is closed in $\mathbb{I}_u$. Suppose $y \in \mathbb{I}_u \setminus [0_u, L_x]$. Then there are intervals $[a_u, b_u]$ containing x and $[c_u, d_u]$ containing y such that $[a_u, b_u] < [c_u, d_u]$. Then $B(\bigcup_{n\in\omega}\{n\} \times (b_n, 1_n])$ separates y from $[0_u, L_x]$.

(iii)The mapping $x/u \mapsto \{x_u\}$ identifies $\mathbb{R}^\omega/u$ with a subset of $\mathbb{I}_u/\sim$. If $a_u < b_u \in \mathbb{R}^\omega/u$ then it is an easy matter to select intervals $[c_u, d_u]$ and $[e_u, f_u]$ such that $a_u \in [c_u, d_u]$, $b_u \in [e_u, f_u]$, and $[c_u, d_u] < [e_u, f_u]$. Thus $\prec$ extends the ordering on $\mathbb{R}^\omega/u$. That $\mathbb{R}^\omega/u$ is dense in this ordering follows from Theorem 5.4 and the continuity of π. □

Theorem 5.9. $(\mathbb{I}_u/\sim, \prec)$ *is the completion of* $\mathbb{R}^\omega/u$.

Proof. $\mathbb{I}_u/\sim$ is compact in the order topology since π is continuous. Apply Theorems 2.20 and 5.8. □

We already know that there are non-$\mathbb{R}^\omega/u$ points in $\mathbb{I}_u$. The layers of these points can be large.

Theorem 5.10. *Countable cofinality layers in* $\mathbb{I}_u$ *contain copies of* ω^*.

Proof. Let $(a_u^n)_{n\in\omega}$ be any strictly increasing sequence of elements in $\mathbb{R}^\omega/u$. Let $A = \{a_u^n : n \in \omega\}$. There exists $L_x \in \mathbb{I}_u/\sim$ such that $L_x = \sup_{\mathbb{I}_u/\sim} A = (\mathrm{cl}_{\mathbb{I}_u/\sim} A) \setminus A$. Then $L_x = \pi^{-1}[(\mathrm{cl}_{\mathbb{I}_u/\sim} A) \setminus A] \supseteq (\mathrm{cl}_{\mathbb{I}_u} A) \setminus A$ (where $\pi : \mathbb{I}_u \to \mathbb{I}_u/\sim$ is the decomposition map). As A is relatively discrete in $\mathbb{I}_u$, Theorem 3.27 says $\mathrm{cl}_{\mathbb{I}_u} A = \mathrm{cl}_{\beta\mathbb{M}\setminus\mathbb{M}} A \simeq \beta A \simeq \beta\omega$. We have $L_x \supseteq (\mathrm{cl}_{\mathbb{I}_u} A) \setminus A \simeq \omega^*$. □

Theorem 5.11. *If* $u, v \in \omega^*$, *then a homeomorphism* $\mathbb{I}_u \to \mathbb{I}_v$ *induces an isomorphism* $(\mathbb{I}_u/\sim, \prec) \to (\mathbb{I}_v/\sim, \prec)$ *which is either order preserving or order reversing.*

Proof. Suppose $h : \mathbb{I}_u \to \mathbb{I}_v$ is a homeomorphism. Define $\varphi : (\mathbb{I}_u/\sim, \prec) \to (\mathbb{I}_v/\sim, \prec)$ by $\varphi(L_x) = L_{h(x)}$.

$$\begin{array}{ccc} \mathbb{I}_u & \xrightarrow{h} & \mathbb{I}_v \\ \downarrow & & \downarrow \\ \mathbb{I}_u/\sim & \xrightarrow{\varphi} & \mathbb{I}_v/\sim \end{array}$$

Figure 5.1: Theorem 5.11

First we show φ is a well-defined bijection by proving h maps layers to layers.

Claim (1): If $x \in \mathbb{I}_u \setminus \mathbb{R}^\omega/u$ then $h[L_x] \subseteq L_{h(x)}$. Well, clearly $h(x) \in h[L_x] \cap L_{h(x)}$. Now suppose for a contradiction that $h[L_x]$ intersects two layers $L_{y'} \prec L_{y''}$ in $\mathbb{I}_v$. Then by connectedness of L_x, continuity of h, and continuity of the canonical epimorphism from $\mathbb{I}_v$ into $\mathbb{I}_v/\sim$, $(L_{y'}, L_{y''})$ is a nonempty open subset of $h[L_x]$. So L_x contains a nonempty open subset of $\mathbb{I}_u$, thus L_x contains a point $x_u \in \mathbb{R}^\omega/u$. But then $L_x = \{x_u\}$, contradicting our assumption about x. Note (2): Arguing as above, if $h(x) \in \mathbb{I}_v \setminus \mathbb{R}^\omega/v$ then $h^{-1}[L_{h(x)}] \subseteq L_{h^{-1}h(x)} = L_x$, so $L_{h(x)} = hh^{-1}[L_{h(x)}] \subseteq h[L_x]$.

We may now prove (3) $h[L_x] = L_{h(x)}$ for each $x \in \mathbb{I}_u$. This is clear if $x \in \mathbb{R}^\omega/u$ and $h(x) \in \mathbb{R}^\omega/v$. If $x \in \mathbb{I}_u \setminus \mathbb{R}^\omega/u$ and $h(x) \in \mathbb{R}^\omega/v$, apply (1). If $x \in \mathbb{R}^\omega/u$ and $h(x) \in \mathbb{I}_v \setminus \mathbb{R}^\omega/v$, apply (2). If $x \in \mathbb{I}_u \setminus \mathbb{R}^\omega/u$ and $h(x) \in \mathbb{I}_v \setminus \mathbb{R}^\omega/v$, apply (1) and (2). Thus,

$$\varphi(L_x) = \varphi(L_y) \Leftrightarrow h[L_x] = h[L_y] \Leftrightarrow L_x = L_y \Leftrightarrow x \sim y.$$

This proves φ is well-defined and injective. φ is surjective by (3) and the fact that h is surjective. This completes our proof that φ is a well-defined bijection. Since h and the epimorphisms are continuous and closed, φ is a homeomorphism. By Theorem 2.22, φ must be order preserving or order reversing. □

The preceding theorem will be used to find nonhomeomorphic $\mathbb{I}_u$ when CH fails. This approach would fail badly under CH (see Corollary 4.4). In fact, if CH is assumed then all $\mathbb{I}_u$ have isomorphic closed lattice bases and are thus homeomorphic (see Theorem 3.22(ii)).

Chapter 6

Constructing Ultrafilters

6.1 Invariant Embeddings

Suppose $\lambda, \theta > \omega$ are regular cardinals. (A, B) is a (λ, θ)-*cut* of L if $A < B$, $L = A \cup B$, $\mathrm{cf}(A) = \lambda$, and $\mathrm{coi}(B) = \theta$. An order preserving map $\varphi : L \to L'$ is an *invariant embedding* if every (λ, θ)-cut (A, B) of L maps to a cut of L', in the sense that there is no $x \in L'$ with $\varphi[A] < x < \varphi[B]$. The main result of this section is that every linear order of cardinality $\mathfrak{c}$ admits an invariant embedding into some ultrapower ω^ω/u.

Suppose D is a filter over ω. Let $I_D = \{X \subseteq \omega : \omega \setminus X \in D\}$ be the corresponding dual ideal. Then I_D contains $\varnothing$ but not X, and is closed under finite unions and subsets. Define

$$A \subseteq B \bmod D \Leftrightarrow A \setminus B \in I_D$$

$$A = B \bmod D \Leftrightarrow A \Delta B \in I_D.$$

Suppose $\mathcal{G} \subseteq \omega^\omega$ is a family of surjective functions. $\mathcal{G}$ is *independent* mod D if for all distinct $g_1, ..., g_l \in \mathcal{G}$ and $j_1, ..., j_l \in \omega$ (not necessarily distinct), we have

$$\{n \in \omega : g_k(n) = j_k \text{ for all } k \leq l\} \neq \varnothing \bmod D.$$

Note that "$A \neq \varnothing \bmod D$" means A is not a subset of a complement of a set in D, i.e., A intersects every set in D.

Let

$$FI(\mathcal{G}) = \{h : h \text{ is a function, } \mathrm{dom}(h) \text{ is a finite subset of } \mathcal{G}, \, \mathrm{ran}(h) \subset \omega\}$$

be the set of all finite partial functions from $\mathcal{G}$ to ω. For each $h \in FI(\mathcal{G})$ let

$$A_h = \{n \in \omega : g(n) = h(g) \text{ for all } g \in \mathrm{dom}(h)\},$$

and

$$FI_s(\mathcal{G}) = \{A_h : h \in FI(\mathcal{G})\}.$$

Theorem 6.1. *(i) $\mathcal{G}$ is independent mod D iff $A_h \neq \varnothing$ mod D for all $h \in FI(\mathcal{G})$. (ii) If $\mathcal{G}$ is independent mod D, then there exists a maximal filter $D^* \supseteq D$ modulo which $\mathcal{G}$ is independent.*

Proof of (i). ($\Rightarrow$) Let $h \in FI(\mathcal{G})$. Enumerate $\mathrm{dom}(h) = \{g_1, ..., g_k\}$. For each $i \leq k$ let $j_i = h(g_i)$. Then

$$\begin{aligned} A_h &= \{n \in \omega : h(g_i) = g_i(n) \text{ for all } i \leq k\} \\ &= \{n \in \omega : g_i(n) = j_i \text{ for all } i \leq k\} \neq \varnothing \text{ mod } D. \end{aligned}$$

($\Leftarrow$) Let $g_1, ..., g_l \in \mathcal{G}$ distinct and $j_1, ..., j_l \in \omega$. Define $h \in FI(\mathcal{G}\}$ by $\mathrm{dom}(h) = \{g_1, ..., g_l\}$ and $h(g_i) = j_i$ for each $i \leq l$. Then $\{n \in \omega : g_i(n) = j_i \text{ for all } i \leq l\} = A_h \neq \varnothing$ mod D. $\square$

Proof of (ii). Let $\mathbb{P} = \{P \subseteq \mathcal{P}(\omega) : P \text{ is a filter, } D \subseteq P, \text{ and } \mathcal{G} \text{ is independent mod } P\}$, partially ordered by inclusion. Suppose $(P_\delta)_{\delta<\alpha}$ is a chain of filters ($P_\gamma \subseteq P_\eta$ for $\gamma \leq \eta < \alpha$) in $\mathbb{P}$. It is easily seen that $\bigcup_{\delta<\alpha} P_\delta \in \mathbb{P}$, thus every chain in $\mathbb{P}$ has an upper bound. The existence of D^* follows from Zorn's Lemma. $\square$

$\mathcal{A} \subseteq \mathcal{P}(\omega)$ is a *partition* mod D if

(i) $A \neq \varnothing$ mod D for all $A \in \mathcal{A}$

(ii) if $A, A' \in \mathcal{A}$ with $A \neq A'$, then $A \cap A' = \varnothing$ mod D

(iii) for all $B \in \mathcal{P}(\omega)$ with $B \neq \varnothing$ mod D, there exists $A \in \mathcal{A}$ s.t. $A \cap B \neq \varnothing$ mod D.

Suppose $\mathcal{A}$ is a partition mod D and $B \in \mathcal{P}(\omega)$. B is *based on* $\mathcal{A}$ mod D if for all $A \in \mathcal{A}$, either $A \subseteq B$ mod D or $A \cap B = \varnothing$ mod D. Suppose $\mathscr{A} \subseteq \mathcal{P}(\omega)$. B is *supported by* $\mathscr{A}$ mod D if it is based on some partition $\mathcal{A} \subseteq \mathscr{A}$, mod D.

Theorem 6.2. *Suppose D is a maximal filter over ω modulo which $\mathcal{G}$ is independent. For every $B \in \mathcal{P}(\omega)$ there exists a countable $\mathcal{G}_0 \subseteq \mathcal{G}$ such that B is supported by $FI_s(\mathcal{G}_0)$ mod D.*

Proof. Claim: For every $X \in \mathcal{P}(\omega)$ with $X \neq \varnothing$ mod D, there exists $A_h \in FI_s(\mathcal{G})$ such that $A_h \subseteq X$ mod D. Well, for a contradiction suppose $X \in \mathcal{P}(\omega)$, $X \neq \varnothing$ mod D, and (*) $A_h \cap (\omega \setminus X) \neq \varnothing$ mod D for all $A_h \in FI_s(\mathcal{G})$. Then $\omega \setminus X \neq \varnothing$ mod D, so $\omega \setminus X$ (which is not in D) intersects every set in D. The filter generated by $D \cup \{\omega \setminus X\}$ properly extends D, and $\mathcal{G}$ is independent mod $(D \cup \{\omega \setminus X\})$ by (*) and Theorem 6.1(i). This contradicts the maximality of D.

Now let $B \in \mathcal{P}(\omega)$. Enumerate $\mathcal{P}(\omega) = \{X_\alpha : \alpha < 2^\omega\}$. Using the claim we may define a collection $\{A_\delta : \delta < 2^\omega\}$ inductively so that for each $\delta < 2^\omega$ either

(i) $A_\delta = \varnothing$ if $X_\delta \cap B = \varnothing$ mod D or $(X_\delta \cap B) \cap A_\xi \neq \varnothing$ mod D for some $\xi < \delta$, or

(ii) $A_\delta \in FI(\mathcal{G})$ with $A_\delta \subseteq (X_\delta \cap B)$ mod D.

Define $\{A'_\delta : \delta < 2^\omega\}$ similarly for $\omega \setminus B$. Then $\mathcal{A} = (\{A_\delta : \delta < 2^\omega\} \cup \{A'_\delta : \delta < 2^\omega\}) \setminus \{\varnothing\}$ is a partition on which B is based mod D. We now show $\mathcal{A}$ is countable. Suppose not, and assume $|\mathcal{A}| = \omega_1$. Enumerate $\mathcal{A} = \{A_{h_\xi} : \xi < \omega_1\}$. Then $\{dom(h_\xi) : \xi < \omega_1\}$ is an uncountable collection of finite sets. By the Δ-system Lemma there exists an uncountable $S \subseteq \omega_1$ and a finite set $r \subseteq \mathcal{G}$ such that $dom(h_\delta) \cap dom(h_\gamma) = r$ whenever $\delta \neq \gamma \in S$. Enumerate $r = \{g_1, ..., g_n\}$. There are only countably many tuples $(h_\xi(g_1), ..., h_\xi(g_n))$, $\xi \in S$, so there must exist $\delta \neq \gamma \in S$ such that h_δ and h_γ agree on the common part of their domains. But then $A_{h_\delta} \cap A_{h_\gamma} \neq \varnothing$ mod D, contradicting partition property (ii). So $\mathcal{A}$ is countable, which implies $\mathcal{G}_0 = \{g \in \mathcal{G} : (\exists h \in FI(\mathcal{G}))(A_h \in \mathcal{A} \text{ and } g \in \mathrm{dom}(h)\}$ is countable. Clearly $\mathcal{A} \subseteq FI_s(\mathcal{G}_0)$, so B is supported by $FI_s(\mathcal{G}_0)$ mod D. □

Theorem 6.3. *There is a family of $\mathfrak{c}$ surjective functions in ω^ω which is independent modulo the cofinite filter F.*

Proof. Consider the triples $(A, \langle C_k : k < n\rangle, \langle j_k : k < n\rangle)$, where A is a finite subset of ω, $n \in \omega$, the sets C_k are distinct subsets of A, and $j_k \in \omega$ for each $k < n$. The collection of all such triples is countable. Let $\{(A^i, \langle C_k^i : k < n_i\rangle, \langle j_k^i : k < n_i\rangle) : i \in \omega\}$ be an enumeration. For each $B \subseteq \omega$ define a function $f_B : \omega \to \omega$ by

$$f_B(i) = \begin{cases} j_k^i & \text{if } B \cap A_i = C_k^i \\ 0 & \text{otherwise} \end{cases}.$$

Claim $\{f_B : B \subseteq \omega\}$ is as desired. Let $\{B_k : k < n\}$ be a finite collection of distinct subsets of ω and $\{j_k : k < n\}$ a finite set of values.

First we show there exists $i \in \omega$ such that $f_{B_k}(i) = j_k$ for each k. For each $l \neq m < n$ let $a_{lm} \in B_l \Delta B_m$. Let $A = \{a_{lm} : l \neq m < n\}$. Then A is a finite subset of ω such that the $B_k \cap A$ are distinct . Let $i \in \omega$ such that $n_i = n$, $A_i = A$, and for each k, $C_k^i = B_k \cap A_i$ and $j_k^i = j_k$. Then $f_{B_k}(i) = j_k$ for each k.

$|\{f_B : B \subseteq \omega\}| = 2^\omega$: If $B_1 \neq B_2 \subseteq \omega$ and $j_1 \neq j_2 \in \omega$, then there exists $i \in \omega$ s.t. $f_{B_1}(i) = j_1$ and $f_{B_2}(i) = j_2$, so that $f_{B_1} \neq f_{B_2}$. It is also clear that the functions f_B are surjective (i.e. for any $j \in \omega$ there exists $i \in \omega$ such that $f_B(i) = j$).

Now we show $\{i \in \omega : f_{B_k}(i) = j_k \text{ for each } k \leq n\} \neq \varnothing \bmod F$. Note that "$\neq \varnothing \bmod F$" simply means "infinite." Suppose not. Enumerate the set $\{i_k : n+1 \leq k \leq n+m\}$ and let $\{B_k : n+1 \leq k \leq n+m\}$ be collection of distinct subsets of ω, distinct from the B_k $(1 \leq k \leq n)$. For each $k \in \{n+1, ..., n+m\}$ let $j_k \in \omega$ with $j_k \neq f_{B_k}(i_k)$. Then

$$\{i \in \omega : f_{B_k}(i) = j_k \text{ for each } k \leq n+m\} = \varnothing,$$

a contradiction. $\square$

Theorem 6.4. *If I is a linear order with $|I| = \mathfrak{c}$, then there exists a free ultrafilter u on ω such that I admits an invariant embedding into ω^ω/u.*

Proof. Let F be the cofinite filter on ω. Let $\mathcal{G} \subseteq \omega^\omega$ be a family of $\mathfrak{c}$ surjective function such that $\mathcal{G}$ is independent mod F (Theorem 6.3). Let $D \supseteq F$ be a maximal filter modulo which $\mathcal{G}$ is independent (Theorem 6.1 (ii)). Enumerate $\mathcal{G} = \{f_t : t \in I\}$. For each pair $s < t \in I$ let

$$B_{s,t} = \{n \in \omega : f_s(n) < f_t(n)\}.$$

For each pair $r < s \in I$ and each function $g \in \omega^\omega$ such that $g^{-1}(l)$ is supported by $FI_s(\{f_t : t \in I \setminus [r,s]\})$ mod D for all $l \in \omega$, let

$$C_{g,r,s} = \{n \in \omega : g(n) < f_r(n) \text{ or } f_s(n) < g(n)\}.$$

Claim. The sets $B_{s,t}, C_{g,r,s}$ have the finite intersection property (any finite intersection contains A_h (mod D) for some $h \in FI(\mathcal{G})$).

Assuming the claim holds, there is an ultrafilter u containing all of the sets $B_{s,t}, C_{g,r,s}$. The map $s \mapsto f_s/u$ is an invariant embedding from I into $\omega^\omega/\mathcal{U}$:

(i) Suppose $s < t \in I$. Then $B_{s,t} \in u$, so $f_s/u < f_t/u$.

(ii) Suppose (I_1, I_2) is a (λ,θ)-cut of I and $g \in \omega^\omega$. For each $l \in \omega$ there exists a countable $\mathcal{G}_l \subseteq \mathcal{G}$ such that $g^{-1}(l)$ is supported by $FI_s(\mathcal{G}_l)$ mod D. Then $g^{-1}(l)$ is supported by $\bigcup_{l\in\omega} \mathcal{G}_l$ for all $l \in \omega$. Since $\lambda, \theta > \omega$ and $\left|\bigcup_{l\in\omega} \mathcal{G}_l\right| = \omega$, there exist $r \in I_1$ and $s \in I_2$ such that $g^{-1}(l)$ is supported by $FI_s(\{f_t : t \in I \setminus [r,s]\})$ mod D for all $l \in \omega$. Thus $C_{g,r,s} \in u$, so either $g/u < f_r/u$ or $f_s/u < g/u$. $\square$

Proof of Claim. Consider a finite intersection

$$\bigcap_{i=1}^{n} B_{u_i,v_i} \cap \bigcap_{k=1}^{b} C_{g_k,r_k,s_k}.$$

Re-label the indices u_i, v_i, r_k, s_k from t_0 to t_a in increasing order. For each $k \leq b$ let $\iota(k) \leq a$ such that $t_{\iota(k)} = r_k$ and $\tau(k) \leq a$ such that $t_{\tau(k)} = s_k$. To show $\bigcap_{i=1}^{n} B_{u_i,v_i} \cap \bigcap_{k=1}^{b} C_{g_k,r_k,s_k} \neq \varnothing$, it suffices to show

$$\bigcap_{i<j\leq a} B_{t_i,t_j} \cap \bigcap_{k\leq b} C_{g_k,t_{\iota(k)},t_{\tau(k)}} \neq \varnothing.$$

Let $\mathcal{T} = \{f_{t_i} : i \leq a\}$ and for $k \leq b$ let $\mathcal{T}_k = \{f_{t_i} : i \notin [\iota(k), \tau(k)]\}$. We define a sequence of functions $h_m \in FI(\mathcal{G})$ so that

(1) $h_m \subseteq h_{m+1}$

(2) $\mathrm{dom}(h_m) \cap \mathcal{T} = \varnothing$

(3) if $h^* : \mathcal{T} \to \omega$ and $k \leq b$, then for m sufficiently large either

(i) there exists $l \in \omega$ such that $A_{h_m \cup h^* \restriction \mathcal{T}_k} \subseteq g_k^{-1}(l) \bmod D$, or

(ii) $A_{h_m \cup h^*} \cap g_k^{-1}(l) = \varnothing \bmod D$ for all $l \in \omega$.

Enumerate the countably many pairs (h^*, k) where $h^* : \mathcal{T} \to \omega$ and $k \leq b$. Suppose $m \in \omega$ and h_{m-1} has been defined for (h^*_{m-1}, k_{m-1}) (to define h_0, follow the cases below and ignore "h_{m-1}"). We now define h_m for (h^*_m, k_m):

Case 1: There exists $l \in \omega$ such that $A_{h_{m-1} \cup h^*_m} \cap g_{k_m}^{-1}(l) \neq \varnothing \bmod D$. By assumption $g_{k_m}^{-1}(l)$ is supported by $FI_s(\{f_t : t \in I \setminus [\iota(k_m), \tau(k_m)]\}) \bmod D$. So there exists $\tilde{h} \in FI(\{f_t : t \in I \setminus [\iota(k_m), \tau(k_m)]\})$ such that $A_{\tilde{h}} \subseteq g_{k_m}^{-1}(l) \bmod D$ and $A_{\tilde{h}} \cap A_{h_{m-1} \cup h^*_m} \neq \varnothing \bmod D$. In particular, $\mathrm{dom}(\tilde{h}) \cap (\mathcal{T} \setminus \mathcal{T}_{k_m}) = \varnothing$, and we may assume $h_{m-1} \cup h^*_m \restriction \mathcal{T}_{k_m} \subseteq \tilde{h}$. Let $h_m = \tilde{h} \setminus (h^*_m \mid \mathcal{T}_{k_m})$. Then $A_{h_m \cup h^*_m \restriction \mathcal{T}_{k_m}} = A_{\tilde{h}} \subseteq g_{k_m}^{-1}(l) \bmod D$, $\mathrm{dom}(h_m) \cap \mathcal{T} = \varnothing$, and $h_{m-1} \subseteq h_m$.

Case 2: $A_{h_{m-1} \cup h^*_m} \cap g_{k_m}^{-1}(l) = \varnothing \bmod D$ for all $l \in \omega$. Let $h_m = h_{m-1}$.

Our goal now is to find $h^* : \mathcal{T} \to \omega$ and $M \in \omega$ such that

$$A_{h_M \cup h^*} \subseteq \bigcap_{i<j\leq a} B_{t_i,t_j} \cap \bigcap_{k\leq b} C_{g_k,t_{\iota(k)},t_{\tau(k)}} \text{ mod } D. \tag{6.1}$$

For each infinite $W \subseteq \omega$ let $W^{(\mathcal{T})}$ denote the set of all increasing functions $h^* : \mathcal{T} \to W$. For each $k \leq b$ define $\varphi_k : \omega^{(\mathcal{T})} \to \{1,2,3,4\}$ by

$$\varphi_k(h^*) = \begin{cases} 1 & \text{if (i) and } h^*(f_{t_{\iota(k)}}) \leq l \leq h^*(f_{t_{\tau(k)}}) \\ 2 & \text{if (i) and } l < h^*(f_{t_{\iota(k)}}) \\ 3 & \text{if (i) and } l > h^*(f_{t_{\tau(k)}}) \\ 4 & \text{if (ii)} \end{cases}.$$

Define $\psi : \omega^{(\mathcal{T})} \to \{1,2,3,4\}^{\{1,\dots,b\}}$ by $\psi(h^*) = (\varphi_1(h^*),...,\varphi_b(h^*))$. Then ψ is an a-place function on ω with finite range. By Ramsey's Theorem there exists an infinite $W \subseteq \omega$ such that $\psi \restriction W^{(\mathcal{T})}$ is constant, i.e., such that $\varphi_k \restriction W^{(\mathcal{T})}$ is constant for each $k \leq b$. Let $h^* \in W^{(\mathcal{T})}$. For any m we have

$$A_{h_m \cup h^*} \subseteq A_{h^*} \subseteq \bigcap_{i<j\leq a} B_{t_i,t_j}.$$

Now fix $k \leq b$.

Case 1: $\varphi_k(h^*) = 1$. This case may be ruled out by our selection of W: Supposing $\varphi_k(h^*) = 1$, we have $\varphi_k \restriction W^{(\mathcal{T})} \equiv 1$. Using the fact that W is infinite, define $h' \in W^{(\mathcal{T})}$ so that $\left|\{w \in W : h'(f_{t_{\iota(k)}}) \leq w \leq h'(f_{t_{\tau(k)}})\}\right| = 2\,|i \in \omega : \iota(k) \leq i \leq \tau(k)|$. Then $\varphi_k(h') = 1$. Let $l_1 \in \omega$ satisfying (i), such that $h'(f_{t_{\iota(k)}}) \leq l_1 \leq h'(f_{t_{\tau(k)}})$. We may modify h' on functions f_{t_i}, $i \in [\iota(k), \tau(k)]$, to get $h'' \in W^{(\mathcal{T})}$ so that either $h''(f_{t_{\iota(k)}}) > l_1$ or $l_1 < h''(f_{t_{\tau(k)}})$. Again, $\varphi_k(h'') = 1$. Let $l_2 \in \omega$ satisfying (i), such that $h''(f_{t_{\iota(k)}}) \leq l_2 \leq h''(f_{t_{\tau(k)}})$. We have $l_1 \neq l_2$,

$A_{h_m\cup h'\restriction\mathcal{T}_k} \subseteq g_k^{-1}(l_1) \bmod D$, $A_{h_m\cup h''\restriction\mathcal{T}_k} \subseteq g_k^{-1}(l_2) \bmod D$. But $A_{h_m\cup h'\restriction\mathcal{T}_k} = A_{h_m\cup h''\restriction\mathcal{T}_k}$. Contradiction.

Case 2: $\varphi_k(h^*) = 2$. Let $l \in \omega$ such that $A_{h_m\cup h^*\restriction\mathcal{T}_k} \subseteq g_k^{-1}(l) \bmod D$ for m sufficiently large and $l < h^*(f_{t_{\iota(k)}})$. Then

$$A_{h_m\cup h^*} \subseteq A_{h_m\cup h^*\restriction(\mathcal{T}_k\cup\{f_{t_{\iota(k)}}\})} \subseteq \{n \in \omega : g_k(n) = l < h^*(f_{t_{\iota(k)}}) \wedge h^*(f_{t_{\iota(k)}}) = f_{t_{\iota(k)}}(n)\} \bmod D \tag{6.2}$$

Case 3: $\varphi_k(h^*) = 3$. Similar to Case 2. For m sufficiently large,

$$A_{h_m\cup h^*} \subseteq \{n \in \omega : f_{t_{\tau(k)}}(n) < g_k(n)\} \bmod D. \tag{6.3}$$

Case 4: $\varphi_k(h^*) = 4$. For m sufficiently large, $A_{h_m\cup h^*} \cap g_k^{-1}(l) = \varnothing \bmod D$ for all $l \in \omega$. As I_D is closed under finite unions,

$$A_{h_m\cup h^*} \cap \bigcup_{l \le h^*(f_{t_{\tau(k)}})} g_k^{-1}(l) = \varnothing \bmod D.$$

That is, almost none of the points in $A_{h_m\cup h^*}$ map under g_k to $\le h^*(f_{t_{\tau(k)}})$. Thus,

$$A_{h_m\cup h^*} \subseteq \{n \in \omega : f_{t_{\tau(k)}}(n) = h^*(f_{t_{\tau(k)}}) \wedge h^*(f_{t_{\tau(k)}}) < g_k(n)\} \bmod D. \tag{6.4}$$

Each set in (6.2)-(6.4) containing $A_{h_m\cup h^*}$ (mod D) is contained in $C_{g_k,t_{\iota(k)},t_{\tau(k)}}$. For each $k \le b$ let m_k be sufficiently large for one of (6.2)-(6.4) to hold. Letting $M = \max_{k\le b}(m_k)$, we have (6.1). □

Corollary 6.5. *There exists $u \in \omega^*$ such that ω^ω/u has a (λ,θ)-cut for each pair of uncountable regular $\lambda, \theta \le \mathfrak{c}$.* □

Corollary 6.6. *If I is a linear order with $|I| = \mathfrak{c}$, then there exists a free ultrafilter u on ω such that I admits an invariant embedding into $\mathbb{R}^\omega/u$.*

Proof. We show the inclusion $\omega^\omega/u \hookrightarrow \mathbb{R}^\omega/u$ is invariant. Suppose (A, B) is a (λ, θ)-cut of ω^ω/u. Let $(f_\alpha/u)_{\alpha<\lambda}$ and $(g_\beta/u)_{\beta<\theta}$ be cofinal and coinitial in A and B, respectively. For a contradiction, suppose there exists $h/u \in \mathbb{R}^\omega/u$ with $f_\alpha/u < h/u < g_\beta/u$ for all $\alpha < \lambda$ and $\beta < \theta$. We may assume $h(n) \geq 0$ for all $n \in \omega$. Let $h^- \in \omega^\omega$ be defined by $h^-(n) = \lfloor h(n) \rfloor$. It must be the case that $E_1 = \{n \in \omega : h(n) \in \mathbb{R} \setminus \omega\} \in u$, so $h^-/u < h/u$. There exists $\alpha_0 < \lambda$ and $E_2 \in u$ s.t. $h^-(n) < f_{\alpha_0}(n) < h(n)$ for all $n \in E_1 \cap E_2 \neq \varnothing$. This is impossible, as $h(n) - h^-(n) < 1$ for all n. □

6.2 The linear orders J_α ($\alpha < 2^\mathfrak{c}$)

Theorem 6.7. *If $\lambda > \omega_1$ is a regular cardinal, then there exists a set $\{I_\alpha : \alpha < 2^\lambda\}$ of linear orders satisfying*

(i) $\mathrm{cf}(I_\alpha) = |I_\alpha| = \lambda$.

(ii) If $\alpha \neq \beta$ and $\varphi_\alpha : I_\alpha \to L$, $\varphi_\beta : I_\beta \to L'$ are cofinal invariant embeddings, then L and L' have no isomorphic final segments.

Proof. There exists a partition $\{S_\tau : \tau < \lambda\}$ of the stationary set $S = \{\delta < \lambda : \mathrm{cf}(\delta) = \omega_1\}$ into λ pairwise disjoint stationary subsets (Theorem 2.30). Fix $X \subseteq \lambda$. For each $\alpha < \lambda$, define

$$\lambda_\alpha^X = \begin{cases} \omega_1 & \text{if } \alpha \in \bigcup_{\tau \in X} S_\tau \\ \omega_2 & \text{otherwise} \end{cases}.$$

Define $I_X = \{(\alpha, \beta) : \alpha < \lambda, \beta < \lambda_\alpha^X\}$. Give I_X the lexicographic ordering, with the order reversed in the second factor.

$$(\lambda_0^X, 0](\lambda_1^X, 0] \cdots\cdots\cdots\cdots (\lambda_\delta^X, 0] \cdots\cdots\cdots\cdots\cdots$$

Figure 6.1: I_X

Suppose $X \neq Y \subseteq \lambda$ and $\varphi_X : I_X \to L$, $\varphi_Y : I_Y \to L'$ are cofinal invariant embeddings into linear orders L, L', respectively. For a contradiction, suppose $\psi : M \to M'$ is an isomorphism between final segments M, M' of L, L', resp. For each $\delta < \lambda$, let

$$M_\delta = \{m \in M : m < \varphi_X(\gamma, 0) \text{ for some } \gamma < \delta\}$$

$$M'_\delta = \{m' \in M' : m' < \varphi_Y(\gamma, 0) \text{ for some } \gamma < \delta\}.$$

Let $C = \{\delta < \lambda : \psi[M_\delta] = M'_\delta\}$. Claim. C is closed unbounded in λ. Assuming the claim holds, let $\tau \in X \Delta Y$. Without loss of generality assume $\tau \in X \setminus Y$. There exists $\delta \in C \cap S_\tau$. We have $\psi[M_\delta] = M'_\delta$ and $\lambda^X_\delta = \omega_1$. Since the stationary sets S_ρ are pairwise disjoint, $\delta \in S_\tau$, and $\tau \in X \setminus Y$, we have $\delta \notin S_\rho$ for any $\rho \in Y$. Thus $\lambda^Y_\delta = \omega_2$. Since φ_X maps cofinally into L, $M_\delta \neq \varnothing$ for δ sufficiently large. As $C \cap S_\tau$ is unbounded in λ, we may assume δ was chosen so that $M_\delta \neq \varnothing$.

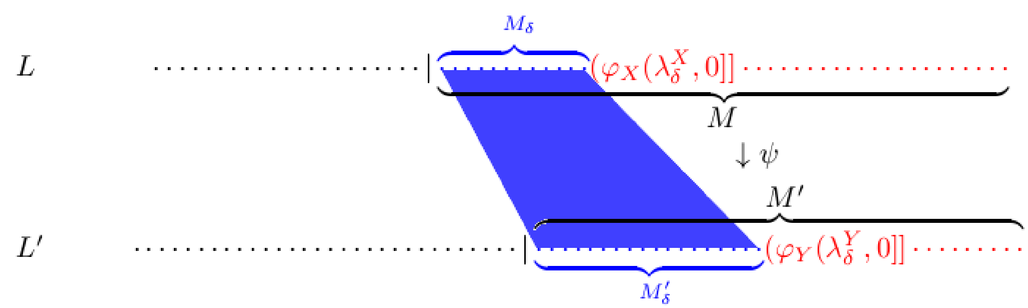

Figure 6.2: Theorem 6.7

Cut I_X and I_Y directly below $(\omega_1, 0]_\delta$ and $(\omega_2, 0]_\delta$, respectively. Since $cf(\delta) = \omega_1$, these are (ω_1, ω_1) and (ω_1, ω_2) cuts of I_X and I_Y, respectively. The assumption that φ_X is an invariant embedding implies M has no elements between M_δ and $\varphi_X(\omega_1, 0]_\delta$. Similarly, there are no elements of M' between M'_δ and $\varphi_Y(\omega_2, 0]_\delta$. So $\mathrm{coi}(M \setminus M_\delta) = \omega_1$ and $\mathrm{coi}(M' \setminus M'_\delta) = \omega_2$. But $\psi[M_\delta] = M'_\delta$ implies $\psi[M \setminus M_\delta] = M' \setminus M'_\delta$. Contradiction. $\square$

Proof of Claim. First we show that $M_\gamma = \bigcup_{\delta<\gamma} M_\delta$ (and $M'_\gamma = \bigcup_{\delta<\gamma} M'_\delta$) when γ is a limit ordinal. We just need to show $M_\gamma \subseteq \bigcup_{\delta<\gamma} M_\delta$. To that end suppose $m \in M_\gamma$. Then there exists $\alpha < \gamma$ such that $m < \varphi_X(\alpha, 0)$. Since γ is a limit ordinal there exists $\alpha < \delta < \gamma$. We have $m \in M_\delta$.

Now we can show C is unbounded in λ. To that end, let $\rho < \lambda$. Since φ_X and φ_Y are cofinal and order preserving, we may choose a strictly increasing sequence $(\rho_i)_{i\in\omega}$ of elements in λ such that $\rho < \rho_0$ and $M'_{\rho_i} \subseteq \psi(M_{\rho_j}) \subseteq M'_{\rho_k}$ for all $i < j < k \in \omega$. Let $\gamma = \sup_{i\in\omega}\rho_i$. Then

$$\psi[M_\gamma] = \bigcup_{i\in\omega} \psi[M_{\rho_i}] = \bigcup_{i\in\omega} M'_{\rho_i} = M'_\gamma.$$

So $\rho < \gamma \in C$, proving C is unbounded in λ. Now suppose γ is a limit point of C (a limit ordinal to which elements in C limit). Then

$$\psi[M_\gamma] = \psi[\bigcup_{\delta<\gamma} M_\delta] = \psi[\bigcup_{\delta\in C\cap\gamma} M_\delta] = \bigcup_{\delta\in C\cap\gamma} \psi[M_\delta] = \bigcup_{\delta\in C\cap\gamma} M'_\delta = M'_\gamma,$$

so $\gamma \in C$. This proves C is closed. □

Theorem 6.8. *If $\kappa > \omega_1$ then there exists a set $\{J_\alpha : \alpha < 2^\kappa\}$ of linear orders satisfying*

(i) $|J_\alpha| = \kappa$

(ii) $\mathrm{coi}(J_\alpha) = \mathrm{cf}(\kappa) + \omega_2$

(iii) if $\alpha \neq \beta$ and $\varphi_\alpha : I_\alpha \to L$, $\varphi_\beta : I_\beta \to L'$ are coinitial invariant embeddings, then L and L' have no isomorphic initial segments.

Proof. If κ is regular, this follows from the previous theorem. Suppose κ is singular. By Theorem 2.27 there exists a set $\{\kappa_i : i < \mathrm{cf}(\kappa)\}$ of regular cardinals, each $\kappa_i > \omega_1$, such that $\sup_{i<\mathrm{cf}(\kappa)}\kappa_i = \kappa$ and

$$\prod_{i<\mathrm{cf}(\kappa)} 2^{\kappa_i} = 2^\kappa.$$

Let $\theta = \mathrm{cf}(\kappa) + \omega_2$. There exists a partition $\{S_\tau : \tau < \mathrm{cf}(\kappa)\}$ of the stationary set $S = \{\delta < \theta : \mathrm{cf}(\delta) = \omega_1\}$ into $\mathrm{cf}(\kappa)$ pairwise disjoint stationary subsets. For each $\delta \in S$, let $h(\delta) \in \mathrm{cf}(\kappa)$ such that $\delta \in S_{h(\delta)}$, and for $\delta \in \theta \setminus S$, let $h(\delta) = 0$.

For each $i < \mathrm{cf}(\kappa)$ let $\{I_{i,\alpha} : \alpha < 2^{\kappa_i}\}$ be the set of linear orders of cardinality κ_i given by Theorem 6.7. For each $v \in \prod_{i<\mathrm{cf}(\kappa)} 2^{\kappa_i}$, define $J_v = \{(\alpha, x) : \alpha < \theta,\, x \in I_{h(\alpha),v(h(\alpha))}\}$

with the lexicographic order reversed in the first factor. Note: J_v needs at least one element when $\alpha \in \theta \setminus S$, to ensure its coinitiality is θ instead of only $cf(\kappa)$.

$$\cdots\cdots\cdots\cdots(I_{h(\delta),v(h(\delta))})\cdots\cdots\cdots(I_{h(1),v(h(1))})(I_{h(0),v(h(0))})$$

Figure 6.3: J_v

Suppose $u, v \in \prod_{i<\mathrm{cf}(\kappa)} 2^{\kappa_i}$ with $u \neq v$ and $\varphi_u : J_u \to L$, $\varphi_v : J_v \to L'$ are invariant coinitial embeddings. For a contradiction suppose there is an isomorphism $\psi : M \to M'$ between initial segments M and M' of L and L', respectively. For each $\delta < \theta$, let

$$M_\delta = \{m \in M : m > \varphi_u(\gamma, x) \text{ for some } \gamma < \delta \text{ and } x \in I_{h(\gamma),u(h(\gamma))}\}$$

$$M'_\delta = \{m' \in M' : m' > \varphi_v(\gamma, x) \text{ for some } \gamma < \delta \text{ and } x \in I_{h(\gamma),v(h(\gamma))}\}.$$

There exists $\alpha \in S$ s.t. $v(h(\alpha)) \neq u(h(\alpha))$ ($h \restriction S$ maps onto $\mathrm{cf}(\kappa)$). Choose $\delta \in S_{h(\alpha)} \cap \{\delta < \theta : \psi[M_\delta] = M'_\delta\}$ with $M_\delta \neq \varnothing$. Note that $h(\delta) = h(\alpha)$. Consider the cuts in L and L' below M_δ and M'_δ respectively.

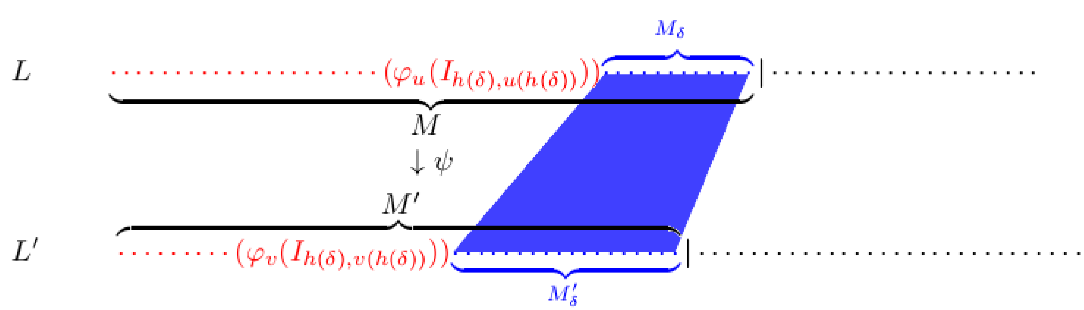

Figure 6.4: Theorem 6.8

Recall $\mathrm{cf}(I_{h(\delta),\xi}) = \kappa_{h(\delta)}$ and $\mathrm{cf}(\delta) = \omega_1$ are regular uncountable. By the invariant property of φ_X and φ_Y, $M \setminus M_\delta$ and $M' \setminus M'_\delta$ have $I_{h(\delta),v(h(\delta))}$ and $I_{h(\delta),u(h(\delta))}$ *cofinally* invariantly embedded. But $\psi[M_\delta] = M'_\delta$, so $M \setminus M_\delta \approx M' \setminus M'_\delta$. This contradicts a property of the linear orders $\{I_{h(\delta),\xi} : \xi < 2^{\kappa_{h(\delta)}}\}$. $\square$

We conclude this section by making a slight modification to Theorems 6.7 and 6.8. Suppose L and L' are linear orders. An order preserving map $L \to L'$ is an *invariant-1 embedding* if the image of each (λ, θ)-cut of L is filled by precisely one element of L'. Theorems 6.7 and 6.8 hold if we replace "invariant" with "invariant-1." For instance, if in the proof of Theorem 6.7 there are unique $l \in L$ and $l' \in L'$ such that $M_\delta < l < \varphi_X(\omega_1, 0]_\delta$ and $M'_\delta < l' < \varphi_Y(\omega_2, 0]_\delta$, then ψ must map l to l'. The contradiction follows as before. In the next sections we will apply Theorem 6.8 with this modification, when $\kappa = \mathfrak{c}$ and CH fails. It is consistent that $\mathfrak{c}$ is not regular, so the singular case provided by Theorem 6.8 is of use.

6.3 A Quick Proof of Theorem 1.1

Let $\overline{\mathbb{R}^\omega/u}$ denote the completion of $\mathbb{R}^\omega/u$ (i.e., $\overline{\mathbb{R}^\omega/u} = \mathbb{I}_u/\sim$).

Theorem 6.9 ($\neg$ CH)**.** *There exists a family $\{D_\alpha : \alpha < 2^{\mathfrak{c}}\}$ of free ultrafilters on ω and a collection $\{[L^1_\alpha, L^2_\alpha] : \alpha < 2^{\mathfrak{c}}\}$ of continua, $[L^1_\alpha, L^2_\alpha] \subseteq \mathbb{I}_{D_\alpha}$ for each $\alpha < 2^{\mathfrak{c}}$, such that $[L^1_\alpha, L^2_\alpha] \not\approx [L^1_\beta, L^2_\beta]$ for any $\alpha < \beta < 2^{\mathfrak{c}}$.*

Proof. Let $\{J_\alpha : \alpha < 2^{\mathfrak{c}}\}$ be the family of linear orders of cardinality $\mathfrak{c}$ given by Theorem 6.8. By Corollary 6.6, for each $\alpha < 2^{\mathfrak{c}}$ there exists $D_\alpha \in \omega^*$ such that $J_\alpha + \omega_1$ has an invariant embedding φ_α into $\mathbb{R}^\omega/D_\alpha$ (an invariant-1 embedding into $\overline{\mathbb{R}^\omega/D_\alpha}$). Let $L^1_\alpha = \inf \varphi_\alpha[J_\alpha + \omega_1]$ and $L^2_\alpha = \sup \varphi_\alpha[J_\alpha + \omega_1]$. Then $[L^1_\alpha, L^2_\alpha]$, the union of the layers in $\mathbb{I}_{D_\alpha}$ between L^1_α and L^2_α, is a subcontinuum of $\mathbb{I}_{D_\alpha}$ ($[L^1_\alpha, L^2_\alpha] = \bigcap\{[a_{D_\alpha}, b_{D_\alpha}] : a_{D_\alpha} < L^1_\alpha \text{ and } L^2_\alpha < b_{D_\alpha}\}$).

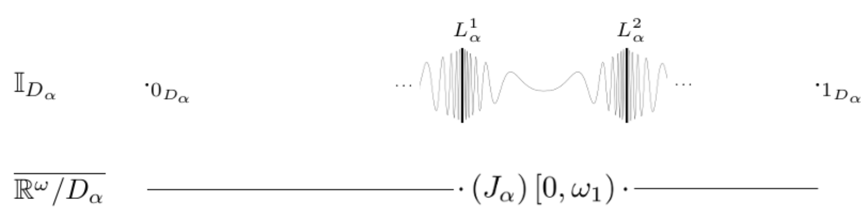

Figure 6.5: $[L^1_\alpha, L^2_\alpha]$

Suppose $\alpha < \beta < 2^{\mathfrak{c}}$ and $[L^1_\alpha, L^2_\alpha] \simeq [L^1_\beta, L^2_\beta]$. Arguing as in the proof of Theorem 5.11, there is an order preserving or order reversing isomorphism between $[L^1_\alpha, L^2_\alpha]$ and $[L^1_\beta, L^2_\beta]$ (their linearizations). Then (L^1_α, L^2_α) and (L^1_β, L^2_β) must be isomorphic via an order preserving map, since their coinitialities are $\mathrm{cf}(\mathfrak{c}) + \omega_2$ and their cofinalities are ω_1. This contradicts a property of the linear orders J_α (J_α and J_β are coinitially invariantly-1 embedded into (L^1_α, L^2_α) and (L^1_β, L^2_β), respectively). $\square$

Note also that $J_0 + \omega_1 + J_1 + \omega_1 + \ldots + J_{\mathfrak{c}} + \omega_1$ may be invariantly embedded into some $\mathbb{R}^\omega/u$, yielding a "chain" of $\mathfrak{c}$ pairwise nonhomeomorphic subcontinua of $\mathbb{I}_u$.

6.4 The Proof of Theorem 1.3

We require one additional result from [3].

Theorem 6.10. *If $\lambda \neq \theta$, then the number of (λ, θ)-cuts of a linear order I is at most $|I|$.*

Proof. Suppose I is a counterexample of minimal cardinality and let $\{(A_i, B_i) : i < |I|^+\}$ be a set of $|I|^+$ distinct (λ, θ)-cuts of I. Let $\kappa = \mathrm{cf}(|I|)$. Enumerate $I = \{i_\alpha : \alpha < |I|\}$. $|I|$ is the supremum of κ many α_γ with $|\alpha_\gamma| < |I|$ for each $\gamma < \kappa$ (Theorem 2.24). For each $\gamma < \kappa$ let $I_\gamma = \{i_\alpha : \alpha < \alpha_\gamma\}$. Then $|I_\gamma| < |I|$, $I = \bigcup_{\gamma<\kappa} I_\gamma$, and $I_\xi \subseteq I_\gamma$ for all $\xi < \gamma < \kappa$.

Case 1: $\kappa \neq \lambda, \theta$. Claim: For each $i < |I|^+$ there exists $\gamma_i < \kappa$ such that $A_i \cap I_{\gamma_i}$ is cofinal in A_i. Let $i < |I|^+$. Suppose $\kappa < \lambda$. Let $(a_\alpha)_{\alpha<\lambda}$ be cofinal in A_i. Since $\lambda = \bigcup_{\gamma<\kappa}\{\alpha < \lambda : a_\alpha \in I_\gamma\}$ and λ is regular, there exists $\gamma_i < \kappa$ such that $|\{\alpha < \lambda : a_\alpha \in I_{\gamma_i}\}| = \lambda$ (Theorem 2.23). Then $A_i \cap I_{\gamma_i}$ is cofinal in A_i. Suppose $\lambda < \kappa$ and there no $\gamma < \kappa$ such that $A_i \cap I_\gamma$ is cofinal in A_i. We may recursively define strictly increasing sequences $(a_\alpha)_{\alpha<\kappa}$ and $(\gamma_\alpha)_{\alpha<\kappa}$ such that for each $\alpha < \kappa$, $a_\alpha \in A_i \cap I_{\gamma_\alpha}$ and $A_i \cap I_{\gamma_\delta} < a_\alpha$ for all $\delta < \alpha$. Then $(a_\alpha)_{\alpha<\kappa}$ is cofinal in A_i, a contradiction. This completes or proof of the claim. Similarly, for each $i < |I|^+$ there exists $\gamma_i < \kappa$ such that $B_i \cap I_{\gamma_i}$ is cofinal in B_i. Thus, for each $i < |I|^+$ there exists $\gamma_i < \kappa$ such that $A_i \cap I_{\gamma_i}$ is cofinal in A_i and $B_i \cap I_{\gamma_i}$ is cofinal in B_i. Because $\kappa < |I|^+$, $|I|^+$ is regular, and $|I|^+ = \bigcup_{\alpha<\kappa}\{i < |I|^+ : \gamma_i < \alpha\}$, there exists $X \subseteq |I|^+$ and $\gamma < \kappa$ such that

$|X| = |I|^+$ and $\gamma_i = \gamma$ for all $i \in X$ (Theorem 2.23). But then $\{(A_i \cap I_\gamma, B_i \cap I_\gamma) : i \in X\}$ is a set of $|I|^+$ distinct λ, θ cuts of I_γ, contradicting minimality of $|I|$.

Case 2: $\kappa = \lambda$. Then $\kappa \neq \theta$. Arguing as in the previous case, there exists a subset $X \subseteq |I|^+$ of cardinality $|I|^+$ and $\gamma < \kappa$ such that $B_i \cap I_\gamma$ is coinitial in B_i for each $i \in X$. We may assume there are less than $|I|^+$ many i's for which $A_i \cap I_\gamma$ is cofinal in A_i, otherwise a contradiction follows as in the previous case. Therefore we may assume for each $i \in X$, $A_i \cap I_\gamma$ is *not* cofinal in A_i. Then for each $i \in X$ there exists $a_i \in A_i \setminus I_\gamma$ such that $A_i \cap I_\gamma < a_i < B_i$. Suppose $i \neq j \in X$. The cuts (A_i, B_i), (A_j, B_j) are distinct, so one of B_i and B_j must be a proper subset of the other. Assume $B_i \subset B_j$, so that $B_j \setminus B_i$ is a nonempty subset of A_i. Then there exists $c \in (B_j \setminus B_i) \cap I_\gamma \subseteq A_i \cap I_\gamma$. We have $a_j < c < a_i$, thus $a_i \neq a_j$. So $\{a_i : i \in X\}$ is a collection of $|I|^+$ distinct elements of I. The case $\kappa = \theta$ yields a contradiction similarly. □

Suppose I is a linear order, $\lambda, \theta > \omega$ are regular cardinals, $A, B \subseteq I$ and $x \in I$. Then (A, x, B) is a $(\lambda, 1, \theta)$-*cut* of I if $A < x < B$, $I = A \cup \{x\} \cup B$, $\mathrm{cf}(A) = \lambda$, and $\mathrm{coi}(B) = \theta$.

Proof of Theorem 1.3. Let $\{J_\alpha : \alpha < 2^{\mathfrak{c}}\}$ be the family of linear orders of cardinality $\mathfrak{c}$ given by Theorem 6.8. By Corollary 6.6, for each $\alpha < 2^{\mathfrak{c}}$ there exists $D_\alpha \in \omega^*$ such that $\omega_1 + J_\alpha$ invariantly embeds into $\mathbb{R}^\omega / D_\alpha$ (invariantly-1 embeds into $\overline{\mathbb{R}^\omega / D_\alpha}$).

Fix $\alpha < 2^{\mathfrak{c}}$ and let

$$E_\alpha = \{\beta < 2^\kappa : \omega_1 + J_\beta \text{ has an invariant-1 embedding into } \overline{\mathbb{R}^\omega / D_\alpha}\}.$$

We show $|E_\alpha| \leq \mathfrak{c}$. For each $\beta \in E_\alpha$ let $\varphi_\beta : \omega_1 + J_\beta \to \overline{\mathbb{R}^\omega / D_\alpha}$ be an invariant-1 embedding. Let $\lambda = \omega_1$ and $\theta = \mathrm{cf}(\mathfrak{c}) + \omega_2$. For each $\beta \in E_\alpha$, the image under φ_β of the cut (ω_1, J_β) of $\omega_1 + J_\beta$ produces a $(\lambda, 1, \theta)$-cut $(A_\alpha, x_\alpha, B_\alpha)$ of $\overline{\mathbb{R}^\omega / D_\alpha}$. Each $(\lambda, 1, \theta)$-cut of $\overline{\mathbb{R}^\omega / D_\alpha}$ corresponds to either a (λ, θ)-cut or a $(\lambda, 1, \theta)$-cut of $\mathbb{R}^\omega / D_\alpha$, each type of which there are only $\mathfrak{c}$ many (Theorem 6.10). So if $|E_\alpha| > \mathfrak{c}$, there exist $\beta_1 \neq \beta_2 \in E_\alpha$ such that $(A_{\beta_1}, x_{\beta_1}, B_{\beta_1}) = (A_{\beta_2}, x_{\beta_2}, B_{\beta_2})$. In particular $B_{\beta_1} = B_{\beta_2}$. But $\varphi_{\beta_1} \restriction J_{\beta_1}$ and $\varphi_{\beta_2} \restriction J_{\beta_2}$

are coinitial invariant-1 embeddings of J_{β_1} and J_{β_2} into B_{β_1} and B_{β_2}, respectively. This contradicts a property of the orders J_α.

We may now recursively define $X \subseteq 2^\kappa$, $|X| = 2^\kappa$, such that $\omega_1 + J_\beta$ admits an invariant-1 embedding into $\overline{\mathbb{R}^\omega/D_\beta}$ but not $\overline{\mathbb{R}^\omega/D_\alpha}$ for any $\alpha < \beta \in X$. Thus there is no order preserving isomorphism between $\overline{\mathbb{R}^\omega/D_\beta}$ and $\overline{\mathbb{R}^\omega/D_\alpha}$, for $\alpha < \beta \in X$. For a fixed $\alpha \in X$, there is at most one $\beta \in X$ such that there exists an order *reversing* isomorphism between $\overline{\mathbb{R}^\omega/D_\alpha}$ and $\overline{\mathbb{R}^\omega/D_\beta}$, so there exists $S \subseteq X$, $|S| = 2^\mathfrak{c}$, such that there is no order preserving or order reversing isomorphism between $\overline{\mathbb{R}^\omega/D_\alpha}$ and $\overline{\mathbb{R}^\omega/D_\beta}$, for $\alpha < \beta \in S$. $\square$

By Theorem 5.11, we have the following.

Theorem 6.11 ($\neg$ CH)**.** *There exists a family $\{D_\alpha : \alpha < 2^\mathfrak{c}\}$ of free ultrafilters on ω such that $\mathbb{I}_{D_\alpha} \not\approx \mathbb{I}_{D_\beta}$ for any $\alpha < \beta < 2^\mathfrak{c}$.* $\square$

6.5 Concluding Remarks

In [10] it is shown that there are also $2^\mathfrak{c}$ subcontinua of $\mathbb{H}^*$ when CH holds. Combined with Theorem 1.1, we have the following theorem of ZFC.

Theorem 6.12. *There exist $2^\mathfrak{c}$ pairwise nonhomeomorphic subcontinua of $\mathbb{H}^*$.*

Prior to this result approximately 20 subcontinua were found in the ZFC setting, many by M. Smith in [7]. In [7] it is also shown that the layers of $\mathbb{I}_u$ are *indecomposable* continua unlike $\mathbb{I}_u$. A. Dow indicates in [10] that the following question remains open: If CH fails, are there $2^\mathfrak{c}$ pairwise nonhomeomorphic indecomposable subcontinua of $\mathbb{H}^*$? In particular, it is not known if one can produce $2^\mathfrak{c}$ pairwise nonhomeomorphic layers by invariantly embedding different linear orders into the linearizations of $\mathbb{I}_u$'s (note that each (λ, θ)-cut of $\mathbb{R}^\omega/u$ corresponds to a layer in $\mathbb{I}_u$).

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