# Nonlocal Dispersal Equations and Convergence to Random Dispersal Equations

by

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Abstract

This dissertation is devoted to the study of the dynamics of nonlocal and random dispersal evolution equations. Dispersal evolution equations are widely used to model the diffusions of organisms or individuals in many biological and ecological systems. More precisely, random and nonlocal dispersal equations arise in modeling the dynamics of diffusive systems which exhibit random or local, and nonlocal internal interactions, respectively. In this dissertation, we study the dynamics of such equations complemented with Dirichlet, Neumann, and periodic types of boundary condition in a unified way. It is mainly concerned with principal spectral theory of nonlocal dispersal operators and the approximations of random dispersal operators/equations by nonlocal dispersal operators/equations.

Regarding the principal spectral theory of nonlocal dispersal operators, we investigate the dependence of the principal spectrum points of nonlocal dispersal operators on the underlying parameters and its applications. In particular, we study the effects of the spatial inhomogeneity, the dispersal rate, and the dispersal distance on the existence of the principal eigenvalues, the magnitude of the principal spectrum points, and the asymptotic behaviors of the principal spectrum points of time homogeneous nonlocal dispersal operators with Dirichlet type, Neumann type, and periodic boundary conditions. We also discuss the applications of the principal spectral theory of nonlocal dispersal operators to the asymptotic dynamics of two species competition systems.

About the approximations of random dispersal operators/equations by nonlocal dispersal operators/equations, we first prove that the solutions of properly rescaled nonlocal dispersal initial-boundary value problems converge to the solutions of the corresponding random dispersal initial-boundary value problems. Next, we prove that the principal spectrum points of time periodic nonlocal dispersal operators with properly rescaled kernels converge

to the principal eigenvalues of the corresponding random dispersal operators. Thirdly, we prove that the unique positive time periodic solutions of nonlocal dispersal KPP type evolution equations with properly rescaled kernels converge to the unique positive time periodic solutions of the corresponding random dispersal KPP type evolution equations. We also discuss the applications of the approximation results to the effects of the rearrangements with equimeasurability on principal spectrum point of nonlocal dispersal operators.

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# Chapter 1

# Introduction

This dissertation is devoted to the study of principal spectral theory of nonlocal dispersal operators and the approximations of random dispersal operators/equations by nonlocal dispersal operators/equations with different boundary conditions in a unified way.

First, let us introduce the prototype of nonlocal problems that will be considered. Let $k : \mathbb{R}^N \to \mathbb{R}$ be a nonnegative, continuous function with unit integral. Nonlocal dispersal evolution equations of the form

$$\partial_t u(t,x) = \nu \left[ \int_{\mathbb{R}^N} k(x-y)u(t,y)dy - u(t,x) \right] + F(t,x,u), \quad x \in \bar{D}, \tag{1.1}$$

and variations of it, have been widely used to model diffusive processes. More precisely, if $u(t,x)$ is thought of as a density at time $t$ and spatial location $x$ of a species and $k(x-y)$ is thought of as the probability distribution of jumping from location $y$ to location $x$, then $\int_{\mathbb{R}^N} k(x-y)u(t,y)dy$ is the rate at which individuals are arriving at position $x$ from all other places and $u(t,x) = \int_{\mathbb{R}^N} k(x-y)u(t,x)dy$ is the rate at which they are leaving location $x$ to travel to all other sites. This consideration leads to the fact that $\nu \left[ \int_{\mathbb{R}^N} k(x-y)u(t,y)dy - u(t,x) \right]$ is a dispersal operator which measures the diffusion or redistribution of the species with $\nu > 0$ being the dispersal rate. In (1.1), $F(t,x,u)$ is the external or internal sources, and $D \subset \mathbb{R}^N$ is the habitat which is not necessarily bounded. Throughout the dissertation, we have the following assumptions for the kernel function $k(\cdot)$.

**(H0)** $k(\cdot) \in C_c^1(\mathbb{R}^N, \mathbb{R}^+), \quad \int_{\mathbb{R}^N} k(z)dz = 1, \quad \textit{and} \quad k(0) > 0$

If there is $\delta > 0$ such that $\text{supp}(k(\cdot)) \subset B(0,\delta) := \{z \in \mathbb{R}^N | \|z\| < \delta\}$ and for any $0 < \tilde{\delta} < \delta$, $\text{supp}(k(\cdot)) \cap \left(B(0,\delta)\backslash B(0,\tilde{\delta})\right) \neq \emptyset$, $\delta$ is called the dispersal distance of the nonlocal dispersal operators.

The operator $u(\cdot) \mapsto \nu\left[\int_{\mathbb{R}^N} k(\cdot - y)u(y)dy - u(\cdot)\right]$ (and variations of it), and equation (1.1) (and its variations) are called the nonlocal dispersal operator, and nonlocal dispersal evolution equation, respectively, since the diffusion of the density $u(t,x)$ at time $t$ and some location $x \in \bar{D}$ depends not only on the values of $u(t,x)$ and its derivatives in an immediate neighborhood of $x$, but also on the values of $u(t,y)$ with $y$ being far away from $x$ through the convolution term $\int_{\mathbb{R}^N} k(x-y)u(t,y)dy$. Thus nonlocal dispersal is widely used to model the population dynamics of a species in which the movements or interactions of the organisms occur between non-adjacent spatial locations.

Classically, one assumes that the internal interactions of the organisms or individuals of some species are random and local, which leads to the well-known reaction-diffusion equations of the following form,

$$\partial_t u(t,x) = \nu\Delta u(t,x) + F(t,x,u), \quad x \in D, \tag{1.2}$$

where $u \mapsto \Delta u$ is the so-called Laplacian operator in literature, which characterizes the diffusion of organisms moving randomly between adjacent spatial locations. And $\nu$, $D \subset \mathbb{R}^N$, and $F(t,x,u)$ have the same meanings as in (1.1). Thus, (1.2) as well as its variations has been extensively studied in modeling the population dynamics of species. In contrast to the nonlocal counterparts, $u \mapsto \Delta u$ (and variations of it) and (1.2) (and its variations) are called random dispersal operator and random dispersal evolution equation, respectively.

Both nonlocal and random dispersal evolution equations are then of great interests in their own. And they are related to each other. In order to indicate some relationship between nonlocal and random dispersal operators, we assume that $k(\cdot)$ is of the form,

$$k(z) = k_\delta(z) := \frac{1}{\delta^N} k_0\left(\frac{z}{\delta}\right) \tag{1.3}$$

for some $k_0(\cdot)$ satisfying that $k_0(\cdot)$ is a smooth, nonnegative, and symmetric (in the sense that $k_0(z) = k_0(z')$ whenever $|z| = |z'|$) function supported on the unit ball $B(0,1)$ and $\int_{\mathbb{R}^N} k_0(z)dz = 1$, where $\delta(>0)$ is the dispersal distance. We also assume that

$$\nu = \nu_\delta := \frac{C}{\delta^2}, \tag{1.4}$$

where $C = \left(\frac{1}{2}\int_{\mathbb{R}^N} k_0(z)z_N^2 dz\right)^{-1}$. Then for any smooth function $u(x)$, we have

$$\begin{aligned}
&\nu_\delta \int_{\mathbb{R}^N} k_\delta(x-y)[u(y)-u(x)]dy \\
&= \frac{C}{\delta^2}\int_{\mathbb{R}^N} \frac{1}{\delta^N} k_0\left(\frac{x-y}{\delta}\right)[u(y)-u(x)]dy \\
&= \frac{C}{\delta^2}\int_{\mathbb{R}^N} k_0(z)[u(x+\delta z)-u(x)]dz \\
&= \frac{C}{\delta^2}\int_{\mathbb{R}^N} k_0(z)\left[\delta(\nabla u(x)\cdot z) + \frac{\delta^2}{2}\sum_{i,j=1}^N u_{x_i x_j} z_i z_j + O(\delta^3)\right]dz \\
&= \Delta u(x) + O(\delta).
\end{aligned}$$

Hence, the nonlocal dispersal operator $u(\cdot) \mapsto \nu_\delta \int_{\mathbb{R}^N} k_\delta(\cdot - y)[u(y)-u(\cdot)]dy$ "behaves" the same as the random dispersal operator $u \mapsto \Delta u$ for $\delta \ll 1$.

Next, let us consider the general boundary value problems with nonlocal dispersal operators in a bounded domain $D$ or unbounded domain $\mathbb{R}^N$. For random dispersal evolution equations, the two most common boundary conditions on a bounded domain are Neumann's and Dirichlet's. When looking at boundary conditions for nonlocal problems on a bounded domain, one has to modify the usual formulations for random problems.

The nonlocal dispersal equation with homogeneous Dirichlet type boundary condition is

$$\begin{cases} \partial_t u(t,x) = \nu\left[\int_{\mathbb{R}^N} k(x-y)u(t,y)dy - u(t,x)\right] + F(t,x,u), & x \in \bar{D}, \\ u(t,x) = 0, & x \notin D, \end{cases} \tag{1.5}$$

or equivalently

$$\partial_t u(t,x) = \nu \left[ \int_D k(x-y)u(t,y)dy - u(t,x) \right] + F(t,x,u), \quad x \in \bar{D}. \tag{1.6}$$

In the model described by (1.5), diffusion takes place in the whole $\mathbb{R}^N$, but we assume that $u$ vanishes outside $D$. The biological interpretation is that we have a hostile environment outside $D$, and any individual that jumps outside dies instantaneously. This is an analog of what is called homogeneous Dirichlet boundary condition in literature, that is,

$$\begin{cases} \partial_t u(t,x) = \nu \Delta u(t,x) + F(t,x,u), & x \in D, \;\; t > 0, \\ u(t,x) = 0, & x \in \partial D. \end{cases} \tag{1.7}$$

However, the boundary datum is not understood in the classical sense for (1.5), since we are not imposing that $u|_{\partial D} = 0$. In the model described by (1.6), the integral $\int_D k(x-y)u(t,y)dy$ takes into account the individuals arriving at position $x \in \bar{D}$ from other places in $D$, which indicates that individuals arriving at $x \in \bar{D}$ are not from the outside of $D$, because there is nothing living outside of $D$. However, all individuals can leave $D$ and travel to all other places, which are represented by $-u(x)$. That's why (1.5) and (1.6) are equivalent.

The nonlocal dispersal equation with homogeneous Neumann type boundary condition is

$$\partial_t u(t,x) = \nu \int_D k(x-y)[u(t,y) - u(t,x)]dy + F(t,x,u), \quad x \in \bar{D}. \tag{1.8}$$

In this model, the integral term takes into account the diffusion inside $D$. In fact, as we have explained, the integral $\int_{\mathbb{R}^N} k(x-y)[u(t,y) - u(t,x)]dy$ takes into account the individuals arriving at or leaving position $x$ from or to other places. Since we are integrating over $D$, we are assuming that diffusion takes place only in $D$. Biologically, the individuals may not enter or leave the domain $D$. This is analogous to the so-called homogeneous Neumann boundary condition in the literature, which is

$$
\begin{cases}
\partial_t u(t,x) = \nu \Delta u(t,x) + F(t,x,u), & x \in D, \\
\frac{\partial u}{\partial \mathbf{n}}(t,x) = 0, & x \in \partial D,
\end{cases}
\tag{1.9}
$$

where $\mathbf{n}$ is the exterior unit normal vector of $\partial D$.

The nonlocal dispersal equation on unbounded domain is prescribed with the periodic boundary condition

$$
\begin{cases}
\partial_t u(t,x) = \nu \left[\int_{\mathbb{R}^N} k(x-y)u(t,y)dy - u(t,x)\right] + F(t,x,u), & x \in \mathbb{R}^N, \\
u(t,x) = u(t, x + p_j \mathbf{e_j}), & x \in \mathbb{R}^N
\end{cases}
\tag{1.10}
$$

($j = 1, 2, \cdots, N$), where $p_j > 0$ and $\mathbf{e_j}$ denotes the vector with a 1 in the $j$th coordinate and 0's elsewhere, and $F(t,x,u) = F(t, x + p_j \mathbf{e_j}, u)$ for $j = 1, 2, \cdots, N$. We remark that heterogeneities are present in many biological end ecological models. The periodicity of the unbounded domain takes into account the periodic heterogeneities of the media. The random dispersal equation with periodic boundary condition is

$$
\begin{cases}
\partial_t u(t,x) = \nu \Delta u(t,x) + F(t,x,u), & x \in \mathbb{R}^N, \\
u(t,x) = u(t, x + p_j \mathbf{e_j}), & x \in \mathbb{R}^N.
\end{cases}
\tag{1.11}
$$

In order to study the three types of boundary condition in a unified way, we summarize (1.5) or (1.6), (1.8) and (1.10) as follows:

$$
\begin{cases}
\partial_t u(t,x) = \nu \int_{D \cup D_c} k(x-y)[u(t,y) - u(t,x)]dy + F(t,x,u), & x \in \bar{D}, \\
B_{n,b} u(t,x) = 0, & x \in D_c \ \ (x \in \mathbb{R}^N \text{ if } D = \mathbb{R}^N),
\end{cases}
\tag{1.12}
$$

where $D$ is a smooth bounded domain of $\mathbb{R}^N$ or $D = \mathbb{R}^N$; $D_c = \mathbb{R}^N \setminus D$ or $D_c = \emptyset$. When $D$ is bounded and $D_c = \mathbb{R}^N \setminus D$, $B_{n,b} u = B_{n,D} := u$ (in such case, $B_{n,D} u = 0$ on $D_c$ represents homogeneous Dirichlet type boundary condition); when $D$ is bounded and $D_c = \emptyset$, $B_{n,b}u = 0$ on $D_c$ trivially holds (we denote $B_{n,b}u$ by $B_{n,N}u$ for convenience) and indicates that nonlocal diffusion takes place only in $D$ (hence $B_{n,N}u = 0$ on $D_c$ represents homogeneous Neumann type boundary condition); when $D = \mathbb{R}^N$, it is assumed that $F(t, x + p_j \mathbf{e_j}, u) = F(t, x, u)$ and $B_{n,b}u = B_{n,P}u := u(t, x + p_j \mathbf{e_j}) - u(t, x)$ for $j = 1, 2, \cdots, N$ (hence $B_{n,P}u = 0$ represents periodic boundary condition). Analogously, (1.7), (1.9) and (1.11) can be written as

$$\begin{cases} \partial_t u(t,x) = \nu \Delta u(t,x) + F(t,x,u), & x \in D, \\ B_{r,b}u(t,x) = 0 & x \in \partial D \;\; (x \in \mathbb{R}^N \text{ if } D = \mathbb{R}^N), \end{cases} \tag{1.13}$$

where $D$ is a smooth bounded domain or $D = \mathbb{R}^N$. When $D$ is a bounded domain, $B_{r,b}u = B_{r,D}u := u$ (in such case, $B_{r,D}u = 0$ on $\partial D$ represents homogeneous Dirichlet boundary condition) or $B_{r,b}u = B_{r,N}u := \frac{\partial u}{\partial \mathbf{n}}$ (in such case, $B_{r,N}u = 0$ on $\partial D$ represents homogeneous Neumann boundary condition), and when $D = \mathbb{R}^N$, it is assumed that $F(t, x, u)$ is periodic in $x_j$ with period $p_j$ and $B_{r,b}u = B_{r,P}u := u(t, x + p_j \mathbf{e_j}) - u(t, x)$ for $j = 1, 2, \cdots, N$ (in such case, $B_{r,P}u = 0$ represents periodic boundary condition).

Finally, let us recall some existing results, and briefly introduce the main objective of this dissertation. Toward various dynamical aspects of random dispersal evolution equations of the form (1.2), a huge amount of research has been carried out (see [3, 4, 5, 10, 28, 29, 34, 44, 52, 64, 68], etc.). And there are many research works toward various dynamical aspects of nonlocal dispersal evolution equations of the form (1.1) (see [7, 11, 12, 14, 17, 19, 20, 27, 31, 38, 41, 46, 47, 66], etc.). It has been seen that random dispersal evolution equations with Dirichlet, or Neumann, or period boundary condition and nonlocal dispersal evolution equations with the corresponding boundary condition share many similar properties. For example, a comparison principle holds for both equations. There are also many differences between these two types of dispersal evolution equations.

For example, solutions of random dispersal evolution equations have smoothness and certain compactness properties, but solutions of nonlocal dispersal evolution equations do not have such properties. Many fundamental dynamical issues for nonlocal dispersal evolution equations are far away from being well understood. The objective of this dissertation is to investigate two dynamical issues, one is the principal spectral theory of nonlocal dispersal operators (see Chapter 4), and the other is the approximations of random dispersal operators/equations by nonlocal dispersal operators/equations (see Chapter 5).

Spectral theory for random and nonlocal dispersal operators is a basic technical tool for the study of nonlinear evolution equations with random and nonlocal dispersals. The following is the eigenvalue problem of time homogeneous nonlocal dispersal operator with Dirichlet, Neumann or periodic types of boundary condition

$$\begin{cases} \nu \int_{D\cup D_c} k(x-y)[u(y)-u(x)]dy + a(x)u(x) = \lambda u(x), & x \in \bar{D}, \\ B_{n,b}u(x) = 0, & x \in \partial D \ (x \in \mathbb{R}^N \text{ if } x = \mathbb{R}^N), \end{cases} \tag{1.14}$$

where $k(\cdot)$ are as in (H0), and $a(x + p_j\mathbf{e_j}) = a(x)$ ( $j = 1, 2, \cdots, N$) in the case of periodic boundary condition. Observe that the eigenvalue problems (1.14) can be viewed as the nonlocal counterpart of the following eigenvalue problems associated with random dispersal operators,

$$\begin{cases} \nu\Delta u(x) + a(x)u(x) = \lambda u(x), & x \in D, \\ B_{r,b}u(x) = 0, & x \in \partial D \ \ (x \in \mathbb{R}^N \text{ if } D = \mathbb{R}^N), \end{cases} \tag{1.15}$$

where $a(x + p_j\mathbf{e_j}) = a(x)$ ( $j = 1, 2, \cdots, N$) in the case of periodic boundary condition.

The eigenvalue problem (1.15) and in particular, its associated principal eigenvalue problem, are well understood. For example, it is known that the largest real part, denoted by $\lambda^{\mathcal{R}}(\nu, a)$, of the spectrum set of (1.15) is an isolated algebraically simple eigenvalue with a positive eigenfunction, and for any other $\lambda$ in the spectrum set, $\text{Re}\lambda < \lambda^{\mathcal{R}}(\nu, a)$ ($\lambda^{\mathcal{R}}(\nu, a)$ is called the *principal eigenvalue* of the random operator in literature).

The principal eigenvalue problem (1.14) has also been studied recently by many people (see [17], [30], [37], [41], [61], [60], and references therein). Let $\tilde{\lambda}^{\mathcal{N}}(\nu, a)$ be the largest real part of the spectrum set of (1.14) (in case that the kernel function $k(\cdot)$ depends on $\delta$, we use $\tilde{\lambda}^{\mathcal{N}}(\nu, a, \delta)$). $\tilde{\lambda}^{\mathcal{N}}(\nu, a)$ is called the *principal spectrum point* of the nonlocal dispersal operator, $\tilde{\lambda}^{\mathcal{N}}(\nu, a)$ is also called the *principal eigenvalue* of (1.14), if it is an isolated algebraically simple eigenvalue with a positive eigenfunction (see Definition 2.1 and Remark 2.2(2) for detail). It is known that a nonlocal dispersal operator may not have a principal eigenvalue (see [17], [61] for examples), which reveals some essential difference between nonlocal and random dispersal operators. Some sufficient conditions are provided in [17], [41], and [61] for the existence of principal eigenvalue of (1.14). Such sufficient conditions have been found important in the study of nonlinear evolution equations with nonlocal dispersals (see [17], [35], [37], [41], [42], [45], [61], [62], [63]). However, the understanding is still little to many interesting questions regarding the principal spectrum points/principal eigenvalues of nonlocal dispersal operators, including the dependence of principal spectrum point or principal eigenvalue (if exists) of nonlocal dispersal operators on the underlying parameters.

In Chapter 4, we study the effects of the spatial inhomogeneity, the dispersal rate, and the dispersal distance on the existence of principal eigenvalues, on the magnitude of the principal spectrum points, and on the asymptotic behavior of the principal spectrum points of nonlocal dispersal operators. Among others, we obtain the following:

- *criteria for $\tilde{\lambda}^{\mathcal{N}}(\nu, a)$ to be the principal eigenvalue of* (1.14) (see Theorem 2.4 (1), (2), Theorem 2.6 (3), and Theorem 2.8 (3) for detail);
- *lower bounds of $\tilde{\lambda}^{\mathcal{N}}(\nu, a)$ in terms of $\hat{a}$ (where $\hat{a}$ is the spatial average of $a(x)$) in the Neumann and periodic boundary cases* (see Theorem 2.4 (4) for detail);
- *monotonicity of $\tilde{\lambda}^{\mathcal{N}}(\nu, a)$ with respect to $a(x)$ and $\nu$* (see Theorem 2.4 (5) and Theorem 2.6 (1) for detail);

• *limits of* $\tilde{\lambda}^{\mathcal{N}}(\nu, a)$ *as* $\nu \to 0$ *and* $\nu \to \infty$ (see Theorem 2.6 (4), (5) for detail);

• *limits of* $\tilde{\lambda}^{\mathcal{N}}(\nu, a, \delta)$ *as* $\delta \to 0$ *and* $\delta \to \infty$ *in the case* $k(\cdot) = k_\delta(\cdot)$, *where* $k_\delta(\cdot)$ *is as in* (1.3). (see Theorem 2.8 (1), (2) for detail).

In Chapter 4, we also discuss the applications of principal spectral theory of nonlocal dispersal operators to the asymptotic dynamics of the following two species competition system,

$$\begin{cases} u_t = \nu[\int_D k(x-y)u(t,y)dy - u(t,x)] + uf(x, u+v), & x \in \bar{D}, \\ v_t = \nu \int_D k(x-y)[u(t,y) - u(t,x)]dy + vf(x, u+v), & x \in \bar{D}, \end{cases} \tag{1.16}$$

where $D$ and $k(\cdot)$ are as in (1.14) and $f(\cdot,\cdot)$ is a $C^1$ function satisfying that $\tilde{\lambda}(\nu, f(\cdot, 0)) > 0$, $f(x, w) < 0$ for $w \gg 1$, and $\partial_2 f(x, w) < 0$ for $w > 0$. (1.16) models the population dynamics of two competing species with the same local population dynamics (i.e. the same growth rate function $f(\cdot,\cdot)$), the same dispersal rate (i.e. $\nu$), but one species adopts nonlocal dispersal with Dirichlet type boundary condition and the other adopts nonlocal dispersal with Neumann type boundary condition, where $u(t, x)$ and $v(t, x)$ are the population densities of two species at time $t$ and space location $x$. We show

• *the species diffusing nonlocally with Neumann type boundary condition drives the species diffusing nonlocally with Dirichlet type boundary condition extinct (see Theorem 2.12 for detail).*

As mentioned in the above, nonlocal dispersal operators/equations and random dispersal operators/equations share many properties and there are also many differences between them. Thanks to the formal relation between the random operator $u \mapsto \Delta u$ and nonlocal dispersal operator $u(\cdot) \mapsto \nu \int_{\mathbb{R}^N} k_\delta(\cdot - y)[u(y) - u(\cdot)]dy$ for sufficiently small $\delta$ with $k_\delta$ and $\nu_\delta$ being as in (1.3) and (1.4), respectively, it is expected that nonlocal dispersal evolution equations with Dirichlet, or Neumann, or periodic boundary condition and small dispersal distance $\delta$ possess similar dynamical behaviors as those of random dispersal evolution equations with the corresponding boundary condition and that certain dynamics of random dispersal evolution equations with Dirichlet, or Neumann, or periodic boundary condition can be approximated by the dynamics of nonlocal dispersal evolution equations with the corresponding boundary condition and properly rescaled kernels. It is of great theoretical and practical importance to investigate whether such naturally expected properties actually hold or not.

Regarding the approximations of dynamics of random dispersal operators or equations by those of nonlocal dispersal operators or equations, we investigate from three different points of view, that is, from initial-boundary value problem point of view, from spectral problem point of view, and from asymptotic behavior point of view. To this end, throughout Chapter 5, we assume

**(H1)** *$D \subset \mathbb{R}^N$ is either a bounded $C^{2+\alpha}$ domain for some $0 < \alpha < 1$ or $D = \mathbb{R}^N$; $k(\cdot) = k_\delta(\cdot)$ defined as in* (1.3) *and $\nu = \nu_\delta$ defined as in* (1.4)*.*

We first explore the approximation in terms of solutions of initial-boundary value problems. Consider (1.13) and (1.12) with the assumption (H1) for random and nonlocal cases, respectively. To be more precise, let $F(t, x, u)$ be $C^1$ in $t \in \mathbb{R}$ and $C^3$ in $(x, u) \in \mathbb{R} \times \mathbb{R}^N$, and $F(t, x + p_j \mathbf{e_j}, u) = F(t, x, u)$ $(j = 1, 2, \cdots, N)$ in case of $D = \mathbb{R}^N$. With an initial value $u_0(x)$ at $t = s$, (1.13) in case of $\nu = 1$ is

$$\begin{cases} \partial_t u(t,x) = \Delta u(t,x) + F(t,x,u), & x \in D, \\ B_{r,b} u(t,x) = 0, & x \in \partial D \ (x \in \mathbb{R}^N \text{ if } D = \mathbb{R}^N). \end{cases} \tag{1.17}$$

By general semigroup theory, for any $u_0 \in C(\bar{D})$ with $B_{r,b} u_0 = 0$ on $\partial D$, (1.17) has a unique (local) solution, denoted by $u(t, \cdot; s, u_0)$, such that $u(t, \cdot; s, u_0) = u_0(\cdot)$. Similarly, with the same initial value $u_0(x)$ at $t = s$, (1.12) in case of $\nu = \nu_\delta$ and $k(\cdot) = k_\delta(\cdot)$ is

$$\begin{cases} \partial_t u(t,x) = \nu_\delta \int_{D \cup D_c} k_\delta(x-y)[u(t,y) - u(t,x)]dy + F(t,x,u), & x \in \bar{D}, \\ B_{n,b} u(t,x) = 0, & x \in D_c \ (x \in \mathbb{R}^N \text{ if } D = \mathbb{R}^N). \end{cases} \tag{1.18}$$

By general semigroup theory, for any $u_0 \in C(\bar{D})$, (1.18) has a unique (local) solution, denoted by by $u^\delta(t,\cdot;s,u_0)$, such that $u^\delta(s,\cdot;s,u_0)=u_0(\cdot)$.

Among others, we prove that for any $u_0 \in C^3(\bar{D})$ with $B_{r,b}u_0=0$, and any $T>0$ satisfying that $u(t,\cdot;s,u_0)$ and $u^\delta(t,\cdot;s,u_0)$ exists on $[s,s+T]$, we have

- $\lim\limits_{\delta\to 0}\sup\limits_{t\in[s,s+T]}\|u^\delta(t,\cdot;s,u_0)-u(t,\cdot;s,u_0)\|_{C(\bar{D})}=0$ (see Theorem 2.13 for details).

We remark that Theorem 2.13 is fundamental in the study of approximation results. And in fact, the smoothness of the initial value $u_0$ is not optimal. But as the optimal smoothness is not what we are seeking for, we assume $u_0 \in C^3(\bar{D})$ technically.

Secondly, we investigate the following eigenvalue problem with time periodic random dispersal

$$\begin{cases} -\partial_t u+\Delta u+a(t,x)u=\lambda u, & x\in D,\\ B_{r,b}u=0, & x\in\partial D\ \ (x\in\mathbb{R}^N \text{ if } D=\mathbb{R}^N),\\ u(t+T,x)=u(t,x), & x\in D,\end{cases} \tag{1.19}$$

and the nonlocal counterpart are as follows

$$\begin{cases} -\partial_t u+\nu_\delta\int_{D\cup D_c}k_\delta(x-y)\,[u(t,y)-u(t,x)]\,dy+a(t,x)u=\lambda u, & x\in\bar{D},\\ B_{n,b}u=0, & x\in D_c\ \ (x\in\mathbb{R}^N \text{ if } D=\mathbb{R}^N),\\ u(t+T,x)=u(t,x), & x\in\bar{D},\end{cases} \tag{1.20}$$

where $B_{r,b}=B_{r,D}$ (resp. $B_{n,b}=B_{n,D}$) or $B_{r,b}=B_{r,N}$ (resp. $B_{n,b}=B_{n,N}$) or $B_{r,b}=B_{r,P}$ (resp. $B_{n,b}=B_{n,P}$). We assume that $a(t,x)$ is a $C^1$ function in $(t,x)\in\mathbb{R}\times\mathbb{R}^N$, $a(t+T,x)=a(t,x)$, and $a(t+T,x+p_j\mathbf{e_j})=a(t,x)$ $(j=1,2,\cdots,N)$ in case of $D=\mathbb{R}^N$.

The eigenvalue problem of (1.19) with $a(t,x)\equiv a(x)$ reduces to (1.15) with $\nu=1$. The principal eigenvalue problem associated to (1.19) has been extensively studied and is quite well understood (see [2, 22, 23, 33, 37, 39, 54, 58], etc.). For example, with any one of the three boundary conditions, it is known that the largest real part, denoted by $\lambda^{\mathcal{R}}(1,a)$, of the spectrum set of (1.19) is an isolated algebraically simple eigenvalue of (1.19) with a positive eigenfunction, and for any other $\lambda$ in the spectrum set of (1.19), $\mathrm{Re}\lambda \le \lambda^{\mathcal{R}}(1,a)$ ($\lambda^{\mathcal{R}}(1,a)$ is called the *principal eigenvalue* in literature).

The eigenvalue problem (1.20) with $a(t,x) \equiv a(x)$ reduces to (1.14) with $\nu = \nu_\delta$ and $k(\cdot) = k_\delta(\cdot)$. The principal spectrum problem associated to (1.20) has also been studied recently by many people (see [8, 17, 39, 56, 58, 61, 62, 59], etc.). The largest real part of the spectrum set of (1.20) with any one of the three boundary conditions, denoted by $\tilde{\lambda}^{\mathcal{N}}(\nu_\delta, a, \delta)$ is called the *principal spectrum point* of (1.20). $\tilde{\lambda}^{\mathcal{N}}(\nu_\delta, a, \delta)$ is also called the *principal eigenvalue* of (1.20), if it is an isolated algebraically simple eigenvalue of (1.20) with a positive eigenfunction (see Definition 2.1 for detail). For simplicity, we put $\tilde{\lambda}^{\mathcal{N}}(\nu_\delta, a, \delta) = \tilde{\lambda}^{\delta}(a)(\lambda^{\mathcal{N}}(\nu_\delta, a, \delta) = \lambda^{\delta}(a)$ if $\lambda^{\mathcal{N}}(\nu_\delta, a, \delta)$ exists), and $\lambda^{\mathcal{R}}(1,a) = \lambda^{r}(a)$ (see Remark 2.2 and Remark 2.19 for detail) and show that the principal eigenvalue of (1.19) can be approximated by the principal spectrum point of (1.20) in case that $\delta$ goes to zero, that is,

- $\lim_{\delta \to 0} \tilde{\lambda}^{\delta}(a) = \lambda^{r}(a)$ (see Theorem 2.15 for details).

We remark that some necessary and sufficient conditions are provided in [56] and [57] for the existence of principal eigenvalues of (1.20) (see Remark 2.11 for detail). This together with Theorem 2.15 implies the following remark.

**Remark 1.1.** *The principal eigenvalue $\lambda^{\delta}(a)$ of* (1.20) *exists provided $\delta \ll 1$.*

We also remark that Theorem 2.15 is another basis for the study of approximations of various dynamics of random dispersal evolution equations by those of nonlocal dispersal evolution equations.

Thirdly, we explore the asymptotic dynamics of the following time periodic KPP type evolution equation with random dispersal

$$\begin{cases} \partial_t u = \Delta u + u f(t,x,u), & x \in D, \\ B_{r,b} u = 0, & x \in \partial D \ \ (x \in \mathbb{R}^N \text{ if } D = \mathbb{R}^N), \end{cases} \tag{1.21}$$

and the time periodic KPP type evolution equation with nonlocal dispersal

$$\begin{cases} \partial_t u = \nu_\delta \int_{D\cup D_c} k_\delta(x-y)[u(t,y)-u(t,x)]dy + uf(t,x,u), & x\in \bar{D}, \\ B_{n,b}u = 0, & x\in D_c\,(x\in\mathbb{R}^N \text{ if } D=\mathbb{R}^N). \end{cases} \tag{1.22}$$

We assume the following monostable assumptions on $f$:

**(H2)** *$f$ is $C^1$ in $t\in\mathbb{R}$ and $C^3$ in $(x,u)\in\mathbb{R}\times\mathbb{R}^N$; $f(t,x,u)<0$ for $u\gg 1$ and $\partial_u f(t,x,u)<0$ for $u\geq 0$; $f(t+T,x,u)=f(t,x,u)$;* and when $D=\mathbb{R}^N$, $f(t+T,x,u)=f(t,x+p_j\mathbf{e_j},u)=f(t,x,u)$ for $j=1,2,\cdots,N$.

**(H3)** *For* (1.21)*, $\lambda^r(f(\cdot,\cdot,0))>0$, where $\lambda^r(f(\cdot,\cdot,0))$ is the principle eigenvalue of* (1.19) *with $a(t,x)=f(t,x,0)$.*

**(H3)$_\delta$** *For* (1.22)*, $\tilde{\lambda}^\delta(f(\cdot,\cdot,0))>0$, where $\tilde{\lambda}^\delta(f(\cdot,\cdot,0))$ is the principle spectrum point of* (1.20) *with $a(t,x)=f(t,x,0)$.*

Equations (1.21) and (1.22) are widely used to model population dynamics of species exhibiting random interactions and nonlocal interactions, respectively (see [7, 20, 53], etc. for (1.21) and [56] for (1.22)). Thanks to the pioneering works of Fisher [29] and Kolmogorov et al. [44] on the following special case of (1.21),

$$\partial_t u = u_{xx} + u(1-u), \quad x\in\mathbb{R},$$

(1.21) and (1.22) are referred to as Fisher type or KPP type evolution equations.

The dynamics of (1.21) and (1.22) have been studied in many papers (see [34, 53, 67] and references therein for (1.21), and [56] and references therein for (1.22)). With conditions (H2) and (H3), it is proved that (1.21) has exactly two nonnegative time periodic solutions, one is $u\equiv 0$ which is unstable and the other one, denoted by $u^*(t,x)$, is asymptotically stable and strictly positive (see [67, Theorem 3.1], see also [53, Theorems 1.1, 1.3]). Similar results for (1.22) under the assumptions (H2) and (H3)$_\delta$ are proved in [56, Theorem E]. We denote the strictly positive time periodic solution of (1.22) by $u_\delta^*(t,x)$. In Chapter 5, we show

- *If (H2) and (H3) hold,* $\sup_{t\in[0,T]}\|u_\delta^*(t,\cdot)-u^*(t,\cdot)\|_{C(\bar{D},\mathbb{R})}\to 0$*, as* $\delta\to 0$ (see Theorem 2.16 for detail).

Theorems 2.13-2.16 show that many important dynamics of random dispersal equations can be approximated by the corresponding dynamics of nonlocal dispersal equations, which is of both great theoretical and practical importance. At the end of Chapter 5, we apply the approximation theorems to the effect of rearrangement with equimeasurability on principal spectrum point of nonlocal dispersal operators.

The rest of the dissertation is organized as follows. In Chapter 2, we state some standing notations, assumptions, definitions, and the main results. In Chapter 3, we develop some basic tools for fundamental theory to be used in later Chapters, such as semigroup theory, comparison principle, sub- and super-solutions. We will investigate the spectral theory of time homogeneous nonlocal dispersal operators in Chapter 4. In Chapter 5, we study the approximations of random dispersal evolution operators/equations by the nonlocal dispersal evolution operators/equations. The dissertation will end with concluding remarks, several problems which are not well understood yet, and future plan in Chapter 6.

## Chapter 2

## Notations, Assumptions, Definitions and Main Results

In this chapter, we introduce first the standing notations, assumptions, and the definitions to be used in the rest of the dissertation. We then state the main results of the dissertation.

### 2.1 Notations, Assumptions and Definitions

Throughout this section, we will distinguish the three boundary conditions by $i = 1, 2, 3$. We first introduce the spaces of time independent functions and their norms. Let

$$X_1 = X_2 = C(\bar{D}) \tag{2.1}$$

with norm $\|u\|_{X_i} = \max_{x\in\bar{D}} |u(x)|$ for $i = 1, 2$,

$$X_3 = \{u \in C(\mathbb{R}^N, \mathbb{R}) \,|\, u(x + p_j\mathbf{e_j}) = u(x), \quad x \in \mathbb{R}^N, j = 1, 2, \cdots, N\} \tag{2.2}$$

with norm $\|u\|_{X_3} = \max_{x\in\mathbb{R}^N} |u(x)|$. And

$$X_i^+ = \{u \in X_i \,|\, u(x) \geq 0, \quad x \in \bar{D}\}, \tag{2.3}$$

$$X_i^{++} = \mathrm{Int}(X_i^+) = \{u \in X_i^+ \,|\, u(x) > 0, \quad x \in \bar{D}\} \tag{2.4}$$

$(i = 1, 2, 3)$. For $u^1(\cdot), u^2(\cdot) \in X_i$, we define

$$u^1 \leq u^2 (u^1 \geq u^2), \qquad \text{if } u^2 - u^1 \in X_i^+ (u^1 - u^2 \in X_i^+), \tag{2.5}$$

$$u^1 \ll u^2 (u^1 \gg u^2), \quad \text{if } u^2 - u^1 \in X_i^{++} (u^1 - u^2 \in X_i^{++}) \tag{2.6}$$

$(i = 1, 2, 3)$. For time periodic functions, we introduce the following spaces, together with their norms. Let

$$\mathcal{X}_1 = \mathcal{X}_2 = \{u \in C(\mathbb{R} \times \bar{D}, \mathbb{R}) | u(t+T, x) = u(t, x)\}$$

with norm $\|u\|_{\mathcal{X}_i} = \sup_{t \in [0,T]} \|u(t, \cdot)\|_{X_i} (i = 1, 2)$,

$$\mathcal{X}_3 = \{u \in C(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}) | u(t+T, x) = u(t, x + p_j \mathbf{e_j}) = u(t, x)\}$$

with norm $\|u\|_{\mathcal{X}_3} = \sup_{t \in [0,T]} \|u(t, \cdot)\|_{X_3}$. And

$$\mathcal{X}_i^+ = \{u \in \mathcal{X}_i | u(t, x) \geq 0\}, \tag{2.7}$$

$$\mathcal{X}_i^{++} = \text{Int}(\mathcal{X}_i^+) = \{u \in \mathcal{X}_i^+ | u(t, x) > 0\} \tag{2.8}$$

$(i = 1, 2, 3)$. For $u^1, u^2 \in \mathcal{X}_i$, we define

$$u^1 \leq u^2 (u^1 \geq u^2), \qquad \text{if } u^2 - u^1 \in \mathcal{X}_i^+ \ (u^1 - u^2 \in \mathcal{X}_i^+), \tag{2.9}$$

$$u^1 \ll u^2 (u^1 \gg u^2), \quad \text{if } u^2 - u^1 \in \mathcal{X}_i^{++} (u^1 - u^2 \in \mathcal{X}_i^{++}) \tag{2.10}$$

$(i = 1, 2, 3)$. The introduction of $X_2$ and $\mathcal{X}_2$ is for convenience.

Next, we introduce the definitions of principal spectrum point and principal eigenvalues for nonlocal dispersal operators.

For $i = 1, 2, 3$, let $a_i(\cdot, \cdot) \in \mathcal{X}_i \cap C^1(\mathbb{R} \times \mathbb{R}^N)$, $\nu_i > 0$, and $\mathcal{N}_i(\nu_i, a_i) : \mathcal{D}(\mathcal{N}_i(\nu_i, a_i)) \subset \mathcal{X}_i \to \mathcal{X}_i$ be defined as follows,

$$(\mathcal{N}_1(\nu_1, a_1)u)(t, x) = -\partial_t u(t, x) + \nu_1 \left[\int_D k(x-y) u(t, y) dy - u(t, x)\right] + a_1(t, x) u(t, x) \tag{2.11}$$

for $(t,x) \in \mathbb{R} \times \bar{D}$,

$$(\mathcal{N}_2(\nu_2, a_2)u)(t,x) = -\partial_t u(t,x) + \nu_2 \int_D k(x-y)[u(t,y) - u(t,x)]dy + a_2(t,x)u(t,x) \quad (2.12)$$

for $(t,x) \in \mathbb{R} \times \bar{D}$, and

$$(\mathcal{N}_3(\nu_3, a_3)u)(t,x) = -\partial_t u(t,x) + \nu_3 \int_{\mathbb{R}^N} k(x-y)[u(t,y) - u(t,x)]dy + a_3(t,x)u(t,x) \quad (2.13)$$

for $(t,x) \in \mathbb{R} \times \mathbb{R}^N$.

**Definition 2.1** (Principal Eigenvalue). *For $i = 1, 2, 3$, let $\sigma(\mathcal{N}_i(\nu_i, a_i))$ be the spectrum of $\mathcal{N}_i(\nu_i, a_i)$ on $\mathcal{X}_i$*

*(1) $\tilde{\lambda}_i^{\mathcal{N}}(\nu_i, a_i) := \sup\{\mathrm{Re}\lambda | \lambda \in \sigma(\mathcal{N}_i(\nu_i, a_i))\}$ is called the* principal spectrum point *of $\mathcal{N}_i(\nu_i, a_i)$.*

*(2) A real number $\lambda_i^{\mathcal{N}}(\nu_i, a_i)$ is called the* principal eigenvalue *of* (1.20) *or $\mathcal{N}_i(\nu_i, a_i)$ if it is an isolated algebraically simple eigenvalue of $\mathcal{N}_i(\nu_i, a_i)$ with an eigenfunction $v \in \mathcal{X}_i^+$, and for every $\lambda \in \sigma(\mathcal{N}_i(\nu_i, a_i)) \backslash \{\lambda_i^{\mathcal{N}}(\nu_i, a_i)\}$,* $\mathrm{Re}\lambda \leq \lambda_i^{\mathcal{N}}(\nu_i, a_i)$.

Observe that if the principal eigenvalue $\lambda_i^{\mathcal{N}}(\nu_i, a_i)$ exists, then $\tilde{\lambda}_i^{\mathcal{N}}(\nu_i, a_i) = \lambda_i^{\mathcal{N}}(\nu_i, a_i)$. If $k(\cdot)$ depends on $\delta$, we put

$$\tilde{\lambda}_i^{\mathcal{N}}(\nu_i, a_i) = \tilde{\lambda}_i^{\mathcal{N}}(\nu_i, a_i, \delta). \quad (2.14)$$

**Remark 2.2.** *(1) We use the super script $\mathcal{N}$ to indicate that both principal eigenvalue and principal spectrum point are for nonlocal operators. If there is no confusion, the notation can be simplified. For example, in Chapter 4, we only focus on nonlocal dispersal operators and consider the dependence of their principal spectrum points and principal eigenvalues on underlying parameters $\nu_i, a_i$ and $\delta$, so we put $\tilde{\lambda}_i^{\mathcal{N}}(\nu_i, a_i, \delta) = \tilde{\lambda}_i(\nu_i, a_i, \delta)$ for principal spectrum point and $\lambda_i^{\mathcal{N}}(\nu_i, a_i, \delta) = \lambda_i(\nu_i, a_i, \delta)$ for principal eigenvalue, respectively. In Chapter 5, we consider the approximation of random dispersal operators by nonlocal dispersal operators as* *the parameter $\delta$ goes to zero. More precisely, in* (1.20)*,* $k(\cdot) = k_\delta(\cdot)$ *is defined as in* (1.3) *and* $\nu_i = \nu_\delta$ *is defined as* (1.4)*, so we put* $\tilde{\lambda}_i^{\mathcal{N}}(\nu_\delta, a, \delta) = \tilde{\lambda}_i^\delta(a)$ *for principal spectrum point and* $\lambda_i^{\mathcal{N}}(\nu_i, a) = \lambda_i^\delta(a)$ *for principal eigenvalue, respectively.*

*(2) In the case* $a_i(t,x) \equiv a_i(x)$ $(i = 1, 2, 3)$*, let*

$$\mathcal{K}_i : X_i \to X_i,\ (\mathcal{K}_i u)(x) = \int_D k(x-y)u(y)dy \quad \forall\, u \in X_i, \quad i = 1, 2, \tag{2.15}$$

*and*

$$\mathcal{K}_3 : X_3 \to X_3,\ (\mathcal{K}_3 u)(x) = \int_{\mathbb{R}^N} k(x-y)u(y)dy \quad \forall\, u \in X_3. \tag{2.16}$$

*Let*

$$\begin{cases} h_1(x) = -\nu_1 + a_1(x), \\ h_2(x) = -\nu_2 \int_D k(x-y)dy + a_2(x), \\ h_3(x) = -\nu_3 + a_3(x). \end{cases} \tag{2.17}$$

*Then we have*

$$\tilde{\lambda}_i^{\mathcal{N}}(\nu_i, a_i) = \sup\{\mathrm{Re}\mu \,|\, \mu \in \sigma(\nu_i \mathcal{K}_i + h_i(\cdot)\mathcal{I})\}, \tag{2.18}$$

*where* $\mathcal{I}$ *is the identity map on* $X_i$*. Moreover, a real number* $\lambda \in \mathbb{R}$ *is called the* principal eigenvalue *of* $\nu_i \mathcal{K}_i + h_i(\cdot)\mathcal{I}$ *if it is an isolated algebraically simple eigenvalue of* $\nu_i \mathcal{K}_i + h_i(\cdot)\mathcal{I}$ *with a positive eigenfunction and for any* $\mu \in \sigma(\nu_i \mathcal{K}_i + h_i(\cdot)\mathcal{I}) \setminus \{\lambda\}$, $\mathrm{Re}\mu < \lambda$*. The principal eigenvalue of* $\mathcal{N}_i(\nu_i, a_i)$ *exists iff the principal eigenvalue of* $\nu_i \mathcal{K}_i + h_i(\cdot)\mathcal{I}$ *exists.*

The spectral theory of random dispersal operators is well known. For the time periodic random dispersal operators, let $a(\cdot,\cdot) \in \mathcal{X}_i \cap C^1(\mathbb{R} \times \mathbb{R}^N)$, and $\mathcal{R}_i(a) : \mathcal{D}(\mathcal{R}_i(\nu_i, a_i)) \subset \mathcal{X}_i \to \mathcal{X}_i$ be defined as follows,

$$(\mathcal{R}_i(\nu_i, a_i)u)(t,x) = -\partial_t u(t,x) + \nu_i \Delta u(t,x) + a_i(t,x)u(t,x)$$

for $i = 1, 2, 3$. Note that for $u \in \mathcal{D}(\mathcal{R}_1(\nu_1, a_1))$, $B_{r,D}u = 0$ on $\partial D$ and for $u \in \mathcal{D}(\mathcal{R}_2(\nu_2, a_2))$, $B_{r,N}u = 0$ on $\partial D$. Let

$$\lambda_i^{\mathcal{R}}(\nu_i, a_i) = \sup\{\mathrm{Re}\lambda | \lambda \in \sigma(\mathcal{R}_i(\nu_i, a_i))\}.$$

It is well known that $\lambda_i^{\mathcal{R}}(\nu_i, a_i)$ is an isolated algebraically simple eigenvalue of $\mathcal{R}_i(\nu_i, a_i)$ with a positive eigenfunction (see [33]) and $\lambda_i^{\mathcal{R}}(\nu_i, a_i)$ is called the *principal eigenvalue* of $\mathcal{R}_i(\nu_i, a_i)$ in literature. Recently, the principal eigenvalue problem for nonlocal dispersal operators has been studied by several authors (see [41] for time homogeneous case; see [39] for time-periodic and almost time-periodic cases; see [58] for general time-periodic cases).

**Remark 2.3.** *In Chapter 5, we consider the approximation of principal eigenvalues* $\lambda^{\mathcal{R}}(1, a)$ *of random dispersal operators in* (1.19) *by the principal spectrum point* $\tilde{\lambda}_i^{\mathcal{N}}(\nu_\delta, a, \delta)$ *(*$i = 1, 2, 3$*) of nonlocal dispersal operators in* (1.20)*. We simplified the notation in the nonlocal case in Remark 2.2, so for our convenience, we put*

$$\lambda_i^{\mathcal{R}}(1, a) = \lambda_i^r(a) \quad \text{for } i = 1, 2, 3. \tag{2.19}$$

## 2.2 Main Results

In this section, we state the main results of this dissertation.

We first state the results of the dependence of principal spectrum points/principal eigenvalues on the underlying parameters. In the following, we put

$$D = [0, p_1] \times [0, p_2] \times \cdots \times [0, p_N], \tag{2.20}$$

in periodic boundary condition case. For given $a_i \in X_i$, let

$$\hat{a}_i = \frac{1}{|D|} \int_D a_i(x) dx, \;\; i = 1, 2, 3, \tag{2.21}$$

where $|D|$ is the Lebesgue measure of $D$. Let

$$a_{i,\max} = \max_{x\in\bar{D}} a_i(x), \quad a_{i,\min} = \min_{x\in\bar{D}} a_i(x),$$

and

$$h_{i,\max} = \max_{x\in\bar{D}} h_i(x), \quad h_{i,\min} = \min_{x\in\bar{D}} h_i(x),$$

where $h_i(\cdot)$ is as in (2.17). If no confusion occurs, we put $\tilde{\lambda}_i(\nu_i, a_i) = \tilde{\lambda}_i^{\mathcal{N}}(\nu_i, a_i)$ and $\lambda_i(\nu_i, a_i) = \lambda_i^{\mathcal{N}}(\nu_i, a_i)$ if $\lambda_i^{\mathcal{N}}(\nu_i, a_i)$ exists.

**Theorem 2.4** (Effects of spatial variation)**.** *Let $1 \le i \le 3$ and $a_i(\cdot) \in X_i$ be given.*

*(1)* (Existence of principal eigenvalues) *For given $1 \le i \le 2$, $\lambda_i(\nu_i, a_i)$ exists if $a_{i,\max} - a_{i,\min} < \nu_i \inf_{x\in\bar{D}} \int_D k(x-y)dy$.*

*(2)* (Existence of principal eigenvalues) *For given $1 \le i \le 2$, $\lambda_i(\nu_i, a_i)$ exists if $h_i(\cdot)$ is in $C^N(\bar{D})$, there is some $x_0 \in \mathrm{Int}(D)$ satisfying that $h_i(x_0) = h_{i,\max}$, and the partial derivatives of $h_i(x)$ up to order $N-1$ at $x_0$ are zero.*

*(3)* (Upper bounds) *For given $1 \le i \le 3$ and $c_i \in \mathbb{R}$,* $\sup\{\tilde{\lambda}_i(\nu_i, a_i) \,|\, a_i \in X_i,\ \hat{a}_i = c_i\} = \infty$.

*(4)* (Lower bounds) *Assume that $k(\cdot)$ is symmetric with respect to $0$ (i.e. $k(-z) = k(z)$) and $i = 2$. For given $c_i \in \mathbb{R}$,*

$$\inf\{\tilde{\lambda}_i(\nu_i, a_i) \,|\, a_i \in X_i,\ \hat{a}_i = c_i\} = \lambda_i(\nu_i, c_i)(= c_i)$$

*(hence $\tilde{\lambda}_i(\nu_i, a_i) \ge \tilde{\lambda}_i(\nu_i, \hat{a}_i)$). If the principal eigenvalue of $\nu_i\mathcal{K}_i + h_i(\cdot)\mathcal{I}$ exists, then the infimum is attained by the constant function (i.e. $a_i(\cdot) \equiv \hat{a}_i$).*

*(5)* (Monotonicity) *For given $a_i^1, a_i^2 \in X_i$, if $a_i^1(x) \le a_i^2(x)$, then $\tilde{\lambda}_i(a_i^1, \nu_i) \le \tilde{\lambda}_i(a_i^2, \nu_i)$ $(i = 1, 2, 3)$.*

**Remark 2.5.** *(1) For the case $i = 3$, similar result to Theorem 2.4(1) is proved in [61]. To be more precise, it is proved in [61] that if $a_{3,\max} - a_{3,\min} < \nu_3$, then $\lambda_3(\nu_3, a_3)$ exists.*

*(2) For the case $i = 3$, similar result to Theorem 2.4(2) is also proved in [61]. Actually it is proved in [61] that if $a_3(\cdot)$ is $C^N$ and there is $x_0 \in \mathbb{R}^N$ such that $a_3(x_0) = a_{3,\max}$ and the partial derivatives of $a_3(x)$ up to order $N - 1$ at $x_0$ are zero, then $\lambda_3(\nu_3, a_3)$ exists.*

*(3) For one space dimensional random dispersal operators, for given $c_i \in \mathbb{R}$,*

$$\sup\{\lambda_i^{\mathcal{R}}(\nu_i, a_i) \,|\, a_i \in X_i^{++},\ \hat{a}_i = c_i\} < \infty$$

*(see Remark 4.8 for detail). Theorem 2.4(3) hence reflects some difference between random dispersal operators and nonlocal dispersal operators.*

*(4) Similar result to Theorem 2.4(4) holds for $i = 3$. To be more precise, it is proved in [63] that for any given $c_3 \in \mathbb{R}$,*

$$\inf\{\tilde{\lambda}_3(\nu_3, a_3) \,|\, a_3 \in X_3,\ \hat{a}_3 = c_3\} = \lambda_3(\nu_3, c_3)(= c_3).$$

*But Theorem 2.4(4) may not hold for the case $i = 1$ (see Remark 4.8 for detail).*

**Theorem 2.6** (Effects of dispersal rate)**.** *Assume that $1 \le i \le 3$ and $k(\cdot)$ is symmetric with respect to 0. Let $a_i \in X_i$ be given.*

*(1)* (Monotonicity) *Assume $a_i(\cdot) \not\equiv$ constant. If $\nu_i^1 < \nu_i^2$, then $\tilde{\lambda}_i(\nu_i^1, a_i) > \tilde{\lambda}_i(\nu_i^2, a_i)$.*

*(2)* (Existence of principal eigenvalue) *If $i = 1$ or 3 and $\lambda_i(\nu_i, a_i)$ exists for some $\nu_i > 0$, then $\lambda_i(\tilde{\nu}_i, a_i)$ exists for all $\tilde{\nu}_i > \nu_i$.*

*(3)* (Existence of principal eigenvalue) *There is $\nu_i^0 > 0$ such that the principal eigenvalue $\lambda_i(\nu_i, a_i)$ of $\nu_i\mathcal{K}_i + h_i(\cdot)\mathcal{I}$ exists for $\nu_i > \nu_i^0$.*

*(4)* (Limits as the dispersal rate goes to 0) $\lim_{\nu_i \to 0+} \tilde{\lambda}_i(\nu_i, a_i) = a_{i,\max}$.

*(5)* (Limits as the dispersal rate goes to $\infty$) $\lim_{\nu_i\to\infty}\tilde{\lambda}_i(\nu_i,a_i) = -\infty$ *for* $i=1$ *and* $\lim_{\nu_i\to\infty}\tilde{\lambda}_i(\nu_i,a_i) = \hat{a}_i$ *for* $i=2$ *and* 3.

**Remark 2.7.** *(1) It is open whether Theorem 2.6 (2) holds for the case* $i=2$.

*(2) Theorem 2.6 (3) and (4) still hold if* $k(\cdot)$ *is not symmetric.*

In the case that $k(\cdot)=k_\delta(\cdot)$ defined as in (1.3) for $\delta>0$, to indicate the dependence of $\tilde{\lambda}_i(\nu_i,a_i)$ on $\delta$, put

$$\tilde{\lambda}_i(\nu_i,a_i,\delta)=\tilde{\lambda}_i(\nu_i,a_i).$$

**Theorem 2.8** (Effects of dispersal distance)**.** *Suppose that* $k(z)=k_\delta(z)$*, where* $k_\delta(z)$ *is defined as in* (1.3) *and* $k(z)=k(-z)$*. Let* $1\le i\le 3$ *and* $a_i\in X_i$ *be given.*

*(1)* (Limits as dispersal distance goes to 0) $\lim_{\delta\to 0}\tilde{\lambda}_i(\nu_i,a_i,\delta)=a_{i,\max}$.

*(2)* (Limits as dispersal distance goes to $\infty$)

$$\lim_{\delta\to\infty}\tilde{\lambda}_1(\nu_1,a_1,\delta)=-\nu_1+a_{1,\max},$$

$$\lim_{\delta\to\infty}\tilde{\lambda}_2(\nu_2,a_2,\delta)=a_{2,\max},$$

*and*

$$\lim_{\delta\to\infty}\tilde{\lambda}_3(\nu_3,a_3,\delta)=\bar{\lambda}_3(\nu_3,a_3),$$

*where*

$$\bar{\lambda}_3(\nu_3,a_3)=\max\{\mathrm{Re}\lambda \mid \lambda\in\sigma(\nu_3\bar{\mathcal{I}}+h_3(\cdot)\mathcal{I})\},$$

*and*

$$\bar{\mathcal{I}}u=\frac{1}{|D|}\int_D u(x)dx.$$

*(3)* (Existence of principal eigenvalue) *There is* $\delta_0>0$ *such that the principal eigenvalue* $\lambda_i(\nu_i,a_i)$ *of* $\nu_i\mathcal{K}_i+h_i(\cdot)\mathcal{I}$ *exists for* $0<\delta<\delta_0$.

**Remark 2.9.** *(1) For $i = 1$ or $3$, Theorem 2.8 (1) is proved in [41, Theorem 2.6].*

*(2) For $i = 1$ or $3$, Theorem 2.8 (3) is proved in [41] (see also [61] for the case $i = 3$).*

**Corollary 2.10** (Criteria for the existence of principal eigenvalues)**.** *Let $1 \le i \le 3$ and $a_i \in X_i$ be given.*

*(1) $\lambda_i(\nu_i, a_i)$ exists provided that $\max_{x\in\bar{D}} a_i(x) - \min_{x\in\bar{D}} a_i(x) < \nu_i \inf_{x\in\bar{D}} \int_D k(x-y)dy$ in the case $i = 1, 2$ and $\max_{x\in\bar{D}} a_i(x) - \min_{x\in\bar{D}} a_i(x) < \nu_i$ in the case $i = 3$.*

*(2) $\lambda_i(\nu_i, a_i)$ exists provided that $h_i(\cdot)$ is in $C^N(\bar{D})$, there is some $x_0 \in \mathrm{Int}(D)$ satisfying that $h_i(x_0) = h_{i,\max}$, and the partial derivatives of $h_i(x)$ up to order $N-1$ at $x_0$ are zero.*

*(3) There is $\nu_i^0 > 0$ such that the principal eigenvalue $\lambda_i(\nu_i, a_i)$ of $\nu_i\mathcal{K}_i + h_i(\cdot)\mathcal{I}$ exists for $\nu_i > \nu_i^0$.*

*(4) Suppose that $k(z) = k_\delta(z)$, where $k_\delta(z)$ is defined as in (1.3) and $\tilde{k}(\cdot)$ is symmetric with respect to 0. Then there is $\delta_0 > 0$ such that the principal eigenvalue $\lambda_i(\nu_i, a_i)$ of $\nu_i\mathcal{K}_i + h_i(\cdot)\mathcal{I}$ exists for $0 < \delta < \delta_0$.*

*Proof.* (1) and (2) are Theorem 2.4(1) and (2), respectively.

(3) is Theorem 2.6(3).

(4) is Theorem 2.8(3). □

**Remark 2.11** (Conditions for the existence of principal eigenvalue in time periodic cases)**.** *The results of conditions for the existence of principal eigenvalue have been extended to time periodic nonlocal dispersal operators of $\mathcal{N}_i(\nu_i, a_i)$ ($i = 1, 2, 3$) in [56]. More precisely, for given $1 \le i \le 3$, and $a_i(\cdot,\cdot) \in \mathcal{X}_i$, let*

$$\bar{a}_i(x) = \frac{1}{T}\int_0^T a_i(t,x)dt, \quad b_1 = -\nu_1, \quad b_2 = -\nu_2\int_D k(x-y)dy, \quad \text{and} \quad b_3 = -\nu_3.$$

*The following conditions for the existence of principal eigenvalues of the nonlocal dispersal operators of* $\mathcal{N}_i(\nu_i, a_i)$ *have already been proved in [56].*

*(1) (Necessary and sufficient condition)* $\tilde{\lambda}_i^{\mathcal{N}}(\nu_i, a_i)$ *is the principal eigenvalue of* $\mathcal{N}_i(\nu_i, a_i)$ *if and only if*

$$\tilde{\lambda}_i^{\mathcal{N}}(\nu_i, a_i) > \max_{x\in\bar{D}_i}(b_i(x) + \bar{a}_i(x)),$$

*where* $D_1 = D_2 = D$ *and* $D_3 = [0, p_1] \times [0, p_2] \times \cdots \times [0, p_N]$ *as in* (2.20).

*(2) (Sufficient condition)* $\tilde{\lambda}_i^{\mathcal{N}}(\nu_i, a_i)$ *is the principal eigenvalue of* $\mathcal{N}_i(\nu_i, a_i)$*, provided that*

*(a)* $\max_{x\in\bar{D}} \bar{a}_i(x) - \min_{x\in\bar{D}} \bar{a}_i(x) < \nu_i \mathrm{Inf}_{x\in\bar{D}} \int_D k(x-y)dy$ *in the case of* $i = 1, 2$ *and* $\max_{x\in\bar{D}} \bar{a}_i(x) - \min_{x\in\bar{D}} \bar{a}_i(x) - \min_{x\in\bar{D}} \bar{a}_i(x) < \nu_i$ *in the case* $i = 3$ *(which extends the result in Theorem 2.4(1));*

*or*

*(b)* $b_i(x) + \bar{a}_i(x)$ *is in* $C^N$*, there is some* $x_0 \in \mathrm{Int}(D_i)$ *in the case of* $i = 1, 2$*, and* $x_0 \in D_3$ *in the case of* $i = 3$ *satisfying that* $b_i(x_0) + \bar{a}_i(x_0) = \max_{x\in\bar{D}}(b_i(x) + \bar{a}_i(x))$*, and the partial derivatives of* $b_i(x)+\bar{a}_i(x)$ *up to order* $N-1$ *at* $x_0$ *is zero(which extends the result in Theorem 2.4(2));*

*or*

*(c)* $0 < \delta \ll 1$ *for* $\mathcal{N}(\nu_i, a_i, \delta)$*, where* $\delta > 0$ *is the dispersal distance and* $k(\cdot) = k_\delta(\cdot)$ *as in* (1.3) *(which extends the result in Theorem 2.8(3)).*

The following is an application of the above stated theorems to a two-species competition system.

**Theorem 2.12.** *(1) There are* $u^*(\cdot) \in X_1^{++}$ *and* $v^*(\cdot) \in X_2^{++}$ *such that* $(u^*(\cdot), 0)$ *and* $(0, v^*(\cdot))$ *are stationary solutions of* (1.16)*. Moreover, for any* $(u_0, v_0) \in X_1^+ \times X_2^+$ *with* $u_0 \neq 0$ *and* $v_0 = 0$ *(resp.* $u_0 = 0$ *and* $v_0 \neq 0$*),* $(u(t, \cdot; u_0, v_0), v(t, \cdot; u_0, v_0)) \to (u^*(\cdot), 0)$ *(resp.* $(u(t, \cdot; u_0, v_0), v(t, \cdot; u_0, v_0)) \to (0, v^*(\cdot))$*) as* $t \to \infty$*.*

*(2) For any $(u_0, v_0) \in (X_1^+ \setminus \{0\}) \times (X_2^+ \setminus \{0\})$, $\lim_{t\to\infty}(u(t,\cdot;u_0,v_0), v(t,\cdot;u_0,v_0)) = (0, v^*(\cdot))$.*

Next, we state the main results on the approximations of random dispersal operators or equations by nonlocal dispersal operators or equations. Recall that $u^\delta(t,x;s,u_0)$ is the solution of (1.18) with $u(s,x;s,u_0) = u_0(x)$ and $u(t,x;s,u_0)$ is the solution of (1.17) with $u(s,x;s,u_0) = u_0(x)$.

**Theorem 2.13** (Approximations of initial-boundary value problems)**.** *For any given $s \in \mathbb{R}$, any $u_0 \in C^3(\bar{D})$ with $B_{r,b}u_0 = 0$, and any $T > 0$ satisfying that $u(t,x;s,u_0)$ and $u^\delta(t,x;s,u_0)$ exist on $[s, s+T]$,*

$$\lim_{\delta\to 0} \sup_{t\in[s,s+T]} \|u^\delta(t,\cdot;s,u_0) - u(t,\cdot;s,u_0)\|_{C(\bar{D})} = 0.$$

**Remark 2.14.** *In the Dirichlet and Neumann boundary condition cases with $F(t,x,u) \equiv 0$ in (1.17) and (1.18), Theorem 2.13 has been proved in [15] and [16], respectively.*

**Theorem 2.15** (Approximation of principal eigenvalues)**.** *For given $1 \le i \le 3$, and $a(\cdot,\cdot) \in \mathcal{X}_i \cap C^1(\mathbb{R}\times\mathbb{R}^N)$, $\lim_{\delta\to 0} \tilde{\lambda}_i^\delta(a) = \lambda_i^r(a)$, where $\tilde{\lambda}^\delta(a)$ and $\lambda^r(a)$ are the principal spectrum point of the nonlocal dispersal operator $\mathcal{N}_i(\nu_\delta, a, \delta)$(see Remark 2.2), and the principal eigenvalue of the random dispersal operator $\mathcal{R}_i(1,a)$ (see Remark 2.19), respectively.*

**Theorem 2.16.** *Consider (1.22) and (1.21). If (H2) and (H3) hold, then for any $\epsilon > 0$, there exists $\delta_0 > 0$, such that for all $0 < \delta < \delta_0$, we have*

$$\sup_{t\in[0,T]} \|u_\delta^*(t,\cdot) - u^*(t,\cdot)\|_{C(\bar{D},\mathbb{R})} \le \epsilon,$$

*where $u_\delta^*(\cdot,\cdot)$ and $u^*(\cdot,\cdot)$ are the strictly positive, asymptotically stable, and time periodic solutions of (1.22), and (1.21), respectively.*

**Remark 2.17.**

*(1) The existence, uniqueness, and asymptotic stability of $u^*(t,x)$ have been proved in [67].*

*(2) The existence, uniqueness, and asymptotic stability of $u^*_\delta(t,x)$ have been proved in [56].*

Finally, we present an application of approximation theorems to the effect of the rearrangements with equimeasurability on principal spectrum point of nonlocal dispersal operators. Consider the restriction of the eigenvalue problem (1.19) on $X_i$ $(i=1,2,3)$, that is,

$$\begin{cases} \Delta u + a(x)u = \lambda u, & x \in D, \\ B_{r,b}u(x) = 0, & x \in \partial D\, (x \in \mathbb{R}^N \text{ if } D = \mathbb{R}^N). \end{cases} \tag{2.22}$$

Note that the principal eigenvalues of (1.19) and (2.22) are the same. Consider also the symmetrized problem

$$\begin{cases} \Delta u + a_\sharp(x)u = \lambda u, & x \in D^\sharp, \\ B_{r,b}u(x) = 0, & x \in \partial D^\sharp \left(x \in \mathbb{R}^N \text{ if } D = \mathbb{R}^N\right), \end{cases} \tag{2.23}$$

where $B_{r,b}u$ denotes the boundary condition as in (1.19), and $D^\sharp$ and $a_\sharp(\cdot)$ are the Schwarz symmetrization of $D$ and $a(\cdot)$, respectively (see [1] for details of the Schwarz symmetrization). It is well-known that

$$\lambda_i^r(a_\sharp) \geq \lambda_i^r(a), \tag{2.24}$$

which simply follows from the following inequality

$$\int_{D^\sharp} a_\sharp(x) u_\sharp^2(x) dx \geq \int_D a(x) u^2(x) dx, \tag{2.25}$$

and the variational characterization of $\lambda_i^r(a_\sharp)$ and $\lambda_i^r(a)$, where $\lambda_i^r(a_\sharp)$ is the principal eigenvalue of (2.23), and $\lambda_i^r(a)$ is the principal eigenvalue of (2.22) respectively. What's more, the "=" in (2.24) holds if and only if both the domain and functions are symmetric, that is $D = D^\sharp$, $a(\cdot) = a_\sharp(\cdot)$, and $u(\cdot) = u_\sharp(\cdot)$ (see [21] for details).

By Theorem 2.15, the principal eigenvalues of random dispersal operators can be approximated by the principal spectrum point of nonlocal dispersal operators. So it is natural to expect that the the relation like (2.24) holds for principal spectrum point of nonlocal dispersal operator. So next, we consider the eigenvalue problems of the nonlocal counterparts of (2.22),

$$\begin{cases} \nu_\delta \left[ \int_{D \cup D_c} k_\delta(x-y)u(y)dy - u(x) \right] + a(x)u(x) = \lambda u(x), & x \in \bar{D}, \\ B_{n,b}u(x) = 0, & x \in D_c \ \ (x \in \mathbb{R}^N \text{ if } D = \mathbb{R}^N), \end{cases} \tag{2.26}$$

and its symmetrized problem

$$\begin{cases} \nu_\delta \left[ \int_{D^\sharp \cup (D^\sharp)_c} k_\delta(x-y)u(y)dy - u(x) \right] + a_\sharp(x)u(x) = \lambda u(x), & x \in \bar{D}^\sharp, \\ B_{n,b}u(x) = 0, & x \in (D^\sharp)_c \ \ (x \in \mathbb{R}^N \text{ if } D = \mathbb{R}^N), \end{cases} \tag{2.27}$$

where the kernel function $k(\cdot)$ is symmetric with respect to 0, and $B_{n,b}u$ denotes the boundary condition as in (1.20), $a_\sharp(\cdot)$, $k_{\delta\sharp}(\cdot)$, and $\bar{D}^\sharp$ are the Schwarz symmetrization of $a(\cdot)$, $k_\delta(\cdot)$ and $\bar{D}$, respectively. We denote the principal spectrum point of (2.26), and (2.27) by $\tilde{\lambda}_i^\delta(a)$ and $\tilde{\lambda}_i^\delta(a_\sharp)$ for $i = 1, 2, 3$, respectively. We have the following comparison relation between $\tilde{\lambda}_i^\delta(a)$ and $\tilde{\lambda}_i^\delta(a_\sharp)$.

**Theorem 2.18.** *For $1 \le i \le 3$, assume $a(\cdot) \in X_i$, $k_\delta(\cdot)$ and $\nu_\delta$ are as in (1.3) and (1.4), respectively. Let $a_\sharp(\cdot)$, $k_{\delta\sharp}$ and $\bar{D}^\sharp$ be the Schwarz symmetrization of $a(\cdot)$, $k_\delta(\cdot)$ and $\bar{D}$. Then there exists $\delta_0 > 0$, such that*

$$\tilde{\lambda}_i^\delta(a_\sharp) \ge \tilde{\lambda}_i^\delta(a) \quad \text{for } \delta \ll \delta_0,$$

*where $\tilde{\lambda}_i^\delta(a)$ and $\tilde{\lambda}_i^\delta(a_\sharp)$ are the principle spectrum points of the eigenvalue problems (2.26) and (2.27), respectively.*

Chapter 3

Preliminary

In this Chapter, we establish some basic properties of solutions of nonlocal evolution equations, including the comparison principle and monotonicity of solutions with respect to initial conditions.

## 3.1 Solutions of Evolution Equation and Semigroup Theory

For given $1 \leq i \leq 3$, and $a_i(\cdot,\cdot) \in \mathcal{X}_i$, consider the following evolution equation

$$\begin{cases} \partial_t u(t,x) = \nu_i \int_{D\cup D_c} k(x-y)[u(t,y)-u(t,x)]dy + a_i(t,x)u(t,x), & x \in \bar{D}, \\ B_{n,b}u(t,x) = 0, & x \in D_c \ (x \in \mathbb{R}^N \text{ if } D = \mathbb{R}^N), \\ u(s,x) = u_0(x), \end{cases} \tag{3.1}$$

where $D \subset \mathbb{R}^N$, $k(\cdot)$ and $B_{n,b}u(t,x) = 0$ are the same as in (1.12). By general linear semigroup theory (see [32] and [55]), for any $u_0 \in X_i$ with $B_{n,b}u_0 = 0$ on $D_c$ ($D_c = \mathbb{R}^N \setminus \bar{D}$ and $b = D$ when $i = 1$, $D_c = \emptyset$ and $b = N$ when $i = 2$, and $D_c = \mathbb{R}^N$ and $b = P$ when $i = 3$), and $s \in \mathbb{R}$, (3.1) has a unique (local) solution, we denote it by $u_i^{\mathcal{N}}(t,\cdot;s,u_0,\nu_i,a_i)$. We put

$$\Phi_i^{\mathcal{N}}(t,s;\nu_i,a_i,u_0) = u_i^{\mathcal{N}}(t,\cdot;s,u_0,\nu_i,a_i), \quad u_0 \in X_i.$$

Note that if $\nu = \nu_\delta$, $a_i(\cdot,\cdot) = a(\cdot,\cdot)$ and $k(\cdot) = k_\delta(\cdot)$, (3.1) is the evolution equation associated to the eigenvalue problem (1.20). For $i = 1,2,3$, we put

$$(\Phi_i^\delta(t,s;a)u_0)(\cdot) = u_i^{\mathcal{N}}(t,\cdot;s,u_0,\nu_\delta,a_i), \quad u_0 \in X_i. \tag{3.2}$$

For evolution equations with random dispersal operators, let $A$ be $-\Delta$ with Dirichlet boundary condition acting on $X_1 \cap C_0(D)$, and put

$$X_1^r = \mathcal{D}(A^\alpha) \tag{3.3}$$

for some $0 < \alpha < 1$ such that $C^1(\bar{D}) \subset X_1^r$ with $\|u\|_{X_1^r} = \|A^\alpha u\|_{X_1}$, and

$$X_i^r = X_i \quad \text{for} \quad i = 2, 3 \tag{3.4}$$

with $\|u\|_{X_i^r} = \|u\|_{X_i}$. And

$$X_i^{r,+} = \{u \in X_i^r | u(x) \geq 0\}$$

(i=1, 2, 3). The random counterpart of (3.1) is

$$\begin{cases} \partial_t u(t,x) = \nu_i \Delta u(t,x) + a_i(t,x)u(t,x), \quad x \in \bar{D}, \\ B_{r,b}u(t,x) = 0, \qquad x \in D_c \ (x \in \mathbb{R}^N \text{ if } D = \mathbb{R}^N), \\ u(s,x) = u_0(x), \end{cases} \tag{3.5}$$

where $D \subset \mathbb{R}^N$ and $B_{n,b}u(t,x)$ are the same as in (1.13). By general linear semigroup theory, for any $u_0 \in X_i$, $B_{r,b}u_0 = 0$ on $\partial D$ ($b = D$ when $i = 1$, $b = N$ when $i = 2$, and $b = P$ when $i = 3$) and $s \in \mathbb{R}$, (3.5) has a unique (local) solution, we denote it by $u^{\mathcal{R}}(t,x;s,u_0,\nu_i,a_i)$. And we put

$$\Phi_i^{\mathcal{R}}(t,s;\nu_i,a_i,u_0) = u_i^{\mathcal{R}}(t,\cdot;s,u_0,\nu_i,a_i), \quad u_0 \in X_i.$$

Note that if $\nu_i = 1$, and $a_i(\cdot,\cdot) = a(\cdot,\cdot)$, (3.5) is the evolution equation associated to the eigenvalue problem (1.19). Similarly, for $i = 1,2,3$, define $\Phi_i^r(t,s;a) : X_i^r \to X_i^r$ by

$$(\Phi_i^r(t,s;a)u_0)(\cdot) = u_i^{\mathcal{R}}(t,\cdot;s,u_0,1,a), \quad u_0 \in X_i^r.$$

By general nonlinear semigroup theory (see [32] and [55]), (1.18) and (1.17) has a unique (local) solution $u^{\mathcal{N}}(t,x;s,u_0)$ with $u^{\mathcal{N}}(s,x;s,u_0) = u_0(x)$ for every $u_0 \in X_i (i = 1,2,3)$ and $u^{\mathcal{R}}(t,x;s,u_0)$ with $u^{\mathcal{R}}(s,x;s,u_0) = u_0(x)$ for every $u_0 \in X_i^r (i = 1,2,3)$, respectively.

Also by general semigroup theory for equation systems (see [32] and [55]), for any given $(u_0,v_0) \in X_1 \times X_2$, (1.16) also has a unique (local) solution $(u(t,\cdot;u_0,v_0), v(t,\cdot;u_0,v_0))$ with $(u(0,x;u_0,v_0), v(0,x;u_0,v_0)) = (u_0(x), v_0(x))$.

## 3.2 Sub- and Super-Solutions

**Definition 3.1** (Sub- and Super- solutions)**.** *A continuous function $u(t,x)$ on $[s,s+T)\times\mathbb{R}^N$ is called a sub-solution (super-solution) of* (1.12) *on $(s,s+T)$ if for any $x \in \bar{D}$, $u(t,x)$ is differentiable on $(s,s+T)$ and satisfies that*

$$\begin{cases} \partial_t u(t,x) \le (\ge) \nu \int_{D\cup D_c} k(x-y)[u(t,y)-u(t,x)]dy + F(t,x,u), & x \in \bar{D}, t > s, \\ B_{n,b}u(t,x) \le (\ge) 0, & x \notin D, t > s, \\ u(s,x) \le (\ge) u_0(x), & x \in \bar{D}, \end{cases}$$

*where $u_0(\cdot) \in X_i$ $(i = 1,2,3)$ is the initial value of the solution of* (1.12) *at $t = s$.*

**Remark 3.2.** *The sub- and super-solutions of evolution equation with random operator* (1.13) *are defined similarly.*

**Remark 3.3.** *In the Dirichlet boundary case with nonlocal kernel $k(\cdot)$ being $k_\delta(\cdot)$, we have the following equivalent definition for a continuous function $u(t,x)$ on $[s,s+T)\times\mathbb{R}^N$ to be the super-solution (sub-solution) of* (1.5).

*For any $x \in \bar{D}$, $u(t,x)$ is differentiable on $(s, s+T)$ and satisfies that*

$$\begin{cases} \partial_t u(t,x) \geq (\leq) \nu \int_{\mathbb{R}^N} k_\delta(x-y)[u(t,y)-u(t,x)]dy + F(t,x,u), & x \in \bar{D}, \\ u(t,x) \geq (\leq) 0, & x \in D_c,\ \operatorname{dist}(x, \partial D) \leq \delta, \\ u(s,x) \geq (\leq) u_0(x), & x \in \bar{D}. \end{cases} \tag{3.6}$$

*where $\delta$ is the dispersal distance and $D_c = \mathbb{R}^N \setminus D$. In fact,* (3.1) *and* (3.6) *are equivalent, since* $\operatorname{supp}(k_\delta(\cdot)) \subset B(0,\delta)$, *and hence*

$$k_\delta(x-y) = 0 \qquad \textit{for } x \in D_c \cap \{x | \mathrm{dist(x, \partial D)} \geq \delta\}, \ \textit{and } \mathrm{y \in D}.$$

*We will use the above definitions for sub- and super-solutions in the proof of Theorem 2.13.*

Next, we consider (1.16) and present some basic properties for solutions of the two species competition system.

For given $(u^1, v^1), (u^2, v^2) \in X_1 \times X_2$, we define

$$(u^1, v^1) \leq_1 (u^2, v^2), \quad \text{if} \quad u^1(x) \leq u^2(x),\ v^1(x) \leq v^2(x),$$

and

$$(u^1, v^1) \leq_2 (u^2, v^2), \quad \text{if} \quad u^1(x) \leq u^2(x),\ v^1(x) \geq v^2(x).$$

**Definition 3.4.** *Let $T > 0$ and $(u(t,x), v(t,x)) \in C([0,T) \times \bar{D}, \mathbb{R}^2)$ with $(u(t,\cdot), v(t,\cdot)) \in X_1^+ \times X_2^+$. Then $(u(t,x), v(t,x))$ is called a super-solution (sub-solution) of* (1.16) *on* $[0,T)$ *if*

$$\begin{cases} \partial_t u(t,x) \geq (\leq) \nu [\int_D k(x-y) u(t,y) dy - u(t,x)] + u(t,x) f(x, u(t,x)+v(t,x)), & x \in \bar{D}, \\ \partial_t v(t,x) \leq (\geq) \nu \int_D k(x-y)[v(t,y)-v(t,x)]dy + v(t,x) f(x, u(t,x)+v(t,x)), & x \in \bar{D}, \end{cases}$$

*for $t \in [0,T)$.*

## 3.3 Comparison Principle and Monotonicity

We will introduce the comparison principle and strong monotonicity for general linear and nonlinear evolution equations, and systems.

**Proposition 3.5** (Comparison principle for evolution equations)**.**

*(1) (Comparison principle for linear evolution equations) If* $u^1(t,x)$ *and* $u^2(t,x)$ *are bounded sub- and super-solution of* (3.1) *(resp.* (3.5)*) on* $(s, s+T)$*, respectively, and* $u^1(s,\cdot) \le u^2(0,\cdot)$*, then* $u^1(t,\cdot) \le u^2(t,\cdot)$ *for* $t \in [s,T)$*.*

*(2) (Comparison principle for nonlinear evolution equations)If* $u^1(t,x)$ *and* $u^2(t,x)$ *are bounded sub- and super-solution of* (1.18) *(resp.* (1.17)*), on* $(s, s+T)$*, respectively, and* $u^1(0,\cdot) \le u^2(0,\cdot)$*, then* $u^1(t,\cdot) \le u^2(t,\cdot)$ *for* $t \in [s, s+T)$*.*

*Proof.* It follows from the arguments in [61, Proposition 2.1]. □

The following remarks follows by the arguments similar to those in Proposition 3.5.

**Remark 3.6.** *For given* $1 \le i \le 3$*,* $u_0 \in X_i^+$*, and* $a_i^1(t,\cdot), a_i^2(t,\cdot) \in X_i$*, if* $a_i^1(t,\cdot) \le a_i^2(t,\cdot)$*, then*

$$u_i^{\mathcal{N}}(t,\cdot;s,u_0,\nu_i,a_i^1) \le u_i^{\mathcal{N}}(t,\cdot;s,u_0,\nu_i,a_i^2) \quad \textit{for } t \ge s,$$

*where* $u_i^{\mathcal{N}}(t,\cdot;s,u_0,\nu_i,a_i^1)$ *and* $u_i^{\mathcal{N}}(t,\cdot;s,u_0,\nu_i,a_i^2)$ *are solutions of* (3.1) *with* $u_i^{\mathcal{N}}(s,\cdot;s,u_0,\nu_i,a_i^1) = u_0$ *and* $u_i^{\mathcal{N}}(s,\cdot;s,u_0,\nu_i,a_i^2) = u_0$*, respectively. And*

$$u_i^{\mathcal{R}}(t,\cdot;s,u_0,\nu_i,a_i^1) \le u_i^{\mathcal{R}}(t,\cdot;s,u_0,\nu_i,a_i^2) \quad \textit{for all } t > s,$$

*where* $u_i^{\mathcal{R}}(t,\cdot;s,u_0,\nu_i,a_i^1)$ *and* $u_i^{\mathcal{R}}(t,\cdot;s,u_0,\nu_i,a_i^2)$ *are solutions of* (3.5) *with* $u_i^{\mathcal{R}}(s,\cdot;s,u_0,\nu_i,a_i^1) = u_0$ *and* $u_i^{\mathcal{R}}(s,\cdot;s,u_0,\nu_i,a_i^2) = u_0$*, respectively.*

*Proof.* We consider the case $i = 1$ for (3.1). Other cases can be proved similarly.

Note that $u_1(t,x;s,u_0,\nu_1,a_1^2)$ is a super-solution of (3.1) in the case $i=1$ with $a_1(\cdot,\cdot)$ being replaced by $a_1^1(\cdot,\cdot)$. Then by Proposition 3.5 (1),

$$u_1^{\mathcal{N}}(t,\cdot;s,u_0,\nu_1,a_1^1) \le u_1^{\mathcal{N}}(t,\cdot;s,u_0,\nu_1,a_1^2) \quad \forall\ t \ge s.$$

□

**Remark 3.7.**

*(1) Suppose that $u^-(t,x)$ and $u^+(t,x)$ are sub-solution and super-solution of* (1.17) *on* $(s,s+T)$*, respectively, then*

$$u^-(t,x) \le u^+(t,x) \quad \forall\ t \in [s,s+T),\ x \in \bar{D}.$$

*(2) Suppose that $u^-(t,x)$ and $u^+(t,x)$ are sub-solution and super-solution of* (1.18) *on* $(s,s+T)$*, respectively, then*

$$u^-(t,x) \le u^+(t,x) \quad \forall\ t \in [s,s+T),\ x \in \bar{D}.$$

*Proof.* (1) It follows from comparison principle for parabolic equations.

(2) It follows from [56, Proposition 3.1]. □

**Proposition 3.8** (Strong monotonicity)**.** *For given $1 \le i \le 3$, if $u^1, u^2 \in X_i$, $u^1 \le u^2$ and $u^1 \not\equiv u^2$, then for all $t > s$,*

*(1) (Strong monotonicity for linear evolution equations)*

$\Phi_i^{\mathcal{N}}(t,s;\nu_i,a_i,u^1) \ll \Phi_i^{\mathcal{N}}(t,s;\nu_i,a_i,u^2)$, *and* $\Phi_i^{\mathcal{R}}(t,s;\nu_i,a_i,u^1) \ll \Phi_i^{\mathcal{R}}(t,s;\nu_i,a_i,u^2)$.

*(2) (Strong monotonicity for linear evolution equations)*

$u_i^{\mathcal{N}}(t,\cdot;s,u^1) \ll u_i^{\mathcal{N}}(t,\cdot;s,u^2)$, *and* $u_i^{\mathcal{R}}(t,\cdot;s,u^1) \ll u_i^{\mathcal{R}}(t,\cdot;s,u^2)$.

*Proof.* (1) It follows from the arguments in [61, Proposition 2.2]. (2) We show the proof of evolution equations in the Dirichlet boundary condition case with nonlocal dispersal operator. Other cases can be proved similarly .

Let $v(t,x) = u_1^{\mathcal{N}}(t,x;s,u^2) - u_1^{\mathcal{N}}(t,x;s,u^1)$ for $t \geq s$ at which both $u_1^{\mathcal{N}}(t,x;s,u^2)$and $u_1^{\mathcal{N}}(t,x;s,u^1)$ exist. Then $v(0,\cdot) = u^2 - u^1 \geq 0$ and $v(t,x)$ satisfies

$$\begin{aligned}\partial_t v =& \nu \left[\int_D k(x-y)v(t,y)dy - v(t,x)\right] + F(t,x,u(t,x;s,u^2))v(t,x) \\ &+ \left[u(t,x;s,u_1) \cdot \int_0^1 F_u(t,x,su(t,x;s,u^1) + (1-s)u(t,x;s,u^2))ds\right] v(t,x), \quad x \in \bar{D}.\end{aligned}$$

(2) then follows from the argument similar to those in (1). □

**Proposition 3.9** (Comparison principle for systems)**.**

(1) *If* $(0,0) \leq_1 (u_0,v_0)$*, then* $(0,0) \leq_1 (u(t,\cdot;u_0,v_0), v(t,\cdot;u_0,v_0))$ *for all* $t > 0$ *at which* $(u(t,\cdot;u_0,v_0), v(t,\cdot;u_0,v_0))$ *exists.*

(2) *If* $(0,0) \leq_1 (u_i,v_i)$*, for* $i = 1,2$*,* $(u_1(0,\cdot), v_1(0,\cdot)) \leq_2 (u_2(0,\cdot),\ v_2(0,\cdot))$*, and* $(u_1(t,x), v_1(t,x))$ *and* $(u_2(t,x), v_2(t,x))$ *are a sub-solution and a super-solution of* (1.16) *on* $[0,T)$ *respectively, then* $(u_1(t,\cdot), v_1(t,\cdot)) \leq_2 (u_2(t,\cdot),\ v_2(t,\cdot))$ *for* $t \in [0,T)$*.*

(3) *If* $(0,0) \leq_1 (u_i,v_i)$*, for* $i = 1,2$*, and* $(u_1,v_1) \leq_2 (u_2,v_2)$*, then*

$$(u(t,\cdot;u_1,v_1), v(t,\cdot;u_1,v_1)) \leq_2 (u(t,\cdot;u_2,v_2), v(t,\cdot;u_2,v_2))$$

*for all* $t>0$ *at which both* $(u(t,\cdot;u_1,v_1),\ v(t,\cdot;u_1,v_1))$ *and* $(u(t,\cdot;u_2,v_2),\ v(t,\cdot;u_2,v_2))$ *exist.*

(4) *Let* $(u_0,v_0) \in X_1^+ \times X_2^+$*, then* $(u(t,\cdot;u_0,v_0), v(t,\cdot;u_0,v_0))$ *exists for all* $t > 0$*.*

*Proof.* It follows from the arguments in Proposition 3.1 in [35]. □

### 3.4 A Technical Lemma

The technical lemma is for time homogeneous evolution equations with nonlocal dispersal operators. However, similar lemma holds in time periodic case (see [56, Lemma 4.2]).

**Lemma 3.10.** *Let $1 \leq i \leq 3$ and $a_i \in X_i$ be given. For any $\epsilon > 0$, there is $a_i^\epsilon \in X_i$ such that*

$$\|a_i - a_i^\epsilon\| < \epsilon,$$

*$h_i^\epsilon(x) = -\nu_i + a_i^\epsilon(x)$ for $i = 1$ or $3$ and $h_i^\epsilon(x) = -\nu_i \int_D k(x-y)dy + a_i^\epsilon(x)$ for $i = 2$ is in $C^N$, and satisfies the following vanishing condition: there is $x_0 \in \mathrm{Int}(D)$ such that $h_i^\epsilon(x_0) = \max_{x \in \bar{D}} h_i^\epsilon(x)$ and the partial derivatives of $h_i^\epsilon(x)$ up to order $N-1$ at $x_0$ are zero.*

*Proof.* See Lemma 3.1 in [59]. □

Chapter 4

# Principal Spectrum Points/Principal Eigenvalues of Nonlocal Dispersal Operators and Applications

In this chapter, we will focus on eigenvalue problems of nonlocal dispersal operators in the time homogeneous case, that is, (1.14) in case of Dirichlet, Neumann, and periodic types of boundary condition. First of all, let us recall some standard notations in Chapter 2, and introduce some basic properties of principal eigenvalues and principal spectrum points of time homogeneous dispersal operators. Next, we will prove Theorem 2.4, Theorem 2.6, and Theorem 2.8 for all the three boundary conditions in a unified way. Finally, we apply some results derived from the above theorems and prove Theorem 2.12.

Throughout this chapter, we assume $a_i(t,x) \equiv a_i(x) \in X_i$ for $i = 1, 2, 3$. Most results in this chapter are included in [59], which has been submitted for publication.

## 4.1 Basic Properties of Principal Eigenvalues/Principal Spectrum Points of Time Homogeneous Dispersal Operators

In the section, we present some basic properties of principal eigenvalue and principal spectrum points of time homogeneous nonlocal dispersal operators. Let us recall that $\Phi_i^{\mathcal{N}}(t,s;\nu_i,a_i)$ is the solution operator of (3.1) for $i = 1, 2, 3$. Without loss of generality, we set $s = 0$. Since we only focus on nonlocal dispersal operators in this chapter, we do not need to distinguish between nonlocal operators and random operators. For simplicity, throughout this chapter, we put

$$\Phi_i^{\mathcal{N}}(t,0;\nu_i,a_i) = \Phi_i(t;\nu_i,a_i) \quad \text{for } i = 1, 2, 3. \tag{4.1}$$

We have the following propositions.

**Proposition 4.1.** *Let $1 \le i \le 3$ be given.*

*(1) For given $t > 0$, $e^{\tilde{\lambda}_i(\nu_i,a_i)t} = r(\Phi_i(t;\nu_i,a_i))$.*

*(2) $\tilde{\lambda}_i(\nu_i,a_i) \in \sigma(\nu_i\mathcal{K}_i + h_i(\cdot)\mathcal{I})$.*

*Proof.* Observe that $\nu_i\mathcal{K}_i + h_i(\cdot)\mathcal{I} : X_i \to X_i$ is a bounded linear operator. Then by spectral mapping theorem,

$$e^{\sigma(\nu_i\mathcal{K}_i + h_i(\cdot)\mathcal{I})t} = \sigma(\Phi_i(t;\nu_i,a_i)) \setminus \{0\} \quad \forall\ t > 0. \tag{4.2}$$

By Proposition 3.7,

$$\Phi_i(t;\nu_i,a_i)X_i^+ \subset X_i^+ \quad \forall\ t > 0. \tag{4.3}$$

Hence $\Phi_i(t;\nu_i,a_i)$ is a positive operator on $X_i$. Then by [50, Proposition 4.1.1], $r(\Phi_i(t;\nu_i,a_i)) \in \sigma(\Phi_i(t;\nu_i,a_i))$ for any $t > 0$. By (4.2),

$$e^{\tilde{\lambda}_i(\nu_i,a_i)t} = r(\Phi_i(t;\nu_i,a_i)) \quad \forall\ t > 0,$$

and hence $\tilde{\lambda}_i(\nu_i,a_i) \in \sigma(\nu_i\mathcal{K}_i + h_i(\cdot)\mathcal{I})$. □

**Proposition 4.2.** *(1) $\tilde{\lambda}_1(\nu_1,0) < 0$.*

*(2) $\tilde{\lambda}_2(\nu_2,0) = 0$.*

*(3) $\tilde{\lambda}_3(\nu_3,0) = 0$.*

*Proof.* (1) Let $u_0(x) \equiv 1$. Observe that

$$\int_D k(x-y)u_0(y)dy - u_0(x) \le 0,$$

and there is $x_0 \in D$ such that

$$\int_D k(x-y_0)u_0(y)dy - u_0(x_0) < 0.$$

By Proposition 3.7(2),

$$0 \ll \Phi_1(t;\nu_1,0)u_0 \ll u_0 \quad \forall\ t>0,$$

and then

$$\|\Phi_1(t;\nu_1,0)u_0\| < 1 \quad \forall\ t>0.$$

Note that for any $\tilde{u}_0 \in X_1$ with $\|\tilde{u}_0\| \le 1$, by Proposition 3.7(2) again,

$$\|\Phi_1(t;\nu_1,0)\tilde{u}_0\| \le \|\Phi_1(t;\nu_1,0)u_0\| < 1 \quad \forall\ t>0.$$

This implies that

$$r(\Phi_1(t;\nu_1,0)) < 1 \quad \forall\ t>0,$$

and then $\tilde{\lambda}_1(\nu_1,0) < 0$.

(2) Let $u_0(\cdot) \equiv 1$. Observe that

$$\Phi_2(t;\nu_2,0)u_0 = u_0 \quad \forall\ t \ge 0,$$

and

$$\|\Phi_2(t;\nu_2,0)\tilde{u}_0\| \le \|\Phi_2(t;\nu_2,0)u_0\| = 1$$

for all $t \ge 0$ and $\tilde{u}_0 \in X_2$ with $\|\tilde{u}_0\| \le 1$. It then follows that

$$r(\Phi_2(t;\nu_2,0)) = 1 \quad \forall\ t \ge 0,$$

and then $\tilde{\lambda}_2(\nu_2,0) = 0$.

(3) It can be proved by the similar arguments as in (2).

□

Next, we prove some properties of principal spectrum points of nonlocal dispersal operators by using the spectral radius of the induced nonlocal operators $U^i_{a_i,\nu_i,\alpha_i}$ and $V^i_{a_i,\nu_i,\alpha_i}$

$(i=1,2,3)$, where $\alpha_i > \max_{x\in\bar{D}} h_i(x)$ $(i=1,2,3)$,

$$(U^i_{a_i,\nu_i,\alpha_i}u)(x) = \int_D \frac{\nu_i k(x-y)u(y)}{\alpha_i - h_i(y)}dy, \quad i=1,2, \tag{4.4}$$

$$(U^3_{a_3,\nu_3,\alpha_3}u)(x) = \int_{\mathbb{R}^N} \frac{\nu_3 k(x-y)u(y)}{\alpha_3 - h_3(y)}dy, \tag{4.5}$$

and

$$(V^i_{a_i,\nu_i,\alpha_i}u)(x) = \frac{\nu_i \int_D k(x-y)u(y)dy}{\alpha_i - h_i(x)} = \frac{\nu_i(\mathcal{K}_i u)(x)}{\alpha_i - h_i(x)}, \quad i=1,2, \tag{4.6}$$

$$(V^3_{a_3,\nu_3,\alpha_3}u)(x) = \frac{\nu_3 \int_{\mathbb{R}^N} k(x-y)u(y)dy}{\alpha_3 - h_3(x)} = \frac{\nu_3(\mathcal{K}_3 u)(x)}{\alpha_3 - h_3(x)}. \tag{4.7}$$

Observe that $U^i_{a_i,\nu_i,\alpha_i}$ and $V^i_{a_i,\nu_i,\alpha_i}$ are positive and compact operators on $X_i$ $(i=1,2,3)$. Moreover, there is $n \geq 1$ such that

$$\big(U^i_{a_i,\nu_i,\alpha_i}\big)^n (X_i^+ \setminus \{0\}) \subset X_i^{++}, \quad i=1,2,3,$$

and

$$\big(V^i_{a_i,\nu_i,\alpha_i}\big)^n (X_i^+ \setminus \{0\}) \subset X_i^{++}, \quad i=1,2,3.$$

Then by Krein-Rutman Theorem,

$$r(U^i_{a_i,\nu_i,\alpha_i}) \in \sigma(U^i_{a_i,\nu_i,\alpha_i}), \ \ r(V^i_{a_i,\nu_i,\alpha_i}) \in \sigma(V^i_{a_i,\nu_i,\alpha_i}), \tag{4.8}$$

and $r(U^i_{a_i,\nu_i,\alpha_i})$ and $r(V^i_{a_i,\nu_i,\alpha_i})$ are isolated algebraically simple eigenvalues of $U^i_{a_i,\nu_i,\alpha_i}$ and $V^i_{a_i,\nu_i,\alpha_i}$ with positive eigenfunctions, respectively.

**Proposition 4.3.** *(1) $\alpha_i > h_{i,\max}$ is an eigenvalue of $\nu_i\mathcal{K}_i + h_i(\cdot)\mathcal{I}$ with $\phi(x)$ being an eigenfunction iff $1$ is an eigenvalue of $U^i_{a_i,\nu_i,\alpha_i}$ with $\psi(x) = (\alpha_i - h_i(x))\phi(x)$ being an eigenfunction.*

*(2)* $\alpha_i > h_{i,\max}$ *is an eigenvalue of* $\nu_i\mathcal{K}_i + h_i(\cdot)\mathcal{I}$ *with* $\phi(x)$ *being an eigenfunction iff* $1$ *is an eigenvalue of* $V^i_{a_i,\nu_i,\alpha_i}$ *with* $\phi(x)$ *being an eigenfunction.*

*Proof.* It follows directly from the definitions of $U^i_{a_i,\nu_i,\alpha_i}$ and $V^i_{a_i,\nu_i,\alpha_i}$. □

**Proposition 4.4.** *Let* $1 \le i \le 3$ *be given.*

*(a)* $r(U^i_{a_i,\nu_i,\alpha_i})$ *is continuous in* $\alpha_i(> h_{i,\max})$*, strictly decreases as* $\alpha_i$ *increases, and*

$$r(U^i_{a_i,\nu_i,\alpha_i}) \to 0 \quad \text{as} \quad \alpha_i \to \infty.$$

*(b)* $r(V^i_{a_i,\nu_i,\alpha_i})$ *is continuous in* $\alpha_i(> h_{i,\max})$*, strictly decreases as* $\alpha_i$ *increases, and*

$$r(V^i_{a_i,\nu_i,\alpha_i}) \to 0 \quad \text{as} \quad \alpha_i \to \infty.$$

*Proof.* We prove (a) in the case $i = 1$. The other cases can be proved similarly.

First, note that $r(U^1_{a_1,\nu_1,\alpha_1})$ is an isolated algebraically simple eigenvalue of $U^1_{a_1,\nu_1,\alpha_1}$. It then follows from the perturbation theory of the spectrum of bounded operators that $r(U^1_{a_1,\nu_1,\alpha_1})$ is continuous in $\alpha_1(> h_{1,\max})$.

Next, we prove that $r(U^1_{a_1,\nu_1,\alpha_1})$ is strictly decreasing as $\alpha_1$ increases. To this end, fix any $\alpha_1 > h_{1,\max}$. Let $\phi_1(\cdot)$ be a positive eigenfunction of $U^1_{a_1,\nu_1,\alpha_1}$ corresponding to the eigenvalue $r(U^1_{a_1,\nu_1,\alpha_1})$. Note that for any given $\tilde{\alpha}_1 > \alpha_1$, there is $\delta_1 > 0$ such that

$$\frac{\tilde{\alpha}_1 - \alpha_1}{\alpha_1 - h_1(x)} > \delta_1 \quad \forall\, x \in \bar{D}.$$

This implies that

$$
\begin{aligned}
\big(U^1_{a_1,\nu_1,\tilde{\alpha}_1}\phi_1\big)(x) &= \int_D \frac{\nu_1 k(x-y)\phi_1(y)}{\tilde{\alpha}_1 - h_1(y)} dy \\
&= \int_D \frac{\nu_1 k(x-y)\phi_1(y)}{\alpha_1 - h_1(y)} \cdot \frac{1}{1 + \frac{\tilde{\alpha}_1 - \alpha_1}{\alpha_1 - h_1(y)}} dy \\
&\le \frac{1}{1+\delta_1} \int_D \frac{\nu_1 k(x-y)\phi_1(y)}{\alpha_1 - h_1(y)} dy \\
&= \frac{r(U^1_{a_1,\nu_1,\alpha_1})}{1+\delta_1}\phi_1(x) \quad \forall\ x \in \bar{D}.
\end{aligned}
$$

It then follows that

$$
r(U^1_{a_1,\nu_1,\tilde{\alpha}_1}) \le \frac{r(U^1_{a_1,\nu_1,\alpha_1})}{1+\delta_1} < r(U^1_{a_1,\nu_1,\alpha_1}),
$$

and hence $r(U^1_{a_1,\nu_1,\alpha_1})$ is strictly decreasing as $\alpha_1$ increases.

Finally, we prove that $r(U^1_{a_1,\nu_1,\alpha_1}) \to 0$ as $\alpha_1 \to \infty$. Note that for any $\epsilon > 0$, there is $\alpha_1^* > 0$ such that for $\alpha_1 > \alpha_1^*$,

$$
\int_D \frac{\nu_1 k(x-y)}{\alpha_1 - h_1(y)} dy < \epsilon \quad \forall\ x \in \bar{D}.
$$

This implies that

$$
\|U^1_{a_1,\nu_1,\alpha_1}\| < \epsilon \quad \forall\ \alpha_1 > \alpha_1^*.
$$

Hence $r(U^1_{a_1,\nu_1,\alpha_1}) \to 0$ as $\alpha_1 \to \infty$. $\square$

**Proposition 4.5.** *Let $1 \le i \le 3$ be given.*

*(a) If there is $\alpha_i > h_{i,\max}$ such that $r(U^i_{a_i,\nu_i,\alpha_i}) > 1$, then $\tilde{\lambda}_i(\nu_i, a_i) > h_{i,\max}$.*

*(b) If there is $\alpha_i > h_{i,\max}$ such that $r(V^i_{a_i,\nu_i,\alpha_i}) > 1$, then $\tilde{\lambda}_i(\nu_i, a_i) > h_{i,\max}$.*

*Proof.* We prove (b). Part (a) can be proved similarly.

Fix $1 \leq i \leq 3$. Suppose that there is $\alpha_i > h_{i,\max}$ such that $r(V^i_{a_i,\nu_i,\alpha_i}) > 1$. Then by Proposition 4.4, there is $\alpha_0 > h_{i,\max}$ such that

$$r(V^i_{a_i,\nu_i,\alpha_0}) = 1. \tag{4.9}$$

By Proposition 4.3, $\alpha_0 \in \sigma(\nu_i \mathcal{K}_i + h_i(\cdot)\mathcal{I})$. This implies that $\tilde{\lambda}_i(\nu_i, a_i) \geq \alpha_0 > h_{i,\max}$. $\square$

**Proposition 4.6** (Necessary and sufficient condition)**.** *For given* $1 \leq i \leq 3$*,* $\lambda_i(\nu_i, a_i)$ *exists if and only if* $\tilde{\lambda}_i(\nu_i, a_i) > h_{i,\max}$*.*

*Proof.* For $1 \leq i \leq 3$, $\nu_i \mathcal{K}_i$ is a compact operator. Hence $\nu_i \mathcal{K}_i + h_i(\cdot)\mathcal{I}$ can be viewed as compact perturbation of the operator $h_i(\cdot)\mathcal{I}$. Clearly, the essential spectrum $\sigma_{\text{ess}}(h_i\mathcal{I})$ of $h_i(\cdot)\mathcal{I}$ is given by

$$\sigma_{\text{ess}}(h_i\mathcal{I}) = [h_{i,\min}, h_{i,\max}].$$

Since the essential spectrum is invariant under compact perturbations (see [25]), we have

$$\sigma_{\text{ess}}(\nu_i \mathcal{K}_i + h_i\mathcal{I}) = [h_{i,\min}, h_{i,\max}],$$

where $\sigma_{ess}(\nu_i \mathcal{K}_i + h_i\mathcal{I})$ is the essential spectrum of $\nu_i \mathcal{K}_i + h_i(\cdot)\mathcal{I}$. Let

$$\sigma_{disc}(\nu_i \mathcal{K}_i + h_i\mathcal{I}) = \sigma(\nu_i \mathcal{K}_i + h_i\mathcal{I}) \backslash \sigma_{ess}(\nu_i \mathcal{K}_i + h_i\mathcal{I}).$$

Note that if $\lambda \in \sigma_{disc}(\nu_i \mathcal{K}_i + h_i\mathcal{I})$, then it is an isolated eigenvalue of finite multiplicity.

On the one hand, if $\tilde{\lambda}_i(\nu_i, a_i) > h_{i,\max}(x)$, then $\tilde{\lambda}_i(\nu_i, a_i) \in \sigma_{\text{disc}}(\nu_i \mathcal{K}_i + h_i\mathcal{I})$. By Proposition 4.3, $1 \in \sigma\Big(U^i_{a_i,\nu_i,\tilde{\lambda}_i(\nu_i,a_i)}\Big)$. Hence

$$r\Big(U^i_{a_i,\nu_i,\tilde{\lambda}_i(\nu_i,a_i)}\Big) \geq 1.$$

By Proposition 4.4, there is $\tilde{\tilde{\lambda}} \geq \tilde{\lambda}_i(\nu_i, a_i)$ such that

$$r\left(U^i_{a_i,\nu_i,\tilde{\tilde{\lambda}}}\right) = 1.$$

This together with Proposition 4.3 implies that $\tilde{\tilde{\lambda}}$ is an isolated algebraically simple eigenvalue of $\nu_i\mathcal{K}_i + h_i(\cdot)\mathcal{I}$ with a positive eigenfunction. By Definition 2.1 (2), $\lambda_i(\nu_i, a_i)$ exists.

On the other hand, if $\lambda_i(\nu_i, a_i)$ exists, then $\tilde{\lambda}_i(\nu_i, a_i) = \lambda_i(\nu_i, a_i) \in \sigma_{\text{disc}}(\nu_i\mathcal{K}_i + h_i\mathcal{I})$. This implies that $\tilde{\lambda}_i(\nu_i, a_i) > h_{i,\max}(x)$. □

Finally, we present some variational characterization of the principal spectrum points of nonlocal dispersal operators when the kernel function is symmetric. In the rest of this subsection, we assume that $k(\cdot)$ is symmetric with respect to 0. Recall

$$\mathcal{K}_3 : X_3 \to X_3, \;\; (\mathcal{K}_3 u)(x) = \int_{\mathbb{R}^N} k(x-y)u(y)dy \quad \forall\; u \in X_3.$$

For given $a \in X_3$, let

$$\hat{k}(z) = \sum_{j_1,j_2,\cdots,j_N\in\mathbb{Z}} k(z + (j_1p_1, j_2p_2, \cdots, j_Np_N)), \tag{4.10}$$

where $p_1, p_2, \cdots p_N$ are periods of $a(x)$. Then $\hat{k}(\cdot)$ is also symmetric with respect to 0 and

$$(\mathcal{K}_3 u)(x) = \int_D \hat{k}(x-y)u(y)dy \quad \forall\; u \in X_3, \tag{4.11}$$

where $D = [0, p_1] \times [0, p_2] \times \cdots \times [0, p_N]$ (see (2.20)).

**Proposition 4.7.** *Assume that $k(\cdot)$ is symmetric with respect to 0. Then*

$$\tilde{\lambda}_i(\nu_i, a_i) = \sup_{u\in L^2(D), \|u\|_{L^2(D)}=1} \int_D [\nu_i(\mathcal{K}_i u)(x)u(x) + h_i(x)u^2(x)]dx \quad (i = 1, 2, 3).$$

*Proof.* First of all, note that $\nu_i\mathcal{K}_i + h_i(\cdot)\mathcal{I}$ is also a bounded operator on $L^2(D)$ and $\nu_i\mathcal{K}_i$ is a compact operator on $L^2(D)$, where $\mathcal{K}_i$ is defined as in (4.11) when $i = 3$. Let $\sigma(\nu_i\mathcal{K}_i + h_i\mathcal{I}, L^2(D))$ be the spectrum of $\nu_i\mathcal{K}_i + h_i(\cdot)\mathcal{I}$ considered on $L^2(D)$ and

$$\tilde{\lambda}(\nu_i, a_i, L^2(D)) = \sup\{\mathrm{Re}\lambda \mid \lambda \in \sigma(\nu_i\mathcal{K}_i + h_i\mathcal{I}, L^2(D))\}.$$

Then we also have

$$\tilde{\lambda}(\nu_i, a_i, L^2(D)) \in \sigma(\nu_i\mathcal{K}_i + h_i\mathcal{I}, L^2(D)),$$

$$[h_{i,\min}, h_{i,\max}] \subset \sigma(\nu_i\mathcal{K}_i + h_i\mathcal{I}, L^2(D)),$$

and

$$\tilde{\lambda}(\nu_i, a_i, L^2(D)) \geq h_{i,\max}.$$

Moreover, if $\tilde{\lambda}_i(\nu_i, a_i) > h_{i,\max}$ (resp. $\tilde{\lambda}_i(\nu_i, a_i, L^2(D)) > h_{i,\max}$), then $\tilde{\lambda}_i(\nu_i, a_i)$ (resp. $\tilde{\lambda}_i(\nu_i, a_i, L^2(D))$) is an eigenvalue of $\nu_i\mathcal{K}_i + h_i\mathcal{I}$ considered on $L^2(D)$ (resp. $C(\bar{D})$) and hence $\tilde{\lambda}_i(\nu_i, a_i, L^2(D)) \geq \tilde{\lambda}_i(\nu_i, a_i)$ (resp. $\tilde{\lambda}_i(\nu_i, a_i) \geq \tilde{\lambda}_i(\nu_i, a_i, L^2(D))$). We then must have

$$\tilde{\lambda}_i(\nu_i, a_i) = \tilde{\lambda}_i(\nu_i, a_i, L^2(D)).$$

Assume now that $k(\cdot)$ is symmetric with respect to 0, that is, $k(-z) = k(z)$ for any $z \in \mathbb{R}^N$. Then for any $u, v \in L^2(D)$, in the case $i = 1, 2$,

$$\begin{aligned}
\int_D (\mathcal{K}_i u)(x)v(x)dx &= \int_D \int_D k(x-y)u(y)v(x)dydx \\
&= \int_D \int_D k(x-y)u(x)v(y)dxdy \\
&= \int_D \int_D k(x-y)v(y)u(x)dydx \\
&= \int_D (\mathcal{K}_i v)(x)u(x)dx,
\end{aligned}$$

and in the case $i = 3$,

$$\begin{aligned}\int_D (\mathcal{K}_3 u)(x)v(x)dx &= \int_D \int_D \hat{k}(x-y)u(y)v(x)dydx \\ &= \int_D \int_D \hat{k}(x-y)u(x)v(y)dxdy \\ &= \int_D \int_D \hat{k}(x-y)v(y)u(x)dydx \\ &= \int_D (\mathcal{K}_3 v)(x)u(x)dx.\end{aligned}$$

Therefore $\mathcal{K}_i : L^2(D) \to L^2(D)$ is self-adjoint. By classical variational formula (see [24]), we have

$$\tilde{\lambda}_i(\nu_i, a_i, L^2(D)) = \sup_{u \in L^2(D), \|u\|_{L^2(D)}=1} \int_D [\nu_i(\mathcal{K}_i u)(x)u(x) + h_i(x)u^2(x)]dx.$$

The proposition then follows. □

### 4.2 Effects of Spatial Variations and the Proof of Theorem 2.4

In this section, we investigate the effects of spatial variations on the principal spectrum points/principal eigenvalues of nonlocal dispersal operators and prove Theorem 2.4.

First of all, for given $1 \le i \le 3$ and $c_i \in \mathbb{R}$, let

$$X_i(c_i) = \{a_i \in X_i \,|\, \hat{a}_i = c_i\}$$

(see (2.21) for the definition of $\hat{a}_i$). For given $x_0 \in \mathbb{R}^N$ and $\sigma > 0$, let

$$B(x_0, \sigma) = \{y \in \mathbb{R}^N \,|\, \|x - y_0\| < \sigma\}.$$

*Proof of Theorem 2.4.* (1) We first prove the case $i = 1$. Let $x_0 \in \bar{D}$ be such that

$$h_1(x_0) = h_{1,\max}.$$

Note that there is $\epsilon_0 > 0$ such that

$$0 \le a_1(x_0) - a_1(x) < \nu_1 \inf_{x\in\bar D} \int_D k(x-y)dy - \epsilon_0 \le \nu_1 \int_D k(x-y)dy - \epsilon_0 \quad \forall\, x \in \bar D.$$

For any $0 < \epsilon < \epsilon_0$, put

$$\lambda_\epsilon = h_1(x_0) + \epsilon (= -\nu_1 + a_1(x_0) + \epsilon).$$

Then

$$\begin{aligned}\frac{\nu_1 \int_D k(x-y)dy}{\lambda_\epsilon - h_1(x)} &= \frac{\nu_1 \int_D k(x-y)dy}{a_1(x_0) - a_1(x) + \epsilon} \\ &\ge \frac{\nu_1 \int_D k(x-y)dy}{\nu_1 \int_D k(x-y)dy + \epsilon - \epsilon_0} \\ &> 1 \quad \forall x \in \bar D.\end{aligned}$$

This implies

$$r(V^1_{a_1,\nu_1,\lambda_\epsilon}) > 1 \quad \forall\, 0 < \epsilon \ll 1.$$

Then by Proposition 4.5 (b), $\tilde\lambda_1(\nu_1, a_1) > h_{1,\max}$. By Proposition 4.6, $\lambda_1(\nu_1, a_1)$ exists.

We now prove the case $i = 2$. Similarly, let $x_0 \in \bar D$ be such that

$$h_2(x_0) = h_{2,\max}.$$

Note that there is $\epsilon_0 > 0$ such that

$$0 \le a_2(x_0) - a_2(x) < \nu_2 \inf_{x\in\bar D} \int_D k(x-y)dy - \epsilon_0 \le \nu_2 \int_D k(x-y_0)dy - \epsilon_0.$$

For any $0 < \epsilon < \epsilon_0$, put

$$\lambda_\epsilon = h_2(x_0) + \epsilon (= -\nu_2 \int_D k(x-y_0)dy + a_2(x_0) + \epsilon).$$

Then

$$
\begin{aligned}
\frac{\nu_2 \int_D k(x-y)dy}{\lambda_\epsilon - h_2(x)} &= \frac{\nu_2 \int_D k(x-y)dy}{a_2(x_0) - \nu_2 \int_D k(x-y_0)dy + \nu_2 \int_D k(x-y)dy - a_2(x) + \epsilon} \\
&\geq \frac{\nu_2 \int_D k(x-y)dy}{\nu_2 \int_D k(x-y)dy + \epsilon - \epsilon_0} \\
&> 1 \quad \forall x \in \bar{D}.
\end{aligned}
$$

This again implies that

$$
r(V^2_{a_2,\nu_2,\lambda_\epsilon}) > 1 \quad \forall\ 0 < \epsilon \ll 1.
$$

Then by Proposition 4.5 (b), $\tilde{\lambda}_2(\nu_2, a_2) > h_{2,\max}$. By Proposition 4.6, $\lambda_2(\nu_2, a_2)$ exists.

(2) It can be proved by the similar arguments as in [61, Theorem B(2)]. For the completeness, we provide a proof below.

Let $x_0 \in \mathrm{Int}(D)$ be such that $h_i(x_0) = h_{i,\max}$ and the partial derivatives of $h_i(x)$ up to order $N-1$ at $x_0$ are zero. Then there is $M > 0$ such that

$$
h_i(x_0) - h_i(y) \leq M||x_0 - y||^N \quad \forall\ y \in D.
$$

Fix $\sigma > 0$ such that $B(x_0, 2\sigma) \subset D$ and $B(0, 2\sigma) \Subset \mathrm{supp}(k(\cdot))$. Let $v^* \in X_i^+$ be such that

$$
v^*(x) = \begin{cases} 1, & x \in B(x_0, \sigma), \\ 0, & x \in D \backslash B(x_0, 2\sigma). \end{cases}
$$

Clearly, for every $x \in D \backslash B(x_0, 2\sigma)$ and $\gamma > 1$, we have

$$
(U^i_{a_i,\nu_i,h_i(x_0)+\epsilon} v^*)(x) \geq \gamma v^*(x) = 0 \quad \forall\ \epsilon > 0. \tag{4.12}
$$

Note that there is $\tilde{M} > 0$ such that for any $x \in B(x_0, 2\sigma)$,

$$k(x-y) \geq \tilde{M} \quad \forall \ y \in B(x_0, \sigma).$$

It then follows that for $x \in B(x_0, 2\sigma)$

$$\begin{aligned}(U^i_{a_i,\nu_i,h_i(x_0)+\epsilon}v^*)(x) &= \int_D \frac{\nu_i k(x-y)v^*(y)}{h_i(x_0)+\epsilon-h_i(y)}dy \\ &\geq \int_{B(x_0,\sigma)} \frac{\nu_i k(x-y)}{M||x_0-y||^N+\epsilon}dy \\ &\geq \int_{B(x_0,\sigma)} \frac{\nu_i \tilde{M}}{M||x_0-y||^N+\epsilon}dy.\end{aligned}$$

Notice that $\int_{B(x_0,\sigma)} \frac{\tilde{M}}{M||x_0-y||^N}dy = \infty$. This implies that for $0 < \epsilon \ll 1$, there is $\gamma > 1$ such that

$$(U^i_{a_i,\nu_i,h_i(x_0)+\epsilon}v^*)(x) > \gamma v^*(x) \quad \forall \ x \in B(x_0, 2\sigma). \tag{4.13}$$

By (4.12) and (4.13),

$$U^i_{a_i,\nu_i,h_i(x_0)+\epsilon}v^*(x) \geq \gamma v^*(x) \quad \forall \ x \in D.$$

Hence, $r(U^i_{a_i,\nu_i,h_i(x_0)+\epsilon}) > 1$. By Proposition 4.5(a), $\tilde{\lambda}_i(\nu_i, a_i) > h_i(x_0) = h_{i,\max}$. By Proposition 4.6, the principle eigenvalue $\lambda_i(\nu_i, a_i)$ exists.

(3) Recall that $\tilde{\lambda}_i(\nu_i, \tilde{a}) = \sup\{\text{Re}\mu | \mu \in \sigma(\nu_i \mathcal{K}_i + \tilde{h}_i(\cdot)\mathcal{I})\}$ with $\tilde{h}_i(x) = -\nu_i + \tilde{a}(x)$ for $i = 1, 3$ and $\tilde{h}_i(x) = -\nu_2 \int_D k(x-y)dy + \tilde{a}(x)$ for $i = 2$. By the arguments of Proposition 4.6,

$$\sigma_{\text{ess}}(\nu_i \mathcal{K}_i + \tilde{h}_i \mathcal{I}) = [\min_{x \in \bar{D}} \tilde{h}_i(x), \max_{x \in \bar{D}} \tilde{h}_i(x)].$$

Note that

$$\sup_{\tilde{a} \in X_i(c_i)} (\max_{x \in \bar{D}} \tilde{a}(x)) = \infty.$$

Then

$$\sup_{\tilde{a} \in X_i(c_i)} \tilde{\lambda}_i(\nu_i, \tilde{a}) \geq \sup_{\tilde{a} \in X_i(c_i)} (\max_{x \in D} \tilde{h}_i(x)) \geq -\nu_i + \sup_{\tilde{a} \in X_i(c_i)} (\max_{x \in D} \tilde{a}(x)) = \infty.$$

(4) We first assume that the principal eigenvalue $\lambda_2(\nu_2, a_2)$ exists. Suppose that $u_2(x)$ is a strictly positive principal eigenfunction with respect to the eigenvalue $\lambda_2(\nu_2, a_2)$. We divide both sides of (1.2) by $u_2(x)$ and integrate with respect to $x$ over $D$ to obtain

$$\int_D \left[\frac{\nu_2[\int_D k(x-y)(u_2(y)-u_2(x))dy] + a_2(x)u_2(x)}{u_2(x)}\right] dx = \int_D \lambda_2(\nu_2, a_2)dx,$$

or

$$\begin{aligned}\lambda_2(\nu_2, a_2) &= \frac{\nu_2}{|D|}\int_D\int_D k(x-y)\frac{u_2(y)-u_2(x)}{u_2(x)}dydx + \frac{1}{|D|}\int_D a_2(x)dx \\ &= \frac{\nu_2}{|D|}\int_D\int_D k(x-y)\frac{u_2(y)-u_2(x)}{u_2(x)}dydx + \hat{a}_2.\end{aligned}$$

By the symmetry of $k(\cdot)$,

$$\begin{aligned}&\int_D\int_D k(x-y)\frac{u_2(y)-u_2(x)}{u_2(x)}dydx \\ &= \frac{1}{2}\int\int_{D\times D} k(x-y)\frac{u_2(y)-u_2(x)}{u_2(x)}dydx + \frac{1}{2}\int\int_{D\times D} k(x-y)\frac{u_2(y)-u_2(x)}{u_2(x)}dydx \\ &= \frac{1}{2}\int\int_{D\times D} k(x-y)\frac{u_2(y)-u_2(x)}{u_2(x)}dydx + \frac{1}{2}\int\int_{D\times D} k(x-y)\frac{u_2(x)-u_2(y)}{u_2(y)}dydx \\ &= \frac{1}{2}\int\int_{D\times D} k(x-y)\frac{(u_2(y)-u_2(x))^2}{u_2(x)u_2(y)}dydx \\ &\geq 0. \end{aligned} \tag{4.14}$$

So,

$$\inf\{\lambda_2(\nu_2, a_2)|a_2 \in X_2, \hat{a}_2 = c_2\} \geq \hat{a}_2 = c_2.$$

And clearly, $\lambda_2(\nu_2, \hat{a}_2) = \hat{a}_2$. Hence,

$$\inf\{\lambda_2(\nu_2, a_2)|a_2 \in X_2, \hat{a}_2 = c_2\} = \lambda_2(\nu_2, \hat{a}_2) = c_2.$$

Second, by Lemma 3.1, for any $\epsilon > 0$, there is $a_2^\epsilon \in X_2 \cap C^N$, such that

$$\|a_2 - a_2^\epsilon\| < \epsilon,$$

and $h_2^\epsilon(\cdot) \in C^N (= -\nu_2 \int_D k(x-y)dy + a_2^\epsilon)$ satisfies the vanishing condition in Theorem 2.1 (2). So, the principal eigenvalue $\lambda_2(\nu_2, a_2^\epsilon)$ exists and $\tilde{\lambda}_2(\nu_2, a_2^\epsilon) = \lambda_2(\nu_2, a_2^\epsilon)$. By the above arguments,

$$\tilde{\lambda}_2(\nu_2, a_2^\epsilon) = \lambda_2(\nu_2, a_2^\epsilon) \geq \lambda_2(\nu_2, \hat{a}_2^\epsilon) = \hat{a}_2^\epsilon. \tag{4.15}$$

We claim that

$$\lim_{\epsilon \to 0} \tilde{\lambda}_2(\nu_2, a_2^\epsilon) = \tilde{\lambda}_2(\nu_2, a_2).$$

In fact, $\|a_2^\epsilon - a_2\| \leq \epsilon$, that is

$$a_2(x) - \epsilon \leq a_2^\epsilon(x) \leq a_2(x) + \epsilon \quad \forall \ x \in \bar{D}.$$

Note that $\Phi_2(t; \nu_2, a_2 + \epsilon)u_0 = e^{\epsilon t}\Phi_2(t; \nu_2, a_2)u_0$, where $\Phi_2(t; \nu_2, a_2)u_0$ is the solution of (3.2) with the initial value $u_0(\cdot)$. Similarly, we have $\Phi_2(t; \nu_2, a_2 - \epsilon)u_0 = e^{-\epsilon t}\Phi_2(t; \nu_2, a_2)u_0$. So

$$r(\Phi_2(t; \nu_2, a_2 \pm \epsilon)) = e^{\pm \epsilon t} r(\Phi_2(t; \nu_2, a_2)).$$

Hence

$$\tilde{\lambda}_2(\nu_2, a_2 \pm \epsilon) = \tilde{\lambda}_2(\nu_2, a_2) \pm \epsilon. \tag{4.16}$$

By Remark 3.6, we have

$$\Phi_2(t; \nu_2, a_2 - \epsilon)u_0 \leq \Phi_2(t; \nu_2, a_2^\epsilon)u_0 \leq \Phi_2(t; \nu_2, a_2 + \epsilon)u_0.$$

Hence

$$r(\Phi_2(t; \nu_2, a_2 - \epsilon)) \leq r(\Phi_2(t; \nu_2, a_2^\epsilon)) \leq r(\Phi_2(t; \nu_2, a_2 + \epsilon)).$$

By(4.16),

$$\tilde{\lambda}_2(\nu_2, a_2 - \epsilon) \le \tilde{\lambda}_2(\nu_2, a_2^\epsilon) \le \tilde{\lambda}_2(\nu_2, a_2 + \epsilon).$$

Taking the limit of (4.15) as $\epsilon \to 0$, we have

$$\tilde{\lambda}_2(\nu_2, a_2) \ge \hat{a}_2$$

So, $\inf\{\tilde{\lambda}_2(\nu_2, \mathrm{a}_2)|\mathrm{a}_2 \in \mathrm{X}_2, \hat{\mathrm{a}}_2 = \mathrm{c}_2\} = \lambda_2(\nu_2, \mathrm{c}_2)(= \mathrm{c}_2)$.

When the principal eigenvalue exists, it is not difficult to prove that the infimum is attained by the constant function $a_2(\cdot) \equiv c_2$. In fact, suppose that $\lambda_2(\nu_2, a_2)$ exists and $u_2(\cdot)$ is a corresponding positive eigenfunction. By (4.14), $\lambda_2(\nu_2, a_2) = \hat{a}_2 (= c_2)$ iff $u_2(x) = u_2(y)$ for all $x, y \in \bar{D}$. Hence $\lambda_2(\nu_2, a_2) = \hat{a}_2(= c_2)$ iff $u_2(\cdot) \equiv$constant, which implies that $a_2(x) = \lambda_2(\nu_2, a_2) = \hat{a}_2$.

(5) Suppose that $a_i^1, a_i^2 \in X_i$ and $a_i^1 \le a_i^2$. By Remark 3.7, for any $u_0 \in X_i^+$ and $t \ge 0$,

$$\Phi_i(t; \nu_i, a_i^1)u_0 \le \Phi_i(t; \nu_i, a_i^2)u_0.$$

This implies that

$$r(\Phi_i(t; \nu_i, a_i^1)) \le r(\Phi_i(t; \nu_i, a_i^2)).$$

By Proposition 4.1, we have

$$\tilde{\lambda}_i(\nu_i, a_i^1) \le \tilde{\lambda}_i(\nu_i, a_i^2).$$

□

**Remark 4.8.** *(1) Theorem 2.1 (3) is not true in the random dispersal case when the space dimension is one. In fact, for* $1 \le i \le 3$*, we have* $\lambda_{R,i} \le c_i + {c_i}^2 L^2$ *for any* $a_i(\cdot) \in X_i^{++}$*,* $\hat{a}_i = c_i$ *and* $D = (0, L)$*. For the periodic boundary case, see Lemma 4.1 in [48]. The proof of Neumann or Dirichlet boundary case is similar to that of the periodic boundary case.*

We give a proof for the Neumann boundary case. Let $\psi(x)$ be the eigenvalue function of the operator $\Delta + a_2(\cdot)\mathcal{I}$ defined on $C^2([0,L])$ with Neumann boundary condition. So $\psi(x) > 0$ and we have

$$\begin{cases} \psi''(x) + a_2(x)\psi(x) = \lambda_{R,2}\psi(x), & x \in (0,L), \\ \frac{\partial \psi}{\partial n}(x) = 0, & x = 0 \text{ or } L. \end{cases}$$

Multiplying this by $\psi(x)$ and integrating it from 0 to $L$, we have

$$-\int_0^L \psi'^2(x)dx + \int_0^L a_2(x)\psi^2(x)dx = \lambda_{R,2}\int_0^L \psi^2(x)dx.$$

Hence

$$\lambda_{R,2} = \frac{-\int_0^L \psi'^2(x)dx + \int_0^L a_2(x)\psi^2(x)dx}{\int_0^L \psi^2(x)dx}.$$

Take $x_1, x_2 \in [0, L)$, we have

$$\psi^2(x_2) - \psi^2(x_1) = \int_{x_1}^{x_2} 2\psi(x)\psi'(x)dx.$$

Hence, for any positive number $k > 0$,

$$\psi^2(x_2) - \psi^2(x_1) \leq \frac{1}{k}\int_0^L \psi'^2(x)dx + k\int_0^L \psi^2(x)dx.$$

Multiplying the above inequality by $a_2(x_2)$ and integrating it with respect to $x_1 \in [0, L)$ and $x_2 \in [0, L)$, we get

$$L\int_0^L a_2(x_2)\psi^2(x_2)dx_2 - c_2 L\int_0^L \psi^2(x_1)dx_1 \leq c_2 L^2\left(\frac{1}{k}\int_0^L \psi'^2(x)dx + k\int_0^L \psi^2(x)dx\right),$$

where $c_2 = \int_0^L a_2(x)dx$. This is equivalent to

$$L\int_0^L a_2(x)\psi^2(x)dx - c_2 L\int_0^L \psi^2(x)dx \le c_2 L^2\left(\frac{1}{k}\int_0^L \psi'^2(x)dx + k\int_0^L \psi^2(x)dx\right).$$

Letting $k = c_2 L$, we obtain

$$-\int_0^L \psi'^2(x)dx + \int_0^L a_2(x)\psi^2(x)dx \le (c_2 + c_2^2 L^2)\int_0^L \psi^2(x)dx.$$

So, we have

$$\lambda_{R,2} \le c_2 + c_2^2 L^2.$$

*(2) Theorem 2.1 (4) may not be true for the Dirichlet type boundary condition. That is,* $\tilde{\lambda}_1(\nu_1, a_1) \ge \lambda_1(\nu_1, \hat{a}_1)$ *may not be true, where* $a_1 \in X_1$.

*In the random dispersal case, there is an example in [60] which shows that the principal eigenvalue* $\lambda_{R,1}(\nu_1, a_1)$ *of (1.4) is smaller than the principal eigenvalue* $\lambda_{R,1}(\nu_1, c_1)$ *of (1.4) with* $a_1(x)$ *being replaced by* $c_1 (= \hat{a}_1)$. *It is proved in Theorem 2.15 that*

$$\tilde{\lambda}_1(\nu_1, a_1, \delta) \to \lambda_{R,1}(\nu_1, a_1)$$

*as* $\delta \to 0$. *So, for any* $0 < \delta \ll 1$, $\tilde{\lambda}_1(\nu_1, a_1, \delta)$ *is close to* $\lambda_{R,1}(\nu_1, a_1)$, *and* $\tilde{\lambda}_1(\nu_1, c_1, \delta)$ *is close to* $\lambda_{R,1}(\nu_1, c_1)$. *Hence* $\tilde{\lambda}_1(\nu_1, a_1, \delta)$ *can be smaller than* $\tilde{\lambda}_1(\nu_1, c_1, \delta) = \lambda_1(\nu_1, c_1, \delta)$ *for* $\delta \ll 1$.

*(3) Theorem 2.1 (4) holds for periodic case (see [63]). When* $\lambda_i(\nu_i, a_i)$ *does not exist* $(i = 2, 3)$, *we may have* $\tilde{\lambda}_i(\nu_i, a_i) = \hat{a}_i$, *but* $a_i(\cdot)$ *is not a constant function. For example, let* $X_3 = \{u(x) \in C(\mathbb{R}^N, \mathbb{R}) | u(x + \mathbf{e_j}) = u(x)), x \in \mathbb{R}^N, j = 1, 2, \cdots, N\}$, *and* $q \in X_3$ *with*

$$q(x) = \begin{cases} e^{\frac{\|x\|^2}{\|x\|^2 - \sigma^2}} & \text{if } \|\mathrm{x}\| < \sigma, \\ 0 & \text{if } \sigma \le \|\mathrm{x}\| \le \frac{1}{2}. \end{cases}$$

*Then $\mathcal{K}_3 + h_3(\cdot)\mathcal{I}$ with $k(z) = k_\delta(z)$ has no principal eigenvalue for $M > 1$, $0 < \sigma \ll 1$, $\delta \gg 1$ and $h_3(x) = -1 + Mq(x)$ where $x \in \mathbb{R}^N$ and $N \geq 3$ (see [61]). Hence $\tilde{\lambda}_3 = \max_{x\in\bar{D}} h_3(x) = -1 + M \max_{x\in\bar{D}} q(x) = -1 + M$. Choosing $M = \frac{1}{1-\hat{q}}$, we have $M\hat{q} = -1 + M$, that is $\hat{a}_3 = \tilde{\lambda}_3$, but $a_3(x) = Mq(x)$ is not a constant function.*

### 4.3 Effects of Dispersal Rates and the Proof of Theorem 2.6

In this section, we investigate the effects of the dispersal rates on the principal spectrum points and the existence of principal eigenvalues of nonlocal dispersal operators and prove Theorem 2.6.

*Proof of Theorem 2.6.* (1) Assume that $k(\cdot)$ is symmetric. Observe that for any $u(\cdot) \in L^2(D)$,

$$
\begin{aligned}
&\int\int_{D\times D} k(x-y)u(x)u(y)dydx - \int_D u^2(x)dx \\
&\leq \int_D\int_D k(x-y)u(y)u(x)dydx - \int_D\int_D k(x-y)dyu^2(x)dx \\
&= \int_D\int_D k(x-y)(u(y)-u(x))u(x)dydx \\
&= \frac{1}{2}\int\int_{D\times D} k(x-y)(u(y)-u(x))u(x)dydx + \frac{1}{2}\int\int_{D\times D} k(x-y)(u(y)-u(x))u(x)dydx \\
&= \frac{1}{2}\int\int_{D\times D} k(x-y)(u(y)-u(x))u(x)dydx + \frac{1}{2}\int\int_{D\times D} k(x-y)(u(x)-u(y))u(y)dydx \\
&= -\frac{1}{2}\int\int_{D\times D} k(x-y)(u(y)-u(x))^2dydx \\
&\leq 0.
\end{aligned}
$$

Then (1) follows from the following facts: $\forall\, \nu_i > 0$,

$$
\tilde{\lambda}_i(\nu_i, a_i) = \sup_{u\in L^2(D), ||u||_{L^2(D)}=1} \left[\nu_i\left(\int_D\int_D k(x-y)u(y)u(x)dydx - \int_D u^2(x)dx\right) + \int_D a_i(x)u^2(x)dx\right]
$$

in the case $i = 1$,

$$\tilde{\lambda}_i(\nu_i, a_i) = \sup_{u \in L^2(D), ||u||_{L^2(D)}=1} \left[ -\frac{\nu_i}{2} \int\int_{D\times D} k(x-y)(u(y)-u(x))^2 dy dx + \int_D a_i(x) u^2(x) dx \right]$$

in the case $i = 2$, and

$$\tilde{\lambda}_i(\nu_i, a_i) = \sup_{u \in L^2(D), ||u||_{L^2(D)}=1} \left[ \nu_i \left( \int_D \int_D \hat{k}(x-y) u(y) u(x) dy dx - \int_D u^2(x) dx \right) + \int_D a_i(x) u^2(x) dx \right]$$

in the case $i = 3$ (see (4.11)).

(2) We prove the case $i = 1$. The case $i = 3$ can be proved similarly.

Without loss of generality, assume $a_1(x) > 0$ for $x \in \bar{D}$. Assume that $\nu_1 > 0$ is such that $\lambda_1(\nu_1, a_1)$ exists and $\tilde{\nu}_1 > \nu_1$. By proposition 4.6, $\lambda_1(\nu_1, a_1) > \max_{x\in\bar{D}} h_1(x)$, that is,

$$\lambda_1(\nu_1, a_1) > \max_{x\in\bar{D}}(-\nu_1 + a_1(x)).$$

Let $\phi_1(\cdot)$ be a positive principal eigenfunction with $||\phi_1||_{L^2(D)} = 1$. Then

$$\lambda_1(\nu_1, a_1) = \nu_1 \int\int_{D\times D} k(x-y)\phi_1(y)\phi_1(x) dy dx - \nu_1 + \int_D a_1(x)\phi_1^2(x) dx > \max_{x\in\bar{D}}(-\nu_1 + a_1(x)).$$

By Proposition 4.7,

$$\begin{aligned} \tilde{\lambda}_1(\tilde{\nu}_1, a_1) &\geq \tilde{\nu}_1 \int\int_{D\times D} k(x-y)\phi_1(y)\phi_1(x) dy dx - \tilde{\nu}_1 + \int_D a_1(x)\phi_1^2(x) dx \\ &= \lambda_1(\nu_1, a_1) + (\tilde{\nu}_1 - \nu_1) \int\int_{D\times D} k(x-y)\phi_1(y)\phi_1(x) dy dx + \nu_1 - \tilde{\nu}_1 \\ &> \max_{x\in\bar{D}}(-\nu_1 + a_1(x)) + \nu_1 - \tilde{\nu}_1 + (\tilde{\nu}_1 - \nu_1) \int\int_{D\times D} k(x-y)\phi_1(y)\phi_1(x) dy dx \\ &> \max_{x\in\bar{D}}(-\tilde{\nu}_1 + a_1(x)). \end{aligned}$$

By proposition 4.6 again, $\lambda_1(\tilde{\nu}_1, a_1)$ exists.

(3) It follows from Theorem 2.1(1) and can also be proved as follows.

To show $\lambda_i(\nu_i, a_i)$ exists, we only need to show $\tilde{\lambda}_i(\nu_i, a_i) > \max_{x\in\bar{D}} h_i(x)$, where $h_i(x) = -\nu_i + a_i(x)$ for $i = 1$ and 3 and $h_i(x) = -\nu_i \int_D k(x-y)dy + a_i(x)$ for $i = 2$. In the case $i = 2$ or 3, $\tilde{\lambda}_i(\nu_i, a_i) \geq \hat{a}_i$ by theorem 2.4(4). This implies that

$$\tilde{\lambda}_i(\nu_i, a_i) > h_{i,\max} \quad \forall\ \nu_i \gg 1.$$

In the case $i = 1$, note that $\lambda_1(1, 0)$ exists and

$$-1 < \lambda_1(1, 0) < 0.$$

This implies that $\lambda_1(1, \frac{a_1}{\nu_1})$ exists for $\nu_1 \gg 1$ and then $\lambda_1(\nu_1, a_1)$ exists for $\nu_1 \gg 1$.

(4) On the one hand, we have

$$\tilde{\lambda}_i(\nu_i, a_i) \geq h_{i,\max} \geq -\nu_i + a_{i,\max}.$$

On the other hand, for any $\lambda > a_{i,\max}$, $\lambda\mathcal{I} - a_i(\cdot)\mathcal{I}$ has bounded inverse. This implies that

$$a_{i,\max} + \epsilon > \tilde{\lambda}_i(\nu_i, a_i) \quad \forall\ 0 < \nu_i \ll 1.$$

Therefore,

$$\lim_{\nu_i\to 0} \tilde{\lambda}_i(\nu_i, a_i) = a_{i,\max}.$$

(5) We prove the cases $i = 1$ and $i = 2$. The case $i = 3$ can be proved by the similar arguments as in the case $i = 2$.

First, we prove the case $i = 1$. By Proposition 4.2,

$$\tilde{\lambda}_1(1, 0) < 0.$$

Observe that

$$\tilde{\lambda}_1(\nu_1, a_1) = \nu_1 \tilde{\lambda}_1\left(1, \frac{a_1}{\nu_1}\right) \quad \text{and} \quad \tilde{\lambda}_1\left(1, \frac{a_1}{\nu_1}\right) \to \tilde{\lambda}_1(1, 0)$$

as $\nu_1 \to \infty$. It then follows that

$$\tilde{\lambda}_1(\nu_1, a_1) \le \frac{\nu_1}{2}\tilde{\lambda}_1(1, 0) \quad \forall\ \nu_1 \gg 1.$$

This implies that

$$\lim_{\nu_1 \to \infty} \tilde{\lambda}_1(\nu_1, a_1) = -\infty.$$

Second, we prove the case $i = 2$. By (3), $\lambda_2(\nu_2, a_2)$ exists for $\nu_2 \gg 1$. In the following, we assume $\nu_2 \gg 1$ such that $\lambda_2(\nu_2, a_2)$ exists. Let $\phi_{2,\nu_2}(x)$ be a positive principal eigenfunction with $\int_D \phi_{2,\nu_2}^2(x)dx = 1$.

Note that

$$\hat{a}_2 \le \lambda_2(\nu_2, a_2) \le a_{2,\max},$$

and

$$\nu_2 \int_D \int_D k(x-y)(\phi_{2,\nu_2}(y) - \phi_{2,\nu_2}(x))\phi_{2,\nu_2}(x)dydx + \int_D a_2(x)\phi_{2,\nu_2}^2(x)dx = \lambda_2(\nu_2, a_2).$$

This implies that

$$\frac{\nu_2}{2} \int_D \int_D k(x-y)(\phi_{2,\nu_2}(y) - \phi_{2,\nu_2}(x))^2 dydx = \int_D a_2(x)\phi_{2,\nu_2}^2(x)dx - \lambda_2(\nu_2, a_2) \le a_{2,\max} - \hat{a}_2,$$

and then

$$\int_D \int_D k(x-y)(\phi_{2,\nu_2}(y) - \phi_{2,\nu_2}(x))^2 dydx \le \frac{2(a_{2,\max} - \hat{a}_2)}{\nu_2}. \tag{4.17}$$

Let $\psi_{2,\nu_2}(x) = \phi_{2,\nu_2}(x) - \hat{\phi}_{2,\nu_2}$. Then

$$\nu_2 \int_D \int_D k(x-y)(\phi_{2,\nu_2}(y) - \phi_{2,\nu_2}(x))dydx + \int_D a_2(x)\phi_{2,\nu_2}(x)dx = \int_D a_2(x)(\psi_{2,\nu_2}(x) + \hat{\phi}_{2,\nu_2})dx,$$

and hence

$$\lambda_2(\nu_2, a_2) \int_D \phi_{2,\nu_2}(x)dx = \hat{\phi}_{2,\nu_2} \int_D a_2(x)dx + \int_D a_2(x)\psi_{2,\nu_2}(x)dx.$$

This implies that

$$\lambda_2(\nu_2, a_2)\hat{\phi}_{2,\nu_2} = \hat{a}_2\hat{\phi}_{2,\nu_2} + \frac{1}{|D|} \int_D a_2(x)\psi_{2,\nu_2}(x)dx. \tag{4.18}$$

To show $\lambda_2(\nu_2, a_2) \to \hat{a}_2$ as $\nu_2 \to \infty$, we first show that $\int_D a_2(x)\psi_{2,\nu_2}(x)dx \to 0$ as $\nu_2 \to \infty$.

Note that $\tilde{\lambda}_2(1, 0) = 0$ and $\tilde{\lambda}_2(1, 0)$ is the principal eigenvalue of $\mathcal{K}_2 + b_0(\cdot)\mathcal{I}$ with $\phi(\cdot) \equiv 1$ being a principal eigenfunction, where

$$b_0(x) = -\int_D k(x - y)dy.$$

Moreover, $\tilde{\lambda}_2(1, 0)$ is also an isolated algebraically simple eigenvalue of $\mathcal{K}_2 + b_0(\cdot)\mathcal{I}$ on $L^2(D)$.

Note also that

$$\int_D \Big((-\mathcal{K}_2 - b_0\mathcal{I})u\Big)(x)u(x)dx = \frac{1}{2} \int_D \int_D k(x - y)(u(y) - u(x))^2 dydx \geq 0 \tag{4.19}$$

for any $u(\cdot) \in L^2(D)$ and $-\mathcal{K}_2 - b_0(\cdot)\mathcal{I}$ is a self-adjoint operator on $L^2(D)$. Then there is a bounded linear operator $A : L^2(D) \to L^2(D)$ such that

$$\int_D \Big((-\mathcal{K}_2 - b_0\mathcal{I})u\Big)(x)u(x)dx = \int_D (Au)(x)(Au)(x)dx \quad \forall\ u \in L^2(D). \tag{4.20}$$

Let

$$E_1 = \mathrm{span}\{\phi(\cdot)\},$$

and

$$E_2 = \{u(\cdot) \in L^2(D) \mid \int_D u^2(x)dx = 0\}.$$

Then

$$L^2(D) = E_1 \oplus E_2$$

and

$$\Big(\mathcal{K}_2 + b_0(\cdot)\mathcal{I}\Big)(E_2) \subset E_2.$$

Moreover, $(\mathcal{K}_2 + b_0(\cdot)\mathcal{I})|_{E_2}$ is invertible. We claim that there is $C > 0$ such that

$$\int_D (Au)(x)(Au)(x)dx \geq C \int_D u^2(x)dx \quad \forall\, u \in E_2. \tag{4.21}$$

For otherwise, there is $u_n \in E_2$ with $\int_D u_n^2(x)dx = 1$ such that

$$\int_D (Au_n)(x)(Au_n)(x)dx \to 0$$

as $n \to \infty$. It then follows that $0 \in \sigma((\mathcal{K}_2 + b_0(\cdot)\mathcal{I})|_{E_2})$, a contradiction. Hence (4.21) holds.

By (4.19), (4.20) and (4.21), for any $\nu_2 \gg 1$,

$$\int_D \psi_{2,\nu_2}^2(x)dx \leq \frac{1}{2C}\int_D\int_D k(x-y)(\psi_{2,\nu_2}(y) - \psi_{2,\nu_2}(x))^2 dydx. \tag{4.22}$$

Observe that

$$\int_D\int_D k(x-y)(\phi_{2,\nu_2}(y) - \phi_{2,\nu_2}(x))^2 dydx = \int_D\int_D k(x-y)(\psi_{2,\nu_2}(y) - \psi_{2,\nu_2}(x))^2 dydx.$$

This together with (4.17) and (4.22) implies that

$$\int_D \psi_{2,\nu_2}^2(x)dx \to 0 \quad \text{as} \quad \nu_2 \to \infty,$$

and then

$$\int_D a_2(x)\psi_{2,\nu_2}(x)dx \to 0 \quad \text{as} \quad \nu_2 \to \infty.$$

Second, assume $\lambda_2(\nu_2, a_2) \not\to \hat{a}_2$ as $\nu_2 \to \infty$. By (4.18), we must have $\hat{\phi}_{2,\nu_{2,n}} \to 0$ for some sequence $\nu_{2,n} \to \infty$. This and (4.17) implies that

$$\begin{aligned}
\int_D \phi^2_{2,\nu_{2,n}}(x)dx &\le C_0 \int_D \int_D k(x-y)\phi^2_{2,\nu_{2,n}}(x)dydx \\
&= C_0 \int_D \int_D k(x-y)(\phi^2_{2,\nu_{2,n}}(x) - \phi_{2,\nu_{2,n}}(x)\phi_{2,\nu_{2,n}}(y))dydx \\
&\quad + C_0 \int_D \int_D k(x-y)\phi_{2,\nu_{2,n}}(y)\phi_{2,\nu_{2,n}}(x)dydx \\
&\le \frac{C_0}{2} \int_D \int_D k(x-y)(\phi_{2,\nu_{2,n}}(y) - \phi_{2,\nu_{2,n}}(x))^2 dydx + |D|^2 C_0 M \hat{\phi}_{2,\nu_{2,n}} \hat{\phi}_{2,\nu_{2,n}} \\
&\le \frac{C_0(a_{2,\max} - \hat{a}_2)}{\nu_2} + |D|^2 C_0 M \hat{\phi}_{2,\nu_{2,n}} \hat{\phi}_{2,\nu_{2,n}}
\end{aligned}$$

where $C_0 = (\min_{x\in\bar{D}} \int_D k(x-y)dy)^{-1}$ and $M = \sup_{x,y\in\bar{D}} k(x-y)$. That is

$$\int_D \phi^2_{2,\nu_{2,n}}(x)dx \to 0 \qquad \text{as } \nu_{2,n} \to \infty.$$

This is a contradiction. Therefore

$$\lambda_2(\nu_2, a_2) \to \hat{a}_2$$

as $\nu_2 \to \infty$. □

## 4.4 Effects of Dispersal Distance and the Proof of Theorem 2.8

In this section, we investigate the effects of the dispersal distance on the principal spectrum points and the existence of principal eigenvalues and prove Theorem 2.8.

*Proof of Theorem 2.8.* (1) As mentioned in Remark 2.9, the cases $i = 1$ and 3 are proved in [41, Theorem 2.6]. The case $i = 2$ can be proved by the similar arguments as in [41, Theorem 2.6]. For completeness, we provide a proof for the case $i = 2$ in the following.

By Proposition 4.7,

$$\tilde{\lambda}_i(\nu_i, a_i, \delta) = \sup_{u \in L^2(D), \|u\|_{L^2(D)}=1} \int_D \left[ \nu_i \int_D k_\delta(x-y)(u(y) - u(x))dy + a_i(x)u(x) \right] u(x)dx.$$

On the one hand,

$$\begin{aligned}
\tilde{\lambda}_i(\nu_i, a_i, \delta) &= \sup_{u \in L^2(D), \|u\|_{L^2(D)}=1} \int_D \left[ \nu_i \int_D k_\delta(x-y)(u(y) - u(x))dy + a_i(x)u(x) \right] u(x)dx \\
&= \sup_{u \in L^2(D), \|u\|_{L^2(D)}=1} \left[ -\frac{\nu_i}{2} \int_D \int_D k_\delta(x-y)(u(y) - u(x))^2 dydx + \int_D \int_D a_i(x)u^2(x)dx \right] \\
&\leq a_{i,\max}.
\end{aligned}$$

On the other hand, assume that $x_0 \in \bar{D}$ is such that $a_i(x_0) = a_{i,\max}$. Then for any $0 < \epsilon < 1$, there are $\sigma_0^* > 0$ and $x_0^* \in \mathrm{Int}D$ such that $B(x_0^*, \sigma_0^*) \subset \bar{D}$ and

$$a_i(x_0) - a_i(x) < \epsilon/2 \quad \text{for} \quad x \in B(x_0^*, \sigma_0^*).$$

Let $u_0(\cdot)$ be a smooth function with $\mathrm{supp}(u_0(\cdot)) \cap D \subset B(x_0^*, \sigma_0^*)$ and $\|u_0\|_{L^2(D)} = 1$. Then

$$\begin{aligned}
\tilde{\lambda}_i(\nu_i, a_i, \delta) &\geq \int_D \left( \nu_i \int_D k_\delta(x-y)(u_0(y) - u_0(x))dy + a_i(x)u_0(x) \right) u_0(x)dx \\
&\geq \nu_i \int_D \left( \int_D k_\delta(x-y)(u_0(y) - u_0(x))dy \right) u_0(x)dx + \left( a_{i,\max} - \frac{\epsilon}{2} \right).
\end{aligned}$$

Note that

$$\int_D k_\delta(x-y)(u_0(y) - u_0(x))dy \to 0 \quad \forall x \in \mathrm{Int}(D)$$

as $\delta \to 0$. And

$$\left| \int_D k_\delta(x-y)(u_0(y)-u_0(x))dy \right| \leq 2 \max_{y \in \bar{D}} |u_0(y)| \quad \forall\, x \in D.$$

Hence, there exists $\delta_0 > 0$, such that for any $\delta < \delta_0$, we have

$$\left| \nu_i \int_D \left( \int_D k_\delta(x-y)(u_0(y)-u_0(x))dy \right) u_0(x)dx \right| \leq \frac{\epsilon}{2}.$$

It then follows that

$$a_{i,\max} \geq \tilde{\lambda}_i(\nu_i, a_i, \delta) \geq a_{i,\max} - \epsilon.$$

This implies that $\tilde{\lambda}_i(\nu_i, a_i, \delta) \to a_{i,\max}$ as $\delta \to 0$.

(2) First, for $i = 1$,

$$\left| \int_D k_\delta(x-y)u(y)dy \right| \leq \|u\| \int_D k_\delta(x-y)dy \to 0,$$

as $\delta \to \infty$ uniformly in $u \in X_1$ with $\|u\| \leq 1$. Therefore,

$$\tilde{\lambda}_1(\nu_1, a_1, \delta) \to \sup\{\mathrm{Re}\lambda | \lambda \in \sigma((-\nu_1 + a_1(\cdot))\mathcal{I})\} = -\nu_1 + a_{1,\max},$$

as $\delta \to \infty$.

For $i = 2$,

$$\left| \int_D k_\delta(x-y)(u(y)-u(x))dy \right| \leq 2\|u\| \int_D k_\delta(x-y)dy \to 0,$$

as $\delta \to \infty$ uniformly in $u \in X_2$ with $\|u\| \leq 1$. Hence

$$\tilde{\lambda}_2(\nu_2, a_2, \delta) \to \sup\{\mathrm{Re}\lambda | \lambda \in \sigma(a_2(\cdot)\mathcal{I})\} = a_{2,\max},$$

as $\delta \to \infty$.

For $i = 3$, recall that

$$\bar{\lambda}_3(\nu_3, a_3) = \sup\{\mathrm{Re}\lambda \,|\, \lambda \in \sigma(\nu_2 \bar{\mathcal{I}} + h_3(\cdot)\mathcal{I})\},$$

where

$$\bar{\mathcal{I}}u = \frac{1}{p_1 p_2 \cdots p_N} \int_0^{p_1} \int_0^{p_2} \cdots \int_0^{p_N} u(x)dx.$$

We first assume that $a_3(\cdot)$ satisfies the conditions in Remark 2.5 (2). Then by similar arguments as in Theorem 2.4 (2), $\bar{\lambda}_3(\nu_3, a_3)$ is the principal eigenvalue of $\nu_3 \bar{\mathcal{I}} + h_3(\cdot)\mathcal{I}$. Let $\phi_3(\cdot)$ be the positive principal eigenfunction of $\nu_3 \bar{\mathcal{I}} + h_3(\cdot)\mathcal{I}$ with $\hat{\phi}_3 = \frac{1}{|D|}\int_D \phi_3(x)dx = 1$. We then have $\bar{\lambda}_3(\nu_3, a_3) > h_{3,\max}$ and

$$\frac{1}{|D|} \int_D \frac{\nu_3 \psi_3(x)}{\bar{\lambda}_3(\nu_3, a_3) + \nu_3 - a_3(x)} dx = 1, \tag{4.23}$$

where

$$\psi_3(x) = (\bar{\lambda}_3(\nu_3, a_3) + \nu_3 - a_3(x))\phi_3(x).$$

Fix $0 < \epsilon < \bar{\lambda}_3(\nu_3, a_3) - h_{i,\max}$. Then

$$\frac{1}{|D|} \int_D \frac{\nu_3 \psi_3(x)}{\bar{\lambda}_3(\nu_3, a_3) - \epsilon + \nu_3 - a_3(x)} dx > 1. \tag{4.24}$$

Observe that for any $\mathbf{k} = (k_1, k_2, \cdots, k_N) \in \mathbb{Z}^N \setminus \{0\}$,

$$\int_{\mathbb{R}^N} \tilde{k}(z) \cos\Big(\sum_{i=1}^N k_i p_i x_i + \delta \sum_{i=1}^N k_i p_i z_i\Big) dz \to 0,$$

and

$$\int_{\mathbb{R}^N} \tilde{k}(z) \sin\Big(\sum_{i=1}^N k_i p_i x_i + \delta \sum_{i=1}^N k_i p_i z_i\Big) dz \to 0$$

as $\delta \to \infty$. This implies that for any $a \in X_3$,

$$\int_{\mathbb{R}^N} \tilde{k}(z)a(x+\delta z)dz \to \hat{a}$$

as $\delta \to \infty$ and then

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{\nu_3 k_\delta(x-y)\psi_3(y)}{\bar{\lambda}_3(\nu_3,a_3)-\epsilon+\nu_3-a_3(y)}dy &= \int_{\mathbb{R}^N} \frac{\nu_3 \tilde{k}(z)\psi_3(x+\delta z)}{\bar{\lambda}_3(\nu_3,a_3)-\epsilon+\nu_3-a_3(x+\delta z)}dz \\ &\to \frac{1}{|D|}\int_D \frac{\nu_3\psi_3(x)}{\bar{\lambda}_3(\nu_3,a_3)-\epsilon+\nu_3-a_3(x)}dx\end{aligned}$$

as $\delta \to \infty$ uniformly in $x \in \mathbb{R}^N$. This together with (4.24) implies that

$$\int_{\mathbb{R}^N} \frac{\nu_3 k_\delta(x-y)\psi_3(y)}{\bar{\lambda}_3(\nu_3,a_3)-\epsilon+\nu_3-a_3(y)}dy > 1 \quad \forall\, x \in \mathbb{R}^N, \;\; \delta \gg 1.$$

It then follows that

$$\tilde{\lambda}_3(\nu_3,a_3,\delta) > \bar{\lambda}_3(\nu_3,a_3)-\epsilon > h_{i,\max} \quad \forall\, \delta \gg 1 \tag{4.25}$$

and $\lambda_3(\nu_3,a_3,\delta)$ exists for $\delta \gg 1$.

Now for any $\epsilon > 0$, by (4.23),

$$\frac{1}{|D|}\int_D \frac{\nu_3\psi_3(x)}{\bar{\lambda}_3(\nu_3,a_3)+\epsilon+\nu_3-a_3(x)}dx < 1. \tag{4.26}$$

Then by the similar arguments in the above,

$$\tilde{\lambda}_3(\nu_3,a_3,\delta) < \bar{\lambda}_3(\nu_3,a_3)+\epsilon \quad \forall\, \delta \gg 1. \tag{4.27}$$

By (4.25) and (4.27),

$$\tilde{\lambda}_3(\nu_3,a_3,\delta) \to \bar{\lambda}_3(\nu_3,a_3) \quad \text{as} \quad \delta \to \infty.$$

Now for general $a_3 \in X_3$, and for any $\epsilon > 0$, there is $a_{3,\epsilon} \in X_3$ such that

$$\|a_3 - a_{3,\epsilon}\| < \epsilon \quad \forall\, x \in \mathbb{R}^N,$$

and $a_{3,\epsilon}(\cdot)$ satisfies the conditions in Remark 2.5 (2). By Theorem 2.4 (5),

$$\tilde{\lambda}_3(\nu_3, a_{3,\epsilon}, \delta) - \epsilon \le \tilde{\lambda}_3(\nu_3, a_3, \delta) \le \tilde{\lambda}_3(\nu_3, a_{3,\epsilon}, \delta) + \epsilon.$$

By the above arguments,

$$\bar{\lambda}_3(\nu_3, a_3) - 3\epsilon \le \bar{\lambda}_3(\nu_3, a_{3,\epsilon}) - 2\epsilon \le \tilde{\lambda}_3(\nu_3, a_3, \delta) \le \bar{\lambda}_3(\nu_3, a_{3,\epsilon}) + 2\epsilon \le \bar{\lambda}_3(\nu_3, a_3) + 3\epsilon \quad \forall\, \delta \gg 1.$$

We hence also have

$$\tilde{\lambda}_3(\nu_3, a_3, \delta) \to \bar{\lambda}_3(\nu_3, a_3) \quad \text{as} \quad \delta \to \infty.$$

(3) By (1), for any $\epsilon > 0$,

$$\tilde{\lambda}_i(\nu_i, a_i, \delta) > a_{i,\max} - \epsilon \quad \forall\, 0 < \delta \ll 1.$$

This implies that there is $\delta_0 > 0$ such that

$$\tilde{\lambda}_i(\nu_i, a_i, \delta) > h_{i,\max} \quad \forall\;\, 0 < \delta < \delta_0.$$

Then by Proposition 4.7, $\lambda_i(\nu_i, a_i)$ exists for $0 < \delta < \delta_0$. □

## 4.5 Applications to the Asymptotic Dynamics of Two Species Competition System

In this section, we consider the asymptotic dynamics of the two species competition system (1.16) and prove Theorem 2.12 by applying some of the principal spectrum properties developed in previous sections. Throughout this section, we assume that $k(-z) = k(z)$, $\tilde{\lambda}_1(\nu, f(\cdot, 0)) > 0$, $f(x, w) < 0$ for $w \gg 1$, and $\partial_2 f(x, w) < 0$ for $w \geq 0$.

### 4.5.1 Asymptotic Dynamics of KPP Type Competition Systems

In this subsection, we present some basic properties about the asymptotic dynamics of the time homogeneous two species competition system (1.16). Throughout this subsection, we assume that $k(-z) = k(z)$, $\tilde{\lambda}_1(\nu, f(\cdot, 0)) > 0$, $f(x, w) < 0$ for $w \gg 1$ and $\partial_2 f(x, w) < 0$ for $w > 0$.

**Proposition 4.9.** *For any given $\nu > 0$ and $a \in X_1 (= X_2)$,*

$$\tilde{\lambda}_1(\nu, a) \leq \tilde{\lambda}_2(\nu, a)$$

*and if $\lambda_1(\nu, a)$ exists, then*

$$\tilde{\lambda}_1(\nu, a) (= \lambda_1(\nu, a)) < \tilde{\lambda}_2(\nu, a)$$

*Proof.* First, assume that $\lambda_1(\nu, a)$ exists. Let $\phi(\cdot)$ be the positive principal eigenfunction of $\nu\mathcal{K}_1 - \nu\mathcal{I} + a(\cdot)\mathcal{I}$ with $\|\phi\| = 1$. Then

$$\Phi_1(t; \nu, a)\phi = e^{\lambda_1(\nu, a)t}\phi, \quad \text{and} \quad \Phi_2(t; \nu, a)\phi = e^{\tilde{\lambda}_2(\nu, a)t}\phi \quad \forall\ t > 0.$$

By Remark 3.7,

$$\Phi_2(t; \nu, a)\phi \gg \Phi_1(t; \nu, a)\phi \quad \forall\ t > 0.$$

This implies that

$$\tilde{\lambda}_2(\nu, a) > \lambda_1(\nu, a).$$

In general, by Lemma 3.10 and Theorem 2.4 (2), for any $\epsilon > 0$, there is $a_\epsilon \in X_1$ such that $\lambda_1(\nu, a_\epsilon)$ exists and

$$a_\epsilon(x) - \epsilon \leq a(x) \leq a_\epsilon(x) + \epsilon.$$

By the above arguments,

$$\tilde{\lambda}_2(\nu, a_\epsilon) > \lambda_1(\nu, a_\epsilon).$$

Observe that

$$\tilde{\lambda}_2(\nu, a) \geq \tilde{\lambda}_2(\nu, a_\epsilon) - \epsilon \quad \text{and} \quad \lambda_1(\nu, a_\epsilon) \geq \tilde{\lambda}_1(\nu, a) - \epsilon.$$

Hence

$$\tilde{\lambda}_2(\nu, a) \geq \tilde{\lambda}_1(\nu, a) - 2\epsilon.$$

Letting $\epsilon \to 0$, we have

$$\tilde{\lambda}_2(\nu, a) \geq \tilde{\lambda}_1(\nu, a).$$

□

Consider

$$u_t = \nu \left[ \int_D k(x-y)u(t,y)dy - u(t,x) \right] + u(t,x)g(x, u(t,x)), \quad x \in \bar{D} \tag{4.28}$$

and

$$v_t = \nu \int_D k(x-y)[v(t,y) - v(t,x)]dy + v(t,x)g(x, v(t,x)), \quad x \in \bar{D}, \tag{4.29}$$

where $g$ is a $C^1$ function, $g(x,w) < 0$ for $w \gg 1$, and $\partial_w g(x,w) < 0$ for $w \geq 0$.

**Proposition 4.10.**

*(1) If $\lambda_1(\nu, g(\cdot, 0)) > 0$, then there is $u^* \in X_1^{++}$ such that $u = u^*$ is a stationary solution of* (4.28) *and for any solution $u(t,x)$ of* (4.28) *with $u(0,\cdot) \in X_1^+ \setminus \{0\}$, $u(t,\cdot) \to u^*(\cdot)$ in $X_1$.*

*(2) If $\lambda_2(\nu, g(\cdot, 0)) > 0$, then there is $v^* \in X_2^{++}$ such that $v = v^*$ is a stationary solution of* (4.29) *and for any solution $v(t,x)$ of* (4.29) *with $v(0,\cdot) \in X_2^+ \setminus \{0\}$, $v(t,\cdot) \to v^*(\cdot)$ in $X_2$.*

*Proof.* It follows from [56, Theorem E]. □

### 4.5.2 Proof of Theorem 2.12

In this subsection, we prove Theorem 2.12.

*Proof of Theorem 2.12.* (1) By $\tilde{\lambda}_1(\nu, f(\cdot, 0)) > 0$ and Proposition 4.9, we have $\tilde{\lambda}_2(\nu, f(\cdot, 0)) > 0$. Then by Lemma 4.10, there are $u^* \in X_1^{++}$ and $v^* \in X_2^{++}$ such that $(u^*, 0)$ and $(0, v^*)$ are stationary solutions of (1.16). Moreover, for any $(u_0, v_0) \in X_1^+ \times X_2^+$ with $u_0 \neq 0$ and $v_0 = 0$ (resp. $u_0 = 0$ and $v_0 \neq 0$), $(u(t, \cdot; u_0, v_0), v(t, \cdot; u_0, v_0)) \to (u^*(\cdot), 0)$ (resp. $(u(t, \cdot; u_0, v_0), v(t, \cdot; u_0, v_0)) \to (0, v^*(\cdot))$ as $t \to \infty$.

(2) Observe that

$$\nu \left[ \int_D k(x-y) u^*(y) dy - u^*(x) \right] + f(x, u^*(x)) u^*(x) = 0, \quad x \in \bar{D}. \tag{4.30}$$

This implies that $\lambda_1(\nu, f(\cdot, u^*(\cdot)))$ exists and $\lambda_1(\nu, f(\cdot, u^*(\cdot))) = 0$. By Proposition 4.9, we have

$$\tilde{\lambda}_2(\nu, f(\cdot, u^*(\cdot))) > 0.$$

By Lemma 3.10, there are $\epsilon > 0$ and $a \in X_1$ such that $\lambda_2(\nu, a)$ exists,

$$a(x) \leq f(x, u^*(x)) - \epsilon, \quad \lambda_2(\nu, a) > 0,$$

and

$$\tilde{\lambda}_2(\nu, f(\cdot, u^*(\cdot) + \epsilon)) > 0.$$

Let $\phi(\cdot)$ be the positive eigenfunction of $\nu\mathcal{K}_2 - \nu b(\cdot)\mathcal{I} + a(\cdot)\mathcal{I}$ with $\|\phi\| = 1$, where $b(x) = \int_D k(x-y)dy$. Let

$$u_\delta(x) = u^*(x) + \delta^2 \quad \text{and} \quad v_\delta(x) = \delta\phi(x).$$

Then

$$
\begin{aligned}
0 &= \nu\left[\int_D k(x-y)u^*(y)dy - u^*(x)\right] + u^*(x)f(x,u^*(x))\\
&= \nu\left[\int_D k(x-y)u_\delta(y)dy - u_\delta(x)\right] + u_\delta(x)f(x,u_\delta(x)+v_\delta(x))\\
&\quad + \nu\delta^2\left(1-\int_D k(x-y)dy\right) - \delta^2 f(x,u^*(x))\\
&\quad + u_\delta\left[f(x,u^*(x)) - f(x,u_\delta(x)+v_\delta(x))\right]\\
&\geq \nu\left[\int_D k(x-y)u_\delta(y)dy - u_\delta(x)\right] + u_\delta(x)f(x,u_\delta(x)+v_\delta(x))
\end{aligned}
$$

for $0<\delta\ll 1$, and

$$
\begin{aligned}
0 &\leq \lambda_2(\nu,a)v_\delta(x)\\
&= \nu\int_D k(x-y)[v_\delta(y)-v_\delta(x)]dy + a(x)v_\delta(x)\\
&\leq \nu\int_D k(x-y)[v_\delta(y)-v_\delta(x)]dy + [f(x,u^*(x))-\epsilon]v_\delta(x)\\
&= \nu\int_D k(x-y)[v_\delta(y)-v_\delta(x)]dy + v_\delta(x)f(x,u_\delta(x)+v_\delta(x))\\
&\quad + v_\delta(x)\left[f(x,u^*(x)) - f(x,u_\delta(x)+v_\delta(x)) - \epsilon\right]\\
&\leq \nu\int_D k(x-y)[v_\delta(y)-v_\delta(x)]dy + v_\delta(x)f(x,u_\delta(x)+v_\delta(x))
\end{aligned}
$$

for $0<\delta\ll 1$. It then follows that for $0<\delta\ll 1$, $(u_\delta(x),v_\delta(x))$ is a super-solution of (1.16). By Proposition 3.9,

$$
(u(t_2,\cdot;u_\delta,v_\delta),v(t_2,\cdot;u_\delta,v_\delta)) \leq_2 (u(t_1,\cdot;u_\delta,v_\delta),v(t_1,\cdot;u_\delta,v_\delta)) \quad \forall\ 0<t_1<t_2. \tag{4.31}
$$

Let

$$
(u_\delta^{**}(x),v_\delta^{**}(x)) = \lim_{t\to\infty}(u(t,x;u_\delta,v_\delta),v(t,x;u_\delta,v_\delta)) \quad \forall\ x\in\bar{D}
$$

(this pointwise limit exists because of (4.31)).

We claim that $(u_\delta^{**}(\cdot), v_\delta^{**}(\cdot)) = (0, v^*(\cdot))$. Observe that $u_\delta^{**}(\cdot)$ and $v_\delta^{**}(\cdot)$ are semi-continuous and $(u_\delta^{**}(\cdot), v_\delta^{**}(\cdot))$ satisfies that

$$\begin{cases} \nu[\int_D k(x-y)u_\delta^{**}(y)dy - u_\delta^{**}(x)] + u_\delta^{**}(x)f(x, u_\delta^{**}(x) + v_\delta^{**}(x)) = 0, \quad x \in \bar{D}, \\ \nu \int_D k(x-y)[v_\delta^{**}(y) - v_\delta^{**}(x)]dy + v_\delta^{**}(x)f(x, u_\delta^{**}(x) + v_\delta^{**}(x)) = 0, \quad x \in \bar{D} \end{cases} \tag{4.32}$$

(see the arguments in [35, Theorem A]). Multiplying the first equation in (4.32) by $v_\delta^{**}(x)$, second equation by $u_\delta^{**}(x)$, and integrating over $D$, we have

$$\int_D u_\delta^{**}(x)v_\delta^{**}(x)dx = \int_D \left( \int_D k(x-y)dy \right) u_\delta^{**}(x)v_\delta^{**}(x)dx.$$

This together with $v_\delta^{**}(x) \geq \delta\phi(x) > 0$ implies that

$$\left[ 1 - \int_D k(x-y)dy \right] u_\delta^{**}(x) = 0 \quad \forall\, x \in \bar{D}.$$

Note that $\int_D k(x-y)dy < 1$ for $x$ near $\partial D$. This together with the first equation in (4.32) implies that $u_\delta^{**}(x) = 0$ for all $x \in \bar{D}$. We then must have $v_\delta^{**}(x) = v^*(x)$ for all $x \in \bar{D}$. Moreover, by (4.31) and Dini's theorem,

$$\lim_{t\to\infty} (u(t,\cdot; u_\delta, v_\delta), v(t,\cdot; u_\delta, v_\delta)) = (0, v^*(\cdot)) \quad \text{in} \quad X_1 \times X_2. \tag{4.33}$$

Now, for any $(u_0, v_0) \in (X_1^+ \setminus \{0\}) \times (X_2^+ \setminus \{0\})$, there is $M_0 > 0$ such that

$$(u_0, v_0) \leq_2 (M, 0).$$

Then by Proposition 3.9,

$$(u(t,\cdot; u_0, v_0), v(t,\cdot; u_0, v_0)) \leq_2 (u(t,\cdot; M, 0), v(t,\cdot; M, 0)) \quad \forall\, t > 0.$$

Since $(u(t,\cdot;M,0),v(t,\cdot;M,0)) \to (u^*(\cdot),0)$ in $X_1 \times X_2$ for $0<\delta \ll 1$, there is $T>0$ such that

$$(u(t,\cdot;u_0,v_0),v(t,\cdot;u_0,v_0)) \le_2 (u_\delta(\cdot),0) \quad \forall\, t \ge T.$$

Then $v(t,\cdot;u_0,v_0)$ satisfies

$$v_t(t,x) \ge \nu \int_D k(x-y)[v(t,y)-v(t,x)]dy + v(t,x) f(x,u^*(x)+\epsilon+v(t,x))$$

for $t \ge T$. Note that $\tilde{\lambda}_2(\nu, f(\cdot,u^*(\cdot)+\epsilon)) > 0$. By Lemma 4.10, for $0<\delta \ll 1$, there is $\tilde{T} \ge T$ such that

$$v(t,\cdot;u_0,v_0) \ge v_\delta(\cdot) \quad \forall\, t \ge 0.$$

We then have

$$(u(t+\tilde{T},\cdot;u_0,v_0),v(t+\tilde{T},\cdot;u_0,v_0)) \le_2 (u(t,\cdot;u_\delta,v_\delta),v(t,\cdot;u_\delta,v_\delta)) \quad \forall\, t \ge 0.$$

By (4.33),

$$\lim_{t\to\infty}(u(t,\cdot;u_0,v_0),v(t,\cdot;u_0,v_0)) = (0,v^*(\cdot)).$$

The theorem is thus proved. $\square$

## Chapter 5

## Approximations of Random Dispersal Operators/Equations by Nonlocal Dispersal Operators/Equations and Applications

In this chapter, we prove Theorem 2.13, Theorem 2.15, and Theorem 2.16 with Dirichlet, Neumann, and periodic types of boundary condition by making use of the comparison principle and other results in the Preliminary. In particular, Theorem 2.13 is fundamental to Theorem 2.15 and Theorem 2.16. Finally, we apply the above approximation results to prove Theorem 2.18. Most results in this chapter are included in [60], which has been submitted for publication.

### 5.1 Approximations of Solutions of Random Dispersal Initial-Boundary Value Problems by Nonlocal Dispersal Initial-Boundary Value Problems

In this section, we explore the approximation of solutions to (1.17) by the solutions to (1.18). We first present some basic properties of solutions to (1.17) and (1.18). Then we prove Theorem 2.13. Though the ideas of the proofs of Theorem 2.13 for different types of boundary conditions are the same, different techniques are needed for different boundary conditions. We hence give proofs of Theorem 2.13 for different boundary conditions in different subsections.

#### 5.1.1 Proof of Theorem 2.13 in the Dirichlet Boundary Condition Case

In this subsection, we prove Theorem 2.13 in the Dirichlet boundary case. Throughout this subsection, we assume (H0), and $B_{r,b}u = B_{r,D}u$ in (1.17), and $D_c = \mathbb{R}^N \setminus \bar{D}$ and $B_{n,b}u = B_{n,D}u$ in (1.18). Without loss of generality, we assume $s = 0$.

*Proof of Theorem 2.13 in the Dirichlet boundary condition case.* Let $u_0 \in C^3(\bar{D})$ with $u_0(x) = 0$ for $x \in \partial D$. Let $u_1^\delta(t,x)$ be the solution of (1.18) with $s = 0$ and $u_1(t,x)$ be the solution of (1.17) with $s = 0$. Suppose that $u_1(t,x)$ and $u_1^\delta(t,x)$ exist on $[0,T]$. By regularity of solutions for parabolic equations, $u_1 \in C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{D} \times (0,T]) \cap C^{2+\alpha,0}(\bar{D} \times [0,T])$. Let $\tilde{u}_1$ be an extension of $u_1$ to $\mathbb{R}^N \times [0,T]$ satisfying that $\tilde{u}_1 \in C^{2+\alpha,0}(\mathbb{R}^N \times [0,T])$. Define

$$L_\delta(z)(t,x) = \nu_\delta \int_{D \cup D_c} k_\delta(x-y)[z(t,y) - z(t,x)]dy.$$

Let $G(t,x) = \tilde{u}_1(t,x)$. Then $\tilde{u}_1$ verifies

$$\begin{cases} \partial_t \tilde{u}_1(t,x) = L_\delta(\tilde{u}_1)(t,x) + F_\delta(t,x) + F(t,x,\tilde{u}_1(t,x)), & x \in \bar{D},\ t \in (0,T], \\ \tilde{u}_1(t,x) = G(t,x), & x \in D_c, t \in [0,T], \\ \tilde{u}_1(0,x) = u_0(x), & x \in \bar{D}, \end{cases}$$

where

$$\begin{aligned} F_\delta(t,x) &= \Delta \tilde{u}_1(t,x) - L_\delta(\tilde{u}_1)(t,x) \\ &= \Delta \tilde{u}_1(t,x) - \nu_\delta \int_{D \cup D_c} k_\delta(x-y)(\tilde{u}_1(t,y) - \tilde{u}_1(t,x))dy. \end{aligned}$$

Let $w_1^\delta = \tilde{u}_1 - u_1^\delta$. We then have

$$\begin{cases} \partial_t w_1^\delta(t,x) = L_\delta(w_1^\delta)(t,x) + F_\delta(t,x) + a_1(t,x) w_1^\delta(t,x), & x \in \bar{D},\ t \in (0,T], \\ w_1^\delta(t,x) = G(t,x), & x \in D_c,\ t \in [0,T], \\ w_1^\delta(0,x) = 0, & x \in \bar{D}, \end{cases} \tag{5.1}$$

where $a_1(t,x) = \int_0^1 F_u[t,x,u_1^\delta(t,x) + \theta(\tilde{u}_1(t,x) - u_1^\delta(t,x))]d\theta$.

We claim that

$$\begin{cases} \sup_{t\in[0,T]} \|F_\delta(t,\cdot)\|_{X_1} = O(\delta^\alpha), \\ \sup_{t\in[0,T],x\in\mathbb{R}^N\setminus\bar{D},\mathrm{dist}(x,\partial D)\le\delta} |G(t,x)| = O(\delta). \end{cases} \tag{5.2}$$

In fact,

$$\begin{aligned}
&\Delta\tilde{u}_1(t,x) - \nu_\delta \int_{D\cup D_c} k_\delta(x-y)(\tilde{u}_1(t,y)-\tilde{u}_1(t,x))dy \\
&= \Delta\tilde{u}_1(t,x) - \nu_\delta \int_{\mathbb{R}^N} \frac{1}{\delta^N} k_0\left(\frac{x-y}{\delta}\right)(\tilde{u}_1(t,y)-\tilde{u}_1(t,x))dy \\
&= \Delta\tilde{u}_1(t,x) - \nu_\delta \int_{\mathbb{R}^N} k_0(z)(\tilde{u}_1(t,x+\delta z)-\tilde{u}_1(t,x))dz \\
&= \Delta\tilde{u}_1(t,x) - \nu_\delta \int_{\mathbb{R}^N} k_0(z)\left[\frac{\delta^2 z_N^2}{2!}\Delta\tilde{u}_1(t,x) + O(\delta^{2+\alpha})\right]dz \\
&= \Delta\tilde{u}_1(t,x) - \left[\nu_\delta\delta^2 \int_{\mathbb{R}^N} k_0(z)\frac{z_N^2}{2}dz\right]\Delta\tilde{u}_1(t,x) + O(\delta^\alpha) \\
&= \Delta\tilde{u}_1(t,x) - \Delta\tilde{u}_1(t,x) + O(\delta^\alpha) \\
&= O(\delta^\alpha) \quad \forall\ x\in\bar{D},
\end{aligned}$$

and

$$\begin{aligned}
|G(t,x)| &= |\tilde{u}_1(t,x)| \\
&\le \sup_{t\in[0,T],x\in\mathbb{R}^N\setminus D,z\in\partial D,\mathrm{dist}(x,z)\le\delta} |\tilde{u}_1(t,x)-u_1(t,z)| \\
&= O(\delta) \quad \forall\ x\in D_c,\ \mathrm{dist}(x,\partial D)\le\delta.
\end{aligned}$$

Therefore, (5.2) holds.

Next, let $\bar{w}$ be given by

$$\bar{w}(t,x) = e^{At}(K_1\delta^\alpha t) + K_2\delta,$$

where $A = \max_{x \in \bar{D}, t \in [0,T]} a_1(t,x)$. By direct calculation, we have

$$\begin{cases} \partial_t \bar{w}(t,x) = L_\delta(\bar{w}) + a_1(t,x)\bar{w} + \bar{F}_\delta(t,x) & x \in \bar{D}, \quad t \in (0,T], \\ \bar{w}(t,x) = e^{At}(K_1\delta^\alpha t) + K_2\delta, & x \in D_c, \quad t \in [0,T], \\ \bar{w}(0,x) = K_2\delta, & x \in \bar{D}, \end{cases} \tag{5.3}$$

where

$$\bar{F}_\delta(t,x) = e^{At}K_1\delta^\alpha + [A - a_1(t,x)]e^{At}K_1\delta^\alpha t - a_1(t,x)K_2\delta.$$

By (5.2), there are $\delta_0 > 0$ and $K_1, K_2 > 0$ such that

$$\begin{cases} F_\delta(t,x) \le \bar{F}_\delta(t,x), & x \in \bar{D}, \quad t \in [0,T], \\ G(t,x) \le e^{At}(K_1\delta^\alpha t) + K_2\delta, & x \in D_c, \ \operatorname{dist}(x, \partial D) \le \delta, \ t \in [0,T], \end{cases} \tag{5.4}$$

when $0 < \delta < \delta_0$. By (5.1), (5.3), (5.4), and Remark 3.7, we obtain

$$w^\delta(t,x) \le \bar{w}(t,x) = e^{At}(K_1\delta^\alpha t) + K_2\delta \quad \forall\, x \in \bar{D}, \ t \in [0,T] \tag{5.5}$$

for $0 < \delta < \delta_0$.

Similarly, let $\underline{w}(t,x) = e^{At}(-K_1\delta^\alpha t) - K_2\delta$. We can prove that for $0 < \delta < \delta_0$,

$$w^\delta(t,x) \ge \underline{w}(t,x) = -e^{At}(K_1\delta^\alpha t) - K_2\delta \quad \forall\ x \in \bar{D}, \ t \in [0,T]. \tag{5.6}$$

By (5.5) and (5.6) we have

$$|w^\delta(t,x)| \le e^{At}K_1\delta^\alpha t + K_2\delta \quad \forall\ x \in \bar{D}, \ t \in [0,T],$$

which implies that there is $C(T) > 0$ such that for any $0 < \delta < \delta_0$,

$$\sup_{t\in(0,T]} \|u_1(\cdot,t) - u_1^\delta(\cdot,t)\|_{X_1} \leq C(T)\delta^\alpha.$$

Theorem 2.13 in the Dirichlet boundary condition case then follows. $\square$

**Remark 5.1.** *If the homogeneous Dirichlet boundary conditions $B_{r,D}u = u = 0$ on $\partial D$ and $B_{n,D}u = u = 0$ on $D_c = \mathbb{R}^N \setminus \bar{D}$ are changed to nonhomogeneous Dirichlet boundary conditions $B_{r,D}u = u = g(t,x)$ on $\partial D$ and $B_{n,D}u = u = g(t,x)$ on $D_c = \mathbb{R}^N \setminus \bar{D}$, Theorem 2.13 also holds, which can be proved by the similar arguments as above.*

### 5.1.2 Proof of Theorem 2.13 in the Neumann Boundary Condition Case

In this subsection, we prove Theorem 2.13 in the Neumann boundary condition case. Throughout this subsection, we assume (H1), and $B_{r,b}u = B_{r,N}u$ in (1.17), and $D_c = \emptyset$ and $B_{n,b}u = B_{n,N}u$ in (1.18). Without loss of generality, we assume $s = 0$.

We first introduce two lemmas. To this end, for given $\delta > 0$ and $d_0 > 0$, let $D_\delta = \{z \in D | \mathrm{dist}(z, \partial D) < d_0\delta\}$.

**Lemma 5.2.** *Let $\theta \in C^{2+\alpha,1+\frac{\alpha}{2}}(\mathbb{R}^N \times (0,T]) \cap C^{2+\alpha,0}(\mathbb{R}^N \times [0,T])$ and $\frac{\partial\theta}{\partial \mathbf{n}} = h$ on $\partial D$, then for $x \in D_\delta$ and $\delta$ small,*

$$\begin{aligned}
&\frac{1}{\delta^2}\int_{\mathbb{R}^N\setminus D} k_\delta(x-y)(\theta(t,y)-\theta(t,x))dy \\
&= \frac{1}{\delta}\int_{\mathbb{R}^N\setminus D} k_\delta(x-y)\mathbf{n}(\bar{x})\cdot\frac{x-y}{\delta}h(\bar{x},t)dy \\
&+ \int_{\mathbb{R}^N\setminus D} k_\delta(x-y)\sum_{|\beta|=2}\frac{D^\beta\theta}{2}(\bar{x},t)\left[\left(\frac{y-\bar{x}}{\delta}\right)^\beta - \left(\frac{x-\bar{x}}{\delta}\right)^\beta\right]dy + O(\delta^\alpha),
\end{aligned}$$

*where $\bar{x}$ is the orthogonal projection of $x$ on the boundary of $D$ so that $\|\bar{x} - y\| \leq 2d_0\delta$ and $\mathbf{n}(\bar{x})$ is the exterior unit normal vector of $\partial D$ at $\bar{x}$.*

*Proof.* See [15, Lemma 3]. $\square$

**Lemma 5.3.** *There exist $K > 0$ and $\bar{\delta} > 0$ such that for $\delta < \bar{\delta}$,*

$$\int_{\mathbb{R}^N \setminus D} k_\delta(x-y)\mathbf{n}(\bar{x})\frac{x-y}{\delta}dy \geq K \int_{\mathbb{R}^N \setminus D} k_\delta(x-y)dy.$$

*Proof.* See [15, Lemma 4]. □

*Proof of Theorem 2.13 in the Neumann boundary condition case.* Suppose that $u_0 \in C^3(\bar{D})$. Let $u_2^\delta(t,x)$ be the solution to (1.18) with $s = 0$ and $u_2(t,x)$ be the solution to (1.17) with $s = 0$. Assume that $u_2(t,x)$ and $u_2^\delta(t,x)$ exist on $[0,T]$. Then $u_2 \in C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{D} \times (0,T])$. Let $\tilde{u}_2$ be an extension of $u_2$ to $\mathbb{R}^N \times [0,T]$ satisfying that $\tilde{u}_2 \in C^{2+\alpha,1+\frac{\alpha}{2}}(\mathbb{R}^N \times (0,T]) \cap C^{2+\alpha,0}(\mathbb{R}^N \times [0,T])$. Define

$$L_\delta(z)(t,x) = \nu_\delta \int_D k_\delta(x-y)(z(t,y) - z(t,x))dy,$$

and

$$\tilde{L}_\delta(z)(t,x) = \nu_\delta \int_{\mathbb{R}^N} k_\delta(x-y)(z(t,y) - z(t,x))dy.$$

Set $w_2^\delta = u_2^\delta - \tilde{u}_2$. Then

$$\begin{aligned}
\partial_t w_2^\delta(t,x) &= \partial_t u_2^\delta(t,x) - \partial_t \tilde{u}_2(t,x) \\
&= [L_\delta(u_2^\delta)(t,x) + F(t,x,u_2^\delta)] - [\Delta \tilde{u}_2(t,x) + F(t,x,\tilde{u}_2)] \\
&= L_\delta(w_2^\delta)(t,x) + a_2(t,x)w_2^\delta(t,x) + F_\delta(t,x),
\end{aligned}$$

where $a_2(t,x) = \int_0^1 F_u(t,x,\tilde{u}_2(t,x) + \theta(u_2^\delta(t,x) - \tilde{u}_2(t,x)))d\theta$ and

$$F_\delta(t,x) = \tilde{L}_\delta(\tilde{u}_2)(t,x) - \Delta\tilde{u}_2(t,x) - \nu_\delta \int_{\mathbb{R}^N \setminus D} k_\delta(x-y)(\tilde{u}_2(t,y) - \tilde{u}_2(t,x))dy.$$

Hence $w_2^\delta$ verifies

$$\begin{cases} \partial_t w_2^\delta(t,x) = L_\delta(w_2^\delta)(t,x) + a_2(t,x)w_2^\delta(t,x) + F_\delta(t,x), & x \in \bar{D}, \\ w_2^\delta(0,x) = 0, & x \in \bar{D}. \end{cases} \tag{5.7}$$

To prove the theorem, let us pick an auxiliary function $v$ as a solution to

$$\begin{cases} \partial_t v(t,x) = \Delta v(t,x) + a_2(t,x)v + h(t,x), & x \in D, \quad t \in (0,T], \\ \frac{\partial v}{\partial \mathbf{n}}(t,x) = g(t,x), & x \in \partial D,\, t \in [0,T], \\ v(0,x) = v_0(x), & x \in D \end{cases}$$

for some smooth functions $h(t,x) \geq 1$, $g(t,x) \geq 1$ and $v_0(x) \geq 0$ such that $v(t,x)$ has an extension $\tilde{v}(t,x) \in C^{2+\alpha,1+\frac{\alpha}{2}}(\mathbb{R}^N \times (0,T]) \cap C^{2+\alpha,0}(\mathbb{R}^N \times [0,T])$. Then $v$ is a solution to

$$\begin{cases} \partial_t v(t,x) = L_\delta(v)(t,x) + a_2(t,x)v(t,x) + H(t,x,\delta), & x \in \bar{D}, t \in (0,T], \\ v(0,x) = v_0(x), & x \in \bar{D}, t \in [0,T], \end{cases} \tag{5.8}$$

where

$$H(t,x,\delta) = \Delta\tilde{v}(t,x) - \tilde{L}_\delta(v)(t,x) + \nu_\delta \int_{\mathbb{R}^N \setminus D} k_\delta(x-y)(\tilde{v}(t,y) - \tilde{v}(t,x))dy + h(t,x).$$

By Lemma 5.2 and the first estimate in (5.2), we have the following estimate for $H(x,t,\delta)$:

$$\begin{aligned} H(t,x,\delta) &= \Delta\tilde{v}(t,x) - \tilde{L}_\delta(v)(t,x) + \frac{C}{\delta^2}\int_{\mathbb{R}^N\setminus D} k_\delta(x-y)(\tilde{v}(t,y)-\tilde{v}(t,x))dy + h(t,x) \\ &\geq \frac{C}{\delta}\int\limits_{\mathbb{R}^N\setminus D} k_\delta(x-y)\mathbf{n}(\bar{x})\frac{x-y}{\delta}g(\bar{x},t)dy \\ &\quad + C\int\limits_{\mathbb{R}^N\setminus D} k_\delta(x-y)\sum_{|\beta|=2}\frac{D^\beta\tilde{v}}{2}(\bar{x},t)\left[\left(\frac{y-\bar{x}}{\delta}\right)^\beta - \left(\frac{x-\bar{x}}{\delta}\right)^\beta\right]dy + 1 - C_1\delta^\alpha \\ &\geq \frac{C}{\delta}g(\bar{x},t)\int\limits_{\mathbb{R}^N\setminus D} k_\delta(x-y)\mathbf{n}(\bar{x})\frac{x-y}{\delta}dy - D_1C\int\limits_{\mathbb{R}^N\setminus D} k_\delta(x-y)dy + \frac{1}{2} \end{aligned} \tag{5.9}$$

for some constants $D_1$ and $C_1$ and $\delta$ sufficiently small such that $C_1\delta^\alpha \leq \frac{1}{2}$. Then Lemma 5.3 implies that there exist $C' > 0$ and $\delta'$ such that

$$\frac{1}{\delta}\int\limits_{\mathbb{R}^N\setminus D} k_\delta(x-y)\mathbf{n}(\bar{x})\frac{x-y}{\delta}dy \geq \frac{C'}{\delta}\int\limits_{\mathbb{R}^N\setminus D} k_\delta(x-y)dy,$$

if $\delta < \delta'$. This implies that

$$H(x,t,\delta) \geq \left[\frac{CC'g(\bar{x},t)}{\delta} - D_1\right]\int\limits_{\mathbb{R}^N\setminus D} k_\delta(x-y)dy + \frac{1}{2}, \tag{5.10}$$

if $\delta < \delta'$.

We estimate now $F_\delta(t,x)$. By Lemmas 5.2, 5.3, the first estimate in (5.2), and the fact that $\frac{\partial \tilde{u}_2}{\partial \mathbf{n}} = 0$, we have

$$
\begin{aligned}
F_\delta(t,x) &= O(\delta^\alpha) + \nu_\delta \int\limits_{\mathbb{R}^N\setminus D} k_\delta(x-y)(\tilde{u}_2(t,y) - \tilde{u}_2(t,x))dy \\
&= O(\delta^\alpha) + C \int\limits_{\mathbb{R}^N\setminus D} k_\delta(x-y) \sum_{|\beta|=2} \frac{D^\beta\theta}{2}(\bar{x},t) \left[ \left(\frac{y-\bar{x}}{\delta}\right)^\beta - \left(\frac{x-\bar{x}}{\delta}\right)^\beta \right] dy \\
&\le C_2\delta^\alpha + D_1 C \int\limits_{\mathbb{R}^N\setminus D} k_\delta(x-y)dy \\
&= C_2\delta^\alpha + D_2 \int\limits_{\mathbb{R}^N\setminus D} k_\delta(x-y)dy
\end{aligned}
\tag{5.11}
$$

for some $C_2 > 0$ and $D_2 > 0$. Given $\epsilon > 0$, let $v_\epsilon = \epsilon v$. By (5.8), $v_\epsilon$ satisfies

$$
\begin{cases}
\partial_t v_\epsilon(t,x) - L_\delta(v_\epsilon)(t,x) - a(t,x)v_\epsilon(t,x) = \epsilon H(t,x,\delta), & x \in \bar{D}, \\
v_\epsilon(0,x) = \epsilon v_0(x), & x \in \bar{D}.
\end{cases}
\tag{5.12}
$$

By (5.10) and (5.11), there exist $C_3 > 0$ and $\delta_0 \le \delta'$ such that for $0 < \delta \le \delta_0$,

$$
\begin{aligned}
F_\delta(t,x) &\le C\delta^\alpha + D_2 \int\limits_{\mathbb{R}^N\setminus D} k_\delta(x-y)dy \\
&\le \frac{\epsilon}{2} + \frac{C_3\epsilon}{\delta} \int\limits_{\mathbb{R}^N\setminus D} k_\delta(x-y)dy \\
&= \epsilon H(x,t,\delta) \quad \forall\, x \in \bar{D},\ t \in [0,T].
\end{aligned}
\tag{5.13}
$$

Then by (5.7), (5.12), (5.13), and Remark 3.7, we have

$$
-M\epsilon \le -v_\epsilon \le w_2^\delta \le v_\epsilon \le M\epsilon \quad \forall\, \delta \le \delta_0,
$$

where $M = \max_{t\in[0,T],x\in\bar{D}} v(t,x)$. This implies

$$\sup_{t\in[0,T]} \|u_2^\delta(t,\cdot) - u_2(t,\cdot)\|_{X_2} \to 0, \quad \text{as } \delta \to 0.$$

Theorem 2.13 in the Neumann boundary condition is thus proved. □

### 5.1.3 Proof of Theorem 2.13 in the Periodic Boundary Condition Case

In this subsection, we prove Theorem 2.13 in the periodic boundary condition case. Throughout this subsection, we assume (H1), and $B_{r,b}u = B_{r,P}u$ in (1.17), and $B_{n,b}u = B_{n,P}u$ in (1.18). Without loss of generality again, we assume $s = 0$.

*Proof of Theorem 2.13 in the periodic boundary case.* Suppose that $u_0 \in X_3 \cap C^3(\mathbb{R}^N)$. Let $u_3^\delta(t,x)$ be the solution to (1.18) with $s = 0$ and $u_3(t,x)$ be the solution to (1.17) with $s = 0$. Suppose that $u_3(t,x)$ and $u_3^\delta(t,x)$ exist on $[0,T]$. Set $w_3^\delta = u_3^\delta - u_3$. Then $w_3^\delta$ satisfies

$$\begin{cases} \partial_t w_3^\delta(t,x) = \nu_\delta \int_{\mathbb{R}^N} k_\delta(x-y)(w_3^\delta(t,y) - w_3^\delta(t,x))dy & \\ \qquad\qquad +a_3(t,x)w_3^\delta(t,x) + F_\delta(t,x), & x \in \mathbb{R}^N,\ t \in (0,T], \\ w_3^\delta(t,x) = w_3^\delta(t, x + p_j \mathbf{e}_j), & x \in \mathbb{R}^N,\ t \in [0,T], \\ w_3^\delta(0,x) = 0, & x \in \mathbb{R}^N, \end{cases} \tag{5.14}$$

where $a_3(t,x) = \int_0^1 F_u(t,x,u_3(t,x) + \theta(u_3^\delta(t,x) - u_3(t,x)))d\theta$ and $F_\delta(t,x) = \nu_\delta \int_{\mathbb{R}^N} k_\delta(x-y)[u_3(t,y) - u_3(t,x)]dy - \Delta u_3$. Let

$$\bar{w}(t,x) = e^{At}(K_1\delta^\alpha t) + K_2\delta,$$

where $A = \max_{x\in\mathbb{R}^N,t\in[0,T]} a_3(t,x)$. Applying the similar approach as in the Dirichlet boundary condition case, we can show that there are $K_1 > 0$, $K_2 > 0$, and $\delta_0 > 0$ such that for

$0 < \delta < \delta_0$,

$$-\bar{w}(t,x) \le w_3^\delta(t,x) \le \bar{w}(t,x) \quad \forall\, x \in \mathbb{R}^N,\ t \in [0,T].$$

Theorem 2.13 in the periodic boundary condition case then follows. □

## 5.2 Approximations of Principal Eigenvalues of Time Periodic Random Dispersal Operators by Time Periodic Nonlocal Dispersal Operators

In this section, we investigate the approximation of principal eigenvalues of time periodic random dispersal operators by the principal spectrum points of time periodic nonlocal dispersal operators. We first recall some basic properties of principal eigenvalues of time periodic random dispersal operators, and basic properties of principal spectrum points of time periodic nonlocal dispersal operators to be used in the proof of Theorem 2.15.

### 5.2.1 Basic Properties of Principal Eigenvalues/Principal Spectrum Points of Time Periodic Dispersal Operators

In this subsection, for $i = 1, 2, 3$, we focus on the time-periodic evolution equations (3.1) with $\nu_i = \nu_\delta$ and $k(\cdot) = k_\delta(\cdot)$, and (3.5) with $\nu_i = 1$.

First of all, let us recall that $\Phi_i^\delta(t,s;a)$ is the solution operator of (3.1) with $\nu_i = \nu_\delta$, $k(\cdot) = k_\delta(\cdot)$ and $a_i(\cdot,\cdot) = a(\cdot,\cdot)$ for $i = 1, 2, 3$. And let $r(\Phi_i^\delta(T,0;a))$ be the spectral radius of $\Phi_i^\delta(T,0;a)$, and $\tilde{\lambda}_i^\delta(a)$ be the principal spectrum point of $\mathcal{N}_i(\nu_\delta, a, \delta)$, respectively. We have the following propositions.

**Proposition 5.4.** *Let* $1 \le i \le 3$ *be given. Then*

$$r(\Phi_i^\delta(T,0;a)) = e^{\tilde{\lambda}_i^\delta(a)T}.$$

*Proof.* See [59, Proposition 3.3]. □

We remark that Proposition 4.1 (1) is a special case of Proposition 5.4.

Next, for $1 \le i \le 3$, recall that $\Phi_i^r(t,s;a)$ is the solution operator of (3.5) with $\nu_i = 1$ and $a_i(\cdot,\cdot) = a(\cdot,\cdot)$. Similarly, let $r(\Phi_i^r(T,0;a))$ be the spectral radius of $\Phi_i^r(T,0;a)$ and $\lambda_i^r(a)$ be the principal eigenvalue of $\mathcal{R}_i(1,a)$. Note that $X_i^r$ is a strongly ordered Banach space with the positive cone $C = \{u \in X_i^r \,|\, u(x) \ge 0\}$ and by the regularity, a priori estimate, and comparison principle for parabolic equations, $\Phi_i^r(T,0;a) : X_i^r \to X_i^r$ is strongly positive and compact. Then by the Kreĭn-Rutman Theorem (see [65]), $r(\Phi_i^r(T,0;a))$ is an isolated algebraically simple eigenvalue of $\Phi_i^r(T,0;a)$ with a positive eigenfunction $u_i^r(\cdot)$ and for any $\mu \in \sigma(\Phi_i^r(T,0;a)) \setminus \{r(\Phi_i^r(T,0;a))\}$,

$$\mathrm{Re}\mu < r(\Phi_i^r(T,0;a)).$$

The following propositions then follow.

**Proposition 5.5.** *Let $1 \le i \le 3$ be given. Then*

$$r(\Phi_i^r(T,0;a)) = e^{\lambda_i^r(a)T}.$$

*Moreover, there is a codimension one subspace $Z_i$ of $X_i^r$ such that*

$$X_i^r = Y_i \oplus Z_i,$$

*where $Y_i = \mathrm{span}\{u_i^r(\cdot)\}$, and there are $M > 0$ and $\gamma > 0$ such that for any $w_i \in Z_i$, there holds*

$$\frac{\|\Phi_i(nT,0;a)w_i\|_{X_i^r}}{\|\Phi_i(nT,0;a)u_i^r\|_{X_i^r}} \le Me^{-\gamma nT}.$$

**Proposition 5.6.** *For given $1 \le i \le 3$ and $a_1, a_2 \in \mathcal{X}_i \cap C^1(\mathbb{R} \times \mathbb{R}^N)$,*

$$|\tilde{\lambda}_i^\delta(a_1) - \tilde{\lambda}_i^\delta(a_2)| \le \max_{x \in \bar{D}, t \in [0,T]} |a_1(t,x) - a_2(t,x)|, \tag{5.15}$$

*and*

$$|\lambda_i^r(a_1) - \lambda_i^r(a_2)| \le \max_{x\in\bar{D}, t\in[0,T]} |a_1(t,x) - a_2(t,x)|. \tag{5.16}$$

*Proof.* Let $a_0 = \max_{x\in\bar{D}, t\in[0,T]} |a_1(t,x) - a_2(t,x)|$ and

$$a_1^{\pm}(t,x) = a_1(t,x) \pm a_0.$$

It is not difficult to see that

$$\Phi_i^\delta(t,s;a_1^{\pm}) = e^{\pm a_0(t-s)}\Phi_i^\delta(t,s;a_1).$$

It then follows that

$$r(\Phi_i^\delta(T,0;a_1^{\pm})) = e^{(\tilde{\lambda}_i^\delta(a_1)\pm a_0)T}. \tag{5.17}$$

Observe that by Remark 3.7, for any $u_0 \in X_i^{r,+}$,

$$\Phi_i^\delta(T,0;a_1^-)u_0 \le \Phi_i^\delta(T,0;a_2)u_0 \le \Phi_i^\delta(T,0;a_1^+)u_0.$$

This implies that

$$r(\Phi_i^\delta(T,0;a_1^-)) \le r(\Phi_i^\delta(T,0;a_2)) \le r(\Phi_i^\delta(T,0;a_1^+)).$$

This together with (5.17) implies that

$$\tilde{\lambda}_i^\delta(a_1) - a_0 \le \tilde{\lambda}_i^\delta(a_2) \le \tilde{\lambda}_i^\delta(a_1) + a_0, \tag{5.18}$$

that is, (5.15) holds.

Similarly, we can prove that (5.16) holds. $\square$

### 5.2.2 Proof of Theorem 2.15 in the Dirichlet Boundary Condition Case

In this subsection, we prove Theorem 2.15 in the Dirichlet boundary condition case. Throughout this subsection, we assume $B_{r,b}u = B_{r,D}u$ in (1.19), and $D_c = \mathbb{R}^N \setminus \bar{D}$ and $B_{n,b}u = B_{n,D}u$ in (1.20). Note that for any $a \in \mathcal{X}_1 \cap C^1(\mathbb{R}\times\mathbb{R}^N)$, there are $a_n \in \mathcal{X}_1 \cap C^3(\mathbb{R}\times\mathbb{R}^N)$ such that $\sup_{t\in[0,T]} \|a_n(t,\cdot) - a(t,\cdot)\|_{X_1} \to 0$ as $n \to \infty$. By Proposition 5.6, without loss of generality, we may assume that $a \in \mathcal{X}_1 \cap C^3(\mathbb{R}\times\mathbb{R}^N)$.

*Proof of Theorem 2.15 in the Dirichlet boundary condition case.* First of all, for the simplicity in notation, we put

$$\Phi^r(T,0) = \Phi_1^r(T,0;a), \qquad \lambda^r = \lambda_1^r(a),$$

and

$$\Phi^\delta(T,0) = \Phi_1^\delta(T,0;a), \quad \tilde{\lambda}^\delta = \tilde{\lambda}_1^\delta(a).$$

Let $u^r(\cdot)$ be a positive eigenfunction of $\Phi^r(T,0)$ corresponding to $r(\Phi^r(T,0))$. Without loss of generality, we assume that $\|u^r\|_{X_1^r} = 1$.

We first show that for any $\epsilon > 0$, there is $\delta_1 > 0$ such that for $0 < \delta < \delta_1$,

$$\tilde{\lambda}^\delta \geq \lambda^r - \epsilon. \tag{5.19}$$

In order to do so, choose $D_0 \subset\subset D$ and $u_0 \in X_1^r \cap C^3(\bar{D})$ such that $u_0(x) = 0$ for $x \in D\backslash D_0$, and $u_0(x) > 0$ for $x \in \mathrm{Int}D_0$. By Proposition 5.5, there exist $\alpha > 0$, $M > 0$, and $u' \in Z_1$, such that

$$u_0(x) = \alpha u^r(x) + u'(x), \tag{5.20}$$

and

$$\|\Phi^r(nT,0)u'\|_{X_1^r} \leq M e^{-\gamma nT} e^{\lambda^r nT}. \tag{5.21}$$

By Theorem 2.13, there is $\delta_0 > 0$ such that for $0 < \delta < \delta_0$, there hold

$$\big(\Phi^\delta(nT,0)u^r\big)(x) \geq \big(\Phi^r(nT,0)u^r\big)(x) - C^1(nT,\delta), \tag{5.22}$$

and

$$\big(\Phi^\delta(nT,0)u'\big)(x) \leq \big(\Phi^r(nT,0)u'\big)(x) + C^2(nT,\delta), \tag{5.23}$$

where $C^i(nT,\delta) \to 0$ as $\delta \to 0$ $(i = 1,2)$. Hence for $0 < \delta < \delta_0$,

$$\begin{aligned}
\big(\Phi^\delta(nT,0)u_0\big)(x) =& \alpha\big(\Phi^\delta(nT,0)u^r\big)(x) + \big(\Phi^\delta(nT,0)u'\big)(x) \\
\geq& \alpha\big(\Phi^r(nT,0)u^r\big)(x) - \alpha C^1(nT,\delta) - C^2(nT,\delta) - \|\Phi^r(nT,0)u'\|_{X_1^r} \\
\geq& \alpha e^{\lambda^r nT}u^r(x) - \alpha C^1(nT,\delta) - C^2(nT,\delta) - Me^{-\gamma nT}e^{\lambda^r nT} \\
=& e^{(\lambda^r-\epsilon)nT}e^{\epsilon nT}(\alpha u^r(x) - Me^{-\gamma nT}) - \alpha C^1(nT,\delta) - C^2(nT,\delta).
\end{aligned} \tag{5.24}$$

Note that there exists $m > 0$ such that

$$u^r(x) \geq m > 0 \quad \text{ for } x \in \bar{D}_0.$$

Hence for any $0 < \epsilon < \gamma$, there is $n_1 > 0$ such that for $n \geq n_1$,

$$e^{\epsilon nT}(\alpha u^r(x) - Me^{-\gamma nT}) \geq u_0(x) + 1 \quad \text{for} \quad x \in \bar{D}_0, \tag{5.25}$$

and there is $\delta_1 \leq \delta_0$ such that for $0 < \delta < \delta_1$,

$$C^1(n_1T,\delta) + C^2(n_1T,\delta) \leq e^{(\lambda^r-\epsilon)n_1T}. \tag{5.26}$$

Note that $u_0(x) = 0$ for $x \in D \setminus D_0$ and $\big(\Phi^\delta(n_1T, 0)u_0\big)(x) \geq 0$ for all $x \in \bar{D}$. This together with (5.24)-(5.26) implies that for $\delta < \delta_1$,

$$\big(\Phi^\delta(n_1T, 0)u_0\big)(x) \geq e^{(\lambda^r - \epsilon)n_1T}u_0(x), \quad x \in \bar{D}. \tag{5.27}$$

By (5.27) and Remark 3.7, for any $0 < \delta < \delta_1$ and $n \geq 1$,

$$(\Phi^\delta(nn_1T, 0)u_0)(\cdot) \geq e^{(\lambda^r - \epsilon)nn_1T}u_0(\cdot).$$

This together with Proposition 5.4 implies that for $0 < \delta < \delta_1$,

$$e^{\tilde{\lambda}^\delta T} = r(\Phi^\delta(T, 0)) \geq e^{(\lambda^r - \epsilon)T}.$$

Hence (5.19) holds.

Next, we prove that for any $\epsilon > 0$, there is $\delta_2 > 0$ such that for $0 < \delta < \delta_2$,

$$\tilde{\lambda}^\delta \leq \lambda^r + \epsilon. \tag{5.28}$$

To this end, first, choose a sequence of smooth domains $\{D_m\}$ with $D_1 \supset D_2 \supset D_3 \cdots \supset D_m \supset \cdots \supset \bar{D}$, and $\cap_{m=1}^\infty D_m = \bar{D}$. Consider the following evolution equation

$$\begin{cases} \partial_t u(t,x) = \Delta u(t,x) + a(t,x)u(t,x), & x \in D_m, \\ u(t,x) = 0, & x \in \partial D_m. \end{cases} \tag{5.29}$$

Let

$$X_{1,m} = \{u \in C(\bar{D}_m, \mathbb{R})\},$$

and

$$X_{1,m}^r = \mathcal{D}(A_m^\alpha),$$

where $A_m$ is $-\Delta$ with Dirichlet boundary condition acting on $X_{1,m} \cap C_0(D_m)$ and $0 < \alpha < 1$. We denote the solution of (5.29) by $u_m(t,\cdot;s,u_0) = (\Phi^r_m(t,s)u_0)(\cdot)$ with $u(s,\cdot;s,u_0) = u_0(\cdot) \in X^r_{1,m}$. By Proposition 5.5, we have

$$r(\Phi^r_m(T,0)) = e^{\lambda^r_m T},$$

where $\lambda^r_m$ is the principal eigenvalue of the following eigenvalue problem,

$$\begin{cases} -\partial_t u + \Delta u + a(t,x)u = \lambda u, & x \in D_m, \\ u(t+T,x) = u(t,x), & x \in D_m, \\ u(t,x) = 0, & x \in \partial D_m. \end{cases}$$

By the dependence of the principle eigenvalue on the domain perturbation (see [22]), for any $\epsilon > 0$, there exists $m_1$ such that

$$\lambda^r_{m_1} \le \lambda^r + \frac{\epsilon}{2}. \tag{5.30}$$

Second, let $u^r_{m_1}(\cdot)$ be a positive eigenfunction of $\Phi^r_{m_1}(T,0)$ corresponding to $r(\Phi^r_{m_1}(T,0))$. By regularity for parabolic equations, $u^r_{m_1} \in C^3(\bar{D}_{m_1})$. Let $(\Phi^\delta_{m_1}(t,0)u^r_{m_1})(x)$ be the solution to

$$\begin{cases} u_t = \nu_\delta \left[\int_{D_{m_1}} k_\delta(x-y)u(t,y)dy - u(t,x)\right] + a(t,x)u(t,x), & x \in \bar{D}_{m_1}, \\ u(0,x) = u^r_{m_1}(x). \end{cases} \tag{5.31}$$

Then by Theorem 2.13,

$$\big(\Phi^\delta_{m_1}(nT,0)u^r_{m_1}\big)(x) \le \big(\Phi_{m_1}(nT,0)u^r_{m_1}\big)(x) + C(nT,\delta) \quad \forall\, x \in \bar{D}_{m_1},$$

where $C(nT,\delta)\to 0$ as $\delta\to 0$. By Remark 3.7,

$$\big(\Phi^\delta(nT,0)u^r_{m_1}|_{\bar D}\big)(x)\le\big(\Phi^\delta_{m_1}(nT,0)u^r_{m_1}\big)(x)\quad\forall\, x\in\bar D.$$

It then follows that for $x\in\bar D$,

$$\begin{aligned}\big(\Phi^\delta(nT,0)u^r_{m_1}|_{\bar D}\big)(x)&\le\big(\Phi^r_{m_1}(nT,0)u^r_{m_1}\big)(x)+C(nT,\delta)\\&=e^{\lambda^r_{m_1}nT}u^r_{m_1}(x)+C(nT,\delta)\\&\le e^{(\lambda^r+\frac{\epsilon}{2})nT}u^r_{m_1}(x)+C(nT,\delta)\\&=e^{(\lambda^r+\epsilon)nT}e^{-\frac{\epsilon}{2}nT}u^r_{m_1}(x)+C(nT,\delta).\end{aligned}\tag{5.32}$$

Note that

$$\min_{x\in\bar D}u^r_{m_1}(x)>0.$$

Hence for any $\epsilon>0$, there is $n_2\ge 1$ such that

$$e^{-\frac{\epsilon}{2}n_2T}\le\frac{1}{2},\tag{5.33}$$

and there is $\delta_2>0$ such that for $0<\delta<\delta_2$,

$$C(n_2T,\delta)\le\frac{1}{2}e^{(\lambda^r+\epsilon)n_2T}u^r_{m_1}(x)\quad\forall\,x\in\bar D.\tag{5.34}$$

By (5.32)-(5.34),

$$\big(\Phi^\delta(n_2T,0)u^r_{m_1}|_{\bar D}\big)(x)\le e^{(\lambda^r+\epsilon)n_2T}u^r_{m_1}(x)\quad\forall\, x\in\bar D.$$

This together with Remark 3.7 implies that for $0<\delta<\delta_2$,

$$\big(\Phi^\delta(nn_2T,0)u^r_{m_1}|_{\bar D}\big)(x)\le e^{(\lambda^r+\epsilon)nn_2T}u^r_{m_1}(x)\quad\forall\, x\in\bar D.\tag{5.35}$$

This together with Proposition 5.4 implies that

$$\tilde{\lambda}^\delta \le \lambda^r + \epsilon$$

for $0 < \delta < \delta_2$, that is, (5.28) holds.

Theorem 2.15 in the Dirichlet boundary condition case then follows from (5.19) and (5.28). □

### 5.2.3 Proof of Theorem 2.15 in the Neumann Boundary Condition Case

*Proof of Theorem 2.15 in the Neumann boundary condition case.* We assume $B_{r,b}u = B_{r,N}u$ in (1.19), and $D_c = \emptyset$ and $B_{n,b}u = B_{n,N}u$ in (1.20). The proof in the Neumann boundary condition case is similar to the arguments in the Dirichlet boundary condition case (it is simpler). For the completeness, we give a proof in the following. Without loss of generality, we may also assume that $a \in \mathcal{X}_2 \cap C^3(\mathbb{R} \times \mathbb{R}^N)$.

For the simplicity in notation, put

$$\Phi^r(nT, 0) = \Phi_2^r(nT, 0; a), \qquad \lambda^r = \lambda^r(a),$$

and

$$\Phi^\delta(nT, 0) = \Phi_2^\delta(nT, 0; a), \quad \tilde{\lambda}^\delta = \tilde{\lambda}^\delta(a).$$

By Propositions 5.4 and 5.5,

$$r(\Phi^r(T, 0)) = e^{\lambda^r T}, \tag{5.36}$$

and

$$r(\Phi^\delta(T, 0)) = e^{\tilde{\lambda}^\delta T}. \tag{5.37}$$

Let $u^r(\cdot)$ be a positive eigenfunction of $\Phi^r(T,0)$ corresponding to $r(\Phi^r(T,0))$. By regularity for parabolic equations, $u^r \in C^3(\bar{D})$. By Theorem 2.13, we have

$$\|\Phi^\delta(nT,0)u^r - \Phi^r(nT,0)u^r\|_{X_2} \leq C(nT,\delta),$$

where $C(nT,\delta) \to 0$ as $\delta \to 0$. This implies that for all $x \in \bar{D}$,

$$\begin{aligned}
\big(\Phi^\delta(nT,0)u^r\big)(x) &\geq \big(\Phi^r(nT,0)u^r\big)(x) - C(nT,\delta) \\
&= e^{\lambda^r nT}u^r(x) - C(nT,\delta) \\
&= e^{(\lambda^r-\epsilon)nT}e^{\epsilon nT}u^r(x) - C(nT,\delta),
\end{aligned} \tag{5.38}$$

and

$$\begin{aligned}
\big(\Phi^\delta(nT,0)u^r\big)(x) &\leq \big(\Phi^r(nT,0)u^r\big)(x) + C(nT,\delta) \\
&= e^{\lambda^r nT}u^r(x) + C(nT,\delta) \\
&= e^{(\lambda^r+\epsilon)nT}e^{-\epsilon nT}u^r(x) + C(nT,\delta).
\end{aligned} \tag{5.39}$$

Note that

$$\min_{x\in\bar{D}} u^r(x) > 0. \tag{5.40}$$

Hence for any $\epsilon > 0$, there is $n_1 > 1$ such that

$$\begin{cases}
e^{\epsilon n_1 T}u^r(x) \geq \frac{3}{2}u^r(x) \quad \forall\, x \in \bar{D}, \\
\\
e^{-\epsilon n_1 T}u^r(x) \leq \frac{1}{2}u^r(x) \quad \forall\, x \in \bar{D},
\end{cases} \tag{5.41}$$

and there is $\delta_0 > 0$ such that for any $0 < \delta < \delta_0$,

$$C(n_1T)\delta < \frac{1}{2}e^{(\lambda^r - \epsilon)n_1T}u^r(x) \quad \forall\, x \in \bar{D}. \tag{5.42}$$

By (5.38)-(5.42), we have that for any $0 < \delta < \delta_0$,

$$e^{(\lambda^r-\epsilon)n_1T}u^r(x) \le \big(\Phi^\delta(n_1T,0)u^r\big)(x) \le e^{(\lambda^r+\epsilon)n_1T}u^r(x) \quad \forall\, x \in \bar{D}.$$

This together with Remark 3.7 implies that for all $n \ge 1$,

$$e^{(\lambda^r-\epsilon)n_1nT}u^r(x) \le \big(\Phi^\delta(n_1nT,0)u^r\big)(x) \le e^{(\lambda^r+\epsilon)n_1nT}u^r(x) \quad \forall\, x \in \bar{D}.$$

It then follows that for any $0 < \delta < \delta_0$,

$$e^{(\lambda^r-\epsilon)T} \le r\big(\Phi^\delta(T,0)\big) \le e^{(\lambda^r+\epsilon)T}.$$

By Proposition 5.4, we have

$$|\tilde{\lambda}^\delta - \lambda^r| < \epsilon \quad \forall\, 0 < \delta < \delta_0.$$

Theorem 2.15 in the Neumann boundary condition case is thus proved. □

### 5.2.4 Proof of Theorem 2.15 in the Periodic Boundary Condition Case

*Proof of Theorem 2.15 in the periodic boundary condition case.* We assume $D = \mathbb{R}^N$, and $B_{r,b}u = B_{r,P}u$ in (1.19), and $B_{n,b}u = B_{n,P}u$ in (1.20). It can be proved by the same arguments as in the Neumann boundary condition case. □

## 5.3 Approximations of Positive Time Periodic Solutions of Random Dispersal KPP Type Evolution Equations by Nonlocal Dispersal KPP Type Evolution Equations

In this section, we study the approximation of the asymptotic dynamics of time periodic KPP type evolution equations with random dispersal by those of time periodic KPP type evolution equations with nonlocal dispersal. We first recall the existing results about time periodic positive solutions of KPP type evolution equations with random as well as nonlocal dispersal. Then we prove Theorem 2.16. Throughout this section, we assume that $D \subset \mathbb{R}^N$ is a bounded $C^{2+\alpha}$ domain or $D = \mathbb{R}^N$, and (H2), (H3) and $(\mathrm{H3})_\delta$ hold.

### 5.3.1 Asymptotic Behavior of KPP Type Evolution Equations

In this subsection, we present some basic known results for (1.21) and (1.22). Let $X_1^r$ and $X_i^r$ $(i = 2, 3)$ be defined as in (3.3) and (3.4), respectively. For $u_0 \in X_i^r$, let $(U(t,0)u_0)(\cdot) = u(t,\cdot;u_0)$, where $u(t,\cdot;u_0)$ is the solution to (1.21) with $u(0,\cdot;u_0) = u_0(\cdot)$ and $B_{r,b}u = B_{r,D}u$ when $i = 1$, $B_{r,b}u = B_{r,N}u$ when $i = 2$, and $B_{r,b}u = B_{r,P}u$ when $i = 3$. Similarly, for $u_0 \in X_i$, let $(U^\delta(t,0)u_0)(\cdot) = u^\delta(t,\cdot;u_0)$, where $u^\delta(t,\cdot;u_0)$ is the solution to (1.22) with $u^\delta(0,\cdot;u_0) = u_0(\cdot)$ and $D_c = \mathbb{R}^N \setminus \bar{D}$, $B_{n,b}u = B_{n,D}u$ when $i = 1$, $D_c = \emptyset$, $B_{n,b}u = B_{n,N}u$ when $i = 2$, and $B_{n,b}u = B_{n,P}u$ when $i = 3$.

**Proposition 5.7.** *(1) If $u_0 \ge 0$, solution $u(t,\cdot;u_0)$ to* (1.21) *with $u(0,\cdot;u_0) = u_0(\cdot)$ exists for all $t \ge 0$ and $u(t,\cdot;u_0) \ge 0$ for all $t \ge 0$.*

*(2) If $u_0 \ge 0$, solution $u(t,\cdot;u_0)$ to* (1.22) *with $u(0,\cdot;u_0) = u_0(\cdot)$ exists for all $t \ge 0$ and $u(t,\cdot;u_0) \ge 0$ for all $t \ge 0$.*

*Proof.* (1) Note that $u(\cdot) \equiv 0$ is a solution of (1.21) and $u(\cdot) \equiv M$ is a super-solution of (1.21) for $M \gg 1$. Then by Remark 3.7, there is $M \gg 1$ such that

$$0 \le u(t,x;u_0) \le M \quad \forall\, x \in \bar{D},\ t \in (0,t_{\max}),$$

where $(0, t_{\max})$ is the existence interval of $u(t, \cdot; u_0)$. This implies that we must have $t_{\max} = \infty$ and hence (1) holds.

(2) It can be proved by similar arguments as in (1). $\square$

**Proposition 5.8.** *(1)* (1.21) *has a unique globally stable positive time periodic solution* $u^*(t, x)$.

*(2)* (1.22) *has a unique globally stable time periodic positive solution* $u_\delta^*(t, x)$.

*Proof.* (1) See [67, Theorem 3.1] (see also [53, Theorems 1.1, 1.3]).

(2) See [56, Theorem E]. $\square$

**Remark 5.9.** *By Proposition 5.8(2), if there is* $u_0 \in X_i^+ \setminus \{0\}$ *such that* $(U^\delta(nT, 0)u_0)(\cdot) \geq u_0(\cdot)$ *for some* $n \geq 1$, *then we must have* $\lim_{n\to\infty}(U^\delta(nT, 0)u_0)(\cdot) = u_\delta^*(0, \cdot)$ *and hence*

$$(U^\delta(nT, 0)u_0)(\cdot) \leq u_\delta^*(0, \cdot).$$

*Similarly, if there is* $u_0 \in X_i^+ \setminus \{0\}$ *such that* $(U^\delta(nT, 0)u_0)(\cdot) \leq u_0(\cdot)$ *for some* $n \geq 1$, *then*

$$(U^\delta(nT, 0)u_0)(\cdot) \geq u_\delta^*(0, \cdot).$$

### 5.3.2 Proof of Theorem 2.16 in the Dirichlet Boundary Condition Case

In this subsection, we prove Theorem 2.16 in the Dirichlet boundary condition case. Throughout this subsection, we assume that $B_{r,b}u = B_{r,D}u$ in (1.21), and $D_c = \mathbb{R}^N \setminus \bar{D}$ and $B_{n,b}u = B_{n,D}u$ in (1.22).

*Proof of Theorem 2.16 in the Dirichlet boundary condition case.* First of all, note that it suffices to prove that for any $\epsilon > 0$, there is $\delta_0 > 0$ such that for $0 < \delta < \delta_0$,

$$u_\delta^*(t, x) - \epsilon \leq u^*(t, x) \leq u_\delta^*(t, x) + \epsilon \quad \forall\ t \in [0, T],\ x \in \bar{D}.$$

We first show that for any $\epsilon > 0$, there is $\delta_1 > 0$ such that for $0 < \delta < \delta_1$,

$$u^*(t,x) \le u^*_\delta(t,x) + \epsilon \quad \forall\, t \in [0,T],\ x \in \bar{D}. \tag{5.43}$$

To this end, choose a smooth function $\phi_0 \in C_c^\infty(D)$ satisfying that $\phi_0(x) \ge 0$ for $x \in D$ and $\phi_0(\cdot) \not\equiv 0$. Let $0 < \eta \ll 1$ be such that

$$u_-(x) := \eta\phi_0(x) < u^*(0,x) \quad \text{for} \quad x \in \bar{D}.$$

Then there is $\epsilon_0 > 0$ such that

$$u^*(0,x) \ge u_-(x) + \epsilon_0 \quad \text{for} \quad x \in \operatorname{supp}(\phi_0). \tag{5.44}$$

By Proposition 5.8, there is $N \gg 1$ such that

$$\big(U(NT,0)u_-\big)(x) \ge u^*(NT,x) - \epsilon_0/2 = u^*(0,x) - \epsilon_0/2 \quad \forall\, x \in \bar{D}.$$

By Theorem 2.13, there is $\bar{\delta}_1 > 0$ such that for $0 < \delta < \bar{\delta}_1$, we have

$$\big(U^\delta(NT,0)u_-\big)(x) \ge \big(U(NT,0)u_-\big)(x) - \epsilon_0/2 \quad \forall\, x \in \bar{D}.$$

Hence for $0 < \delta < \bar{\delta}_1$,

$$\big(U^\delta(NT,0)u_-\big)(x) \ge u^*(0,x) - \epsilon_0 \quad \forall\, x \in \bar{D}. \tag{5.45}$$

Note that

$$\big(U^\delta(NT,0)u_-\big)(x) \ge 0 \quad \forall\, x \in \bar{D}.$$

It then follows from (5.44) and (5.45) that for $0 < \delta < \bar{\delta}_1$,

$$\big(U^\delta(NT,0)u_-\big)(x) \geq u_-(x) \quad \forall\, x \in \bar{D}.$$

This together with Proposition 5.8 (2) implies that

$$\big(U^\delta(NT,0)u_-\big)(x) \leq u_\delta^*(0,x) \quad \forall\, x \in \bar{D} \tag{5.46}$$

(see Remark 5.9).

By Proposition 5.8 (1) again, for $n \gg 1$,

$$u^*(t,x) \leq (U(nNT+t,0)u_-)(x) + \epsilon/2 \ \ \forall\, t \in [0,T],\ x \in \bar{D}. \tag{5.47}$$

Fix an $n \gg 1$ such that (5.47) holds. By Theorem 2.13, there is $0 < \tilde{\delta}_1 \leq \bar{\delta}_1$ such that for $0 < \delta < \tilde{\delta}_1$,

$$(U(nNT+t,0)u_-)(x) \leq (U^\delta(nNT+t,0)u_-)(x) + C_1(\delta), \tag{5.48}$$

where $C_1(\delta) \to 0$ as $\delta \to 0$. By (5.46), Remark 3.7, and Proposition 5.8 (2),

$$\big(U^\delta(nNT+t,0)u_-\big)(x) \leq \big(U^\delta(t,0)u_\delta^*(0,\cdot)\big)(x) = u_\delta^*(t,x) \tag{5.49}$$

for $t \in [0,T]$ and $x \in \bar{D}$. Let $0 < \delta_1 \leq \tilde{\delta}_1$ be such that

$$C_1(\delta) < \epsilon/2 \quad \forall\, 0 < \delta < \delta_1. \tag{5.50}$$

(5.43) then follows from (5.47)-(5.50).

Next, we need to show for any $\epsilon > 0$, there is $\delta_2 > 0$ such that for $0 < \delta < \delta_2$,

$$u^*(t,x) \geq u^*_\delta(t,x) - \epsilon \quad \forall\, t \in [0,T],\ x \in \bar{D}. \tag{5.51}$$

To this end, choose a sequence of open sets $\{D_m\}$ with smooth boundaries such that $D_1 \supset D_2 \supset D_3 \cdots \supset D_m \supset \cdots \supset \bar{D}$, and $\bar{D} = \cap_{m\in\mathcal{N}} D_m$. According to Corollary 5.11 in [2], $D_m \to D$ regularly and all assertions of Theorem 5.5 in [2] hold.

Consider

$$\begin{cases} \partial_t u = \Delta u + uf(t,x,u), & x \in D_m, \\ u(t,x) = 0, & x \in \partial D_m. \end{cases} \tag{5.52}$$

Let $U_m(t,0)u_0 = u(t,\cdot;u_0)$, where $u(t,\cdot;u_0)$ is the solution to (5.52) with $u(0,\cdot;u_0) = u_0(\cdot)$. By Proposition 5.8, (5.52) has a unique time periodic positive solution $u^*_m(t,x)$. We first claim that

$$\lim_{m\to\infty} u^*_m(t,x) \to u^*(t,x) \text{ uniformly in } t \in [0,T] \text{ and } x \in \bar{D}. \tag{5.53}$$

In fact, it is clear that $u^* \in C(\mathbb{R}\times\bar{D},\mathbb{R})$ and $u^*_m \in C(\mathbb{R}\times\bar{D}_m,\mathbb{R})$. By [22, Theorem 7.1],

$$\sup_{t\in\mathbb{R}} \|u^*_m(t,\cdot) - u^*(t,\cdot)\|_{L^q(D)} \to 0 \quad \text{as} \quad m \to \infty$$

for $1 \leq q < \infty$. Let $a(t,x) = f(t,x,u^*(t,x))$ and $a_m(t,x) = f(t,x,u^*_m(t,x))$. Then $u^*(t,x)$ and $u^*_m(t,x)$ are time periodic solutions to the following linear parabolic equations,

$$\begin{cases} u_t = \Delta u + a(t,x)u, & x \in D, \\ u(t,x) = 0, & x \in \partial D, \end{cases} \tag{5.54}$$

and

$$\begin{cases} u_t = \Delta u + a_m(t,x)u, & x \in D_m, \\ u(t,x) = 0, & x \in \partial D_m, \end{cases} \tag{5.55}$$

respectively.

Observe that there is $M > 0$ such that $\|a\|_{L^\infty(D)} < M$, $\|a_m\|_{L^\infty(D_m)} < M$, $\|u^*(0,\cdot)\|_{L^\infty(D)} < M$, and $\|u_m^*(0,\cdot)\|_{L^\infty(D_m)} < M$. By [3, Theorem D(1)], $\{u_m^*(t,x)\}$ is equi-continuous on $[T,2T]\times\bar{D}$. Without loss of generality, we may then assume that $u_m^*(t,x)$ converges uniformly on $[T,2T]\times\bar{D}$. But $u_m^*(t,\cdot)\to u^*(t,\cdot)$ in $L^q(D)$ uniformly in $t$. We then must have

$$u_m^*(t,x)\to u^*(t,x) \quad \text{as} \quad n\to\infty$$

uniformly in $(t,x)\in[T,2T]\times\bar{D}$. This together with the time periodicity of $u_m^*$ shows that (5.53) holds.

Next, for any $\epsilon > 0$, fix $m \gg 1$ such that

$$u^*(t,x)\geq u_m^*(t,x)-\epsilon/3 \quad \forall\ t\in[0,T],\ x\in\bar{D}. \tag{5.56}$$

Choose $M \gg 1$ such that for $0<\delta\leq 1$,

$$Mu_m^*(t,x)\geq u_\delta^*(t,x) \ \ \forall\ t\in[0,T],\ x\in\bar{D}. \tag{5.57}$$

Let

$$u_m^+(x)=Mu_m^*(0,x), \quad u^+(x)=u_m^+(x)|_{\bar{D}}.$$

By Proposition 5.8, for fixed $m$ and $\epsilon$, there exists $N\gg 1$, such that

$$u_m^*(t,x)\geq \big(U_m(NT+t,0)u_m^+\big)(x)-\epsilon/3 \quad \forall\ t\in[0,T],\ x\in\bar{D}. \tag{5.58}$$

By Theorem 2.13, there is $0<\tilde{\delta}_2<1$ such that for $0<\delta<\tilde{\delta}_2$,

$$(U_m(NT+t,0)u_m^+)(x)\geq (U_m^\delta(NT+t,0)u_m^+)(x)-C_2(\delta) \quad \forall t\in[0,T],\ x\in D_m, \tag{5.59}$$

where $C_2(\delta) \to 0$ as $\delta \to 0$ and $(U_m^\delta(t,0)u_0)(\cdot) = u(t,\cdot;u_0)$ is the solution to

$$\begin{cases} u_t(t,x) = \nu_\delta \left[\int_{D_m} k_\delta(x-y)u(t,y)dy - u(t,x)\right] + u(t,x)f(t,x,u(t,x)), & x \in \bar{D}_m \\ u(0,x) = u_0(x), & x \in \bar{D}_m. \end{cases}$$

Let $0 < \delta_2 < \tilde{\delta}_2$ be such that for $0 < \delta < \delta_2$,

$$C_2(\delta) < \epsilon/3. \tag{5.60}$$

By Remark 3.7, for $x \in \bar{D}$ we have

$$(U_m^\delta(NT+t,0)u_m^+)(x) \geq (U^\delta(NT+t,0)u^+)(x),$$

and

$$(U^\delta(NT+t,0)u^+)(x) = (U^\delta(t,0)U^\delta(NT,0)u^+)(x) \geq (U^\delta(t,0)u_\delta^*(0,\cdot))(x) = u_\delta^*(t,x).$$

This together with (5.56), (5.58), (5.59), and (5.60) implies (5.51).

So, for any $\epsilon > 0$, there exists $\delta_0 = \min\{\delta_1, \delta_2\}$, such that for any $\delta < \delta_0$, we have

$$|u^*(t,x) - u_\delta^*(t,x)| \leq \epsilon \quad \text{uniformly in } t > 0 \text{ and } x \in \bar{D}.$$

□

### 5.3.3 Proof of Theorem 2.16 in the Neumann Boundary Condition Case

We assume $B_{r,b}u = B_{r,N}u$ in (1.19), and $D_c = \emptyset$ and $B_{n,b}u = B_{n,N}u$ in (1.20). The proof in the Neumann boundary condition case is similar to the arguments in the Dirichlet boundary condition case (it is indeed simpler). For completeness, we provide a proof.

*Proof of Theorem 2.16 in the Neumann boundary condition case.* For completeness, we provide a proof.

First, we show that for any $\epsilon > 0$, there is $\delta_1 > 0$ such that

$$u^*(t,x) \le u_\delta^*(t,x) + \epsilon \quad \forall\, t \in [0,T],\ x \in \bar{D}. \tag{5.61}$$

Choose a smooth function $u_- \in C^\infty(\bar{D})$ with $u_-(\cdot) \ge 0$ and $u_-(\cdot) \not\equiv 0$ such that

$$u_-(x) < u^*(0,x) \quad \forall\, x \in \bar{D}.$$

Then there is $\epsilon_0 > 0$ such that

$$u^*(0,x) \ge u_-(x) + \epsilon_0 \quad \forall\, x \in \bar{D}. \tag{5.62}$$

By Proposition 5.8 (1), there is $N \gg 1$ such that

$$\big(U(NT,0)u_-\big)(x) \ge u^*(0,x) - \epsilon_0/2 \quad \forall\, x \in \bar{D}. \tag{5.63}$$

By Theorem 2.13, there is $\bar{\delta}_1 > 0$ such that for $0 < \delta < \bar{\delta}_1$,

$$(U^\delta(NT,0)u_-)(x) \ge (U(NT,0)u_-)(x) - \epsilon_0/2 \quad \forall\, x \in \bar{D}. \tag{5.64}$$

By (5.62), (5.63) and (5.64),

$$\big(U^\delta(NT,0)u_-\big)(x) \ge u_-(x) \quad \forall\, x \in \bar{D},$$

and then by Proposition 5.8 (2),

$$\big(U^\delta(NT,0)u_-\big)(x) \le u_\delta^*(0,x) \quad \forall\, x \in \bar{D}. \tag{5.65}$$

By Proposition 5.8 (1) again, for any given $\epsilon > 0$, $n \gg 1$, and $0 < \delta < \bar{\delta}_1$,

$$u^*(t,x) \le (U(nNT+t,0)u_-)(x) + \epsilon/2 \quad \forall\, t \in [0,T],\ x \in \bar{D}. \tag{5.66}$$

By Theorem 2.13, there is $0 < \delta_1 \le \bar{\delta}_1$ such that for $\delta < \delta_1$,

$$(U(nNT+t,0)u_-)(x) \le (U^\delta(nNT+t,0)u_-)(x) + \frac{\epsilon}{2} \quad \forall\, t \in [0,T],\ x \in \bar{D}. \tag{5.67}$$

By Remark 3.7 and (5.65), we have

$$(U^\delta(nNT+t,0)u_-)(x) = (U^\delta(t,0)U^\delta(nNT,0)u_-)(x) \le (U^\delta(t,0)u_\delta^*(t,\cdot))(x) = u_\delta^*(t,x) \tag{5.68}$$

for $t \in [0,T]$ and $x \in \bar{D}$. (5.61) then follows from (5.66)-(5.68).

Next, we show that for any $\epsilon > 0$, there is $\delta_2 > 0$ such that for $0 < \delta < \delta_2$,

$$u^*(t,x) \ge u_\delta^*(t,x) - \epsilon \quad \forall\, t \in [0,T],\ x \in \bar{D}. \tag{5.69}$$

Choose $M \gg 1$ such that $f(t,x,M) < 0$ for $t \in \mathbb{R}$ and $x \in \bar{D}$. Put

$$u^+(x) = M \quad \forall\, x \in \bar{D}.$$

Then for all $\delta > 0$,

$$u_\delta^*(0,x) \le u^+(x) \quad \forall\, x \in \bar{D}. \tag{5.70}$$

By Proposition 5.8, there is $N \gg 1$ such that

$$u^*(t,x) \ge (U(NT+t,0)u^+)(x) - \epsilon/2 \quad \forall\, t \in [0,T],\ x \in \bar{D}. \tag{5.71}$$

By Theorem 2.13, there are $\delta_2 > 0$ such that for $0 < \delta < \delta_2$,

$$(U(NT+t,0)u^+)(x) \geq (U^\delta(NT+t,0)u^+)(x) - \frac{\epsilon}{2} \quad \forall\, t \in [0,T],\ x \in \bar{D}. \tag{5.72}$$

By (5.70),

$$(U^\delta(NT+t,0)u^+)(x) = (U^\delta(t,0)U^\delta(NT,0)u^+)(x) \geq (U^\delta(t,0)u^*_\delta(t,\cdot))(x) = u^*_\delta(t,x) \tag{5.73}$$

for $t \in [0,T]$ and $x \in \bar{D}$. (5.69) then follows from (5.71)-(5.73).

So, for any $\epsilon > 0$, there exists $\delta_0 = \min\{\delta_1, \delta_2\}$, such that for any $\delta < \delta_0$, we have

$$|u^*(t,x) - u^*_\delta(t,x)| \leq \epsilon \quad \text{uniform in } t > 0 \text{ and } x \in \bar{D}.$$

□

### 5.3.4 Proof of Theorem 2.16 in the Periodic Boundary Condition Case

*Proof of Theorem 2.16 in the periodic boundary condition case.* We assume $D = \mathbb{R}^N$, and $B_{r,b}u = B_{r,P}u$ in (1.19), and $B_{n,b}u = B_{n,P}u$ in (1.20). It can be proved by the similar arguments as in the Neumann boundary condition case. □

## 5.4 Applications to the Effect of the Rearrangements with Equimeasurability on Principal Spectrum Point of Nonlocal Dispersal Operators

In this section, we will apply the approximation results established in this Chapter to the effect of the rearrangements with equimeasurability on principal spectrum point of nonlocal dispersal operators. First, we show the proof of Theorem 2.18.

*Proof of Theorem 2.18.* In the case of $D = D^\sharp$, $a(\cdot) = a_\sharp(\cdot)$, and $u(\cdot) = u_\sharp(\cdot)$, Theorem 2.18 holds trivially. Otherwise, by (2.24), we have

$$\tilde{\lambda}_i^r(a_\sharp) > \tilde{\lambda}_i^r(a) \quad \text{for } \delta \ll \delta_0.$$

And by Theorem 2.15, we have

$$\lim_{\delta \to 0} \tilde{\lambda}_i^\delta(a) = \lambda_i^r(a)$$

and

$$\lim_{\delta \to 0} \tilde{\lambda}_i^\delta(a_\sharp) = \lambda_i^r(a_\sharp).$$

Hence, we have

$$\tilde{\lambda}_i^\delta(a_\sharp) \geq \tilde{\lambda}_i^\delta(a) \quad \text{for } \delta \ll \delta_0.$$

$\square$

**Remark 5.10** (Effect of the rearrangements with equimeasurability on principal spectrum point of general nonlocal dispersal operators)**.**

*(1) Consider* (2.26) *and* (2.27) *for general kernel $k(\cdot)$ and dispersal rate $\nu$ in the Dirichlet boundary condition case. We denote the principal spectrum point of* (2.26) *(independent of $\delta$) and* (2.27) *(independent of $\delta$) by $\tilde{\lambda}_1(a)$ and $\tilde{\lambda}_1(a_\sharp)$. Assume that $k(\cdot)$ is symmetric with respect to* 0*. Let $a_\sharp(\cdot)$, $k_\sharp(\cdot)$ and $\bar{D}^\sharp$ be the Schwarz symmetrization of $a(\cdot)$, $k(\cdot)$ and $\bar{D}$, respectively. Then we have*

$$\tilde{\lambda}_1(a) \leq \tilde{\lambda}_1(a_\sharp). \tag{5.74}$$

*In fact, by Proposition 4.7 and rearrangement inequalities (see [1] for detail), we have*

$$
\begin{aligned}
\tilde{\lambda}_1(a) &= \sup_{\{u \mid \|u\|_{X_1}=1\}} \nu \int\int_{D\times D} k(x-y)u(y)u(x)dydx - \nu + \int_D a(x)u^2(x)dx \\
&\leq \sup_{\{u_\sharp \mid \|u_\sharp\|_{X_1}=1\}} \int\int_{D^\sharp\times D^\sharp} k(x-y)u_\sharp(y)u_\sharp(x)dydx - \nu + \int_{D^\sharp} a_\sharp(x)u_\sharp^2(x)dx \\
&= \tilde{\lambda}_1(a_\sharp).
\end{aligned}
$$

*(2) For* (2.26) *and* (2.27) *with general kernels* $k(\cdot)$ *and dispersal rate* $\nu$ *in the Neumann boundary condition case, we have similar result as in the Dirichlet boundary condition case.*

*(3) For* (2.26) *and* (2.27) *with general kernels* $k(\cdot)$ *and dispersal rate* $\nu$ *in the periodic boundary condition case, it is open to get similar result as in* (5.74)*.*

# Chapter 6

## Concluding Remarks, Problems, and Future Plans

In this dissertation, I studied two dynamical issues. One is about the principal spectrum of nonlocal dispersal operators and its applications in nonlocal dispersal evolution equations, and the other is about the approximations of random dispersal operators and equations by nonlocal dispersal operators and equations from three points of view. Both are theoretically and practically important. The results of eigenvalue problems of nonlocal dispersal operators are applied to a two species competition system, the approximation results are applied to the effects of rearrangement with nonlocal dispersals. The two applications cast a new light on diffusive systems arising in ecology or biology.

More precisely, regarding to the first dynamical issue, we prove Theorem 2.4, Theorem 2.6, Theorem 2.8 and Theorem 2.12 as an application. Although the semigroups generated by nonlocal operators are not compact, we are able to convert the time homogeneous nonlocal operator into a compact operator and study the existence of its principal eigenvalue. There are examples showing that there is no principal eigenvalue to some nonlocal operator. However, in some circumstances, the principal spectrum plays the same role as the principal eigenvalue. So we focus on the dependence of the principal spectrum points of nonlocal dispersal operators on the underlying parameter with Dirichlet, Neumann, and periodic types of boundary condition in a unified way. Finally, in the model of population dynamics of two species competing system, we show that the species diffusing nonlocally with Neumann type boundary condition drives the species adopting Dirichlet type boundary condition extinct. Biologically, individuals diffusing inside $D$ (Neumann type boundary condition) are more likely to survive than those living in a habitat surrounded by a hostile environment (Dirichlet type boundary condition).

On the second dynamical issue, we prove Theorem 2.13, Theorem 2.15, Theorem 2.16 and Theorem 2.18 as an application. From the formal relation between nonlocal operators and Laplacian operators, we are inspired to study the approximation of random dispersal equations by its nonlocal counterparts from other perspectives. Theorem 2.13 is fundamental to the investigation of other approximations, since theorem 2.13 build the connection of solution operators with random dispersal and nonlocal dispersal. By the spectral mapping theorem, the principal spectrum points and principal eigenvalues are related to the solution operators. Hence, we have the approximations of principal eigenvalues of random dispersal operators by principal spectrum points of nonlocal dispersal operators. Next, based on the previous two theorems, we show the approximation of asymptotic dynamics of KPP type evolution equations with random dispersal by that with nonlocal dispersal. Finally, to see the advantage of approximation results, we apply them to the effect of the rearrangements on principal spectrum point of nonlocal dispersal operators, and prove Theorem 2.18. Hence, as long as we know some results in the random models, we should have the similar results in the nonlocal models, when the dispersal distance $\delta$ of the nonlocal kernel is small.

Along the line of my dissertation, there are several important problems which are not well understood yet. We discuss the following three problems.

**Problem 1** *In [57], the authors proved the spreading speeds and traveling waves of nonlocal monostable equations in time and space periodic habitats, so it is natural to ask whether the results hold in a cylindrical domain, such that in one direction, it is periodic and in the other direction, either Dirichlet or Neumann type boundary condition is prescribed.*

It seems like there should be no difficulty in extending the results to the cylinder domain. But it will be interesting to prove the existence of traveling waves with speed $c = c^*(\xi)$ and uniqueness and stability of traveling waves in the case that $f$ is both space and time periodic.

**Problem 2** *In [43], the authors studied the principal eigenvalue of a general random operator with indefinite weight on cylindrical domains. Biologically, this problem is motivated by the question of determining the optimal spatial arrangement of favorable and unfavorable regions* *for a species to survive. So it will be worthwhile to study the principal spectrum point of a nonlocal operator and find the optimal spatial arrangement for a species to survive.*

The principal spectrum point plays the same role as the principle eigenvalue in some situations. The survival of a species is determined by the magnitude of the principal spectrum point of nonlocal dispersal operators.

**Problem 3** *In [40], authors study an evolution equation with nonlinear nonlocal operators as follows*

$$\partial_t u = \int_{\mathbb{R}^N} k(x-y)|u(t,y)-u(t,x)|^{p-2}(u(t,y)-u(t,x))dy, \quad x \in \mathbb{R}^N, \tag{6.1}$$

*and they study the decay estimates for* (6.1) *in the whole space. We can consider the random counterparts*

$$\partial_t u = \nabla \cdot |\nabla u(t,x)|^{p-2}, \quad x \in \mathbb{R}^N, \tag{6.2}$$

*and investigate the approximations of nonlinear random dispersal operators/equations by nonlinear nonlocal dispersal operators/equations from many other points of view.*

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