Efficient Numerical Algorithms for Solving Nonlinear Filtering Problems

by

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Abstract

Examples of nonlinear filtering problems arise in biology, mathematical finance, signal processing, image processing, target tracking and many engineering applications. Commonly used numerical simulation methods are the Bayesian filter which is derived from the Bayesian formula and the Zakai filter which is related to a system of stochastic partial differential equation known as "the Zakai equation".

This dissertation mainly focuses on developing and analysing novel, efficient numerical algorithms for solving nonlinear filtering problems. We first introduce a novel numerical algorithm which lies in the general framework on the Bayesian filter. The algorithm is constructed based on samples of the current state obtained by solving the state equation implicitly. We call this algorithm the "implicit filter method". Rigorous analysis has been done to prove the convergence of the algorithm. Through numerical experiments we show that our algorithm is more accurate than the Kalman filter and more stable than the particle filter.

In the second effort of this work, we propose a hybrid numerical algorithm for the Zakai filter to solve nonlinear filtering problems efficiently. The algorithm combines the splitting-up finite difference scheme and hierarchical sparse grid method to solve moderately high dimensional nonlinear filtering problems. When applying hierarchical sparse grid method to approximate bell-shaped solutions in most applications of nonlinear filtering problem, we introduce a logarithmic approximation to reduce the approximation errors. Some space adaptive methods are also introduced to make the algorithm more efficient. Numerical experiments are carried out to demonstrate the performance and efficiency of our algorithm.

In this dissertation, we also develop high order numerical approximation methods for backward doubly stochastic differential equations (BDSDEs). One of the most important properties of BDSDEs is it's equivalence to the Zakai equation. In this connection, our numerical approximation methods for BDSDEs can be considered as efficient numerical approaches to solving nonlinear filtering problems. The convergence order is proved through rigorous error analysis for each algorithm. Numerical experiments are carried out to verify the theoretical results and to demonstrate the efficiency of the proposed numerical scheme.

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Chapter 1

Introduction

Stochastic filtering theory was first established in the early 1940s due to the pioneering work by Norbert Wiener [79, 80] and Andrey N. Kolmogorov [46]. The goal of data assimilation of the stochastic filtering problem is to obtain good estimates of the state of a stochastic dynamic system model based on noisy partial observations. Examples of stochastic filtering problems arise in biology [4], mathematical finance [11, 31, 37, 69], signal processing [2, 3, 74], image processing [12, 71, 91], target tracking [13, 20, 26, 29, 62, 85] and many engineering applications. The major breakthrough of the classic filtering problem was the landmark work of Kalman and Bucy in 1960s [45], and subsequent Kalman-Bucy filter in 1961. Most people now call this type of filter theory "Kalman filter"(see also [28, 40, 49, 56, 57, 66, 72, 77]).

For decades, Kalman filter has been one of the dominant tools in solving filtering problems. According to the Kalman filter, under standard assumption of linearity and Gaussinity of the noise, the conditional distribution of the state, given the observations, is Gaussian. This conditional distribution gives the best estimate of the statistical description of the state of the system based on all the available observation information up to the current time. The power of Kalman filter lies in it's simplicity and accuracy. Largely because of the success of Kalman filters, linear and nonlinear filters have been applied in the various engineering and scientific areas, including communications such as positioning in wireless networks, signal processing such as tracking and navigation, economics and business, and many others.

In most of the practical application problems, however, linearity assumption is not valid because of the nonlinearity in the model specification process as well as the observation process. For example, in typical atmospheric data assimilation problems, the dynamic system is described by a rather complex system of equations, which is highly nonlinear. In bearing-only tracking problems, depending on how the measurements are obtained, the observation could also have a nonlinear relation with the state process.

Numerous nonlinear filtering methods have been proposed and developed to overcome the linearity limitation of Kalman filter. Two of the widely used methods for nonlinear filtering problems are the Bayesian filter and Zakai filter.

The Bayesian filter is based on the Baysian theory. It is originally discovered by the British researcher Thomas Bayes in 1763. The most well-known Bayesian filter methods include the extended Kalman filter (EKF) and Particle Filter method (PFM). For the EKF, the state equation and the observation equation are linearized so that the standard Kalman filter can be applied [7, 14, 27, 33, 42, 43, 44, 47, 63, 73]. The central theme behind the PFM involves the representation of the desired probability density function (PDF) of the system state with a set of adaptively selected random samples [15, 18, 21, 24, 38, 59, 58, 65, 76, 78]. Since PFMs are essentially sequential Monte Carlo methods, with sufficiently large number of samples, PFMs can provide an accurate representation of the PDF for the nonlinear filtering solution. While the aforementioned methods have been remarkably successful in attacking the nonlinear filtering problem, each of them has its drawbacks and limitations. For instance, when the state equation describing the signal process and the observation equation are highly non-linear, the EKF can give particularly poor performance. PFM has a number of advantages over EFK, including its ability to represent arbitrary densities, adaptively focus on the most probable regions of state-space. However, it also has a number of disadvantages, including high computational complexity, degeneracy for long period simulation and its difficulty of determining optimal number of particles. The first effort of this work is to construct a new algorithm for numerical simulations of nonlinear filtering problems using the Bayesian filtering theory. For each time recursive step, we have two stages: the prediction state and the update stage. The prediction stage gives the estimation for the prior PDF of the future state based on the currently available observation information while update stage gives the posterior PDF from the updated observation information and the the result obtained in the prediction stage. However, instead of attempting to search for a representation of the PDF as in PFM, we approximate the PDF as a function over a grid in state space. Specifically, at the prediction stage, we attempt to seek the predicted pdf of the future state variable through a Monte Carlo method by evaluating the conditional expectation of the future state with respect to the current stage. Since the sample points for the current state is computed by solving the state equation implicitly, we name our method as an “implicit filter method”. The following two items summarize the novelty of our approach.

(i) We propose an accurate implicit scheme for prediction purpose. The implicit scheme has stabilizing effect on the proposed numerical algorithm. This is verified in our numerical experiments.

(ii) Based on the Bayesian theory, we apply a novel Monte-Carlo like method to approximate the conditional expectation in the update stage to compute the prior PDF.

The Zakai filter represents the PDF of the nonlinear filtering solution through the solution of a parabolic-type stochastic partial differential equation, known as the Zakai equation [82]. Similar to the PFM, the Zakai filter allows one to accurately compute conditional distributions. A number of numerical algorithms have been proposed to solve Zakai equations [1, 10, 22, 23, 30, 32, 36, 39, 67, 75, 85]. One of the most effective methods is the splitting-up approximation scheme [10, 30, 85] where the original Zakai equation is split into a second order deterministic PDE, related with the prediction step, and a degenerated second order stochastic PDE in the update step. In the numerical simulation process, a prior PDF is obtained by solving the deterministic PDE at the prediction; then this prior PDF is updated following a posteriori criterion. The main drawback of the Zakai filter is that the numerical approximation is grid based, thus it suffers the so-called “curse of dimensionality” since the computing cost increases exponentially as the dimension of the system increases. Another difficulty related to solutions of the Zakai equation is that the domain is the whole space

$\mathbb{R}^d$. To address these challenges, we propose the construction of an efficient hybrid numerical algorithm which combines the advantages of the splitting-up approximation scheme for the Zakai equation, a hierarchical sparse grid method [41, 50, 55, 68, 70, 81, 83, 84] for moderately high dimensional nonlinear filtering problems to compute the numerical solution of the Zakai filter, and an importance sampling method to adaptively construct a bounded domain at each time step of the temporal discretization. Specifically, this enables us to use the splitting-up finite difference scheme to solve the Zakai equation on the sparse grid of the bounded domain. The hierarchical sparse grid method, which was originally created to approximate multi-variable functions, uses only $O(n(\log n)^{d-1})$ number of grid points instead of $O(n^d)$ number of grid points, required by the standard full-grid approximation.

Backward doubly stochastic differential Equations (BDSDEs) were introduced as Feynman-Kac type probabilistic representations of semi-linear parabolic stochastic partial differential equations (SPDEs), which are generalized Zakai equations [61]. In this connection, numerical approximation methods for simulating solutions of BDSDEs are also numerical approximation methods for solving Zakai type equations. Thus it can be considered as alternative numerical approaches for nonlinear filtering problems. In this work, we propose efficient numerical algorithms for approximating solutions of BDSDEs. Several effective numerical approaches for backward stochastic differential equations (BSDEs) and forward backward stochastic differential equations (FBSDEs) have been proposed in the last decade, including primary schemes for BSDEs ([5, 19, 52, 86]), the four-step scheme for FBSDEs [9, 53, 54, 51]), and the θ-scheme with high convergence rate for BSDEs ([87, 88, 89, 90]). In comparison, efficient high order numerical algorithms for BDSDEs are not well developed. Obviously solving BDSDEs numerically is more difficult than solving SDEs and BSDEs as they contain two Brownian motions. In this work, we first construct a half order algorithm using the simple Euler method. It is much more involved to construct higher order algorithms. The bottle neck is the difficulty in approximating the forward and backward Itô integrals with high order quadratures. To tackle this difficulty, we propose to use an Itô-Taylor formula for two-sided stochastic integrals [60, 61] in order to obtain a higher order quadrature for the backward Itô integral; for the forward stochastic integral, we propose to use the variational equations for BDSDEs [61] to derive high order quadrature rule. It is worth noting that although our focus is the on construction of a first order algorithm, the methodology developed in this work can be used to obtain even higher order algorithms.

The outline of this work is as follows. In Chapter 2, we give a brief overview of the probability theory that will be used throughout the rest of the chapters. We will also introduce the mathematical definition of the nonlinear filtering problems and some existing numerical approximation methods for nonlinear filtering problems. In Chapter 3, we present our novel "implicit filter algorithm" for nonlinear filtering problems. We also give convergence analysis which shows weak convergence of our algorithm. Numerical experiments demonstrate that our algorithm is more accurate than the Kalman filter and more stable than the particle filter. In Chapter 4, we construct a hybrid finite difference algorithm for the Zakai equation to solve nonlinear filtering problems. The algorithm combines the splitting-up finite difference scheme and hierarchical sparse grid method to solve moderately high dimensional nonlinear filtering problems. A space adaptive method is introduced to make the algorithm more efficient. Numerical experiments are carried out to demonstrate the performance and efficiency of our algorithm. In Chapter 5, we focus on numerical algorithms of BDSDEs, which can be considered as an alternative numerical approach for nonlinear filtering problems. We first introduce a half order convergence algorithm based on Euler approximation for stochastic integrals. Then, we develop a first order convergence scheme by using two-sided Itô Taylor expansion. For each algorithm, we give rigorous error analysis and numerical experiments to verify the convergence rates of our algorithms.

Chapter 2

Nonlinear Filtering problems and Existing Numerical Approaches

In this Chapter, we first give some definitions from general probability theory. Then, we introduce the mathematical formulations of nonlinear filtering problems and some of the most well-known numerical schemes that have been studied in the literature. There are mainly two numerical approaches to solve the nonlinear filtering problem. One is the Bayesian filter, which is based on the Bayes rule; the other is the Zakai filter, which is based on the Zakai equation, a stochastic evolution equation.

2.1 Mathematical Preliminaries

Definition 1 *If Ω is a given set, then a* ***σ-algebra*** *$\mathcal{F}$ on Ω is a family $\mathcal{F}$ of subsets of Ω with the following properties:*

(i) $\emptyset \in \mathcal{F}$

(ii) $F \in \mathcal{F} \Rightarrow F^C \in \mathcal{F}$*, where* $F^C = \Omega \backslash F$ *is complement of* F *in* Ω

(iii) $A_1, A_2, \cdots \in \mathcal{F} \Rightarrow A := \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

The pair $(\Omega, \mathcal{F})$ is called a measurable space. A ***probability measure*** *P on a measurable space $(\Omega, \mathcal{F})$ is a function $P : \mathcal{F} \to [0, 1]$ such that*

(a) $P(\emptyset) = 0$, $P(\Omega) = 1$

(b) if $A_1, A_2, \cdots \in \mathcal{F}$ *and* $\{A_i\}_{i=1}^{\infty}$ *is disjoint (i. e.* $A_i \cap A_j = \emptyset$ *if* $i \neq j$*) then*

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

The triple $(\Omega, \mathcal{F}, P)$ *is called a* ***probability space****.p*

Given any family $\mathcal{U}$ of subsets of Ω there is a smallest σ-algebra $\mathcal{H}_{\mathcal{U}}$ containing $\mathcal{U}$, namely

$$\mathcal{H}_{\mathcal{U}} = \cap\{\mathcal{H}; \mathcal{H} \ \sigma\text{-algebra of } \Omega,\ \mathcal{U} \subset \mathcal{H}\}.$$

We call $\mathcal{H}_{\mathcal{U}}$ the σ-algebra generated by $\mathcal{U}$.

If $(\Omega, \mathcal{F}, P)$ is a given probability space, then a function $f : \Omega \to \mathbb{R}^n$ is called **$\mathcal{F}$-measurable** if

$$f^{-1}(U) := \{\omega \in \Omega;\ f(\omega) \in U\} \in \mathcal{F}$$

for all open sets $U \in \mathbb{R}^n$.

If $X : \Omega \to \mathbb{R}^n$ is any function, then the σ-algebra $\mathcal{H}_X$ generated by X is the smallest σ-algebra on Ω containing all sets

$$X^{-1}(U); \quad U \subset \mathbb{R}^n \text{ open}.$$

Definition 2 *A* ***random variable*** *X is an $\mathcal{F}$-measurable function $X : \Omega \to \mathbb{R}^n$. Every random variable induces a probability measure μ_X on $\mathbb{R}^n$ defined by*

$$\mu_X(B) = P(X^{-1}(B)).$$

μ_X is called the distribution of X.

If $\int_\Omega |X(\omega)| dP(\omega) < \infty$ then the number

$$E[X] := \int_\Omega X(\omega) dP(\omega) = \int_{\mathbb{R}^n} x d\mu_X(x)$$

is called the ***expectation*** *of X (w.r.t. P).*

Definition 3 *A* ***stochastic process*** *is a parameterized collection of random variables $\{X_t\}_{t\in T}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ and assuming values in $\mathbb{R}^n$.*

Next, we introduce a general form of the stochastic differential equations. First, we need the following definition of Brownian motion

Definition 4 ***Brownian motion*** *(or* ***Wienner process****)* B_t *starting at* x *is a stochastic process satisfying the following properties*

(i) $B_0 = x$

(ii) B_t *is almost surely continuous*

(iii) B_t *has independent increments*

(iv) $B_t - B_s \sim N(0, t-s)$ *(for* $0 \leq s \leq t$ *)*

$N(\mu, \sigma^2)$ *denotes the normal distribution with mean value* μ *and variance* σ^2*.*

To proceed, we show how to define the Itô integral $\int_S^T f(t,\omega) dB_t(\omega)$.

A function ψ is called **elementary** if it has the form

$$\phi(t,\omega) = \sum_j e_j(\omega) \cdot \mathcal{X}_{[t_j, t_{j+1})}(t).$$

For elementary function ϕ, we define the Itô integral by

$$\int_S^T \phi(t,\omega) dB_t(\omega) = \sum_{j \geq 0} e_j(\omega)[B_{t_{j+1}} - B_{t_j}](\omega).$$

Then we define the Itô integral as follows

Definition 5 *Let* f *be a measurable function. Then the Itô integral of* f *(from* S *to* T*) is defined by*

$$\int_S^T f(t,\omega) dB_t(\omega) = \lim_{n \to \infty} \int_S^T \phi_n(t,\omega) dB_t(\omega)$$

where $\{\phi_n\}$ *is a sequence of elementary functions such that*

$$E[\int_S^T (f(t,\omega) - \phi_n(t,\omega))^2 dt] \to 0 \qquad \text{as } n \to \infty.$$

With the definition of Itô integral, a general form of **stochastic differential equation** is given by

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \qquad X_0 = x,$$

where X_t is the solution of the stochastic differential equation.

2.2 Nonlinear Filtering Problems

Now, let us consider the following generic stochastic filtering problem in a dynamic state-space form

$$\begin{aligned} \frac{X_t}{dt} &= f(t, X_t, \dot{W}_t) && (2.1) \\ \frac{Y_t}{dt} &= g(t, X_t, \dot{B}_t). && (2.2) \end{aligned}$$

Equations (2.1) and (2.2) are called signal state equation and measurement equation (or observation equation), respectively; X_t represents the state vector, Y_t is the measurement vector; $f : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$ and $g : \mathbb{R}^{N_x} \to \mathbb{R}^{N_y}$ are two vector valued functions, which are potentially time-varying; W_t and B_t are two independent Wienner processes, with covariance I and R, respectively, which represent process noise and measurement noise, respectively. When f and g are both linear functions, the filtering problem (2.1)-(2.2) is called the linear filtering problem, otherwise, it's called the nonlinear filtering problem.

In many applications, the noise can be assumed to be additive and (2.1)-(2.2) become

$$\begin{aligned} dX_t &= b(X_t)dt + \sigma(X_t)dW_t && (2.3) \\ dY_t &= h(X_t)dt + dB_t, && (2.4) \end{aligned}$$

where, $b : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$ and $h : \mathbb{R}^{N_x} \to \mathbb{R}^{N_y}$ are two vector valued functions, which are potentially time-varying, and $\sigma : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x \times n_w}$ is a matrix valued function.

The main purpose of numerical simulations of a filtering process is to obtain, recursively in time, the best estimate for the probability density function (pdf) of the state X_t based on observations up until time t, i.e.$\{Y_s : 0 \leq s \leq t\}$. This can be expressed as finding stochastic process $\tilde{X}_t$ such that

$$\tilde{X}_t = E[(X_t|\mathcal{Y}_t)] = \inf\{E[|X - Y|^2]; Y \in \mathcal{K}\}$$

where $\mathcal{Y}_t$ is the σ-algebra generated by the observation process up to t, and $\mathcal{K}$ is the space of all $\mathcal{Y}_t$-measurable and square integrable random variables.

In typical applications, observations are available only at a discrete sequence of time instants. We assume that the observations are collected at time instants $\Delta t, 2\Delta t, \cdots$, where Δt is a given time stepsize. Then denoting

$$Y_n \doteq Y(n\Delta t) - Y((n-1)\Delta t),$$

we have

$$Y_n = \int_{(n-1)\Delta t}^{n\Delta t} h(X_s)ds + B_{n\Delta t} - B_{(n-1)\Delta t} \approx h(X_{n\Delta t})\Delta t + \theta\xi_n,$$

where ξ_n is an i.i.d. sequence of standard Gaussian variables and $\theta \doteq \sqrt{R\Delta t}$. Denoting $h(x)\Delta t$ by $g(x)$, we rewrite a discrete analog of the observation equation (2.4) as follows.

$$Y_n = g(X_{n\Delta t}) + \theta\xi_n, \quad n \geq 1. \tag{2.5}$$

Alternatively, one can consider (2.5) as the original underlying observation model and not a discrete approximation to (2.4).

2.3 Bayesian Filters

We first review the extended Kalman filter (EKF), the most commonly used numerical approach for solving the nonlinear filter. Then, we discuss a popular class of approximation methods, the particle filter.

2.3.1 Extended Kalman Filter

The basic idea of EKF is to linearize the nonlinear functions in both the state equation and observation equation and as a result approximate the conditional distribution P_n by a normal distribution with parameters (μ_n, σ_n). Given the approximation P^*_{n-1} to P_{n-1} as a normal distribution with mean μ_{n-1} and covariance σ_{n-1}. The approximation P^*_n to P_n is obtained by the following two steps.

Prediction step. Let $\{x^*(t), t \geq 0\}$ be the trajectory about which linearization is performed. Typically, the trajectory x^* is chosen as the solution of the deterministic ODE

$$\dot{x}_t = b(x_t)$$

with initial condition μ_{n-1}. We denote $\Delta X_t \doteq X_t - x^*(t)$. Then, the linearized dynamic signal state is given as

$$\Delta \dot{X}_t = [\frac{\partial f}{\partial x}]_{x=x^*} \Delta X_t + \sigma(x^*(t))dB_t.$$

The prediction step of the standard continuous time Kalman filter is then applied with this linearized state model to give the approximation $\hat{P}^*_n$ to $\hat{P}_n$ via a normal distribution with parameters $(\hat{\mu}_n, \hat{\sigma}_n)$.

Update step. Similar to the state model, the observation model (2.5) is also linearized and is given as follows.

$$Y_n - g(\hat{\mu}_n) = [\frac{\partial g}{\partial x}]_{x=\hat{m}_n}(X_{t_n} - \hat{m}_n) + \theta\xi_n$$

The update step of the usual Kalman filter [45] is used with this linearized observation model to obtain P_n^* as a normal distribution with parameter μ_n, σ_n.

2.3.2 Particle filter

The particle filter, also known as sequential Monte-Carlo method is a recursive Bayesian filter based on the idea of approximating expected values by suitable Monte-Carlo sample averages. It is also called bootstrap filter, due to Gordon, Salmond and Smith [38]. The basic idea of the particle filter is as follows. The state space is partitioned into subregions, in which some random samples, called particles, are filled according to some probability measure. The higher the probability of a subregion, the denser the particles are concentrated. Since the pdf can be approximated by the point-mass histogram, by random sampling of the state space, we get a number of particles representing the pdf.

To be more specific, the approximation P_n^* to P_n is given by a discrete probability measure supported on random points $x_1^{(n)}, \cdots, x_L^{(n)}$ and corresponding weights $p_1^{(n)}, \cdots, p_L^{(n)}$. Here, $\{x_l^{(n)}\}_{l=1,\cdots,L}$ are called particles and L represents the number of particles that are used to approximate the distribution P_n. Given the approximation P_n^* to P_n by the discrete probability measure $\{(x_1^{(n-1)}, p_1^{((n-1))}), \cdots (x_L^{(n-1)}, p_L^{((n-1))})\}$ the two key steps of the algorithm are as follows.

Prediction step. Propagate each of the particles according to the state equation, i.e. $x_j^{(n-1)} \to \hat{x}_j^{(n)}$. This requires simulating the SDE (2.3) with initial condition $x_j^{(n-1)}$ by some discretization scheme. For example, we can use Euler scheme to get

$$\hat{x}_j^{(n)} = x_j^{(n-1)} + b(x_j^{(n-1)})\Delta t + \sigma(x_j^{(n-1)})\sqrt{\Delta t}\gamma_{n-1},$$

where $\{\gamma_n\}_{n=0,1,2,\ldots}$ is an i.i.d sequence of standard Gaussian random variables. This gives an approximation $\hat{P}_n^*$ to $\hat{P}_n$ as the discrete probability distribution $\{(\hat{x}_1^{(n)}, \hat{p}_1^{(n)}), \cdots (\hat{x}_L^{(n)}, \hat{p}_L^{(n)})\}$ where we set $\hat{p}_j^{(n)} \doteq p_j^{(n-1)}$.

Update step. Update the weights $\hat{p}_j^{(n)} \to p_j^{(n)}$ using the observation Y_n by setting

$$p_j^{(n)} = c\hat{p}_j^{(n)} R(\hat{x}_j^{(n)}, Y_n),$$

where c is a normalization constant and

$$R(\hat{x}_j^{(n)}, Y_n) \doteq \exp(\frac{g(\hat{x}_j^{(n)})Y_n - \frac{1}{2}|g(\hat{x}_j^{(n)})|^2}{\theta^2}).$$

The approximation P_n^* is now given as $\{(x_1^{(n)}, p_1^{(n)}), \cdots (x_L^{(n)}, p_L^{(n)})\}$ where we set $x_j^{(n)} \doteq \hat{x}_j^{(n)}$. Although the scheme is very easy to implement, it suffers from several degeneracy problems, especially in high dimensions. The main difficulty is that after a few time steps all the weights tend to concentrate on a very few particles which drastically reduces the effective sample size. A common remedy for this problem is to re-sample all the particles in order to rejuvenate the particle cloud.

2.4 Zakai Filter

An alternative approach to the computation of the nonlinear filter is by developing an evolution equation for the conditional pdf $p(t, X_t) \doteq p(X_t|\mathcal{Y}_t)$ (see [82]). To be more specific, under suitable regularity conditions one can show that the conditional pdf $p(t, x)$ is the unique solution of the following stochastic partial differential equation (SPDE):

$$dp(t, x) = L^* p(t, x)dt + h(x)p(t, x)dY_t, \tag{2.6}$$

where L is the infinitesimal generator of the state equation X_t and L^* is the adjoint of L. The above SPDE is also called the Zakai equation. The Zakai equation gives a recursive way for evaluating $p(t, x)$. Several works [10, 23, 30, 36, 85] have been done to develop suitable time and space discretization schemes for the Zakai equation for the purpose of obtaining numerical approximations for the nonlinear filter. This type of approximation methods are called "the Zakai filters".

Chapter 3

An Implicit Algorithm of Solving Nonlinear Filtering Problems

In this Chapter, we consider the following state and observation equations in the dynamic state-space form:

$$\begin{aligned} \frac{dX_t}{dt} &= f(t, X_t; W_t) && (3.1) \\ Y_t &= g(t, X_t; V_t) && (3.2) \end{aligned}$$

where $X_t \in \mathbb{R}^{n_x}$ denotes the state vector, $Y_t \in \mathbb{R}^{n_y}$ denotes the measurement vector, $W_t \in \mathbb{R}^{n_w}$ is a random vector representing the uncertainties in the model, and $V_t \in \mathbb{R}^{n_v}$ denotes the random measurement error. In many applications, the noise from measurement can be assumed to be additive, and the problem can be formulated in a discrete manner as

$$\begin{aligned} X_{t+1} &= f_t(X_t, W_t) && (3.3) \\ Y_t &= g_t(X_t) + V_t, && (3.4) \end{aligned}$$

where $\{W_t\}_{t\in\mathbb{N}} \in \mathbb{R}^{n_w}$ and $\{V_t\}_{t\in\mathbb{N}\setminus\{0\}} \in \mathbb{R}^{n_v}$ are mutually independent white noises and the subscript t indexes the discrete time level at which the functions are evaluated. In data assimilation, the observation Y_t arrives sequentially in time and the goal is to estimate the state vector X_t given the information of $\{Y_s,\ 0 < s \le t\}$.

3.1 Methodology and the implicit filter algorithm

In this section, we provide a brief review on the formulation of Bayesian optimal filter and introduce the implicit filter algorithm.

3.1.1 Bayesian Optimal Filter

First we adoptp some notations that will be used throughoutp the rest of this Chapter. Denote $Z_{m:n}$ as $(Z_m, Z_{m+1}, \cdots, Z_n)$ and denote $X_t \sim p(x_t)$ if the pdf of a random variable X_t is $p(x_t)$. Write

$$X_t \mid (X_{t-1} = x_{t-1}) \sim p(x_t \mid x_{t-1}) \tag{3.5}$$

where $X_t \mid X_{t-1}$ denotes the conditional expectation. When the context is clear, notations similar to (3.5) will be introduced without formal explanations.

The dynamical model is Markovian such that any future X_t is independent of the past given the present X_{t-1}:

$$p(x_t|x_{1:t-1}, y_{1:t-1}) = p(x_t|x_{t-1})$$

and the measurements are conditionally independent given x_t:

$$p(y_t|x_{1:t}, y_{1:t-1}) = p(y_t|x_t)$$

In this Chapter, we denote by $\mathcal{I}_t \doteq \{y_1, y_2, \cdots, y_t\}$ the information observed before time t. Given a prior distribution $p(x_0)$, Bayesian optimal filter is to construct the distribution $p(x_t|\mathcal{I}_t)$ recursively in two stages: prediction and update.

Assume that the required pdf $p(x_{t-1}|\mathcal{I}_{t-1})$ of previous step $t-1$ is available. The Chapman-Kolmogorov equation gives the prediction step of

$$p(x_t|\mathcal{I}_{t-1}) = \int_{\mathbb{R}^{n_x}} p(x_t|x_{t-1})p(x_{t-1}|\mathcal{I}_{t-1})dx_{t-1}. \tag{3.6}$$

At time t, as measurement y_t becomes available, the prior distribution from (3.6) can then be updated via Bayesian's formula

$$p(x_t|\mathcal{I}_t) = \frac{p(y_t|x_t)p(x_t|\mathcal{I}_{t-1})}{p(y_t|\mathcal{I}_{t-1})} = \frac{p(y_t|x_t)p(x_t|\mathcal{I}_{t-1})}{\int_{\mathbb{R}^{n_x}} p(y_t|x_t)p(x_t|\mathcal{I}_{t-1})\, dx_t}. \tag{3.7}$$

The exact computation of (3.6) and (3.7) is generally not possible. Exception exists where all $p(x_t|\mathcal{I}_t)$ are Gaussian and the model is linear, in which case the moments can be obtained using Kalman filter. In practically all other cases, approximate solutions are sought by numerical methods. Traditional particle filtering methods recursively generate samples (particles) following $p(x_t|\mathcal{I}_t)$ and use these samples to approximate moments of $p(x_t|\mathcal{I}_t)$. We propose here an inverse method which generates samples from the white noise and use them to approximate $p(x_t|\mathcal{I}_t)$, by utilizing inverse solutions to discretized stochastic differential equations.

3.1.2 An inverse algorithm

Our proposed method is based on the the fact that

$$p(x_t|x_{t-1}) = \int_{\mathbb{R}^{n_w}} p(x_t|x_{t-1}, w_{t-1}) \cdot p(w_{t-1}) dw_{t-1} = \mathbb{E}[p(x_t|x_{t-1}, W_{t-1})],$$

and therefore the term $p(x_t|\mathcal{I}_{t-1})$ in (3.6) can be written as

$$\begin{aligned} p(x_t|\mathcal{I}_{t-1}) &= \int_{\mathbb{R}^{n_x}} p(x_t|x_{t-1}) p(x_{t-1}|\mathcal{I}_{t-1}) dx_{t-1} \\ &= \int_{\mathbb{R}^{n_x}} \mathbb{E}[p(x_t|x_{t-1}, W_{t-1})] p(x_{t-1}|\mathcal{I}_{t-1}) dx_{t-1}. \end{aligned} \tag{3.8}$$

We assume that a compact domain, $\mathcal{B} \subset \mathbb{R}^{n_x}$, is the region of interest. Assuming that the pdf $p(x_{t-1}|\mathcal{I}_{t-1})$ is given, to achieve the prediction step (3.8) from time $t-1$ to time t:

1. Generate M particles/paths $\{w_{t-1}^{(j)}\}_{j=1\cdots M}$ according to the pdf of W_{t-1} and approximate $p(\cdot|W_{t-1})$ by its empirical pdf, denoted by $\pi^M(\cdot|W_{t-1}) \doteq \frac{1}{M}\sum_{j=1}^{M} \delta_{\cdot|w_{t-1}^{(j)}}$, where δ_x denotes the delta-Dirac mass located in x. In fact, according to Bayesian formula,

$$p(x_t|x_{t-1}, w_{t-1}) = \frac{p(x_{t-1}, w_{t-1}|x_t)}{p(x_{t-1}, w_{t-1})} \cdot p(x_t),$$

and it follows immediately that $p(x_t|x_{t-1}, W_{t-1})$ is a random variable and the randomness comes from the white noise W_{t-1}, for each given x_t and $x_{t-1} \in \mathbb{R}^{n_x}$. Therefore the term $\mathbb{E}[p(x_t|x_{t-1}, W_{t-1})]$ in (3.8) can be approximated by

$$\mathbb{E}[p(x_t|x_{t-1}, W_{t-1})] \approx \mathbb{E}[\pi^M(x_t|x_{t-1}, W_{t-1})],$$

in which $\pi^M(x_t|x_{t-1}, W_{t-1})$ has the probability distribution

$$Pr(\pi^M(x_t|x_{t-1}, w_{t-1}^{(j)})) = \frac{1}{M}, \quad j = 1, \ldots, M.$$

2. Partition region $\mathcal{B}$ by using N nodes: $\{u^{(i)}\}_{i=1\cdots N}$ and approximate $\pi^M(x_t|\cdot)$ by $\pi^M(u^{(i)}|\cdot)$. Step 1 together with (3.8) gives

$$p(u^{(i)}|\mathcal{I}_{t-1}) \approx \int_{\mathcal{B}} \mathbb{E}[\pi^M(u^{(i)}|x_{t-1}, W_{t-1})]p(x_{t-1}|\mathcal{I}_{t-1})dx_{t-1}. \tag{3.9}$$

3. Assume that f_t is invertible. Let $x_{t-1}^{(i,j)}$ be the solution to the equation $f_t(x_{t-1}^{(i,j)}, w_{t-1}^{(j)}) = u^{(i)}$ for $j = 1, \cdots, M$ and $i = 1, \cdots N$. Then

$$\begin{aligned} \mathbb{E}[\pi^M(u^{(i)}|x_{t-1}, W_{t-1})] &= \frac{1}{M}\sum_{j=1}^{M} \pi^M(u^{(i)}|x_{t-1}, w_{t-1}^{(j)}) \\ &= \frac{1}{M}\sum_{j=1}^{M} \delta(x_{t-1} - x_{t-1}^{(i,j)}), \quad i = 1, \ldots, N \end{aligned} \tag{3.10}$$

Hence the integral on the right hand side of (3.9) can be simplified and gives that for each $i = 1, \ldots, N$,

$$\begin{aligned} p(u^{(i)}|\mathcal{I}_{t-1}) &\approx \int_{\mathcal{B}} \mathbb{E}[\pi^M(u^{(i)}|x_{t-1}, W_{t-1})]p(x_{t-1}|\mathcal{I}_{t-1})dx_{t-1} \\ &= \frac{1}{M}\sum_{j=1}^{M} p\left(x_{t-1}^{(i,j)}|\mathcal{I}_{t-1}\right). \end{aligned} \tag{3.11}$$

4. The last step of prediction is to use interpolation to construct a piecewise approximation $\rho(x_t|\mathcal{I}_{t-1})$ of $p(x_t|\mathcal{I}_{t-1})$, from $p(u^{(i)}|\mathcal{I}_{t-1})$ obtained in step 3. Denote by $T[\{\cdot\}]$ the piecewise linear function that connects the points $(u^{(i)}, \{\cdot\})$, then we obtain the approximation of $p(x_t|\mathcal{I}_{t-1})$ via

$$\begin{aligned} p(x_t|\mathcal{I}_{t-1}) &\approx T\left[\{p(u^{(i)}|\mathcal{I}_{t-1})\}_{i=1}^{N}\right] \\ &\approx T\left[\left\{\frac{1}{M}\sum_{j=1}^{M} p\left(x_{t-1}^{(i,j)}|\mathcal{I}_{t-1}\right)\right\}_{i=1}^{N}\right] \doteq \rho(x_t|\mathcal{I}_{t-1}). \end{aligned} \tag{3.12}$$

where $p\left(x_{t-1}^{(i,j)}|\mathcal{I}_{t-1}\right)$ is the value of the pdf $p(x_{t-1}|\mathcal{I}_{t-1})$ at the point $x_{t-1} = x_{t-1}^{(i,j)}$.

Finally in the update step, we update the prior pdf $\rho(x_t|\mathcal{I}_{t-1})$ at $x_t \in \mathcal{B}$ by using the Bayes formula. Thus the approximation of $p(x_t|\mathcal{I}_t)$, denoted by $\rho(x_t|\mathcal{I}_t)$ is given as

$$p(x_t|\mathcal{I}_t) \approx \rho(x_t|\mathcal{I}_t) = \frac{p(y_t|x_t)\rho(x_t|\mathcal{I}_{t-1})}{\int_{\mathcal{B}} p(y_t|x_t)\rho(x_t|\mathcal{I}_{t-1})dx_t}. \tag{3.13}$$

3.2 Weak Convergence

Now, we study the convergence of the pdf obtained by our algorithm converges to the Bayesian optimal filter on $\mathcal{B}$. In general, given a measure μ and a function φ, we define

$$\langle \mu(\cdot), \varphi \rangle = \int \varphi(x)\mu(\cdot)\, dx.$$

Definition 6 *Let $\{\mu_n\}_{n=1}^{\infty}$ be a sequence of probability densities on $\mathcal{P}(\mathcal{B})$, where $\mathcal{P}(\mathcal{B})$ is the space of all probability measures over $\mathcal{B}$. We say that*

- *μ_n converges to $\mu \in \mathcal{P}(\mathcal{B})$ uniformly and write $\lim_{n\to\infty} \mu_n = \mu$ if for any $\varepsilon > 0$, there exists N_0, such that $|\mu_n(z) - \mu(z)| < \varepsilon$ for all $z \in \mathcal{B}$ and $n > N_0$.*

- *μ_n converges to $\mu \in \mathcal{P}(\mathcal{B})$ weakly and write $\lim_{n\to\infty} \mu_n \stackrel{\varphi}{=} \mu$ if*

$$\lim_{n\to\infty} \langle \mu_n, \varphi \rangle = \langle \mu, \varphi \rangle, \quad \forall \varphi \in C_b(\mathcal{B}),$$

where $C_b(\mathcal{B})$ is the set of all continuous bounded functions on $\mathcal{B}$.

We will prove the weak convergence of $\rho(x_t|\mathcal{I}_t)$ to $p(x_t|\mathcal{I}_t)$, i.e., the convergence of $\langle \rho(x_t|\mathcal{I}_t), \varphi \rangle$ to $\langle p(x_t|\mathcal{I}_t), \varphi \rangle$.

To guarantee that the Bayes' formula in (3.7) is well defined and can be fulfilled in our algorithm, we make the following standing assumptions:

(A1) For given $\mathcal{I}_t$, the denominator in (3.7) (normalization constants) satisfies

$$\int_{\mathcal{B}} p(y_t|x_t)p(x_t|\mathcal{I}_{t-1})\, dx_t > \xi > 0$$

(A2) The conditional kernel densities $p(x_t|x_{t-1})$ and $p(y_t|x_t)$ are uniformly continuous, bounded and strictly positive, i.e., given $\mathcal{I}_t$,

$$0 < p(x_t|x_{t-1}) < 1, \quad 0 < p(y_t|x_t) < 1.$$

For simplicity we denote $K_{t|t-1} := \mathbb{E}[p(x_t|x_{t-1}, W_{t-1})]$, denote the true pdf's by

$$p_{t-1|t-1} := p(x_{t-1}|\mathcal{I}_{t-1}), \quad p_{t|t-1} := p(x_t|\mathcal{I}_{t-1}), \quad p_{t|t} := p(x_t|\mathcal{I}_t);$$

and similarly denote the simulated pdf's by

$$\rho_{t-1|t-1} := \rho(x_{t-1}|\mathcal{I}_{t-1}), \quad \rho_{t|t-1} := \rho(x_t|\mathcal{I}_{t-1}), \quad \rho_{t|t} := \rho(x_t|\mathcal{I}_t).$$

We first define $a_t : \mathcal{P}(\mathcal{B}) \to \mathcal{P}(\mathcal{B})$ to be the mapping

$$a_t(\mu)(x_t) = \int_{\mathcal{B}} K_{t|t-1}\mu(x_{t-1})\, dx_{t-1}, \quad \forall \mu \in \mathcal{P}(\mathcal{B}). \tag{3.14}$$

Then we have

$$\langle a_t(\mu), \varphi \rangle = \left\langle \mu, K_{t|t-1}\varphi \right\rangle, \quad \forall \varphi \in C_b(\mathcal{B}), \tag{3.15}$$

and it holds that

$$a_t(p_{t-1|t-1}) = p_{t|t-1}.$$

It's natural to assume that a_t is continuous, since in the context of filtering two realizations of the signal that start from "close" positions will remain "close" at subsequent times. In fact, when the transition kernel $p(x_t|x_{t-1})$ is Feller, i.e., $p(x_t|x_{t-1})\varphi$ is a continuous bounded function for any continuous bounded function φ, we have according to [25] that if $\lim_{n\to\infty} \mu_n \overset{\varphi}{=} \mu$ then

$$\lim_{n\to\infty} \langle a_t(\mu_n), \varphi \rangle = \lim_{n\to\infty} \left\langle \mu_n, K_{t|t-1}\varphi \right\rangle = \left\langle \mu, K_{t|t-1}\varphi \right\rangle = \langle a_t(\mu), \varphi \rangle, \quad \forall\ \varphi \in C_b(\mathcal{B}). \tag{3.16}$$

Define $b_t : \mathcal{P}(\mathcal{B}) \to \mathcal{P}(\mathcal{B})$ to be the mapping

$$b_t(\mu) = \frac{p(y_t|x_t)\mu(x_t)}{\int_{\mathcal{B}} p(y_t|x_t)\mu(x_t)dx_t}, \quad \forall \mu \in \mathcal{P}(\mathcal{B}). \tag{3.17}$$

Then we have

$$\left\langle b_t(p_{t|t-1}), \varphi \right\rangle = \left\langle p_{t|t-1}, p(y_t|x_t) \right\rangle^{-1} \cdot \left\langle p_{t|t-1}, \varphi p(y_t|x_t) \right\rangle, \quad \forall \varphi \in C_b(\mathcal{B}), \tag{3.18}$$

and it holds that

$$b_t(p_{t|t-1}) = p_{t|t}.$$

It is also natural to assume that b_t is continuous, which means that a slight variation in two distributions will not result in a large variation in the distributions when observations are taken into account. In fact, assuming that $p(y_t|\cdot)$ is a continuous bounded strictly positive function, we have according to [25] that $\lim_{n\to\infty}\mu_n \overset{\varphi}{=} \mu$ then

$$\begin{aligned}\lim_{n\to\infty}\langle b_t(\mu_n),\varphi\rangle &= \lim_{n\to\infty}\langle \mu_n, p(y_t|x_t)\rangle^{-1}\cdot\langle \mu_n, \varphi p(y_t|x_t)\rangle \\ &= \langle \mu, p(y_t|x_t)\rangle^{-1}\cdot\langle \mu, \varphi p(y_t|x_t)\rangle = \langle b_t(\mu),\varphi\rangle, \quad \forall\, \varphi\in C_b(\mathcal{B}). \quad (3.19)\end{aligned}$$

We next define two approximation operators, the sampling operator, and the interpolation operator that appear in the prediction step.

1. Denote by ψ a function of W_{t-1} from $\mathcal{P}(\mathcal{B})$ to $\mathcal{P}(\mathcal{B})$. At each step t, we draw M samples, $w_{t-1}^{(1)},\cdots,w_{t-1}^{(M)}$, which are i.i.d. random variables with common distribution W_{t-1}. The Monte Carlo estimate of $\mathbb{E}[\psi(W_{t-1})]$ can be obtained to be

$$\tilde{\psi}(w_{t-1}) = \frac{1}{M}\sum_{j=1}^{M}\psi(w_{t-1}^{(j)}).$$

Define the sampling operator $s^M : \mathcal{P}(\mathcal{B}) \to \mathcal{P}(\mathcal{B})$ to be

$$s^M(\langle \mathbb{E}[\psi],\mu\rangle) = \langle \mathbb{E}[\tilde{\psi}],\mu\rangle, \quad \forall\mu\in\mathcal{P}(\mathcal{B}), \tag{3.20}$$

Then

$$s^M\circ a_t(\mu) = \int_{\mathcal{B}}\frac{1}{M}\sum_{j=1}^{M}\pi^M(x_t|x_{t-1},w_{t-1}^{(j)})\cdot\mu(x_{t-1})\,dx_{t-1}, \quad \forall\mu\in\mathcal{P}(\mathcal{B}),$$

2. Given the rectangular spatial partition nodes $\{u^{(i)}\}_{i=1}^{N}$. Define $T^N : \mathcal{P}(\mathcal{B}) \to \mathcal{P}(\mathcal{B})$ to be the interpolation operator

$$T^N(\mu)(x_t) = \sum_{i=1}^{N} \psi_i^N(x_t)\mu(u^{(i)}),$$

for each $x_t \in \mathcal{B}$, where $\{\psi_i\}_{i=1}^{N}$ are basis functions for interpolation operator T^N. Therefore

$$T^N \circ s^M \circ a_t(p_{t-1|t-1}) = \rho_{t|t-1},$$

where $\rho_{t|t-1}$ is the piecewise linear approximation of $p_{t|t-1}$ satisfying

$$\rho_{t|t-1}(u^{(i)}) = \frac{1}{M}\sum_{j=1}^{M} p\left(x_{t-1}^{(i,j)}|\mathcal{I}_{t-1}\right), \quad \text{for each } \ i = 1, \ldots, N.$$

Denote by $\kappa_t \doteq b_t \circ a_t$, $\kappa_t^{M,N} \doteq b_t \circ T^N \circ s^M \circ a_t$, $\kappa_{1:t} = \kappa_t \circ \kappa_{t-1} \circ \cdots \circ \kappa_1$ and $\kappa_{1:t}^{M,N} = \kappa_t^{M,N} \circ \kappa_{t-1}^{M,N} \circ \cdots \circ \kappa_1^{M,N}$, we have

$$\kappa_t(p_{t-1|t-1}) = p_{t|t}, \quad \kappa_t^{M,N}(p_{t-1|t-1}) = \rho_{t|t}, \quad \text{and} \quad \kappa_{1:t}(p_{0|0}) = p_{t|t}, \quad \kappa_{1:t}^{M,N}(p_{0|0}) = \rho_{t|t}.$$

Our goal is to show that $\kappa_{1:t}^{M,N} \xrightarrow{\varphi} \kappa_{1:t}$. This can be done by showing $\kappa_t^{M,N} \xrightarrow{\varphi} \kappa_t$ for each step t and induction.

Recalling that $\mathcal{P}(\mathcal{B})$ is the set of all probability measures on $\mathcal{B}$, we denote by $\mathcal{P}_U(\mathcal{B})$ be the set of all uniformly continuous probability measures on $\mathcal{B}$ and $\mathcal{P}_C(\mathcal{B})$ the set of all a.e. continuous probability measures on $\mathcal{B}$, for latter use.

Lemma 1 *$s^M \circ a_t$ converges to a_t weakly, i.e., for any $\mu_M, \mu \in \mathcal{P}(\mathcal{B})$ with $\lim_{M\to\infty} \mu_M \overset{\varphi}{=} \mu$, it holds that*

$$\lim_{M\to\infty} s^M \circ a_t(\mu_M) \overset{\varphi}{=} a_t(\mu). \tag{3.21}$$

p *Proof.* For any $t \in \mathbb{N}$, by the Strong Law of Large Numbers,

$$\lim_{M\to\infty} \frac{1}{M} \sum_{j=1}^{M} \pi^M(x_t|x_{t-1}, w_{t-1}^{(j)}) = \mathbb{E}[p(x_t|x_{t-1}, W_{t-1})], \quad a.s. \tag{3.22}$$

Therefore, for any $\varphi \in C_b(\mathcal{B})$ we have

$$\begin{aligned} & \left\langle s^M \circ a_t(\mu_M), \varphi \right\rangle - \langle a_t(\mu), \varphi \rangle \\ = \; & \left\langle s^M \circ a_t(\mu_M), \varphi \right\rangle - \langle a_t(\mu_M), \varphi \rangle + \langle a_t(\mu_M), \varphi \rangle - \langle a_t(\mu), \varphi \rangle \\ \leq \; & \int_{\mathcal{B}} \left(\frac{1}{M} \sum_{j=1}^{M} \pi^M(x_t|x_{t-1}, w_{t-1}^{(j)}) \mu_M(x_{t-1}) dx_{t-1} - \mathbb{E}[p(x_t|x_{t-1}, W_{t-1})] \right) \mu_M(x_{t-1}) \varphi dx_{t-1} \\ & + \langle a_t(\mu_M), \varphi \rangle - \langle a_t(\mu), \varphi \rangle . \end{aligned} \tag{3.23}$$

It then follows directly from equation (3.22) and (3.16) that

$$\lim_{M\to\infty} s^M \circ a_t(\mu_M) \stackrel{\varphi}{=} a_t(\mu).$$

The proof is complete.

Lemma 2 *Assume that $\{\mu_{M,N}\}_{M,N=1}^{\infty} \in \mathcal{P}_C(\mathcal{B})$ and $\mu_M \in \mathcal{P}_U(\mathcal{B})$ with $\lim_{N\to\infty} \mu_{M,N} = \mu_M$ for each $M \in \mathbb{N}$. Then, $\lim_{N\to\infty} s^M \circ a_t(\mu_{M,N}) = s^M \circ a_t(\mu_M)$ for each $M \in \mathbb{N}$. Moreover, if there exists $\lambda > 0$ such that $\left\| \frac{\partial}{\partial x} f_t^{-1} \right\| < \lambda$, then $s^M \circ a_t(\mu_{M,N}) \in \mathcal{P}_C(\mathcal{B})$ and $s^M \circ a_t(\mu_M) \in \mathcal{P}_U(\mathcal{B})$.*

Proof. For any $x \in \mathcal{B}$, by the definition of s^M and a_t we have

$$s^M \circ a_t(\mu_{M,N})(x) = \frac{1}{M} \sum_{j=1}^{M} \mu_{M,N} \left(x_{t-1}^{(x,j)} \right)$$

and

$$s^M \circ a_t(\mu_M)(x) = \frac{1}{M} \sum_{j=1}^{M} \mu_M \left(x_{t-1}^{(x,j)} \right).$$

Since $\lim_{N\to\infty} \mu_{M,N}=\mu_M$ for each $M \in \mathbb{N}$, given any $\varepsilon > 0$, there exists N_0, such that for all $z \in \mathcal{B}$, $|\mu_{M,N}(z) - \mu_M(z)| < \varepsilon$ for each $M \in \mathbb{N}$. Therefore, for all $x \in \mathcal{B}$,

$$\begin{aligned}
\left|s^M \circ a_t(\mu_{M,N})(x) - s^M \circ a_t(\mu_M)(x)\right| &= \left|\frac{1}{M}\sum_{j=1}^{M}\left(\mu_{M,N}\left(x_{t-1}^{(x,j)}\right) - \mu_M\left(x_{t-1}^{(x,j)}\right)\right)\right| \\
&\leq \frac{1}{M}\sum_{j=1}^{M}\left|\left(\mu_{M,N}\left(x_{t-1}^{(x,j)}\right) - \mu_M\left(x_{t-1}^{(x,j)}\right)\right)\right| \\
&< \frac{1}{M}\sum_{j=1}^{M}\varepsilon = \varepsilon.
\end{aligned}$$

This proves that $\lim_{N\to\infty} s^M \circ a_t(\mu_{M,N})=s^M \circ a_t(\mu_M)$ for all $M \in \mathbb{N}$.

We next prove that $s^M \circ a_t(\mu_{M,N}) \in \mathcal{P}_C(\mathcal{B})$. In fact, for any $\varepsilon > 0$ and $z_0 \in \mathcal{B}$, since $\mu_{M,N} \in \mathcal{P}_C(\mathcal{B})$, there exists $\delta > 0$ such that when $|z - z_0| < \delta$,

$$|\mu_{M,N}(z) - \mu_{M,N}(z_0)| < \varepsilon.$$

Fix arbitrary $x_0 \in \mathcal{B}$, for any $x \in \mathcal{B}$ satisfying $|x - x_0| < \delta/\lambda$, using that $f_t(x_{t-1}^{(x,j)}, w_{t-1}^{(j)}) = x$ and $f_t(x_{t-1}^{(x_0,j)}, w_{t-1}^{(j)}) = x_0$ we have

$$\left|x_{t-1}^{(x,j)} - x_{t-1}^{(x_0,j)}\right| = |f_t^{-1}(x, w_{t-1}^{(j)}) - f_t^{-1}(x_0, w_{t-1}^{(j)})| \leq \left\|\frac{\partial}{\partial x}f_t^{-1}\right\| \cdot |x - x_0| < \delta,$$

and thus

$$\left|s^M \circ a_t(\mu_{M,N})(x) - s^M \circ a_t(\mu_{M,N})(x_0)\right| \leq \frac{1}{M}\sum_{j=1}^{M}\left|\mu_{M,N}\left(x_{t-1}^{(x,j)}\right) - \mu_{M,N}\left(x_{t-1}^{(x_0,j)}\right)\right| < \varepsilon.$$

It remains to show that $s^M \circ a_t(\mu_M) \in \mathcal{P}_U(\mathcal{B})$. In fact, given any $x_1, x_2 \in \mathcal{B}$,

$$s^M \circ a_t(\mu_M)(x_1) - s^M \circ a_t(\mu_M)(x_2) = \frac{1}{M}\sum_{j=1}^{M}\left(\mu_M\left(x_{t-1}^{(x_1,j)}\right) - \mu_M\left(x_{t-1}^{(x_2,j)}\right)\right).$$

For any $\varepsilon > 0$, from the uniformly continuity of μ_M, there exists $\delta > 0$, such that for any $z_1, z_2 \in \mathcal{B}$ with $|z_1 - z_2| < \delta$, $|\mu_M(z_1) - \mu_M(z_2)| < \varepsilon$. Let $\tilde{\delta} = \frac{\delta}{\lambda}$, then $|x_1 - x_2| < \tilde{\delta}$ implies that

$$\left|x_{t-1}^{(x_1,j)} - x_{t-1}^{(x_2,j)}\right| = \left|f_t^{-1}(x_1, w_{t-1}^{(j)}) - f_t^{-1}(x_2, w_{t-1}^{(j)})\right| \leq \left\|\frac{\partial}{\partial x}(f_t^{-1})\right\| \cdot |x_1 - x_2| < \delta.$$

Hence

$$\left|s^M \circ a_t(\mu_M)(x_1) - s^M \circ a_t(\mu_M)(x_2)\right| = \left|\frac{1}{M}\sum_{j=1}^{M}\left(\mu_M\left(x_{t-1}^{(x_1,j)}\right) - \mu_M\left(x_{t-1}^{(x_2,j)}\right)\right)\right| < \varepsilon.$$

The proof is complete.

Lemma 3 *For $\{\nu_{M,N}\}_{M,N=1}^{\infty} \in \mathcal{P}_C(\mathcal{B})$ and $\nu_M \in \mathcal{P}_U(\mathcal{B})$ with $\lim_{N\to\infty}\nu_{M,N}=\nu_M$ for each $M \in \mathbb{N}$, it holds that*

$$\lim_{N\to\infty} T^N(\nu_{M,N})=\nu_M, \quad \forall M \in \mathbb{M}. \tag{3.24}$$

Proof. For any $x_t \in \mathcal{B}$,

$$\left|T^N(\nu_{M,N})(x_t) - \nu_M(x_t)\right| \leq \left|T^N(\nu_{M,N})(x_t) - T^N(\nu_M)(x_t)\right| + \left|T^N(\nu_M)(x_t) - \nu_M(x_t)\right|. \tag{3.25}$$

Since $\lim_{N\to\infty}\nu_{M,N}=\nu_M$, for any $\varepsilon > 0$, there exists $N_1 = N_1(M) > 0$ such that when $N > N_1$,

$$|\nu_{M,N}(x_t) - \nu_M(x_t)| < \frac{\varepsilon}{2}.$$

Thus because of the linearity of T^N we have

$$\left|T^N(\nu_{M,N})(x_t) - T^N(\nu_M)(x_t)\right| = \left|T^N(\nu_{M,N} - \nu_M)(x_t)\right| < \frac{\varepsilon}{2}. \tag{3.26}$$

For the second term on the right hand side of inequality (3.25), since T^N is the linear interpolation operator and ν_M is uniformly continuous, for any $\varepsilon > 0$, there exists $N_2 =$

$N_2(M) > 0$ such that when $N > N_2$ we have

$$\left|T^N(\nu_M)(x_t) - \nu_M(x_t)\right| < \frac{\varepsilon}{2}. \tag{3.27}$$

In summary letting $N_0 = \max\{N_1, N_2\}$ we have by (3.25), (3.26) and (3.27) that for any $\varepsilon > 0$,

$$\left|T^N(\nu_{M,N})(x_t) - \nu_M(x_t)\right| < \varepsilon, \quad \forall N > N_0, \quad \forall x_t \in \mathcal{B}, \quad \forall M \in \mathbb{N}. \tag{3.28}$$

The proof is complete.

We next prove the weak convergence of the operator $\kappa_t^{M,N}$ to κ_t. Letting $\kappa_t^{M,N}$ and κ_t be the composition operators defined as above, we have the following theorem.

Theorem 3.1 *(Local convergence) Assume that the transition kernel $p(x_t|x_{t-1})$ is Feller and $p(y_t|x_t)$ is bounded, uniformly continuous, and strictly positive. Also assume that $\left\|\frac{\partial}{\partial x} f_t^{-1}\right\|$ is bounded. Then, for any $\{\mu_{M,N}\}_{M,N=1}^{\infty} \in \mathcal{P}_C(\mathcal{B})$ and $\mu_M, \mu \in \mathcal{P}_U(\mathcal{B})$ with $\lim_{N\to\infty} \mu_{M,N}=\mu_M$ for each $M \in \mathbb{N}$ and $\lim_{M\to\infty} \mu_M \overset{\varphi}{=} \mu$, it holds that*

$$\lim_{M\to\infty} \lim_{N\to\infty} \kappa_t^{M,N}(\mu_{M,N}) \overset{\varphi}{=} \kappa_t(\mu). \tag{3.29}$$

Proof. Given $\lim_{M\to\infty} \mu_M \overset{\varphi}{=} \mu$, by Lemma 1,

$$\lim_{M\to\infty} s^M \circ a_t(\mu_M) \overset{\varphi}{=} a_t(\mu). \tag{3.30}$$

Given $\lim_{N\to\infty} \mu_{M,N}=\mu_M$ for each $M \in \mathbb{N}$, by Lemma 2 we have $\lim_{N\to\infty} s^M \circ a_t(\mu_{M,N})=s^M \circ a_t(\mu_M)$ and $s^M \circ a_t(\mu_{M,N}) \in \mathcal{P}_C(\mathcal{B})$, $s^M \circ a_t(\mu_M) \in \mathcal{P}_U(\mathcal{B})$. Thus by letting $\nu_{M,N} \doteq s^M \circ a_t(\mu_{M,N})$ and $\nu_M \doteq s^M \circ a_t(\mu_M)$ in Lemma 3 we get

$$\lim_{N\to\infty} T^N \circ s^M \circ a_t(\mu_{M,N})=s^M \circ a_t(\mu_M). \tag{3.31}$$

Equations (3.30) and (3.31) together give

$$\lim_{M\to\infty}\lim_{N\to\infty} T^N \circ s^M \circ a_t(\mu_{M,N}) \stackrel{\varphi}{=} a_t(\mu).$$

Therefore it follows directly from (3.19) that

$$\lim_{M\to\infty}\lim_{N\to\infty} b_t \circ T^N \circ s^M \circ a_t(\mu_{M,N}) \stackrel{\varphi}{=} b_t \circ a_t(\mu).$$

The proof is complete.

To prove the global weak convergence result, we also need the following Lemma.

Lemma 4 *Assume $p(y_t|x_t)$ is bounded, uniformly continuous, and strictly positive. For $\{\gamma_{M,N}\}_{M,N=1}^{\infty} \in \mathcal{P}_C(\mathcal{B})$ and $\gamma_M \in \mathcal{P}_U(\mathcal{B})$ with $\lim_{N\to\infty}\gamma_{M,N}=\gamma_M$ for each $M \in \mathbb{N}$, if there exists a $\xi_0 > 0$ such that $\int_{\mathcal{B}} p(y_t|x_t)\gamma_M(x_t)dx_t \geq \xi_0$, then we have*

$$\lim_{N\to\infty} b_t(\gamma_{M,N})=b_t(\gamma_M) \in \mathcal{P}_U(\mathcal{B}), \quad \forall M \in \mathbb{N}.$$

Proof. Since $\lim_{N\to\infty}\gamma_{M,N}=\gamma_M$ for each $M \in \mathbb{N}$, then for any $0 < \varepsilon < \frac{\xi_0}{2}$, there exists N_0, such that when $N > N_0$,

$$|\gamma_{M,N}(x_t) - \gamma_M(x_t)| < \varepsilon, \quad \forall x_t \in \mathcal{B} \text{ and } \forall M \in \mathbb{N}.$$

It then follows that $\left|\int_{\mathcal{B}} p(y_t|x_t)(\gamma_{M,N}(x_t) - \gamma_M(x_t))dx_t\right| < \varepsilon$ and

$$\int_{\mathcal{B}} p(y_t|x_t)\gamma_{M,N}(x_t)dx_t > \int_{\mathcal{B}} p(y_t|x_t)\gamma_M(x_t)dx_t - \varepsilon > \frac{\xi_0}{2}.$$

Thus for any $x_t \in B$ when $N > N_0$ we have

$$\begin{aligned}
&|b_t(\gamma_{M,N})(x_t) - b_t(\gamma_M)(x_t)| \\
&= p(y_t|x_t) \cdot \left| \frac{\gamma_{M,N}(x_t) \cdot \int_{\mathcal{B}} p(y_t|x_t)\gamma_M(x_t)dx_t - \gamma_M(x_t) \cdot \int_{\mathcal{B}} p(y_t|x_t)\gamma_{M,N}(x_t)dx_t}{\int_{\mathcal{B}} p(y_t|x_t)\gamma_{M,N}(x_t)dx_t \cdot \int_{\mathcal{B}} p(y_t|x_t)\gamma_M(x_t)dx_t} \right| \\
&\leq \left| \frac{\int_{\mathcal{B}} p(y_t|x_t)\left(\gamma_M(x_t) - \gamma_{M,N}(x_t)\right)dx_t}{\int_{\mathcal{B}} p(y_t|x_t)\gamma_{M,N}(x_t)dx_t \cdot \int_{\mathcal{B}} p(y_t|x_t)\gamma_M(x_t)dx_t} \right| + \left| \frac{\gamma_M(x_t) - \gamma_{M,N}(x_t)}{\int_{\mathcal{B}} p(y_t|x_t)\gamma_M(x_t)dx_t} \right| \\
&< \left(\frac{4}{\xi_0^2} + \frac{2}{\xi_0} \right) \varepsilon.
\end{aligned} \tag{3.32}$$

Therefore $\lim_{N\to\infty} b_t(\gamma_{M,N}) = b_t(\gamma_M)$. It remains to show that $b_t(\gamma_M) \in \mathcal{P}_U(\mathcal{B})$.

In fact, by the definition of b_t, we have for any $x_t^{(1)}, x_t^{(2)} \in \mathcal{B}$,

$$\left| b_t(\gamma_M)(x_t^{(1)}) - b_t(\gamma_M)(x_t^{(2)}) \right| \leq \frac{2}{\xi_0} \left| p(y_t|x_t^{(1)})\gamma_M(x_t^{(1)}) - p(y_t|x_t^{(2)})\gamma_M(x_t^{(2)}) \right|.$$

From the uniformly continuity property and the boundedness of γ_M and $p(x_t|x_{t-1})$ that for any $\varepsilon > 0$, there exists $\delta > 0$, such that when $x_t^{(1)}, x_t^{(2)} \in \mathcal{B}$ with $|x_t^{(1)} - x_t^{(2)}| < \delta$, we have $|\gamma_M(x_t^{(1)}) - \gamma_M(x_t^{(2)})| < \frac{\varepsilon}{2}$ and $|p(x_t^{(1)}) - p(x_t^{(2)})| < \frac{\varepsilon}{2}$. Thus,

$$\left| b_t(\gamma_M)(x_t^{(1)}) - b_t(\gamma_M)(x_t^{(2)}) \right| < C\varepsilon.$$

This completes the proof.

Note 1 When the standing assumption (A1) holds, it follows immediately by the definition of s^M that there exists a $\xi_0 > 0$ such that $\int_{\mathcal{B}} p(y_t|x_t)s^M \circ p(x_t|\mathcal{I}_{t-1})dx_t > \xi_0$ for M sufficient large.

Applying Theorem 3.1 to the context of filtering problems, we can obtain the weak convergence of our implicit filtering simulation to the Bayesian optimal filter. Our main result of this work is stated in the following theorem. For simplicity, we define two new operators θ_t^M and $\theta_{1:t}^M$ to be

$$\theta_t^M = b_t \circ s^M \circ a_t \quad \text{and} \quad \theta_{1:t}^M = \theta_t \circ \theta_{t-1} \circ \cdots \circ \theta_1.$$

Theorem 3.2 *(Global convergence) Assume that the transition kernel $p(x_t|x_{t-1})$ is Feller and $p(x_t|x_{t-1})$ is bounded, uniformly continuous, and strictly positive. Also assume that $\left\|\frac{\partial}{\partial x} f_t^{-1}\right\|$ is bounded. Then*

$$\lim_{N\to\infty} \kappa_{1:t}^{M,N}(p_{0|0}) = \theta_{1:t}^M(p_{0|0}) \;\; \forall M \in \mathbb{N}, \quad \textit{and} \quad \lim_{M\to\infty} \theta_{1:t}^M(p_{0|0}) \overset{\varphi}{=} \kappa_{1:t}(p_{0|0}),$$

which implies that

$$\lim_{M\to\infty} \lim_{N\to\infty} \rho_{t|t} \overset{\varphi}{=} p_{t|t}.$$

Proof. To prove Theorem 3.2, we use induction method.

(1) $t = 1$: choose $\mu_{M,N} = \mu_M = \mu = p_{0|0}$ in equation (3.29). It is obviously that

$$\lim_{N\to\infty} \mu_{M,N} {=} \mu_M, \quad \lim_{M\to\infty} \mu_M \overset{\varphi}{=} \mu,$$

and $\mu_M = p_{0|0}$ is uniformly continuous. By Lemma 2, Lemma 3 and Lemma 4 and Note 1,

$$\lim_{N\to\infty} \kappa_1^{M,N}(p_{0|0}) = \theta_1^M(p_{0|0}) \in \mathcal{P}_U(\mathcal{B}), \quad \forall M \in \mathbb{N}.$$

By Lemma 1 and the continuity of b_t, we have

$$\lim_{M\to\infty} \theta_1^M(p_{0|0}) \overset{\varphi}{=} \kappa_1(p_{0|0}) = p_{1|1}.$$

It then follows from Theorem 3.1 that

$$\lim_{M\to\infty} \lim_{N\to\infty} \kappa_1^{M,N}(p_{0|0}) \overset{\varphi}{=} \kappa_1(p_{0|0}) = p_{1|1}.$$

(2) Assume that

$$\lim_{N\to\infty} \kappa_{1:t-1}^{M,N}(p_{0|0}) = \theta_{1:t-1}^{M}(p_{0|0}) \in \mathcal{P}_U(\mathcal{B}) \quad \forall M \in \mathbb{N}, \quad \text{and} \quad \lim_{M\to\infty} \theta_{1:t-1}^{M}(p_{0|0}) \stackrel{\varphi}{=} \kappa_{1:t-1}(p_{0|0}).$$

We choose $\mu_{M,N} = \kappa_{1:t-1}^{M,N}(p_{0|0})$ and $\mu = \kappa_{1:t-1}(p_{0|0}) = p_{t-1|t-1}$ in equation (3.29) and $\mu_M = \theta_{1:t-1}^{M}(p_{0|0})$ in Theorem 3.1. From the assumption,

$$\lim_{N\to\infty} \mu_{M,N}{=}\mu_M \quad \text{and} \quad \lim_{M\to\infty} \mu_M \stackrel{\varphi}{=} \mu.$$

By Lemma 2, Lemma 3, Lemma 4 and Note 1 that

$$\lim_{N\to\infty} \kappa_{1:t}^{M,N}(p_{0|0}) = \lim_{N\to\infty} b_t \circ T^N \circ s^M \circ a_t(\kappa_{1:t-1}^{M,N}(p_{0|0})){=}\theta_{1:t}^{M}(p_{0|0}) \in \mathcal{P}_U(\mathcal{B}). \tag{3.33}$$

By Lemma 1 and the continuity of b_t we have

$$\lim_{M\to\infty} \theta_{1:t}^{M}(p_{0|0}) \stackrel{\varphi}{=} p_{t|t}. \tag{3.34}$$

Therefore it follows from Theorem 3.1 that

$$\lim_{M\to\infty} \lim_{N\to\infty} \kappa_t^{M,N}(\kappa_{1:t-1}^{M,N}(p_{0|0})) \stackrel{\varphi}{=} \kappa_t(p_{t-1|t-1}) = \kappa_{1:t}(p_{0|0}) = p_{t|t}.$$

The proof is complete.

3.3 Numerical experiments

In this section, we present two numerical examples to demonstrate the efficiency of our method. The first example involves a one dimensional nonlinear system and measurement equation while the second is a 2-D bearing-only tracking problem. We shall compare our method with the standard EKF and particle filter. Here the particle filter we are using is the sequential important sampling with resampling (SIR).

Example 1

Consider the following nonlinear model

$$\begin{aligned} x_k &= 40 \cdot \tan(x_{k-1} + 10) + 50 w_{k-1}, \\ y_k &= 40 \cdot \frac{x_k}{2000 + x_k} + v_k, \end{aligned} \tag{3.35}$$

where w_k and v_k are two independent zero-mean white noise processes with variance 1.0, y_k is the noise perturbed observation of x_k. The initial position is taken to be $x_0 = 2$ and Figure 3.1 shows a 50 step realization of the state equation in model (3.35).

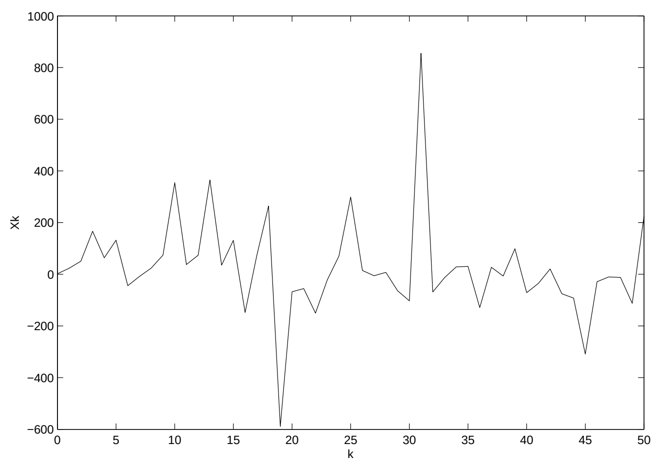


Figure 3.1: Original Position

Figure 3.2, Figure 3.3 and Figure 3.4 are the simulation results obtained by using EKF and particle filter and our implicit particle filter method, respectively. The true state is represented by blue diamonds while simulation results are given as red "stars" and connected by solid lines. The prior pdf $p(x_0)$ is initialized with the standard normal distribution with the mean value x_0 and the variance 1.0.

In particle filter method, we use 500 particles (sample points) to represent the pdf and in our implicit filter, we use 100 nodes to partition the region and the number of Monte-Carlo samples is $M = 10$.

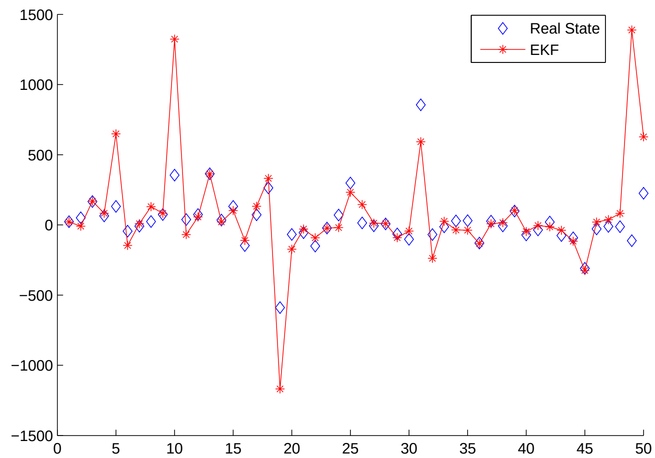


Figure 3.2: Extended Kalman Filter

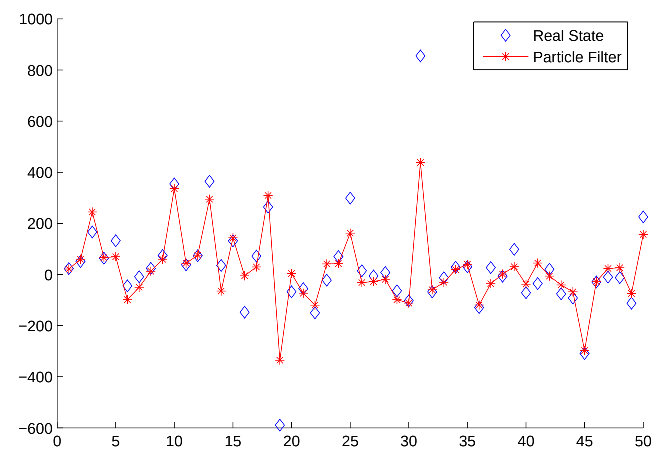


Figure 3.3: Particle Filter

Form the three figures, one can see that when the variation between two consecutive points is not very large, all three methods produce very accurate approximations to the true state. On the other hand, when the true state has very large variations at some time steps, i.e., the state variable has a large jump from its previous state, both EKF and particle filter fail to produce accurate approximations. However, our implicit particle algorithm still produces accurate estimations at these points.

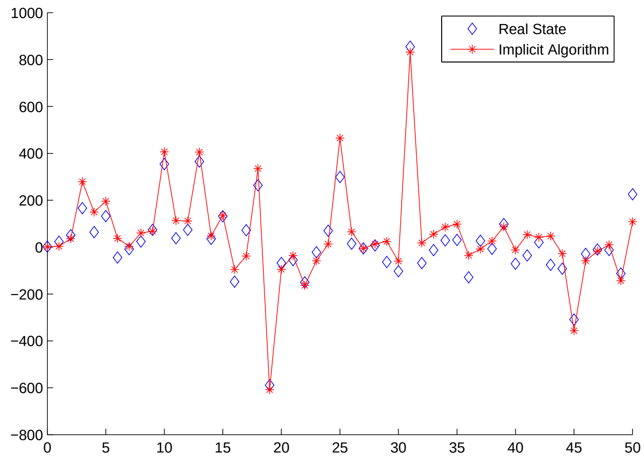


Figure 3.4: implicit Algorithm

Example 2

In this example, we consider the following bearing-only tracking problem.

$$\begin{aligned} dX_1(t) &= -\alpha X_2(t)dt + \beta \frac{X_1(t)}{(X_1(t))^2 + (X_2(t))^2} \\ &\quad + \sigma_1 dW_1(t), \\ dX_2(t) &= \alpha X_1(t)dt + \beta \frac{X_2(t)}{(X_1(t))^2 + (X_2(t))^2} \\ &\quad + \sigma_2 dW_2(t) \end{aligned} \tag{3.36}$$

where $W_1(t)$ and $W_2(t)$ are two independent Brownian Motions. This stochastic dynamical system may serve to model the motion of a ship which moves with a constant radial and angular velocity, perturbed by a white noise. The observations are collected by a detector located at a platform with time intervals of length $\Delta = 0.05$ and the data are angular measurements corrupted by noise.

To approximate the state variables $X = (X_1, X_2)$, we discretize the dynamical system (3.36) in time and obtain a discrete nonlinear filtering problem. Let $x_k = (x_k^1, x_k^2)$. We have the discrete system model

$$\begin{aligned} x_k^1 &= x_{k-1}^1 - \alpha\Delta \cdot x_{k-1}^2 \\ &+\beta\Delta \cdot \frac{x_{k-1}^1}{(x_{k-1}^1)^2+(x_{k-1}^2)^2} + \sigma_1\sqrt{\Delta} \cdot w_{k-1}^1 \\ x_k^2 &= x_{k-1}^2 + \alpha\Delta \cdot x_{k-1}^1 \\ &+\beta\Delta \cdot \frac{x_{k-1}^2}{(x_{k-1}^1)^2+(x_{k-1}^2)^2} + \sigma_2\sqrt{\Delta} \cdot w_{k-1}^2. \end{aligned} \tag{3.37}$$

The mathematical formula for the measurement equation is given by

$$y_k = \arctan(\frac{x_k^2 - x_{platform}^2}{x_k^1 - x_{platform}^1}) + \sqrt{\Delta}v_k, \tag{3.38}$$

where $x_{platform} = (x_{platform}^1, x_{platform}^2)$ is the location of the platform where a detector is placed.

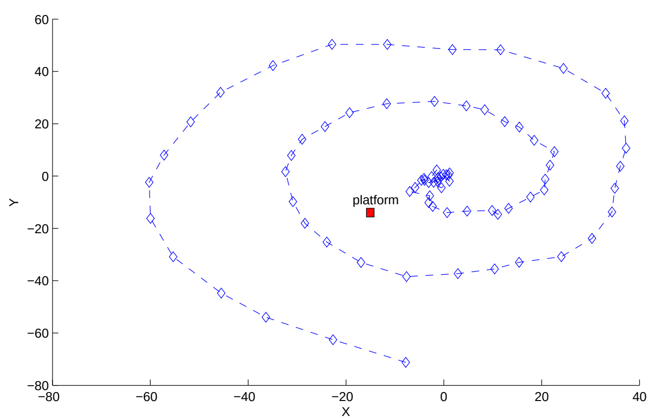


Figure 3.5: Target Positions

In the numerical simulations the model parameters are chosen as $\alpha = 5$, $\beta = 2$ and $\sigma_1 = \sigma_2 = 8$. Figure 3.5 gives the target path in the x-y plan, with the position of the target at each time step shown by a diamond. The location of the detector platform is chosen as $x_{platform} = (-15, -15)$, marked by a red box.

The problem is initialized with a best guess of the target position at the initial time, which is $(x_0^1, x_0^2) = (0.5, 0.5)$. In this example, we use 8000 particles (sample points) to represent the pdf in the particle filter method and in the implicit filter, we use 1600 nodes to partition the region and the number of Monte-Carlo samples is $M = 10$. p Figure 3.6 shows the simulation results of observation angle using EKF, particle filter and our implicit particle filter method. From this figure, one can see that both particle filter and the implicit particle filter produce good approximation for the relative observation angle of the target. Although the EKF provides the trend of the movement of the target, the estimation is a few steps delayed from the true target observation angle.

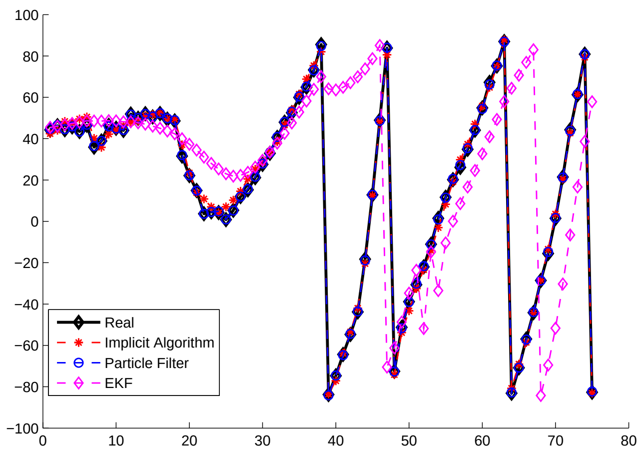


Figure 3.6: Comparison result for the observation angle

Figure 3.7 shows the results of system state simulations using EKF while Figure 3.8 compares the performance between particle filter and the implicit particle filter. Clearly both the particle filter and implicit particle filter outperform EKF. While the accuracy of the particle filter and the implicit particle filter are very close to each other at the initial stage, the implicit particle filter becomes more accurate at the final stage of time interval.

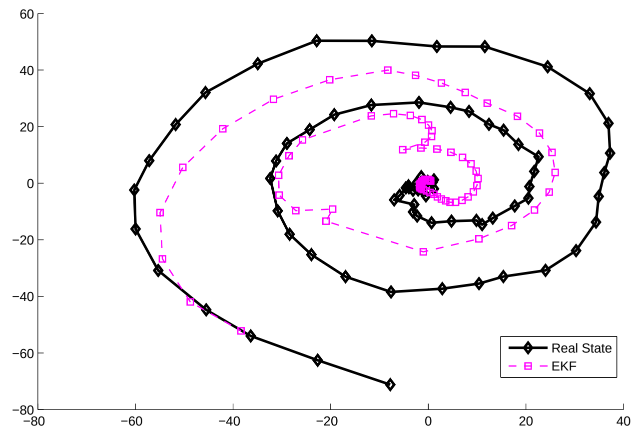


Figure 3.7: Simulation result of EKF

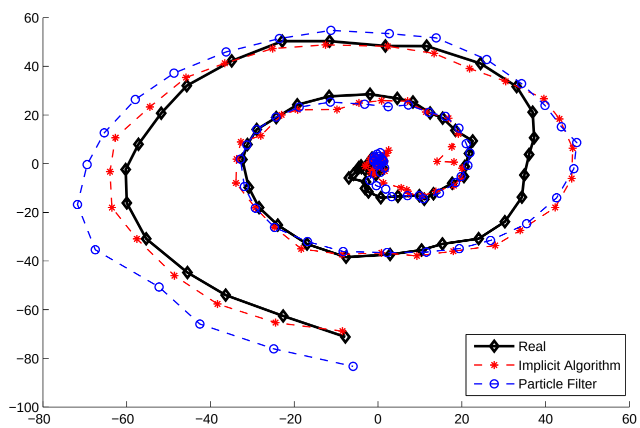


Figure 3.8: Comparison result between Particle Filter and implicit Algorithm

Chapter 4

A Hybrid Sparse Grid Approach for Nonlinear Filtering Problems Based on Adaptive-Domain of the Zakai Equation Approximations

In this Chapter, we develop an efficient hybrid sparse grid approach for nonlinear filtering problems based on numerical approximations of the Zakai equation. Consider the following stochastic differential system that combines an equation for the "state" and for the "observation" defined on the probability space $(\Omega, \mathcal{F}, P)$

$$\begin{cases} dX_t = b(X_t)dt + \sigma(X_t)dW_t, & (state) \\ dY_t = h(X_t)dt + dB_t. & (observation) \end{cases} \tag{4.1}$$

Here $\{X_t \in \mathbb{R}^d, t \geq 0\}$ and $\{Y_t \in \mathbb{R}^r, t \geq 0\}$ are two stochastic processes, $\{W_t, t \geq 0\}$ and $\{B_t, t \geq 0\}$ are independent Brownian Motions in $\mathbb{R}^p$ and $\mathbb{R}^r$, with covariance matrices C_W (identity) and C_B, respectively, and the given initial value X_0 is independent of W_t and B_t with probability distribution $u_0(x)dx$.

4.1 Nonlinear Filtering Problems and Zakai Equations

Now, we outline the derivation of the Zakai equation and its relationship to the nonlinear filtering problem (4.1). Throughout this Chapter, we assume that the coefficients $b : \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times p}$ and $h : \mathbb{R}^d \to \mathbb{R}^r$ in (4.1) are globally Lipschitz continuous functions. Denote

$$\rho(t) \doteq \exp\left\{ \int_0^t h^*(X_s)dY_s - \frac{1}{2}\int_0^t |h(X_s)|^2 ds \right\},$$

then the measure $\tilde{P}$ defined by $\tilde{P} = \rho(t)dP$ is also a probability measure on $(\Omega, \mathcal{F})$ equivalent to P . Furthermore, in the probability space $(\Omega, \mathcal{F}, \tilde{P})$, Y_t is a Browanian motion independent of X_t (for details, see [82]).

Assuming that $u = u(t, x)$ is the conditional density function of the state X_t given an observed path Y_t, then the optimal filtering solution is given by (see [82, 85])

$$E(\Phi(X_t) \mid \mathcal{Y}_t) = \frac{\int \Phi(x)u(t,x)dx}{\int u(t,x)dx}. \tag{4.2}$$

Under regularity assumptions for the coefficients b and h given above, u satisfies the following stochastic partial differential equation, known as Zakai equation

$$du(t,x) = L^* u(t,x)dt + h^*(x)u(t,x)dY_t, \quad x \in R^d, \tag{4.3}$$

with the initial value $u(0, x)$, and L the infinitesimal generator associated with the state process X_t such that

$$Lu = \frac{1}{2}\sum_{i,j}^{d}(\sigma\sigma^*)_{i,j}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i \frac{\partial u}{\partial x_i}, \tag{4.4}$$

and "$*$" is the transpose operator which transforms Lu to be

$$L^* u = \frac{1}{2}\sum_{i,j}^{d}\frac{\partial^2(\sigma\sigma^*)_{i,j}u}{\partial x_i \partial x_j} - \sum_{i=1}^{d}\frac{\partial b_i u}{\partial x_i}. \tag{4.5}$$

The goal of the Zakai filter method is to obtain numerical solutions of the Zakai equation (4.3). However, there are several challenges in the construction of an efficient numerical algorithm for the Zakai equation: (i) high-dimensionality of the state equations; (ii) low regularity of the solution; and (iii) unbounded solution domain. In the next two sections we construct a hybrid algorithm combining the ideas of split-up finite difference method, sparse grid interpolation and the importance sample approach to overcome these obstacles.

4.2 Hierarchical Local Sparse Grid Interpolation

In this section, we introduce a sparse grid interpolation constructed from a local hierarchical basis which will be used in the finite difference approximation of the Zakai equation in the spatial domain.

4.2.1 Standard hierarchical sparse grid ipnterpolation

Assume that we have the following one dimensional interpolation formula at our disposal:

$$\mathcal{Q}^i(u) = \sum_{j=1}^{m_i} u(x_j^i) \cdot \phi_j^i(x), \quad x \in R, \tag{4.6}$$

where $i \in \mathbb{N}$ is the *resolution level* of the interpolant $\mathcal{Q}^i$, m_i is the number of grid points on level i, x_j^i and $\phi_j^i(x)$ for $j = 1, \ldots, m_i$ are the interpolation points and the corresponding basis functions, respectively. In the context of linear hierarchical interpolation, m_i, x_j^i and ϕ_j^i in the standard interval $[-1, 1]$ for $i \in \mathbb{N}, j = 1, \ldots, m_i$ are defined by

$$m_i = \begin{cases} 1, & \text{if } i = 1, \\ 2^{i-1} + 1, & \text{if } i > 1, \end{cases} \tag{4.7}$$

$$x_j^i = \begin{cases} 0, & \text{for } j = 1, & \text{if } m_i = 1, \\ \dfrac{2(j-1)}{m_i - 1} - 1, & \text{for } j = 1, \ldots, m_i, & \text{if } m_i > 1, \end{cases} \tag{4.8}$$

and for $i = 1$, $\phi_1^1 = 1$; for $i > 1$ and $j = 1, \ldots, m_i$,

$$\phi_j^i = \begin{cases} 1 - \dfrac{m_i - 1}{2} \cdot |x - x_j^i|, & \text{if } |x - x_j^i| < \dfrac{2}{m_i - 1}, \\ 0, & \text{otherwise}. \end{cases} \tag{4.9}$$

Note that the nodal basis function ϕ_j^i has local support $[x_j^i - 2^{1-i}, x_j^i + 2^{1-i}]$.

In the multi-dimensional case, i.e. $d > 1$, the tensor-product interpolatant is

$$\left(\mathcal{Q}^{i_1} \otimes \cdots \otimes \mathcal{Q}^{i_d}\right)(u) = \sum_{j_1=1}^{m_{i_1}} \cdots \sum_{j_d=1}^{m_{i_d}} u\left(x_{j_1}^{i_1}, \cdots, x_{j_d}^{i_d}\right) \cdot \phi_{\mathbf{j}}^{\mathbf{i}}(\boldsymbol{x}), \tag{4.10}$$

where $\phi_{\mathbf{j}}^{\mathbf{i}} = \prod_{k=1}^{d} \phi_{j_k}^{i_k}$. Clearly, the above product requires $\Pi_{i=1}^{d} m_i$ function values, which is computationally prohibitive when d is large. The sparse gird interpolation [17] is a linear combination of a series of tensor-product interpolants, each of which is defined on a coarse grid with different resolutions in different dimensions, i.e.,

$$\mathcal{I}^{L,d}(u) = \sum_{L-d+1 \leqslant |\mathbf{i}| \leqslant L} (-1)^{L-|\mathbf{i}|} \begin{pmatrix} d-1 \\ q-|\mathbf{i}| \end{pmatrix} \left(\mathcal{Q}^{i_1} \otimes \cdots \otimes \mathcal{Q}^{i_d}\right)(u), \tag{4.11}$$

where $L \geq d$, the multi-index $\mathbf{i} = (i_1, \ldots, i_d)$ and $|\mathbf{i}| = i_1 + \cdots + i_d$. Here, $i_k (k = 1, \ldots, d)$ is the level of the tensor-product interpolant $\mathcal{Q}^{i_1} \otimes \cdots \otimes \mathcal{Q}^{i_d}$ along the kth direction. The Smolyak algorithm builds the interpolant by adding a combination of all tensor-product interpolants satisfying $L - d + 1 \leq |\mathbf{i}| \leq L$. The structure of the algorithm becomes clearer when one considers the incremental interpolant, Δ^i given in [17]

$$\mathcal{Q}^0(u) = 0, \quad \Delta^i = \mathcal{Q}^i(u) - \mathcal{Q}^{i-1}(u). \tag{4.12}$$

The sparse grid interpolant (4.11) is then equivalent to

$$\mathcal{I}^{L,d}(u) = \sum_{|\mathbf{i}| \leqslant L} (\Delta^{i_1} \otimes \cdots \otimes \Delta^{i_d}) = \mathcal{I}^{L-1,d}(u) + \sum_{|\mathbf{i}| = L} (\Delta^{i_1} \otimes \cdots \otimes \Delta^{i_d})(u). \tag{4.13}$$

The corresponding sparse grid associated with $\mathcal{I}^{L,d}(u)$ is represented by

$$\mathcal{H}^{L,d} = \bigcup_{L-d+1 \leqslant |\mathbf{i}| \leqslant L} (\chi^{i_1} \times \cdots \times \chi^{i_d}), \tag{4.14}$$

where χ^i denotes the set of interpolation points used by $\mathcal{Q}^i$. According to (4.13), to extend the Smolyak interpolant $\mathcal{I}^{L,d}(u)$ from level $L-1$ to L, one only needs to evaluate the function at the incremental grid $\Delta\mathcal{H}^{L,d}$ defined by

$$\Delta\mathcal{H}^{L,d} = \bigcup_{|\mathbf{i}|=L} (\Delta\chi^{i_1} \times \cdots \times \Delta\chi^{i_d}), \tag{4.15}$$

where $\Delta\chi^{i_j} = \chi^{i_j} \setminus \chi^{i_j-1}$, $j = 1, \ldots, d$. According to the nested structure of the one-dimensional hierarchical grid defined by (4.8), it is easy to see that $\chi^{i-1} \subset \chi^i$ and $\Delta\chi^i = \chi^i \backslash \chi^{i-1}$ has $m^i_\Delta = m^i - m^{i-1}$ points. By consecutively numbering the points in $\Delta\chi^i$, and denoting the jth point of $\Delta\chi^i$ as x^i_j, the incremental interpolant in (4.12) can be represented by (see [17, 55] for details)

$$\Delta^i(u) = \sum_{j=1}^{m^i_\Delta} \phi^i_j \cdot \left[u(x^i_j) - \mathcal{Q}^{i-1}(u)(x^i_j)\right], \tag{4.16}$$

where $\omega^i_j = u(x^i_j) - \mathcal{Q}^{i-1}(u)(x^i_j)$ is defined as the one-dimensional hierarchical surplus on level i. This is just the difference between the values of the interpolating polynomials and the function evaluated at x^i_j. From (4.16), the hierarchical sparse grid interpolant (4.13) can be rewritten as

$$\begin{aligned}
\mathcal{I}^{L,d}(u) &= \mathcal{I}^{L-1,d}(u) + \sum_{|\mathbf{i}|=L} (\Delta^{i_1} \otimes \cdots \otimes \Delta^{i_d})(u) \\
&= \mathcal{A}^{L-1,d}(u) + \sum_{\substack{|\mathbf{i}|=L \\ \mathbf{j}\in B_\mathbf{i}}} \omega^\mathbf{i}_\mathbf{j} \cdot \phi^\mathbf{j}_\mathbf{i}(\boldsymbol{x}) \\
&= \sum_{|\mathbf{i}|\leqslant L} \sum_{\mathbf{j}\in B_\mathbf{i}} \omega^\mathbf{i}_\mathbf{j} \cdot \phi^\mathbf{j}_\mathbf{i}(\boldsymbol{x}),
\end{aligned} \tag{4.17}$$

where the multi-index set $B_\mathbf{i}$ is

$$B_\mathbf{i} = \left\{\mathbf{j} \in \mathbb{N}^d : x^{i_k}_{j_k} \in \Delta\chi^{i_k} \text{ for } j_k = 1, \ldots, m^{i_k}_\Delta, k = 1, \ldots, d\right\}, \tag{4.18}$$

and the surpluses $\omega_{\mathbf{j}}^{\mathbf{i}}$ are

$$\omega_{\mathbf{j}}^{\mathbf{i}} = u(x_{j_1}^{i_1}, \ldots, x_{j_d}^{i_d}) - \mathcal{I}^{L-1,d}(u)(x_{j_1}^{i_1}, \ldots, x_{j_d}^{i_d}). \tag{4.19}$$

As proved in [17], for smooth functions, the hierarchical surpluses tend to zero as the interpolation level tends to infinity. On the other hand, the magnitude of the surplus is a good indicator about the smoothness of the interpolated function. In general, the larger the magnitude, the stronger the underlying discontinuity.

For a bounded box domain $\mathcal{D} \subset \mathbb{R}^d$:

$$\mathcal{D} = [\boldsymbol{a}, \boldsymbol{b}] := \Pi_{i=1}^{d}[a_i, b_i], \tag{4.20}$$

we first transform it to $[-1, 1]^d$ through a simple linear transform. The corresponding sparse grid interpolation is then defined according to (4.11).

4.2.2 Hierarchical sparse grid approximation of bell-shaped cpurves

In many nonlinear filtering problems in practical applications, the conditional target state PDF resembles a "bell-shaped" Gaussian curve or surface, if not exactly Gaussian. In such cases, standard hierarchical sparse grid interpolations may lead to large errors. This is especially the case for two or higher dimensional problems. As a demonstration, we consider a Gaussian-type function

$$U(x) \doteq \prod_{i=1}^{d} \exp\left\{-\frac{1}{2}\left(\frac{x_i - \mu}{\sigma}\right)^2\right\}, \quad x \in \mathbb{R}^d, \tag{4.21}$$

where $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}^+$. From [17], we know that the L^2 error between the function U and the standard sparse grid approximation $\bar{U}$ is

$$\|U - \bar{U}\|_{L^2} \leq \frac{2 \cdot |U|_{\mathbf{2},2}}{12^d} \cdot 2^{-2L} \cdot A(d, L), \tag{4.22}$$

where $L \in \mathbb{N}^+$ represents the level of hierarchical sparse grid with

$$A(d, L) = \frac{L^d}{(d-1)!} + \mathcal{O}(k^{d-2}), \text{ and } |U|_{\mathbf{2},2} = \|D^{\mathbf{2}}U\|_{L^2}.$$

It can be shown that $|U|_{\mathbf{2},2}$ in (4.22) is bounded by (see [64] for detailed derivation)

$$|U|_{\mathbf{2},2} \leq \frac{1}{\sigma^{4d}}(\max\{x_i - \mu\}^{2d})\|U\|_{L^2}. \tag{4.23}$$

We can see from the above estimate that when the constant σ is small, the right hand side

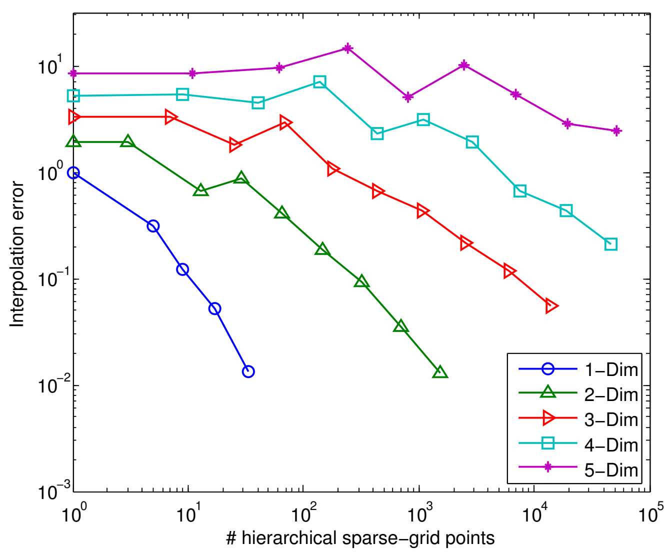


Figure 4.1: The interpolation error for regular sparse grid approximation up to 5 dimensions.

becomes exponentially large as d increases.

Here, we consider a special case of (4.21) with $\mu = 0$ and $\sigma = 0.15$, i.e.,

$$U(x) = \prod_{i=1}^{d} \exp\left(-\frac{x_i^2}{0.045}\right). \tag{4.24}$$

Figure 4.1 shows the interpolation errors using the L_2-norm for regular sparse grid approximations of U. We can see that as the dimension d increases, the interpolation errors barely decrease even, though the number of interpolation points has increased significantly.

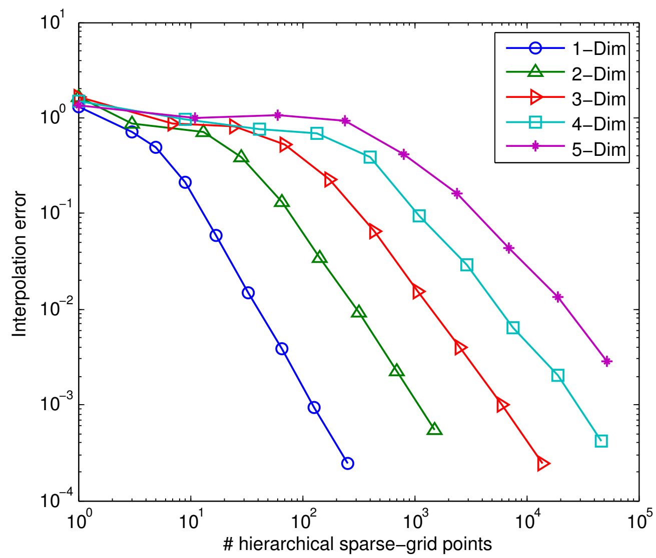


Figure 4.2: The interpolation error for logarithmic sparse grid approximation up to 5 dimensions.

One way of alleviating this poor performance of the sparse grid interpolant when approximating bell-shaped-functions is to utilize logarithmic interpolation (see [64]), in which we take the logarithm of U, i.e. $V \doteq \log(U)$, and build the approximation $\bar{V}$ using the standard sparse grid approach. Then we obtain the approximation of U by $\bar{U} = e^{\bar{V}}$. Figure 4.2 shows the absolute interpolation errors measured in L_2-norm for the logarithmic sparse grid approximation of function U defined in (4.24). Compared with Fig. 4.1, we can see from Figure 4.2 that the convergence is improved as the dimension increases.

A visual demonstration of the efficiency of the logarithmic sparse grid interpolation is shown in Fig. 4.3, where we plot the level 6 approximation for a marginal distribution of both U and $\log(U)$ in $d = 4$ dimensions.

4.3 Hybrid Approach for Numerical Solution of the Nonlinear Filtering Problem.

In this section we describe our hybrid numerical algorithm for the solution of the Zakai equation (4.3).

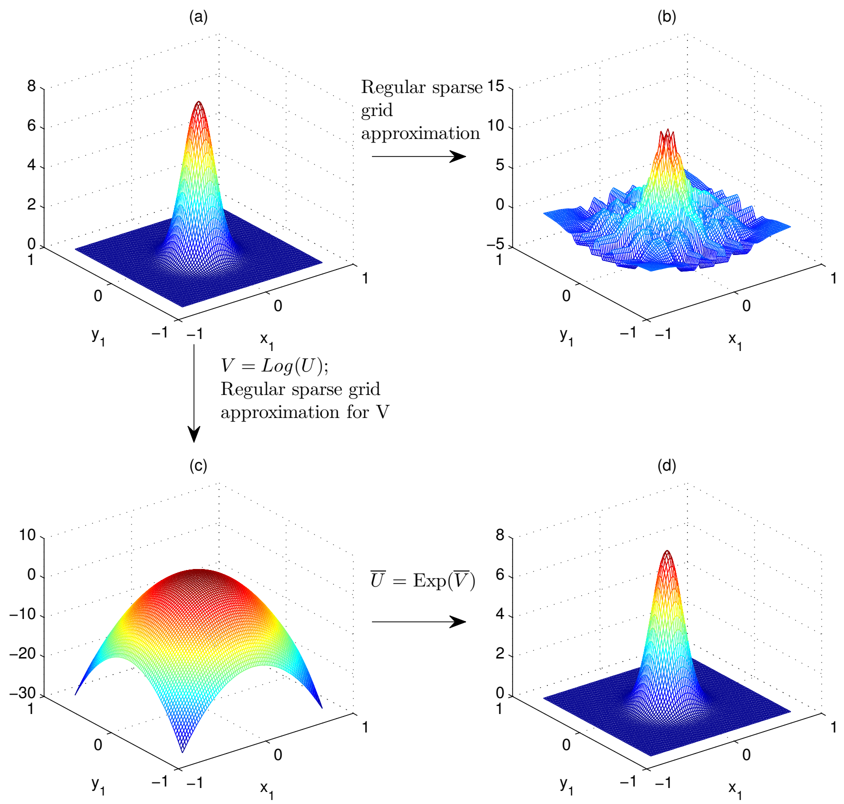


Figure 4.3: (a) is the x_1x_2-marginal surface of U with $x_3 = x_4 = 0$ while (b) is its interpolate approximation obtained by applying the regular sparse grid approximation to function U. We can see from the Figure that the approximation is quite poor and significant oscillations occur at the bottom of the surface. On the other hand, from (d) one can see that the logarithmic interpolation approximation described above is far more accurate.

4.3.1 Adaptive selection of solution domains

Since the Zakai equation (4.3) is defined on the whole space R^d, it is essential to choose a proper bounded domain to achieve an accurate numerical approximation. Motivated by importance sampling and particle filter methods, we adaptively select a hypercube at each time step according to an estimation of the density function of the solution at the next step.

In particular, let $\mathcal{R}_t$ be a partition of $[0, T]$ such that:

$$\mathcal{R}_t = \{t_n | t_n \in [0, T], t_n < t_{n+1}, n = 0, 1, \cdots, N_T - 1, t_0 = 0, t_{N_T} = T\}$$

and denote $\Delta t_n = t_{n+1} - t_n$, $n = 0, 1, \cdots, N_T - 1$. For $n = 0, 1, \cdots N_T$, assume that u_n is the numerical solution of the Zakai equation (4.3) at t_n. We use an importance sampling method to draw M realizations according to the conditional PDF u_n of the state X_n, denoted by $\{p_n^m\}_{m=1,\cdots,M}$, where M is a pre-defined positive integer. We then propagate each of these samples from time step t_n to t_{n+1} using the *state equation* in the nonlinear filtering problem (4.1) to get M updated sample points, denoted by $\{p_{n+1}^m\}_{m=1,\cdots,M}$. To complete our adaptive domain selection, we denote

$$\mathcal{D}_{n+\frac{1}{2}} = [\boldsymbol{a}_{n+\frac{1}{2}}, \boldsymbol{b}_{n+\frac{1}{2}}] \subset \mathbb{R}^d$$

as the smallest hypercube containing all the samples $\{p_{n+1}^m\}_{m=1,\cdots,M}$ and $\boldsymbol{\Sigma} = (\Sigma^1, \ldots, \Sigma^d)$ as the vector of marginal standard deviations of these samples. Then, for a user defined positive constant λ we let

$$\begin{aligned} \boldsymbol{a}_{n+1} &\doteq \boldsymbol{a}_{n+\frac{1}{2}} - \lambda \boldsymbol{\Sigma} \\ \boldsymbol{b}_{n+1} &\doteq \boldsymbol{b}_{n+\frac{1}{2}} + \lambda \boldsymbol{\Sigma} \end{aligned} \tag{4.25}$$

and finally we choose

$$\mathcal{D}_{n+1} = [\boldsymbol{a}_{n+1}, \boldsymbol{b}_{n+1}] \tag{4.26}$$

as the solution domain for u_{n+1}.

Remark 1 *The idea of the adaptive selection of solution domain $\mathcal{D}_n$ is similar to that in the prediction step of particle filter method. As such the region $\mathcal{D}_{n+\frac{1}{2}}$ which includes all updated samples is similar to the domain where particle filter method builds the prior PDF of the target state. In our approach, we choose a confidence region surrounding $\mathcal{D}_{n+\frac{1}{2}}$ as our solution domain. This way, we can use a much smaller number of samples than the particle filter technique and maintain the accuracy of our approximation. In addition, our numerical experiments also indicate that this adaptive domain selection approach is more accurate at predicting the tail distribution than the particle filter method.*

4.3.2 Spliting-up finite difference method on sparse grid

Following [10, 48, 85], the splitting-up scheme for (4.3) consists of prediction and update steps, described in Sections 4.2.1 and 4.2.2 respectively. At each time step t_n, we define the sampled observation Z_n from the observation process Y_t by

$$Z_n \doteq \frac{Y_{t_{n+1}} - Y_{t_n}}{\Delta t_n} = \frac{1}{\Delta t_n} \cdot \left(\int_{t_n}^{t_{n+1}} h(X_s) ds + B_{t_{n+1}} - B_{t_n} \right).$$

In what follows, we denote u_n to be the approximate solution for u_t at $t = t_n$, $n = 0, 1, 2, \cdots, N_T$.

Prediction Step

In the prediction step we solve for $u_{n+\frac{1}{2}}$ from

$$u_{n+\frac{1}{2}} = u_n + L^* u_n \Delta t_n, \tag{4.27}$$

which is equivalent to solving the deterministic PDE, known as the Fokker Plank equation [82, 85],

$$\frac{\partial u_t}{\partial t} = L^* u_t, \quad t_n \leq t \leq t_{n+1} \tag{4.28}$$

with an one-step forward Euler scheme. An efficient spatial discretization technique for the Fokker-Planck equation (4.28) in the prediction step is essential to the success of the splitting-up scheme. In the one-dimensional case, a simple finite difference method to discretize the operator L is sufficient. In a straight forward fashion, such a discretization can be extended to multi-dimensional case with use of a direct tensor product. However, the computational cost of the numerical solution based on a tensor-product approximation increases exponentially as the dimension d increases, known as the "*curse of dimensionality*". To alleviate this numerical challenge and reduce the overall computational complexity, we use the sparse-grid method described in section 3 to construct a finite difference algorithm for solving (4.28).

To see this, we present the upwind finite difference method for approximating the partial differential operator L defined in (4.4) on a sparse grid. First, let u_0 be the initial value of the solution u of the Zakai equation (4.3). For a positive integer n, let $\mathcal{H}_n^{L,d}$ be the set of sparse grid points defined by (4.15) in the hypercube $\mathcal{D}_n$ defined by (4.14). For $n = 0, \cdots, N_T$ and $x \in \mathcal{H}_{n+1}^{L,d}$, we approximate the first order partial derivative with given coefficient function $\mu \in \mathbb{R}^d$ by

$$\mu_i \frac{\partial u_n}{\partial x_i}(x) \approx \mu_i \tilde{D}_{x_i} u_n(x) \doteq \begin{cases} \mu_i \dfrac{\hat{u}_n(x + e_i h_i) - u_n(x)}{h_i} & \text{if } \mu_i \geq 0 \\ \mu_i \dfrac{u_n(x) - \hat{u}_n(x - e_i h_i)}{h_i} & \text{if } \mu_i < 0 \end{cases},$$

where e_i is the unit vector in the ith coordinate direction and h_i is a properly chosen meshsize in the ith coordinate direction.

To approximate second order partial derivatives at the sparse grid point $x \in \mathcal{H}_{n+1}^{L,d}$ with given coefficient function $\alpha \in \mathbb{R}^{d \times d}$, we use central differences to obtain

$$\alpha_{i,i} \frac{\partial^2 u_n}{\partial x_i \partial x_i}(x) \approx \alpha_{i,i} \tilde{D}^2_{x_i x_i} u_n(x) \doteq \alpha_{i,j} \frac{\hat{u}_n(x + e_i h_i) - 2u_n(x) + \hat{u}_n(x - e_i h_i)}{h_i^2}$$

and

$$\begin{aligned} &\alpha_{i,j} \frac{\partial^2 u_n}{\partial x_i \partial x_j}(x) \approx \alpha_{i,j} \tilde{D}^2_{x_i x_j} u_n(x) \\ \doteq &\begin{cases} \dfrac{\alpha_{i,j}}{2h_i} \Bigg[\dfrac{\hat{u}_n(x + e_i h_i + e_j h_j) - \hat{u}_n(x + e_i h_i)}{h_j} - \dfrac{\hat{u}_n(x + e_j h_j) - u_n(x)}{h_j} \\ \quad + \dfrac{u_n(x) - \hat{u}_n(x - e_j h_j)}{h_j} - \dfrac{\hat{u}_n(x - e_i h_i) - \hat{u}_n(x - e_i i - e_j h_j)}{h_j} \Bigg], \text{if } \alpha_{i,j} \geq 0, \\ \dfrac{\alpha_{i,j}}{2h_i} \Bigg[\dfrac{\hat{u}_n(x + e_i h_i) - \hat{u}_n(x + e_i h_i - e_j h_j)}{h_j} - \dfrac{u_n(x) - \hat{u}_n(x - e_j h_j)}{h_j} \\ \quad + \dfrac{\hat{u}_n(x + e_j h_j) - u_n(x)}{h_j} - \dfrac{\hat{u}_n(x - e_i h_i + e_j h_j) - \hat{u}_n(x - e_i h_i)}{h_j} \Bigg], \text{if } \alpha_{i,j} < 0. \end{cases} \end{aligned}$$

With the above finite difference operators in hand, we define the finite difference approximation of the Fokker-Planck equation (4.28) on sparse grid $\mathcal{H}_{n+1}^{L,d}$ as follows.

$$u_{n+\frac{1}{2}}(x) = u_n(x) + L_n^* u_n(x)\Delta t_n, \quad x \in \mathcal{H}_{n+1}^{L,d}, \tag{4.29}$$

where

$$\begin{aligned} L_n^* u_n(x) = \frac{1}{2}\sum_{i,j}^{d} \Bigg\{ & \frac{\partial^2(\sigma\sigma^*)_{i,j}}{\partial x_i \partial x_j} u_n(x) + \frac{\partial(\sigma\sigma^*)_{i,j}}{\partial x_i}\tilde{D}_{x_j} u_n(x) + \frac{\partial(\sigma\sigma^*)_{i,j}}{\partial x_j}\tilde{D}_{x_i} u_n(x) \\ & + (\sigma\sigma^*)_{i,j}\tilde{D}^2_{x_i x_j} u_n(x) \Bigg\} - \sum_{i=1}^{d}\left(\frac{\partial b^i}{\partial x_i} u_n(x) + b^i \tilde{D}_{x_i} u_n(x)\right). \end{aligned}$$

Update Step

In the update step, we use the new observation Z_n and the Bayes formula [**?**] to update the prior $u_{n+\frac{1}{2}}$ to the posterior u_{n+1} as follows.

$$u_{n+1}(x) = C_n \Psi^n(x, Z_n) u_{n+\frac{1}{2}}, \quad x \in \mathcal{H}_{n+1}^{L,d}, \tag{4.30}$$

where C_n is a normalization factor and function Ψ^n is defined by

$$\Psi^n(x, Z_n) = \exp\left\{ -\frac{\Delta t_n}{2} \cdot |Z_n - h(x)|_R^2 \right\}.$$

The norm $|\cdot|_R$ is defined by $|\alpha|_R^2 = \alpha R^{-1}\alpha$, where R is the covariance matrix of $\{B_t, t \geq 0\}$ in (4.1) (Please see Page 2, [85] for more details). Finally following the procedure described in Section 3, we derive the logarithmic sparse grid interpolation $u_{n+1} = u_{n+1}(x),\ x \in \mathbb{R}^d$ using its values on the sparse grid $\mathcal{H}_{n+1}^{L,d}$.

We summarize our hybrid sparse grid adaptive-domain splitting-up finite difference algorithm as follows:

Step 1: Input u_0 as the initial value of the solution u of the Zakai equation (4.3).

Step 2: For $n = 0 \cdots, N_T - 1$,

1 Compute dynamic domain $\mathcal{D}_{n+1}$ for the solution u_{n+1} using the importance sampling method.

2 Generate sparse grid $\mathcal{H}_{n+1}^{L,d}$ on the solution domain $\mathcal{D}_{n+1}$.

3 Evaluate u_{n+1} on the sparse grid $\mathcal{H}_{n+1}^{L,d}$ by using finite difference scheme (4.29).

4 Extend the solution u_{n+1} to the whole space R^d through the logarithmic sparse grid interpolation described in Section 3.

Step 3: Normalization.

Remark 2 *Since we use an explicit finite difference scheme to solve equation (4.28), time step Δt_n must satisfy the following stability condition*

$$\max_{0 \le n \le N_T - 1} \Delta t_n \le \frac{1}{\sum_{i=1}^{d} \frac{|(\sigma\sigma^T)_{i,i}| + |b^i| h_i}{h_i^2}}.$$

4.4 Numerical Experiments

In this section, we present three numerical experiments to demonstrate the effectiveness of our new numerical algorithm, for solving nonlinear filtering problems.

Example 1

In the first example, we use a two dimensional nonlinear filtering problem to illustrate the accuracy of the selection process of the dynamic solution domain $\mathcal{D}_n$. To see this, we consider the following dynamical system

$$dX_t = (40, 2 \cdot (10t)^2)^T \ dt + 0.5 \ dW_t, \tag{4.31}$$

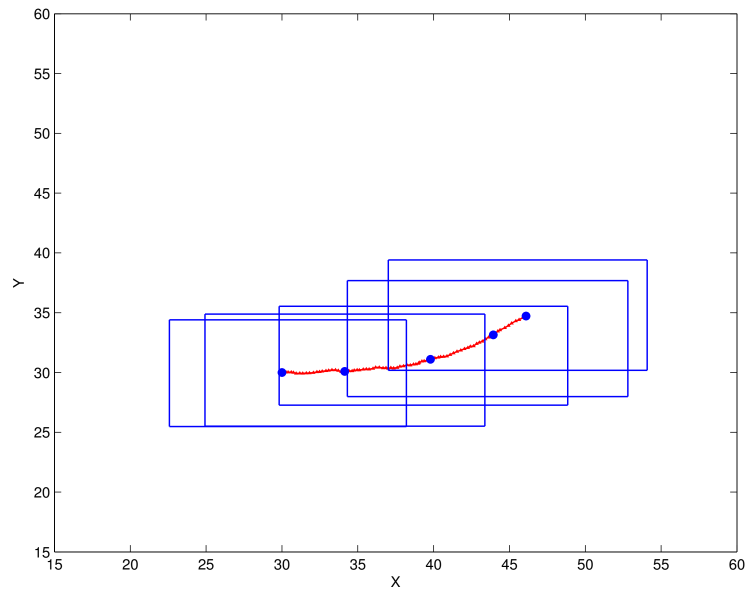


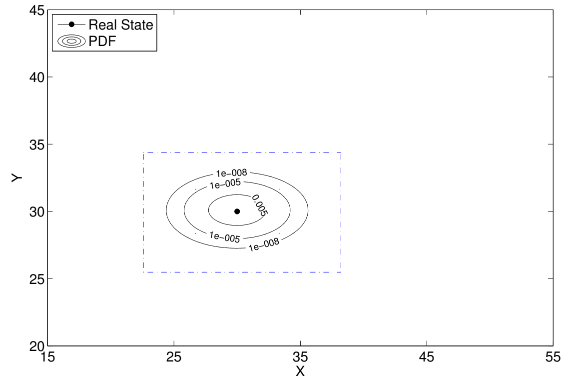
Figure 4.4: Target Trajectory and Adaptive Solution Domain. The red curve shows the real target state. The blue points are actual states of the target and the blue boxes are the corresponding solution domains.

where W_t is a two-dimensional Brownian Motion and the initial state is given by $X_0 = (30, 30)^T$. The observation process is given by

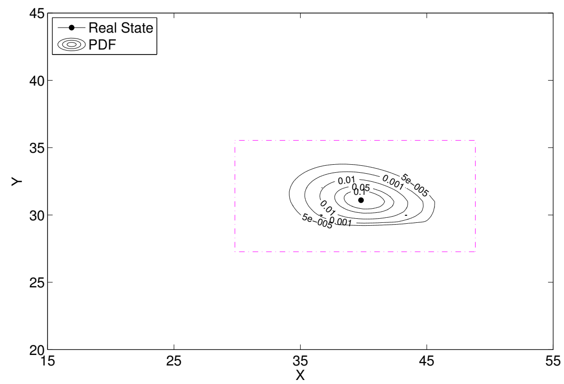
$$dY_t = \sqrt{(X_t^1 - 20)^2 + (X_t^2)^2} \cdot dt + dB_t, \tag{4.32}$$

which measures the perturbed distance between the target state and a reference point $P = (20, 0)$, and B_t is a one-dimensional Brownian motion independent of W_t.

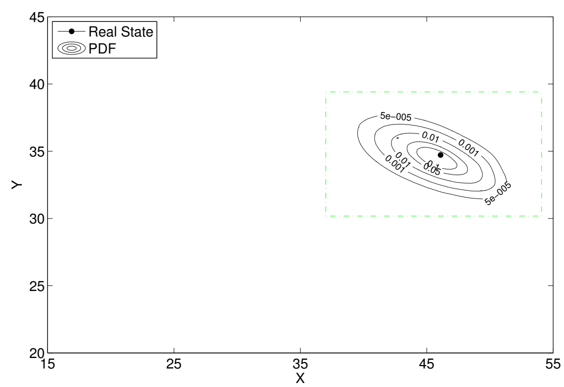
In this numerical simulation, we take $T = 0.4$ and use an uniform partition in time with stepsize $\Delta t_n = 0.005$. The initial value is given by $u_0 \sim N(X_0, \Sigma)$; a normal distribution with mean X_0 and standard deviation $\Sigma = (1, 0.5)^T$. In the adaptive solution domain selection process, we choose the sample size $M = 500$ and the parameter λ in (4.25) as $\lambda = 4$. Figure 4.4 shows the trajectory of the target state and solution domain $\mathcal{D}_n$ for $n = 1, 20, 50, 70, 80$. In Figure 4.5 we show solution domain $\mathcal{D}_n$ with the corresponding contour plot of target



(a)



(b)



(c)

Figure 4.5: Target state PDF at time step: (a) n=1; (b) n=50; (c)=80; in the corresponding solution domain

state PDF for $n = 1, 50, 80$. From this figure one can see the that solution domains are extremely accurate in approximating the high density area of u_n.

Example 2

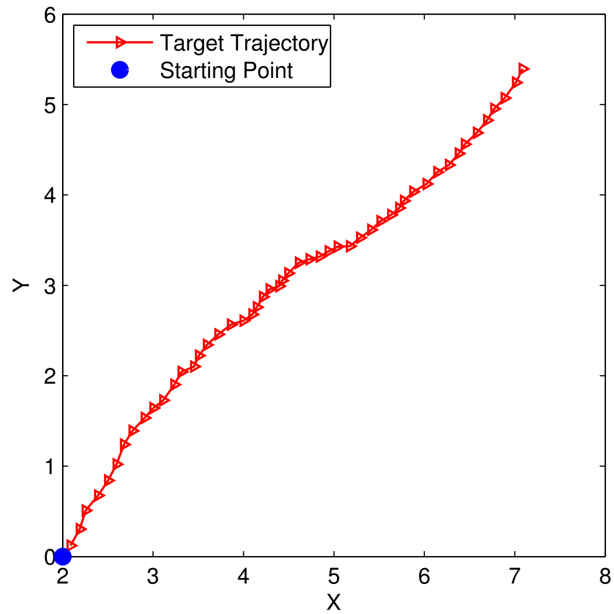
In this example, we consider the following nonlinear filtering problem:

$$\begin{cases} dX_t = (10, 6 \cdot (\sin X_t^1 + 2))^T \, dt + dW_t, & (state) \\ dY_t = (X_t^1, X_t^2)^T \, dt + dB_t. & (observation) \end{cases} \tag{4.33}$$

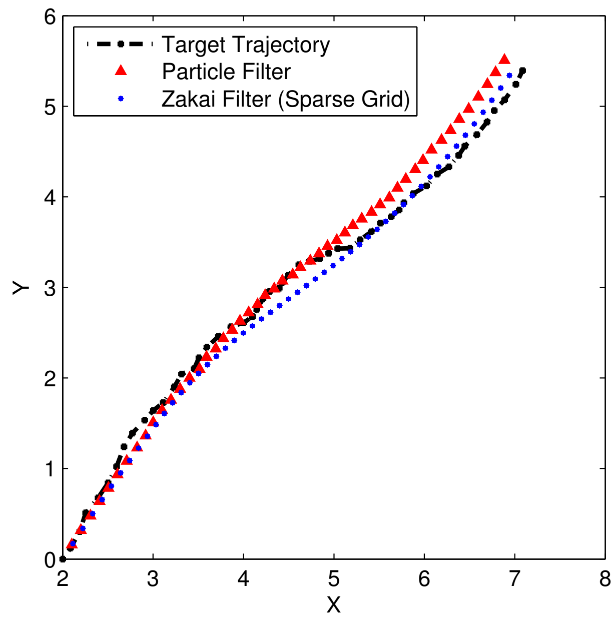
In this numerical experiment, we let $T = 0.5$ and use a uniform partition in time with stepsize $\Delta t_n = 0.01$. An example of the signal trajectory is shown in figure 4.6a. The initial value is given by $u_0 \sim N(X_0, \Sigma)$; a normal distribution with mean $X_0 = (2, 0)^T$ and standard deviation $\Sigma = (0.5, 0.5)^T$. We also choose the hierarchical sparse grid level as $L = 6$. In the adaptive solution domain selection process, the sample size and parameter λ are given by $M = 500$ and $\lambda = 4$, respectively.

Figure 4.6b shows the comparison of the estimated values of the target state between the particle filter and sparse grid Zakai filter. In the particle filter simulation, we use 8000 particles to represent the PDF of the target state. The black dashed line shows the trajectory of the real target state, the red triangles and blue dots show the estimate target state (the mean of the estimate posterior PDF) obtained by using the particle filter and the Zakai filter respectively. As we can see from Figure 4.6b, our hybrid approach is more accurate than the PFM for a longer period of time.

To further examine the performance of the hybrid sparse grid Zakai filter method, in Figure 4.7a and Figure 4.7b we plot the marginal probability density functions at time $T = 0.5$, obtained by using the particle filter and our hybrid with respect to xy-coordinates. From the plots we observe that the convergence of the particle filter as the particle size increases. Moreover, the PDE obtained by our hybrid sparse grid Zakai filter is very close to the one obtained by the particle filter with $160,000$ particles. However, in table 4.1 we can see that with $160,000$ particles the particle filter is far more costly in terms of CPU time than our method. Finally, we also show the confidence bands in Figure 4.8. The blue dashed



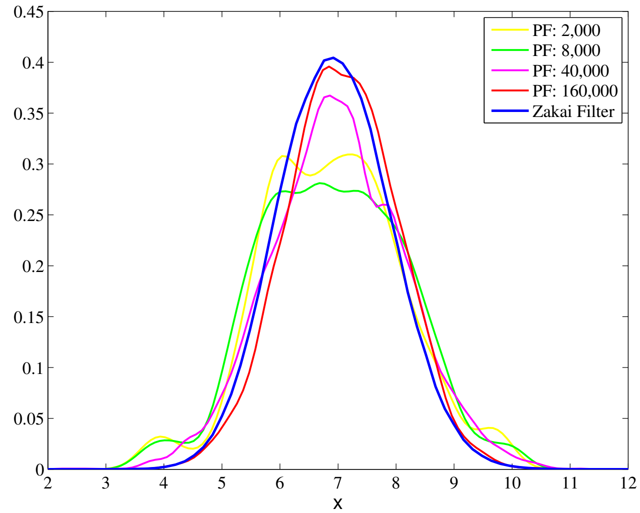
(a) Signal trajectory: Example 2



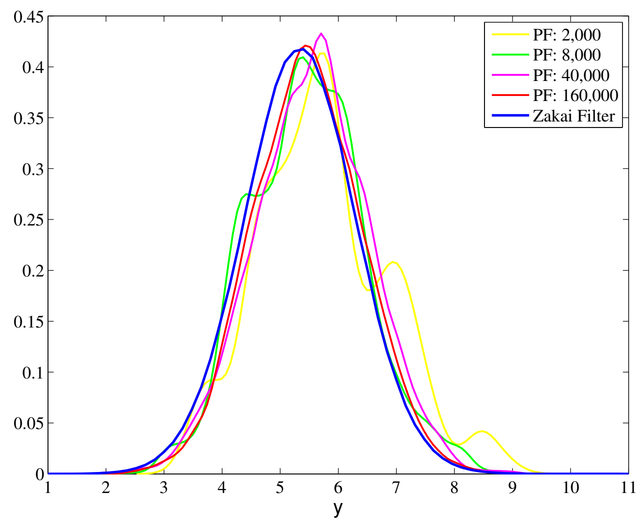
(b) Comparison of estimated mean of the target state between particle filter and sparse grid Zakai filter

Figure 4.6: 2D nonlinear filtering problem

curves show the real target trajectory with respect to xy-coordinates, respectively, the red



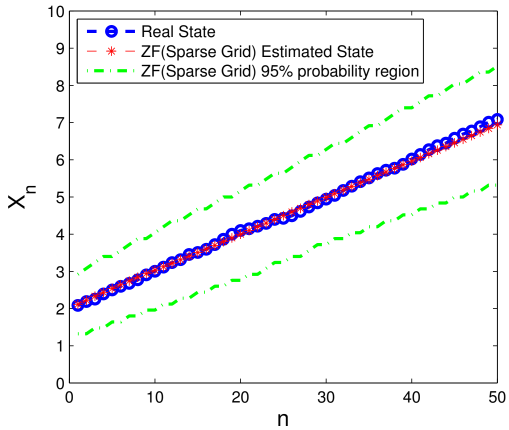
(a) Marginal distribution with respect to X-coordinate



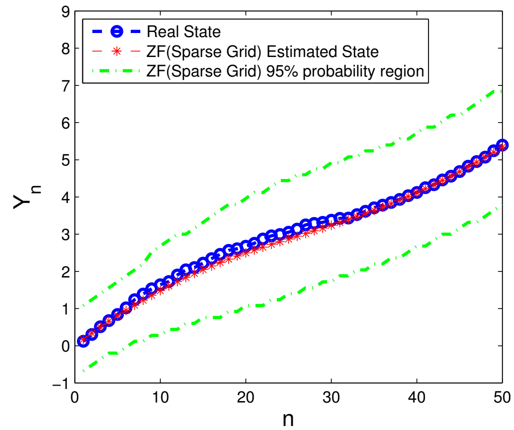
(b) Marginal distribution with respect to Y-coordinate

Figure 4.7: Marginal distributions

curves represent the estimate of posterior means, and the green dashed curves represent the estimated 95% confidence bands.



(a)



(b)

Figure 4.8: Sparse grid Zakai filter estimate of posterior mean with 95% probability region .
(a) Estimate of posterior mean and 95% probability region: X-coordinate
(b) Estimate of posterior mean and 95% probability region: Y-coordinate

Table 4.1: Comparison of computing costs

Particle filter	CPU time (seconds)
2,000 particles	1.49
8,000 particles	32.67
40,000 particles	516.91
160,000 particles	7622.29
Sparse grid Zakai filter	20.45

Example 3

In this example, we consider the following " bearing-only " tracking problem given by:

$$dX_t = bdt + \sigma dW_t, \tag{4.34}$$

where $X_t = (x, y, u, v)_t^T$ is a four-dimensional vector which models the movement of a target ship sailing on the sea plane ($x - y$ plane). Here (x, y) and (u, v) are the position and velocity components respectively, the vector $b = (u, v, 0, 0)_t^T$, the covariance matrix σ^2 is defined by

$$\sigma^2 = \begin{pmatrix} \sigma_1^2 & 0 & 0 & 0 \\ 0 & \sigma_2^2 & 0 & 0 \\ 0 & 0 & \sigma_3^2 & 0 \\ 0 & 0 & 0 & \sigma_4^2 \end{pmatrix}, \tag{4.35}$$

and W_t is a four-dimensional Brownian motion. To estimate the target state, a passive sonar

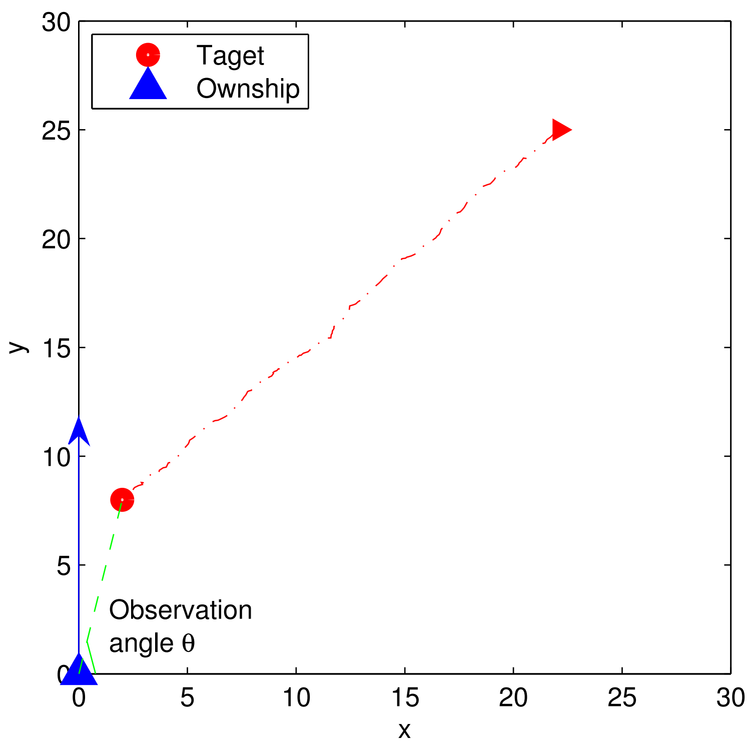


Figure 4.9: Bearing-only tracking

is located on an observation ship, denoted “ownship”. The observation process is given by:

$$dY_t = h(X_t)dt + dB_t,$$

where the observation function h is the angle

$$h(X_t) = \arctan\left(\frac{y_t - y_t^{obs}}{x_t - x_t^{obs}}\right),$$

and B_t is a one-dimensional Brownian motion independent from W_t. Here, $X_t^{obs} = \left(x^{obs}, y^{obs}\right)_t^T$ describes the movement of the ownship given by $dX_t^{obs} = b^{obs}dt$.

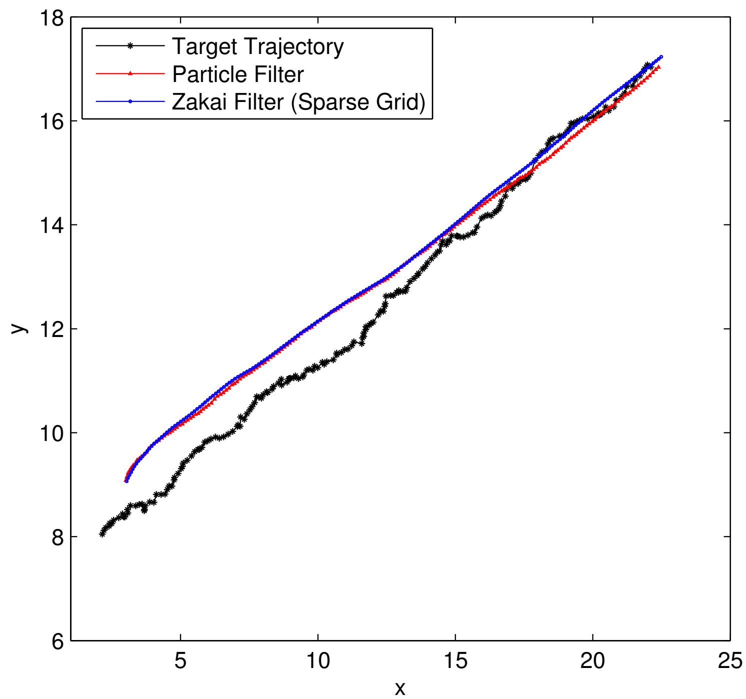


Figure 4.10: Comparison of estimated mean of the target state between particle filter and sparse grid Zakai filter

In the numerical simulations, we choose $\sigma_1 = \sigma_2 = 0.75$, $\sigma_3 = \sigma_4 = 0.05$ in (4.35) and $b^{obs} = (8, 0)^T$; thus the movement of the ownship is along the Y-axis with a constant speed. In addition, we set $T = 1$, the time partition $\Delta t_n = 0.005$, initial target state $X_0 = (2, 8, 20, 8)^T$, and the initial PDF of the target sate $N(\bar{X}, \Sigma)$, where $\bar{X} = (3, 9, 19.8, 7.9)^T$ and $\Sigma = (0.75, 0.75, 0.5, 0.25)^T$.

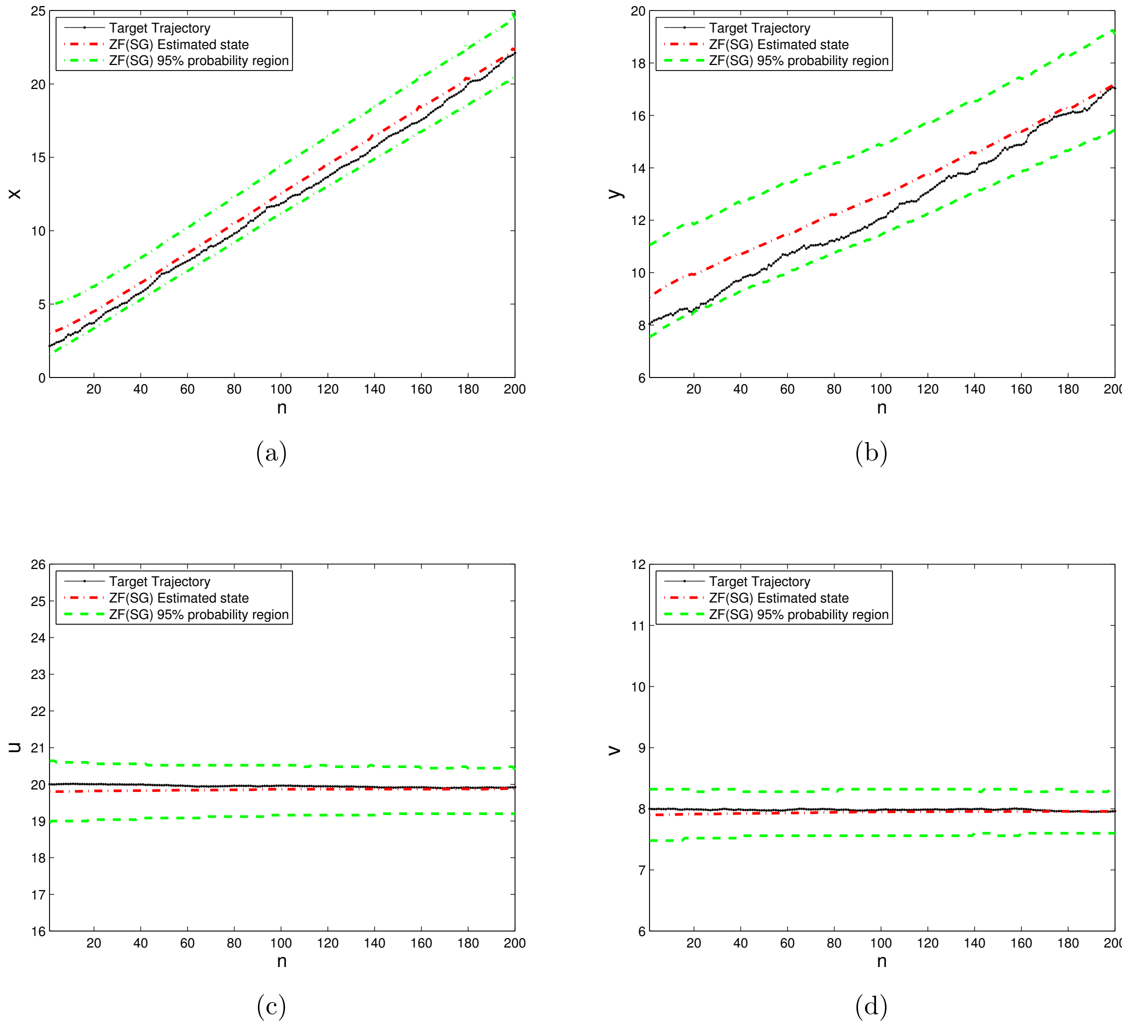
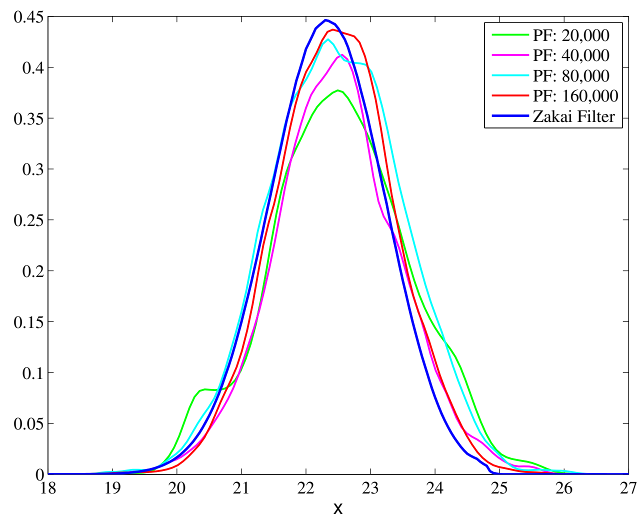


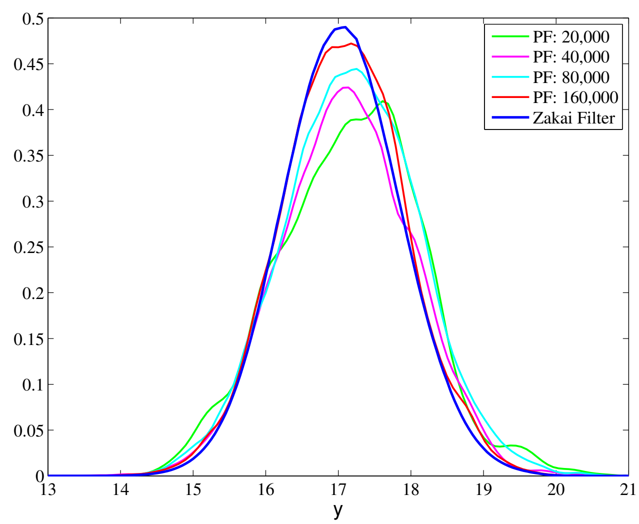
Figure 4.11: Sparse grid Zakai filter estimate of posterior mean with 95% probability region. (a) Estimate of posterior mean of the location x and 95% probability region. (b) Estimate of posterior mean of the location y and 95% probability region. (c) Estimate of posterior mean of the velocity u and 95% probability region. (d) Estimate of posterior mean of the velocity v and 95% probability region

The target-observer plane ($x - y$ plane) is illustrated in Figure 4.9. The red dot shows the initial position of the target ship while the blue triangle shows the initial position of our ownship. The dashed red curve gives a possible trajectory of the target ship and the blue arrow describes the movement of our ownship.

Figure 4.10 shows the comparison of the estimated mean values of the relative target position with respect to the ownship, in the target-observer plane. For the sparse grid Zakai



(a) Marginal distribution of the target state on x-coordinate



(b) Marginal distribution of the target state on y-coordinate

Figure 4.12: Marginal distributions

filter, we let the hierarchical level $L = 6$ and for the adaptive solution domain selection process, we set $M = 1,000$ and $\lambda = 4$. The estimate for the particle filter is obtained by using $160,000$ particles. The actual trajectory of the target position relative to our ownship is given by the black curve. The red curve and blue curve show the estimate target state (the mean of the estimate posterior PDF) obtained by using the particle filter and the Zakai

Table 4.2: Comparison of computing costs

Particle filter	CPU time (seconds)
20,000 particles	554.24
40,000 particles	1976.39
80,000 particles	7702.57
160,000 particles	32380.38
Sparse grid Zakai filter	587.14

filter respectively. We can see from Figure 4.10 that our hybrid sparse grid Zakai filter yields similar estimate results compared with the particle filter with $160,000$ particles.

In Figure 4.12a and 4.12b we compare the marginal probability distribution functions at time $T = 1$ with respect to xy coordinates using our hybrid sparse grid Zakai filter with the particle filter. Similar to Example 2, we see that the convergence of the particle filter with the particle size $160,000$ to the approximate PDE is very close to the approximate PDE obtained by the hybrid sparse grid Zakai filter. In table 4.2, we show the comparison of computational from which one can see that the particle filter with $160,000$ particles takes 60 times more computing time to achieve a similar PDF. We also plot the confidence curves in Figure 4.11 for further examination of the performance of sparse grid Zakai filter. The blue dashed curves show the real target trajectory with respect to xy coordinates. The red curves represent the estimate of posterior means, and the green dashed curves give the estimate 95% confidence curves (± 2 times of the estimated standard deviation).

Chapter 5

Numerical Algorithms for Backward Doubly Stochastic Differential Equations and it's Applications to Nonlinear Filtering Problems

5.1 FBDSDEs and SPDEs

To derive the numerical algorithm and conduct its convergence analysis, we provide a brief introduction to forward backward doubly stochastic differential equations (FBDSDEs) and the relationship between FBDSDEs and the SPDE.

Let $(\Lambda, \mathcal{F}, P)$ be a complete probability space and $T > 0$ be the terminal time, $\{W_t, 0 \leq t \leq T\}$ and $\{B_t, 0 \leq t \leq T\}$ be two mutually independent standard Brownian motions defined on $(\Lambda, \mathcal{F}, P)$ with their values in $\mathbb{R}^d$ and in $\mathbb{R}^l$, respectively. Let $\mathcal{N}$ denote the class of P-null sets of $\mathcal{F}$. For each $t \in [0, T]$, we define

$$\mathcal{F}_t := \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B,$$

where $\mathcal{F}_{s,t}^\eta = \sigma\{\eta_r - \eta_s; s \leq r \leq t\} \vee \mathcal{N}$ the σ-field generated by $\{\eta_r - \eta_s; s \leq r \leq t\}$, and $\mathcal{F}_t^\eta = \mathcal{F}_{0,t}^\eta$ for a stochastic process η. Note that the collection $\{\mathcal{F}_t, t \in [0, T]\}$ is neither increasing nor decreasing and it is not a filtration. For positive integer $n \in \mathbb{N}$, we define spaces $M^2(0, T; \mathbb{R}^n)$ and $S^2([0, T]; \mathbb{R}^n)$ as follows.

$$M^2(0, T; \mathbb{R}^n) := \quad \{\varphi_t | \varphi_t \in \mathbb{R}^n, E \int_0^T |\varphi_t|^2 dt < \infty,\ \varphi_t \in \mathcal{F}_t,\ \text{a.e. } t \in [0, T]\},$$

and

$$S^2([0, T]; \mathbb{R}^n) := \{\varphi_t | \varphi_t \in \mathbb{R}^n, E(\sup_{0 \leq t \leq T} |\varphi_t|^2) < \infty,\ \varphi_t \in \mathcal{F}_t,\ t \in [0, T]\}.$$

Let

$$f : \Lambda \times [0,T] \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \to \mathbb{R}^k$$

and

$$g : \Lambda \times [0,T] \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \to \mathbb{R}^{k\times l}$$

be jointly measurable such that for any $(y,z) \in \mathbb{R}^k \times \mathbb{R}^{k\times d}$,

$$f_t(y,z) \in M^2(0,T;\mathbb{R}^k),$$

$$g_t(y,z) \in M^2(0,T;\mathbb{R}^{k\times l}).$$

We assume moreover that there exist constants $c > 0$ and $0 < \alpha < 1$ such that for any $(\omega, t) \in \Lambda \times [0,T]$, (y_1, z_1), $(y_2, z_2) \in \mathbb{R}^k \times \mathbb{R}^{k\times l}$,

$$|f_t(y_1,z_1) - f_t(y_2,z_2)|^2 \leq c(|y_1 - y_2|^2 + \|z_1 - z_2\|^2),$$

$$\|g_t(y_1,z_1) - g_t(y_2,z_2)\|^2 \leq c|y_1 - y_2|^2 + \alpha\|z_1 - z_2\|^2.$$

From [61], under the above assumptions and standard conditions on b and σ, we know that there exists a pair of processes $\{(Y_s^{t,x}, Z_s^{t,x}); (t,x) \in [0,T] \times \mathbb{R}^d\}$ which is the unique solution to the following FBDSDE: For $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$

$$X_s^{t,x} = x + \int_t^s b(X_r^{t,x})dr + \int_t^s \sigma(X_r^{t,x})dW_r, \quad t \leq s \leq T, \tag{5.1}$$

$$\begin{aligned} Y_s^{t,x} &= h(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{r,x}, Z_r^{t,x})dr \\ &+ \int_s^T g(r, X_r^{t,x}, Y_r^{r,x}, Z_r^{t,x})d\overleftarrow{B}_r - \int_s^T Z_r^{t,x}dW_r, \quad t \leq s \leq T, \end{aligned} \tag{5.2}$$

where $(Y_s^{t,x}, Z_s^{t,x}) \in S^2([0,T];\mathbb{R}^k) \times M^2(0,T;\mathbb{R}^{k\times l})$. Here $d\overleftarrow{B}_r$ denotes the backward Itô integration, i.e., for a $\mathcal{F}_{t,T}^B$ adapted process V_t, and quasi-uniform time partitions Δ: $0 =$

$t_0 < t_1 < t_2 < \cdots < t_{N-1} < t_N = T$,

$$\int_0^T V_t d\overleftarrow{B}_t := \lim_{\Delta t \to 0} \sum_{n=0}^{N} V_{t_{n+1}}(B_{t_{n+1}} - B_{t_n})$$

where $\Delta t = \max_{0 \le i \le N-1}(t_{i+1} - t_i)$. According to Pardoux and Peng ([61]), we have the following nonlinear Feynman-Kac formula.

$$Y_s^{t,x} = u_s(X_s^{t,x}),\ \ Z_s^{t,x} = (\nabla u_s \sigma)(X_s^{t,x}); (t,x) \in [0,T] \times \mathbb{R}^d,$$

where $u = u_t(x) \in R^k$ is the unique solution of the following system of backward stochastic partial differential equation:

$$\begin{aligned} u_t(x) &= h(x) + \int_t^T [\mathcal{L}_s u_s(x) + f(s,x,u_s(x),(\nabla u_s \sigma)(x))]ds \\ &\quad + \int_t^T g(s,x,u_s(x),(\nabla u_s \sigma)(x)) d\overleftarrow{B}_s, \quad 0 \le t \le T. \end{aligned} \tag{5.3}$$

Equation (5.3) is the Zakai type equation (2.6) and (4.3) as used to solve nonlinear filtering problems.

To simplify our presentation, in what follows we assume that $b \equiv 0$ and $\sigma \equiv 1$ in (5.1). Thus we have

$$X_s^{0,x} = x + W_s, \quad x \in \Omega, s \in [0,T],$$

and the elliptic partial differential operator L becomes

$$L = \frac{1}{2} \sum_i^d \frac{\partial^2}{\partial x_i^2}.$$

When the spatial domain Ω of the SPDE (5.3) is a subset of $\mathbb{R}^d$ and the boundary condition

$$u_t(x) = \gamma(t,x), \quad on\ [0,T] \times \partial\Omega$$

is prescribed, the corresponding BDSDE is defined through a stopping time τ defined as

$$\tau = \inf\{s; X_s^{t,x} \in \partial\Omega, s \geq t, x \in \Omega\}. \tag{5.4}$$

Then, we have the BDSDE with stopping time as follows.

$$\begin{aligned} Y_s^{t,x} &= \Phi(X_{\tau\wedge T}^{x}) + \int_s^{T\wedge\tau} f(r, X_r^{t,x}, Y_r^{r,x}, Z_r^{t,x})dr \\ &+ \int_s^{T\wedge\tau} g(r, X_r^{t,x}, Y_r^{r,x}, Z_r^{t,x})d\overleftarrow{B}_r - \int_s^{T\wedge\tau} Z_r^{t,x}dW_r, \quad t \leq s \leq T,\ x \in \Omega, \end{aligned} \tag{5.5}$$

where $\Phi(X_{\tau,T}^{x}) = h(X_T^{t,x})I_{\tau\geq T} + \gamma(\tau, X_\tau^{t,x})I_{\tau\leq T}$. When $t = 0$, the stopping time τ defined in (5.6) becomes

$$\tau = \inf\{s; X_s^{0,x} \in \partial\Omega,\ s \geq 0\, x \in \Omega\}.$$

Thus, BDSDE (5.7) changes to the following equation:

$$\begin{aligned} &Y_t^{0,x} = \Phi(X_{\tau\wedge T}^{0,x}) + \int_t^{T\wedge\tau} f(s, X_s^{0,x}, Y_s^{0,x}, Z_s^{0,x})ds \\ &- \int_t^{T\wedge\tau} Z_s^{0,x}dW_s + \int_t^{T\wedge\tau} g(s, X_s^{0,x}, Y_s^{0,x}, Z_s^{0,x})d\overleftarrow{B}_s, \quad t \in [0,T],\ x \in \Omega, \end{aligned}$$

where for given x, $X_0^{0,x} = x$, $\Phi(X_{\tau\wedge T}^{0,x}) = h(X_T^{0,x})I_{\tau\geq T} + \gamma(\tau, X_\tau^{0,x})I_{\tau\leq T}$. The related SPDE is

$$\begin{cases} u_t(x) &= h(x) + \int_t^T [\frac{1}{2}\sum_{i=1}^{d} \frac{\partial^2 u_s(x)}{\partial x_i^2} + f(s, x, u_s(x), \nabla u_s(x))]ds \\ &\quad + \int_t^T g(s, x, u_s(x), \nabla u_s(x))d\overleftarrow{B}_s, \quad x \in \Omega,\ 0 \leq t \leq T, \\ u_t(x) &= \gamma(t,x), \quad \text{on } [0,T] \times \partial\Omega. \end{cases}$$

5.2 Half Order Numerical Algorithms

We first study numerical algorithms for BDSDEs and assume that $b \equiv 0$ and $\sigma \equiv 1$ in (5.1). Thus we have

$$X_s^{0,x} = x + W_s, \quad x \in \Omega, s \in [0,T],$$

and the elliptic partial differential operator L becomes

$$L = \frac{1}{2}\sum_{i}^{d}\frac{\partial^2}{\partial x_i^2}.$$

When the spatial domain Ω of the SPDE (5.3) is a subset of $\mathbb{R}^d$ and the boundary condition

$$u_t(x) = \gamma(t,x), \quad on\,[0,T]\times\partial\Omega$$

is prescribed, the corresponding BDSDE is defined through a stopping time τ defined as

$$\tau = \inf\{s; X_s^{t,x} \in \partial\Omega, s \geq t, x \in \Omega\}. \tag{5.6}$$

Then, we have the BDSDE with stopping time as follows.

$$\begin{aligned} Y_s^{t,x} &= \Phi(X_{\tau\wedge T}^x) + \int_s^{T\wedge\tau} f(r, X_r^{t,x}, Y_r^{r,x}, Z_r^{t,x})dr \\ &+ \int_s^{T\wedge\tau} g(r, X_r^{t,x}, Y_r^{r,x}, Z_r^{t,x})d\overleftarrow{B}_r - \int_s^{T\wedge\tau} Z_r^{t,x}dW_r, \quad t \leq s \leq T,\ x \in \Omega, \end{aligned} \tag{5.7}$$

where $\Phi(X_{\tau,T}^x) = h(X_T^{t,x})I_{\tau\geq T} + \gamma(\tau, X_\tau^{t,x})I_{\tau\leq T}$. When $t = 0$, the stopping time τ defined in (5.6) becomes

$$\tau = \inf\{s; X_s^{0,x} \in \partial\Omega,\ s \geq 0\, x \in \Omega\}.$$

Thus, BDSDE (5.7) changes to the following equation:

$$\begin{aligned} &Y_t^{0,x} = \Phi(X_{\tau\wedge T}^{0,x}) + \int_t^{T\wedge\tau} f(s, X_s^{0,x}, Y_s^{0,x}, Z_s^{0,x})ds \\ &- \int_t^{T\wedge\tau} Z_s^{0,x}dW_s + \int_t^{T\wedge\tau} g(s, X_s^{0,x}, Y_s^{0,x}, Z_s^{0,x})d\overleftarrow{B}_s, \quad t \in [0,T],\ x \in \Omega, \end{aligned}$$

where for given x, $X_0^{0,x} = x$, $\Phi(X_{\tau\wedge T}^{0,x}) = h(X_T^{0,x})I_{\tau\geq T} + \gamma(\tau, X_\tau^{0,x})I_{\tau\leq T}$. The related SPDE is

$$\begin{cases} u_t(x) &=& h(x) + \int_t^T [\frac{1}{2}\sum_{i=1}^d \frac{\partial^2 u_s(x)}{\partial x_i^2} + f(s,x,u_s(x),\nabla u_s(x))]ds \\ & & + \int_t^T g(s,x,u_s(x),\nabla u_s(x))d\overleftarrow{B}_s, \quad x\in\Omega,\ 0\leq t\leq T, \\ u_t(x) &=& \gamma(t,x), \quad \text{on } [0,T]\times\partial\Omega. \end{cases}$$

5.2.1 Numerical algorithms

For the simplicity of presentation we only consider the one dimensional case. The high dimensional cases can be handled through straightforward generalization of the one dimensional case. To simplify the notations we shall use (y_t, z_t) to denote the solution $(Y_t^{t,x}, Z_t^{t,x})$ of the BDSDE (5.2). We also denote $\mathbb{E}_s^{t,x}[X] = \mathbb{E}[X|\mathcal{F}_s^{W,t,x}]$ where $\mathcal{F}_s^{W,t,x} := \sigma(x + W_s - W_t; t\leq s\leq T)\cup\sigma(B_t; 0\leq t\leq T)$.

Reference equations

To further simplify the notations, we denote $f(s,y_s,z_s) = f(s,X_s^{t,x},y_s,z_s)$ and $g(s,y_s,z_s) = g(s,X_s^{t,x},y_s,z_s)$, knowing that $x\in\Omega\subset\mathbb{R}$. Then we have

$$y_t = y_{t+\delta} + \int_t^{t+\delta} f(s,y_s,z_s)ds - \int_t^{t+\delta} z_s dW_s + \int_t^{t+\delta} g(s,y_s,z_s)\overleftarrow{dB}_s, \tag{5.8}$$

where δ is a deterministic nonnegative number with $t+\delta\leq T$. Taking the conditional expectation $\mathbb{E}_t^{t,x}[\cdot]$ on (5.52), we obtain

$$y_t^{t,x} = \mathbb{E}_t^{t,x}[y_{t+\delta}] + \int_t^{t+\delta}\mathbb{E}_t^{t,x}[f(s,y_s,z_s)]ds + \int_t^{t+\delta}\mathbb{E}_t^{t,x}[g(s,y_s,z_s)]\overleftarrow{dB}_s, \tag{5.9}$$

where $y_t^{t,x} = \mathbb{E}_t^{t,x}[y_t]$, that is, $y_t^{t,x}$ is the value of y_t at the time-space point (t,x). We use the simple right point formula to approximate the integrals in (5.53):

$$\int_t^{t+\delta} \mathbb{E}_t^{t,x}[f(s,y_s,z_s)]ds = \delta \mathbb{E}_t^{t,x}[f(t+\delta, y_{t+\delta}, z_{t+\delta})] + R_y^W, \tag{5.10}$$

$$\int_t^{t+\delta} \mathbb{E}_t^{t,x}[g(s,y_s,z_s)]\overleftarrow{dB}_s = \mathbb{E}_t^{t,x}[g(t+\delta, y_{t+\delta}, z_{t+\delta})]\Delta \overleftarrow{B}_t + R_y^B, \tag{5.11}$$

where R_y^W and R_y^B denote the corresponding errors of approximations. Inserting (5.54) and (5.11) into (5.53), we obtain

$$\begin{aligned} y_t^{t,x} = \quad & \mathbb{E}_t^{t,x}[y_{t+\delta}] + \delta \mathbb{E}_t^{t,x}[f(t+\delta, y_{t+\delta}, z_{t+\delta})] \\ & + \mathbb{E}_t^{t,x}[g(t+\delta, y_{t+\delta}, z_{t+\delta})]\Delta \overleftarrow{B}_t + R_y, \end{aligned} \tag{5.12}$$

where $R_y = R_y^W + R_y^B$ is the truncation error for solving y_t. Let $\Delta W_s = W_s - W_t$ for $t \le s \le t+\delta$. Multiplying by $\Delta W_{t+\delta}$ on (5.52), taking the conditional expectation $\mathbb{E}_t^{t,x}[\cdot]$ and applying the Itô isometry we get

$$\begin{aligned} -\mathbb{E}_t^{t,x}[y_{t+\delta}\Delta W_{t+\delta}] \quad = \quad & \int_t^{t+\delta} \mathbb{E}_t^{t,x}[f(s,y_s,z_s)\Delta W_s]ds \\ & + \int_t^{t+\delta} \mathbb{E}_t^{t,x}[g(s,y_s,z_s)\Delta W_s]\overleftarrow{dB}_s - \int_t^{t+\delta} \mathbb{E}_t^{t,x}[z_s]ds. \end{aligned} \tag{5.13}$$

Similar to (5.12) we approximate the integrals in (5.13) with the right point formula to obtain

$$\begin{aligned} & \int_t^{t+\delta} \mathbb{E}_t^{t,x}[f(s,y_s,z_s)\Delta W_s]ds \\ & = \delta \mathbb{E}_t^{t,x}[f(t+\delta, y_{t+\delta}, z_{t+\delta})\Delta W_{t+\delta}] + R_{z1}^W \end{aligned} \tag{5.14}$$

$$-\int_t^{t+\delta} \mathbb{E}_t^{t,x}[z_s]ds = -\delta z_t^{t,x} + R_{z2}^W \tag{5.15}$$

and

$$\int_t^{t+\delta} \mathbb{E}_t^{t,x}[g(s,y_s,z_s)\Delta W_s]\overleftarrow{dB}_s = \mathbb{E}_t^{t,x}[g(t+\delta, y_{t+\delta}, z_{t+\delta})\Delta W_{t+\delta}]\Delta B_t + R_z^B, \tag{5.16}$$

where $z_t^{t,x}$ is the value of z_t at the time-space point (t, x), and R_{z1}^W, R_{z2}^W and R_z^B are the corresponding approximation errors. Inserting (5.14), (5.15) and (5.16) into (5.13), we get the second approximation equation for (5.52) as follows.

$$\begin{aligned} -\mathbb{E}_t^{t,x}[y_{t+\delta}\Delta W_{t+\delta}] &= \delta\mathbb{E}_t^{t,x}[f(t+\delta, y_{t+\delta}, z_{t+\delta})\Delta W_{t+\delta}] - \delta z_t^{t,x} \\ &\quad + \mathbb{E}_t^{t,x}[g(t+\delta, y_{t+\delta}, z_{t+\delta})\Delta W_{t+\delta}]\Delta B_t + R_z, \end{aligned} \tag{5.17}$$

where $R_z = R_{z1}^W + R_{z2}^W + R_z^B$ is the truncation error for solving z_t. (5.12) and (5.17) are two key equations of solving BDSDE (5.52) numerically. We refer them as reference equations.

Discrete scheme

To derive a numerical algorithm from the reference equations (5.12) and (5.17), we introduce the following time partition on $[0, T]$.

$$\mathcal{R}_{th} = \{t_i | t_i \in [0, T], t_i < t_{i+1}, i = 0, 1, \ldots, N_T - 1, t_0 = 0, t_{N_T} = T\}.$$

Let $\Delta t_n = t_{n+1} - t_n$ and $\Delta t = \max_{0 \le n \le N_T - 1} \Delta t_n$. We discretize (5.12) and (5.17) by substituting t, δ, y_t and z_t with t_n, Δt_n with y^n and z^n respectively and dropping the errors terms in (5.12) and (5.17), to obtain the following numerical algorithm for solving BSDE: given random variable y^N, for $n = N-1, N-2, \ldots, 1, 0$, solve the random variables y^n and z^n backwardly by

$$\begin{aligned} y^n &= \mathbb{E}_{t_n}^{t_n,x}[y^{n+1}] + \Delta t_n \mathbb{E}_{t_n}^{t_n,x} f(t_{n+1}, y^{n+1}, z^{n+1}) \\ &\quad + \mathbb{E}_{t_n}^{t_n,x}[g(t_{n+1}, y^{n+1}, z^{n+1}))]\Delta B_{t_n} \end{aligned} \tag{5.18}$$

and

$$\begin{aligned} 0 &= \mathbb{E}_{t_n}^{t_n}[y^{n+1}\Delta W_{t_{n+1}}] + \Delta t_n \mathbb{E}_{t_n}^{t_n,x}[f(t_{n+1}, y^{n+1}, z^{n+1})\Delta W_{t_{n+1}}] \\ &\quad + \mathbb{E}_{t_n}^{t_n,x}[g(t_{n+1}, y^{n+1}, z^{n+1})\Delta W_{t_{n+1}}]\Delta B_{t_n} - \Delta t_n z^n. \end{aligned} \tag{5.19}$$

Obviously (y^n, z^n) is an approximate solution for (y_t, z_t) at $t = t_n, n = 0, 1, \cdots, N_T$.

5.2.2 Error Estimates

To derive the error estimates, we first need some regularity results for the exact solution.

Regularity of the exact solution

We assume that f and g satisfy the following properties.

$$
\begin{aligned}
&E[(f(s,y_1,z_1)-f(t,y_2,z_2))^2] \le L(|s-t|+|y_1-y_2|^2+|z_1-z_2|^2),\\
&E[(g(s,y_1,z_1)-g(t,y_2,z_2))^2] \le L_1(|s-t|+|y_1-y_2|^2)+L_2|z_1-z_2|^2,
\end{aligned} \tag{5.20}
$$

where L, L_1 and L_2 are positive constants and $0 \le L_2 < 1$ (see [61] for similar assumptions). We also assume that the derivatives f'_x, f'_y, f'_z, g'_x, g'_y and g'_z of f and g are all continuous and bounded. Let $\nabla y_r^{t,x}$, $\nabla z_r^{t,x}$ and $\nabla X_r^{t,x}$ be the variations of $y_r^{t,x}$, $z_r^{t,x}$, $X_r^{t,x}$ with respect to x at time level $t = r$. Then the following equation holds.

$$
\begin{aligned}
\nabla y_s^{t,x} \;=\;& h'(X_T^{t,x})\nabla X_T^{t,x} + \int_s^T [f'_x(r,X_r^{t,x},y_r^{t,x},z_r^{t,x})\nabla X_r^{t,x}\\
&+f'_y(r,X_r^{t,x},y_r^{t,x},z_r^{t,x})\nabla y_r^{t,x} + f'_z(r,X_r^{t,x},y_r^{t,x},z_r^{t,x})\nabla z_r^{t,x}]dr\\
&+\int_s^T [g'_x(r,X_r^{t,x},y_r^{t,x},z_r^{t,x})\nabla X_r^{t,x} + g'_y(r,X_r^{t,x},y_r^{t,x},z_r^{t,x})\nabla y_r^{t,x}\\
&+g'_z(r,X_r^{t,x},y_r^{t,x},z_r^{t,x})\nabla z_r^{t,x}]d\overleftarrow{B}_r - \int_s^T \nabla z_r^{t,x} dW_r,
\end{aligned} \tag{5.21}
$$

where $\nabla X_r^{t,x}$ is the solution of following SDE (see [86], page 464, equation (12)).

$$
\nabla X_s^{t,x} = 1 + \int_t^s \partial_x b(r)\nabla X_r^{t,x} dr + \int_t^s \partial_x \sigma(r) \nabla X_r^{t,x} dW_r.
$$

We have the following result concerning the regularity of the solution (y_t, z_t) of the FBDSDE (5.2).

Proposition 1 *Assume that Hypothesis (5.20) holds and the derivatives* f'_x, f'_y, f'_z, g'_x, g'_y *and* g'_z *of* f *and* g *are all bounded, then we have*

$$E[(y_s^{t,x} - y_t^{t,x})^2] \le C|s-t| \tag{5.22}$$

and

$$E[(z_s^{t,x} - z_t^{t,x})^2] \le C|s-t|. \tag{5.23}$$

Proof: Under the assumptions of the Proposition, Pardoux and Peng ([61] proved the estimate (5.22) and the following estimate

$$\sup_{t\le s\le T} E[(y_s^{t,x})^2] \le C,$$

where C is a constant. To obtain (5.23), we use the fact that (see [61], page 223, Proposition 2.3)

$$z_s^{t,x} = \nabla y_s^{t,x} (\nabla X_s^{t,x})^{-1} \sigma(X_s^{t,x})$$

and

$$z_t^{t,x} = \nabla y_t^{t,x} \sigma(x).$$

Now we treat (5.21) the same as equation (5.2) with f replaced by

$$[f'_x(r, X_r^{t,x}, y_r^{t,x}, z_r^{t,x}) \nabla X_r^{t,x} + f'_y(r, X_r^{t,x}, y_r^{t,x}, z_r^{t,x}) \nabla y_r^{t,x} + f'_z(r, X_r^{t,x}, y_r^{t,x}, z_r^{t,x}) \nabla z_r^{t,x}]$$

and g replaced by

$$[g'_x(r, X_r^{t,x}, y_r^{t,x}, z_r^{t,x}) \nabla X_r^{t,x} + g'_y(r, X_r^{t,x}, y_r^{t,x}, z_r^{t,x}) \nabla y_r^{t,x} + g'_z(r, X_r^{t,x}, y_r^{t,x}, z_r^{t,x}) \nabla z_r^{t,x}],$$

and use the same result of Pardoux and Peng to obtain

$$E[(\nabla y_s^{t,x} - \nabla y_t^{t,x})^2] < C|s-t|$$

and

$$\sup_{t\le s\le T} E[(\nabla y_s^{t,x})^2] \le C.$$

Because of the assumption that $b=0$ and $\sigma=1$, we have that

$$\nabla X_s^{t,x} = (\nabla X_s^{t,x})^{-1} = 1$$

and $\sigma = 1$. Thus

$$\begin{aligned} E[(z_s^{t,x} - z_t^{t,x})^2] &= E[(\nabla y_s^{t,x} - \nabla y_t^{t,x})^2] \\ &\le C|s-t|. \end{aligned}$$

Estimates of truncation errors

For the sake of simplicity of our presentation, in the sequel, we use $E_t[\cdot]$ to denote $E_t^{t_n,x}[\cdot]$. Recall the numerical scheme

$$\begin{aligned} y^n &= E_{t_n}[y^{n+1}] + \Delta t_n E_{t_n}[f(t_{n+1}, y^{n+1}, z^{n+1})] + \Delta B_{t_{n+1}} E_{t_n}[g(t_{n+1}, y^{n+1}, z^{n+1})], \\ z^n &= \frac{1}{\Delta t_n}\left\{E_{t_n}[y^{n+1}\Delta W_{t_{n+1}}] + \Delta t_n E_{t_n}[f(t_{n+1}, y^{n+1}, z^{n+1})\Delta W_{t_{n+1}}]\right. \\ &\quad \left. + \Delta B_{t_{n+1}} E_{t_n}[g(t_{n+1}, y^{n+1}, z^{n+1})\Delta W_{t_{n+1}}]\right\} \end{aligned}$$

and the reference equations

$$\begin{aligned} y_{t_n} &= E_{t_n}[y_{t_{n+1}}] + \Delta t_n E_{t_n}[f(t_{n+1}, y_{t_{n+1}}, z_{t_{n+1}})] + \Delta B_{t_{n+1}} E_{t_n}[g(t_{n+1}, y_{t_{n+1}}, z_{t_{n+1}})] + R_y^n, \\ z_{t_n} &= \frac{1}{\Delta t_n}\left\{E_{t_n}[y_{t_{n+1}}\Delta W_{t_{n+1}}] + \Delta t_n E_{t_n}[f(t_{n+1}, y_{t_{n+1}}, z_{t_{n+1}})\Delta W_{t_{n+1}}]\right. \\ &\quad \left. + \Delta B_{t_{n+1}} E_{t_n}[g(t_{n+1}, y_{t_{n+1}}, z_{t_{n+1}})\Delta W_{t_{n+1}}]\right\} + \frac{1}{\Delta t_n} R_z^n, \end{aligned}$$

where (y_{t_n}, z_{t_n}) is the exact solution. We have truncation errors R_y^n and R_z^n for y and z respectively as

$$
\begin{aligned}
R_y^n &= \int_{t_n}^{t_{n+1}} E_{t_n}[f(s, y_s, z_s) - f(t_{n+1}, y_{t_{n+1}}, z_{t_{n+1}})]ds \\
&\quad + \int_{t_n}^{t_{n+1}} E_{t_n}[g(s, y_s, z_s) - g(t_{n+1}, y_{t_{n+1}}, z_{t_{n+1}})]d\overleftarrow{B}_s
\end{aligned}
$$

and

$$
\begin{aligned}
R_z^n &= \int_{t_n}^{t_{n+1}} E_{t_n}[(f(s, y_s, z_s) - f(t_{n+1}, y_{t_{n+1}}, z_{t_{n+1}}))\Delta W_{t_{n+1}}]ds \\
&\quad + \int_{t_n}^{t_{n+1}} E_{t_n}[(g(s, y_s, z_s) - g(t_{n+1}, y_{t_{n+1}}, z_{t_{n+1}}))\Delta W_{t_{n+1}}]d\overleftarrow{B}_s \\
&\quad + \int_{t_n}^{t_{n+1}} E_{t_n}[z_s - z_{t_n}]ds.
\end{aligned}
$$

Denote $f_t = f(t, y_t, z_t)$ and $g_t = g(t, y_t, z_t)$. By Proposition 1, we have the estimates

$$
\max_{0 \le n \le N-1} E[(y_{t_n} - y_{t_{n-1}})^2] \le C \cdot \Delta t
$$

and

$$
\max_{0 \le n \le N-1} E[(z_{t_n} - z_{t_{n-1}})^2] \le C \cdot \Delta t.
$$

For the truncation error R_y^n, we have the following estimate.

$$
\begin{aligned}
E[(R_y^n)^2] &\le C_1 \Delta t_n \int_{t_n}^{t_{n+1}} E[(f(s, y_s, z_s) - f(t_{n+1}, y_{t_{n+1}}, z_{t_{n+1}}))^2]ds \\
&\quad + C_2 \int_{t_n}^{t_{n+1}} E[(g(s, y_s, z_s) - g(t_{n+1}, y_{t_{n+1}}, z_{t_{n+1}}))^2]ds \\
&\le C_1 \Delta t_n \int_{t_n}^{t_{n+1}} E[(t_{n+1} - s) + (y_s - y_{t_{n+1}})^2 + (z_s - z_{t_{n+1}})^2]ds \\
&\quad + C_2 \int_{t_n}^{t_{n+1}} E[(t_{n+1} - s) + (y_s - y_{t_{n+1}})^2 + (z_s - z_{t_{n+1}})^2]ds \\
&\le C(\Delta t)^2.
\end{aligned}
$$

Similarly, we have

$$
E[(R_z^n)^2] \le K(\Delta t)^3.
$$

Error estimate for y

Denote $e_y^n = y_{t_n} - y^n$, $e_z^n = z_{t_n} - z^n$, $e_f^n = f(t_n, y_{t_n}, z_{t_n}) - f(t_n, y^n, z^n)$, and $e_g^n = g(t_n, y_{t_n}, z_{t_n}) - g(t_n, y^n, z^n)$. We have the estimate of e_y^n for scheme (5.18) and (5.19) in the following theorem.

Theorem 5.1 *Let (y_t, z_t) be the exact solution and (y^n, z^n) be the solution of the scheme* (5.18) *and* (5.19)*. If Hypothesis* (5.20) *is satisfied and the derivatives f'_x, f'_y, f'_z, g'_x, g'_y and g'_z of f and g are all bounded, then*

$$\max_{0 \le n \le N-1} E[e_y^n]^2 \le C\Delta t,$$

where C is a constant.

Proof: We first decompose the error for y as

$$e_y^n = E_{t_n}[e_y^{n+1}] + E_{t_n}[e_f^{n+1}]\Delta t_n + E_{t_n}[e_g^{n+1}]\Delta B_{t_{n+1}} + R_y^n.$$

Taking square on both sides of the above equation and then taking expectation we obtain

$$\begin{aligned}
E|e_y^n|^2 &= E[(E_{t_n}[e_y^{n+1}] + \Delta B_{t_{n+1}} E_{t_n}[e_g^{n+1}] + R_y^n)^2 + (\Delta t_n E_{t_n}[e_f^{n+1}])^2 \\
&\quad +2(E_{t_n}[e_y^{n+1}] + \Delta B_{t_{n+1}} E_{t_n}[e_g^{n+1}] + R_y^n)(\Delta t_n E_{t_n}[e_f^{n+1}])] \\
&= E\left[|E_{t_n}[e_y^{n+1}]|^2 + (\Delta B_{t_{n+1}})^2 |E_{t_n}[e_g^{n+1}]|^2 + (R_y^n)^2\right. \\
&\quad \left.+2E_{t_n}[e_y^{n+1}]\Delta B_{t_{n+1}} E_{t_n}[e_g^{n+1}] + 2E_{t_n}[e_y^{n+1}]R_y^n + 2\Delta B_{t_{n+1}} E_{t_n}[e_g^{n+1}]R_y^n\right] \qquad (5.24) \\
&\quad +E[(\Delta t_n E_{t_n}[e_f^{n+1}])^2] \\
&\quad +E[2(E_{t_n}[e_y^{n+1}] + \Delta B_{t_{n+1}} E_{t_n}[e_g^{n+1}] + R_y^n)(\Delta t_n E_{t_n}[e_f^{n+1}])] \\
&= A_y + B_y + C_y,
\end{aligned}$$

where

$$
\begin{aligned}
A_y &= E\left[|E_{t_n}[e_y^{n+1}]|^2 + (\Delta B_{t_{n+1}})^2|E_{t_n}[e_g^{n+1}]|^2 + (R_y^n)^2\right. \\
&\quad \left. +2E_{t_n}[e_y^{n+1}]\Delta B_{t_{n+1}}E_{t_n}[e_g^{n+1}] + 2E_{t_n}[e_y^{n+1}]R_y^n + 2\Delta B_{t_{n+1}}E_{t_n}[e_g^{n+1}]R_y^n\right],
\end{aligned}
$$

$$
B_y = E[(\Delta t_n E_{t_n}[e_f^{n+1}])^2]
$$

and

$$
C_y = E[2(E_{t_n}[e_y^{n+1}] + \Delta B_{t_{n+1}}E_{t_n}[e_g^{n+1}] + R_y^n)(\Delta t_n E_{t_n}[e_f^{n+1}])].
$$

Next we use Cauchy's inequality and Young's inequality and the facts that

$$
E[E_{t_n}[e_y^{n+1}] \cdot \int_{t_n}^{t_{n+1}} E_{t_n}[g(s, y_s, z_s) - g(t_{n+1}, y_{t_{n+1}}, z_{t_{n+1}})]d\overleftarrow{B}_s] = 0
$$

and

$$
\int_{t_n}^{t_{n+1}} E[f(s, y_s, z_s) - f(t_{n+1}, y_{t_{n+1}}, z_{t_{n+1}})]^2 ds \le C(\Delta t)^2,
$$

to obtain

$$
\begin{aligned}
A_y &= E\left[|E_{t_n}[e_y^{n+1}]|^2 + (\Delta B_{t_{n+1}})^2|E_{t_n}[e_g^{n+1}]|^2 + (R_y^n)^2\right. \\
&\quad \left. +2E_{t_n}[e_y^{n+1}]\Delta B_{t_{n+1}}E_{t_n}[e_g^{n+1}] + 2E_{t_n}[e_y^{n+1}]R_y^n + 2\Delta B_{t_{n+1}}E_{t_n}[e_g^{n+1}]R_y^n\right] \\
&\le E[|E_{t_n}[e_y^{n+1}]|^2] + \Delta t_n E[|E_{t_n}[e_g^{n+1}]|^2] + E[(R_y^n)^2] \\
&\quad +2E[E_{t_n}[e_y^{n+1}] \cdot \int_{t_n}^{t_{n+1}} E_{t_n}^x[f(s, y_s, z_s) - f(t_{n+1}, y_{t_{n+1}}, z_{t_{n+1}})]ds \\
&\quad +E_{t_n}[e_y^{n+1}] \cdot \int_{t_n}^{t_{n+1}} E_{t_n}^x[g(s, y_s, z_s) - g(t_{n+1}, y_{t_{n+1}}, z_{t_{n+1}})]d\overleftarrow{B}_s] \\
&\quad +\Delta t_n \epsilon_1 E[|E_{t_n}[e_g^{n+1}]|^2] + E[\frac{1}{\epsilon_1}(R_y^n)^2] \\
&\le E[|E_{t_n}[e_y^{n+1}]|^2] + \Delta t_n E[|E_{t_n}[e_g^{n+1}]|^2] + E[(R_y^n)^2] \\
&\quad +\Delta t_n E[e_y^{n+1}]^2 + \int_{t_n}^{t_{n+1}} E[f(s, y_s, z_s) - f(t_{n+1}, y_{t_{n+1}}, z_{t_{n+1}})]^2 ds \\
&\quad +\Delta t_n \epsilon_1 E[|E_{t_n}[e_g^{n+1}]|^2] + E[\frac{1}{\epsilon_1}(R_y^n)^2]
\end{aligned}
$$

$$\leq \; E[|E_{t_n}[e_y^{n+1}]|^2] + \Delta t_n E[(1+\epsilon_1)|E_{t_n}[e_g^{n+1}]|^2] + \Delta t_n E[e_y^{n+1}]^2 + C_1(\Delta t)^2, \quad (5.25)$$

where $\epsilon_1 > 0$ is a constant to be determined later. By the Lipschitz continuity of f, we have

$$\begin{aligned} B_y &= E[(\Delta t_n E_{t_n}[e_f^{n+1}])^2] \\ &\leq L(\Delta t_n)^2(E[e_y^{n+1}]^2 + E[e_z^{n+1}]^2). \end{aligned} \quad (5.26)$$

Similarly for C_y, using Cauchy's inequality and Young's inequality, we obtain

$$\begin{aligned} C_y &= E[2(E_{t_n}[e_y^{n+1}] + \Delta B_{t_{n+1}} E_{t_n}[e_g^{n+1}] + R_y^n)(\Delta t_n E_{t_n}[e_f^{n+1}])] \\ &\leq \Delta t_n \frac{1}{\epsilon_2} E[E_{t_n}[e_y^{n+1}] + \Delta B_{t_{n+1}} E_{t_n}[e_g^{n+1}] + R_y^n]^2 \\ &\quad + \Delta t_n \epsilon_2 E[e_f^{n+1}]^2 \\ &\leq \Delta t_n \frac{3}{\epsilon_2} E[E_{t_n}[e_y^{n+1}]^2 + \Delta t_n(L_1 E_{t_n}[e_y^{n+1}]^2 + L_2 E_{t_n}[e_z^{n+1}]^2) + (R_y^n)^2] \\ &\quad + \Delta t_n E[L\epsilon_2 E_{t_n}[e_y^{n+1}]^2 + L\epsilon_2 E_{t_n}[e_z^{n+1}]^2] \\ &\leq \Delta t_n C_2 E[e_y^{n+1}]^2 + (\Delta t_n)^2 C_3(E[e_y^{n+1}]^2 + E[e_z^{n+1}]^2) + \Delta t_n L\epsilon_2 E[e_z^{n+1}]^2 \\ &\quad + C_4(\Delta t)^2, \end{aligned} \quad (5.27)$$

where $\epsilon_2 > 0$ is a constant to be determined later. Combining (5.24), (5.25), (5.26) and (5.27) together, we obtain

$$\begin{aligned} E[e_y^n]^2 &\leq E[|E_{t_n}[e_y^{n+1}]|^2] + \Delta t_n E[(1+\epsilon_1)|E_{t_n}[e_g^{n+1}]|^2] \\ &\quad + \Delta t_n L\epsilon_2 E[e_z^{n+1}]^2 + K_1 \Delta t_n E[e_y^{n+1}]^2 \\ &\quad + K_2(\Delta t_n)^2 E[e_z^{n+1}]^2 + K_3(\Delta t)^2. \end{aligned} \quad (5.28)$$

For e_z^n, we have the identity

$$\begin{aligned} \Delta t_n e_z^n &= E_{t_n}[e_y^{n+1}\Delta W_{t_{n+1}}] + \Delta t_n E_{t_n}[e_f^{n+1}\Delta W_{t_{n+1}}] \\ &\quad + \Delta B_{t_{n+1}} E_{t_n}[e_g^{n+1}\Delta W_{t_{n+1}}] + R_z^n. \end{aligned} \quad (5.29)$$

Taking square on both sides of equation (5.29) and then taking expectation we obtain

$$\begin{aligned}
E[\Delta t_n e_z^n]^2 &= E[E_{t_n}[e_y^{n+1}\Delta W_{t_{n+1}}] + \Delta B_{t_{n+1}} E_{t_n}[e_g^{n+1}\Delta W_{t_{n+1}}] + R_z^n]^2 \\
&\quad + E[(\Delta t_n)^2 E_{t_n}[e_f^{n+1}\Delta W_{t_{n+1}}]^2] \\
&\quad + 2E[(E_{t_n}[e_y^{n+1}\Delta W_{t_{n+1}}] + \Delta B_{t_{n+1}} E_{t_n}[e_g^{n+1}\Delta W_{t_{n+1}}] + R_z^n) \\
&\quad (\Delta t_n E_{t_n}[e_f^{n+1}\Delta W_{t_{n+1}}])] \\
&= E[(E_{t_n}[e_y^{n+1}\Delta W_{t_{n+1}}])^2 + (\Delta B_{t_{n+1}})^2 (E_{t_n}[e_g^{n+1}\Delta W_{t_{n+1}}])^2 + (R_z^n)^2 \\
&\quad + 2(E_{t_n}[e_y^{n+1}\Delta W_{t_{n+1}}])(E_{t_n}[e_g^{n+1}\Delta W_{t_{n+1}}])\Delta B_{t_{n+1}} + 2(E_{t_n}[e_y^{n+1}\Delta W_{t_{n+1}}])R_z^n \\
&\quad + 2E_{t_n}[e_g^{n+1}\Delta W_{t_{n+1}}]\Delta B_{t_{n+1}} R_z^n] \\
&\quad + E[(\Delta t_n)^2 E_{t_n}[e_f^{n+1}\Delta W_{t_{n+1}}]^2] \\
&\quad + 2E[(E_{t_n}[e_y^{n+1}\Delta W_{t_{n+1}}] + \Delta B_{t_{n+1}} E_{t_n}[e_g^{n+1}\Delta W_{t_{n+1}}] + R_z^n) \\
&\quad (\Delta t_n E_{t_n}[e_f^{n+1}\Delta W_{t_{n+1}}])] \\
&= A_z + B_z + C_z,
\end{aligned} \tag{5.30}$$

where

$$\begin{aligned}
A_z &= E[(E_{t_n}[e_y^{n+1}\Delta W_{t_{n+1}}])^2 + (\Delta B_{t_{n+1}})^2 (E_{t_n}[e_g^{n+1}\Delta W_{t_{n+1}}])^2 + (R_z^n)^2 \\
&\quad + 2(E_{t_n}[e_y^{n+1}\Delta W_{t_{n+1}}])(E_{t_n}[e_g^{n+1}\Delta W_{t_{n+1}}])\Delta B_{t_{n+1}} + 2(E_{t_n}[e_y^{n+1}\Delta W_{t_{n+1}}])R_z^n \\
&\quad + 2E_{t_n}[e_g^{n+1}\Delta W_{t_{n+1}}]\Delta B_{t_{n+1}} R_z^n], \\
B_z &= E[(\Delta t_n)^2 E_{t_n}[e_f^{n+1}\Delta W_{t_{n+1}}]^2]
\end{aligned}$$

and

$$\begin{aligned}
C_z &= 2E[(E_{t_n}[e_y^{n+1}\Delta W_{t_{n+1}}] + \Delta B_{t_{n+1}} E_{t_n}[e_g^{n+1}\Delta W_{t_{n+1}}] + R_z^n) \\
&\quad \cdot(\Delta t_n E_{t_n}[e_f^{n+1}\Delta W_{t_{n+1}}])].
\end{aligned}$$

For any $\mathcal{F}_t^W$ adapted process X_t we have

$$\begin{aligned}
(E_{t_n}[X_{t_{n+1}}\Delta W_{t_{n+1}}])^2 &= (E_{t_n}[(X_{t_{n+1}} - E_{t_n}[X_{t_{n+1}}])\Delta W_{t_{n+1}}])^2 \\
&\leq E_{t_n}[(X_{t_{n+1}} - E_{t_n}[X_{t_{n+1}}])]^2 \Delta t_n \\
&= \Delta t_n (E_{t_n}[(X_{t_{n+1}})^2] - |E_{t_n}[X_{t_{n+1}}]|^2).
\end{aligned} \tag{5.31}$$

For A_z, using (5.31), Cauchy's inequality and Young's inequality, we have

$$
\begin{aligned}
A_z &= E[(E_{t_n}[e_y^{n+1}\Delta W_{t_{n+1}}])^2 + (\Delta B_{t_{n+1}})^2(E_{t_n}[e_g^{n+1}\Delta W_{t_{n+1}}])^2 + (R_z^n)^2 \\
&\quad +2(E_{t_n}[e_y^{n+1}\Delta W_{t_{n+1}}])(E_{t_n}[e_g^{n+1}\Delta W_{t_{n+1}}])\Delta B_{t_{n+1}} + 2(E_{t_n}[e_y^{n+1}\Delta W_{t_{n+1}}])R_z^n \\
&\quad +2E_{t_n}[e_g^{n+1}\Delta W_{t_{n+1}}]\Delta B_{t_{n+1}} R_z^n] \\
&\leq E[(E_{t_n}[e_y^{n+1}\Delta W_{t_{n+1}}])^2 + (\Delta B_{t_{n+1}})^2(E_{t_n}[e_g^{n+1}\Delta W_{t_{n+1}}])^2 + (R_z^n)^2 \\
&\quad +\epsilon(E_{t_n}[e_y^{n+1}\Delta W_{t_{n+1}}])^2 + \frac{1}{\epsilon}(R_z^n)^2 \\
&\quad +\epsilon_1(E_{t_n}[e_g^{n+1}\Delta W_{t_{n+1}}]\Delta B_{t_{n+1}})^2 + \frac{1}{\epsilon_1}(R_z^n)^2] \\
&\leq \Delta t_n E[E_{t_n}[e_y^{n+1}]^2 - |E_{t_n}[e_y^{n+1}]|^2] \\
&\quad +(\Delta t_n)^2 E[E_{t_n}[e_g^{n+1}]^2 - |E_{t_n}[e_g^{n+1}]|^2] + E[(R_z^n)^2] \\
&\quad +\epsilon\Delta t_n E[E_{t_n}[e_y^{n+1}]^2 - |E_{t_n}[e_y^{n+1}]|^2] + \frac{1}{\epsilon}E[(R_z^n)^2] \\
&\quad +\epsilon_1(\Delta t_n)^2 E[E_{t_n}[e_g^{n+1}]^2 - |E_{t_n}[e_g^{n+1}]|^2] + \frac{1}{\epsilon_1}E[(R_z^n)^2] \\
&\leq (1+\epsilon)\Delta t_n E[E_{t_n}[e_y^{n+1}]^2 - |E_{t_n}[e_y^{n+1}]|^2] \\
&\quad +(1+\epsilon_1)(\Delta t_n)^2 E[E_{t_n}[e_g^{n+1}]^2 - |E_{t_n}[e_g^{n+1}]|^2] \\
&\quad +C_5(\Delta t)^3.
\end{aligned} \tag{5.32}
$$

Under the conditions in the theorem, we have

$$B_z = E[(\Delta t_n)^2 E_{t_n}[e_f^{n+1}\Delta W_{t_{n+1}}]^2] \leq C_6(\Delta t)^3. \tag{5.33}$$

Similarly, using Cauchy's inequality and Young's inequality, we obtain

$$
\begin{aligned}
C_z &= 2E[(E_{t_n}[e_y^{n+1}\Delta W_{t_{n+1}}] + \Delta B_{t_{n+1}} E_{t_n}[e_g^{n+1}\Delta W_{t_{n+1}}] + R_z^n) \\
&\quad (\Delta t_n E_{t_n}[e_f^{n+1}\Delta W_{t_{n+1}}])] \\
&\leq \Delta t_n \frac{1}{\epsilon_2} E[(E_{t_n}[e_y^{n+1}\Delta W_{t_{n+1}}] + \Delta B_{t_{n+1}} E_{t_n}[e_g^{n+1}\Delta W_{t_{n+1}}] + R_z^n)]^2 \\
&\quad +\Delta t_n \epsilon_2 E_{t_n}[e_f^{n+1}\Delta W_{t_{n+1}}]^2 \\
&\leq (\Delta t_n)^2 \frac{3}{\epsilon_2}(E[e_y^{n+1}]^2 + \Delta t_n L_1 E[e_y^{n+1}]^2 + \Delta t_n L_2 E[e_z^{n+1}]^2 + (R_z^n)^2)
\end{aligned}
$$

$$
\begin{aligned}
& +(\Delta t_n)^2 L\epsilon_2(E[e_y^{n+1}]^2 + E[e_z^{n+1}]^2) \\
\leq\ & C_7\left\{(\Delta t_n)^2 E[e_y^{n+1}]^2 + (\Delta t_n)^3(E[e_y^{n+1}]^2 + E[e_z^{n+1}]^2)\right\} \\
& +(\Delta t_n)^2 L\epsilon_2 E[e_z^{n+1}]^2 + C_8(\Delta t)^3. \qquad (5.34)
\end{aligned}
$$

Here ϵ_1 and ϵ_2 are same as in equation (5.91), and ϵ is a positive constant which will be determined later. Combining (5.30), (5.32), (5.33) and (5.34) together, we get

$$
\begin{aligned}
E[\Delta t_n e_z^n]^2 \leq\ & (1+\epsilon)\Delta t_n E[E_{t_n}[e_y^{n+1}]^2 - |E_{t_n}[e_y^{n+1}]|^2] \\
& +(1+\epsilon_1)(\Delta t_n)^2 E[E_{t_n}[e_g^{n+1}]^2 - |E_{t_n}[e_g^{n+1}]|^2] \\
& +(\Delta t_n)^2 L\epsilon_2 E[e_z^{n+1}]^2 \qquad (5.35) \\
& +K_4(\Delta t_n)^2 E_{t_n}[e_y^{n+1}]^2 + K_5(\Delta t_n)^3 E_{t_n}[e_z^{n+1}]^2 \\
& +K_6(\Delta t)^3.
\end{aligned}
$$

Next we divide by $\Delta t_n(1+\epsilon)$ on both sides of (5.35) to obtain

$$
\begin{aligned}
\frac{\Delta t_n}{1+\epsilon}E[e_z^n]^2 \leq\ & E[E_{t_n}[e_y^{n+1}]^2 - |E_{t_n}[e_y^{n+1}]|^2] \\
& +(\Delta t_n)\frac{1+\epsilon_1}{1+\epsilon}E[E_{t_n}[e_g^{n+1}]^2 - |E_{t_n}[e_g^{n+1}]|^2] \\
& +(\Delta t_n)L\frac{\epsilon_2}{1+\epsilon}E[e_z^{n+1}]^2 \qquad (5.36) \\
& +K_4\Delta t_n E_{t_n}[e_y^{n+1}]^2 + K_5(\Delta t_n)^2 E_{t_n}[e_z^{n+1}]^2 \\
& +K_6(\Delta t)^2.
\end{aligned}
$$

Adding (5.36) to (5.91), we obtain

$$
\begin{aligned}
E[e_y^n]^2 + \frac{\Delta t_n}{1+\epsilon}E[e_z^n]^2 \leq\ & E[|E_{t_n}[e_y^{n+1}]|^2] + \Delta t_n E[(1+\epsilon_1)|E_{t_n}[e_g^{n+1}]|^2] \\
& +\Delta t_n L\epsilon_2 E[e_z^{n+1}]^2 + K_1\Delta t_n E[e_y^{n+1}]^2 + K_2(\Delta t_n)^2 E[e_z^{n+1}]^2 \\
& +K_3(h^2 + E[\int_{t_n}^{t_{n+1}} |z_s - z_{t_n}|^2]ds) \\
& +E[E_{t_n}[e_y^{n+1}]^2 - |E_{t_n}[e_y^{n+1}]|^2] \\
& +(\Delta t_n)\frac{1+\epsilon_1}{1+\epsilon}E[E_{t_n}[e_g^{n+1}]^2 - |E_{t_n}[e_g^{n+1}]|^2]
\end{aligned}
$$

$$
\begin{aligned}
& +(\Delta t_n) L \frac{\epsilon_2}{1+\epsilon} E[e_z^{n+1}]^2 \\
& +K_4 \Delta t_n E_{t_n}[e_y^{n+1}]^2 + K_5 (\Delta t_n)^2 E_{t_n}[e_z^{n+1}]^2 \\
& +K_6((\Delta t)^2 + E[\int_{t_n}^{t_{n+1}} |z_s - z_{t_n}|^2] ds) \\
= \quad & E[e_y^n]^2 + (\Delta t_n) \frac{1+\epsilon_1}{1+\epsilon} E[E_{t_n}[e_g^{n+1}]^2] \\
& +(\Delta t_n) \frac{1+\epsilon_1}{1+\epsilon} \epsilon E[|E_{t_n}[e_g^{n+1}]|^2] \\
& +(\Delta t_n) L \frac{\epsilon_2 + \epsilon_2 + \epsilon\epsilon_2}{1+\epsilon} E[e_z^{n+1}]^2 \\
& +G_1 \Delta t_n E[e_y^{n+1}]^2 + G_2 (\Delta t_n)^2 E[e_z^{n+1}]^2 + G_3 (\Delta t)^2 \\
\leq \quad & E[e_y^n]^2 + (\Delta t_n)(L_2 \frac{1+\epsilon_1}{1+\epsilon} \\
& +L_2 \frac{1+\epsilon_1}{1+\epsilon} \epsilon + L \frac{\epsilon_2 + \epsilon_2 + \epsilon\epsilon_2}{1+\epsilon}) E[e_z^{n+1}]^2 \\
& +G_1 \Delta t_n E[e_y^{n+1}]^2 + G_2 (\Delta t_n)^2 E[e_z^{n+1}]^2 + G_3 (\Delta t)^2.
\end{aligned} \tag{5.37}
$$

Now we choose ϵ, ϵ_1 and ϵ_2, all positive, sufficiently small such that

$$
L_2(1+\epsilon_1) + L_2\epsilon(1+\epsilon_1) + L(2\epsilon_2 + \epsilon\epsilon_2) \leq 1.
$$

This is possible since $L_2 < 1$. Thus, by equation (5.37), we have

$$
\begin{aligned}
E[e_y^n]^2 + \frac{\Delta t_n}{1+\epsilon} E[e_z^n]^2 \quad \leq \quad & E[e_y^n]^2 + \frac{\Delta t_n}{1+\epsilon} E[e_z^{n+1}]^2 \\
& +T_1 \Delta t_n (E[e_y^{n+1}]^2 + \frac{\Delta t_n}{1+\epsilon} E[e_z^{n+1}]^2) \\
& +T_2 (\Delta t)^2.
\end{aligned}
$$

Denote $e_n := E[e_y^n]^2 + \frac{\Delta t_n}{1+\epsilon} E[e_z^n]^2$. Then the above equation becomes

$$
e_n \leq (1 + T_1 \Delta t) e_{n+1} + T_2 (\Delta t)^2.
$$

By Gronwall's inequality, we have

$$\max_{0\leq n\leq N-1}(E[e_y^n]^2+\frac{\Delta t_n}{1+\epsilon}E[e_z^n]^2)\leq C\Delta t$$

as required.

Error estimate for z

We first construct an approximate solution $(\tilde{Y},\tilde{Z})$ with step process as follows. Let

$$\tilde{Y}_{t_{n+1}}=y^{n+1}+\Delta t_n\cdot f^{n+1}+\Delta B_{t_n}\cdot g^{n+1}$$

and $\mathcal{G}_t=\mathcal{F}_t^W\vee\mathcal{F}_T^B$. By an extension of Itô's martingale representation theorem, we can find an $\mathcal{G}_t$ adapted process $\tilde{Z}_t$, such that

$$\tilde{Y}_{t_{n+1}}=E[\tilde{Y}_{t_{n+1}}|\mathcal{G}_{t_n}]+\int_{t_n}^{t_{n+1}}\tilde{Z}_r dW_r. \tag{5.38}$$

Define a continuous approximate process $(\tilde{Y},\tilde{Z})$ as follows.

$$\begin{aligned}\tilde{Y}_t=y^{n+1}+f^{n+1}\cdot(t_{n+1}-t)+g^{n+1}\cdot(B_{t_{n+1}}-B_t)-\int_t^{t_{n+1}}\tilde{Z}_r dW_r,\\ t\in(t_n,t_{n+1}],n=0,\cdots N-1\end{aligned} \tag{5.39}$$

where

$$f^{n+1}=f(t_{n+1},y^{n+1},z^{n+1})$$

and

$$g^{n+1}=g(t_{n+1},y^{n+1},z^{n+1}).$$

By (5.19) and (5.38) it's easy to see that

$$\Delta t_n z^n=\int_{t_n}^{t_{n+1}}\mathbb{E}_{t_n}[\tilde{Z}_r]dr.$$

Thus

$$
\begin{aligned}
&\int_{t_n}^{t_{n+1}} E[(z_s - z^n)^2]ds = \int_{t_n}^{t_{n+1}} E[(z_s - \frac{1}{\Delta t_n}\int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}[\tilde{Z}_r]dr)^2]ds \\
&= \int_{t_n}^{t_{n+1}} E[(\frac{1}{\Delta t_n}\int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}[z_s - \tilde{Z}_r]dr)^2]ds \\
&\leq E\int_{t_n}^{t_{n+1}} \frac{1}{\Delta t_n}\int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}[(z_s - \tilde{Z}_r)^2]drds \\
&\leq 2(\int_{t_n}^{t_{n+1}} E[\mathbb{E}_{t_n}[(z_r - \tilde{Z}_r)^2]]dr + \int_{t_n}^{t_{n+1}} \frac{1}{\Delta t_n}\int_{t_n}^{t_{n+1}} E[\mathbb{E}_{t_n}[(z_s - z_r)^2]]drds) \\
&\leq 2(\int_{t_n}^{t_{n+1}} E[(z_r - \tilde{Z}_r)^2]dr + (\Delta t_n)^2).
\end{aligned}
\tag{5.40}
$$

Now we are ready to prove an error estimate for z.

Theorem 5.2 *Let (y_t, z_t) be the exact solution and (y^n, z^n) be the solution of the scheme* (5.18) *and* (5.19). *Assume that Hypothesis* (5.20) *holds and derivatives f'_x, f'_y, f'_z, g'_x, g'_y and g'_z of f and g are all bounded. Then for Δt sufficiently small, we have*

$$\sum_{n=0}^{N-1} E\int_{t_n}^{t_{n+1}} (z_s - z^n)^2 ds \leq C\Delta t.$$

Proof: For $t \in [t_n, t_{n+1}]$, let $e_y^t = y_t - \tilde{Y}_t$, $e_z^t = z_t - \tilde{Z}_t$, $f_t = f(t, y_t, z_t)$ and $g_t = g(t, y_t, z_t)$. Subtracting BDSDE (5.39) from (5.2) we have that

$$e_y^t = e_y^{t_{n+1}} + \int_t^{t_{n+1}} (f_s - f^{n+1})ds + \int_t^{t_{n+1}} (g_s - g^{n+1})d\overleftarrow{B}_s - \int_t^{t_{n+1}} e_z^s dW_s. \tag{5.41}$$

Taking square on both sides of (5.41), applying Itô's formula (Pardoux and Peng (1994)) and taking expectation, we have

$$
\begin{aligned}
& E[(e_y^t)^2] + E\int_t^{t_{n+1}} (e_z^s)^2 ds \\
= \;& E[(e_y^{t_{n+1}})^2] + 2E\int_t^{t_{n+1}} e_y^s \cdot (f_s - f^{n+1})ds + E\int_t^{t_{n+1}} (g_s - g^{n+1})^2 ds \\
\leq \;& E[(e_y^{t_{n+1}})^2] + \frac{1}{\epsilon_0}E\int_t^{t_{n+1}} (e_y^s)^2 ds + \epsilon_0 E\int_t^{t_{n+1}} (f_s - f^{n+1})^2 ds + E\int_t^{t_{n+1}} (g_s - g^{n+1})^2 ds
\end{aligned}
$$

$$
\begin{aligned}
&\leq E[(e_y^{t_{n+1}})^2] + \frac{1}{\epsilon_0} E\int_t^{t_{n+1}} (e_y^s)^2 ds + E\int_t^{t_{n+1}} 2c(y_s - y^{n+1})^2 + (\epsilon_1+\alpha)(z_s - z^{n+1})^2 ds \\
&\leq E[(e_y^{t_{n+1}})^2] + \frac{1}{\epsilon_0} E\int_t^{t_{n+1}} (e_y^s)^2 ds + CE\int_t^{t_{n+1}} (y_s - y_{t_{n+1}})^2 + (y_{t_{n+1}} - y^{n+1})^2 ds \\
&\quad + E\int_t^{t_{n+1}} (\epsilon_1 + \frac{1}{\epsilon_2} + \alpha)[(z_s - (\Delta t_n)^{-1}\int_{t_{n+1}}^{t_{n+2}} \mathbb{E}_{t_{n+1}}[z_r]dr)^2] \\
&\quad + (\epsilon_1 + \epsilon_2 + \alpha)[((\Delta t_n)^{-1}\int_{t_{n+1}}^{t_{n+2}} \mathbb{E}_{t_{n+1}}[z_r]dr - z^{n+1})^2]ds, \qquad (5.42)
\end{aligned}
$$

where ϵ_0, ϵ_1 and ϵ_2 are positive constants to be determined later. Since $E[(y_s-y_t)^2]+E[(z_s-z_t)^2] \leq C|s-t|$, we have

$$
E\int_t^{t_{n+1}} (y_s - y_{t_{n+1}})^2 ds \leq C(\Delta t)^2
$$

and

$$
\begin{aligned}
&E\int_t^{t_{n+1}} (z_s - (\Delta t_n)^{-1}\int_{t_n+1}^{t_{n+2}} \mathbb{E}_{t_{n+1}}[z_r]dr)^2 ds \\
&= E\int_t^{t_{n+1}} (z_s - z_{t_{n+1}} + z_{t_{n+1}} - (\Delta t_n)^{-1}\int_{t_n+1}^{t_{n+2}} \mathbb{E}_{t_{n+1}}[z_r]dr)^2 ds \\
&= 2E\int_t^{t_{n+1}} (z_s - z_{t_{n+1}})^2 + ((\Delta t_n)^{-1}\int_{t_{n+1}}^{t_{n+2}} \mathbb{E}_{t_{n+1}}[z_{t_{n+1}} - z_r]dr)^2 ds \\
&\leq C(\Delta t)^2.
\end{aligned}
$$

Also, since $\Delta t z^{n+1} = \int_{t_{n+1}}^{t_{n+2}} \mathbb{E}_{t_{n+1}}[\tilde{Z}_r]dr$, we have

$$
((\Delta t_n)^{-1}\int_{t_{n+1}}^{t_{n+2}} \mathbb{E}_{t_{n+1}}[z_r]dr - z^{n+1})^2 \leq (\Delta t_n)^{-1}\int_{t_{n+1}}^{t_{n+2}} \mathbb{E}_{t_{n+1}}[(z_r - \tilde{Z}_r)^2]dr.
$$

Thus we can rewrite (5.42) as

$$
\begin{aligned}
E[(e_y^t)^2] + E\int_t^{t_{n+1}} (e_z^s)^2 ds \quad &\leq \quad E[(e_y^{t_{n+1}})^2] + CE\int_t^{t_{n+1}} (e_y^{t_{n+1}}) ds \\
&\quad + (\epsilon_1 + \epsilon_2 + \alpha)E\int_{t_{n+1}}^{t_{n+2}} (e_z^s)^2 ds + C(\Delta t)^2 \\
&\quad + \frac{1}{\epsilon_0} E\int_t^{t_{n+1}} (e_y^s)^2 ds.
\end{aligned}
$$

Choose ϵ_1 and ϵ_2 small enough such that $(\epsilon_1 + \epsilon_2 + \alpha) = K < 1$ (since $\alpha < 1$). Then

$$
\begin{aligned}
E[(e_y^t)^2] + E\int_t^{t_{n+1}} (e_z^s)^2 ds &\leq C_1 E\int_t^{t_{n+1}} (e_y^s)^2 ds + E[(e_y^{t_{n+1}})^2] + C_2 E\int_{t_n}^{t_{n+1}} (e_y^{t_{n+1}})^2 ds \\
&\quad + KE\int_{t_{n+1}}^{t_{n+2}} (e_z^s)^2 ds + C(\Delta t)^2 \\
&\leq \left\{(1 + C_2\Delta t)E[(e_y^{t_{n+1}})^2] + KE\int_{t_{n+1}}^{t_{n+2}} (e_z^s)^2 ds + C(\Delta t)^2\right\} \\
&\quad + C_1 E\int_t^{t_{n+1}} (e_y^s)^2 ds.
\end{aligned}
\tag{5.43}
$$

By Gronwall's inequality, we get

$$
E[(e_y^s)^2] \leq C((1 + C_2\Delta t)E[(e_y^{t_{n+1}})^2] + E\int_{t_{n+1}}^{t_{n+2}} (e_z^s)^2 ds + C(\Delta t)^2) \tag{5.44}
$$

for $s \in [t_n, t_{n+1}]$. Now we let $t = t_n$ in (5.43) and substitute (5.44) in (5.43) to obtain

$$
\begin{aligned}
E[(e_y^{t_n})^2] + E\int_{t_n}^{t_{n+1}} (e_z^s)^2 ds &\leq (1 + C_1\Delta t)E[(e_y^{t_{n+1}})^2] \\
&\quad + (K + C_2\Delta t)E\int_{t_{n+1}}^{t_{n+2}} (e_z^s)^2 ds + C(\Delta t)^2.
\end{aligned}
$$

Now by Theorem 1, we easily obtain

$$
E[(e_y^{t_n})^2] + E\int_{t_n}^{t_{n+1}} (e_z^s)^2 ds \leq E[(e_y^{t_{n+1}})^2] + (K + C\Delta t)E\int_{t_{n+1}}^{t_{n+2}} (e_z^s)^2 ds + C(\Delta t)^2.
$$

Let Δt be sufficiently small such that $C\Delta t + K \leq L < 1$, where L is a constant. Summing the above equation from $n = 0$ to $n = N - 1$, we obtain

$$
(1 - L)\sum_{n=0}^{N-1} E\int_{t_n}^{t_{n+1}} (e_z^s)^2 ds \leq C\Delta t + L\int_{t_{N-1}}^{t_N} E[(e_z^s)^2] ds \leq C\Delta t. \tag{5.45}
$$

Through a similar argument, it's easy to obtain

$$
\int_{t_{N-1}}^{t_N} E[(e_z^s)^2] ds \leq C\Delta t. \tag{5.46}
$$

By (5.40), (5.45) and (5.46), we conclude that

$$\sum_{n=0}^{N-1} E \int_{t_n}^{t_{n+1}} (z_s - z^n)^2 ds \leq C\Delta t$$

as required in Theorem 2.

5.2.3 Numerical experiments

In this section we carry out numerical experiments to verify the rate of convergence results obtained in Section 3 and compare our numerical method with the finite difference method for stochastic parabolic partial differential equations ([**?**]). The conditional expectations in (5.18) and (5.19) can be evaluated using Monte Carlo method or Gaussian quadratures ([**?**, 90]. In our examples, we use the binomial tree method which is amount to two point Gaussian quadrature ([**?**]).

Example 1: In the first example, we consider the initial boundary value problem.

$$\begin{aligned}
u_t(x) &= \exp(x \cdot T)\sin(\frac{B(T)}{2}) + \int_t^T [\frac{1}{2}\frac{\partial^2}{\partial x^2}u_s(x) - (x+\frac{1}{8})u_s(x) - \frac{s}{2}\frac{\partial}{\partial x}u_s(x)]ds \\
&\quad + \int_t^T \frac{1}{2}\exp(x \cdot s)\cos(B(T) - \frac{B(s)}{2})d\overleftarrow{B}_s, \\
u_t(-1) &= \exp((-1) \cdot t)\sin(B(T) - \frac{B(T)}{2}), \\
u_t(1) &= \exp(1 \cdot t)\sin(B(T) - \frac{B(t)}{2}).
\end{aligned} \tag{5.47}$$

Table 5.1: Example 1 of Section 6.2

J	N_T (FD)	error(FD)	N_T (BDSDE)	error (BDSDE)
2^2	2^5	0.0213	2^4	0.200
2^3	2^7	0.0177	2^6	0.0383
2^4	2^9	0.0112	2^8	0.0106
2^5	2^{11}	0.00587	2^{10}	0.00376
2^6	2^{13}	0.00313	2^{12}	0.00152

We construct the SPDE (5.47) in such a way that $u_t(x) = \exp(x \cdot t)\sin(B(T) - \frac{B(t)}{2})$ is the exact solution. The corresponding BDSDE is given by

$$\begin{aligned} y_0^{0,x} &= \exp(X_T^{0,x} \cdot T)\sin(\frac{B(T)}{2})I_{\tau \geq T} + \exp(X_\tau^{0,x} \cdot \tau)\sin(B(T) - \frac{B(\tau)}{2})I_{\tau \leq T} \\ &+ \int_0^{T\wedge\tau} [-(X_t^{0,x} + \frac{1}{8})y_t^{0,x} - \frac{t}{2}z_t^{0,x}]ds \\ &- \int_0^{T\wedge\tau} z_t^{0,x} dW_t + \int_0^{T\wedge\tau} \frac{1}{2}\exp(X_t^{0,x} \cdot t)\cos(B(T) - \frac{B(t)}{2})d\overleftarrow{B}_t. \end{aligned}$$

The numerical results are shown in Table 1 and Figure 1. Here J denotes the number of spatial partition grids, $N_T(FD)$ the number of time steps used in finite difference method, $N_T(BDSDE)$ the number of time steps used in our method for solving the related BDSDE, and $error(FD)$ and $error(BDSDE)$ the errors of finite difference method and our method, respectively. The results indicate that our algorithm is comparable to the the algorithm of solving the SPDE directly using the finite difference scheme, with a little higher rate of convergence.

Example 2: In this example, we consider the unbounded SPDE initial value problem.

$$\begin{aligned} u_t(x) &= \sin(x+T)\cos(2B_T) + \int_t^T [\frac{1}{2}\frac{\partial^2}{\partial x^2}u_s(x) - \frac{\partial}{\partial x}u_s(x)]ds \\ &+ \int_t^T \sin(W(s)+s)(\sin(B_T+B_s) - \cos(B_T+B_s)) + u_s(x)d\overleftarrow{B}_s, \end{aligned} \tag{5.48}$$

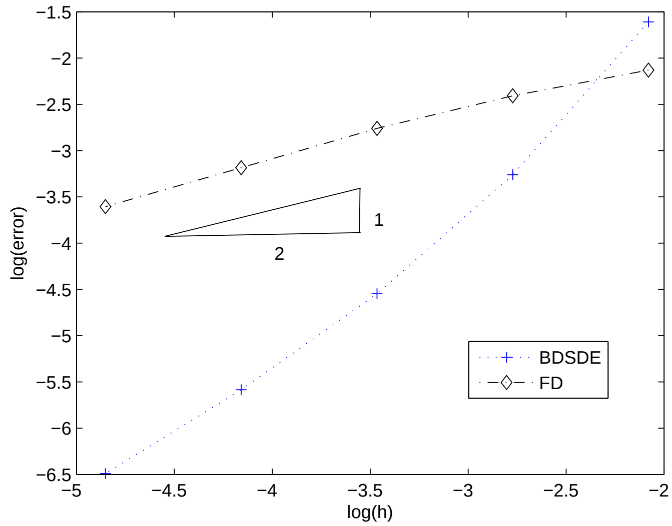


Figure 5.1: Example 1: Convergence comparison between the direct finite difference scheme and our scheme

Table 5.2: Example 2 of Section 6.2

J	N_T	error(Y) (u)	error(Z) (∇u)
2^3	2^3	$6.4096E-002$	0.1188
2^4	2^4	$3.2019E-002$	$9.0028E-002$
2^5	2^5	$1.4426E-002$	$6.3314E-002$
2^6	2^6	$7.2577E-003$	$4.1337E-002$
2^7	2^7	$3.5995E-003$	$2.7149E-002$

where $u_t(x) = \sin(x+t)\cos(B_T + B_t)$ is the solution of the SPDE (5.48). The corresponding FBDSDE is given by

$$
\begin{aligned}
y_0^{0,x} &= \sin(W(T)+T)\cos(2B_T) - \int_0^T z_s^{0,x} ds \\
&\quad + \int_0^t [\sin(W(s)+s)(\sin(B_T+B_s) - \cos(B_T+B_s)) +_s^{0,x}] d\overleftarrow{B}_s \\
&\quad - \int_0^t z_s^{0,x} dW_s.
\end{aligned}
$$

The errors are shown in Table 2, in which $error(Y)$ and $error(Z)$ are errors for Y and Z at time-space point $(t,x) = (0,0)$, respectively. These data also confirm our rate of converence results.

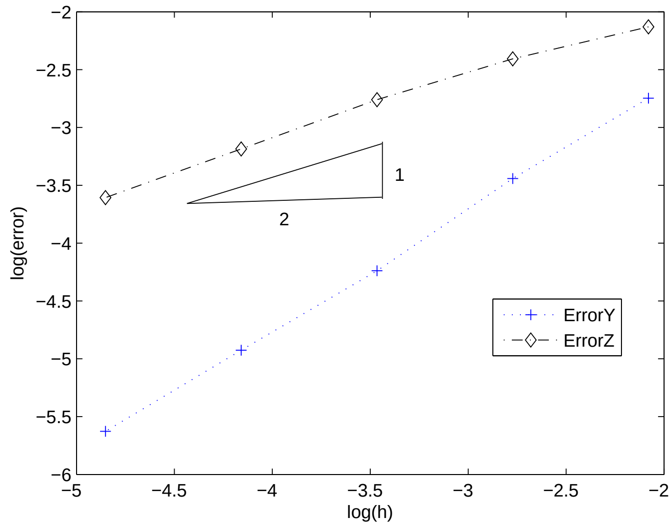


Figure 5.2: Example 2: Convergence comparison between the approximations of y and z

5.3 First Order Numerical Algorithms

To develop first order numerical algorithms for BDSDEs, we simplify the BDSDEs system to

$$\begin{aligned} Y_s^{t,x} &= h(W_T^{t,x}) + \int_s^T f(r, W_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr \\ &\quad + \int_s^T g(r, W_r^{t,x}, Y_r^{t,x})d\overleftarrow{B}_r - \int_s^T Z_r^{t,x} dW_r, \quad s \le t \le T. \end{aligned} \tag{5.49}$$

In addition to the definitions and assumptions in subsection 5.1, we introduce variational equations of BDSDEs.

5.3.1 Variational Equations

Let $\nabla Y_s^{t,x}$, $\nabla Z_s^{t,x}$ and $\nabla W_s^{t,x}$ be the variations of $Y_s^{t,x}$, $Z_s^{t,x}$ and $W_s^{t,x}$, respectively, with respect to x at time level $s = t$. Notice that that $\nabla W_s^{t,x} \equiv 1$. To simplify the presentation, we write $y_s = Y_s^{t,x}$, $z_s = Z_s^{t,x}$, $W_s = W_s^{t,x}$, $\nabla y_s = \nabla Y_s^{t,x}$, $\nabla z_s = \nabla Z_s^{t,x}$ and $\nabla W_s = \nabla W_s^{t,x}$.

From [61], $\{(\nabla y_t, \nabla z_t); 0 \le t \le T\}$, is the unique solution of the equation

$$\nabla y_t = \nabla y_T + \int_t^T \nabla F_s ds + \int_t^T \nabla G_s d\overleftarrow{B}_s - \int_t^T \nabla z_s dW_s, \tag{5.50}$$

where

$$\nabla y_T = h'(X_T)\nabla X_T,$$

$$\nabla F_s = f'_W(s, W_s, y_s, z_s) + f'_y(s, W_s, y_s, z_s) \cdot \nabla y_s + f'_z(s, W_s, y_s, z_s) \cdot \nabla z_s, \quad t \le s \le T$$

and

$$\nabla G_s = g'_W(s, W_s, y_s) + g'_y(s, W_s, y_s) \cdot \nabla y_s, \quad t \le s \le T.$$

Here f'_W, f'_y, f'_z and g'_W, g'_y are first order partial derivatives of functions f and g with respect to W_s, y_s and z_s. By Proposition 2.3 of [61],

$$z_s = \nabla y_s \cdot (\nabla W_s)^{-1} = \nabla y_s, \qquad 0 \le s \le T. \tag{5.51}$$

5.3.2 Reference equations

In this subsection we approximate integrals in (5.49) with appropriate quadratures with first order accuracy and name the resulting equations as reference equations (see [87]). For this purpose, we define $\mathcal{F}_s^{W,t} := \sigma(W_r; t \le r \le s) \vee \sigma(B_p; 0 \le p \le T)$. Let $E[X]$ denote the mathematical expectation of the random variable X and $\mathbb{E}_t^{t,x}[X]$ denote the conditional expectation $E[X|\mathcal{F}_t^{W,t}]$ of the random variable X with $W_t = x$.

To further simplify the notation, we denote

$$\begin{aligned}
&f(s, y_s, z_s) := f(s, W_s, B_s, y_s, z_s), \quad g(s, y_s) := g(s, W_s, B_s, y_s),\\
&g'_t(s, y_s) := g'_t(s, W_s, B_s, y_s), \quad g'_W(s, y_s) := g'_W(s, W_s, B_s, y_s),\\
&g'_B(s, y_s) := g'_B(s, W_s, B_s, y_s), \quad g'_y(s, y_s) := g'_y(s, W_s, B_s, y_s),\\
&g''_{WW}(s, y_s) := g''_{WW}(s, W_s, B_s, y_s), \quad g''_{BB}(s, y_s) := g''_{BB}(s, W_s, B_s, y_s),
\end{aligned}$$

and $g''_{yy}(s, y_s) := g''_{yy}(s, W_s, B_s, y_s)$, where g'_t, g'_W, g'_B, g'_y and g'_z are first order partial derivatives with respect to variables t, W_t, B_t and y_t, respectively; g''_{WW}, g''_{BB} and g''_{yy} are second order partial derivatives with respect to variables W_t, B_t, and y_t, respectively.

With the above notations and from equation (5.49), we have for $n = 1, \cdots, N_T - 1$,

$$y_{t_n} = y_{t_{n+1}} + \int_{t_n}^{t_{n+1}} f(s, y_s, z_s) ds - \int_{t_n}^{t_{n+1}} z_s dW_s + \int_{t_n}^{t_{n+1}} g(s, y_s) \overleftarrow{dB}_s. \tag{5.52}$$

Reference equation for y_s

We first eliminate the forward Itô integral from(5.52). To this end we take the conditional expectation $\mathbb{E}_{t_n}^{t_n,x}[\cdot]$ on both sides of (5.52) to obtain

$$y_{t_n}^{t_n,x} = \mathbb{E}_{t_n}^{t_n,x}[y_{t_{n+1}}] + \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{t_n,x}[f(s, y_s, z_s)] ds + \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{t_n,x}[g(s, y_s, z_s)] \overleftarrow{dB}_s, \tag{5.53}$$

where $y_{t_n}^{t_n,x} = \mathbb{E}_{t_n}^{t_n,x}[y_{t_n}]$ is the value of y_{t_n} at the space point x.

Next we approximate the integrals in the above equation with appropriate quadrature formulas. For the first integral on the right hand side of (5.53), we simply use right point formula to obtain

$$\int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{t_n,x}[f(s, y_s, z_s)] ds = \Delta t_n \mathbb{E}_{t_n}^{t_n,x}[f(t_{n+1}, y_{t_{n+1}}, z_{t_{n+1}})] + R_y^{W,n}; \tag{5.54}$$

where

$$R_y^{W,n} = \int_{t_n}^{t_{n+1}} \{\mathbb{E}_{t_n}^{t_n,x}[f(s, y_s, z_s)] - \mathbb{E}_{t_n}^{t_n,x}[f(t_{n+1}, y_{t_{n+1}}, z_{t_{n+1}})]\} ds. \tag{5.55}$$

For the backward Itô integral term in (5.53), the application of the right point formula will result in only a half order scheme (see [6]) and thus is inadequate for deriving a first order algorithm. To obtain a first order numerical scheme, we introduce Itô's formula for doubly stochastic integrals ([61] Lemma 1.3).

Let $\alpha_t = (t, W_t, B_t, y_t)$, $\beta_t = (1, 0, 0, -f(t, y_t, z_t))$, $\gamma_t = (0, 0, 1, -g(t, y_t))$ and $\delta_t = (0, 1, 0, z_t)$. In light of (5.49), (5.50) and the identity (5.51), we have following SDE for α_t:

$$\alpha_t = \alpha_s + \int_s^t \beta_r dr + \int_s^t \gamma_r d\overleftarrow{B}_r + \int_s^t \delta_r dW_r, \quad 0 \le s < t \le T.$$

Applying Itô's formula for doubly stochastic integrals gives

$$
\begin{aligned}
g(t,y_t) &= g(s,y_s) + \int_s^t \left(g'_t(r,y_r) - g'_y(r,y_r)\cdot f(r,y_r,z_r)\right) dr \\
&\quad + \int_s^t \left(g'_B(r,y_r) - g'_y(r,y_r)\cdot g(r,y_r)\right) d\overleftarrow{B}_r + \int_s^t \left(g'_W(r,y_r) + g'_y(r,y_r)\cdot z_r\right) dW_r \\
&\quad - \frac{1}{2}\int_s^t \left(g''_{BB}(r,y_r) + g''_{yy}(r,y_r)\cdot (g(r,y_r))^2\right) dr \\
&\quad + \frac{1}{2}\int_s^t \left(g''_{WW}(r,y_r) + g''_{yy}(r,y_r)\cdot z_r^2\right) dr.
\end{aligned}
$$

Letting $t = t_{n+1}$ in the above equation we have for $t_n \leq s \leq t_{n+1}$ that

$$
\begin{aligned}
g(s,y_s) = g(t_{n+1}, y_{t_{n+1}}) &- \int_s^{t_{n+1}} \left(g'_B(r,y_r) - g'_y(r,y_r)\cdot g(r,y_r)\right) d\overleftarrow{B}_r \\
&- \int_s^{t_{n+1}} \left(g'_W(r,y_r) + g'_y(r,y_r)\cdot z_r\right) dW_r + R_g^n(s),
\end{aligned}
\tag{5.56}
$$

where

$$
\begin{aligned}
R_g^n(s) &= -\int_s^{t_{n+1}} \left(g'_t(r,y_r) - g'_y(r,y_r)\cdot f(r,y_r,z_r)\right) dr \\
&\quad + \frac{1}{2}\int_s^{t_{n+1}} \left(g''_{BB}(r,y_r) + g''_{yy}(r,y_r)\cdot (g(r,y_r))^2\right) dr \\
&\quad - \frac{1}{2}\int_s^{t_{n+1}} \left(g''_{WW}(r,y_r) + g''_{yy}(r,y_r)\cdot z_r^2\right) dr.
\end{aligned}
$$

Taking conditional expectation $\mathbb{E}_{t_n}^{t_n,x}[\cdot]$ on both sides of (5.56) and noting that

$$
\mathbb{E}_{t_n}^{t_n,x}[\int_s^{t_{n+1}} \left(g'_W(r,y_r) + g'_y(r,y_r)\cdot z_r\right) dW_r] = 0
$$

we obtain

$$
\mathbb{E}_{t_n}^{t_n,x}[g(s,y_s)] \quad = \quad \mathbb{E}_{t_n}^{t_n,x}[g(t_{n+1},y_{t_{n+1}})] - \int_s^{t_{n+1}} \mathbb{E}_{t_n}^{t_n,x}\left[g'_B(r,y_r)\right.
$$

$$-g'_y(r, y_r) \cdot g(r, y_r)\big]\, d\overleftarrow{B}_r + \mathbb{E}_{t_n}^{t_n,x}[R_g^n(s)].$$

Using the right point formula for the backward Itô integral above, we have that

$$\begin{aligned} \mathbb{E}_{t_n}^{t_n,x}[g(s, y_s)] \;&=\; \mathbb{E}_{t_n}^{t_n,x}[g(t_{n+1}, y_{t_{n+1}})] - \mathbb{E}_{t_n}^{t_n,x}\big[g'_B(t_{n+1}, y_{t_{n+1}}) \\ &\quad -g'_y(t_{n+1}, y_{t_{n+1}}) \cdot g(t_{n+1}, y_{t_{n+1}})\big] \cdot \int_s^{t_{n+1}} d\overleftarrow{B}_r + R_{g1}^n(s) + R_{g2}^n(s), \end{aligned} \tag{5.57}$$

where $R_{g1}^n(s) = \mathbb{E}_{t_n}^{t_n,x}[R_g^n(s)]$ and

$$\begin{aligned} R_{g2}^n(s) = &\int_s^{t_{n+1}} \mathbb{E}_{t_n}^{t_n,x}\big[g'_B(r, y_r) - g'_y(r, y_r) \cdot g(r, y_r)\big]\, d\overleftarrow{B}_r \\ &-\mathbb{E}_{t_n}^{t_n,x}\big[g'_B(t_{n+1}, y_{t_{n+1}}) - g'_y(t_{n+1}, y_{t_{n+1}}) \cdot g(t_{n+1}, y_{t_{n+1}})\big] \cdot \int_s^{t_{n+1}} d\overleftarrow{B}_r \end{aligned}$$

Integrating (5.57) from t_n to t_{n+1} with respect to $d\overleftarrow{B}_s$ gives

$$\begin{aligned} \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{t_n,x}[g(s, y_s)]d\overleftarrow{B}_s \;&=\; \mathbb{E}_{t_n}^{t_n,x}[g(t_{n+1}, y_{t_{n+1}})] \cdot \Delta B_{t_n} - \mathbb{E}_{t_n}^{t_n,x}\big[g'_B(t_{n+1}, y_{t_{n+1}}) \\ &\quad -g'_y(t_{n+1}, y_{t_{n+1}}) \cdot g(t_{n+1}, y_{t_{n+1}})\big] \cdot \int_{t_n}^{t_{n+1}} \int_s^{t_{n+1}} d\overleftarrow{B}_r d\overleftarrow{B}_s + R_y^{B,n} \end{aligned} \tag{5.58}$$

where

$$R_y^{B,n} = \int_{t_n}^{t_{n+1}} (R_{g1}^n(s) + R_{g2}^n(s))d\overleftarrow{B}_s. \tag{5.59}$$

Notice that

$$\int_{t_n}^{t_{n+1}} \int_s^{t_{n+1}} d\overleftarrow{B}_r d\overleftarrow{B}_s = \frac{1}{2}((\Delta B_{t_n})^2 - \Delta t_n).$$

Thus from (5.58),

$$\begin{aligned} \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{t_n,x}[g(s, y_s)]d\overleftarrow{B}_s \;&=\; \mathbb{E}_{t_n}^{t_n,x}[g(t_{n+1}, y_{t_{n+1}})\Delta B_{t_n}] - \mathbb{E}_{t_n}^{t_n,x}\big[g'_B(t_{n+1}, y_{t_{n+1}}) \\ &\quad -g'_y(t_{n+1}, y_{t_{n+1}}) \cdot g(t_{n+1}, y_{t_{n+1}})\big] \cdot \frac{1}{2}((\Delta B_{t_n})^2 - \Delta t_n) + R_y^{B,n}, \end{aligned} \tag{5.60}$$

Inserting (5.54) and (5.60) to (5.53) yields

$$
\begin{aligned}
y_{t_n}^{t_n,x} \quad = \quad & \mathbb{E}_{t_n}^{t_n,x}[y_{t_{n+1}}] + \Delta t_n \mathbb{E}_{t_n}^{t_n,x}[f(t_{n+1}, y_{t_{n+1}}, z_{t_{n+1}})] \\
& + \mathbb{E}_{t_n}^{t_n,x}[g(t_{n+1}, y_{t_{n+1}}) \Delta B_{t_n}] - \mathbb{E}_{t_n}^{t_n,x}\left[g_B'(t_{n+1}, y_{t_{n+1}})\right. \qquad (5.61) \\
& \left. - g_y'(t_{n+1}, y_{t_{n+1}}) \cdot g(t_{n+1}, y_{t_{n+1}})\right] \cdot \frac{1}{2}((\Delta B_{t_n})^2 - \Delta t_n) + R_y^n,
\end{aligned}
$$

where

$$
R_y^n = R_y^{B,n} + R_y^{W,n}. \qquad (5.62)
$$

We name (5.61) the reference equation for the solution y_t of the BDSDE (5.49). The term R_y^n in the reference equation may serve as the truncation error of the numerical integrations and the resulting numerical algorithm.

Reference equation for z_s

Next we derive the reference equation for the solution z_t. We multiply ΔW_{t_n} on both sides of (5.52), then take conditional expectation $\mathbb{E}_{t_n}^{t_n,x}[\cdot]$ to get

$$
\int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{t_n,x}[z_s] ds = \mathbb{E}_{t_n}^{t_n,x}[y_{t_{n+1}} \Delta W_{t_n} + \int_{t_n}^{t_{n+1}} f(s, y_s, z_s) \Delta W_{t_n} ds + \int_{t_n}^{t_{n+1}} g(s, y_s) \Delta W_{t_n}] d\overleftarrow{B}_s. \qquad (5.63)
$$

In what follows, we will approximate the integrals in the above equations with appropriate quadratures. The application of either left point formula or even Crank-Nicolson formula will result in a half order algorithm. To obtain a first order numerical scheme, we need to use a more accurate quadrature rule. Here we use the equation (5.50) to achieve this goal. Choosing $t = t_n$, $T = s$ and taking conditional expectation $\mathbb{E}_{t_n}^{t_n,x}[\cdot]$ in (5.50), we obtain

$$
\nabla y_{t_n}^{t_n,x} = E_{t_n}^{t_n,x}[\nabla y_s] + \int_{t_n}^{s} E_{t_n}^{t_n,x}[\nabla F_r] dr + \int_{t_n}^{s} E_{t_n}^{t_n,x}[\nabla G_r] d\overleftarrow{B}_r,
$$

where $\nabla y_{t_n}^{t_n,x} = \mathbb{E}_{t_n}^{t_n,x}[\nabla y_{t_n}]$ is the value of ∇y_{t_n} at the space point x. Hence,

$$E_{t_n}^{t_n,x}[\nabla y_s] = \nabla y_{t_n}^{t_n,x} - (\int_{t_n}^{s} E_{t_n}^{t_n,x}[\nabla F_r]dr + \int_{t_n}^{s} E_{t_n}^{t_n,x}[\nabla G_r]d\overleftarrow{B}_r). \tag{5.64}$$

Integrating (5.64) with respect to s from t_n to t_{n+1} yields

$$\begin{aligned} & \int_{t_n}^{t_{n+1}} E_{t_n}^{t_n,x}[\nabla y_s]ds \\ = & \int_{t_n}^{t_{n+1}} [\nabla y_{t_n}^{t_n,x} - (\int_{t_n}^{s} E_{t_n}^{t_n,x}[\nabla F_r]dr + \int_{t_n}^{s} E_{t_n}^{t_n,x}[\nabla G_r]d\overleftarrow{B}_r)]ds \\ = & \nabla y_{t_n}^{t_n,x} \cdot \Delta t_n - \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} E_{t_n}^{t_n,x}[\nabla F_r]drds - \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} E_{t_n}^{t_n,x}[\nabla G_r]d\overleftarrow{B}_r ds. \end{aligned} \tag{5.65}$$

Replacing ∇G_r in the above equation with $\nabla G_{t_{n+1}}$ gives

$$\int_{t_n}^{t_{n+1}} E_{t_n}^{t_n,x}[\nabla y_s]ds = \nabla y_{t_n}^{t_n,x} \cdot \Delta t_n - E_{t_n}^{t_n,x}[\nabla G_{t_{n+1}}] \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} d\overleftarrow{B}_r ds + R_z^{s,n} \tag{5.66}$$

where

$$R_z^{s,n} = -\int_{t_n}^{t_{n+1}} \int_{t_n}^{s} E_{t_n}^{t_n,x}[\nabla F_r]drds - \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} (E_{t_n}^{t_n,x}[\nabla G_r] - E_{t_n}^{t_n,x}[\nabla G_{t_{n+1}}])d\overleftarrow{B}_r ds.$$

From the identity (5.51),

$$\int_{t_n}^{t_{n+1}} E_{t_n}^{t_n,x}[z_s]ds = \int_{t_n}^{t_{n+1}} E_{t_n}^{t_n,x}[\nabla y_s]ds.$$

Thus, replacing ∇y with z in (5.66), we have

$$\int_{t_n}^{t_{n+1}} E_{t_n}^{t_n,x}[z_s]ds = z_{t_n}^{t_n,x} \cdot \Delta t_n - E_{t_n}^{t_n,x}[\nabla G_{t_{n+1}}] \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} d\overleftarrow{B}_r ds + R_z^{s,n}. \tag{5.67}$$

Notice that $\nabla G_s = g'_W(s, y_s) + g'_y(s, y_s) \cdot z_s$ in (5.67).

For the first integral on the right hand side of (5.63), we use the right point formula to get

$$\int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{t_n,x}[f(s,y_s,z_s)\Delta W_{t_n}]ds = \mathbb{E}_{t_n}^{t_n,x}[f(t_{n+1},y_{t_{n+1}},z_{t_{n+1}})\Delta W_{t_n}]\Delta t_n + R_z^{W,n}, \tag{5.68}$$

where

$$R_z^{W,n} = \int_{t_n}^{t_{n+1}} \{\mathbb{E}_{t_n}^{t_n,x}[f(s,y_s,z_s)\Delta W_{t_n}] - \mathbb{E}_{t_n}^{t_n,x}[f(t_{n+1},y_{t_{n+1}},z_{t_{n+1}})\Delta W_{t_n}]\}ds.$$

To deal with the third term on the right hand side of (5.63), we multiply ΔW_{t_n} and take conditional expectation $\mathbb{E}_{t_n}^{t_n,x}[\cdot]$ on both sides of (5.56) to obtain

$$\begin{aligned}
& \mathbb{E}_{t_n}^{t_n,x}[g(s,y_s)\Delta W_{t_n}] \\
= \; & \mathbb{E}_{t_n}^{t_n,x}[g(t_{n+1},y_{t_{n+1}})\Delta W_{t_n}] - \int_s^{t_{n+1}} \mathbb{E}_{t_n}^{t_n,x}\left[\left(g'_B(r,y_r) - g'_y(r,y_r)\cdot g(r,y_r)\right)\cdot \Delta W_{t_n}\right] d\overleftarrow{B}_r \\
& -\mathbb{E}_{t_n}^{t_n,x}[\int_s^{t_{n+1}} \left(g'_W(r,y_r) + g'_y(r,y_r)\cdot z_r\right) dW_r \cdot \Delta W_{t_n}] + \mathbb{E}_{t_n}^{t_n,x}[R_g^n(s)\Delta W_{t_n}].
\end{aligned}$$

Then, the application of Itô's isometry on the third term of the right hand side of the above equation gives

$$\begin{aligned}
\mathbb{E}_{t_n}^{t_n,x}[g(s,y_s)\Delta W_{t_n}] = \; & \mathbb{E}_{t_n}^{t_n,x}[g(t_{n+1},y_{t_{n+1}})\Delta W_{t_n}] \\
& - \int_s^{t_{n+1}} \mathbb{E}_{t_n}^{t_n,x}\left[\left(g'_B(r,y_r) - g'_y(r,y_r)\cdot g(r,y_r)\right)\cdot \Delta W_{t_n}\right] d\overleftarrow{B}_r \\
& - \int_s^{t_{n+1}} \mathbb{E}_{t_n}^{t_n,x}\left[g'_W(r,y_r) + g'_y(r,y_r)\cdot z_r\right] dr + \mathbb{E}_{t_n}^{t_n,x}[R_g^n(s)\Delta W_{t_n}],
\end{aligned}$$

from which we have

$$
\begin{aligned}
& \mathbb{E}_{t_n}^{t_n,x}[\, g(s,y_s)\Delta W_{t_n}] \\
= \; & \mathbb{E}_{t_n}^{t_n,x}[\, g(t_{n+1},y_{t_{n+1}})\Delta W_{t_n}] - \mathbb{E}_{t_n}^{t_n,x}\left[\big(g_B'(t_{n+1},y_{t_{n+1}})\right. \\
& \left. - g_y'(t_{n+1},y_{t_{n+1}}) \cdot g(t_{n+1},y_{t_{n+1}})\big) \cdot \Delta W_{t_n}\right] \cdot \int_s^{t_{n+1}} d\overleftarrow{B}_r \\
& - \mathbb{E}_{t_n}^{t_n,x}\left[\, g_W'(t_{n+1},y_{t_{n+1}}) + g_y'(t_{n+1},y_{t_{n+1}}) \cdot z_{t_{n+1}}\right] \cdot \int_s^{t_{n+1}} dr \\
& + R_{z1}^{B,n}(s) + R_{z2}^{B,n}(s) + R_{z3}^{B,n}(s),
\end{aligned}
\tag{5.69}
$$

where

$$
R_{z1}^{B,n}(s) = \mathbb{E}_{t_n}^{t_n,x}[R_g^n(s)\Delta W_{t_n}],
$$

$$
\begin{aligned}
R_{z2}^{B,n}(s) = -\int_s^{t_{n+1}} & \left\{\mathbb{E}_{t_n}^{t_n,x}\left[\big(g_B'(r,y_r) - g_y'(r,y_r)\cdot g(r,y_r)\big)\cdot \Delta W_{t_n}\right]\right. \\
& \left. -\mathbb{E}_{t_n}^{t_n,x}\left[\big(g_B'(t_{n+1},y_{t_{n+1}}) - g_y'(t_{n+1},y_{t_{n+1}})\cdot g(t_{n+1},y_{t_{n+1}})\big)\cdot \Delta W_{t_n}\right]\right\} d\overleftarrow{B}_r
\end{aligned}
$$

and

$$
\begin{aligned}
R_{z3}^{B,n}(s) = -\int_s^{t_{n+1}} & \left\{\mathbb{E}_{t_n}^{t_n,x}\left[g_W'(r,y_r) + g_y'(r,y_r)\cdot z_r\right]\right. \\
& \left. -\mathbb{E}_{t_n}^{t_n,x}\left[g_W'(t_{n+1},y_{t_{n+1}}) + g_y'(t_{n+1},y_{t_{n+1}})\cdot z_{t_{n+1}}\right]\right\} dr.
\end{aligned}
$$

Integrating the equation (5.69) with respect to $d\overleftarrow{B}_s$ from t_n to t_{n+1}, we obtain the approximation to the third term on right hand side of (5.63) as

$$
\begin{aligned}
& \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{t_n,x}[g(s,y_s)\Delta W_{t_n}] d\overleftarrow{B}_s \\
& = \mathbb{E}_{t_n}^{t_n,x}[g(t_{n+1},y_{t_{n+1}})\Delta W_{t_n}]\Delta B_{t_n} - \mathbb{E}_{t_n}^{t_n,x}[(g_B'(t_{n+1},y_{t_{n+1}}) \\
& \quad - g_y'(t_{n+1},y_{t_{n+1}})g(t_{n+1},y_{t_{n+1}}))\cdot \Delta W_{t_n}] \int_{t_n}^{t_{n+1}}\int_s^{t_{n+1}} d\overleftarrow{B}_r d\overleftarrow{B}_s \\
& \quad - \mathbb{E}_{t_n}^{t_n,x}\left[g_W'(t_{n+1},y_{t_{n+1}}) + g_y'(t_{n+1},y_{t_{n+1}})\cdot z_{t_{n+1}}\right] \cdot \int_{t_n}^{t_{n+1}}\int_s^{t_{n+1}} dr d\overleftarrow{B}_s + R_z^{B,n},
\end{aligned}
\tag{5.70}
$$

where

$$
R_z^{B,n} = \int_{t_n}^{t_{n+1}} (R_{z1}^{B,n}(s) + R_{z2}^{B,n}(s) + R_{z3}^{B,n}(s)) d\overleftarrow{B}_s.
$$

Inserting (5.67), (5.68) and (5.70) into the equation (5.63), we have

$$\begin{aligned}
&z_{t_n}^{t_n,x} \cdot \Delta t_n - E_{t_n}[\nabla G_{t_{n+1}}] \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} d\overleftarrow{B}_r ds \\
&= \mathbb{E}_{t_n}^{t_n,x}[y_{t_{n+1}} \Delta W_{t_n}] + \mathbb{E}_{t_n}^{t_n,x}[f(t_{n+1}, y_{t_{n+1}}, z_{t_{n+1}}) \Delta W_{t_n}] \Delta t_n \\
&\quad + \mathbb{E}_{t_n}^{t_n,x}[g(t_{n+1}, y_{t_{n+1}}) \Delta W_{t_n}] \Delta B_{t_n} - \mathbb{E}_{t_n}^{t_n,x}\big[\big(g'_B(t_{n+1}, y_{t_{n+1}}) \\
&\quad - g'_y(t_{n+1}, y_{t_{n+1}}) g(t_{n+1}, y_{t_{n+1}})\big) \cdot \Delta W_{t_n}\big] \cdot \int_{t_n}^{t_{n+1}} \int_{s}^{t_{n+1}} d\overleftarrow{B}_r d\overleftarrow{B}_s \\
&\quad - \mathbb{E}_{t_n}^{t_n,x}\left[g'_W(t_{n+1}, y_{t_{n+1}}) + g'_y(t_{n+1}, y_{t_{n+1}}) \cdot z_{t_{n+1}}\right] \cdot \int_{t_n}^{t_{n+1}} \int_{s}^{t_{n+1}} dr d\overleftarrow{B}_s + R_z^n
\end{aligned} \tag{5.71}$$

with the error term

$$R_z^n = R_z^{s,n} + R_z^{W,n} + R_z^{B,n}. \tag{5.72}$$

Since $\int_{t_n}^{t_{n+1}} \int_{s}^{t_{n+1}} d\overleftarrow{B}_r d\overleftarrow{B}_s = \frac{1}{2}[(\Delta B_{t_n})^2 - \Delta t_n]$, we replace $\int_{t_n}^{t_{n+1}} \int_{s}^{t_{n+1}} d\overleftarrow{B}_r d\overleftarrow{B}_s$ by $\frac{1}{2}[(\Delta B_{t_n})^2 - \Delta t_n]$ in the equation (5.71) to obtain

$$\begin{aligned}
&\Delta t_n \cdot z_{t_n}^{t_n,x} = E_{t_n}^{t_n,x}[\nabla G_{t_{n+1}}] \cdot \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} d\overleftarrow{B}_r ds + \mathbb{E}_{t_n}^{t_n,x}[y_{t_{n+1}} \Delta W_{t_n}] \\
&\quad + \mathbb{E}_{t_n}^{t_n,x}[f(t_{n+1}, y_{t_{n+1}}, z_{t_{n+1}}) \Delta W_{t_n}] \cdot \Delta t_n + \mathbb{E}_{t_n}^{t_n,x}[g(t_{n+1}, y_{t_{n+1}}) \Delta W_{t_n}] \Delta B_{t_n} \\
&\quad - \mathbb{E}_{t_n}^{t_n,x}\left[\big(g'_B(t_{n+1}, y_{t_{n+1}}) - g'_y(t_{n+1}, y_{t_{n+1}}) \cdot g(t_{n+1}, y_{t_{n+1}})\big) \cdot \Delta W_{t_n}\right] \cdot \frac{1}{2}[(\Delta B_{t_n})^2 - \Delta t_n] \\
&\quad - \mathbb{E}_{t_n}^{t_n,x}\left[g'_W(t_{n+1}, y_{t_{n+1}}) + g'_y(t_{n+1}, y_{t_{n+1}}) \cdot z_{t_{n+1}}\right] \cdot \int_{t_n}^{t_{n+1}} \int_{s}^{t_{n+1}} dr d\overleftarrow{B}_s + R_z^n.
\end{aligned} \tag{5.73}$$

To proceed, we need the following lemma for the double integrals $\int_{t_n}^{t_{n+1}} \int_{s}^{t_{n+1}} dr d\overleftarrow{B}_s$ and $\int_{t_n}^{t_{n+1}} \int_{t_n}^{s} d\overleftarrow{B}_r ds$ appear in (5.73)

Lemma 5

$$\int_{t_n}^{t_{n+1}} \int_{s}^{t_{n+1}} dr d\overleftarrow{B}_s = \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} d\overleftarrow{B}_r ds.$$

Proof:It is obvious that

$$\int_{t_n}^{t_{n+1}} \int_{s}^{t_{n+1}} dr d\overleftarrow{B}_s = \int_{t_n}^{t_{n+1}} (t_{n+1} - s) d\overleftarrow{B}_s = t_{n+1} \cdot \Delta B_{t_n} - \int_{t_n}^{t_{n+1}} s d\overleftarrow{B}_s, \tag{5.74}$$

$$\int_{t_n}^{t_{n+1}} \int_{t_n}^{s} d\overleftarrow{B}_r ds = \int_{t_n}^{t_{n+1}} (B_s - B_{t_n}) ds = \int_{t_n}^{t_{n+1}} B_s ds - B_{t_n} \cdot \Delta t_n. \tag{5.75}$$

Subtract (5.74) from (5.75) to get

$$\begin{aligned} &\int_{t_n}^{t_{n+1}} \int_{t_n}^{s} d\overleftarrow{B}_r ds - \int_{t_n}^{t_{n+1}} \int_{s}^{t_{n+1}} dr d\overleftarrow{B}_s \\ &= \int_{t_n}^{t_{n+1}} s d\overleftarrow{B}_s + \int_{t_n}^{t_{n+1}} B_s ds - B_{t_n} \cdot \Delta t_n - t_{n+1} \cdot \Delta B_{t_n}. \end{aligned} \tag{5.76}$$

By Itô's formula [61, 60], we have that

$$t_{n+1} B_{t_{n+1}} = t_n B_{t_n} + \int_{t_n}^{t_{n+1}} B_s ds + \int_{t_n}^{t_{n+1}} s d\overleftarrow{B}_s. \tag{5.77}$$

Hence,

$$\int_{t_n}^{t_{n+1}} B_s ds + \int_{t_n}^{t_{n+1}} s d\overleftarrow{B}_s = t_{n+1} B_{t_{n+1}} - t_n B_{t_n}.$$

Then, from (5.76) and above identity, we have that

$$\begin{aligned} &-\int_{t_n}^{t_{n+1}} \int_{s}^{t_{n+1}} dr d\overleftarrow{B}_s \\ &= -\int_{t_n}^{t_{n+1}} \int_{t_n}^{s} d\overleftarrow{B}_r ds + t_{n+1} B_{t_{n+1}} - t_n B_{t_n} - B_{t_n} \cdot (t_{n+1} - t_n) - t_{n+1} \cdot (B_{t_{n+1}} - B_{t_n}) \\ &= -\int_{t_n}^{t_{n+1}} \int_{t_n}^{s} d\overleftarrow{B}_r ds, \end{aligned} \tag{5.78}$$

which means that

$$\int_{t_n}^{t_{n+1}} \int_{s}^{t_{n+1}} dr d\overleftarrow{B}_s = \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} d\overleftarrow{B}_r ds.$$

The proof of the Lemma is completed. □

From the definition of ∇G_s and using Lemma 5, we can rewrite (5.73) as

$$\begin{aligned} \Delta t_n z_{t_n}^{t_n,x} = {} & \mathbb{E}_{t_n}^{t_n,x}[y_{t_{n+1}} \Delta W_t] + \mathbb{E}_{t_n}^{t_n,x}[f(t_{n+1}, y_{t_{n+1}}, z_{t_{n+1}}) \Delta W_{t_n}] \Delta t_n \\ & + \mathbb{E}_{t_n}^{t_n,x}[g(t_{n+1}, y_{t_{n+1}}) \Delta W_{t_n}] \Delta B_{t_n} - \mathbb{E}_{t_n}^{t_n,x} \big[\big(g'_B(t_{n+1}, y_{t_{n+1}}) \\ & - g'_y(t_{n+1}, y_{t_{n+1}}) \cdot g(t_{n+1}, y_{t_{n+1}}) \big) \cdot \Delta W_{t_n} \big] \cdot \frac{1}{2} [(\Delta B_{t_n})^2 - \Delta t_n] + R_z^n, \end{aligned} \tag{5.79}$$

which is the reference equation for z_t.

5.3.3 Numerical scheme

Neglecting the truncation error terms R_y^n and R_z^n in (5.61) and (5.79) respectively, we derive the numerical algorithm for solving the BDSDE (5.49) as follows. Given random variables y^{N_T} and z^{N_T}, for $n = N_T - 1, N_T - 2 \dots 0, 1$, find the approximate solution (y^n, z^n) of (y_t, z_t) backwardly from

$$y^n = \mathbb{E}_{t_n}^{t_n,x}[y^{n+1}] + \Delta t_n \cdot M_{t_n} + \Delta B_{t_n} \cdot N_{t_n}, \tag{5.80}$$

and

$$\Delta t_n z^n = \mathbb{E}_{t_n}^{t_n,x}[y^{n+1} \cdot \Delta W_{t_n}] + \Delta t_n \cdot M_{t_n}^W + \Delta B_{t_{n+1}} \cdot N_{t_n}^W, \tag{5.81}$$

where

$$\begin{aligned} M_{t_n} &= \mathbb{E}_{t_n}^{t_n,x}[f(t_{n+1}, y^{n+1}, z^{n+1})] + \frac{1}{2}\mathbb{E}_{t_n}^{t_n,x}[g_B'(t_{n+1}, y^{n+1})] \\ &\quad - \frac{1}{2}\mathbb{E}_{t_n}^{t_n,x}[g_y'(t_{n+1}, y^{n+1}) \cdot g(t_{n+1}, y^{n+1})], \\ N_{t_n} &= \mathbb{E}_{t_n}^{t_n,x}[g(t_{n+1}, y^{n+1})] - \frac{1}{2}\mathbb{E}_{t_n}^{t_n,x}[g_B'(t_{n+1}, y^{n+1})] \cdot \Delta B_{t_n} \\ &\quad + \frac{1}{2}\mathbb{E}_{t_n}^{t_n,x}[g_y'(t_{n+1}, y^{n+1}) \cdot g(t_{n+1}, y^{n+1})] \cdot \Delta B_{t_n}, \end{aligned}$$

and

$$\begin{aligned} M_{t_n}^W &= \mathbb{E}_{t_n}^{t_n,x}[f(t_{n+1}, y^{n+1}, z^{n+1}) \cdot \Delta W_{t_n}] + \frac{1}{2}\mathbb{E}_{t_n}^{t_n,x}[g_B'(t_{n+1}, y^{n+1}) \cdot \Delta W_{t_n}] \\ &\quad - \frac{1}{2}\mathbb{E}_{t_n}^{t_n,x}[g_y'(t_{n+1}, y^{n+1}) \cdot g(t_{n+1}, y^{n+1}) \cdot \Delta W_{t_n}], \\ N_{t_n}^W &= \mathbb{E}_{t_n}^{t_n,x}[g(t_{n+1}, y^{n+1}) \cdot \Delta W_{t_n}] - \frac{1}{2}\mathbb{E}_{t_n}^{t_n,x}[g_B'(t_{n+1}, y^{n+1}) \cdot \Delta W_{t_n}] \cdot \Delta B_{t_{n+1}} \\ &\quad + \frac{1}{2}\mathbb{E}_{t_n}^{t_n,x}[g_y'(t_{n+1}, y^{n+1}) \cdot g(t_{n+1}, y^{n+1}) \cdot \Delta W_{t_n}] \cdot \Delta B_{t_{n+1}}. \end{aligned}$$

Here, (y^n, z^n) is the approximate solution for (y_t, z_t) at $t = t_n$, $n = N_T - 1, N_T - 2 \dots, 1, 0$.

5.3.4 Error Estimates

In this section, we show the first order convergence for the numerical scheme defined by (5.80) and (5.81). This is done in two steps. In the first step, we derive the upper bounds of $\|y_t - y_n\|$ and $\|z_t - z_n\|$ with respect to the truncation errors R_y^n in (5.61) and R_z^n in (5.79). This amounts to the stability analysis of the algorithm. In the second step we derive the error estimates by estimating the convergence order of the truncation errors.

To simplify the presentation, we introduce the following notations which will be used throughout the rest of the paper. Denote

$$
\begin{aligned}
e_y^n := y_{t_n} - y^n, \quad e_z^n := z_{t_n} - z^n, \quad e_f^n := f(t_n, y_{t_n}, z_{t_n}) - f(t_n, y^n, z^n),\\
e_g^n := g(t_n, y_{t_n}) - g(t_n, y^n), \quad e_{g_B'}^n := g_B'(t_n, y_{t_n}) - g_B'(t_n, y^n),\\
e_{(g_y' \cdot g)}^n := g_y'(t_n, y_{t_n}) \cdot g(t_n, y_{t_n}) - g_y'(t_n, y^n) \cdot g(t_n, y^n).
\end{aligned}
$$

With these notations in hand, we subtract (5.80) and (5.81) from (5.61) and (5.79), respectively to obtain

$$
\begin{aligned}
e_y^n = \mathbb{E}_{t_n}^{t_n,x}[e_y^{n+1}] + \Delta t_n \cdot (\mathbb{E}_{t_n}^{t_n,x}[e_f^{n+1}] + \frac{1}{2}\mathbb{E}_{t_n}^{t_n,x}[e_{g_B'}^{n+1}] - \frac{1}{2}\mathbb{E}_{t_n}^{t_n,x}[e_{(g_y'\cdot g)}^{n+1}])\\
+ \Delta B_{t_n} \cdot (\mathbb{E}_{t_n}^{t_n,x}[e_g^{n+1}] - \frac{1}{2}\mathbb{E}_{t_n}^{t_n,x}[e_{g_B'}^{n+1}] \cdot \Delta B_{t_n} + \frac{1}{2}\mathbb{E}_{t_n}^{t_n,x}[e_{(g_y'\cdot g)}^{n+1}] \cdot \Delta B_{t_n}) + R_y^n
\end{aligned}
\tag{5.82}
$$

and

$$
\begin{aligned}
\Delta t_n e_z^n \quad = \quad & \mathbb{E}_{t_n}^{t_n,x}[e_y^{n+1}\Delta W_{t_n}] + \Delta t_n \cdot (\mathbb{E}_{t_n}^{t_n,x}[e_f^{n+1}\Delta W_{t_n}] + \frac{1}{2}\mathbb{E}_{t_n}^{t_n,x}[e_{g_B'}^{n+1}\Delta W_{t_n}]\\
& -\frac{1}{2}\mathbb{E}_{t_n}^{t_n,x}[e_{(g_y'\cdot g)}^{n+1}\Delta W_{t_n}]) + \Delta B_{t_{n+1}} \cdot (\mathbb{E}_{t_n}^{t_n,x}[e_g^{n+1}\Delta W_{t_n}] - \frac{1}{2}\mathbb{E}_{t_n}^{t_n,x}[e_{g_B'}^{n+1}\Delta W_{t_n}] \cdot \Delta B_{t_{n+1}}\\
& +\frac{1}{2}\mathbb{E}_{t_n}^{t_n,x}[e_{(g_y'\cdot g)}^{n+1}\Delta W_{t_n}] \cdot \Delta B_{t_{n+1}}) + R_z^n.
\end{aligned}
\tag{5.83}
$$

Furthermore, if f, g, g'_B and g'_y are Lipschitz continuous, then

$$\begin{aligned}
&\mathbb{E}_{t_n}^{t_n,x}[(e_f^{n+1})^2] \leq L\left(\mathbb{E}_{t_n}^{t_n,x}[(e_y^{n+1})^2] + \mathbb{E}_{t_n}^{t_n,x}[(e_z^{n+1})^2]\right), \\
&\mathbb{E}_{t_n}^{t_n,x}[(e_{g'_B}^{n+1})^2] \leq L\mathbb{E}_{t_n}^{t_n,x}[(e_y^{n+1})^2], \\
&\mathbb{E}_{t_n}^{t_n,x}[(e_{(g'_y\cdot g)}^{n+1})^2] \leq L\mathbb{E}_{t_n}^{t_n,x}[(e_y^{n+1})^2], \\
&\mathbb{E}_{t_n}^{t_n,x}[(e_g^{n+1})^2] \leq L\mathbb{E}_{t_n}^{t_n,x}[(e_y^{n+1})^2],
\end{aligned} \tag{5.84}$$

where L is a constant depending on the Lipschitz coefficients of f, g, g'_B and g'_y.

Theorem 5.3 *Assume that f, g, g'_B and g'_y are all Lipschitz continuous and g is bounded. Then*

$$\begin{aligned}
\max_{0\leq n\leq N_T-1}(E[(e_y^n)^2] + \frac{\Delta t_n}{1+\epsilon}E[(e_z^n)^2]) \leq & C\cdot(E[(e_y^{N_T})^2] + \frac{\Delta t_{N_T-1}}{1+\epsilon}E[(e_z^{N_T})^2]) \\
& + \sum_{n=0}^{N_T-1}\{3E[(R_y^n)^2] + \frac{(E[R_y^{W,n}])^2}{\Delta t_n} + C_\epsilon(\Delta t_n)^{-1}\cdot E[(R_z^n)^2]\},
\end{aligned} \tag{5.85}$$

where $R_y^{W,n}$, R_y^n and R_z^n are error terms defined in (5.55), (5.62) and (5.72), respectively, ϵ is a positive constant, C is a positive constant depending on functions f, g and constant ϵ, C_ϵ is a constant only depending on constant ϵ.

Proof: We first derive an estimate for e_y^n. Denote

$$\begin{aligned}
H_y^{n+1} &= \mathbb{E}_{t_n}^{t_n,x}[e_f^{n+1}] + \frac{1}{2}\mathbb{E}_{t_n}^{t_n,x}[e_{g'_B}^{n+1}] - \frac{1}{2}\mathbb{E}_{t_n}^{t_n,x}[e_{(g'_y\cdot g)}^{n+1}], \\
G_y^{n+1} &= \mathbb{E}_{t_n}^{t_n,x}[e_g^{n+1}] - \frac{1}{2}\mathbb{E}_{t_n}^{t_n,x}[e_{g'_B}^{n+1}]\cdot\Delta B_{t_n} + \frac{1}{2}\mathbb{E}_{t_n}^{t_n,x}[e_{(g'_y\cdot g)}^{n+1}]\cdot\Delta B_{t_n}.
\end{aligned}$$

Taking the square and then the expectation $E[\cdot]$ on both sides of (5.82) gives

$$\begin{aligned}
E[(e_y^n)^2] = E\big[&(\mathbb{E}_{t_n}^{t_n,x}[e_y^{n+1}])^2 + (\Delta t_n\cdot H_y^{n+1} + \Delta B_{t_n}\cdot G_y^{n+1} + R_y^n)^2 \\
&+2(\mathbb{E}_{t_n}^{t_n,x}[e_y^{n+1}])\cdot(\Delta t_n\cdot H_y^{n+1} + \Delta B_{t_n}\cdot G_y^{n+1} + R_y^n)\ \big].
\end{aligned} \tag{5.86}$$

Using Cauchy's inequality and Young's inequality on the right hand side of the above equation, we have that

$$
\begin{aligned}
E[(e_y^n)^2] \leq E \Big[& (\mathbb{E}_{t_n}^{t_n,x}[e_y^{n+1}])^2 + 3(\Delta t_n)^2 \cdot (H_y^{n+1})^2 + 3(\Delta B_{t_n})^2 \cdot (G_y^{n+1})^2 + 3(R_y^n)^2 \\
& + \frac{C_1}{\epsilon_1}\Delta t_n \cdot (\mathbb{E}_{t_n}^{t_n,x}[e_y^{n+1}])^2 + \frac{\epsilon_1}{C_1}\Delta t_n \cdot (H_y^{n+1})^2 \\
& + 2\mathbb{E}_{t_n}^{t_n,x}[e_y^{n+1}] \cdot (\Delta B_{t_n} \cdot G_y^{n+1} + R_y^n) \Big],
\end{aligned}
$$

where ϵ_1 is a positive constant which will be determined later. It follows from (5.84) and the Cauchy's inequality that there exist positive constants C_1, C_2 such that

$$
\begin{aligned}
& E[(H_y^{n+1})^2] \leq C_1 \cdot \left(E[\mathbb{E}_{t_n}^{t_n,x}[(e_y^{n+1})^2]] + E[\mathbb{E}_{t_n}^{t_n,x}[(e_z^{n+1})^2]] \right), \\
& E[(G_y^{n+1})^2] \leq C_2 \cdot \left(E[\mathbb{E}_{t_n}^{t_n,x}[(e_y^{n+1})^2]] + \Delta t_n E[\mathbb{E}_{t_n}^{t_n,x}[(e_y^{n+1})^2]] \right).
\end{aligned} \tag{5.87}
$$

Thus

$$
\begin{aligned}
E[(e_y^n)^2] \quad \leq \quad & E[(\mathbb{E}_{t_n}^{t_n,x}[e_y^{n+1}])^2] + 3C_1(\Delta t_n)^2 \cdot \left(E[\mathbb{E}_{t_n}^{t_n,x}[(e_y^{n+1})^2]] + E[\mathbb{E}_{t_n}^{t_n,x}[(e_z^{n+1})^2]] \right) \\
& + 3C_2(\Delta t_n) \cdot \left(E[\mathbb{E}_{t_n}^{t_n,x}[(e_y^{n+1})^2]] + \Delta t_n E[\mathbb{E}_{t_n}^{t_n,x}[(e_y^{n+1})^2]] \right) + 3E[(R_y^n)^2] \\
& + \frac{C_1}{\epsilon_1}\Delta t_n \cdot E[(\mathbb{E}_{t_n}^{t_n,x}[e_y^{n+1}])^2] + \epsilon_1 \Delta t_n \cdot \left(E[\mathbb{E}_{t_n}^{t_n,x}[(e_y^{n+1})^2]] + E[\mathbb{E}_{t_n}^{t_n,x}[(e_z^{n+1})^2]] \right) \\
& + 2E[\mathbb{E}_{t_n}^{t_n,x}[e_y^{n+1}] \cdot \Delta B_{t_n} \cdot G_y^{n+1}] + 2E[\mathbb{E}_{t_n}^{t_n,x}[e_y^{n+1}] R_y^n].
\end{aligned} \tag{5.88}
$$

For the term $2E[\mathbb{E}_{t_n}^{t_n,x}[e_y^{n+1}] \cdot \Delta B_{t_n} \cdot G_y^{n+1}]$ in the above inequality, we use the definition of G_y^{n+1}, the Cauchy's inequality and (5.84) to deduce that

$$
\begin{aligned}
& 2E[\mathbb{E}_{t_n}^{t_n,x}[e_y^{n+1}] \cdot \Delta B_{t_n} \cdot G_y^{n+1}] \\
= \; & 2E[\; \mathbb{E}_{t_n}^{t_n,x}[e_y^{n+1}] \cdot \Delta B_{t_n} \cdot (\mathbb{E}_{t_n}^{t_n,x}[e_g^{n+1}] - \frac{1}{2}\mathbb{E}_{t_n}^{t_n,x}[e_{g_B'}^{n+1}]\Delta B_{t_n} + \frac{1}{2}\mathbb{E}_{t_n}^{t_n,x}[e_{(g_y' \cdot g)}^{n+1}]\Delta B_{t_n})] \\
= \; & 0 + \Delta t_n \cdot E[\; \mathbb{E}_{t_n}^{t_n,x}[e_y^{n+1}] \cdot (-\mathbb{E}_{t_n}^{t_n,x}[e_{g_B'}^{n+1}] + \mathbb{E}_{t_n}^{t_n,x}[e_{(g_y' \cdot g)}^{n+1}])] \\
\leq \; & (\frac{1}{2} + 2L)\Delta t_n \cdot E[\mathbb{E}_{t_n}^{t_n,x}[(e_y^{n+1})^2]].
\end{aligned} \tag{5.89}
$$

For the last term on the right hand side of (5.88), we recall that $R_y^n = R_y^{B,n} + R_y^{W,n}$, and due to the property of the backward Itô integral, $E\left[\ \mathbb{E}_{t_n}^{t_n,x}[e_y^{n+1}] \cdot R_y^{B,n}\ \right] = 0$. Therefore,

$$2E[\mathbb{E}_{t_n}^{t_n,x}[e_y^{n+1}] \cdot R_y^n] = 2E[\mathbb{E}_{t_n}^{t_n,x}[e_y^{n+1}] \cdot R_y^{W,n}] \leq \Delta t_n E[\mathbb{E}_{t_n}^{t_n,x}[(e_y^{n+1})^2]] + \frac{(E[R_y^{W,n}])^2}{\Delta t_n}. \quad (5.90)$$

Inserting (5.89) and (5.90) into (5.88) gives

$$\begin{aligned} E[(e_y^n)^2] \leq\ & E[(\mathbb{E}_{t_n}^{t_n,x}[e_y^{n+1}])^2] + C_{\epsilon_1}\Delta t_n \cdot E[\mathbb{E}_{t_n}^{t_n,x}[(e_y^{n+1})^2]] + \epsilon_1 \Delta t_n \cdot E[\mathbb{E}_{t_n}^{t_n,x}[(e_z^{n+1})^2]] \\ & + 3C_1(\Delta t_n)^2 \cdot E[\mathbb{E}_{t_n}^{t_n,x}[(e_z^{n+1})^2]] + 3E[(R_y^n)^2] + \frac{(E[R_y^{W,n}])^2}{\Delta t_n} \end{aligned} \quad (5.91)$$

where C_{ϵ_1} is a constant depending on ϵ_1 and functions f and g.

Similar to the above estimate, we next derive an estimate for e_z^n following a similar procedure. To this end we square both sides of (5.83), and then take the expectation $E[\cdot]$ to obtain

$$\begin{aligned} (\Delta t_n)^2 E[(e_z^n)^2] = E\ \Big[& (\mathbb{E}_{t_n}^{t_n,x}[e_y^{n+1}\Delta W_{t_n}])^2 + \left(\Delta t_n \cdot H_z^{n+1} + \Delta B_{t_n} \cdot G_z^{n+1} + R_z^n\right)^2 \\ & + 2\mathbb{E}_{t_n}^{t_n,x}[e_y^{n+1}\Delta W_{t_n}] \cdot \left(\Delta t_n \cdot H_z^{n+1} + \Delta B_{t_n} \cdot G_z^{n+1} + R_z^n\right)\Big] \end{aligned} \quad (5.92)$$

where

$$\begin{aligned} H_z^{n+1} &= \mathbb{E}_{t_n}^{t_n,x}[e_f^{n+1}\Delta W_{t_n}] + \frac{1}{2}\mathbb{E}_{t_n}^{t_n,x}[e_{g_B'}^{n+1}\Delta W_{t_n}] - \frac{1}{2}\mathbb{E}_{t_n}^{t_n,x}[e_{(g_y'\cdot g)}^{n+1}\Delta W_{t_n}], \\ G_z^{n+1} &= \mathbb{E}_{t_n}^{t_n,x}[e_g^{n+1}\Delta W_{t_n}] - \frac{1}{2}\mathbb{E}_{t_n}^{t_n,x}[e_{g_B'}^{n+1}\Delta W_{t_n}] \cdot \Delta B_{t_n} + \frac{1}{2}\mathbb{E}_{t_n}^{t_n,x}[e_{(g_y'\cdot g)}^{n+1}\Delta W_{t_n}] \cdot \Delta B_{t_n}. \end{aligned}$$

Similarly to (5.87)

$$\begin{aligned} E[(H_z^{n+1})^2] &\leq C_3\Delta t_n \cdot \left(E[\mathbb{E}_{t_n}^{t_n,x}[(e_y^{n+1})^2]] + E[\mathbb{E}_{t_n}^{t_n,x}[(e_z^{n+1})^2]]\right), \\ E[(G_z^{n+1})^2] &\leq C_4\Delta t_n \cdot \left(E[\mathbb{E}_{t_n}^{t_n,x}[(e_y^{n+1})^2]] + \Delta t_n E[\mathbb{E}_{t_n}^{t_n,x}[(e_y^{n+1})^2]]\right) \end{aligned} \quad (5.93)$$

where C_3 and C_4 are positive constants. Applying Cauchy's inequality on the right hand side of (5.92), we deduce that

$$\begin{aligned}(\Delta t_n)^2 E[(e_z^n)^2] \leq\ & E[\mathbb{E}_{t_n}^{t_n,x}[e_y^{n+1}\Delta W_{t_n}])^2 + 3(\Delta t_n)^2(H_z^{n+1})^2 \\ & + 3(\Delta B_{t_n})^2(G_z^{n+1})^2 + 3(R_z^n)^2] + 2E[\mathbb{E}_{t_n}^{t_n,x}[e_y^{n+1}\Delta W_{t_n}] \cdot \Delta t_n \cdot H_z^{n+1}] \\ & + 2E[\mathbb{E}_{t_n}^{t_n,x}[e_y^{n+1}\Delta W_{t_n}]\Delta B_{t_n} \cdot G_z^{n+1}] + 2E[\mathbb{E}_{t_n}^{t_n,x}[e_y^{n+1}\Delta W_{t_n}]R_z^n].\end{aligned} \tag{5.94}$$

Next we estimate the last three terms of the above inequality. Using Young's inequality and (5.93) we have that

$$\begin{aligned}& 2E[\mathbb{E}_{t_n}^{t_n,x}[e_y^{n+1}\Delta W_{t_n}] \cdot \Delta t_n \cdot H_z^{n+1}] \\ \leq\ & \frac{C_3}{\epsilon_2}\Delta t_n \cdot E[(\mathbb{E}_{t_n}^{t_n,x}[e_y^{n+1}\Delta W_{t_n}])^2] + \frac{\epsilon_2}{C_3}\Delta t_n \cdot E[(H_z^{n+1})^2] \\ \leq\ & \frac{C_3}{\epsilon_2}(\Delta t_n)^2 \cdot E[\mathbb{E}_{t_n}^{t_n,x}[(e_y^{n+1})^2]] + \epsilon_2(\Delta t_n)^2 \cdot E[\mathbb{E}_{t_n}^{t_n,x}[(e_y^{n+1})^2] + \mathbb{E}_{t_n}^{t_n,x}[(e_z^{n+1})^2]] \\ =\ & (\frac{C_3}{\epsilon_2} + \epsilon_2)(\Delta t_n)^2 \cdot E[\mathbb{E}_{t_n}^{t_n,x}[(e_y^{n+1})^2]] + \epsilon_2(\Delta t_n)^2 \cdot E[\mathbb{E}_{t_n}^{t_n,x}[(e_z^{n+1})^2]],\end{aligned} \tag{5.95}$$

where ϵ_2 is a positive number which will be determined later. With Cauchy's inequality and the estimates in (5.84), we deduce that

$$\begin{aligned}& 2E[\ \mathbb{E}_{t_n}^{t_n,x}[e_y^{n+1}\Delta W_{t_n}] \cdot \Delta B_{t_n} \cdot G_z^{n+1}\ ] \\ =\ & 2E[\ \mathbb{E}_{t_n}^{t_n,x}[e_y^{n+1}\Delta W_{t_n}] \cdot \Delta B_{t_n} \\ & \cdot (\mathbb{E}_{t_n}^{t_n,x}[e_g^{n+1}\Delta W_{t_n}] - \frac{1}{2}\mathbb{E}_{t_n}^{t_n,x}[e_{g'_B}^{n+1}\Delta W_{t_n}]\Delta B_{t_n} + \frac{1}{2}\mathbb{E}_{t_n}^{t_n,x}[e_{(g'_y\cdot g)}^{n+1}\Delta W_{t_n}]\Delta B_{t_n})\ ] \\ =\ & 0 + \Delta t_n \cdot E[\mathbb{E}_{t_n}^{t_n,x}[e_y^{n+1}\Delta W_{t_n}] \cdot (-\mathbb{E}_{t_n}^{t_n,x}[e_{g'_B}^{n+1}\Delta W_{t_n}] + \mathbb{E}_{t_n}^{t_n,x}[e_{(g'_y\cdot g)}^{n+1}\Delta W_{t_n}])] \\ & \leq (\frac{1}{2} + 2L) \cdot (\Delta t_n)^2 \cdot E[\mathbb{E}_{t_n}^{t_n,x}[(e_y^{n+1})^2]].\end{aligned} \tag{5.96}$$

Also, for an arbitrary given constant $\epsilon > 0$, it follows from Young's inequality that

$$2E[\mathbb{E}_{t_n}^{t_n,x}[e_y^{n+1}\Delta W_{t_n}] \cdot R_z^n] \leq \epsilon E[(\mathbb{E}_{t_n}^{t_n,x}[e_y^{n+1}\Delta W_{t_n}])^2] + \frac{1}{\epsilon}E[(R_z^n)^2]. \tag{5.97}$$

Inserting (5.95), (5.96) and (5.97) into the equation (5.94) and using (5.93), we obtain the estimate

$$
\begin{aligned}
(\Delta t_n)^2 E[(e_z^n)^2] \leq\ & (1+\epsilon) E[(\mathbb{E}_{t_n}^{t_n,x}[e_y^{n+1}\Delta W_{t_n}])^2] + C_{\epsilon_2}(\Delta t_n)^2 \cdot E[\mathbb{E}_{t_n}^{t_n,x}[(e_y^{n+1})^2]] \\
& + \epsilon_2 (\Delta t_n)^2 \cdot E[\mathbb{E}_{t_n}^{t_n,x}[(e_z^{n+1})^2]] + 3C_3(\Delta t_n)^3 \cdot E[\mathbb{E}_{t_n}^{t_n,x}[(e_z^{n+1})^2]] \\
& + C_\epsilon E[(R_z^n)^2],
\end{aligned} \tag{5.98}
$$

where C_{ϵ_2}, C_ϵ are constants depending on ϵ_2, ϵ respectively and functions f and g.

Dividing both sides of equation (5.98) by $(\Delta t_n)(1+\epsilon)$ and noticing that

$$
(\mathbb{E}_{t_n}^{t_n,x}[e_y^{n+1}\Delta W_{t_n}])^2 \leq \Delta t_n \cdot (\mathbb{E}_{t_n}^{t_n,x}[(e_y^{n+1})^2] - (\mathbb{E}_{t_n}^{t_n,x}[e_y^{n+1}])^2),
$$

we obtain

$$
\begin{aligned}
(\frac{\Delta t_n}{1+\epsilon}) E[(e_z^n)^2] \leq\ & E[\mathbb{E}_{t_n}^{t_n,x}[(e_y^{n+1})^2] - (\mathbb{E}_{t_n}^{t_n,x}[e_y^{n+1}])^2] + C_{\epsilon_2}\frac{\Delta t_n}{1+\epsilon} \cdot E[\mathbb{E}_{t_n}^{t_n,x}[(e_y^{n+1})^2]] \\
& + \epsilon_2 \frac{\Delta t_n}{1+\epsilon} \cdot E[\mathbb{E}_{t_n}^{t_n,x}[(e_z^{n+1})^2]] + \frac{3C_3}{1+\epsilon}(\Delta t_n)^2 \cdot E[\mathbb{E}_{t_n}^{t_n,x}[(e_z^{n+1})^2]] \\
& + C_\epsilon (\Delta t_n)^{-1} \cdot \frac{1}{1+\epsilon} \cdot E[(R_z^n)^2].
\end{aligned}
$$

Since $\frac{1}{1+\epsilon} < 1$, the above inequality can be rewritten as

$$
\begin{aligned}
(\frac{\Delta t_n}{1+\epsilon}) E[(e_z^n)^2] \leq\ & E[\mathbb{E}_{t_n}^{t_n,x}[(e_y^{n+1})^2] - (\mathbb{E}_{t_n}^{t_n,x}[e_y^{n+1}])^2] + C_{\epsilon_2}\Delta t_n \cdot E[\mathbb{E}_{t_n}^{t_n,x}[(e_y^{n+1})^2]] \\
& + \epsilon_2 \Delta t_n \cdot E[\mathbb{E}_{t_n}^{t_n,x}[(e_z^{n+1})^2]] + 3C_3(\Delta t_n)^2 \cdot E[\mathbb{E}_{t_n}^{t_n,x}[(e_z^{n+1})^2]] \\
& + C_\epsilon (\Delta t_n)^{-1} \cdot E[(R_z^n)^2].
\end{aligned} \tag{5.99}
$$

With (5.91) and (5.99) in hand, we are ready to drive the result of the theorem. First we add (5.99) to (5.91) to obtain

$$\begin{aligned}
&E[(e_y^n)^2] + \frac{\Delta t_n}{1+\epsilon} E[(e_z^n)^2] \\
\leq\ & E[\mathbb{E}_{t_n}^{t_n,x}[(e_y^{n+1})^2]] + (C_{\epsilon_1} + C_{\epsilon_2})\Delta t_n \cdot E[\mathbb{E}_{t_n}^{t_n,x}[(e_y^{n+1})^2]] \\
&+ (\epsilon_1 + \epsilon_2)\Delta t_n \cdot E[\mathbb{E}_{t_n}^{t_n,x}[(e_z^{n+1})^2]] + 3(C_1 + C_3)(\Delta t_n)^2 \cdot E[\mathbb{E}_{t_n}^{t_n,x}[(e_z^{n+1})^2]] \\
&+ 3E[(R_y^n)^2] + \frac{(E[R_y^{W,n}])^2}{\Delta t_n} + C_\epsilon (\Delta t_n)^{-1} \cdot E[(R_z^n)^2].
\end{aligned} \tag{5.100}$$

For a fixed positive constant ϵ, we choose positive numbers ϵ_1, ϵ_2 so that

$$\epsilon_1 + \epsilon_2 \leq \frac{1}{1+\epsilon}.$$

Then, the application of Jensen's inequality leads to

$$\begin{aligned}
E[(e_y^n)^2] + \frac{\Delta t_n}{1+\epsilon} E[(e_z^n)^2] \leq (1 + C_{\epsilon_1,\epsilon_2}\Delta t_n) \cdot (E[(e_y^{n+1})^2] + \frac{\Delta t_n}{1+\epsilon} \cdot E[(e_z^{n+1})^2]) \\
+ 3E[(R_y^n)^2] + \frac{(E[R_y^{W,n}])^2}{\Delta t_n} + C_\epsilon (\Delta t_n)^{-1} \cdot E[(R_z^n)^2],
\end{aligned} \tag{5.101}$$

where $C_{\epsilon_1,\epsilon_2}$ is a constant depending on ϵ_1, ϵ_2 and functions f and g.

Finally using the discrete Gronwall inequality, we obtain

$$\begin{aligned}
&\max_{0 \leq n \leq N_T - 1} (E[(e_y^n)^2] + \frac{\Delta t_n}{1+\epsilon} E[(e_z^n)^2]) \\
\leq\ & C \cdot (E[(e_y^{N_T})^2] + \frac{\Delta t_{N_T-1}}{1+\epsilon} E[(e_z^{N_T})^2]) \\
&+ \sum_{n=0}^{N_T-1} \{3E[(R_y^n)^2] + \frac{(E[R_y^{W,n}])^2}{\Delta t_n} + C_\epsilon (\Delta t_n)^{-1} \cdot E[(R_z^n)^2]\}
\end{aligned} \tag{5.102}$$

as required. □

We now apply Theorem 5.3 to obtain error estimates for the proposed numerical scheme (5.80) and (5.81). In light of (5.102), it suffices to estimate the truncation errors R_y^n, R_z^n and $R_y^{W,n}$. First we need a couple of regularity results for the exact solution (y_t, z_t).

Lemma 6 *[61] For bounded f, g and h,*

$$E[(y_s^{t,x} - y_t^{t,x})^2] + E[\int_s^t (z_r)^2 dr] \leq C|t-s|, \tag{5.103}$$

and for bounded function Ψ with bounded second-order derivatives,

$$(E[\Psi(t, y_t, z_t) - \Psi(s, y_s, z_s)])^2 \leq C(t-s)^2$$

where C is a constant independent of s and t.

Lemma 7 *[61] Assume that f, g, h, f'_W, f'_y, f'_z, g'_W, g'_y, g'_z, and h' are all bounded, then*

$$E[(z_s^{t,x} - z_t^{t,x})^2] + E[\int_s^t (\nabla z_r)^2 dr] \leq C|t-s|. \tag{5.104}$$

where C is a constant independent of s and t.

Now we are ready to derive estimates for the truncation errors $R_y^{W,n}$ R_y^n and R_z^n.

Proposition 2 *Assume that that functions f, g, g'_B, g'_W, g'_y are Lipschitz continuous, and f, g and h are bounded. Furthermore, assume that for any $s \in [0,T]$, $(x,y,z) \to (f(s,x,y,z), g(s,x,y))$ of class C^3, all derivatives are bounded on $[0,T] \times \mathbb{R}^d \times \mathbb{R}^k$ and $h \in C^3(\mathbb{R}^d; \mathbb{R}^k)$. Then we have the following estimates*

$$\text{(i).} \quad (E[R_y^{W,n}])^2 \leq C(\Delta t_n)^4, \tag{5.105}$$

$$\text{(ii).} \quad E[(R_y^n)^2] \leq C(\Delta t_n)^3, \tag{5.106}$$

$$\text{(iii).} \quad E[(R_z^n)^2] \leq C(\Delta t_n)^4, \tag{5.107}$$

where C is a constant only related to functions f and g.

Proof: (i). By Lemma 6, the definition of $R_y^{W,n}$ given by (5.55), and the assumptions in the Proposition we have that

$$\begin{aligned}(E[R_y^{W,n}])^2 &= (\int_{t_n}^{t_{n+1}} E[f(s,y_s,z_s) - f(t_{n+1},y_{t_{n+1}},z_{t_{n+1}})]ds)^2 \\ &\leq \Delta t_n \cdot \int_{t_n}^{t_{n+1}} (E[f(s,y_s,z_s) - f(t_{n+1},y_{t_{n+1}},z_{t_{n+1}})])^2 ds \\ &\leq C(\Delta t_n)^4,\end{aligned}$$

which is (5.105).

(ii). By definition (5.62), $R_y^n = R_y^{B,n} + R_y^{W,n}$. Thus it suffices to estimate $R_y^{B,n}$ and $R_y^{W,n}$. For $R_g^n(s)$ given by (5.56), it easy to see that

$$E[(R_{g1}^n(s))^2] = E[(\mathbb{E}_t^{t,x}[R_g^n(s)])^2] = O((\Delta t_n)^2), \tag{5.108}$$

From the definition of $R_{g2}^n(s)$ given by (5.57) we have that

$$\begin{aligned}E[(R_{g2}^n(s))^2] &= E[(\mathbb{E}_{t_n}^{t_n,x}[\int_s^{t_{n+1}} (g_B'(r,y_r) - g_y'g(r,y_r))d\overleftarrow{B}_r \\ &\quad - (g_B'(t_{n+1},y_{t_{n+1}}) - g_y'(t_{n+1},y_{t_{n+1}}) \cdot g(t_{n+1},y_{t_{n+1}})) \int_s^{t_{n+1}} d\overleftarrow{B}_r])^2] \\ &\leq 2E[(\mathbb{E}_{t_n}^{t_n,x}[\int_s^{t_{n+1}} (g_B'(r,y_r) - g_B'(t_{n+1},y_{t_{n+1}}))d\overleftarrow{B}_r])^2] \\ &\quad + 2E[(\mathbb{E}_{t_n}^{t_n,x}[\int_s^{t_{n+1}} (g_y'(r,y_r) \cdot g(r,y_r) - g_y'(t_{n+1},y_{t_{n+1}}) \cdot g(t_{n+1},y_{t_{n+1}}))d\overleftarrow{B}_r])^2].\end{aligned}$$

For the first term on the right hand side of the above inequality, we have by Lemma 6, Itô's isometry and the assumptions in the Proposition, that

$$E[(\mathbb{E}_{t_n}^{t_n,x}[\int_s^{t_{n+1}} (g_B'(r,y_r) - g_B'(t_{n+1},y_{t_{n+1}}))d\overleftarrow{B}_r])^2] = O((\Delta t_n)^2), \tag{5.109}$$

Similarly,

$$E[(\mathbb{E}_{t_n}^{t_n,x}[\int_s^{t_{n+1}} (g_y'(r,y_r)\cdot g(r,y_r) - g_y'(t_{n+1},y_{t_{n+1}})\cdot g(t_{n+1},y_{t_{n+1}}))d\overleftarrow{B}_r])^2] = O((\Delta t_n)^2). \tag{5.110}$$

Thus

$$E[(R_{g2}^n(s))^2] \leq O((\Delta t_n)^2). \tag{5.111}$$

It follows from (5.108) and (5.111) that

$$E[(R_y^{B,n})^2] = E[(\int_{t_n}^{t_{n+1}} (R_{g1}^n(s) + R_{g2}^n(s))d\overleftarrow{B}_s)^2] = O((\Delta t_n)^3). \tag{5.112}$$

For $R_y^{W,n}$, it follows from Lemma 6, Lemma 7 and the assumptions of the Proposition that

$$\begin{aligned} E[(R_y^{W,n})^2] &= E[(\int_{t_n}^{t_{n+1}} \{\mathbb{E}_{t_n}^{t_n,x}[f(s,y_s,z_s)] - \mathbb{E}_{t_n}^{t_n,x}[f(t_{n+1},y_{t_{n+1}},z_{t_{n+1}})]\}ds)^2] \\ &\leq \Delta t_n \cdot E[\int_{t_n}^{t_{n+1}} \{\mathbb{E}_{t_n}^{t_n,x}[f(s,y_s,z_s)] - \mathbb{E}_{t_n}^{t_n,x}[f(t_{n+1},y_{t_{n+1}},z_{t_{n+1}})]\}^2 ds] \\ &\leq \Delta t_n \cdot E[\int_{t_n}^{t_{n+1}} (C(\Delta t_n)^2 + L\mathbb{E}_{t_n}^{t_n,x}[(y_s - y_{t_{n+1}})^2] + L\mathbb{E}_{t_n}^{t_n,x}[(z_s - z_{t_{n+1}})^2])ds] \\ &= O((\Delta t_n)^3). \end{aligned} \tag{5.113}$$

Combing (5.111) with (5.113), we obtain the desired estimate (5.106) for R_y^n.

(iii). At last, we derive estimate (5.107) for R_z^n, which is defined by $R_z^n = R_z^{s,n} + R_z^{W,n} + R_z^{B,n}$ in (5.72). To this end, we derive the estimates for $R_z^{s,n}$, $R_z^{W,n}$ and $R_z^{B,n}$.

We first estimate the error term $R_z^{s,n}$. Under the assumptions in the Proposition,

$$E[\int_{t_n}^{t_{n+1}} (\nabla F_s)^2 ds] = O(\Delta t_n)$$

and

$$E[(\mathbb{E}_{t_n}^{t_n,x}[\nabla G_r] - \mathbb{E}_{t_n}^{t_n,x}[\nabla G_{t_{n+1}}])^2] = O(\Delta t_n).$$

It follows from the definition of $R_z^{s,n}$ given by (5.66) and the above two identities that

$$\begin{aligned} E[(R_z^{s,n})^2] &= E[(-\int_{t_n}^{t_{n+1}}\int_{t_n}^{s} E_{t_n}^{t_n,x}[\nabla F_r]drds \\ &\quad -\int_{t_n}^{t_{n+1}}\int_{t_n}^{s}(E_{t_n}^{t_n,x}[\nabla G_r]-E_{t_n}^{t_n,x}[\nabla G_{t_{n+1}}])d\overleftarrow{B}_r ds)^2] \\ &\leq 2E[(\int_{t_n}^{t_{n+1}}\int_{t_n}^{s}\mathbb{E}_{t_n}^{t_n,x}[\nabla F_r]drds)^2] \qquad (5.114) \\ &\quad +2E[(\int_{t_n}^{t_{n+1}}\int_{t_n}^{s}(\mathbb{E}_{t_n}^{t_n,x}[\nabla G_r]-\mathbb{E}_{t_n}^{t_n,x}[\nabla G_{t_{n+1}}])d\overleftarrow{B}_r ds)^2] \\ &= O((\Delta t_n)^4). \end{aligned}$$

Similarly to (5.113),

$$E[(R_z^{W,n})^2] = E[(\int_{t_n}^{t_{n+1}}\mathbb{E}_{t_n}^{t_n,x}[(f(s,y_s,z_s)-f(t_{n+1},y_{t_{n+1}},z_{t_{n+1}}))\cdot\Delta W_{t_n}]ds)^2] = O((\Delta t_n)^4). \tag{5.115}$$

To get the estimate $R_z^{B,n}$ given by (5.70), we need the estimates for $R_{z1}^{B,n}$, $R_{z2}^{B,n}$ and $R_{z3}^{B,n}$. With estimate (5.108), (5.109) and (5.110) in hand, it's easy to obtain

$$\begin{gathered} E[(\mathbb{E}_t^{t,x}[R_g^n(s)\Delta W_{t_n}])^2] = O((\Delta t_n)^3), \\ E[(\mathbb{E}_{t_n}^{t_n,x}[(\int_s^{t_{n+1}}(g_B'(r,y_r)-g_B'(t_{n+1},y_{t_{n+1}}))d\overleftarrow{B}_r)\Delta W_{t_n}])^2] = O((\Delta t_n)^3), \qquad (5.116) \end{gathered}$$

and

$$\begin{aligned} & E[(\mathbb{E}_{t_n}^{t_n,x}[(\int_s^{t_{n+1}}\left(g_y'(r,y_r)\cdot g(r,y_r)-g_y'(t_{n+1},y_{t_{n+1}})\cdot g(t_{n+1},y_{t_{n+1}})\right)d\overleftarrow{B}_r)\Delta W_{t_n}])^2] \\ = \;& O((\Delta t_n)^3). \qquad (5.117) \end{aligned}$$

Thus, we get the estimates for $R_{z1}^{B,n}$ and $R_{z2}^{B,n}$ as following

$$E[(R_{z1}^{B,n}(s))^2] = E[(\mathbb{E}_t^{t,x}[R_g^n(s)\Delta W_{t_n}])^2] = O((\Delta t_n)^3), p$$

and

$$
\begin{aligned}
& E[(R_{z2}^{B,n}(s))^2] \\
= \;& E[(-\int_s^{t_{n+1}} \left\{\mathbb{E}_{t_n}^{t_n,x}\left[\left(g_B'(r,y_r) - g_y'(r,y_r)\cdot g(r,y_r)\right)\cdot \Delta W_{t_n}\right]\right. \\
& \left. -\mathbb{E}_{t_n}^{t_n,x}\left[\left(g_B'(t_{n+1},y_{t_{n+1}}) - g_y'(t_{n+1},y_{t_{n+1}})\cdot g(t_{n+1},y_{t_{n+1}})\right)\cdot \Delta W_{t_n}\right]\right\} d\overleftarrow{B}_r)^2] \\
\leq \;& 2E[(\mathbb{E}_{t_n}^{t_n,x}[(\int_s^{t_{n+1}} (g_B'(r,y_r) - g_B'(t_{n+1},y_{t_{n+1}}))d\overleftarrow{B}_r)\Delta W_{t_n}])^2] \\
& +2E[(\mathbb{E}_{t_n}^{t_n,x}[(\int_s^{t_{n+1}} (g_y'(r,y_r)\cdot g(r,y_r) - g_y'(t_{n+1},y_{t_{n+1}})\cdot g(t_{n+1},y_{t_{n+1}}))d\overleftarrow{B}_r)\Delta W_{t_n}])^2] \\
= \;& O((\Delta t_n)^3).
\end{aligned}
$$

By using the following estimates obtained directly from Lemma 6, Lemma 7 and the assumptions in the Proposition

$$
E[(\mathbb{E}_{t_n}^{t_n,x}[\int_s^{t_{n+1}} (g_W'(r,y_r) - g_W'(t_{n+1},y_{t_{n+1}}))dr])^2] = O((\Delta t_n)^3), \tag{5.118}
$$

$$
E[(\mathbb{E}_{t_n}^{t_n,x}[\int_s^{t_{n+1}} (g_y'(r,y_r)\cdot z_r - g_y'(t_{n+1},y_{t_{n+1}})\cdot z_{t_{n+1}})dr])^2] = O((\Delta t_n)^3). \tag{5.119}
$$

We get an estimate for $R_{z3}^{B,n}$

$$
\begin{aligned}
E[(R_{z3}^{B,n}(s))^2] \;=\;& E[(-\int_s^{t_{n+1}} \left\{\mathbb{E}_{t_n}^{t_n,x}\left[g_W'(r,y_r) + g_y'(r,y_r)\cdot z_r\right]\right. \\
& \left. -\mathbb{E}_{t_n}^{t_n,x}\left[g_W'(t_{n+1},y_{t_{n+1}}) + g_y'(t_{n+1},y_{t_{n+1}})\cdot z_{t_{n+1}}\right]\right\} dr)^2] \\
\leq\;& 2E[(\mathbb{E}_{t_n}^{t_n,x}[\int_s^{t_{n+1}} (g_W'(r,y_r) - g_W'(t_{n+1},y_{t_{n+1}}))dr])^2] \\
& +2E[(\mathbb{E}_{t_n}^{t_n,x}[\int_s^{t_{n+1}} (g_y'(r,y_r)\cdot z_r - g_y'(t_{n+1},y_{t_{n+1}})\cdot z_{t_{n+1}})dr])^2] \\
=\;& O((\Delta t_n)^3).
\end{aligned}
$$

Therefore,

$$E[(R_z^{B,n})^2] = E[\Big(\int_{t_n}^{t_{n+1}} (R_{z1}^{B,n}(s) + R_{z2}^{B,n}(s) + R_{z3}^{B,n}(s))d\overleftarrow{B}_s\Big)^2] = O((\Delta t_n)^4). \tag{5.120}$$

Then, from estimate (5.114), (5.115) and (5.120), we have

$$E[(R_z^n)^2] \leq C(\Delta t_n)^4. \tag{5.121}$$

□

Combining Theorem 5.3 and Proposition 2, we obtain the error estimates for our numerical scheme (5.80), (5.81).

Theorem 5.4 *Under the conditions of Theorem 5.3 and Proposition 2, if $y^{N_T} = y_{t_{N_T}}$ and $z^{N_T} = z_{t_{N_T}}$, we have*

$$\max_{0 \leq n \leq N_T - 1} (E[(e_y^n)^2]) \leq C(\Delta t)^2, \quad \max_{0 \leq n \leq N_T - 1} (E[(e_z^n)^2]) \leq C\Delta t. \tag{5.122}$$

5.3.5 Fully discrete scheme and its error estimate

Fully discrete scheme

The scheme we provided above is a semi-discrete scheme. In order to solve for (y^n, z^n) numerically, spatial approximations are needed. To simplify the presentation, in this section, we only consider the one dimensional case. The multi-dimensional cases can be deduced through a straightforward generalization. In addition to the temporal partition provided in Section 3, we introduce the following spatial partition of the real line $\mathbb{R}$:

$$\mathcal{R}_h = \{x_i | x_i \in \mathbb{R}, i \in \mathbb{Z}, x_i < x_{i+1}, \lim_{i \to +\infty} x_i = +\infty, \lim_{i \to -\infty} x_i = -\infty\}, \tag{5.123}$$

where $\{x_i\}_{i \in \mathbb{Z}}$ are deterministic. We denote $h_i = x_{i+1} - x_i$ as the spatial step and $h = \max_{i \in \mathbb{Z}} h_i$ as the maximum spatial step.

Assume φ is a functional of W_t, it follows from the Markov property that

$$\mathbb{E}[\varphi(W_t)|\mathcal{F}_{t_n}^{W,t_n}] = \mathbb{E}[\varphi(\xi + W_t - W_{t_n})]|_{\xi=W_{t_n}}, \quad t \geq t_n,$$

where $\mathbb{E}[X]$ denotes the conditional expectation $E[X|\sigma(B_p; 0 \leq p \leq T)]$ of the random variable X. We use values of $\mathbb{E}[\varphi(\xi+W_t-W_{t_n})]|_{\xi=W_{t_n}}$ at partition points $\{x_i\}_{i\in\mathbb{Z}}$ to approximate the entire conditional expectation.

Take (y_i^n, z_i^n) $(n = N, N-1, \cdots, 0, i \in \mathbb{Z})$ as an approximation at the time-space point (t_n, x_i). A fully discrete scheme is defined as follows: given the random variable y_i^N, z_i^N, $i \in \mathbb{Z}$, find an approximate solution (y_i^n, z_i^n) $(n = N-1, \cdots, 0, i \in \mathbb{Z})$ satisfying

$$\begin{aligned} \hat{y}_i^n = {} & \hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\hat{y}^{n+1}] + \Delta t_n \cdot (\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\hat{f}(t_{n+1}, \hat{y}^{n+1}, \hat{z}^{n+1})] + \frac{1}{2}\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\hat{g}_B'(t_{n+1}, \hat{y}^{n+1})] \\ & - \frac{1}{2}\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\hat{g}_y'(t_{n+1}, \hat{y}^{n+1}) \cdot \hat{g}(t_{n+1}, \hat{y}^{n+1})]) + \Delta B_{t_n} \cdot (\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\hat{g}(t_{n+1}, \hat{y}^{n+1})] \\ & - \frac{1}{2}\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\hat{g}_B'(t_{n+1}, \hat{y}^{n+1})]\Delta B_{t_n} + \frac{1}{2}\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\hat{g}_y'(t_{n+1}, \hat{y}^{n+1}) \cdot \hat{g}(t_{n+1}, \hat{y}^{n+1})]\Delta B_{t_n}) \end{aligned} \tag{5.124}$$

$$\begin{aligned} \Delta t_n \hat{z}_i^n = {} & \hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\hat{y}^{n+1}\Delta W_{t_n}] + \Delta t_n \cdot \Big(\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\hat{f}(t_{n+1}, \hat{y}^{n+1}, \hat{z}^{n+1})\Delta W_{t_n}] \\ & + \frac{1}{2}\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\hat{g}_B'(t_{n+1}, \hat{y}^{n+1})\Delta W_{t_n}] - \frac{1}{2}\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\hat{g}_y'(t_{n+1}, \hat{y}^{n+1}) \cdot \hat{g}(t_{n+1}, \hat{y}^{n+1})\Delta W_{t_n}]\Big) \\ & + \Delta B_{t_n} \cdot \Big(\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\hat{g}(t_{n+1}, \hat{y}^{n+1}, \hat{z}^{n+1})\Delta W_{t_n}] - \frac{1}{2}\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\hat{g}_B'(t_{n+1}, \hat{y}^{n+1})\Delta W_{t_n}]\Delta B_{t_n} \\ & + \frac{1}{2}\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\hat{g}_y'(t_{n+1}, \hat{y}^{n+1}) \cdot \hat{g}(t_{n+1}, \hat{y}^{n+1})\Delta W_{t_n}]\Delta B_{t_n}\Big) \end{aligned} \tag{5.125}$$

In this section, we denote $\mathbb{E}_{t_n}^{t_n,x_i}[\varphi(W_{t_{n+1}})]$ to be the value of $\mathbb{E}[\varphi(\xi + W_{t_{n+1}} - W_{t_n})]|_{\xi=W_{t_n}}$ at the grid points x_i, i.e. $\xi = x_i$. Here $\hat{y}^{n+1}$ and $\hat{z}^{n+1}$ are corresponding interpolated values at the point $x_i+\Delta W_{t_n}$ $(i \in \mathbb{Z})$ by using the grid values of y_l^{n+1} and z_l^{n+1}, $l \in \mathbb{Z}$, respectively; for any function ϕ, $\hat{\phi}$ is the interpolated value at the point $x_i + \Delta W_{t_n}$ $(i \in \mathbb{Z})$ by using the grid values of $\phi(x_i)$ $(i \in \mathbb{Z})$; $\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\cdot]$ is an approximation to the conditional expectation $\mathbb{E}_{t_n}^{t_n,x_i}[\cdot]$. Many methods are used to approximate $\mathbb{E}_{t_n}^{t_n,x_i}[\cdot]$. For example, $\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\cdot]$ can be approximations of the conditional expectation by using Monte Carlo method (see [16, 34, 35]) or the Gauss quadrature.

In this paper, $\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\cdot]$ is an approximation to $\mathbb{E}_{t_n}^{t_n,x_i}[\cdot]$ using the Gauss-Hermite quadrature. The main reason of doing that is the high-order accuracy of the Gauss-Hermite quadrature when using the values of the integrand at a very few number of points. This will significantly reduce the computation time needed to compute the conditional expectation, especially when the integrand f is very expensive to calculate.

For a given one-dimensional function $g(x)$, the Gauss-Hermite quadrature formula can be written as

$$\int_{-\infty}^{\infty} e^{-x^2} g(x)dx \approx \sum_{i=1}^{K} w_i g(a_i), \tag{5.126}$$

where w_i $(i = 1, \cdots K)$ are weights defined by

$$w_i = \frac{2^{K+1}K!\sqrt{\pi}}{[H_K'(a_i)]^2},$$

and a_i $(i = 1, \cdots K)$ are K roots of the Hermite polynomial of degree K defined by

$$H_K(x) = (-1)^K e^{x^2} \frac{d^K}{dx^K}(e^{-x^2}).$$

The weights $\{w_i\}_{i=1}^K$ and the roots $\{a_i\}_{i=1}^K$ for different positive integers K can be easily found. Let

$$R(g, K) = \int_{-\infty}^{\infty} e^{-x^2} g(x)dx - \sum_{i=1}^{K} w_i g(a_i).$$

Then $R(g, K)$ is the truncation error of the Gauss-Hermite quadrature and

$$R(g, K) = \frac{K!\sqrt{\pi}}{2^K(2K)!} g^{2K}(\xi), \tag{5.127}$$

for some real number ξ.

Now let us define the approximate mathematical expectation $\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\hat{y}^{n+1}]$ on time interval $[t_n, t_{n+1}]$. Since $\Delta W_{t_n} \sim N(0, \Delta t_n)$, we know that

$$\mathbb{E}[y^{n+1}(\xi + \Delta W_{t_n})]|_{\xi = W_{t_n}} = \int_{-\infty}^{\infty} y^{n+1}(\xi + w) \frac{1}{\sqrt{2\pi \Delta t_n}} e^{-\frac{w^2}{2\Delta t_n}} dw$$

From the definition of $\mathbb{E}_{t_n}^{t_n,x_i}[\cdot]$, $\mathbb{E}_{t_n}^{t_n,x_i}[y^{n+1}]$ is the value of $\mathbb{E}[y^{n+1}(\xi + \Delta W_{t_n})]|_{\xi = W_{t_n}}$ when $\xi = x_i$. Then we define

$$\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\hat{y}^{n+1}] = \frac{1}{\sqrt{\pi}} \sum_{j=1}^{K} w_j \hat{y}^{n+1}(x_i + \sqrt{2\Delta t_n} a_j)$$

Similarly, we define $\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\hat{y}^{n+1}\Delta W_{t_n}]$ by

$$\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\hat{y}^{n+1}\Delta W_{t_n}] = \frac{1}{\sqrt{\pi}} \sum_{j=1}^{K} \sqrt{2\Delta t_n} a_j \cdot w_j \hat{y}^{n+1}(x_i + \sqrt{2\Delta t_n} a_j),$$

where $\hat{y}^{n+1}(x_i + \sqrt{2\Delta t_{n+1}} a_j)$ is the interpolated grid function y_i^{n+1} at the spatial point $x_i + \sqrt{2\Delta t_{n+1}} a_j$ by using values of y_i^{n+1} at a finite number of spatial grid points in $\mathcal{R}_h$ near the spatial point $x_i + \sqrt{2\Delta t_{n+1}} a_j$. Approximate functions $\hat{\mathbb{E}}_{t_n}^{x_i}[\hat{f}(t_{n+1}, \hat{y}^{n+1}, \hat{z}^{n+1})]$, $\hat{\mathbb{E}}_{t_n}^{x_i}[\hat{g}(t_{n+1}, \hat{y}^{n+1})]$, $\hat{\mathbb{E}}_{t_n}^{x_i}[\hat{g}'_B(t_{n+1}, \hat{y}^{n+1})]$, $\hat{\mathbb{E}}_{t_n}^{x_i}[\hat{g}'_y(t_{n+1}, \hat{y}^{n+1})]$, $\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\hat{g}'_y(t_{n+1}, \hat{y}^{n+1}) \cdot \hat{g}(t_{n+1}, \hat{y}^{n+1})]$, and $\hat{\mathbb{E}}_{t_n}^{x_i}[\hat{f}(t_{n+1}, \hat{y}^{n+1}, \hat{z}^{n+1})\Delta W_{t_n}]$, $\hat{\mathbb{E}}_{t_n}^{x_i}[\hat{g}(t_{n+1}, \hat{y}^{n+1})\Delta W_{t_n}]$, $\hat{\mathbb{E}}_{t_n}^{x_i}[\hat{g}'_B(t_{n+1}, \hat{y}^{n+1})\Delta W_{t_n}]$, $\hat{\mathbb{E}}_{t_n}^{x_i}[\hat{g}'_y(t_{n+1}, \hat{y}^{n+1})\Delta W_{t_n}]$, $\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\hat{g}'_y(t_{n+1}, \hat{y}^{n+1}) \cdot \hat{g}(t_{n+1}, \hat{y}^{n+1})\Delta W_{t_n}]$ are defined similarly.

Error estimates for the fully discrete scheme

We now present error estimates for the fully discrete scheme defined by (5.124) and (5.125) where linear polynomial interpolation is used for computation of the approximate conditional expectation. To proceed, we assume that the time partition is uniform, that is $\Delta t_n = \Delta t$, for $n = 0, 1, 2, \cdots N_T - 1$.

Letting $e_y^{n,i} = y_{t_n}^{t_n,x_i} - y_i^n$ and subtracting (5.124) from (5.61), we obtain

$$\begin{aligned} e_y^{n,i} \;=\;& \mathbb{E}_{t_n}^{t_n,x_i}[y_{t_{n+1}}] - \hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\hat{y}^{n+1}] + \Delta t \cdot \{\mathbb{E}_{t_n}^{t_n,x_i}[f(t_{n+1}, y_{t_{n+1}}, z_{t_{n+1}})] \\ & - \hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\hat{f}(t_{n+1}, \hat{y}^{n+1}, \hat{z}^{n+1})] + \frac{1}{2}\mathbb{E}_{t_n}^{t_n,x_i}[g'_B(t_{n+1}, y_{t_{n+1}})] - \frac{1}{2}\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\hat{g}'_B(t_{n+1}, \hat{y}^{n+1})] \\ & - \frac{1}{2}\mathbb{E}_{t_n}^{t_n,x_i}[g'_y(t_{n+1}, y_{t_{n+1}}) \cdot g(t_{n+1}, y_{t_{n+1}})] + \frac{1}{2}\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\hat{g}'_y(t_{n+1}, \hat{y}^{n+1}) \cdot \hat{g}(t_{n+1}, \hat{y}^{n+1})]\} \\ & + \Delta B_{t_n} \cdot \{\mathbb{E}_{t_n}^{t_n,x_i}[g(t_{n+1}, y_{t_{n+1}})] - \hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\hat{g}(t_{n+1}, \hat{y}^{n+1})] \qquad (5.128) \\ & - \frac{1}{2}\mathbb{E}_{t_n}^{t_n,x_i}[g'_B(t_{n+1}, y_{t_{n+1}})]\Delta B_{t_n} + \frac{1}{2}\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\hat{g}'_B(t_{n+1}, \hat{y}^{n+1})]\Delta B_{t_n} \\ & + \frac{1}{2}\mathbb{E}_{t_n}^{t_n,x_i}[g'_y(t_{n+1}, y_{t_{n+1}}) \cdot g(t_{n+1}, y_{t_{n+1}})]\Delta B_{t_n} \\ & - \frac{1}{2}\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\hat{g}'_y(t_{n+1}, \hat{y}^{n+1}) \cdot \hat{g}(t_{n+1}, \hat{y}^{n+1})]\Delta B_{t_n}\} + R_y^{n,i}, \end{aligned}$$

where $R_y^{n,i}$ is the truncation error R_y^n at the grid point $x = x_i$.

Similarly, letting $e_z^{n,i} = z_{t_n}^{t_n,x_i} - z_i^n$ and subtracting (5.125) from (5.79), we obtain

$$\begin{aligned} \Delta t e_z^{n,i} \;=\;& \mathbb{E}_{t_n}^{t_n,x_i}[y_{t_{n+1}}\Delta W_{t_n}] - \hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\hat{y}^{n+1}\Delta W_{t_n}] + \Delta t \cdot (\mathbb{E}_{t_n}^{t_n,x_i}[f(t_{n+1}, y_{t_{n+1}}, z_{t_{n+1}})\Delta W_{t_n}] \\ & - \hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\hat{f}(t_{n+1}, \hat{y}^{n+1}, \hat{z}^{n+1})\Delta W_{t_n}] + \frac{1}{2}\mathbb{E}_{t_n}^{t_n,x_i}[g'_B(t_{n+1}, y_{t_{n+1}})\Delta W_{t_n}] \\ & - \frac{1}{2}\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\hat{g}'_B(t_{n+1}, \hat{y}^{n+1})\Delta W_{t_n}] - \frac{1}{2}\mathbb{E}_{t_n}^{t_n,x_i}[g'_y(t_{n+1}, y_{t_{n+1}}) \cdot g(t_{n+1}, y_{t_{n+1}})\Delta W_{t_n}] \\ & + \frac{1}{2}\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\hat{g}'_y(t_{n+1}, \hat{y}^{n+1}) \cdot \hat{g}(t_{n+1}, \hat{y}^{n+1})\Delta W_{t_n}]) \\ & + \Delta B_{t_n} \cdot (\mathbb{E}_{t_n}^{t_n,x_i}[g(t_{n+1}, y_{t_{n+1}})\Delta W_{t_n}] - \hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\hat{g}(t_{n+1}, \hat{y}^{n+1})\Delta W_{t_n}] \qquad (5.129) \\ & - \frac{1}{2}\mathbb{E}_{t_n}^{t_n,x_i}[g'_B(t_{n+1}, y_{t_{n+1}})\Delta W_{t_n}]\Delta B_{t_n} + \frac{1}{2}\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\hat{g}'_B(t_{n+1}, \hat{y}^{n+1})\Delta W_{t_n}]\Delta B_{t_n} \\ & + \frac{1}{2}\mathbb{E}_{t_n}^{t_n,x_i}[g'_y(t_{n+1}, y_{t_{n+1}}) \cdot g(t_{n+1}, y_{t_{n+1}})\Delta W_{t_n}]\Delta B_{t_n} \\ & - \frac{1}{2}\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\hat{g}'_y(t_{n+1}, \hat{y}^{n+1}) \cdot \hat{g}(t_{n+1}, \hat{y}^{n+1})\Delta W_{t_n}]\Delta B_{t_n}) + R_z^{n,i}, \end{aligned}$$

where $R_z^{n,i}$ is the truncation error R_z^n at the grid point $x = x_i$.

We have the following theorem for the fully discretized scheme.

Theorem 5.5 *In addition to the conditions of Theorem 5.3 and Lemma 2, assume that* y, z, f, g, g'_B, $g'_y \in C_b^{(2K)}$, *then:*

$$\max_{0\le n\le N_T-1} E[\max_i (e_y^{n,i})^2] \le C_\epsilon \left[(\Delta t)^{-2}\cdot\left(\frac{K!}{2^K(2K)!}\right)^2 + \frac{h^4}{(\Delta t)^2} + (\Delta t)^2\right],$$
$$\max_{0\le n\le N_T-1} E[\max_i (e_z^{n,i})^2] \le C_\epsilon \left[(\Delta t)^{-3}\cdot\left(\frac{K!}{2^K(2K)!}\right)^2 + \frac{h^4}{(\Delta t)^3} + \Delta t\right],$$

where C_ϵ *is a constant determined by a pre-chosen constant* $\epsilon > 0$ *and functions* f *and* g.

Recall that for a given random variable ξ,

$$\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\xi] = \frac{1}{\sqrt{\pi}}\sum_{j=1}^{K} w_j \xi(x_i + \sqrt{2\Delta t}a_j). \tag{5.130}$$

To present the error estimates for the fully discrete scheme, we need the following Lemma

Lemma 8 *We assume that* u, v *are two functions,* $u, v : \mathbb{R} \longrightarrow \mathbb{R}$, *and* ξ *is any random variable. Then the following inequalities hold*

$$\begin{aligned}(\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\hat{u}-\hat{v}])^2 &\le \max_i (u(x_i)-v(x_i))^2,\\ \hat{\mathbb{E}}_{t_n}^{t_n,x_i}[(\hat{u}-\hat{v})^2] &\le \max_i (u(x_i)-v(x_i))^2,\end{aligned} \tag{5.131}$$

$$(\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\xi\cdot\Delta W_{t_n}])^2 \le \Delta t\cdot\left[\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\xi^2] - (\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\xi])^2\right] \tag{5.132}$$

and

$$(\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\xi\cdot\Delta W_{t_n}])^2 \le \Delta t\cdot \hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\xi^2]. \tag{5.133}$$

Proof: From the definition of $\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\cdot]$ and the fact that $\dfrac{1}{\sqrt{\pi}}\displaystyle\sum_{j=1}^{K} w_j = 1$,

$$\begin{aligned}
(\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\hat{u}-\hat{v}])^2 &= (\frac{1}{\sqrt{\pi}}\sum_{j=1}^{K} w_j \cdot \Big[\hat{u}(x_i+\sqrt{2\Delta t}\ a_j) - \hat{v}(x_i+\sqrt{2\Delta t}\ a_j)\Big])^2 \\
&\le (\frac{1}{\sqrt{\pi}}\sum_{j=1}^{K} w_j \cdot \Big|\hat{u}(x_i+\sqrt{2\Delta t}\ a_j) - \hat{v}(x_i+\sqrt{2\Delta t}\ a_j)\Big|)^2 \\
&\le (\frac{1}{\sqrt{\pi}}\sum_{j=1}^{K} w_j \cdot \max_i |u(x_i)-v(x_i)|)^2 \\
&= \Big(\max_i |u(x_i)-v(x_i)|\Big)^2 = \max_i \left(u(x_i)-v(x_i)\right)^2
\end{aligned}$$

and

$$\begin{aligned}
\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[(\hat{u}-\hat{v})^2]) &= \frac{1}{\sqrt{\pi}}\sum_{j=1}^{K} w_j \Big(\hat{u}(x_i+\sqrt{2\Delta t}\ a_j) - \hat{v}(x_i+\sqrt{2\Delta t}\ a_j)\Big)^2 \\
&\le \frac{1}{\sqrt{\pi}}\sum_{j=1}^{K} w_j \cdot \max_i \left(u(x_i)-v(x_i)\right)^2 \\
&= \max_i \left(u(x_i)-v(x_i)\right)^2 .
\end{aligned}$$

as required in (5.131).

To prove (5.132), we set $\xi_j^i = \xi(x_i+\sqrt{2\Delta t}\ a_j)$, it follows from the Cauchy Schwarz inequality and the fact $\dfrac{1}{\sqrt{\pi}}\displaystyle\sum_{j=1}^{K}\sqrt{2\Delta t}\ a_j \cdot w_j = 0$ that

$$\begin{aligned}
\Big(\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\xi\cdot\Delta W_{t_n}]\Big)^2 &= (\frac{1}{\sqrt{\pi}}\sum_{j=1}^{K}\sqrt{2\Delta t}\ a_j \cdot w_j \xi_j^i)^2 \\
&= (\frac{1}{\sqrt{\pi}}\sum_{j=1}^{K}\sqrt{2\Delta t}\ a_j \cdot w_j \cdot [\ \xi_j^i - \sum_{p=1}^{K}\frac{w_p}{\sqrt{\pi}}\xi_p^i\ ])^2 \\
&\le [\ \sum_{j=1}^{K}\frac{w_j}{\sqrt{\pi}} \cdot (\sqrt{2\Delta t}\ a_j)^2\ ] \cdot [\ \sum_{j=1}^{K}\frac{w_j}{\sqrt{\pi}} \cdot (\xi_j^i - \sum_{p=1}^{K}\frac{w_p}{\sqrt{\pi}}\xi_p^i)^2\ ]
\end{aligned}$$

Since $\sum_{j=1}^{K} \frac{2w_j}{\sqrt{\pi}} \cdot (a_j)^2 = 1$ and $\sum_{j=1}^{K} \frac{w_j}{\sqrt{\pi}} = 1$, we see that

$$
\begin{aligned}
& [\sum_{j=1}^{K} \frac{w_j}{\sqrt{\pi}} \cdot (\sqrt{2\Delta t}\ a_j)^2\ ] \cdot [\ \sum_{j=1}^{K} \frac{w_j}{\sqrt{\pi}} \cdot (\xi_j^i - \sum_{p=1}^{K} \frac{w_p}{\sqrt{\pi}} \xi_p^i)^2\ ] \\
&= \Delta t \cdot [\ \sum_{j=1}^{K} \frac{2w_j}{\sqrt{\pi}} \cdot (a_j)^2\ ] \cdot [\ \sum_{j=1}^{K} \frac{w_j}{\sqrt{\pi}} \cdot (\ (\xi_j^i)^2 - 2 \cdot \xi_j^i \sum_{p=1}^{K} \frac{w_p}{\sqrt{\pi}} \xi_p^i + (\sum_{p=1}^{K} \frac{w_p}{\sqrt{\pi}} \xi_p^i)^2\ )\ ] \\
&= \Delta t \cdot [\ \sum_{j=1}^{K} \frac{w_j}{\sqrt{\pi}} \cdot (\xi_j^i)^2 - (\sum_{j=1}^{K} \frac{w_j}{\sqrt{\pi}} \xi_j^i)^2\ ] \\
&= \Delta t \cdot [\ \hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\xi^2] - (\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\xi])^2\ ],
\end{aligned}
$$

which proves the inequality (5.132).

The inequality (5.133) follows directly from (5.132). $\square$

Similar to the proof of (5.131),

$$
\begin{aligned}
& \Big(\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[(\hat{f}(t_{n+1}, \hat{y}_{t_{n+1}}, \hat{z}_{t_{n+1}}) - \hat{f}(t_{n+1}, \hat{y}^{n+1}, \hat{z}^{n+1}))]\Big)^2 \\
&= [\frac{1}{\sqrt{\pi}} \sum_{j=1}^{K} w_j \cdot \Big(\hat{f}(t_{n+1}, \hat{y}_{t_{n+1}}(x_i + \sqrt{2\Delta t}\ a_j), \hat{z}_{t_{n+1}}(x_i + \sqrt{2\Delta t}\ a_j)) \\
&\qquad - \hat{f}(t_{n+1}, \hat{y}^{n+1}(x_i + \sqrt{2\Delta t}\ a_j), \hat{z}^{n+1}(x_i + \sqrt{2\Delta t}\ a_j))\Big)]^2 \\
&\leq [\ \frac{1}{\sqrt{\pi}} \sum_{j=1}^{K} w_j \cdot \max_i \Big| f(t_{n+1}, y_{t_{n+1}}^{t_{n+1},x_i}, z_{t_{n+1}}^{t_{n+1},x_i}) - f(t_{n+1}, y_i^{n+1}, z_i^{n+1}) \Big|\ ]^2 \\
&= \max_i \Big(f(t_{n+1}, y_{t_{n+1}}^{t_{n+1},x_i}, z_{t_{n+1}}^{t_{n+1},x_i}) - f(t_{n+1}, y_i^{n+1}, z_i^{n+1}) \Big)^2 \\
&\leq L \cdot [\max_i (e_y^{n+1,i})^2 + \max_i (e_z^{n+1,i})^2].
\end{aligned}
$$

Following the above procedure, we see that

$$
\begin{aligned}
& \Big(\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[(\hat{g}(t_{n+1}, \hat{y}_{t_{n+1}}) - \hat{g}(t_{n+1}, \hat{y}^{n+1}))]\Big)^2 \leq L \max_i (e_y^{n+1,i})^2, \\
& \Big(\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[(\hat{g}_B'(t_{n+1}, \hat{y}_{t_{n+1}}) - \hat{g}_B'(t_{n+1}, \hat{y}^{n+1}))]\Big)^2 \leq L \max_i (e_y^{n+1,i})^2, \\
& \Big(\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[(\hat{g}_B'(t_{n+1}, \hat{y}_{t_{n+1}})\hat{g}(t_{n+1}, \hat{y}_{t_{n+1}}) - \hat{g}_B'(t_{n+1}, \hat{y}^{n+1})\hat{g}(t_{n+1}, \hat{y}^{n+1}))]\Big)^2 \leq L \max_i (e_y^{n+1,i})^2.
\end{aligned}
\tag{5.134}
$$

Proof of Theorem 5.5: Notice that for any random variable $\varphi_{t_{n+1}}$ and φ^{n+1},

$$
\begin{aligned}
&\mathbb{E}_{t_n}^{t_n,x_i}[\varphi_{t_{n+1}}] - \hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\hat{\varphi}^{n+1}] \\
&= \mathbb{E}_{t_n}^{t_n,x_i}[\varphi_{t_{n+1}}] - \hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\varphi_{t_{n+1}}] + \ \hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\varphi_{t_{n+1}} - \hat{\varphi}_{t_{n+1}}] + \hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\hat{\varphi}_{t_{n+1}} - \hat{\varphi}^{n+1}].
\end{aligned} \tag{5.135}
$$

From the equation (5.128), we have that

$$
\begin{aligned}
e_y^{n,i} \ = \ & \hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\hat{y}_{t_{n+1}} - \hat{y}^{n+1}] + \Delta t \cdot \{\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\hat{f}(t_{n+1}, \hat{y}_{t_{n+1}}, \hat{z}_{t_{n+1}}) - \hat{f}(t_{n+1}, \hat{y}^{n+1}, \hat{z}^{n+1})] \\
& + \frac{1}{2}\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\hat{g}_B'(t_{n+1}, \hat{y}_{t_{n+1}}) - \hat{g}_B'(t_{n+1}, \hat{y}^{n+1})] - \frac{1}{2}\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\hat{g}_y'(t_{n+1}, \hat{y}_{t_{n+1}}) \cdot \hat{g}(t_{n+1}, \hat{y}_{t_{n+1}}) \\
& - \hat{g}_y'(t_{n+1}, \hat{y}^{n+1}) \cdot \hat{g}(t_{n+1}, \hat{y}^{n+1})]\} + \Delta B_{t_n} \cdot \{\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\hat{g}(t_{n+1}, \hat{y}_{t_{n+1}}) - \hat{g}(t_{n+1}, \hat{y}^{n+1})] \\
& - \frac{1}{2}\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\hat{g}_B'(t_{n+1}, \hat{y}_{t_{n+1}}) - \hat{g}_B'(t_{n+1}, \hat{y}^{n+1})]\Delta B_{t_n} \\
& + \frac{1}{2}\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\hat{g}_y'(t_{n+1}, \hat{y}_{t_{n+1}}) \cdot \hat{g}(t_{n+1}, \hat{y}_{t_{n+1}}) - \hat{g}_y'(t_{n+1}, \hat{y}^{n+1}) \cdot \hat{g}(t_{n+1}, \hat{y}^{n+1})]\Delta B_{t_n}\} \\
& + I_1^y + I_2^y + R_y^{n,i},
\end{aligned} \tag{5.136}
$$

where

$$
\begin{aligned}
I_1^y \ = \ & \mathbb{E}_{t_n}^{t_n,x_i}[y_{t_{n+1}}] - \hat{\mathbb{E}}_{t_n}^{t_n,x_i}[y_{t_{n+1}}] + \Delta t \cdot \{\mathbb{E}_{t_n}^{t_n,x_i}[f(t_{n+1}, y_{t_{n+1}}, z_{t_{n+1}})] \\
& - \hat{\mathbb{E}}_{t_n}^{t_n,x_i}[f(t_{n+1}, y_{t_{n+1}}, z_{t_{n+1}})] + \frac{1}{2}\mathbb{E}_{t_n}^{t_n,x_i}[g_B'(t_{n+1}, y_{t_{n+1}})] - \frac{1}{2}\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[g_B'(t_{n+1}, y_{t_{n+1}})] \\
& - \frac{1}{2}\mathbb{E}_{t_n}^{t_n,x_i}[g_y'(t_{n+1}, y_{t_{n+1}}) \cdot g(t_{n+1}, y_{t_{n+1}})] + \frac{1}{2}\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[g_y'(t_{n+1}, y_{t_{n+1}}) \cdot g(t_{n+1}, y_{t_{n+1}})]\} \\
& + \Delta B_{t_n} \cdot \{\mathbb{E}_{t_n}^{t_n,x_i}[g(t_{n+1}, y_{t_{n+1}})] - \hat{\mathbb{E}}_{t_n}^{t_n,x_i}[g(t_{n+1}, y_{t_{n+1}})] \\
& - \frac{1}{2}\mathbb{E}_{t_n}^{t_n,x_i}[g_B'(t_{n+1}, y_{t_{n+1}})]\Delta B_{t_n} + \frac{1}{2}\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[g_B'(t_{n+1}, y_{t_{n+1}})]\Delta B_{t_n} \\
& + \frac{1}{2}\mathbb{E}_{t_n}^{t_n,x_i}[g_y'(t_{n+1}, y_{t_{n+1}}) \cdot g(t_{n+1}, y_{t_{n+1}})]\Delta B_{t_n} \\
& - \frac{1}{2}\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[g_y'(t_{n+1}, y_{t_{n+1}}) \cdot g(t_{n+1}, y_{t_{n+1}})]\Delta B_{t_n}\},
\end{aligned}
$$

and

$$
I_2^y \ = \ \hat{\mathbb{E}}_{t_n}^{t_n,x_i}[y_{t_{n+1}} - \hat{y}_{t_{n+1}}] + \Delta t \cdot \{\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[f(t_{n+1}, y_{t_{n+1}}, z_{t_{n+1}}) - \hat{f}(t_{n+1}, \hat{y}_{t_{n+1}}, \hat{z}_{t_{n+1}})]
$$

$$
\begin{aligned}
&+\frac{1}{2}\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[g_B'(t_{n+1},y_{t_{n+1}})-\hat{g}_B'(t_{n+1},\hat{y}_{t_{n+1}})]-\frac{1}{2}\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[g_y'(t_{n+1},y_{t_{n+1}})\cdot g(t_{n+1},y_{t_{n+1}})\\
&-\hat{g}_y'(t_{n+1},\hat{y}_{t_{n+1}})\cdot\hat{g}(t_{n+1},\hat{y}_{t_{n+1}})]\}+\Delta B_{t_n}\cdot\{\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[g(t_{n+1},y_{t_{n+1}})-\hat{g}(t_{n+1},\hat{y}_{t_{n+1}})]\\
&-\frac{1}{2}\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[g_B'(t_{n+1},y_{t_{n+1}})-\hat{g}_B'(t_{n+1},\hat{y}_{t_{n+1}})]\Delta B_{t_n}\\
&+\frac{1}{2}\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[g_y'(t_{n+1},y_{t_{n+1}})\cdot g(t_{n+1},y_{t_{n+1}})-\hat{g}_y'(t_{n+1},\hat{y}_{t_{n+1}})\cdot\hat{g}(t_{n+1},\hat{y}_{t_{n+1}})]\Delta B_{t_n}\},
\end{aligned}
$$

Similarly, the equation (5.129) becomes

$$
\begin{aligned}
\Delta te_z^{n,i} \;=\;& \hat{\mathbb{E}}_{t_n}^{t_n,x_i}[(\hat{y}_{t_{n+1}}-\hat{y}^{n+1})\cdot\Delta W_{t_{n+1}}]\\
&+\Delta t\cdot\{\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[(\hat{f}(t_{n+1},\hat{y}_{t_{n+1}},\hat{z}_{t_{n+1}})-\hat{f}(t_{n+1},\hat{y}^{n+1},\hat{z}^{n+1}))\cdot\Delta W_{t_{n+1}}]\\
&+\frac{1}{2}\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[(\hat{g}_B'(t_{n+1},\hat{y}_{t_{n+1}})-\hat{g}_B'(t_{n+1},\hat{y}^{n+1}))\cdot\Delta W_{t_{n+1}}]\\
&-\frac{1}{2}\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[(\hat{g}_y'(t_{n+1},\hat{y}_{t_{n+1}})\cdot\hat{g}(t_{n+1},\hat{y}_{t_{n+1}})-\hat{g}_y'(t_{n+1},\hat{y}^{n+1})\cdot\hat{g}(t_{n+1},\hat{y}^{n+1}))\cdot\Delta W_{t_{n+1}}]\}\\
&+\Delta B_{t_n}\cdot\{\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[(\hat{g}(t_{n+1},\hat{y}_{t_{n+1}})-\hat{g}(t_{n+1},\hat{y}^{n+1}))\cdot\Delta W_{t_{n+1}}]\\
&-\frac{1}{2}\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[(\hat{g}_B'(t_{n+1},\hat{y}_{t_{n+1}})-\hat{g}_B'(t_{n+1},\hat{y}^{n+1}))\cdot\Delta W_{t_{n+1}}]\Delta B_{t_n} \qquad (5.137)\\
&+\frac{1}{2}\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[(\hat{g}_y'(t_{n+1},\hat{y}_{t_{n+1}})\cdot\hat{g}(t_{n+1},\hat{y}_{t_{n+1}})\\
&-\hat{g}_y'(t_{n+1},\hat{y}^{n+1})\cdot\hat{g}(t_{n+1},\hat{y}^{n+1}))\cdot\Delta W_{t_{n+1}}]\Delta B_{t_n}\}+I_1^z+I_2^z+R_z^{n,i},
\end{aligned}
$$

where

$$
\begin{aligned}
I_1^z \;=\;& \mathbb{E}_{t_n}^{t_n,x_i}[y_{t_{n+1}}\Delta W_{t_{n+1}}]-\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[y_{t_{n+1}}\Delta W_{t_{n+1}}]+\Delta t\cdot\{\mathbb{E}_{t_n}^{t_n,x_i}[f(t_{n+1},y_{t_{n+1}},z_{t_{n+1}})\Delta W_{t_{n+1}}]\\
&-\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[f(t_{n+1},y_{t_{n+1}},z_{t_{n+1}})\Delta W_{t_{n+1}}]\\
&+\frac{1}{2}\mathbb{E}_{t_n}^{t_n,x_i}[g_B'(t_{n+1},y_{t_{n+1}})\Delta W_{t_{n+1}}]-\frac{1}{2}\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[g_B'(t_{n+1},y_{t_{n+1}})\Delta W_{t_{n+1}}]\\
&-\frac{1}{2}\mathbb{E}_{t_n}^{t_n,x_i}[g_y'(t_{n+1},y_{t_{n+1}})\cdot g(t_{n+1},y_{t_{n+1}})\Delta W_{t_{n+1}}]\\
&+\frac{1}{2}\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[g_y'(t_{n+1},y_{t_{n+1}})\cdot g(t_{n+1},y_{t_{n+1}})\Delta W_{t_{n+1}}]\}\\
&+\Delta B_{t_n}\cdot\{\mathbb{E}_{t_n}^{t_n,x_i}[g(t_{n+1},y_{t_{n+1}})\Delta W_{t_{n+1}}]-\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[g(t_{n+1},y_{t_{n+1}})\Delta W_{t_{n+1}}]\\
&-\frac{1}{2}\mathbb{E}_{t_n}^{t_n,x_i}[g_B'(t_{n+1},y_{t_{n+1}})\Delta W_{t_{n+1}}]\Delta B_{t_n}+\frac{1}{2}\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[g_B'(t_{n+1},y_{t_{n+1}})\Delta W_{t_{n+1}}]\Delta B_{t_n}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}\mathbb{E}_{t_n}^{t_n,x_i}[g'_y(t_{n+1},y_{t_{n+1}})\cdot g(t_{n+1},y_{t_{n+1}})\Delta W_{t_{n+1}}]\Delta B_{t_n}\\
&-\frac{1}{2}\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[g'_y(t_{n+1},y_{t_{n+1}})\cdot g(t_{n+1},y_{t_{n+1}})\Delta W_{t_{n+1}}]\Delta B_{t_n}\},
\end{aligned}
$$

$$
\begin{aligned}
I_2^z \;=\;& \hat{\mathbb{E}}_{t_n}^{t_n,x_i}[(y_{t_{n+1}}-\hat{y}_{t_{n+1}})\cdot \Delta W_{t_{n+1}}]\\
&+\Delta t\cdot\{\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[(f(t_{n+1},y_{t_{n+1}},z_{t_{n+1}})-\hat{f}(t_{n+1},\hat{y}_{t_{n+1}},\hat{z}_{t_{n+1}}))\cdot\Delta W_{t_{n+1}}]\\
&+\frac{1}{2}\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[(g'_B(t_{n+1},y_{t_{n+1}})-\hat{g}'_B(t_{n+1},\hat{y}_{t_{n+1}}))\cdot\Delta W_{t_{n+1}}]\\
&-\frac{1}{2}\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[(g'_y(t_{n+1},y_{t_{n+1}})\cdot g(t_{n+1},y_{t_{n+1}})-\hat{g}'_y(t_{n+1},\hat{y}_{t_{n+1}})\cdot\hat{g}(t_{n+1},\hat{y}_{t_{n+1}}))\cdot\Delta W_{t_{n+1}}]\}\\
&+\Delta B_{t_n}\cdot\{\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[(g(t_{n+1},y_{t_{n+1}})-\hat{g}(t_{n+1},\hat{y}_{t_{n+1}}))\cdot\Delta W_{t_{n+1}}]\\
&-\frac{1}{2}\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[(g'_B(t_{n+1},y_{t_{n+1}})-\hat{g}'_B(t_{n+1},\hat{y}_{t_{n+1}}))\cdot\Delta W_{t_{n+1}}]\Delta B_{t_n}\\
&+\frac{1}{2}\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[(g'_y(t_{n+1},y_{t_{n+1}})\cdot g(t_{n+1},y_{t_{n+1}})-\hat{g}'_y(t_{n+1},\hat{y}_{t_{n+1}})\cdot\hat{g}(t_{n+1},\hat{y}_{t_{n+1}}))\cdot\Delta W_{t_{n+1}}]\Delta B_{t_n}\},
\end{aligned}
$$

For any function φ,

$$
|\mathbb{E}_{t_n}^{t_n,x_i}[\varphi]-\hat{\mathbb{E}}_{t_n}^{t_n,x_i}[\varphi]|\le D_\varphi\frac{K!}{2^K(2K)!},
$$

where D_φ is the upper bound for $|\varphi_x^{(2K)}|$.

Since y, z, f, g, g'_B, $g'_y\in C_b^{(2K)}$, we have

$$
E[(I_1^y])^2\le C\left(\frac{K!}{2^K(2K)!}\right)^2
$$

and

$$
\begin{aligned}
&E[\max_i(e_y^{n,i})^2]+\frac{\Delta t}{1+\epsilon}E[\max_i(e_z^{n,i})^2]\\
\le\;&(1+C_\epsilon\Delta t)\cdot(E[\max_i(e_y^{n+1,i})^2]+\frac{\Delta t}{1+\epsilon}\cdot E[\max_i(e_z^{n+1,i})^2])\\
&+3E[(I_1^y+I_2^y+\max_i R_y^{n,i})^2]+\frac{(E[I_1^y+I_2^y+\max_i R_y^{W,n,i}])^2}{\Delta t}\\
&+C_\epsilon(\Delta t)^{-1}\cdot E[(I_1^z+I_2^z+\max_i R_z^{n,i})^2].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&E[\max_i(e_y^{n,i})^2] + \frac{\Delta t}{1+\epsilon}E[\max_i(e_z^{n,i})^2] \\
&\leq (1 + C_\epsilon \Delta t)\cdot (E[\max_i(e_y^{n+1,i})^2] + \frac{\Delta t}{1+\epsilon}\cdot E[\max_i(e_z^{n+1,i})^2]) \\
&\quad +3C\left[\left(\frac{K!}{2^K(2K)!}\right)^2 + h^4 + (\Delta t)^3\right] + C(\Delta t)^{-1}\cdot\left[\left(\frac{K!}{2^K(2K)!}\right)^2 + h^4 + (\Delta t)^4\right] \\
&\quad +C_\epsilon(\Delta t)^{-1}\cdot C\left[\left(\frac{K!}{2^K(2K)!}\right)^2 + h^4 + (\Delta t)^4\right] \\
&= (1 + C_\epsilon \Delta t)\cdot (E[\max_i(e_y^{n+1,i})^2] + \frac{\Delta t}{1+\epsilon}\cdot E[\max_i(e_z^{n+1,i})^2]) \\
&\quad +C_\epsilon(\Delta t)^{-1}\cdot C\left[\left(\frac{K!}{2^K(2K)!}\right)^2 + h^4 + (\Delta t)^4\right].
\end{aligned}
\tag{5.138}
$$

By applying discrete Grownwall's inequality, we obtain

$$
E[\max_i(e_y^{n,i})^2] + \frac{\Delta t}{1+\epsilon}E[\max_i(e_z^{n,i})^2] \leq \sum_{j=n}^{N_T} C_\epsilon(\Delta t)^{-1}\cdot C\left[\left(\frac{K!}{2^K(2K)!}\right)^2 + h^4 + (\Delta t)^4\right].
$$

Thus,

$$
\max_{0\leq n\leq N_{T-1}} E[\max_i(e_y^{n,i})^2] \leq C_\epsilon\left[(\Delta t)^{-2}\cdot\left(\frac{K!}{2^K(2K)!}\right)^2 + \frac{h^4}{(\Delta t)^2} + (\Delta t)^2\right]
$$

and

$$
\max_{0\leq n\leq N_{T-1}} E[\max_i(e_z^{n,i})^2] \leq C_\epsilon\left[(\Delta t)^{-3}\cdot\left(\frac{K!}{2^K(2K)!}\right)^2 + \frac{h^4}{(\Delta t)^3} + \Delta t\right],
$$

as required.

Remark 3 *Notice that the error terms* $(\Delta t)^{-2}\cdot\left(\frac{K!}{2^K(2K)!}\right)^2$ *and* $(\Delta t)^{-3}\cdot\left(\frac{K!}{2^K(2K)!}\right)^2$ *decrease very rapidly as the parameter* K *becomes large.*

5.3.6 Numerical experiment

We now demonstrate the effectiveness and accuracy of our method for solving the BDSDE (5.49). We will compare our first order scheme with the half order scheme (see [6]) in

Table 5.3: Example 1 of Section 6.3

N_T	$Error_{\frac{1}{2}}(Y)$	$Error_{\frac{1}{2}}(Z)$	$Error_1(Y)$	$Error_1(Z)$
2^4	0.2899	0.4197	0.1036	0.1486
2^5	0.2162	0.3193	$5.35E-02$	$7.74E-02$
2^6	0.13478	0.1942	$2.52E-02$	$3.74E-02$
2^7	$8.76E-02$	0.1292	$1.21E-02$	$1.81E-02$
CR	0.5864	0.5817	1.0380	1.0162

example1. In example 2, we show an application of BDSDE in solving a nonlinear filtering problem.

Example 1. Consider the BDSDE

$$Y_t^{t,x} = \exp(T + W_T^{t,x}) + \int_t^T \big(\sin(s)\cdot Y_s^{t,x} - Z_s^{t,x} - \sin(s)\cdot\exp(s + W_s^{t,x} + B_s - B_T)\big)\,ds \\ - \int_t^T Z_s^{t,x} dW_s - \int_t^T Y_s^{t,x} d\overleftarrow{B}_s.$$

The exact solution is $Y_s^{t,x} = \exp(s + W_s^{t,x} + B_s - B_T)$, $Z_s^{t,x} = \exp(s + W_s^{t,x} + B_s - B_T)$.

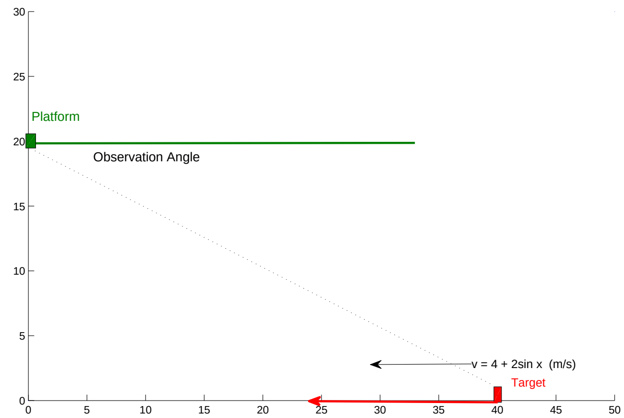
Numerical results are shown in Table 5.3. Here, the integer N_T is the number of temporal partitions, CR is the convergence rate, $Error_1(Y)$ and $Error_1(Z)$ represent, respectively, the errors in the approximation of $Y_t^{t,x}$ and $Z_t^{t,x}$ of the first order scheme. Also, $Error_{\frac{1}{2}}(Y)$ and $Error_{\frac{1}{2}}(Z)$ are respectively, the errors in approximation of $Y_t^{t,x}$ and $Z_t^{t,x}$ of the half order scheme. The results verify the theoretical error estimates we obtained in Section 4.

Example 2. We present a practical example which illustrates the application of our numerical method to solve a nonlinear filtering problem. This example is a classical "bearing-only tracking" problem (see [8]).

The simulation scenario is shown in Figure 5.3. Consider a target moving along the x-axis, according to a state equation

$$dX_t = -(4 + 2\sin X_t)dt + d\tilde{W}_t, \qquad X_0 = 40.$$

Figure 5.3: Target tracking by using one detector

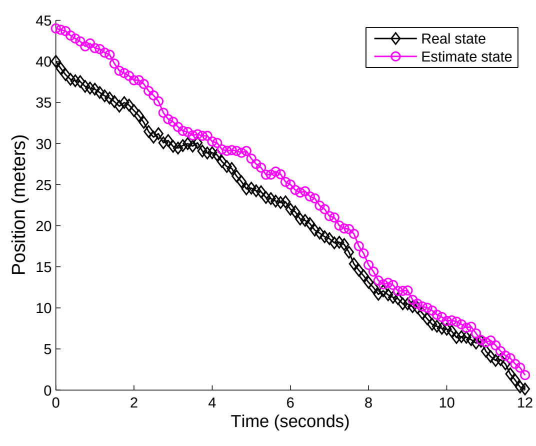


A fixed observer located on a platform on the y-axis takes noisy measurement O_t of the target bearing

$$dO_t = \tan^{-1}(\frac{20}{X_t}) + d\tilde{B}_t, \qquad O_0 = 0.$$

Here, $\tilde{W}_t$ and $\tilde{B}_t$ are two independent standard Brownian Motions. The goal is to find

Figure 5.4: Tracking Estimate

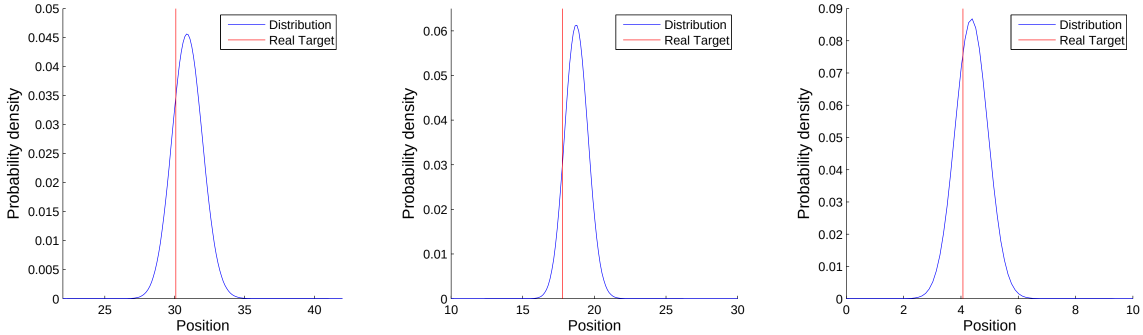


the best estimate of the target position X_t based on the observations up to time t, i.e. for $E[X_t|\mathcal{F}(O_0^t)]$, where $\mathcal{F}(O_0^t)$ denotes the σ-algebra generated by the observation $\{O_s\}_{0\leq s\leq t}$. The initial guess of the position of the target is $X_0 = 44$. One has the conditional density

Figure 5.5: Probability Distribution

(a) 30 steps (b) 60 steps (c) 90 steps

p



$p(X_t|\mathcal{F}(O_0^t)) = Y_{T-t}^{T-t,X_t}$, where $Y_s^{T-t,x}$ satisfies the following BDSDE(see [61, ?])

$$\begin{aligned} Y_s^{T-t,x} &= Y_T^{T-t,x} + \int_s^T \left[2\cos(W_r^{T-t,x}) \cdot Y_r^{T-t,x} + (4 + 2\sin(W_r^{T-t,x})) \cdot Z_r^{T-t,x}\right] dr \\ &\quad - \int_s^T Z_r^{T-t,x} dW_r + \int_s^T \tan^{-1}(\frac{20}{W_r^{T-t,x}}) \cdot Y_r^{T-t,x} d\overleftarrow{O}_{T-r}, \end{aligned}$$

where $T - t \leq s \leq T$, and $x \in \mathbb{R}$. The initial condition $Y_T^{T-t,x}$ of the above equation is the normal distribution $N(44, 1)$. The stochastic process W_t is a standard Brownian motion independent of the observation process O_t. Define the equivalent probability measure Q by

$$\frac{dQ}{dP}|_{\mathcal{G}_t} = \exp\left(\int_0^t \tan^{-1}(\frac{20}{X_t})dO_s - \frac{1}{2}\int_0^t |\tan^{-1}(\frac{20}{X_t})|^2 ds\right), \quad t \in \mathbb{R}_+,$$

where $\mathcal{G}_t$ is the σ-algebra generated by $\{X_s\}_{0 \leq s \leq t}$ and $\{O_s\}_{0 \leq s \leq t}$, $t \in \mathbb{R}_+$. One has under probability measure Q, that W_t and O_t are two independent standard Brownian Motions(see [?]).

In our simulation example, we assume that the observations are collected at intervals of length $\Delta t = 0.125s$ and we track this target for $12s$. Figure 5.4 illustrates the comparison of simulation results and the real target state. Figure 5.5 shows distributions of the simulated conditional probability $p(X_t|\mathcal{F}(O_0^t))$. We present the simulation results to three time levels:

$t = 3.75s(30 \cdot \Delta t), 7.5s(60 \cdot \Delta t), 11.25s(90 \cdot \Delta t)$. The red lines represent the real target positions and the blue curves show the simulated distribution.

Chapter 6

Concluding Remarks

The major contribution of this work is the development of three novel numerical approximation methods for the solution of nonlinear filtering problems: (i) an implicit filter method; (ii) a hybrid sparse grid Zakai filter method; (iii) the FBDSDE filter method.

Our first effort in this work is the development of an implicit filter algorithm for nonlinear filtering problems. This method is under the general framework of the Bayesian filter. The essential idea is to evaluate the probability distribution of the current state in the prediction step by evaluating the previous state through the state equation, given the value of the current state and a sample of the noise. Through rigorous analysis we proved the convergence of the algorithm. Numerical experiments indicate that our method is more accurate than the standard Kalman filter and is more stable than the standard particle filter method for long term simulations. It needs to point out that our method is a grid method in which the probability distributions are evaluated at all grid points. As such, the computing cost will increase exponentially as the dimension increases. Thus our method at current form is suitable for only low dimension problems such as target tracking problems.. In the future, we plan to improve the algorithm by adding an adaptive mechanism and a new interpolation method called the "radial basis approximations" to it so that it will be more efficient in solving higher dimensional problems.

The second effort in this work focuses on the Zakai filter. We proposed the construction of a hybrid numerical algorithm to improve the efficiency of the Zakai filter for moderately high dimensional nonlinear filtering problems. Our algorithm combines the advantages of the splitting-up approximation scheme for the Zakai filter, a hierarchical sparse grid method for moderately high dimensional problems to compute the numerical solution of the Zakai filter, and an importance sampling method to adaptively construct a bounded domain at each time step of the temporal discretization. The hierarchical sparse grid method reduces the number of grid points we need to solve in the splitting-up approximation scheme for the Zakai filter. The application of the solution domain adaptive method allows us to solve the Zakai filter in the high density region of the target state pdf which reduces the size of domains that we build the sparse grid points. We used one numerical experiment to show the effectiveness of our solution domain adaptive method and two other examples to demonstrate that our algorithm is more efficient than the standard Zakai filter while maintaining its high accuracy.

In addition, we proposed two numerical schemes for computing backward doubly s-tochastic differential equations (BDSDEs). Becasue of the equivalent relation between BDS-DEs and SPDEs, our schemes can also be used to find numerical solutions of the Zakai equation. Thus our algorithms also provide numerical methods for solving nonlinear filtering problems. It is worthy to mention that when solving the BDSDEs, we only need to solve stochastic ordinary differential equations instead of solving SPDEs in the Zakai filter. In this connection, high order numerical approximation methods for stochastic ordinary differential equations can be applied to develop high order numerical schemes for BDSDEs. The first numerical scheme we proposed in this work provides a half order numerical approximation method for BDSDEs which is derived by using the Euler scheme to discretize the stochastic integrals. The second numerical scheme gives a first order numerical approximation method. The main idea of the first order scheme is to use the two-sided Itô-Taylor expansion for forward and backward stochastic differentials to construct high order quadratures for the stochastic integrals involving both backward and forward Brownian motions. Through rigorous analysis, we proved the convergence rates for both the half order scheme and the first order scheme. Although we only introduced a first order numerical scheme in this work, higher order schemes can also be developed using higher order Itô Taylor expansions.

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