Principal Eigenvalue Theroy for Time Periodic Nonlocal Dispersal Operators and Applications

by

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Abstract

The dissertation is concerned with the spectral theory, in particular, the principal eigenvalue theory for nonlocal dispersal operators with time periodic dependence, and its applications. Nonlocal and random dispersal operators are widely used to model diffusion systems in applied sciences and share many properties. There are also some essential differences between nonlocal and random dispersal operators, for example, a smooth random dispersal operator always has a principal eigenvalue, but a smooth nonlocal dispersal operator may not have a principal eigenvalue.

In this dissertation, we first establish criteria for the existence of principal eigenvalues of time periodic nonlocal dispersal operators with Dirichlet type, Neumann type, or periodic type boundary conditions. Among others, it is shown that a time periodic nonlocal dispersal operator possesses a principal eigenvalue provided that the nonlocal dispersal distance is sufficiently small, or the time average of the underlying media satisfies some vanishing condition with respect to the space variable at a maximum point or is nearly globally homogeneous with respect to the space variable. We also obtain lower bounds of the principal spectrum points of time periodic nonlocal dispersal operators in terms of the corresponding time averaged problems.

Next, we discuss the applications of the established principal eigenvalue theory to the existence, uniqueness, and stability of time periodic positive solutions to Fisher or KPP type equations with nonlocal dispersal in periodic media. We prove that such equations are of monostable feature, that is, if the trivial solution is linearly unstable, then there is a unique time periodic positive solution $u^+(t,x)$ which is globally asymptotically stable.

Finally, we discuss the application of the established principal eigenvalue theory to the spatial spreading and front propagation dynamics of KPP equations with nonlocal dispersal in periodic media. We show that such an equation has a spatial spreading speed $c^*(\xi)$ in the direction of any given unit vector ξ. A variational characterization of $c^*(\xi)$ is given. Under the assumption that the nonlocal dispersal operator associated to the linearization of the monostable equation at the trivial solution 0 has a principal eigenvalue, we also show that the monostable equation has a periodic traveling wave solution connecting $u^+(\cdot,\cdot)$ and 0 propagating in any given direction of ξ with speed $c > c^*(\xi)$.

Key words. Nonlocal dispersal, random dispersal, principal eigenvalue, principal spectrum point, vanishing condition, lower bound, monostable equation, spatial spreading speed, traveling wave solution.

Mathematics subject classification. 35K55, 35K57, 45C05, 45M15, 45M20, 47G10, 92D25.

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Chapter 1

Introduction

Both random dispersal evolution equations and nonlocal dispersal evolution equations are widely used to model diffusive systems in applied sciences. Classically, one assumes that the internal interaction of organisms in a diffusive system is infinitesimal or the internal dispersal is random, which leads to a diffusion operator, e.g., Δu as dispersal operator. Many diffusive systems in real world exhibit long range internal interaction or dispersal, which can be modeled by nonlocal dispersal operators such as $\int_{\mathbb{R}^N} \kappa(y-x)u(t,y)dy - u(t,x)$, here $\kappa(\cdot)$ is a convolution kernel supported on the ball centered at the origin with radius r, the interaction range. As a basic technical tool for the study of nonlinear evolution equations with random and nonlocal dispersals, it is of great importance to investigate aspects of spectral theory for random and nonlocal dispersal operators.

This dissertation is devoted to the study of principal eigenvalues of the following three eigenvalue problems associated to nonlocal dispersal operators with time periodic dependence,

$$\begin{cases} -u_t + \nu_1[\int_D \kappa(y-x)u(t,y)dy - u(t,x)] + a_1(t,x)u = \lambda u, & x \in \bar{D} \\ u(t+T,x) = u(t,x) \end{cases} \tag{1.1}$$

where $D \subset \mathbb{R}^N$ is a smooth bounded domain and $a_1(t,x)$ is a continuous function with $a_1(t+T,x) = a_1(t,x)$,

$$\begin{cases} -u_t + \nu_2[\int_D \kappa(y-x)(u(t,y) - u(t,x))dy] + a_2(t,x)u = \lambda u, & x \in \bar{D} \\ u(t+T,x) = u(t,x) \end{cases} \tag{1.2}$$

where $D \subset \mathbb{R}^N$ is as in (1.1) and $a_2(t,x)$ is a continuous function with $a_2(t+T,x) = a_2(t,x)$, and

$$\begin{cases} -u_t + \nu_3[\int_{\mathbb{R}^N} \kappa(y-x)u(t,y)dy - u(t,x)] + a_3(t,x)u = \lambda u, \quad x \in \mathbb{R}^N \\ u(t+T,x) = u(t,x+p_j\mathbf{e_j}) = u(t,x), \quad x \in \mathbb{R}^N \end{cases} \tag{1.3}$$

where $p_j > 0$, $\mathbf{e_j} = (\delta_{j1}, \delta_{j2}, \cdots, \delta_{jN})$ with $\delta_{jk} = 1$ if $j = k$ and $\delta_{jk} = 0$ if $j \neq k$, and $a_3(t,x)$ is a continuous function with $a_3(t+T,x) = a_3(t,x+p_j\mathbf{e_j}) = a_3(t,x)$, $j = 1, 2, \cdots, N$. $\kappa(\cdot)$ in (1.1)-(1.3) is a nonnegative C^1 with compact support, $\kappa(0) > 0$, and $\int_{\mathbb{R}^N} \kappa(z)dz = 1$.

This dissertation is also devoted to the applications of the principal eigenvalue theory for (1.1)-(1.3) to be developed.

The eigenvalue problems (1.1), (1.2), and (1.3) can be viewed as the nonlocal dispersal counterparts of the following eigenvalue problems associated to random dispersal operators,

$$\begin{cases} -u_t + \nu_1 \Delta u + a_1(t,x)u = \lambda u, \quad x \in D \\ u(t+T,x) = u(t,x), \quad x \in D \\ u = 0, \quad x \in \partial D, \end{cases} \tag{1.4}$$

$$\begin{cases} -u_t + \nu_2 \Delta u + a_2(t,x)u = \lambda u, \quad x \in D \\ u(t+T,x) = u(t,x), \quad x \in D \\ \frac{\partial u}{\partial n} = 0, \quad x \in \partial D, \end{cases} \tag{1.5}$$

and

$$\begin{cases} -u_t + \nu_3 \Delta u + a_3(t,x)u = \lambda u, \quad x \in \mathbb{R}^N \\ u(t+T,x) = u(t,x+p_j\mathbf{e_j}) = u(t,x), \quad x \in \mathbb{R}^N, \end{cases} \tag{1.6}$$

respectively. It is in fact proved in [53] that the principal eigenvalues of (1.4), (1.5), and (1.6) can be approximated by the principal spectrum points of (1.1), (1.2), and (1.3) with properly rescaled kernels, respectively (see Definition 2.1 for the definition of principal spectrum points of (1.1), (1.2), and (1.3)). We may then say that (1.1), (1.2), and (1.3) are of the Dirichlet type boundary condition, Neumann type bounday condition, and periodic boundary condition, respectively. The reader is referred to [8], [9], and [53] about the approximations of the initial value problems of the random dispersal operators associated to (1.4), (1.5), and

(1.6) by the initial value problems of the nonlocal dispersal operators with properly rescaled kernels associated to (1.1), (1.2), and (1.3), respectively.

The eigenvalue problems (1.4), (1.5), and (1.6), in particular, their associated principal eigenvalue problems, are well understood. For example, it is known that there is $\lambda_{R,1} \in \mathbb{R}$ such that $\lambda_{R,1}$ is an isolated algebraic simple eigenvalue of (1.4) with a positive eigenfunction, and for any other eigenvalues λ of (1.4), $\mathrm{Re}\lambda \le \lambda_{R,1}$ ($\lambda_{R,1}$ is called the *principal eigenvalue* of (1.4)) (see [22]).

The principal eigenvalue problem for time independent nonlocal dispersal operators with Dirichlet type, or Neumann type, or periodic boundary condition has been recently studied by many people (see [11], [18], [23], [30], [49], [52], and references therein) and is quite well understood now. For example, the following criteria for the existence of principal eigenvalues for nonlocal dispersal operators are established in [49] and [52] ([49] is on the periodic boundary condition case and [52] is on Dirichlet type and Neumann type boundary conditions) (see Definition 2.1 for the definition of principal eigenvalues of nonlocal dispersal operators),

(i) *If* $a_1(t,x) \equiv a_1(x)$ *(resp.* $a_2(t,x) \equiv a_2(x)$, $a_3(t,x) \equiv a_3(x)$*) and* $\kappa(z) = \frac{1}{\delta^N}\tilde{\kappa}(\frac{z}{\delta})$ *for some* $\delta > 0$ *and* $\tilde{\kappa}(\cdot)$ *with* $\tilde{\kappa}(z) \ge 0$, $\mathrm{supp}(\tilde{k}) = B(0,1) := \{z \in \mathbb{R}^N \mid \|z\| < 1\}$, *and* $\int_{\mathbb{R}^N} \tilde{\kappa}(z)dz = 1$, *then (1.1) (resp. (1.2), (1.3)) admits a principal eigenvalue provided that* δ *is sufficiently small.*

(ii) *If* $a_1(t,x) \equiv a_1(x)$ *(resp.* $a_2(t,x) \equiv a_2(x)$, $a_3(t,x) \equiv a_3(x)$*) is* C^N *and there is some* $x_0 \in \mathrm{Int}(D)$ *(resp.* $x_0 \in \mathrm{Int}(D)$, $x_0 \in \mathbb{R}^N$*) satisfying that* $a_1(x_0) = \max_{x\in\bar{D}} a_1(x)$ *(resp.* $-\nu_2\int_D \kappa(y-x_0)dy + a_2(x_0) = \max_{x\in\bar{D}}(-\nu_2\int_D \kappa(y-x)dy + a_2(x))$, $a_3(x_0) = \max_{x\in\mathbb{R}^N} a_3(x)$*) and the partial derivatives of* $a_1(x)$ *(resp.* $-\nu_2\int_D \kappa(y-x)dy + a_2(x)$, $a_3(x)$*) up to order* $N-1$ *at* x_0 *are zero, then (1.1) (resp. (1.2), (1.3)) admits a principal eigenvalue.*

(iii) *If* $a_1(t,x) \equiv a_1(x)$ *(resp.* $a_2(t,x) \equiv a_2(x)$, $a_3(t,x) \equiv a_3(x)$*) and* $\max_{x\in\bar{D}} a_1(x) - \min_{x\in\bar{D}} a_1(x) < \nu_1 \inf_{x\in\bar{D}} \int_D \kappa(y-x)dy$ *(resp.* $\max_{x\in\bar{D}} a_2(x) - \min_{x\in\bar{D}} a_2(x) < \nu_2 \inf_{x\in\bar{D}} \int_D \kappa(y-$

$x)dy$, $\max_{x\in\mathbb{R}^N} a_3(x) - \min_{x\in\mathbb{R}^N} a_3(x) < \nu_3$), then (1.1) (resp. (1.2), (1.3)) admits a principal eigenvalue.

It should be pointed out that [30] contains some results similar to (i) in the Dirichlet type boundary condition case and [11] contains some results similar to (ii). It should also be pointed out that a nonlocal dispersal operator may not have a principal eigenvalue (see [49] for an example), which reveals some essential difference between nonlocal and random dispersal operators. Methologically, due to the lack of regularity and compactness of the solutions of nonlocal evolution equations, some difficulties, which do not arise in the study of spectral theory of random dispersal operators, arise in the study of spectral theory of nonlocal dispersal operators.

Regarding nonlocal dispersal operators with time periodic dependence, in [28], the authors studied the existence of principal eigenvalues of (1.1) in the case that $N = 1$. In [28] and [48], the influence of temporal variation on the principal eigenvalue of (1.1) (if exists) is investigated. In general, the understanding to the principal eigenvalue problems associated to (1.1), (1.2), and (1.3) is very little.

The first objective of this dissertation is to develop criteria for the existence of principal eigenvalues of (1.1), (1.2), and (1.3) and to explore fundamental properties of principal eigenvalues of (1.1), (1.2), and (1.3). Many existing results on principal eigenvalues of time independent and some special time periodic nonlocal dispersal operators are extended to general time periodic nonlocal dispersal operators. For example, the following result is established in this dissertation, which extends (ii) in the above for time independent nonlocal dispersal operators to time periodic ones,

• *If $a_1(t,x)$ (resp. $a_2(t,x)$, $a_3(t,x)$) is in C^N in x and there is some $x_0 \in \mathrm{Int}(D)$ (resp. $x_0 \in \mathrm{Int}(D)$, $x_0 \in \mathbb{R}^N$) such that that $\hat{a}_1(x_0) = \max_{x\in\bar{D}} \hat{a}_1(x)$ (resp. $-\int_D \kappa(y-x_0)dy + \hat{a}_2(x_0) = \max_{x\in\bar{D}} (-\int_D \kappa(y-x)dy + \hat{a}_2(x)$, $\hat{a}_3(x_0) = \max_{x\in\mathbb{R}^N} \hat{a}_3(x)$) and the partial derivatives of $\hat{a}_1(x)$ (resp. $-\int_D \kappa(y-x)dy + \hat{a}_2(x)$, $\hat{a}_3(x)$) up to order $N-1$ at x_0 are zero, then (1.1)* *(resp. (1.2), (1.3)) admits a principal eigenvalue, where* $\hat{a}_i(x)$ *is the time average of* $a_i(t,x)$ $(i = 1,2,3)$ *(see (2.1) for the definition of* $\hat{a}_i(\cdot)$ *for* $i = 1,2,3$*).*

The reader is referred to Theorems A-C in section 2 for the principal eigenvalue theories established in this dissertation for general time periodic nonlocal dispersal operators.

The second objective of this dissertation is to consider applications of the established principal theories to the following time periodic KPP type or Fisher type equations with nonlocal dispersal,

$$u_t = \nu_1[\int_D \kappa(y-x)u(t,y)dy - u(t,x)] + uf_1(t,x,u), \quad x \in \bar{D}, \tag{1.7}$$

$$u_t = \nu_2[\int_D \kappa(y-x)(u(t,y) - u(t,x))dy] + uf_2(t,x,u), \quad x \in \bar{D}, \tag{1.8}$$

and

$$\begin{cases} u_t = \nu_3[\int_{\mathbb{R}^N} \kappa(y-x)u(t,y)dy - u(t,x)] + uf_3(t,x,u), & x \in \mathbb{R}^N \\ u(t,x+p_j\mathbf{e_j}) = u(t,x), & x \in \mathbb{R}^N, \end{cases} \tag{1.9}$$

where $f_i(t,x)$ $(i = 1,2,3)$ are C^1 functions, $f_i(t+T,x,u) = f_i(t,x,u)$ $(i = 1,2,3)$, $f_3(t,x+p_j\mathbf{e_j},u) = f_3(t,x,u)$ $(j = 1,2,\cdots,N)$, and $f_i(t,x,u) < 0$ for $u \gg 1$ and $\partial_u f_i(t,x,u) < 0$ for $u \geq 0$ $(i = 1,2,3)$.

Equations (1.7), (1.8), and (1.9) are the nonlocal counterparts of the following reaction diffusion equations,

$$\begin{cases} u_t = \nu_1 \Delta u + uf_1(t,x,u), & x \in D \\ u(t,x) = 0, & x \in \partial D, \end{cases} \tag{1.10}$$

$$\begin{cases} u_t = \nu_2 \Delta u + uf_2(t,x,u), & x \in D \\ \frac{\partial u}{\partial n} = 0, & x \in \partial D, \end{cases} \tag{1.11}$$

and

$$\begin{cases} u_t = \nu_3 \Delta u + uf_3(t,x,u), & x \in \mathbb{R}^N \\ u(t,x+p_j\mathbf{e_j}) = u(t,x), & x \in \mathbb{R}^N, \end{cases} \tag{1.12}$$

respectively (see [53] for the approximations of the solutions of (1.7), (1.8), and (1.9) to (1.10), (1.11), and (1.12), respectively).

Equations (1.7)-(1.9) and (1.10)-(1.12) are widely used to model population dynamics of species exhibiting nonlocal internal interactions and random internal interactions, respectively. Thanks to the pioneering works of Fisher ([17]) and Kolmogorov, Petrowsky, Piscunov ([31]) on the following special case of (1.12),

$$u_t = u_{xx} + u(1-u), \quad x \in \mathbb{R}, \tag{1.13}$$

(1.7)-(1.9) and (1.10)-(1.12) are referred to as Fisher type or KPP type equations.

One of the central problems for (1.7)-(1.9) and (1.10)-(1.12) is about the existence, uniqueness, and stability of positive time periodic solutions. This problem has been extensively studied and is well understood for (1.10)-(1.12). For example, it is known that (1.10) exhibits the following monostable feature: if the trivial solution $u \equiv 0$ is a linearly unstable solution of (1.10), then (1.10) has a unique stable time periodic positive solution. Again, some difficulties, which do not arise in the study of (1.10)-(1.12), aries in the study of (1.7)-(1.9) due to the lack of compactness and regularities of the solutions of nonlocal dispersal evolution equations. In [51], the authors proved that time independent KPP equations with nonlocal dispersal also exhibit monostable feature (see also [2], [11] for the study of positive stationary solutions of time independent KPP equations with nonlocal dispersal). But it is hardly studied whether a general time periodic KPP equation with nonlocal dispersal is of the monostable feature. In this dissertation, by applying the principal eigenvalue theories for time periodic nonlocal dispersal operators to be established, we prove

- *A time periodic KPP equations with nonlocal dispersal is of the monostable feature, that is, if $u \equiv 0$ is a linearly unstable solution of a time periodic KPP equation with nonlocal dispersal, then the equation has a unique stable time periodic positive solution $u^+(\cdot,\cdot)$* (see Theorem E in Section 2).

Consider (1.9) without the periodic condition $u(t,x+p_i\mathbf{e}_i)=u(t,x)$, that is,

$$u_t=\nu_3[\int_{\mathbb{R}^N}\kappa(y-x)u(t,y)dy-u(t,x)]+uf_3(t,x,u),\quad x\in\mathbb{R}^N \tag{1.14}$$

where $f_3(t,x,u)<0$ for $u\gg 1$, $\partial_u f_3(t,x,u)<0$ for $u\geq 0$, and $f_3(t,x,u)$ is of certain recurrent property in t and x. The spatial spreading and front propagation dynamics is also among central problems. This problem has been studied by many people for the random dispersal counterpart of (1.14) since the pioneering work by Fisher ([17]) and Kolmogorov, Petrowsky, and Piscunov ([31]). Fisher in [17] found traveling wave solutions $u(t,x)=\phi(x-ct)$, $(\phi(-\infty)=1,\phi(\infty)=0)$ of all speeds $c\geq 2$ and showed that there are no such traveling wave solutions of slower speed. He conjectured that the take-over occurs at the asymptotic speed 2. This conjecture was proved in [31] by Kolmogorov, Petrowsky, and Piscunov, that is, they proved that for any nonnegative solution $u(t,x)$ of (1.13), if at time $t=0$, u is 1 near $-\infty$ and 0 near ∞, then $\lim_{t\to\infty}u(t,ct)$ is 0 if $c>2$ and 1 if $c<2$ (i.e. the population invades into the region with no initial population with speed 2). The number 2 is called the *spatial spreading speed* of (1.13) in literature.

The results of Fisher and Kolmogorov, Petrowsky, Piscunov [31] for (1.13) have been extended by many people to quite general reaction diffusion equations of the form,

$$u_t=\Delta u+uf_3(t,x,u),\quad x\in\mathbb{R}^N, \tag{1.15}$$

where $f_3(t,x,u)<0$ for $u\gg 1$, $\partial_u f_3(t,x,u)<0$ for $u\geq 0$, and $f_3(t,x,u)$ is of certain recurrent property in t and x. For example, assume that $f_3(t,x,u)$ is periodic in t with period T and periodic in x_i with period p_i ($p_i>0$, $i=1,2,\cdots,N$) (i.e. $f_3(\cdot+T,\cdot,\cdot)=f_3(\cdot,\cdot+p_i\mathbf{e_i},\cdot)=f_3(\cdot,\cdot,\cdot)$, $\mathbf{e_i}=(\delta_{i1},\delta_{i2},\cdots,\delta_{iN})$, $\delta_{ij}=1$ if $i=j$ and 0 if $i\neq j$, $i,j=1,2,\cdots,N$), and that $u\equiv 0$ is a linearly unstable solution of (1.15) with respect to periodic perturbations. Then it is known that (1.15) has a unique positive periodic solution $u^+(t,x)$ ($u^+(t+T,x)=u^+(t,x+p_i\mathbf{e_i})=u^+(t,x)$) which is asymptotically stable with respect to periodic perturbations and it has been proved that for every $\xi \in S^{N-1} := \{x \in \mathbb{R}^N \mid \|x\| = 1\}$, there is a $c^*(\xi) \in \mathbb{R}$ such that for every $c \geq c^*(\xi)$, there is a traveling wave solution connecting u^+ and $u^- \equiv 0$ and propagating in the direction of ξ with speed c, and there is no such traveling wave solution of slower speed in the direction of ξ. Moreover, the minimal wave speed $c^*(\xi)$ is of some important spreading properties. The reader is referred to [3], [4], [5], [32], [33], [38], [39], [55], [56] and references therein for the above mentioned properties and to [24], [37], [46], [47] for the extensions of the above results to the cases that $f_3(t, x, u)$ is almost periodic in t and periodic in x and that $f_3(t, x, u) \equiv f_3(t, u)$ is recurrent in t.

Recently, the spatial spreading and front propagation dynamics for (1.14) with $f_3(t, x, u) = f_3(x, u)$ has been studied by many authors. See, for example, [10], [12], [13], [14], [23], [34], [36], [40], [49], [50], [51] for the study of the existence of spreading speeds and traveling wave solutions of (1.14) connecting the trivial solution $u = 0$ and a nontrivial positive stationary solution in the case that $f_3(t, x, u) \equiv f_3(x, u)$. However, in contrast to (1.15), the spatial spreading and front propagation dynamics of (1.14) with both time and space periodic dependence or with general time and/or space dependence is much less understood. The results on spatial spreading speeds and traveling wave solutions established in [33] and [56] for quite general periodic monostable evolution equations cannot be applied to time and space periodic nonlocal monostable equations because of the lack of certain compactness of the solution operators for such equations. In this dissertation, by applying the principal eigenvalue theories for time periodic nonlocal dispersal operators to be established, we obtain

- *For any given unit vector $\xi \in \mathbb{R}^N$, (1.14) has a spatial spreading speed $c_*(\xi)$ in the direction of ξ. Moreover, some variational characterization for $c_*(\xi)$ is given and the spreading speed $c_*(\xi)$ is of some important spreading features (see Theorems G and H for detail).*

- *If for given $\xi \in \mathbb{R}^N$ with $\|\xi\| = 1$, the following eigenvalue problem*

$$\begin{cases} -u_t + \int_{\mathbb{R}^N} e^{-\mu(y-x)\cdot\xi}\kappa(y-x)u(t,y)dy - u(t,x) + a_0(t,x)u(t,x) = \lambda u(t,x) \\ u(t+T,x) = u(t,x+p_i\mathbf{e_i}) = u(t,x) \end{cases} \tag{1.16}$$

has a principal eigenvalue $\lambda_0(\xi, \mu, a_0)$ *for each* $\mu > 0$*, where* $a_0(t, x) = f_3(t, x, 0)$*, then for any* $c > c_*(\xi)$*, (1.14) has a (periodic) traveling wave solution* $u(t, x) = \Phi(x - ct, t, ct)$ *connecting* u^+ *and* 0 *(see Theorem I for detail).*

The results stated above cover most of the results in literature when $f_3(t, x, u) \equiv f_3(x, u)$. It should be pointed out that if (1.16) has no principal eigenvalue for some $\mu > 0$, it remains open whether (1.14) has a traveling wave solution connecting $u^+(\cdot, \cdot)$ and 0 in the direction of ξ with speed $c > c_*(\xi)$ (this remains open even when $f_3(t, x, u) \equiv f_3(x, u)$ is time independent but space periodic).

Nonlocal evolution equations have been attracting more and more attention due to the presence of nonlocal interaction in many diffusive systems in applied sciences. The reader is referred to [7], [10], [12], [14], [16], [18], [19], [30], [34], [36], [40], [48], [50], etc., for the study of various aspects of nonlocal dispersal evolution equations.

The rest of the dissertation is organized as follows. In Chapter 2, we introduce standing notations and definitions and state the main results of the dissertation. We present basic properties needed in the proofs of the main results in Chapter 3. The principal eigenvalue theory is developed in Chapter 4. The chapter 5 is about time periodic positive solutions of nonlocal KPP equations in periodic media. In Chapters 6 and 7, the spatial spreading speeds and traveling wave solutions of nonlocal KPP equations in periodic media are presented, respectively.

Chapter 2

Notations, Definitions, and Main Results

This chapter begins with standing notations that are used in this chapter and beyond. Following the notations, the definitions of principal spectrum points and principal eigenvalues of (1.1), (1.2), and (1.3) are given. We then state the main results concerning the existence of principal eigenvalues, its applications to time periodic KPP equations with nonlocal dispersal.

Let

$$\mathcal{X}_1 = \mathcal{X}_2 = \{u \in C(\mathbb{R} \times \bar{D}, \mathbb{R}) \,|\, u(t+T, x) = u(t, x)\}$$

with norm $\|u\|_{\mathcal{X}_i} = \sup_{t\in\mathbb{R}, x\in\bar{D}} |u(t,x)|$ $(i = 1, 2)$,

$$\mathcal{X}_3 = \{u \in C(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}) \,|\, u(t+T, x) = u(t, x + p_i\mathbf{e_i}) = u(t, x)\}$$

with norm $\|u\|_{\mathcal{X}_3} = \sup_{t\in\mathbb{R}, x\in\mathbb{R}^N} |u(t,x)|$, and

$$\mathcal{X}_i^+ = \{u \in \mathcal{X}_i \,|\, u \geq 0\}$$

$(i = 1, 2, 3)$.

For given $a_i \in \mathcal{X}_i$, let $L_i(a_i) : \mathcal{D}(L_i(a_i)) \subset \mathcal{X}_i \to \mathcal{X}_i$ be defined as follows,

$$(L_1(a_1)u)(t,x) = -u_t(t,x) + \nu_1[\int_D \kappa(y-x)u(t,y)dy - u(t,x)] + a_1(t,x)u(t,x),$$

$$(L_2(a_2)u)(t,x) = -u_t(t,x) + \nu_2 \int_D \kappa(y-x)(u(t,y) - u(t,x))dy + a_2(t,x)u(t,x),$$

and

$$(L_3(a_3)u)(t,x) = -u_t(t,x) + \nu_3[\int_{\mathbb{R}^N} \kappa(y-x)u(t,y)dy - u(t,x)] + a_3(t,x)u(t,x).$$

Definition 2.1. *Let*

$$s(L_i, a_i) = \sup\{\text{Re}\lambda \,|\, \lambda \in \sigma(L_i(a_i))\}$$

for $i = 1,2,3$. Then, $s(L_i, a_i)$ is called the principal spectrum point *of $L(a_i)$ $(i = 1,2,3)$. If $s(L_i, a_i)$ is an isolated eigenvalue of $L(a_i)$ with a positive eigenfunction ϕ (i.e. $\phi \in \mathcal{X}_i^+$), then $s(L_i, a_i)$ is called the* principal eigenvalue *of $L_i(a_i)$ or it is said that $L_i(a_i)$* has a principal eigenvalue *$(i = 1,2,3)$.*

For given $1 \le i \le 3$ and $a \in \mathcal{X}_i$, let

$$\hat{a}_i(x) = \frac{1}{T}\int_0^T a_i(t,x)dt, \tag{2.1}$$

$$b_i(x) = \begin{cases} -\nu_i \quad \text{for} \quad i = 1,3; \\ -\nu_i \int_D \kappa(y-x)dy \quad \text{for} \quad i = 2, \end{cases} \tag{2.2}$$

and

$$D_i = \begin{cases} D \quad \text{for} \quad i = 1,2; \\ [0,p_1] \times [0,p_2] \times \cdots \times [0,p_N] \quad \text{for} \quad i = 3. \end{cases} \tag{2.3}$$

2.1 Principal eigenvalues and principal spectrum points

Our main results on the principal spectrum points and principal eigenvalues of nonlocal dispersal operators can then be stated as follows.

Theorem A. (Necessary and sufficient condition)

Let $1 \le i \le 3$ be given. Then, $s(L_i, a_i)$ is the principal eigenvalue of $L_i(a_i)$ iff $s(L_i, a_i) > \max_{x \in \bar{D}_i}(b_i(x) + \hat{a}_i(x))$.

Theorem B. (Sufficient conditions)

Let $1 \leq i \leq 3$ be given.

(1) Suppose that $\kappa(z) = \frac{1}{\delta^N}\tilde{\kappa}(\frac{z}{\delta})$ for some $\delta > 0$ and $\tilde{\kappa}(\cdot)$ with $\tilde{\kappa}(z) \geq 0$, $\mathrm{supp}(\tilde{\kappa}) = B(0,1) := \{z \in \mathbb{R}^N \mid \|z\| < 1\}$, and $\int_{\mathbb{R}^N} \tilde{\kappa}(z)dz = 1$. Then the principal eigenvalue of $L_i(a_i)$ exists for $0 < \delta \ll 1$.

(2) The principal eigenvalue of $L_i(a_i)$ exists if $a_i(t,x)$ is in C^N in x, there is some $x_0 \in \mathrm{Int}(D_i)$ in the case $i = 1, 2$ and $x_0 \in D_3$ in the case $i = 3$ satisfying that $b_i(x_0)+\hat{a}_i(x_0) = \max_{x\in\bar{D}_i}(b_i(x) + \hat{a}_i(x))$, and the partial derivatives of $b_i(x) + \hat{a}_i(x)$ up to order $N - 1$ at x_0 are zero.

(3) The principal eigenvalue of $L_i(a_i)$ exists if

$$\max_{x\in\bar{D}_i} \hat{a}_i(x) - \min_{x\in\bar{D}_i} \hat{a}_i(x) < \nu_i \inf_{x\in\bar{D}_i} \int_{D_i} \kappa(y - x)dy$$

in the case $i = 1, 2$ and

$$\max_{x\in\bar{D}_i} \hat{a}_i(x) - \min_{x\in\bar{D}_i} \hat{a}_i(x) < \nu_i$$

in the case $i = 3$.

Theorem C. (Influence of temporal variation)

For given $1 \leq i \leq 3$, $s(L_i, a_i) \geq s(L_i, \hat{a}_i)$.

Corollary D. *If $s(L_i, \hat{a}_i)$ is the principal eigenvalue of $L_i(\hat{a}_i)$, then $s(L_i, a_i)$ is the principal eigenvalue of $L_i(a_i)$.*

Proof. Assume that $s(L_i, \hat{a}_i)$ is the principal eigenvalue of $L_i(\hat{a}_i)$. Then by Theorem A,

$$s(L_i, \hat{a}_i) > \max_{x\in\bar{D}_i}(b_i(x) + \hat{a}_i(x)).$$

This together with Theorem C implies that

$$s(L_i, a_i) > \max_{x \in \bar{D}_i} (b_i(x) + \hat{a}_i(x)).$$

Then by Theorem A again, $s(L_i, a_i)$ is the principal eigenvalue of $L_i(a_i)$. $\square$

Observe that when $a_i(t, x) \equiv a_i(x)$ $(i = 1, 2, 3)$, Theorems A and B recover the existing results for time independent nonlocal dispersal operators (see [49], [52], and references therein). Theorem B (2) extends a result in [28] for the case $i = 1$ and $N = 1$ to time periodic nonlocal dispersal operators in higher space dimension domains. In the case $i = 1$ and both $s(L_1, a_1)$ and $s(L_1, \hat{a}_1)$ are eigenvalues of $L_1(a_1)$ and $L_1(\hat{a}_1)$, it is shown in [28] that $s(L_1, a_1) \geq s(L_1, \hat{a}_1)$. Theorem C extends this result to general time periodic nonlocal dispersal operators and shows that temporal variation does not reduce the principal spectrum point of a general time periodic nonlocal dispersal operator.

Theorems A-C and Corollary D establish some fundamental principal eigenvalue theory for general time periodic nonlocal dispersal operators and provide a basic tool for the study of nonlinear evolution equations with nonlocal dispersal. In the following, we consider their applications to the study of the asymptotic dynamics of (1.7)-(1.9).

2.2 Time periodic positive solutions of nonlocal KPP equations

Let

$$X_1 = X_2 = \{u \in C(\bar{D}, \mathbb{R})\}$$

with norm $\|u\|_{X_i} = \sup_{x \in \bar{D}} |u(x)|$ $(i = 1, 2)$,

$$X_3 = \{u \in C(\mathbb{R}^N, \mathbb{R}) \,|\, u(x + p_j \mathbf{e_j}) = u(x)\}$$

with norm $\|u\|_{X_3} = \sup_{x\in\mathbb{R}^N} |u(x)|$, and

$$X_i^+ = \{u \in X_i \mid u \geq 0\},$$

$(i = 1, 2, 3)$ and

$$X_i^{++} = \begin{cases} u \in X_i^+ \mid u(x) > 0 \quad \forall\, x \in \bar{D}, \quad i = 1, 2 \\ u \in X_i^+ \mid u(x) > 0 \quad \forall x \in \mathbb{R}^N, \quad i = 3. \end{cases}$$

.

By general semigroup theory, for any $s \in \mathbb{R}$ and $u_0 \in X_1$ (resp. $u_0 \in X_2$, $u_0 \in X_3$), (1.7) (resp. (1.8), (1.9)) has a unique (local) solution $u_1(t, x; s, u_0)$ (resp. $u_2(t, x; s, u_0)$, $u_3(t, x; s, u_0)$) with $u_1(s, x; s, u_0) = u_0(x)$ (resp. $u_2(s, x; s, u_0) = u_0(x)$, $u_3(s, x; s, u_0) = u_0(x)$) (see Proposition 3.1). Moreover, if $u_0 \in X_i^+$, then $u_i(t, x; s, u_0)$ exists and $u_i(t, \cdot; s, u_0) \in X_i^+$ for all $t \geq s$ $(i = 1, 2, 3)$ (see Proposition 3.3).

Theorem E. (Existence, uniqueness, and stability of time periodic positive solutions)

Let $a_i(t, x) = f_i(t, x, 0)$ $(i = 1, 2, 3)$. If $s(L_1, a_i) > 0$ (resp. $s(L_2, a_2) > 0$, $s(L_3, a_3) > 0$), then (1.7) (resp. (1.8), (1.9)) has a unique time periodic solution solution $u_1^*(t, \cdot) \in X_1^{++}$ (resp. $u_2^*(t, \cdot) \in X_2^{++}$, $u_3^*(t, \cdot) \in X_3^{++}$). Moreover, for any $u_0 \in X_i^+ \setminus \{0\}$,

$$\|u_i(t, \cdot; 0, u_0) - u_i^*(t, \cdot)\|_{X_i} \to 0$$

as $t \to \infty$ $(i = 1, 2, 3)$.

Corollary F. *Let $a_i(t, x) = f_i(t, x, 0)$ $(i = 1, 2, 3)$. If $s(L_1, \hat{a}_1) > 0$ (resp. $s(L_2, \hat{a}_2) > 0$, $s(L_3, \hat{a}_3) > 0$), then (1.7) (resp.(1.8), (1.9)) has a unique time periodic solution solution $u_1^*(t, \cdot) \in X_1^{++}$ (resp. $u_2^*(t, \cdot) \in X_2^{++}$, $u_3^*(t, \cdot) \in X_3^{++}$). Moreover, for any $u_0 \in X_i^+ \setminus \{0\}$,*

$$\|u_i(t, \cdot; 0, u_0) - u_i^*(t, \cdot)\|_{X_i} \to 0$$

as $t \to \infty$ $(i = 1, 2, 3)$.

Proof. Assume $s(L_1, \hat{a}_1) > 0$ (resp. $s(L_2, \hat{a}_2) > 0$, $s(L_3, \hat{a}_3) > 0$). By Theorem C, $s(L_1, a_i) > 0$ (resp. $s(L_2, a_2) > 0$, $s(L_3, a_3) > 0$). The corollary then follows from Theorem E. $\square$

2.3 Spatial spreading speeds of time and space periodic KPP equations

For simplicity in notation, when considering the spatial spreading and front propagation dynamics of (1.14), we drop the sub-index 3, that is, we write (1.14) as

$$\frac{\partial u}{\partial t} = \int_{\mathbb{R}^N} \kappa(y-x)u(t,y)dy - u(t,x) + u(t,x)f(t,x,u(t,x)), \quad x \in \mathbb{R}^N, \tag{2.4}$$

where $\kappa(\cdot)$ is as in (1.14) and $f(t,x,u)$ is periodic in t and x and satisfies proper monostablility assumptions. More precisely, let (H0) stands for the following assumption.

(H0) *$f(t,x,u)$ is C^1 in $(t,x,u) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}$, and $f(\cdot + T, \cdot, \cdot) = f(\cdot, \cdot + p_i \mathbf{e_i}, \cdot) = f(\cdot, \cdot, \cdot)$, $\mathbf{e_i} = (\delta_{i1}, \delta_{i2}, \cdots, \delta_{iN})$, $\delta_{ij} = 1$ if $i = j$ and 0 if $i \neq j$, $i, j = 1, 2, \cdots, N$.*

Let

$$\mathcal{X}_p = \{u \in C(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}) | u(\cdot + T, \cdot) = u(\cdot, \cdot + p_i \mathbf{e_i}) = u(\cdot, \cdot), \quad i = 1, \cdots, N\} \tag{2.5}$$

with norm $\|u\|_{\mathcal{X}_p} = \sup_{(t,x) \in \mathbb{R} \times \mathbb{R}^N} |u(t,x)|$ (note that $\mathcal{X}_p = \mathcal{X}_3$), and

$$\mathcal{X}_p^+ = \{u \in \mathcal{X}_p \,|\, u(t,x) \geq 0 \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N\}. \tag{2.6}$$

Let I be the identity map on $\mathcal{X}_p$, and $\mathcal{K}$, $a_0(\cdot,\cdot)I : \mathcal{X}_p \to \mathcal{X}_p$ be defined by

$$(\mathcal{K}u)(t,x) = \int_{\mathbb{R}^N} \kappa(y-x)u(t,y)dy, \tag{2.7}$$

$$(a_0(\cdot,\cdot)Iu)(t,x) = a_0(t,x)u(t,x), \tag{2.8}$$

where $a_0(t,x) = f(t,x,0)$. Let $\sigma(-\partial_t + \mathcal{K} - I + a_0(\cdot,\cdot)I)$ be the spectrum of $-\partial_t + \mathcal{K} - I + a_0(\cdot,\cdot)I$ acting on $\mathcal{X}_p$. The monostablility assumptions are then stated as follows:

(H1) $\frac{\partial f(t,x,u)}{\partial u} < 0$ *for* $t \in \mathbb{R}$, $x \in \mathbb{R}^N$ *and* $u \in \mathbb{R}$ *and* $f(t,x,u) < 0$ *for* $t \in \mathbb{R}$, $x \in \mathbb{R}^N$ *and* $u \gg 1$.

(H2) $u \equiv 0$ *is linearly unstable in* X_p, *that is,* $\lambda_0(a_0) > 0$, *where* $\lambda_0(a_0) := \sup\{\mathrm{Re}\lambda \mid \lambda \in \sigma(-\partial_t + \mathcal{K} - I + a_0(\cdot,\cdot))$.

Let

$$X = \{u \in C(\mathbb{R}^N, \mathbb{R}) \mid u \text{ is uniformly continuous and bounded}\} \tag{2.9}$$

with supremum norm and

$$X^+ = \{u \in X \mid u(x) \geq 0 \quad \forall\, x \in \mathbb{R}^N\}. \tag{2.10}$$

By general semigroup theory, for any $u_0 \in X$, (2.4) has a unique solution $u(t,x;u_0)$ with $u(0,x;u_0) = u_0(x)$. By comparison principle, if $u_0 \in X^+$, then $u(t,\cdot;u_0)$ exists for all $t \geq 0$ and $u(t,\cdot;u_0) \in X^+$ (see Proposition 3.3 for detail).

By Theorem E, (H1) and (H2) imply that (2.4) has exactly two time periodic solutions in $\mathcal{X}_p^+$, $u = 0$ and $u = u^+(t,x)$, and $u = 0$ is linearly unstable and $u = u^+(t,x)$ is asymptotically stable with respect to positive perturbations in X_p^+, where

$$X_p = \{u \in C(\mathbb{R}^N, \mathbb{R}) \mid u(\cdot + p\mathbf{e_i}) = u(\cdot)\} \tag{2.11}$$

with maximum norm (note that $X_p = X_3$) and

$$X_p^+ = \{u \in X_p \mid u(x) \geq 0 \;\forall\; x \in \mathbb{R}^N\}. \tag{2.12}$$

Hence (H1) and (H2) are called monostability assumptions.

For given $\xi \in S^{N-1}$ and $\mu \in \mathbb{R}$, let $\lambda_0(\xi,\mu,a_0)$ be the principal spectrum point of the eigenvalue problem

$$\begin{cases} -u_t + \int_{\mathbb{R}^N} e^{-\mu(y-x)\cdot\xi}\kappa(y-x)u(t,y)dy - u(t,x) + a_0(t,x)u(t,x) = \lambda u(t,x) \\ u(\cdot,\cdot) \in \mathcal{X}_p \end{cases} \tag{2.13}$$

(defined as in Definition 2.1 with $\kappa(y-x)$ being replaced by $e^{-\mu(y-x)\cdot\xi}\kappa(y-x)$). Let $X^+(\xi)$ be defined by

$$X^+(\xi) = \{u \in X^+ \mid \inf_{x\cdot\xi \ll -1} u(x) > 0,\ \sup_{x\cdot\xi\gg 1} u(x) = 0\}. \tag{2.14}$$

Definition 2.2. *For a given vector $\xi \in S^{N-1}$, let*

$$C^*_{\inf}(\xi) = \Big\{\ c \mid \forall\ u_0 \in X^+(\xi),\ \limsup_{t\to\infty} \sup_{x\cdot\xi\le ct} |u(t,x;u_0) - u^+(t,x)| = 0\Big\}$$

and

$$C^*_{\sup}(\xi) = \Big\{\ c \mid \forall\ u_0 \in X^+(\xi),\ \limsup_{t\to\infty} \sup_{x\cdot\xi\ge ct} u(t,x;u_0) = 0\Big\}.$$

Define

$$c^*_{\inf}(\xi) = \sup\ \{\ c \mid c \in C^*_{\inf}(\xi)\},\quad c^*_{\sup}(\xi) = \inf\ \{\ c \mid c \in C^*_{\sup}(\xi)\}.$$

We call $[c^*_{\inf}(\xi), c^*_{\sup}(\xi)]$ the spreading speed interval *of (2.4) in the direction of ξ. If $c^*_{\inf}(\xi) = c^*_{\sup}(\xi)$, we call $c^*(\xi) := c_{\inf}(\xi)$ the* spreading speed *of (2.4) in the direction of ξ.*

Theorem G. (Existence of spreading speeds) Assume (H1) and (H2). For any given $\xi \in S^{N-1}$, $c^*_{\inf}(\xi) = c^*_{\sup}(\xi)$ and hence the spreading speed $c^*(\xi)$ of (2.4) in the direction of ξ exists. Moreover,

$$c^*(\xi) = \inf_{\mu>0} \frac{\lambda_0(\xi,\mu,a_0)}{\mu},$$

where $a_0(t,x) = f(t,x,0)$.

Theorem H. (Spreading features of spreading speeds) *Assume (H1) and (H2).*

(1) If $u_0 \in X^+$ satisfies that $u_0(x) = 0$ for $x \in \mathbb{R}^N$ with $|x \cdot \xi| \gg 1$, then for each $c > \max\{c^*(\xi), c^*(-\xi)\}$,

$$\limsup_{t\to\infty} \sup_{|x\cdot\xi|\geq ct} u(t,x;u_0) = 0.$$

(2) Assume that $\xi \in S^{N-1}$ and $0 < c < \min\{c^*(\xi), c^*(-\xi)\}$. Then for any $\sigma > 0$ and $r > 0$,

$$\liminf_{t\to\infty} \inf_{|x\cdot\xi|\leq ct} (u(t,x;u_0) - u^+(t,x)) = 0$$

for every $u_0 \in X^+$ satisfying $u_0(x) \geq \sigma$ for all $x \in \mathbb{R}^N$ with $|x \cdot \xi| \leq r$.

(3) If $u_0 \in X^+$ satisfies that $u_0(x) = 0$ for $x \in \mathbb{R}^N$ with $\|x\| \gg 1$, then

$$\limsup_{t\to\infty} \sup_{\|x\|\geq ct} u(t,x;u_0) = 0$$

for all $c > \sup_{\xi\in S^{N-1}} c^*(\xi)$.

(4) Assume that $0 < c < \inf_{\xi\in S^{N-1}}\{c^*(\xi)\}$. Then for any $\sigma > 0$ and $r > 0$,

$$\liminf_{t\to\infty} \inf_{\|x\|\leq ct} (u(t,x;u_0) - u^+(t,x)) = 0$$

for every $u_0 \in X^+$ satisfying $u_0(x) \geq \sigma$ for $x \in \mathbb{R}^N$ with $\|x\| \leq r$.

2.4 Traveling wave solutions of time and space periodic KPP equations

Definition 2.3 (Traveling wave solution)**.** *(1) An entire solution $u(t,x)$ of (2.4) is called a* traveling wave solution *connecting $u^+(\cdot,\cdot)$ and 0 and propagating in the direction of ξ with speed c if there is a bounded function $\Phi : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^+$ such that $\Phi(\cdot,\cdot,\cdot)$ is Lebesgue measurable, $u(t,\cdot;\Phi(\cdot,0,z),z)$ exists for all $t \in \mathbb{R}$,*

$$u(t,x) = u(t,x;\Phi(\cdot,0,0),0) = \Phi(x - ct\xi, t, ct\xi) \quad \forall\ t \in \mathbb{R},\ x \in \mathbb{R}^N, \tag{2.15}$$

$$u(t,x;\Phi(\cdot,0,z),z) = \Phi(x - ct\xi, t, z + ct\xi) \quad \forall t \in \mathbb{R},\ x,z \in \mathbb{R}^N, \tag{2.16}$$

$$\lim_{x\cdot\xi\to-\infty} (\Phi(x,t,z) - u^+(t,x+z)) = 0, \qquad \lim_{x\cdot\xi\to\infty} \Phi(x,t,z) = 0 \tag{2.17}$$

uniformly in $(t,z) \in \mathbb{R} \times \mathbb{R}^N$,

$$\Phi(x,t,z-x) = \Phi(x',t,z-x') \quad \forall x,x' \in \mathbb{R}^N \text{ with } \mathrm{x}\cdot\xi = \mathrm{x}'\cdot\xi, \tag{2.18}$$

and

$$\Phi(x,t+T,z) = \Phi(x,t,z+p_i\mathbf{e_i}) = \Phi(x,t,z) \quad \forall x,z \in \mathbb{R}^N. \tag{2.19}$$

(2) A bounded function $\Phi : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^+$ *is said to* generate a traveling wave solution of (2.4) in the direction of ξ with speed c *if it is Lebesgue measurable and satisfies (2.16) - (2.19).*

Remark 2.4. *Suppose that* $u(t,x) = \Phi(x - ct\xi, t, ct\xi)$ *is a traveling wave solution of (2.4) connecting* $u^+(\cdot)$ *and* 0 *and propagating in the direction of* ξ *with speed* c*. Then* $u(t,x)$ *can be written as*

$$u(t,x) = \Psi(x\cdot\xi - ct, t, x) \tag{2.20}$$

for some $\Psi : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ *satisfying* $\Psi(\eta,t+T,z) = \Psi(\eta,t,z+p_i\mathbf{e_i}) = \Psi(\eta,t,z)$, $\lim_{\eta\to-\infty}\Psi(\eta,t,z) = u^+(t,z)$*, and* $\lim_{\eta\to\infty}\Psi(\eta,t,z) = 0$ *uniformly in* $(t,z) \in \mathbb{R}\times\mathbb{R}^N$*. In fact, let* $\Psi(\eta,t,z) = \Phi(x,t,z-x)$ *for* $x \in \mathbb{R}^N$ *with* $x\cdot\xi = \eta$*. Observe that* $\Psi(\eta,t,z)$ *is well defined and has the above mentioned properties.*

For convenience, we introduce the following assumption:

(H3) *For every* $\xi \in S^{N-1}$ *and* $\mu \geq 0$, $\lambda_0(\xi,\mu,a_0)$ *is the principal eigenvalue of* $-\partial_t + \mathcal{K}_{\xi,\mu} - I + a_0(\cdot,\cdot)I$*, where* $a_0(t,x) = f(t,x,0)$.

We now state the main results of this section. For given $\xi \in S^{N-1}$ and $c > c^*(\xi)$, let $\mu \in (0,\mu^*(\xi))$ be such that

$$c = \frac{\lambda_0(\xi,\mu,a_0)}{\mu}.$$

Let $\phi(\mu,\cdot,\cdot) \in \mathcal{X}_p^+$ be the positive principal eigenfunction of $-\partial_t + \mathcal{K}_{\xi,\mu} - I + a_0(\cdot)I$ with $\|\phi(\mu,\cdot,\cdot)\|_{\mathcal{X}_p} = 1$.

Theorem I. (Existence of traveling wave solutions) *Assume (H1)-(H3). For any $\xi \in S^{N-1}$ and $c > c^*(\xi)$, there is a bounded function $\Phi : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^+$ such that $\Phi(\cdot,\cdot,\cdot)$ generates a traveling wave solution connecting $u^+(\cdot,\cdot)$ and 0 and propagating in the direction of ξ with speed c. Moreover, $\lim_{x\cdot\xi\to\infty} \frac{\Phi(x,t,z)}{e^{-\mu x\cdot\xi}\phi(\mu,t,x+z)} = 1$ uniformly in $t \in \mathbb{R}$ and $z \in \mathbb{R}^N$.*

Chapter 3

Basic Properties

In this chapter, we present basic properties to be used in the following chapters.

3.1 Basic properties for solutions of nonlocal evolution equations

In this section, we present some basic properties for solutions of (1.7)-(1.9) and linear nonlocal evolution equations,

$$u_t = \nu_1[\int_D \kappa(y-x)u(y)dy - u(x)] + a_1(t,x)u, \quad x \in \bar{D}, \tag{3.1}$$

$$u_t = \nu_2[\int_D \kappa(y-x)(u(y)-u(x))dy] + a_2(t,x)u, \quad x \in \bar{D}, \tag{3.2}$$

and

$$u_t = \nu_3[\int_{\mathbb{R}^N} \kappa(y-x)u(y)dy - u(x)] + a_3(t,x)u, \quad x \in \mathbb{R}^N, \tag{3.3}$$

where $a_i \in \mathcal{X}_i$ $(i = 1,2,3)$.

Throughout this chapter, i denotes any integer with $1 \le i \le 3$, unless specified otherwise and $\mathcal{X}_i$, $\mathcal{X}_i^+$, and X_i, X_i^+, X_i^{++} are as in section 2. D_i is as in (2.3). For $u_1, u_2 \in \mathcal{X}_i$, we define

$$u_1 \le u_2 \ \ (u_1 \ge u_2) \ \text{ if } \ u_2 - u_1 \in \mathcal{X}_i^+ \ \ (u_1 - u_2 \in \mathcal{X}_i^+).$$

For $u_1, u_2 \in X_i$, we define

$$u_1 \le u_2 \ \ (u_1 \ge u_2) \ \text{ if } \ u_2 - u_1 \in X_i^+ \ \ (u_1 - u_2 \in X_i^+),$$

and

$$u_1 \ll u_2 \ \ (u_1 \gg u_2) \ \text{ if } \ u_2 - u_1 \in X_i^{++} \ \ (u_1 - u_2 \in X_i^{++}).$$

Proposition 3.1. *(1) For any $u_0 \in X_1$ (resp. $u_0 \in X_2$, $u_0 \in X_3$) and $s \in \mathbb{R}$, (3.1) (resp. (3.2), (3.3)) has a unique solution $u(t, \cdot; s, u_0)$, denoted by $\Phi_1(t, s)u_0$ (resp. $\Phi_2(t, s)u_0$, $\Phi_3(t, s)u_0$) with $u(s, x; s, u_0) = u_0(x)$.*

(2) For any $u_0 \in X_1$ (resp. $u_0 \in X_2$, $u_0 \in X_3$) and $s \in \mathbb{R}$, (1.7) (resp. (1.8), (1.9)), has a unique (local) solution $u_1(t, \cdot; s, u_0)$ (resp. $u_2(t, \cdot; s, u_0)$, $u_3(t, \cdot; s, u_0)$) with $u_1(s, x; s, u_0) = u_0(x)$ (resp. $u_2(s, x; s, u_0) = u_0(x)$, $u_3(s, x; s, u_0) = u_0(x)$).

Proof. (1) We prove the existence of a unique solution of the initial value problem associated to (3.1). The existence of unique solutions of the initial value problems associated to (3.2) and (3.3) can be proved similarly.

Assume $0 \le s < t \le T$. Define $K_1 : X_1 \to X_1$ and $A_1(t) : X_1 \to X_1$ by $(K_1 u)(x) = \nu_1[\int_D \kappa(y - x)u(y)dy - u(x)]$ and $(A_1(t)u)(x) = a_1(t, x)u(x)$. Then, K_1 and for every t, $A_1(t)$ are linear, bounded operators on X_1. Assume $A(t) := K_1 + A_1(t)$. Then, for every t, $0 \le t \le T$, $A(t)$ is a bounded linear operator on X_1. The function $t \to A(t)$ is continuous in the uniform operator topology. Then, by [[41], Chapter 5,Theorem 5.1], for every $u_0 \in X_1$, the initial value problem, $\frac{du(t)}{dt} = A(t)u(t), \quad 0 \le s < t \le T$ with $u(s) = u_0$ has a unique classical solution $u(t, \cdot; s, u_0)$.

(2) Write (1.7) as $u_t = A_1 u + g_1(t, x, u)$ where $A_1 u = \nu_1 \int_D \kappa(y - x)u(y)dy - u(x)$ and $g_1(t, x, u) = u(x)f_1(t, x, u)$. Then A_1 is bounded linear operator and hence generates a C_0 semigroup on X_1 and g_1 is continuous in t and Lipschitz continuos in u because of f_1. By [[41], Chapter 6, Theorem 1.4], for any $u_0 \in X_1$, (1.7) has a unique local solution $u_1(t, \cdot; s, u_0)$ with $u_1(s, \cdot; s, u_0) = u_0(\cdot)$.

The existence of unique solutions of the initial value problems associated to (1.7) and (1.9) can be proved analogously. □

Definition 3.2. *A continuous function $u(t, x)$ on $[0, \tau) \times \bar{D}$ is called a* super-solution *(or* sub-solution*) of (1.7) if for any $x \in \bar{D}$, u(t,x) is differentiable on $[0, \tau)$ and satisfies that for* *each* $x \in \bar{D}$,

$$\frac{\partial u}{\partial t} \geq (or \leq)\ \nu_1 \Big[\int_D \kappa(y-x)u(t,y)dy - u(t,x) \Big] + u(t,x)f_1(t,x,u)$$

for $t \in [0, \tau)$.

Super-solutions and sub-solutions of (1.8), (1.9), and (3.1)-(3.3) are defined in an analogous way.

Proposition 3.3 (Comparison principle)**.**

(1) If $u^1(t,x)$ *and* $u^2(t,x)$ *are sub-solution and super-solution of (3.1) ((resp. 3.2) , (3.3)) on* $[0,T)$*, respectively,* $u^1(0,\cdot) \leq u^2(0,\cdot)$*, and* $u^2(t,x) - u^1(t,x) \geq -\beta_0$ *for* $(t,x) \in [0,T) \times \bar{D}_i$ *and some* $\beta_0 > 0$*, then* $u^1(t,\cdot) \leq u^2(t,\cdot) \quad \text{for} \quad t \in [0,T)$.

(2) If $u^1(t,x)$ *and* $u^2(t,x)$ *are bounded sub- and super-solutions of (1.7) (resp. (1.8), (1.9)) on* $[0,T)$*, respectively, and* $u^1(0,\cdot) \leq u^2(0,\cdot)$*, then* $u^1(t,\cdot) \leq u^2(t,\cdot)$ *for* $t \in [0,T)$.

(3) For every $u_0 \in X_i^+$*,* $u_i(t,x;s,u_0)$ *exists for all* $t \geq s$.

Proof. (1) We prove the case that $u^1(t,x)$ and $u^2(t,x)$ are sub-solution and super-solution of (3.1). Other cases can be proved similarly.

Let $u^1(t,x)$ and $u^2(t,x)$ be sub-solution and super-solution of (3.1) respectively. Define $v(t,x) = e^{\alpha t}(u^2(t,x) - u^1(t,x))$ and $p_1 = \alpha - \nu_1 + a_1(t,x)$. Then v satisfies

$$\frac{\partial v}{\partial t} \geq \nu_1 \int_D \kappa(y-x)v(t,y)dy + p_1(t,x)v(t,x), \qquad x \in \overline{D}.$$

Choose $\alpha > 0$ so large enough that $p_1(t,x) \geq 0$ for $(t,x) \in (0,T) \times \overline{D}$. We need to prove $v(t,.) \geq 0$ for $t \in (0,T)$. It suffices to prove $v(t,.) \geq 0$ for $t \in (0,T_0)$ where $T_0 = min\{T, \frac{1}{k_0+p_0}\}$, $k_0 = max_{x\in\overline{D}} \int_D \kappa(y-x)dy$ and $p_0 = sup_{(t,x)\in(0,T)\times\overline{D}} p(t,x)$.

Suppose not. Then there exists $(t^0, x^0) \in (0,T_0) \times \overline{D}$ such that $v(t^0,x^0) < 0$. Let $v_{inf} = inf_{(t,x)\in(0,t^0]\times\overline{D}} v(t,x)$. Then $v_{inf} < 0$. Choose the sequence $(t_n, x_n) \in (0,t^0] \times \overline{D}$ such that $v(t_n, x_n) \to v_{inf}$ as $n \to \infty$. Then we have,

$$v(t_n, x_n) - v(0, x_n) \geq \int_0^{t_n} [\int_D \kappa(y - x_n) v(t, y) dy + p_1(t, x_n) v(t, x_n)].$$

This implies,

$$v(t_n, x_n) - v(0, x) \geq (k_0 + p_0) t_n v_{inf} \geq (k_0 + p_0) t^0 v_{inf}.$$

Letting $n \to \infty$, we get

$$v_{inf} \geq (k_0 + p_0) t^0 v_{inf} > v_{inf},$$

which is a contradiction.

(2) We prove the case that $u^1(t, x)$ and $u^2(t, x)$ are bounded sub- and super-solutions of (1.7). Other cases can be proved similarly.

Let $u^1(t, x)$ and $u^2(t, x)$ be bounded sub-solution and super-solution of (1.7) respectively. Define $v(t, x) = e^{\alpha t}(u^2(t, x) - u^1(t, x))$ and

$$p = \alpha - 1 + f_1(x, u^2(t, x)) + [u^1(t, x). \int_0^1 \frac{\partial f_1}{\partial u}(x, su^1(t, x) + (1 - s)u^2(t, x)) ds] v(t, x)$$

for $t \in [0, T)$. Then v satisfies,

$$\frac{\partial v}{\partial t} \geq \nu_1 \int_D \kappa(y - x) v(t, y) dy + p(t, x) v(t, x), \quad x \in \overline{D}.$$

By the boundedness of u^1 and u^2, there is $\alpha > 0$ such that $\inf_{t \in [0,T), x \in \overline{D}} \; p(t, x) > 0$. Proof of (2) then follows from the arguments in (1) with $p(x)$ and $p_0(x)$ being replaced by $p(t, x)$ and $sup_{(t,x) \in [0,T) \times \overline{D}} p(t, x)$ respectively.

(3) We prove the case that $i = 1$. Other cases can be proved similarly.

There is $L > 0$ such that $u_0(x) \leq L$ and $f_1(t, x, L) < 0$ for $x \in \overline{D}$. Let $u_L(t, x) \equiv L$ for $x \in \overline{D}$ and $t \in \mathbb{R}$. Then u_L is a super solution of (1.7) on $[0, \infty)$. Let $I(u_0) \subset \mathbb{R}$ be the maximal interval of existence of the solution $u_1(t,.;s,u_0)$ of (1.7) with $u_1(s,\cdot;s,u_0) = u_0(\cdot)$. Then by (2), $0 \le u_1(t,x;s,u_0) \le L$ for $x \in \overline{D}, t \in I(u_0) \cap [s,\infty)$. It then follows that $[s,\infty) \subset I(u_0)$, and hence $u_1(t,x;s,u_0)$ exists for all $t \ge s$. □

Proposition 3.4 (Strong monotonicity). *(1) If $u^1, u^2 \in X_i$, $u^1 \le u^2$ and $u^1 \ne u^2$, then $\Phi_i(t,s)u^1 \ll \Phi_i(t,s)u^2$ for all $t > s$.*

(2) If $u^1, u^2 \in X_i$, $u^1 \le u^2$ and $u^1 \ne u^2$, then $u_i(t,\cdot;s,u^1) \ll u_i(t,\cdot;s,u^2)$ for every $t > s$ at which both $u_i(t,\cdot;s,u^1)$ and $u_i(t,\cdot;s,u^2)$ exist.

Proof. (1) We prove the case $i = 1$. The cases $i = 2$ and $i = 3$ can be proved analogously. First we prove $\Phi_1(t,s)u_0 \gg 0$ if $u_0 \in X_1 \backslash \{0\}$. We claim that $e^{\nu_1 K_1 t}u_0 \gg 0$ for $t > 0$, where $(K_1 u)(s,x) = \int_D \kappa(y-x)u(s,y)dy$.

Note that

$$e^{\nu_1 K_1 t}u_0 = u_0 + \nu_1 t K_1 u_0 + \frac{(\nu_1 t K_1)^2}{2} u_0 + ...$$

Let $x_0 \in \bar{D}$ be such that $u_0(x_0) > 0$. Then there is $r > 0, \delta > 0$ such that $u_0(x_0) > 0$ for $x \in B(x_0, r) := \{y \in D \,|\|\, y - x_0 \,\|< r$, which implies that

$$(\nu_1 K_1 u_0)(x) = \int_D \kappa(y-x)u_0(y)dy > 0$$

for $x \in B(x_0, r+\delta)$. By induction $(\nu_1 K_1)^n u_0 > 0$ for $x \in B(x_0, r+n\delta), n \in \mathbb{N}$. Therefore, $e^{\nu_1 K_1 t}u_0 \gg 0$ for $t > 0$. Let $m > \nu_1 - \min_{x\in\bar{D}, t\in\mathbb{R}} a_1(t,x)$. Then,

$$\begin{aligned}
\Phi_1(t,s)u_0 &= e^{-m(t-s)}e^{\nu_1 K_1(t-s)}u_0 \\
&\quad + \int_s^t e^{-m(t-\tau)}e^{\nu_1 K_1(t-\tau)}(m - \nu_1 + a_1(\tau,\cdot))u_1(\tau,\cdot;s,u_0)d\tau \\
&\ge e^{-m(t-s)}e^{\nu_1 K_1(t-s)}u_0 \\
&\gg 0
\end{aligned}$$

for $t > s$ It then follows that $\Phi_1(t,s)u_0 \gg 0$ for all $t > s$. Now let $u_0 = u^2 - u^1$. Then $u_0 \in X_1^+\backslash\{0\}$. Hence $\Phi_1(t,s)u_0 \gg 0$ for $t > s$, which implies $\Phi_1(t,s)u^1 \ll \Phi_1(t,s)u^2$ for all $t > s$.

(2) We prove the case $i = 1$. Other cases can be proved analogously.

Let $v(t,x) = u_1(t,x;s,u^2) - u_1(t,x;s,u^1)$ for $t \geq s$ at which both $u_1(t,x;s,u^1)$ and $u_1(t,x;s,u^2)$ exist. Then $v(t,\cdot) \geq 0$ and $v(t,x)$ satisfies

$$\begin{aligned}\frac{\partial v}{\partial t} &= \nu_1 \int_D \kappa(y-x)v(t,y)dy - \nu_1 v(t,x) + f(x,u_1(t,x;u^2))v(t,x)\\ &\quad +[u_1(t,x,u^1).\int_0^1 f_u(x,su_1(t,x;u^1) + (1-s)u_1(t,x;u^2))ds]v(t,x),\end{aligned}$$

$x \in \bar{D}$ and $t \geq s$. Proof of (2) then follows from the arguments similar to those in proof of (1). □

Observe that when considering the spatial spreading and front propagation dynamics of (2.4), we need to consider (2.4) in X and also need to consider the following nonlocal linear evolution equation,

$$\frac{\partial u}{\partial t} = \int_{\mathbb{R}^N} e^{-\mu(y-x)\cdot\xi}\kappa(y-x)u(t,y)dy - u(t,x) + a(t,x)u(t,x), \quad x \in \mathbb{R}^N \tag{3.4}$$

where $\mu \in \mathbb{R}$, $\xi \in S^{N-1}$, and $a(t,\cdot) \in X_p$ and $a(t+T,x) = a(t,x)$. Note that if $\mu = 0$ and $a(t,x) = a_0(t,x)(:= f(t,x,0))$, (3.4) is the linearization of (2.4) at $u \equiv 0$.

Remark 3.5. *In space X_p, (3.4) share the same properties as (3.3).*

Throughout the rest of this section, we assume that $\xi \in S^{N-1}$ and $\mu \in \mathbb{R}$ are fixed, unless otherwise specified.

By the same arguments as in Proposition 3.1, for every $u_0 \in X$, (3.4) has a unique solution $u(t,\cdot;u_0,\xi,\mu,a) \in X$ with $u(0,x;u_0,\xi,\mu,a) = u_0(x)$. Put

$$\Phi(t;\xi,\mu,a)u_0 = u(t,\cdot;u_0,\xi,\mu,a). \tag{3.5}$$

Note that if $u_0 \in X_p$, then $\Phi(t;\xi,\mu,a)u_0 \in X_p$ for $t \geq 0$. Similarly, (2.4) has a unique (local) solution $u(t,x;u_0)$ with $u(0,x;u_0) = u_0(x)$ for every $u_0 \in X$. Also if $u_0 \in X_p$, then $u(t,x;u_0) \in X_p$ for t in the existence interval of the solution $u(t,x;u_0)$.

A continuous function $u(t,x)$ on $[0,T) \times \mathbb{R}^N$ is called a *super-solution* or *sub-solution* of (3.4) if $\frac{\partial u}{\partial t}$ exists and is continuous on $[0,T) \times \mathbb{R}^N$ and satisfies

$$\frac{\partial u}{\partial t} \geq \int_{\mathbb{R}^N} e^{-\mu(y-x)\cdot\xi} k(y-x)u(t,y)dy - u(t,x) + a(t,x)u(t,x), \quad x \in \mathbb{R}^N$$

or

$$\frac{\partial u}{\partial t} \leq \int_{\mathbb{R}^N} e^{-\mu(y-x)\cdot\xi)} k(y-x)u(t,y)dy - u(t,x) + a(t,x)u(t,x), \quad x \in \mathbb{R}^N$$

for $t \in (0,T)$.

For convenience, we would like to restate some comparison properties of solutions to (2.4) and (3.4) in the following.

Proposition 3.6 (Comparison principle)**.**

(1) If $u_1(t,x)$ and $u_2(t,x)$ are sub-solution and super-solution of (3.4) on $[0,T)$, respectively, $u_1(0,\cdot) \leq u_2(0,\cdot)$, and $u_2(t,x) - u_1(t,x) \geq -\beta_0$ for $(t,x) \in [0,T) \times \mathbb{R}^N$ and some $\beta_0 > 0$, then $u_1(t,\cdot) \leq u_2(t,\cdot) \quad \text{for} \quad t \in [0,T)$.

(2) Suppose that $u_1, u_2 \in X_p$ and $u_1 \leq u_2$, $u_1 \neq u_2$. Then $\Phi(t;\xi,\mu,a)u_1 \ll \Phi(t;\xi,\mu,a)u_2$ for all $t > 0$.

Proof. It follows from Propositions 3.3 and 3.4. □

For given $\rho \geq 0$, let

$$X(\rho) = \{u \in C(\mathbb{R}^N, \mathbb{R}) \mid \ x \mapsto e^{-\rho\|x\|}u(x) \ \in \ X\} \tag{3.6}$$

equipped with the norm $\|u\|_{X(\rho)} = \sup_{x\in\mathbb{R}^N} e^{-\rho\|x\|}|u(x)|$.

Remark 3.7. *For every $u_0 \in X(\rho)$ $(\rho \geq 0)$,the equation (3.4) has a unique solution $u(t,\cdot;u_0,\xi,\mu) \in X(\rho)$ with $u(0,x;u_0,\xi,\mu) = u_0(x)$. Moreover, Proposition 3.6 holds for such solutions of (3.4).*

Proposition 3.8 (Comparison principle)**.**

(1) If $u_1(t,x)$ and $u_2(t,x)$ are bounded sub- and super-solutions of (2.4) on $[0,T)$, respectively, and $u_1(0,\cdot) \leq u_2(0,\cdot)$, then $u_1(t,\cdot) \leq u_2(t,\cdot)$ for $t \in [0,T)$.

(2) If $u_1, u_2 \in X_p$ with $u_1 \leq u_2$ and $u_1 \neq u_2$, then $u(t,\cdot;u_1) \ll u(t,\cdot;u_2)$ for every $t > 0$ at which both $u(t,\cdot;u_1)$ and $u(t,\cdot;u_2)$ exist.

(3) For every $u_0 \in X^+$, $u(t,x;u_0)$ exists for all $t \geq 0$.

Proof. If follows from the arguments in Propositions 3.3 and 3.4. □

Remark 3.9. *Let*

$$\tilde{X} = \{u : \mathbb{R}^N \to \mathbb{R} \,|\, u \text{ is Lebesgue measurable and bounded}\}$$

equipped with the norm $\|u\| = \sup_{x\in\mathbb{R}^N} |u(x)|$, and

$$\tilde{X}^+ = \{u \in \tilde{X} \,|\, u(x) \geq 0 \ \forall \ x \in \mathbb{R}^N\}.$$

By general semigroup theory, for any $u_0 \in X$, (2.4) has also a unique (local) solution $u(t,\cdot;u_0) \in \tilde{X}$ with $u(0,x;u_0) = u_0(x)$. Similarly, we can define measurable sub- and super-solutions of (2.4). Proposition 3.8 (1) and (3) also hold for bounded measurable sub-, super-solutions and solutions.

3.2 Basic properties of principal eigenvalues and principal spectrum points of nonlocal dispersal operators

Let $K_i : \mathcal{X}_i \to \mathcal{X}_i$ and $H_i : \mathcal{D}(H_i) \subset \mathcal{X}_i \to \mathcal{X}_i$ be defined as follows,

$$(K_1 u)(s,x) = (K_2 u)(s,x) = \int_D \kappa(y-x)u(s,y)dy,$$

$$(K_3 u)(s,x) = \int_{\mathbb{R}^N} \kappa(y-x)u(s,y)dy,$$

$$(H_1(a_1)u)(s,x) = -u_s - \nu_1 u(s,x) + a_1(s,x)u(s,x),$$

$$(H_2(a_2)u)(s,x) = -u_s - \nu_2 \int_D \kappa(y-x)dy u(s,x) + a_2(s,x)u(s,x),$$

and

$$(H_3(a_3)u)(s,x) = -u_s - \nu_3 u(s,x) + a_3(s,x)u(s,x).$$

Then,

$$L_i(a_i)u = (\nu_i K_i + H_i(a_i))u, \quad i = 1,2,3.$$

We denote I as an identity map from $\mathcal{X}_i$ to $\mathcal{X}_i$ and may write $\alpha I u$ as αu and $\alpha I - H_i(a_i)$ as $\alpha - H_i(a_i)$, etc.. If no confusion occurs, we may write $L_i(a_i)$ and $H_i(a_i)$ as L_i and H_i, respectively.

Observe that if $\alpha \in \mathbb{R}$ is such that $(\alpha - H_i)^{-1}$ exists, then

$$(\nu_i K_i + H_i)u = \alpha u$$

has nontrivial solutions in $\mathcal{X}_i$ is equivalent to

$$\nu_i K_i(\alpha - H_i)^{-1} v = v$$

has nontrivial solutions in $\mathcal{X}_i$. Moreover, it can be claimed that α is an eigenvalue of $L_i(a_i)$ if and only if 1 is an eigenvalue of $\nu_i K_i(\alpha - H_i)^{-1}$.

In fact, if α is an eigenvalue of $L_i(a_i)$, then there exists nonzero v such that $L_i(a_i)v = (\nu_i K_i + H_i(a_i))v = \alpha v$. There exists nonzero u such that $v = (\alpha - H_i)^{-1}u$. This implies, $\nu_i K_i(\alpha - H_i)^{-1}u = u$, showing that 1 is an eigenvalue of $\nu_i K_i(\alpha - H_i)^{-1}$. Conversely, if 1 is an eigenvalue of $\nu_i K_i(\alpha - H_i)^{-1}$, then there exists nonzero w such that $\nu_i K_i(\alpha - H_i)^{-1}w = w$. Let $v = (\alpha - H_i)^{-1}w$. Then $(\nu_i K_i)v = (\alpha - H_i)v$, which implies $L_i(a_i)v = (\nu_i K_i + H_i(a_i))v = \alpha v$, showing that α is an eigenvalue of $L_i(a_i)$.

Lemma 3.10. *Let $\{u_n\}$ be any bounded sequence in $\mathcal{X}_1$. Then for any $\alpha > \max_{x\in\bar{D}}(b_1(x) + \hat{a}_1(x))$, $\int_{-\infty}^t exp(\int_s^t(-\nu_1 + a_1(\tau, y) - \alpha)u_n(\tau, y)d\tau)ds$* is bounded.

Proof. First of all, it is clear that for any $y \in \bar{D}$ and $t \in \mathbb{R}$, $\int_{-\infty}^t exp(\int_s^t(-\nu_1 + a_1(\tau, y) - \alpha)u_n(\tau, y)d\tau)ds$ exists. Suppose that $\|u_n\| \leq M$ for all $n \geq 1$. Then

$$|\int_{-\infty}^t exp(\int_s^t(-\nu_1 + a_1(\tau, y) - \alpha)u_n(\tau, y)d\tau)ds| \leq M\int_{-\infty}^t exp(\int_s^t(-\nu_1 + a_1(\tau, y) - \alpha)d\tau)ds$$

To prove the boundedness of $\int_{-\infty}^t exp(\int_s^t(-\nu_1 + a_1(\tau, y) - \alpha)u_n(\tau, y)d\tau)ds$, let $f(t, y) = \int_{-\infty}^t exp(\int_s^t(-\nu_1 + a_1(\tau, y) - \alpha)d\tau)ds$. Then,

$$\begin{aligned} f(t+T, y) &= \int_{-\infty}^{t+T} exp(\int_s^{t+T}(\nu_1 + a_1(\tau, y) - \alpha)d\tau)ds \\ &= \int_{-\infty}^{t} exp(\int_{s+T}^{t+T}(-\nu_1 + a_1(\tau, y) - \alpha)d\tau)ds \\ &= \int_{-\infty}^{t} exp(\int_s^{t}(-\nu_1 + a_1(\tau, y) - \alpha)d\tau)ds \\ &= f(t, y). \end{aligned}$$

Note that $f(t, y)$ is continuous and being the continuous periodic function, it is bounded. This implies that $\int_{-\infty}^t exp(\int_s^t(-\nu_1 + a_1(\tau, y) - \alpha)u_n(\tau, y)d\tau)ds$ is bounded. □

Proposition 3.11. *Let $1 \le i \le 3$ be given. H_i generates a positive semigroup of contractions on $\mathcal{X}_i$ and for any $\alpha > \max_{x\in\bar{D}}(b_i(x) + \hat{a}_i(x))$, $\nu_i K_i(\alpha - H_i)^{-1}$ is a compact operator on $\mathcal{X}_i$.*

Proof. We will prove that H_1 generates a positive semigroup $\phi_1(s)$ of contraction on $\mathcal{X}_1$. The remaining cases can be proved similarly. Define $\phi_1(s) : \mathcal{X}_1 \to \mathcal{X}_1$ by

$$(\phi_1(s)u)(t,x) = e^{\int_{t-s}^{t} h_1(\tau,x)d\tau}u(t-s,x)$$

where $h_1(t,x) = a_1(t,x) - \nu_1$. Then, we claim the following

Claim 1. $\phi_1(s_1 + s_2) = \phi_1(s_1)\phi_1(s_2)$.

Claim 2. $\phi_1(0) = I$.

Proof of claim 1. Note that,

$$\begin{aligned}
& (\phi_1(s_1)\phi_1(s_2)u)(t,x) \\
= \;& \phi_1(s_1)e^{\int_{t-s_2}^{t} h_1(\tau,x)d\tau}u(t-s_2,x) \\
= \;& e^{\int_{t-s_1}^{t} h_1(\tau,x)d\tau}w(t-s_1,x)[where \;\; w(t,x) = e^{\int_{t-s_2}^{t} h_1(\tau,x)d\tau}u(t-s_2,x)] \\
= \;& e^{\int_{t-s_1}^{t} h_1(\tau,x)d\tau}e^{\int_{t-s_1-s_2}^{t-s_1} h_1(\tau,x)d\tau}u(t-s_1-s_2,x) \\
= \;& e^{\int_{t-s_1-s_2}^{t} h_1(\tau,x)d\tau}u(t-s_1-s_2,x) \\
= \;& (\phi_1(s_1+s_2)u)(t,x).
\end{aligned}$$

Proof of claim 2. Note that,

$$\begin{aligned}
(\phi_1(0)u)(t,x) &= e^{\int_{t}^{t} h_1(\tau,x)d\tau}u(t,x) \\
&= u(t,x).
\end{aligned}$$

Now, let $U(s,t,x;u)$ be solution of

$$U_s = -U_t + h_1(t,x)U$$

with initial condition $U(0,t,x;u) = u(t,x)$.

Then by direct computation

$$U(s,t,x;u) = (\phi_1(s)u)(t,x)$$

and

$$\{u \in \mathcal{X}_1 | \lim_{s \to 0^+} \frac{\phi_1(s)u - u}{s} exists\} = D(H_1).$$

Also, from the definition of $\phi_1(s)$, positivity is obvious. Moreover, $\|\phi_1(s)u\| \leq \|u\|, (s \geq 0)$. Thus, H_1 generates positive semigroup of contraction $\phi_1(s)$ on $\mathcal{X}_1$.

Next, we prove that for any $\alpha > max_{x \in \bar{D}}(b_1(x) + \hat{a}_1(x)),\ \nu_1 K_1(\alpha - H_1)^{-1}$ is a compact operator on $\mathcal{X}_1$. The other cases $(i = 2, i = 3)$ can be proved analogously.

Note that,

$$\begin{aligned} & \nu_1 K_1(\alpha - H_1)^{-1} u(t,x) \\ = \ & \nu_1 \int_D \kappa(y-x)(\alpha - H_1)^{-1} u(t,y) dy \\ = \ & \nu_1 \int_D \{\kappa(y-x) \int_{-\infty}^{t} exp(\int_s^t (-\nu_1 + a_1(\tau,y) - \alpha) u(\tau,y) d\tau) ds\} dy. \end{aligned}$$

To show the compactness, let $\{u_n\}$ be any bounded sequence in $\mathcal{X}_1$ and let $v_n = \nu_1 K_1(\alpha - H_1)^{-1} u_n$. By the smoothness property of $\kappa(y-x)$ and Lemma 3.10,

$$\begin{aligned} & |v_n(t,x_1) - v_n(t,x_2)| \\ = \ & |\nu_1 K_1(\alpha - H_1)^{-1} u_n(t,x_1) - \nu_1 K_1(\alpha - H_1)^{-1} u_n(t,x_2)| \\ = \ & \nu_1 | \int_D [\kappa(y-x_1) - \kappa(y-x_2)](\alpha - H_1)^{-1} u_n(t,y) dy | \\ \leq \ & \nu_1 \int_D |\{\kappa(y-x_1) - \kappa(y-x_2)\}| \int_{-\infty}^{t} exp(\int_s^t (\nu_1 + a_1(\tau,y) - \alpha) u_n(\tau,y) d\tau) ds] dy \\ \leq \ & M(x_2 - x_1). \end{aligned}$$

Then for every $\epsilon > 0$ there is $\delta > 0$ such that if $|x_1 - x_2| < \delta$, $|v_n(t, x_1) - v_n(t, x_2)| < \epsilon$. Clearly, for every $\epsilon > 0$, there is also $\delta > 0$ such that if $|t_1 - t_2| < \delta$, then $|v_n(t_2, x) - v_n(t_1, x)| < \epsilon$. Therefore, $\{v_n\}$ is equicontinuous. The compactness of $\nu_i K_i(\alpha - H_i)^{-1}$ then follows by using Arzela Ascoli theorem. □

Put

$$\Phi_i(T; a_i) = \Phi_i(T, 0), \quad i = 1, 2, 3,$$

and let $r(\Phi_i(T; a_i))$ be the spectral radius of $\Phi_i(T; a_i)$.

Proposition 3.12. *For give* $1 \le i \le 3$,

$$\frac{\ln r(\Phi_i(T; a_i))}{T} = \limsup_{t-s\to\infty} \frac{\ln \|\Phi_i(t, s)\|}{t - s}.$$

Proof. First, by $(\Phi_i(T; a_i))^n = \Phi_i(nT, 0)$. it is clear that

$$\frac{\ln r(\Phi_i(T; a_i))}{T} = \frac{\ln \left\{ \lim_{n\to\infty} \left(\|(\Phi_i(T; a_i))^n\| \right)^{1/n} \right\}}{T} \le \limsup_{t-s\to\infty} \frac{\ln \|\Phi_i(t, s)\|}{t - s}.$$

Next, for any $\epsilon > 0$, there is $K \ge 1$ such that

$$\|(\Phi_i(T; a_i))^n\| = \|\Phi_i(nT, 0)\| \le (r(\Phi_i(T; a_i)) + \epsilon)^n \quad \forall\ n \ge K.$$

Note that there is $M > 0$ such that

$$\|\Phi_i(t, s)\| \le M \quad \forall t > s,\ t - s < 1.$$

For any $s < t$ with $t - s \ge (K + 2)T$, let $n_1, n_2 \in \mathbb{Z}$ be such that $0 \le s - n_1 T < T$ and $0 \le t - n_2 T < T$. Then

$$n_2 - n_1 \ge K$$

and

$$
\begin{aligned}
\|\Phi_i(t,s)\| &= \|\Phi_i(t,n_2T)\circ\Phi_i(n_2T,n_1T)\circ\Phi_i(n_1T,s)\| \\
&\le \|\Phi_i(t,n_2T)\|\cdot\|\Phi_i((n_2-n_1)T,0)\|\cdot\|\Phi_i(n_1T,s)\| \\
&\le M^2(r(\Phi_i(T;a_i))+\epsilon)^{n_2-n_1}.
\end{aligned}
$$

This implies that

$$
\frac{\ln\|\Phi_i(t,s)\|}{t-s}\le\frac{\ln M^2+(n_2-n_1)\ln(r(\Phi_i(T;a_i))+\epsilon)}{(n_2-n_1)T}
$$

and hence

$$
\limsup_{t-s\to\infty}\frac{\ln\|\Phi_i(t,s)\|}{t-s}\le\frac{\ln(r(\Phi_i(T;a_i))+\epsilon)}{T}.
$$

Now making $\epsilon\to 0$, we have

$$
\limsup_{t-s\to\infty}\frac{\ln\|\Phi_i(t,s)\|}{t-s}\le\frac{\ln r(\Phi_i(T;a_i))}{T}.
$$

□

Let

$$
\lambda_i(x)=b_i(x)+\hat{a}_i(x) \tag{3.7}
$$

for $i=1,2,3$.

Proposition 3.13. *Let $1\le i\le 3$ be given. Then $[\min_{x\in\bar{D}}\lambda_i(x),\max_{x\in\bar{D}}\lambda_i(x)]\subset\sigma(H_i)$ and for any $\alpha\in\mathbb{R}\setminus[\min_{x\in\bar{D}}\lambda_i(x),\max_{x\in\bar{D}}\lambda_i(x)]$, $(\alpha-H_i)^{-1}$ exists.*

Proof. It follows from the arguments in [28, Lemma 3.7]. However, for the reader's convenience, we provide a proof in the following.

Fix any $x_0\in\bar{D}_i$. By Floquet theory for time periodic ordinary differential equations, the equation

$$\dot{\phi} = b_i(x_0)\phi + a_i(t,x_0)\phi - \lambda_i(x_0)\phi \tag{3.8}$$

has a nontivial solution $\phi^*(t)$ with $\phi^*(t+T) = \phi^*(t)$. Similarly, the equation

$$\dot{\psi} = -b_i(x_0)\psi - a_i(t,x_0)\psi + \lambda_i(x_0)\psi \tag{3.9}$$

has a nontivial solution $\psi^*(t)$ with $\psi^*(t+T) = \psi^*(t)$. Assume that $\lambda_i(x_0) \in \rho(H_i)$. Then for any $v \in \mathcal{X}_i$ with $v(t,x) \equiv v(t)$, there is a unique $u(\cdot,\cdot;v) \in \mathcal{X}_i$ such that

$$u_t(t,x;v) = b_i(x)u(t,x;v) + a_i(t,x)u(t,x;v) - \lambda_i(x_0)u(t,x;v) + v(t) \tag{3.10}$$

This implies that

$$u_t(t,x_0;\psi^*) = b_i(x_0)u(t,x_0;\psi^*) + a_i(t,x_0)u(t,x_0;\psi^*) - \lambda_i(x_0)u(t,x_0;\psi^*) + \psi^*(t). \tag{3.11}$$

Put

$$\tilde{\phi}^*(t) = u(t,x_0;\psi^*).$$

Then,

$$\begin{aligned}
\int_0^T \psi^*(t)\psi^*(t)dt &= \int_0^T [\frac{d\tilde{\phi}^*(t)}{dt} - b_i(x_0)\tilde{\phi}^*(t) - a_i(t,x_0)\tilde{\phi}^*(t) + \lambda_i(x_0)\tilde{\phi}^*(t)]\psi^*(t)dt \\
&= \int_0^T [-\frac{d\psi^*(t)}{dt} - b_i(x_0)\psi^*(t) - a_i(t,x_0)\psi^*(t) + \lambda_i(x_0)\psi^*(t)]\tilde{\phi}^*(t)dt \\
&= 0,
\end{aligned}$$

which is a contradiction. Therefore $\lambda_i(x_0) \in \sigma(H_i)$ and the proposition follows. □

Let

$$\lambda_{i,\max} = \max_{x\in D_i} \lambda_i(x), \quad \lambda_{i,\min} = \min_{x\in D_i} \lambda_i(x)$$

Proposition 3.14. *Let $1 \le i \le 3$ be given. For any $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha > \lambda_{i,\max}$, $(\alpha - H_i)^{-1}$ exists. Moreover,*

$$\Big((\alpha - H_i)^{-1} v\Big)(t,x) \ge \frac{M}{\alpha - \lambda_i(x)} v(x)$$

for any $\lambda_{i,\max} < \alpha \le \lambda_{i,\max} + 1$ and any $v \in \mathcal{X}_i^+$ with $v(t,x) \equiv v(x)$, where

$$M = \inf_{s \le t \le s+T, s,t\in\mathbb{R}} \exp\Big(\int_s^t (\min_{x\in D_i}(b_i(x) + a_i(\tau,x)) - \lambda_{i,\max} - 1) d\tau\Big).$$

Proof. First of all, by Floquet theory for periodic ordinary differential equations, for any $\alpha \in C$ with $\operatorname{Re}\alpha > \lambda_{i,\max}$, $(\alpha - H_i)^{-1}$ exists. Moreover, for any $v \in \mathcal{X}_i \oplus i\mathcal{X}_i$, we have

$$\Big((\alpha - H_i)^{-1} v\Big)(t,x) = \int_{-\infty}^t \exp\Big(\int_s^t (b_i(x) + a_i(\tau,x) - \alpha) v(\tau,x) d\tau\Big) ds.$$

Hence for any $v \in \mathcal{X}_i$ with $v(t,x) \equiv v(x)$, we have

$$\Big((\alpha - H_i)^{-1} v\Big)(t,x) = \Big\{\int_{-\infty}^t \exp\Big(\int_s^t (b_i(x) + a_i(\tau,x) - \alpha) d\tau\Big) ds\Big\} v(x).$$

If $\lambda_{i,\max} < \alpha \le \lambda_{i,\max} + 1$, then

$$\int_{-\infty}^t \exp\Big(\int_s^t (b_i(x) + a_i(\tau,x) - \alpha) d\tau\Big) ds \ge \frac{M}{\alpha - \lambda_i(x)},$$

where

$$M = \inf_{s \le t \le s+T, s,t\in R} \exp\Big(\int_s^t (\min_{x\in D_i}(b_i(x) + a_i(\tau,x)) - \lambda_{i,\max} - 1) d\tau\Big)$$

(see the arguments of [28, Lemma 3.6]). It then follows that for any $\lambda_{i,\max} < \alpha \le \lambda_{i,\max} + 1$ and $v \in \mathcal{X}_i^+$ with $v(t,x) \equiv v(x)$,

$$\Big((\alpha - H_i)^{-1} v\Big)(t,x) \ge \frac{M}{\alpha - \lambda_i(x)} v(x).$$

The proposition is thus proved.

□

Proposition 3.15. *For given $1 \le i \le 3$, $s(L_i, a_i) > \max_{x \in \bar{D}} \lambda_i(x)$ iff there is $\alpha > s(L_i, a_i)$ such that $r(\nu_i K_i (\alpha - H_i)^{-1}) > 1$.*

Proof. By Propositions 3.13,

$$\lambda_{i,\max} = \sup \sigma(H_i).$$

By Proposition 3.11, $\nu_i K_i (\alpha - H_i)^{-1}$ is a compact operator for any $\alpha \in C$ with $\operatorname{Re}\alpha > \lambda_{i,\max}$. It then follows from [6, Theorem 2.2] that $s(L_i, a_i) > \lambda_{i,\max}$ iff there is $\alpha > \lambda_{i,\max}$ such that $r(\nu_i K_i (\alpha - H_i)^{-1}) > 1$. □

Proposition 3.16. *For given $1 \le i \le 3$, if there is $\alpha_0 > \max_{x \in \bar{D}_i} \lambda_i(x)$ such that $r(\nu_i K_i (\alpha_0 - H)^{-1}) > 1$, then there is $\alpha_i > \alpha_0 (> \max_{x \in \bar{D}} \lambda_i(x))$ such that $r(\nu_i K_i (\alpha_i - H)^{-1}) = 1$ and α_i is an isolated eigenvalue of $\nu_i K_i + H_i$ of finite multiplicity with a positive eigenfunction.*

Proof. Suppose that there is $\alpha_0 > \lambda_{i,\max}$ such that $r(\nu_i K_i (\alpha_0 - H)^{-1}) > 1$. Then by Proposition 3.15, $s(L_i, a_i) > \lambda_{i,\max}$. Moreover, by [6, Theorem 2.2], $r(\nu_i K_i (s(L_i, a_i) - H)^{-1}) = 1$, and $s(L_i, a_i)$ is an isolated eigenvalue of $\nu_i K_i + H_i$ of finite multiplicity with a positive eigenfunction. □

Proposition 3.17. *For given $1 \le i \le 3$, if $\lambda \in \mathbb{R}$ is an eigenvalue of $L_i(a_i)$ with a positive eigenfunction, then it is geometric simple.*

Proof. Suppose that $\phi(t,x)$ is a positive eigenfunction of L_i associated with λ. By Proposition 3.3, $\phi(t,x) > 0$ for $t \in \mathbb{R}$ and $x \in \bar{D}_i$. Assume that $\psi(t,x)$ is also an eigenfunction of L_i associated with λ. Then there is $a \in \mathbb{R}$ such that $w(t,x) = \phi(t,x) - a\psi(t,x)$ satisfies

$$w(t,x) \geq 0 \quad \forall t \in \mathbb{R},\ x \in \bar{D}_i \ \text{ and } \ w(t_0,x_0) = 0$$

for some $t_0 \in \mathbb{R}$ and $x_0 \in \bar{D}_i$. By Proposition 3.3 again, $w(t,x) \equiv 0$ and then $\phi(t,x) = a\psi(t,x)$. This implies that λ is geometric simple. □

Proposition 3.18. *For* $1 \leq i \leq 3$, $s(L_i,a_i) = \frac{\ln r(\Phi_i(T;a_i))}{T}$.

Proof.

$$s(L_i,a_i) = \limsup_{t-s\to\infty} \frac{\ln\|\Phi_i(t,s;a_i)\|}{t-s}.$$

By Proposition 3.12,

$$\limsup_{t-s\to\infty} \frac{\ln\|\Phi_i(t,s;a_i)\|}{t-s} = \frac{\ln r(\Phi_i(T;a_i))}{T}.$$

The proposition thus follows. □

Proposition 3.19. *For* $1 \leq i \leq 3$, *if* $a_i^n \in \mathcal{X}_i$ *and* $a_i^n \to a_i$ *in* $\mathcal{X}_i$ *as* $n \to \infty$, *then* $s(L_1,a_i^n) \to s(L_i,a_i)$ *as* $n \to \infty$.

Proof. We prove the case $i = 1$. The remaining cases can be proved similarly.

By Propositions 3.18, $s(L_1,a_1) = \limsup_{t-s\to\infty} \frac{ln\|\phi_1(t,s)\|}{t-s}$.

First, for given a_1^1 and a_1^2 with $a_1^1 \leq a_1^2$, let $\phi^i(t,s), i = 1,2$, be the evolution operators generated by (3.1) with $a_1(t,x)$ replaced by $a_1^i(t,x), i = 1,2$ respectively. We claim that

$$\|\phi^1(t,s)\| \leq \|\phi^2(t,s)\|.$$

In fact, for any given $u_0 \in X_1$ with $u_0 \geq 0$, by Proposition 3.3 , $\phi^i(t,s)u_0 \geq 0$ for $i = 1,2$ and $s \leq t$. Assume, $v(t,x) = \phi^2(t,s)u_0 - \phi^1(t,s)u_0$. Then v satisfies,

$$
\begin{aligned}
v_t &= \nu_1 \int_D \kappa(y-x)v(t,y)dy - \nu_1 v(t,x) + a_1^2(t,x)v(t,x) + (a_1^2 - a_1^1)\phi^1(t,s)u_0 \\
&\geq \nu_1 \int_D \kappa(y-x)v(t,y)dy - \nu_1 v(t,x) + a_1^2(t,x)v(t,x).
\end{aligned}
$$

By Proposition 3.3, $v(t,x) \geq 0$ and claim is thus proved.

Next, let $\phi^{\pm\epsilon}(t,s)$ be the evolution operators generated by (3.1) with $a_1(t,x)$ being replaced by $a_1(t,x) \pm \epsilon$. Then we have $\phi^{\pm\epsilon}(t,s) = e^{\pm\epsilon(t-s)}\phi(t,s)$. Therefore,

$$
s(L_1, a_1 \pm \epsilon) = s(L_1, a_1) \pm \epsilon.
$$

By the first and next arguments it follows that $s(L_1, a_1^n) \to s(L_1, a_1)$ as $n \to \infty$ whenever $a_1^n \to a_1$ as $n \to \infty$.

□

Chapter 4

Principal Eigenvalue and Principal Spectrum Point Theory

This chapter contains two sections. In the first section, we investigate the existence and lower bounds of principal eigenvalues of nonlocal dispersal operators with time periodic dependence and prove Theorems A-C. Most results in this section have been published (see [42]). In the sequel section, we explore some other important properties about principal spectrum point and principal eigenvalues of nonlocal dispersal operators. Most results in this section are submitted for publication (see [43]).

4.1 Proofs of Theorems A-C

First of all, we prove an important technical lemma, which will also be used in next section.

Lemma 4.1. *For any $a_i \in \mathcal{X}_i$ and any $\epsilon > 0$, there is $a_{i,\epsilon} \in \mathcal{X}_i$ satisfying that*

$$\|a_i - a_{i,\epsilon}\|_{\mathcal{X}_i} < \epsilon,$$

$b_i + a_{i,\epsilon}$ is C^N, $b_i + \hat{a}_{i,\epsilon}$ attains its maximum at some point $x_0 \in \mathrm{Int}(D_i)$, and the partial derivatives of $b_i + \hat{a}_{i,\epsilon}$ up to order $N-1$ at x_0 are zero.

Proof. We prove the case $i = 1$ or 2. The case $i = 3$ can be proved similarly (it is simpler). First, let $\tilde{x}_0 \in \bar{D}_i$ be such that

$$\lambda_i(\tilde{x}_0) = \max_{x \in \bar{D}} \lambda_i(x).$$

For any $\epsilon > 0$, there is $\tilde{x}_\epsilon \in \mathrm{Int}(D_i)$ such that

$$\lambda_i(\tilde{x}_0) - \lambda(\tilde{x}_\epsilon) < \epsilon. \tag{4.1}$$

Let $\tilde{\sigma} > 0$ be such that

$$B(\tilde{x}_\epsilon, \tilde{\sigma}) \subset D_i,$$

where $B(\tilde{x}_\epsilon, \tilde{\sigma})$ denotes the open ball with center $\tilde{x}_\epsilon$ and radius $\tilde{\sigma}$.

Note that there is $\tilde{h}_i \in C(\bar{D}_i)$ such that $0 \leq \tilde{h}_i(x) \leq 1$, $\tilde{h}_i(\tilde{x}_\epsilon) = 1$, and $\text{supp}(\tilde{h}_i) \subset B(\tilde{x}_\epsilon, \tilde{\sigma})$. Let

$$\tilde{a}_{i,\epsilon}(t, x) = a_i(t, x) + \epsilon \tilde{h}_i(x)$$

and

$$\tilde{\lambda}_{i,\epsilon}(x) = b_i(x) + \hat{a}_i(x) + \epsilon \tilde{h}_i(x).$$

Then $\tilde{a}_{i,\epsilon}$ and $\tilde{\lambda}_{i,\epsilon}$ are continuous on $\bar{D}_i$,

$$\|\tilde{a}_{i,\epsilon} - a_i\| \leq \epsilon \tag{4.2}$$

and $\tilde{\lambda}_{i,\epsilon}$ attains its maximum in $\text{Int}(D_i)$.

Let $\tilde{D}_i \subset \mathbb{R}^N$ be such that $D_i \subset \tilde{D}_i$. Note that $\tilde{\lambda}_{i,\epsilon}$ can be continuously extended to $\tilde{D}_i$. Without loss of generality, we may then assume that $\tilde{\lambda}_{i,\epsilon}$ is a continuous function on $\tilde{D}_i$ and assume that there is $x_0 \in \text{Int}(D_i)$ such that $\tilde{\lambda}_{i,\epsilon}(x_0) = \sup_{x \in \tilde{D}_i} \tilde{\lambda}_{i,\epsilon}(x)$. Observe that there is $\sigma > 0$ and $\bar{\lambda}_{i,\epsilon} \in C(\tilde{D}_i)$ such that $B(x_0, \sigma) \subset D_i$,

$$0 \leq \bar{\lambda}_{i,\epsilon}(x) - \tilde{\lambda}_{i,\epsilon}(x) \leq \epsilon \quad \forall \quad x \in \tilde{D}_i, \tag{4.3}$$

and

$$\bar{\lambda}_{i,\epsilon}(x) = \tilde{\lambda}_{i,\epsilon}(x_0) \quad \forall \quad x \in B(x_0, \sigma).$$

Let

$$\eta(x) = \begin{cases} C \exp(\frac{1}{\|x\|^2 - 1}) & \text{if} \quad \|x\| < 1 \\ \\ 0 \quad \text{if} \quad \|x\| \geq 1, \end{cases}$$

where $C > 0$ is such that $\int_{\mathbb{R}^N} \eta(x)dx = 1$. For given $\delta > 0$, set

$$\eta_\delta(x) = \frac{1}{\delta^N}\eta(\frac{x}{\epsilon}).$$

Let

$$\lambda_{i,\epsilon,\delta}(x) = \int_{\tilde{D}_i} \eta_\delta(y-x)\bar{\lambda}_{i,\epsilon}(y)dy.$$

By [15, Theorem 6, Appendix C], $\bar{\lambda}_{i,\epsilon,\delta}$ is in $C^\infty(\tilde{D}_i)$ and when $0 < \delta \ll 1$,

$$|\lambda_{i,\epsilon,\delta}(x) - \bar{\lambda}_{i,\epsilon}(x)| < \epsilon \quad \forall \quad x \in \bar{D}_i.$$

It is not difficult to see that for $0 < \delta \ll 1$,

$$\bar{\lambda}_{i,\epsilon,\delta}(x) \le \bar{\lambda}_{i,\epsilon}(x_0) \quad \forall x \in B(x_0, \sigma)$$

and

$$\bar{\lambda}_{i,\epsilon,\delta}(x) = \bar{\lambda}_{i,\epsilon}(x_0) \quad \forall x \in B(x_0, \sigma/2).$$

Fix $0 < \delta \ll 1$, and let

$$\lambda_{i,\epsilon}(x) = \lambda_{i,\epsilon,\delta}(x).$$

Then $\lambda_{i,\epsilon}$ attains its maximum at some $x_0 \in \mathrm{Int}(D_i)$, and the partial derivatives of $\lambda_{i,\epsilon}$ up to order $N-1$ at x_0 are zero, Let

$$a_{i,\epsilon} = \tilde{a}_{i,\epsilon} + \lambda_{i,\epsilon} - \tilde{\lambda}_{i,\epsilon}.$$

Then $a_{i,\epsilon}$ is $C^N(\bar{D}_i)$,

$$\|a_i - a_{i,\epsilon}\| \le \|a_i - \tilde{a}_{i,\epsilon}\| + \|\lambda_{i,\epsilon} - \tilde{\lambda}_{i,\epsilon}\| < 2\epsilon$$

and

$$b_i(x) + \hat{a}_{i,\epsilon}(x) = \lambda_{i,\epsilon}(x).$$

Therefore, $b_i + \hat{a}_{i,\epsilon}$ attains its maximum at some point $x_0 \in \mathrm{Int}(D)$, and the partial derivatives of $b_i + \hat{a}_{i,\epsilon}$ up to order $N-1$ at x_0 are zero. The lemma is thus proved. □

Next, we recall some results proved in [49] and [52].

Lemma 4.2. *If*

$$\max_{x\in\bar{D}_i} \hat{a}_i(x) - \min_{x\in\bar{D}_i} \hat{a}_i(x) < \nu_i \inf_{x\in\bar{D}_i} \int_{D_i} \kappa(y-x)dy$$

in the case $i = 1, 2$ *and*

$$\max_{x\in\bar{D}_i} \hat{a}_i(x) - \min_{x\in\bar{D}_i} \hat{a}_i(x) < \nu_i$$

in the case $i = 3$*, then* $s(L_i, \hat{a}_i) > \max_{x\in\bar{D}_i} \lambda_i(x)$ $(1 \le i \le 3)$.

Proof. See [49] in the case $i = 3$ and [52] in the case $i = 2, 3$.

□

Proof of Theorem A. We prove the case $i = 1$. The other cases can be proved similarly.

First, we assume that $s(L_1, a_1)$ is an isolated eigenvalue of L_1 with a positive eigenfunction $\phi(t,x)$. Let $u(t,x) = e^{s(L_1,a_1)t}\phi(t,x)$. Then $u(t,x)$ is the solution of (3.1) with $u(0,\cdot) = \phi(0,\cdot) \in X_1^+$. By Proposition 3.3, we must have $\phi(t,x) > 0$ for $t \in \mathbb{R}$ and $x \in \bar{D}$. Then

$$-\frac{\phi_t(t,x)}{\phi(t,x)} + \frac{\nu_1 \int_D \kappa(y-x)\phi(t,y)dy}{\phi(t,x)} - \nu_1 + a_1(t,x) = s(L_1, a_1) \quad \forall x \in \bar{D},\ t \in \mathbb{R}.$$

This implies that

$$s(L_1, a_1) = -\nu_1 + \hat{a}_1(x) + \frac{\nu_1}{T}\int_0^T \frac{\int_D \kappa(y-x)\phi(t,y)dy}{\phi(t,x)}dt \quad \forall x \in \bar{D}$$

and hence

$$s(L_1, a_1) > -\nu_1 + \max_{x\in\bar{D}} \hat{a}_1(x).$$

Conversely, assume that $s(L_1, a_1) > -\nu_1 + \max_{x\in\bar{D}} \hat{a}_1(x)$. By Proposition 3.15, there is $\alpha > s(L_1, a_1)$ such that $r(\nu_1 K_1(\alpha - H_1)^{-1}) > 1$. By Proposition 3.16, $s(L_1, a_1)$ is the isolated eigenvalue of $L_1(a_1)$ of finite multiplicity with a positive eigenfunction. Thus $s(L_1, a_1)$ is the principal eigenvalue of $L_1(a_1)$.

□

Next, we prove Theorem B(1) and (2).

Proof of Theorem B. (1) We prove the case $i = 3$. The other cases can be proved similarly. Put

$$(K_\delta u)(t, x) = \int_{D_3} \frac{1}{\delta^N} \tilde{\kappa}(\frac{y - x}{\delta})u(t, y)dy.$$

Assume $x_0 \in \bar{D}_3$ is such that $\lambda_3(x_0) = \max_{x\in\bar{D}_3} \lambda_3(x)$. By Proposition 3.14, for any $\epsilon > 0$, there is $M > 0$ such that for any $\alpha > \lambda_3(x_0)$ with $\alpha - \lambda_3(x_0) < \epsilon$ and any $v \in \mathcal{X}_3^+$ with $v(t, x) \equiv v(x)$ and $\text{supp}(v) \subset \{x \in \bar{D}_3 \,|\, \alpha - \lambda_3(x) < \epsilon\}$,

$$(\lambda - H_3)^{-1} v \geq \frac{M}{\alpha - \lambda_3(x_0)} v.$$

This implies that

$$\nu_3 K_\delta(\alpha - H_1)v \geq \int_D \frac{\nu_3 M \kappa_\delta(y - x)}{\alpha - \lambda_3(y)} v(y)dy$$

It then follows from the arguments in [49, Theorem A] that there is $\delta_0 > 0$ such that for $0 < \delta < \delta_0$, $s(L_3, a_3)$ is the principal eigenvalue of $L_3(a_3)$.

(2) We prove the case when $i = 2$. The other cases can be proved similarly. Let $x_0 \in \text{Int}(D)$ be such that $\lambda_2(x_0) = \max_{x\in\bar{D}} \lambda(x)$. By Proposition 3.14, there is $M > 0$ such that

$$(\alpha - H_2)^{-1} v \geq \frac{M}{\alpha - \lambda_2(x)} v$$

where $v(t,x) \equiv 1$. This implies that

$$\nu_2 K_2(\alpha - H_2)^{-1} v \geq \int_D \frac{\nu_2 M \kappa(y-x)}{\alpha - \lambda_2(y)} dy.$$

By the arguments in [49, Theorem B] (see also [52]), for $\alpha - \lambda_2(x_0) \ll 1$,

$$\nu_2 K_2(\alpha - H_2)^{-1} v > v.$$

This implies that $r(\nu_2 K_2(\alpha - H_2)^{-1}) > 1$. By Propositions 3.16 and 3.17, $s(L_2, a_2)$ is the principal eigenvalue of $L_2(a_2)$.

□

Before proving Theorem B(3), we first prove Theorem C.

Proof of Theorem C. We prove the case $i = 2$. Other cases can be proved similarly.

First of all, if both $L_2(a_2)$ and $L_2(\hat{a}_2)$ have principal eigenvalues, then by the arguments in [28, Theorem 4.1],

$$s(L_2, a_2) \geq s(L_2, \hat{a}_2).$$

[For the detailed proof of the last statement, we need a lemma which we will state without proof. Before stating the lemma, we state the Jensen inequality which will be useful in proving the lemma. Jensen Inequality: If f is a positive,continuous function defined on $[0, T]$ then,

$$\frac{1}{T} \int_0^T f(t) dt \geq exp\{\frac{1}{T} \int_0^T ln[f(t)] dt\}$$

with equality if and only if f is a constant function. Now we state the lemma whose detailed proof can be found in [28, Theorem 4.1].

Lemma: Let $w(x,t)$ be a positive continuous function defined on Ω x $[0, T]$ where Ω is compact . Let $\theta(x,y) = \frac{1}{T} \int_0^T \frac{w(y,t)}{w(x,t)} dt$. Then either $w(x,t)$ is independent of x or there exists $x^* \in \Omega$ such that $\theta(x^*, y) \geq 1$ for all $y \in \Omega$ with strict inequality for some y.

Proof of the last statement: Assume $s(L_2, a_2) = \beta$ and $s(L_2, \hat{a}_2) = \beta^*$. There exists eigenfunctions $\phi(t,x)$ and $\psi(x)$ with $\phi(t,x) > 0$ for all t and x and $\psi(x) > 0$ for all x such that

$$-\phi_t + \nu_2 \int_D \kappa(y-x)\phi(t,y)dy - \nu_2\phi(t,x) + a_2(t,x)\phi(t,x) = \beta\phi(t,x)$$

and

$$\nu_2 \int_D \kappa(y-x)\psi(y)dy - \nu_2\psi(x) + \hat{a}_2\psi(x) = \beta^*\psi(x),$$

which implies,

$$-\frac{\phi_t}{\phi} + \nu_2 \int_D \kappa(y-x)\frac{\phi(t,y)}{\phi(t,x)}dy - \nu_2 + a_2 = \beta$$

and

$$\nu_2 \int_D \kappa(y-x)\frac{\psi(y)}{\psi(x)}dy - \nu_2\psi(x) + \hat{a}_2(x) = \beta^*.$$

Integrating the second last equations with respect to t from 0 to T and then mulplying by $\frac{1}{T}$ we get

$$\beta = \nu_2 \int_D \kappa(y-x)\frac{1}{T}\int_0^T \frac{\phi(t,y)}{\phi(t,x)}dtdy - \nu_2 + \hat{a}_2$$

Now,

$$\begin{aligned}\beta - \beta^* &= \nu_2 \int_D \kappa(y-x)\{\frac{1}{T}\int_0^T \frac{\phi(t,y)}{\phi(t,x)}dt - \frac{\psi(y)}{\psi(x)}\}dy \\ &= \nu_2 \int_D \kappa(y-x)\frac{\psi(y)}{\psi(x)}\{\frac{1}{T}\int_0^T \frac{w(t,y)}{w(t,x)}dt - 1\}dy\end{aligned}$$

where $w(t,x) = \frac{\phi(t,x)}{\psi(x}$.

From the lemma mentioned above, the expression within {} of the above expression is positive for all y. Since, $\kappa(y-x)$ and $\psi(x)$ are also nonnegative, it follows that $\beta \geq \beta^*$.]

In general, $s(L_2, a_2)$ (resp. $s(L_2, \hat{a}_2)$) may not be the principal eigenvalue of $L_2(a_2)$ (resp. $L_2(\hat{a}_2)$). By Lemma 4.1, for any $\epsilon > 0$, there is $a_{2,\epsilon} \in \mathcal{X}_2$ such that

$$\|a_{2,\epsilon} - a_2\|_{\mathcal{X}_2} < \epsilon$$

$s(L_2, a_{2,\epsilon})$ and $s(L_2, \hat{a}_{2,\epsilon})$ are principal eigenvalues of $L_2(a_{2,\epsilon})$ and $L_2(\hat{a}_{2,\epsilon})$, respectively. By the above arguments,

$$s(L_2, a_{2,\epsilon}) \geq s(L_2, \hat{a}_{2,\epsilon}).$$

Clearly,

$$s(L_2, a_2) \geq s(L_2, a_{2,\epsilon}) - \epsilon, \quad s(L_2, \hat{a}_2) \leq s(L_2, \hat{a}_{2,\epsilon}) + \epsilon.$$

It then follows that

$$s(L_2, a_2) \geq s(L_2, \hat{a}_2) - 2\epsilon$$

for any $\epsilon > 0$ and hence

$$s(L_2, a_2) \geq s(L_2, \hat{a}_2).$$

□

Finally, we prove Theorem B(3).

Theorem B(3). By Lemma 4.2, $s(L_i, \hat{a}_i)$ is the principal eigenvalue of $L_i(\hat{a}_i)$. By Theorem A,

$$s(L_i, \hat{a}_i) > \max_{x \in \bar{D}_i} \lambda_i(x).$$

By Theorem C,

$$s(L_i, a_i) > \max_{x \in \bar{D}_i} \lambda_i(x).$$

By Theorem A again, $s(L_i, a_i)$ is the principal eigenvalue of $L_i(a_i)$. □

4.2 Other important properties

In this section, we present some other properties of principal spectrum points and principal eigenvalues for time periodic nonlocal dispersal operators. Throughout this section, $r(A)$ denotes the spectral radius of an operator A on some Banach space.

Let $\mathcal{X}_p$ be as in (2.5). Consider the following eigenvalue problem

$$-v_t + (\mathcal{K}_{\xi,\mu} - I + a(\cdot,\cdot)I)v = \lambda v, \quad v \in \mathcal{X}_p, \tag{4.4}$$

where $\xi \in S^{N-1}$, $\mu \in \mathbb{R}$, and $a(\cdot,\cdot) \in \mathcal{X}_p$. The operator $a(\cdot,\cdot)I$ has the same meaning as in (2.8) with $a_0(\cdot,\cdot)$ being replaced by $a(\cdot,\cdot)$, and $\mathcal{K}_{\xi,\mu} : X_p \to X_p$ is defined by

$$(\mathcal{K}_{\xi,\mu}v)(t,x) = \int_{\mathbb{R}^N} e^{-\mu(y-x)\cdot\xi}\kappa(y-x)v(t,y)dy. \tag{4.5}$$

We point out the following relation between (2.4) and (4.4): if $u(t,x) = e^{-\mu(x\cdot\xi - \frac{\lambda}{\mu}t)}\phi(t,x)$ with $\phi \in \mathcal{X}_p \setminus \{0\}$ is a solution of the linearization of (2.4) at $u = 0$,

$$\frac{\partial u}{\partial t} = \int_{\mathbb{R}^N} \kappa(y-x)u(t,y)dy - u(t,x) + a_0(t,x)u(t,x), \quad x \in \mathbb{R}^N, \tag{4.6}$$

where $a_0(t,x) = f(t,x,0)$, then λ is an eigenvalue of (4.4) with $a(t,\cdot) = a_0(t,\cdot)$ or $-\partial_t + \mathcal{K}_{\xi,\mu} - I + a_0(\cdot,\cdot)I$ and $v = \phi(t,x)$ is a corresponding eigenfunction.

Let $\sigma(-\partial_t + \mathcal{K}_{\xi,\mu} - I + a(\cdot,\cdot)I)$ be the spectrum of $-\partial_t + \mathcal{K}_{\xi,\mu} - I + a(\cdot,\cdot)I$ on $\mathcal{X}_p$. Let

$$\lambda_0(\xi,\mu,a) := \sup\{\mathrm{Re}\lambda \,|\, \lambda \in \sigma(-\partial_t + \mathcal{K}_{\xi,\mu} - I + a(\cdot,\cdot)I)\}.$$

Observe that if $\mu = 0$, (4.4) is independent of ξ and hence we put

$$\lambda_0(a) := \lambda_0(\xi, 0, a) \quad \forall\, \xi \in S^{N-1}. \tag{4.7}$$

Observe that $-\partial_t+\mathcal{K}_{\xi,\mu}-I+a(\cdot,\cdot)I$ may not have a principal eigenvalue (see an example in [49]). Recall that

$$\hat{a}(x)=\frac{1}{T}\int_0^T a(t,x)dt.$$

The following proposition provides necessary and sufficient condition for $-\partial_t+\mathcal{K}_{\xi,\mu}-I+a(\cdot,\cdot)I$ to have a principal eigenvalue.

Proposition 4.3. *$\lambda_0(\xi,\mu,a)$ is the principal eigenvalue of $-\partial_t+\mathcal{K}_{\xi,\mu}-I+a(\cdot,\cdot)I$ if and only if $\lambda_0(\xi,\mu,a)>-1+\max_{x\in\mathbb{R}^N}\hat{a}(x)$.*

Proof. It follows from Theorem A. □

The following proposition provides a very useful sufficient condition for $\lambda_0(\xi,\mu,a)$ to be the principal eigenvalue of $-\partial_t+\mathcal{K}_{\xi,\mu}-I+a(\cdot,\cdot)I$.

Proposition 4.4. *If $a(t,\cdot)$ is C^N and the partial derivatives of $\hat{a}(x)$ up to order $N-1$ at some x_0 are zero* (we refer this to as a vanishing condition), *where x_0 is such that $\hat{a}(x_0)=\max_{x\in\mathbb{R}^N}\hat{a}(x)$, then $\lambda_0(\xi,\mu,a)$ is the principal eigenvalue of $-\partial_t+\mathcal{K}_{\xi,\mu}-I+a(\cdot,\cdot)I$ for all $\xi\in S^{N-1}$ and $\mu\in\mathbb{R}$.*

Proof. It follows from the arguments of Theorem B(2). □

Proposition 4.5. *Each $\lambda\in\sigma(-\partial_t+\mathcal{K}_{\xi,\mu}-I+a(\cdot,\cdot)I)$ with $\mathrm{Re}\lambda>-1+\max_{x\in\mathbb{R}^N}\hat{a}(x)$ is an isolated eigenvalue with finite algebraic multiplicity.*

Proof. It follows from [6, Proposition 2.1(ii)]. □

The following theorem shows that the principal eigenvalue of $-\partial_t+\mathcal{K}_{\xi,\mu}-I+a(\cdot,\cdot)I$ (if it exists) is algebraically simple, which plays an important role in the proof of the existence of spreading speeds of (2.4).

Theorem 4.6. *Suppose that $\lambda_0(\xi,\mu,a)$ is the principal eigenvalue of $-\partial_t+\mathcal{K}_{\xi,\mu}-I+a(\cdot,\cdot)I$. Then $\lambda_0(\xi,\mu,a)$ is isolated and algebraically simple with a positive eigenfunction $\phi(\cdot,\cdot;\xi,\mu)$, $\|\phi(\cdot,\cdot;\xi,\mu)\|=1$, and $\lambda_0(\xi,\mu,a)$ and $\phi(\cdot,\cdot;\xi,\mu)$ are smooth in ξ and μ.*

Proof. First of all, note that for $\alpha > -1 + \max_{x\in\mathbb{R}^N} \hat{a}(x)$, $(\alpha I + \partial_t + I - aI)^{-1}$ exists (see [42, Proposition 3.5]). For given $\alpha > -1 + \max_{x\in\mathbb{R}^N} \hat{a}(x)$, let

$$(U_{\alpha,\xi,\mu}u)(t,x) = \int_{\mathbb{R}^N} e^{-\mu(y-x)\cdot\xi}\kappa(y-x)(\alpha + \partial_t + I - aI)^{-1}u(t,y)dy$$

and

$$r(\alpha) = r(U_{\alpha,\xi,\mu}).$$

By [42, Proposition 3.6], $U_{\alpha,\xi,\mu} : \mathcal{X}_p \to \mathcal{X}_p$ is a positive and compact operator.

Next, by [42, Proposition 3.9], $\lambda_0(\xi,\mu,a)$ is an isolated geometrically simple eigenvalue of $-\partial_t + \mathcal{K}_{\xi,\mu} - I + a(\cdot,\cdot)I$. Let $\alpha_0 = \lambda_0(\xi,\mu,a)$. This implies that $r(\alpha_0) = 1$ and $r(\alpha_0)$ is an isolated geometrically simple eigenvalue of $U_{\alpha_0,\xi,\mu}$ with $\phi(\cdot,\cdot;\xi,\mu)$ being a positive eigenfunction. We claim that $r(\alpha_0)$ is an algebraically simple isolated eigenvalue of $U_{\alpha_0,\xi,\mu}$ with a positive eigenfunction $\phi(\cdot,\cdot)$, or equivalently, $(I - U_{\alpha_0,\xi,\mu})^2\psi = 0$ ($\psi \in \mathcal{X}_p$) iff $\psi \in \text{span}\{\phi\}$. In fact, suppose that $\psi \in \mathcal{X}_p \setminus \{0\}$ is such that $(I - U_{\alpha_0,\xi,\mu})^2\psi = 0$. Then

$$(I - U_{\alpha_0,\xi,\mu})\psi = \gamma\phi, \tag{4.8}$$

for some $\gamma \in \mathbb{R}$. We prove that $\gamma = 0$. Assume that $\gamma \neq 0$. Without loss of generality, we assume that $\gamma > 0$. By (4.8) and $U_{\alpha_0,\xi,\mu}\phi = \phi$, we have

$$\psi = U_{\alpha_0,\xi,\mu}\psi + \gamma\phi = U_{\alpha_0,\xi,\mu}(\psi + \gamma\phi). \tag{4.9}$$

Then by (4.9) and $U_{\alpha_0,\xi,\mu}\phi = \phi$, we have

$$\psi + \gamma\phi = U_{\alpha_0,\xi,\mu}(\psi + \gamma\phi) + \gamma\phi, = U_{\alpha_0,\xi,\mu}(\psi + 2\gamma\phi)$$

and hence

$$\psi = U_{\alpha_0,\xi,\mu}(\psi + \gamma\phi) = U_{\alpha_0,\xi,\mu}\Big(U_{\alpha_0,\xi,\mu}(\psi + 2\gamma\phi)\Big) = U^2_{\alpha_0,\xi,\mu}(\psi + 2\mu\phi).$$

By induction, we have

$$\psi = U^n_{\alpha_0,\xi,\mu}(\psi + n\gamma\phi), \quad \forall\ n \geq 1.$$

This implies that

$$\frac{\psi}{n} = U^n_{\alpha_0,\xi,\mu}(\frac{\psi}{n} + \gamma\phi).$$

Note that $\phi(t,x) > 0$ and then

$$\frac{\psi(t,x)}{n} + \gamma\phi(t,x) > 0, \quad \forall\ n \gg 1.$$

By the positivity of $U_{\alpha_0,\xi,\mu}$, we then have

$$\frac{\psi(t,x)}{n} > 0, \quad \forall\ n \gg 1$$

and then

$$\frac{\psi(t,x)}{n} - \mu\phi(t,x) = (U^n_{\alpha_0,\xi,\mu}(\frac{\psi}{n}))(t,x) > 0, \quad \forall\ n \gg 1.$$

It then follows that

$$-\gamma\phi(t,x) \geq 0$$

and so

$$\gamma \leq 0,$$

whis is a contradiction. Therefore, $\gamma = 0$ and hence by (4.8),

$$\psi \in \text{span}\{\phi\}.$$

The claim is thus proved.

Now, we prove that $\lambda_0(\xi,\mu,a)$ is an algebraically simple eigenvalue of $-\partial_t+\mathcal{K}_{\xi,\mu}-I+a(\cdot,\cdot)I$ or equivalently, $(-\partial_t+\mathcal{K}_{\xi,\mu}-I+a(\cdot,\cdot)I-\alpha_0 I)^2\psi=0$ iff $\psi\in\text{span}\{\phi\}$. By the above arguments, there are one dimensional subspace $\mathcal{X}_{1,p}=\text{span}(\phi)$ and one-codimensional subspace $\mathcal{X}_{2,p}$ of $\mathcal{X}_p$ such that

$$\mathcal{X}_p=\mathcal{X}_{1,p}\oplus\mathcal{X}_{2,p},$$

$$U_{\alpha_0,\xi,\mu}\mathcal{X}_{1,p}=\mathcal{X}_{1,p},\quad U_{\alpha_0,\xi,\mu}\mathcal{X}_{2,p}\subset\mathcal{X}_{2,p}, \tag{4.10}$$

and

$$1\notin\sigma(U_{\alpha_0,\xi,\mu}|_{\mathcal{X}_{2,p}}).$$

Suppose that $\psi\in\mathcal{X}_p$ is such that

$$(-\partial_t+\mathcal{K}_{\xi,\mu}-I+a(\cdot,\cdot)I-\alpha_0 I)^2\psi=0.$$

Then there is $\gamma\in\mathbb{R}$ such that

$$(-\partial_t+\mathcal{K}_{\xi,\mu}-I+a(\cdot,\cdot)I-\alpha_0 I)\psi=\gamma\phi. \tag{4.11}$$

Let $\psi_i\in\mathcal{X}_{i,p}$ $(i=1,2)$ be such that

$$\psi=(\alpha_0 I+\partial_t+I-aI)^{-1}\psi_1+(\alpha_0 I+\partial_t+I-aI)^{-1}\psi_2.$$

Then

$$\begin{aligned}(-\partial_t+\mathcal{K}_{\xi,\mu}-I+a(\cdot,\cdot)I-\alpha_0 I)\psi &= (U_{\alpha_0,\xi,\mu}-I)\psi_1+(U_{\alpha_0,\xi,\mu}-I)\psi_2\\ &= (U_{\alpha_0,\xi,\mu}-I)\psi_2\\ &= \gamma\phi.\end{aligned}$$

This together with (4.10) implies that $\gamma\phi \in \mathcal{X}_{2,p}$ and hence $\gamma = 0$. By (4.11), $\psi \in \text{span}\{\phi\}$ and hence $\lambda_0(\xi, \mu, a)$ is an algebraically simple eigenvalue of $-\partial_t + \mathcal{K}_{\xi,\mu} - I + a(\cdot,\cdot)I$. $\square$

Proposition 4.7. *Assume $\lambda_0(\xi, 0, a) > 0$ and $\lambda_0(\xi, \mu, a)$ is the principal eigenvalue for $\mu > 0$. Then there is $\mu^*(\xi) \in (0, \infty)$ such that*

$$\frac{\lambda_0(\xi, \mu^*(\xi), a)}{\mu^*(\xi)} = \inf_{\mu>0} \frac{\lambda_0(\xi, \mu, a)}{\mu}. \tag{4.12}$$

Proof. Note that $\lambda_0(\xi, \mu, a) \geq \lambda_0(\xi, \mu, a_{\min})$, and

$$\lambda_0(\xi, \mu, a_{\min}) = \int_{\mathbb{R}^N} e^{-\mu y\cdot\xi}\kappa(y)dy - 1 + a_{\min}$$

with 1 as an eigenfunction. Note also that there is $k_0 > 0$ such that $\kappa(y) \geq k_0$ for $\|y\| \leq \frac{r_0}{2}$. Let $m_n(\xi) = k_0 \int_{y\cdot\xi<0, \|y\|\leq\frac{r_0}{2}} \frac{(-y\cdot\xi)^n}{n!}dy$. Then, for $\mu > 0$

$$\begin{aligned}
\int_{\mathbb{R}^N} e^{-\mu y\cdot\xi}\kappa(y)dy - 1 + a_{\min} &\geq k_0 \int_{\|y\|\leq\frac{r_0}{2}} e^{-\ y\cdot\xi}dy - 1 + a_{\min} \\
&= k_0 \sum_{n=0}^{\infty} \int_{\|y\|\leq\frac{r_0}{2}} \frac{(-\mu y\cdot\xi)^n}{n!}dy - 1 + a_{\min} \\
&\geq m_0 + m_2(\xi)\mu^2 + \sum_{n=2}^{\infty} m_{2n}(\xi)\mu^{2n} - 1 + a_{\min}
\end{aligned}$$

Let $m := \inf_{\xi\in S^{N-1}} m_2(\xi)(> 0)$. We then have $\frac{\lambda_0(\xi,\mu,a)}{\mu} \geq \frac{m_0+m\mu^2-1+a_{\min}}{\mu} \to \infty$ as $\mu \to \infty$. By $\lambda_0(\xi, 0, a) > 0$, $\frac{\lambda_0(\xi,\mu,a)}{\mu} \to \infty$ as $\mu \to 0+$. This together with the smoothness of $\lambda_0(\xi, \mu, a)$ (see Theorem 4.6) implies that there is $\mu^*(\xi)$ such that (4.12) holds. $\square$

Proposition 4.8. *For given $\xi \in S^{N-1}$, suppose that $\lambda_0(\xi, \mu, a)$ is the principal eigenvalue of $-\partial_t + \mathcal{K}_{\xi,\mu} - I + a(\cdot,\cdot)I$ for all $\mu \in \mathbb{R}$. Then $\lambda_0(\xi, \mu, a)$ is convex in μ.*

Proof. First, recall that $\Phi(t; \xi, \mu, a)$ is the solution operator of (3.4). Let

$$\Phi^p(T; \xi, \mu, a) = \Phi(T; \xi, \mu, a)|_{X_p}.$$

By [42, Proposition 3.10], we have

$$r(\Phi^p(T;\xi,\mu,a)) = e^{\lambda_0(\xi,\mu,a)T}.$$

Note that $\Phi(t;\xi,0,a)$ is independent of $\xi \in S^{N-1}$. We put

$$\tilde{\Phi}(t;a) = \Phi(t;\xi,0,a) \tag{4.13}$$

for $\xi \in S^{N-1}$. For given $u_0 \in X$ and $\mu \in \mathbb{R}$, if we let $u_0^{\xi,\mu}(x) = e^{-\mu x\cdot\xi}u_0(x)$, then $u_0^{\xi,\mu} \in X(|\mu|)$. By the uniqueness of solutions of (3.4), we have that for given $u_0 \in X$, $\xi \in S^{N-1}$, and $\mu \in \mathbb{R}$,

$$\Phi(t;\xi,\mu,a)u_0 = e^{\mu x\cdot\xi}\tilde{\Phi}(t;a)u_0^{\xi,\mu}. \tag{4.14}$$

Next, observe that for each $x \in \mathbb{R}^N$, there is a measure $m(x;y,dy)$ such that

$$(\tilde{\Phi}(T;a)u_0)(x) = \int_{\mathbb{R}^N} u_0(y)m(x;y,dy). \tag{4.15}$$

Moreover, by $(\tilde{\Phi}(T;a)u_0(\cdot - p_ie_i))(x) = (\tilde{\Phi}(T;a)u_0(\cdot))(x - p_ie_i)$ for $x \in \mathbb{R}^N$ and $i = 1,2,\cdots,N$,

$$\int_{\mathbb{R}^N} u_0(y)m(x-p_ie_i;y,dy) = \int_{\mathbb{R}^N} u_0(y-p_ie_i)m(x;y,dy) = \int_{\mathbb{R}^N} u_0(y)m(x;y+p_ie_i,dy)$$

and hence

$$m(x-p_ie_i;y,dy) = m(x;y+p_ie_i,dy) \tag{4.16}$$

for $i = 1,2,\cdots,N$. By (4.14), we have

$$(\Phi(T;\xi,\mu,a)u_0)(x) = \int_{\mathbb{R}^N} e^{\mu(x-y)\cdot\xi}u_0(y)m(x;y,dy), \quad u_0 \in X.$$

Let $\hat{\lambda}_0(\mu_i) := r(\Phi^p(T;\xi,\mu_i))$. By the arguments of [49, Theorem A (2)],

$$\ln[\hat{\lambda}_0(\mu_1)]^{\alpha}[\hat{\lambda}_0(\mu_2)]^{1-\alpha} \geq \ln(r(\Phi^p(T;\xi,\alpha\mu_1+(1-\alpha)\mu_2)).$$

Thus, by $r(\Phi(T;\xi,\mu,a) = e^{\lambda_0(\xi,\mu,a)T}$, we have

$$\alpha\lambda_0(\xi,\mu_1,a)+(1-\alpha)\lambda_0(\xi,\mu_2,a) \geq \lambda_0(\xi,\alpha\mu_1+(1-\alpha)\mu_2,a),$$

that is, $\lambda_0(\xi,\mu,a)$ is convex in μ.

□

For a fixed $\xi \in S^{N-1}$ and $a \in \mathcal{X}_p$, we may denote $\lambda_0(\xi,\mu,a)$ by $\lambda(\mu)$.

Proposition 4.9. *Let $\xi \in S^{N-1}$ and $a \in \mathcal{X}_p$ be given. Assume that (4.4) has the principal eigenvalue $\lambda(\mu)$ for $\mu \in \mathbb{R}$ and that $\lambda(0) > 0$. Then we have:*

(i) $\lambda'(\mu) < \frac{\lambda(\mu)}{\mu}$ for $0 < \mu < \mu^*(\xi)$.

(ii) For every $\epsilon > 0$, there exists some $\mu_\epsilon > 0$ such that for $\mu_\epsilon < \mu < \mu^*(\xi)$,

$$-\lambda'(\mu) < -\frac{\lambda(\mu^*(\xi))}{\mu^*(\xi)} + \epsilon.$$

Proof. It follows from Theorem 4.6, Propositions 4.7, 4.8, and the arguments of [49, Theorem 3.1]. □

Proposition 4.10. *For any $\epsilon > 0$ and $M > 0$, there are $a^{\pm}(\cdot,\cdot)$ satisfying the vanishing condition in Proposition 4.4 such that*

$$a(t,x)-\epsilon \leq a^-(t,x) \leq a(t,x) \leq a^+(t,x) \leq a(t,x)+\epsilon$$

and

$$|r(\Phi^p(T;\xi,\mu,a) - r(\Phi^p(T;\xi,\mu,a^{\pm})| < \epsilon$$

for $\xi \in S^{N-1}$ *and* $|\mu| \leq M$.

Proof. It follows from [42, Lemma 4.1] and the fact that

$$\Phi^p(T; \xi, \mu, a \pm \epsilon) = e^{\pm \epsilon T} \Phi^p(T; \xi, \mu, a).$$

□

Chapter 5

Time Periodic Positive Solutions of Nonlocal KPP Equations in Periodic Media

In this chapter, we consider applications of the principal eigenvalue theory established in the previous section to time periodic KPP equations with nonlocal dispersal. Main results of this chapter have been published (see [42]).

For given $u_1, u_2 \in X_1^{++}(= X_2^{++})$ or $u_1, u_2 \in X_3^{++}$, we define

$$\rho(u_1, u_2) = \inf\{\ln\alpha \,|\, \frac{1}{\alpha}u_1(\cdot) \le u_2(\cdot) \le \alpha u_1(\cdot),\ \alpha \ge 1\}. \tag{5.1}$$

Observe that for $u_1, u_2 \in X_1^{++}(= X_2^{++})$ or $u_1, u_2 \in X_3^{++}$, there is $\alpha \ge 1$ such that

$$\rho(u_1, u_2) = \ln\alpha.$$

Proposition 5.1. *Let $1 \le i \le 3$ be given.*

(1) For any $u_0, v_0 \in X_i^{++}$, $\rho(u_i(t,\cdot;0,u_0), u_i(t,\cdot;0,v_0))$ decreases as t increases.

(2) For any $u_0, v_0 \in X_i^{++}$, if $u_0 \ne v_0$, then $\rho(u_i(t,\cdot;0,u_0), u_i(t,\cdot;0,v_0))$ strictly decreases as t increases.

(3) For any $\epsilon_0 > 0$, there is $\delta_0 > 0$ such that for any $u_0, v_0 \in X_i^{++}$ satisfying that

$$\inf_{0\le t\le T, x\in\bar{D}}\{u_i(t,x;0,u_0), v_i(t,x;0,v_0\} \ge \epsilon_0$$

and

$$\rho(u_0, v_0) \ge 1 + \epsilon_0,$$

there holds

$$\rho(u_i(T,\cdot;0,u_0),u_i(T,\cdot;0,v_0)) \le \rho(u_0,v_0) - \delta_0.$$

Proof. We prove the case $i=1$. The other cases can be proved similarly.

(1) For any $u_0, v_0 \in X_1^{++}$, there is $\alpha \ge 1$ such that

$$\frac{1}{\alpha}v_0 \le u_0 \le \alpha v_0$$

and

$$\rho(u_0,v_0) = \ln \alpha.$$

By Proposition 3.3, for any $t>0$, we have

$$u_1(t,\cdot;0,u_0) \le u_1(t,\cdot;0,\alpha v_0).$$

Let $w(t,x) = \alpha u_1(t,x;0,v_0)$. Then $w(0,x) = \alpha v_0(x)$ and

$$\begin{aligned}
\partial_t w &= \int_D \kappa(y-x)w(t,y)dy - w(t,x) + w(t,x)f(t,x,u_1(t,x;0,v_0)) \\
&= \int_D \kappa(y-x)w(t,y)dy - w(t,x) + wf(t,x,w(t,x)) \\
&\quad + w[f(t,x,u_1(t,x;0,v_0)) - f(t,x,w(t,x))] \\
&\ge \int_D \kappa(y-x)w(t,y)dy - w(t,x) + wf(t,x,w(t,x)).
\end{aligned}$$

This together with Proposition 3.3 implies that

$$w(t,x) = \alpha u_1(t,x;0,v_0) \ge u_1(t,x;0,\alpha v_0) \ge u_1(t,x;0,u_0).$$

Similarly, we can prove that

$$\frac{1}{\alpha}u_1(t,x;0,v_0) \le u_1(t,x;0,u_0).$$

Therefore

$$\rho(u_1(t,\cdot;0,u_0),u_1(t,\cdot;0,v_0)) \le \ln\alpha \le \rho(u_0,v_0)$$

for $t>0$. Repeating the above arguments, we have

$$\begin{aligned} \rho(u_1(t,\cdot;0,u_0),u_1(t,\cdot;0,v_0)) &= \rho(u_1(t-s,\cdot;s,u(s,\cdot;0,v_0)),u_1(t-s,\cdot;s,u(s,\cdot;0,v_0)) \\ &\le \rho(u_1(s,\cdot;0,u_0),u_1(s,\cdot;0,v_0)) \end{aligned}$$

for any $0\le s<t$. It then follows that $\rho(u_1(t,\cdot;0,u_0),u_1(t,\cdot;0,v_0))$ decreases as t increases.

(2) For any $u_0,v_0\in X_1^{++}$ with $u_0\neq v_0$, there is $\alpha>1$ such that $\rho(u_0,v_0)=\ln\alpha$. As in (1), let $w(t,x)=\alpha u_1(t,x;0,v_0)$. Then $w(0,x)=\alpha v_0(x)$ and

$$\begin{aligned} \partial_t w &= \int_D \kappa(y-x)w(t,y)dy - w(t,x) + w(t,x)f(t,x,u_1(t,x;0,v_0)) \\ &= \int_D \kappa(y-x)w(t,y)dy - w(t,x) + wf(t,x,w(t,x)) \\ &\quad + w[f(t,x,u_1(t,x;0,v_0)) - f(t,x,w(t,x))] \\ &\ge \int_D \kappa(y-x)w(t,y)dy - w(t,x) + wf(t,x,w(t,x)) + \delta_0 \end{aligned}$$

for some δ_0. This implies that

$$\partial_t w(0,x) \ge \partial_t u_1(0,x;0,\alpha v_0) + \delta_0.$$

Hence

$$w(t,x) = \alpha u_1(t,x;0,v_0) \ge u_1(t,x;0,\alpha v_0) + \tilde\delta_0$$

for some $\tilde\delta_0>0$ and $0<t\ll 1$. This implies that there is $\tilde\alpha^+<\alpha$ such that

$$\tilde\alpha^+ u_1(t,x;0,v_0) \ge u_1(t,x;0,\alpha v_0)$$

and hence

$$u_1(t, x; 0, u_0) \le \tilde{\alpha}^+ u_1(t, x; 0, v_0)$$

for $0 < t \ll 1$. Similarly, we can prove that

$$\frac{1}{\tilde{\alpha}^-} u_1(t, x; 0, v_0) \le u_1(t, x; 0, u_0)$$

for some $\tilde{\alpha}^- < \alpha$ and $0 < t \ll 1$. Therefore,

$$\rho(u_1(t, \cdot; 0, u_0), v_1(t, \cdot; 0, v_0)) \le \ln \tilde{\alpha} < \rho(u_0, v_0)$$

for $0 < t \ll 1$, where $\tilde{\alpha} = \max\{\tilde{\alpha}^+, \tilde{\alpha}^-\} < \alpha$. This together with (1) implies that $\rho(u_1(t, \cdot; u_0), u_1(t, \cdot; v_0))$ is strictly decreasing as t increases.

Proof of (3). By the arguments in (1) and (2), for any $\epsilon_0 > 0$, there is $\delta_0 > 0$ such that for any $u_0, v_0 \in X_i^{++}$ with $\inf_{0 \le t \le T, x \in \bar{D}} \{u_1(t, x; 0, u_0), v_1(t, x; 0, v_0\} \ge \epsilon_0$ and $\rho(u_0, v_0) \ge 1 + \epsilon_0$, there holds

$$\rho(u_1(T, \cdot; 0, u_0), u_1(T, \cdot; 0, v_0)) \le \rho(u_0, v_0) - \delta.$$

□

Proof of Theorem E. We prove the case when $i = 1$. Other cases can be proved similarly.

First of all, for given $M \gg 1$, $u(t, x) \equiv M$ is a supersolution of (1.7). This together with Proposition 3.3 implies that $u_i(nT, x; 0, M)$ decreases as t increases. Let

$$u^+(x) = \lim_{n \to \infty} u_i(nT, x; 0, M).$$

Next, by Lemma 4.1, there are $a_i^k \in \mathcal{X}_i$ such that $s(L_i, a_i^k)$ is the principal eigenvalue of $L_i(a_i^k)$ with

$$a_i^k(t, x) < f_i(t, x, 0)$$

and

$$a_i^k(t,x) \to f_i(t,x,0) \quad \text{as} \quad k \to \infty.$$

Let ϕ_i^k be the positive principal eigenfunction of $L_i(a_i^k)$ with $\|\phi_i^k\| = 1$. By Proposition 3.19, $s(L_i, a_i^k) > 0$ for $k \gg 1$.

Fix a $k \gg 1$ such that $s(L_i, a_i^k) > 0$. Then $u = \delta\phi_i^k(t,x)$ is a subsolution of (1.7) for $0 < \delta \ll 1$.

This together with Proposition 3.3 implies that $u(nT, x; 0, \delta\phi_i^k(0,\cdot))$ increases as n increases. Let

$$u^-(x) = \lim_{k\to\infty} u(kT, x; 0, \phi_i^n).$$

We claim that

$$u^-(x) \equiv u^+(x).$$

In fact, Assume that $u^-(x) \not\equiv u^+(x)$. Observe that

$$\begin{aligned} \delta\phi_i^k(0,\cdot) &\le u_i(T,\cdot;0,\delta\phi_i^k(0,\cdot)) \le u_i(2T,\cdot;0,\delta\phi_i^k(0,\cdot)) \le \cdots \\ &\le u_i(2T,\cdot;0,M) \le u_i(T,\cdot;0,M) \le M \end{aligned}$$

There are $\alpha_n > 1$ such that

$$\alpha_1 > \alpha_2 > \alpha_3 > \cdots$$

and

$$\rho(u_i(nT,\cdot;0,\delta\phi_i^k(0,\cdot)), u_i(nT,\cdot;0,M)) = \ln \alpha_n.$$

Let

$$\alpha = \lim_{n\to\infty} \alpha_n.$$

Then

$$\frac{1}{\alpha} u^-(x) \le u^+(x) \le \alpha u^-(x)$$

and we must have $\alpha > 1$. Therefore,

$$\inf_{n\geq 1, 0\leq t\leq T, x\in\bar{D}} \{u_i(t,x;0,\delta\phi_i^k(0,\cdot)), u_i(t,x;0,M)\} > 0$$

and

$$\inf_{n\geq 1} \rho(u_i(nT,\cdot;0,\delta\phi_i^k(0,\cdot)), u_i(nT,\cdot;0,M)) > 0.$$

By Proposition 5.1 (3), there is $\delta_0 > 0$ such that

$$\ln\alpha_{n+1} \leq \ln\alpha_n - \delta_0$$

and hence

$$\ln\alpha = \lim_{n\to\infty} \ln\alpha_n = -\infty,$$

which is a contradiction, and therefore

$$u^-(x) \equiv u^+(x).$$

Observe that $u^+(x)$ is upper semicontinuous and $u^-(x)$ is lower semicontinuous.

Hence

$$u_i^*(\cdot) := u^+(\cdot)(= u^-(\cdot)) \in X_i^{++}$$

Moreover, by Dini's Theorem,

$$\lim_{n\to\infty} u_i(nT,x;0,M) = u_i^*(x)$$

uniformly in $x \in \bar{D}$. We then have

$$u_i(T,\cdot;0,u_i^*) = \lim_{n\to\infty} u_i(T,\cdot;0,u_i(nT,\cdot;0,M)) = \lim_{n\to\infty} u_i((n+1)T,\cdot;0,M) = u_i^*(\cdot).$$

This implies that $u_i(t, x; 0, u_i^*)$ is a positive time periodic solution, and the existence of time periodic positive solutions of (1.7) is thus proved.

Now suppose that $u^1(t, x)$ and $u^2(t, x)$ are two time periodic positive solutions of (1.7). Since $\rho(u^1(t, \cdot), u^2(t, \cdot))$ strictly decreases if $u^1 \neq u^2$, we must have $u^1 = u^2$. This proves the uniqueness of time periodic positive solutions.

Finally, for any $u_0 \in X^+ \setminus \{0\}$, $u_i(t, \cdot; u_0) \in \text{Int}(X^+)$ for $t > 0$. Take $0 < \delta \ll 1$, $k \gg 1$, and $M \gg 1$, we have

$$\delta\phi_i^k(0, \cdot) \leq u_0(\cdot) \leq M.$$

Then

$$u_i(t, x; 0, \delta\phi_i^k(0, \cdot)) \leq u_i(t, x; 0, u_0) \leq u(t, x; 0, M)$$

for $t \geq 0$. It then follows that

$$\lim_{t\to\infty}(u_i(t, x; 0, u_0) - u_i^*(t, x)) = 0$$

uniformly in $x \in \bar{D}$. Therefore, the unique time periodic positive solution is asymptotically stable. □

Chapter 6

Spatial Spreading Speed of Nonlocal KPP Equations in Periodic Media

In this chapter, we investigate the existence and characterization of the spreading speeds of (2.4) and prove Theorems G and H. The main results of this chapter have been submitted for publication (see [43]). Throughout this chapter, we assume (H1) and (H2). $u(t,x;u_0)$ denotes the solution of (2.4) with $u(0,x;u_0) = u_0(x)$. By Theorem E, (2.4) has a unique positive periodic solution $u^+(\cdot,\cdot) \in \mathcal{X}_p^+$.

To prove Theorems G and H, we first prove some lemmas.

Consider the space shifted equations of (2.4),

$$\frac{\partial u}{\partial t} = \int_{\mathbb{R}^N} \kappa(y-x)u(t,y)dy - u(t,x) + u(t,x)f(t,x+z,u(t,x)), \quad x \in \mathbb{R}^N, \tag{6.1}$$

where $z \in \mathbb{R}^N$. Let $u(t,x;u_0,z)$ be the solution of (6.1) with $u(0,x;u_0,z) = u_0(x)$ for $u_0 \in X$.

Lemma 6.1. *(1) Let $\xi \in S^{N-1}$, $u_0 \in \tilde{X}^+$ with $\liminf_{x\cdot\xi\to-\infty} u_0(x) > 0$ and $\limsup_{x\cdot\xi\to\infty} u_0(x) = 0$, and $c \in \mathbb{R}$ be given. If there is δ_0 such that*

$$\liminf_{x\cdot\xi \le cnT, n\to\infty} u(nT,x;u_0,z) \ge \delta_0 \quad \text{uniformly in} \quad z \in \mathbb{R}^N, \tag{6.2}$$

then for every $c' < c$,

$$\liminf_{x\cdot\xi\le c't, t\to\infty} (u(t,x;u_0,z) - u^+(t,x+z)) = 0 \quad \text{uniformly in} \quad z \in \mathbb{R}^N.$$

(2) Let $c \in \mathbb{R}$ and $u_0 \in \tilde{X}$ with $u_0 \ge 0$ be given. If there is δ_0 such that

$$\liminf_{|x\cdot\xi|\le cnT_0, n\to\infty} u(nT,x;u_0,z) \ge \delta_0 \quad \text{uniformly in} \quad z \in \mathbb{R}^N, \tag{6.3}$$

then for every $c^{'} < c$,

$$\limsup_{|x\cdot\xi|\le c^{'}t, t\to\infty} |u(t,x;u_0,z) - u^+(t,x+z)| = 0 \quad \text{uniformly in} \quad z\in\mathbb{R}^N.$$

(3) Let $c\in\mathbb{R}$ and $u_0\in\tilde{X}$ with $u_0\ge 0$ be given. If there are δ_0 and $T_0>0$ such that

$$\liminf_{\|x\|\le cnT_0, n\to\infty} u(nT,x;u_0,z)\ge\delta_0 \quad \text{uniformly in} \quad z\in\mathbb{R}^N, \tag{6.4}$$

then for every $c^{'} < c$,

$$\limsup_{\|x\|\le c^{'}t, t\to\infty} |u(t,x;u_0,z) - u^+(t,x+z)| = 0 \quad \text{uniformly in} \quad z\in\mathbb{R}^N.$$

Proof. It follows from the arguments of [49, Proposition 4.4]. □

Lemma 6.2.

$$\int_{\|y-x\|\ge B} e^{\mu\|y-x\|} m(x;y,dy)\to 0 \quad \text{as} \quad B\to\infty$$

uniformly for μ in bounded sets and for $x\in\mathbb{R}^N$.

Proof. For given $\mu_0>0$ and $n\in\mathbb{N}$, let $u_n\in X(\mu_0+1)$ be such that

$$u_n(x)=\begin{cases} e^{\mu_0\|x\|} & \text{for} \quad \|x\|\ge n \\ 0 \quad \text{for} & \|x\|\le n-1\end{cases}$$

and

$$0\le u_n(x)\le e^{\mu_0 n} \quad \text{for} \quad \|x\|\le n.$$

Then $\|u_n\|_{X(\mu_0+1)}\to 0$ as $n\to\infty$. Therefore, $\|\tilde{\Phi}(T)u_n\|_{X(\mu_0+1)}\to 0$ as $n\to\infty$. This implies that

$$\int_{\mathbb{R}^N} u_n(y)m(x;y,dy)\to 0 \quad \text{as} \quad n\to\infty$$

uniformly for x in bounded subsets of $\mathbb{R}^N$ and then

$$\int_{\|y\|\geq n} e^{\mu_0\|y\|} m(x;y,dy) \to 0 \quad \text{as} \quad n\to\infty$$

uniformly for x in bounded subsets of $\mathbb{R}^N$. The later implies that

$$\int_{\|y-x\|\geq n} e^{\mu\|y-x\|} m(x;y,dy) \to 0 \quad \text{as} \quad n\to\infty$$

uniformly for $|\mu|\leq\mu_0$ and x in bounded subset of $\mathbb{R}^N$. By (4.16), for every $1\leq i\leq N$,

$$\begin{aligned}\int_{\|y-(x+p_ie_i)\|\geq n} e^{\mu\|y-(x+p_ie_i)\|} m(x+p_ie_i;y,dy) &= \int_{\|y-x\|\geq n} e^{\mu\|y-x\|} m(x+p_ie_i;y+p_ie_i,dy)\\ &= \int_{\|y-x\|\geq n} e^{\mu\|y-x\|} m(x;y,dy).\end{aligned}$$

We then have

$$\int_{\|y-x\|\geq n} e^{\mu\|y-x\|} m(x;y,dy) \to 0 \quad \text{as} \quad n\to\infty$$

uniformly for $|\mu|\leq\mu_0$ and $x\in\mathbb{R}^N$. The lemma now follows. □

Without loss of generality, in the rest of this section, we assume that the time period $T=1$.

Lemma 6.3. *For given $\xi\in S^{N-1}$, if $\lambda_0(\xi,\mu,a_0)$ is the principal eigenvalue of $-\partial_t+\mathcal{K}_{\xi,\mu}-I+a_0(\cdot,\cdot)I$ for any $\mu>0$, then*

$$c^*_{\sup}(\xi)\leq \inf_{\mu>0}\frac{\lambda_0(\xi,\mu,a_0)}{\mu}. \tag{6.5}$$

Proof. For given $\xi\in S^{N-1}$, put $\lambda(\mu)=\lambda_0(\xi,\mu,a_0)$. For any $\mu>0$, suppose that $\phi(\mu,\cdot,\cdot)\in\mathcal{X}_p^+$ and

$$[-\partial_t+(\mathcal{K}_{\xi,\mu}-I+a_0(\cdot,\cdot)I)]\phi(\mu,t,x)=\lambda(\mu)\phi(\mu,t,x).$$

Since $f(t,x,u) = f(t,x,0) + f_u(t,x,\eta)u$ for some $0 \le \eta \le u$, we have, by assumption (H1), $f(t,x,u) \le f(t,x,0)$ for $u \ge 0$. If $u_0 \in X^+$, then

$$u(t,x;u_0,z) \le (\Phi(t;\xi,0,a_0(\cdot,\cdot+z))u_0)(x) \quad \text{for} \quad x,z \in \mathbb{R}^N. \tag{6.6}$$

It can easily be verified that

$$(\Phi(n;\xi,0,a_0(\cdot,\cdot+z))\tilde{u}_0)(x) = Me^{-\mu(x\cdot\xi-n\tilde{c})}\phi(\mu,1,x+z)\Big(=Me^{-\mu(x\cdot\xi-n\tilde{c})}\phi(\mu,0,x+z)\Big)$$

with $\tilde{u}_0(x) = Me^{-\mu x\cdot\xi}\phi(\mu,0,x+z)$ for $\tilde{c} = \frac{\lambda(\mu)}{\mu}$ and $M > 0$. For any $u_0 \in X^+(\xi)$, choose $M > 0$ large enough such that $\tilde{u}_0 \ge u_0$. Then by Propositions 3.6 and 3.8, we have

$$\begin{aligned} u(n,x;u_0,z) &\le (\Phi(n;\xi,0,a_0(\cdot,\cdot+z))u_0)(x) \\ &\le (\Phi(n;\xi,0,a_0(\cdot,\cdot+z))\tilde{u}_0)(x) \\ &= Me^{-\mu(x\cdot\xi-n\tilde{c})}\phi(\mu,0,x+z). \end{aligned}$$

Hence,

$$\limsup_{x\cdot\xi\ge nc, n\to\infty} u(n,x;u_0,z) = 0 \quad \text{for} \quad \text{every} \quad c > \tilde{c}$$

uniformly in $z \in \mathbb{R}$. This together with Lemma 6.1 implies that $c^*_{\sup}(\xi) \le \frac{\lambda(\mu)}{\mu}$ for any $\mu > 0$ and hence (6.5) holds. □

Lemma 6.4. *For given $\xi \in S^{N-1}$, if $\lambda_0(\xi,\mu,a_0)$ is the principal eigenvalue of $-\partial_t + \mathcal{K}_{\xi,\mu} - I + a_0(\cdot,\cdot)I$ for any $\mu > 0$, then*

$$c^*_{\inf}(\xi) \ge \inf_{\mu>0} \frac{\lambda_0(\xi,\mu,a_0)}{\mu}. \tag{6.7}$$

Proof. We prove (6.7) by modifying the arguments in [35] and [55].

Observe that, for every $\epsilon_0 > 0$, there is $b_0 > 0$ such that

$$f(t, x, u) \geq f(t, x, 0) - \epsilon_0 \quad \text{for} \quad 0 \leq u \leq b_0, \quad x \in \mathbb{R}^N. \tag{6.8}$$

Hence if $u_0 \in X^+$ is so small that $0 \leq u(t, x; u_0, z) \leq b_0$ for $t \in [0, 1]$, $x \in \mathbb{R}^N$ and $z \in \mathbb{R}^N$, then

$$u(1, x; u_0, z) \geq e^{-\epsilon_0}(\Phi(1; \xi, 0, a_0(\cdot, \cdot + z))u_0)(x) \tag{6.9}$$

for $x \in \mathbb{R}^N$ and $z \in \mathbb{R}^N$.

Let $r(\mu)$ be the spectral radius of $\Phi(1; \xi, \mu, 0)$. Then $\lambda(\mu) = \ln r(\mu)$ and $r(\mu)$ is an eigenvalue of $\Phi(1; \xi, \mu, a_0(\cdot, \cdot))$ with a positive eigenfunction $\phi(\mu, x) := \phi(\mu, 1, x)(= \phi(\mu, 0, x))$.

By Proposition 4.9, for any $\epsilon_1 > 0$, there is μ_{ϵ_1} such that

$$-\lambda^{'}(\mu) < -\frac{\lambda(\mu^*(\xi))}{\mu^*(\xi)} + \epsilon_1 \tag{6.10}$$

for $\mu_{\epsilon_1} < \mu < \mu^*(\xi)$. In the following, we fix $\mu \in (\mu_{\epsilon_1}, \mu^*(\xi))$. By Proposition 4.9 again, we can choose $\epsilon_0 > 0$ so small that

$$\lambda(\mu) - \mu\lambda^{'}(\mu) - 3\epsilon_0 > 0. \tag{6.11}$$

Let $\zeta : \mathbb{R} \to [0, 1]$ be a smooth function satisfying that

$$\zeta(s) = \begin{cases} 1 & \text{for} \quad |s| \leq 1 \\ 0 & \text{for} \quad |s| \geq 2. \end{cases} \tag{6.12}$$

By Theorem 4.6, $\phi(\mu, x)$ is smooth in μ. Let

$$\kappa(\mu, z) = \frac{\phi_\mu(\mu, z)}{\phi(\mu, z)}.$$

For given $\gamma > 0,\ B > 0$, and $z \in \mathbb{R}^N$, define

$$\tau(\mu,\gamma,z,B) = \frac{1}{\gamma} tan^{-1} \frac{\int_{\mathbb{R}^N} \phi(\mu,y) e^{-\mu(y-z)\cdot\xi} \sin\gamma(-(y-z)\cdot\xi + \kappa(\mu,y))\zeta(\|y-z\|/B) m(z;y,dy)}{\int_{\mathbb{R}^N} \phi(\mu,y) e^{-\mu(y-z)\cdot\xi} \cos\gamma(-(y-z)\cdot\xi + \kappa(\mu,y))\zeta(\|y-z\|/B) m(z;y,dy)}.$$

By Lemma 6.2, $\tau(\mu,\gamma,z,B)$ is well defined for any $B > 0$ and $0 < \gamma \ll 1$, and

$$\lim_{\gamma\to 0} \tau(\mu,\gamma,z,B) = \frac{\int_{\mathbb{R}^N} \phi(\mu,y) e^{-\mu(y-z)\cdot\xi} \big(-(y-z)\cdot\xi + \kappa(\mu,y)\big)\zeta(\|y-z\|/B) m(z;y,dy)}{\int_{\mathbb{R}^N} \phi(\mu,y) e^{-\mu(y-z)\cdot\xi} \zeta(\|y-z\|/B) m(z;y,dy)}$$

uniformly in $z \in \mathbb{R}^N$ and $B > 0$. By Lemma 6.2 again,

$$\lim_{B\to\infty} \int_{\mathbb{R}^N} \phi(\mu,y) e^{-\mu(y-z)\cdot\xi} \zeta(\|y-z\|/B) m(z;y,dy) = r(\mu)\phi(\mu,z) \tag{6.13}$$

uniformly in $z \in \mathbb{R}^N$ and

$$\begin{aligned} \lim\nolimits_{B\to\infty} \Big[&\int_{\mathbb{R}^N} \phi(\mu,y) e^{-\mu(y-z)\cdot\xi} ((y-z)\cdot\xi) \zeta(\|y-z\|/B) m(z;y,dy) \\ &+ \int_{\mathbb{R}^N} \phi_\mu(\mu,y) e^{-\mu(y-z)\cdot\xi} \zeta(\|y-z\|/B) m(z;y,dy) \Big] \\ = &\int_{\mathbb{R}^N} \phi(\mu,y) e^{-\mu(y-z)\cdot\xi} \big(-(y-z)\cdot\xi\big) m(z;y,dy) + \int_{\mathbb{R}^N} \phi_\mu(\mu,y) e^{-\mu(y-z)\cdot\xi} m(z;y,dy) \\ &= r'(\mu)\phi(\mu,z) + r(\mu)\phi_\mu(\mu,z) \end{aligned} \tag{6.14}$$

uniformly in $z \in \mathbb{R}^N$.

By (6.13) and (6.14), we can choose $B \gg 1$ and fix it so that

$$\int_{\mathbb{R}^N} \phi(\mu,y) e^{-\mu(y-z)\cdot\xi} \zeta(\|y-z\|/B) m(z;y,dy) \ge e^{\lambda(\mu)-\epsilon_0} \phi(\mu,z), \quad z \in \mathbb{R}^N, \tag{6.15}$$

$$\gamma(B + |\tau(\mu,\gamma,z,B)| + |\kappa(\mu,z)|) < \pi, \quad z \in \mathbb{R}^N,\ \ 0 < \gamma \ll 1,$$

$$-\kappa(\mu,z) + \tau(\mu,\gamma,z,B) < \lambda'(\mu) - \frac{\epsilon_0}{\mu}, \quad z \in \mathbb{R}^N,\ \ 0 < \gamma \ll 1, \tag{6.16}$$

and

$$\kappa(\mu,z) - \tau(\mu,\gamma,z,B) < -\lambda'(\mu) + \epsilon_1, \quad z \in \mathbb{R}^N,\ \ 0 < \gamma \ll 1. \tag{6.17}$$

For given $\epsilon_2 > 0$ and $\gamma > 0$, define

$$v(s,z) = \begin{cases} \epsilon_2 \phi(\mu,z) e^{-\mu s} \sin\gamma(s-\kappa(\mu,z)), & 0 \le s-\kappa(\mu,z) \le \frac{\pi}{\gamma} \\ 0, & \text{otherwise.} \end{cases} \tag{6.18}$$

Let

$$v^*(x;s,z) = v(x\cdot\xi + s - \kappa(\mu,z) + \tau(\mu,\gamma,z,B), x+z).$$

Choose $\epsilon_2 > 0$ so small that

$$0 \le u(t,x;v^*(\cdot;s,z),z) \le b_0 \quad \text{for} \quad t\in[0,1], \quad x,z\in\mathbb{R}^N.$$

Let

$$\eta(\gamma,\mu,z,B) = -\kappa(\mu,z) + \tau(\mu,\gamma,z,B).$$

Then for $0 \le s-\kappa(\mu,z) \le \frac{\pi}{\gamma}$, we have

$$\begin{aligned} & u(1,0;v^*(\cdot;s,z),z) \\ & \ge e^{-\epsilon_0}\Phi(1;\xi,0,a_0(\cdot,\cdot+z))v^*(\cdot;s,z) \\ & \ge \epsilon_2 e^{-\epsilon_0}\textstyle\int_{\mathbb{R}^N}\Big[\phi(\mu,y)e^{-\mu[(y-z)\cdot\xi+s+\eta(\gamma,\mu,z.B)]}\cdot\sin\gamma[(y-z)\cdot\xi+s+\eta(\gamma,\mu,z,B)-\kappa(\mu,y)] \\ & \qquad \cdot\zeta(\|y-z\|/B)\Big]m(z;y,dy) \\ & = e^{-\epsilon_0}v(s,z)e^{-\mu\eta(\gamma,\mu,z,B)}\tfrac{\sec\gamma\tau(\mu,\gamma,z,B)}{\phi(\mu,z)}\textstyle\int_{\mathbb{R}^N}\Big[\phi(\mu,y)e^{-\mu(y-z)\cdot\xi}\cdot\cos\gamma(-(y-z)\cdot\xi+\kappa(\mu,y)) \\ & \qquad \cdot\zeta(\|y-z\|/B)\Big]m(z;y,dy). \end{aligned}$$

Observe that

$$\begin{aligned} & \lim_{\gamma\to 0} e^{-\epsilon_0}e^{-\mu\eta(\gamma,\mu,z,B)}\tfrac{\sec\gamma\tau(\mu,\gamma,z,B)}{\phi(\mu,z)}\textstyle\int_{\mathbb{R}^N}\Big[\phi(\mu,y)e^{-\mu(y-z)\cdot\xi}\cdot\cos\gamma(-(y-z)\cdot\xi+\kappa(\mu,y)) \\ & \qquad \cdot\zeta(\|y-z\|/B)\Big]m(z;y,dy) \\ & \ge e^{-\epsilon_0}e^{-\mu\lambda'(\mu)-\epsilon_0}e^{\lambda(\mu)-\epsilon_0} \qquad \text{by (6.15), (6.16)} \end{aligned}$$

$$= e^{\lambda(\mu)-\mu\lambda'(\mu)-3\epsilon_0}$$

$$> 1 \quad (\text{by} \quad (6.11)).$$

It then follows that for $0 \le s - \kappa(\mu, z) \le \frac{\pi}{\gamma}$,

$$u(1, 0; v^*(\cdot; s, z), z) \ge v(s, z) = v^*((\kappa(\mu, z) - \tau(\mu, \gamma, z, B))\xi; s, (-k(\mu, z) + \tau(\mu, \gamma, z, B))\xi + z).$$

Clearly, the above equality holds for all $s \in \mathbb{R}$ (since $v(s, z) = 0$ for $s \le \kappa(\mu, z)$ or $s \ge \kappa(\mu, z) + \frac{\pi}{\gamma}$).

Let $\bar{s}(x)$ be such that $v(\bar{s}(x), x) = \max_{s\in\mathbb{R}} v(s, x)$. Let

$$\bar{v}(s, x) = \begin{cases} v(\bar{s}(x), x), & s \le \bar{s}(x) - \frac{\pi}{\gamma} \\ v(s + \frac{\pi}{\gamma}, x), & s \ge \bar{s}(x) - \frac{\pi}{\gamma}. \end{cases}$$

Set

$$\bar{v}^*(x; s, z) = \bar{v}(x \cdot \xi + s - \kappa(\mu, z) + \tau(\mu, \gamma, z, B), x + z).$$

We then have

$$u(1, 0; \bar{v}^*(\cdot; s, z), z) \ge \bar{v}(s, z) = \bar{v}^*((\kappa(\mu, z) - \tau(\mu, \gamma, z, B))\xi; s, (-\kappa(\mu, z) + \tau(\mu, \gamma, z, B))\xi + z)$$

for $s \in \mathbb{R}$ and $z \in \mathbb{R}^N$.

Let

$$v_0(x; z) = \bar{v}(x \cdot \xi, x + z).$$

Note that $\bar{v}(s, x)$ is non-increasing in s. Hence we have

$$\begin{aligned} u(1, x; v_0(\cdot; z), z) &= u(1, 0; v_0(\cdot + x; z), x + z) \\ &= u(1, 0; \bar{v}^*(\cdot; x \cdot \xi + \kappa(\mu, x + z) - \tau(\gamma, x + z), x + z), x + z) \\ &\ge \bar{v}(x \cdot \xi + \kappa(\mu, x + z) - \tau(\mu, \gamma, x + z, B), x + z) \end{aligned}$$

$$
\begin{aligned}
&\geq \bar{v}(x\cdot\xi-\lambda^{'}(\mu)+\epsilon_1, x+z) \quad \text{(by} \quad (6.17))\\
&\geq \bar{v}\Big(x\cdot\xi-\frac{\lambda(\mu^*(\xi))}{\mu^*(\xi)}+2\epsilon_1, x+z\Big) \quad \text{(by} \quad (6.10))\\
&= v_0\Big(x-[\frac{\lambda(\mu^*(\xi))}{\mu^*(\xi)}-2\epsilon_1]\xi, [\frac{\lambda(\mu^*(\xi))}{\mu^*(\xi)}-2\epsilon_1]\xi+z\Big)
\end{aligned}
$$

for $z \in \mathbb{R}^N$. Let $\tilde{c}^*(\xi) = \frac{\lambda(\mu^*(\xi))}{\mu^*(\xi)} - 2\epsilon_1$. Then

$$
u(1, x; v_0(\cdot, z), z) \geq v_0(x - \tilde{c}^*(\xi)\xi, \tilde{c}^*(\xi)\xi + z)
$$

for all $z \in \mathbb{R}^N$. We also have

$$
\begin{aligned}
u(2, x; v_0(\cdot, z), z) &\geq u(1, x; v_0(\cdot - \tilde{c}^*(\xi)\xi, \tilde{c}^*(\xi)\xi + z), z)\\
&= u(1, x - \tilde{c}^*(\xi)\xi; v_0(\cdot, \tilde{c}^*(\xi)\xi + z), \tilde{c}^*(\xi)\xi + z)\\
&\geq v_0(x - 2\tilde{c}^*(\xi)\xi, 2\tilde{c}^*(\xi) + z)
\end{aligned}
$$

for all $z \in \mathbb{R}^N$. By induction, we have

$$
u(n, x; v_0(\cdot, z), z) \geq v_0(x - n\tilde{c}^*(\xi)\xi, n\tilde{c}^*(\xi) + z)
$$

for $n \geq 1$ and $z \in \mathbb{R}^N$. This together with Lemma 6.1 implies that

$$
c^*_{\inf}(\xi) \geq \tilde{c}^*(\xi) = \frac{\lambda(\mu^*(\xi))}{\mu^*(\xi)} - 2\epsilon_1.
$$

Since ϵ_1 is arbitrary, (6.7) holds. □

Proof of Theorem G. Fix $\xi \in S^{N-1}$. Put $\lambda(\mu) = \lambda_0(\xi, \mu, a_0)$, where $a_0(t, x) = f(t, x, 0)$. By Proposition 4.7, there is $\mu^* = \mu^*(\xi) \in (0, \infty)$ such that

$$
\inf_{\mu>0} \frac{\lambda(\mu)}{\mu} = \frac{\lambda(\mu^*)}{\mu^*}.
$$

It is easy to see that $c^*(\xi)$ exists and $c^*(\xi)=\frac{\lambda(\mu^*)}{\mu^*}$ if and only if $c^*_{\inf}(\xi)=c^*_{\sup}(\xi)=\frac{\lambda(\mu^*)}{\mu^*}$.

If $\lambda_0(\xi,\mu,a_0)$ is the principal eigenvalue of $-\partial_t+\mathcal{K}_{\xi,\mu}-I+a_0(\cdot,\cdot)I$ for all μ, then by Lemmas 6.3 and 6.4, we have $c^*(\xi)$ exists and $c^*(\xi)=\inf_{\mu>0}\frac{\lambda(\mu)}{\mu}$.

In general, let $a^n(\cdot,\cdot)\in C^N(\mathbb{R}\times\mathbb{R}^N,\mathbb{R})\cap\mathcal{X}_p$ be such that a^n satisfies the vanishing condition in Proposition 4.4,

$$a^n\geq a_0\quad\text{for}\quad n\geq 1\quad\text{and}\quad \|a^n-a\|_{\mathcal{X}_p}\to 0\quad\text{as}\quad n\to\infty.$$

Then,

$$\lambda_0(\xi,\mu,a^n)\to\lambda_0(\xi,\mu,a_0)\quad\text{as}\quad n\to\infty.$$

Note that for $0<\epsilon\ll 1$,

$$uf(t,x,u)\leq u(a^n(t,x)-\epsilon u)\quad\text{for}\quad x\in\mathbb{R}^N,\ u\geq 0.$$

By Lemma 6.3 and Proposition 3.8, for any $u_0\in X^+(\xi)$ and $c>\inf_{\mu>0}\frac{\lambda_0(\xi,\mu,a^n)}{\mu}$,

$$\lim_{x\cdot\xi\geq ct,t\to\infty}u(t,x;u_0)\leq\lim_{x\cdot\xi\geq ct,t\to\infty}u^n(t,x;u_0)=0,$$

where $u_n(t,x;u_0)$ is the solution of (6.1) with $f(t,x,u)$ being replaced by $f^n(t,x,u)=a^n(t,x)-\epsilon u$. This implies that

$$c^*_{\sup}(\xi)\leq\frac{\lambda_0(\xi,\mu,a^n)}{\mu}\quad\forall\ \mu>0,\ n\geq 1$$

and then

$$c^*_{\sup}(\xi)\leq\frac{\lambda_0(\xi,\mu,a_0)}{\mu}\quad\forall\mu>0.$$

Therefore,

$$c^*_{\sup}(\xi)\leq\inf_{\mu>0}\frac{\lambda_0(\xi,\mu,a_0)}{\mu}.\tag{6.19}$$

For any $\epsilon > 0$, there is $\delta_0 > 0$ such that

$$f(t,x,u) \geq f(t,x,0) - \epsilon \quad \text{for} \quad t \in \mathbb{R},\ x \in \mathbb{R}^N,\ 0 < u < \delta_0.$$

Let $a_n(\cdot,\cdot) \in C^N(\mathbb{R}\times\mathbb{R}^N)\cap\mathcal{X}_p$ be such that a_n satisfies the vanishing condition in Proposition 4.4 and

$$f(\cdot,\cdot,0) - 2\epsilon \leq a_n(\cdot,\cdot) \leq f(\cdot,\cdot,0) - \epsilon \quad \forall n \geq 1.$$

Note that

$$uf(t,x,u) \geq u(a_n(t,x) - Mu) \quad \forall\ 0 \leq u \leq \delta_0,\ M > 0.$$

Choose $M \geq \frac{\max_{t\in\mathbb{R},x\in\mathbb{R}^N} a_n(t,x)}{\delta_0}$. By Lemma 6.4 and Proposition 3.8, for any $u_0 \in X^+(\xi)$ with $\sup_{x\in\mathbb{R}^N} u_0(x) \leq \delta_0$,

$$\liminf_{x\cdot\xi\leq ct,t\to\infty} u(t,x;u_0,z) \geq \liminf_{x\cdot\xi\leq ct,t\to\infty} u_n(t,x;u_0,z) > 0$$

for any $c < \inf_{\mu>0} \frac{\lambda_0(\xi,\mu,a_n)}{\mu}$, where $u_n(t,x;u_0,z)$ is the solution of (6.1) with $f(t,x,u)$ being replaced by $f_n(t,x,u) = a_n(t,x) - Mu$. This implies that

$$c^*_{\inf}(\xi) \geq \inf_{\mu>0} \frac{\lambda_0(\xi,\mu,a_n)}{\mu}.$$

Thus,

$$c^*_{\inf}(\xi) \geq \inf_{\mu>0} \frac{\lambda_0(\xi,\mu,a_0) - 2\epsilon}{\mu}.$$

Letting $\epsilon \to 0$, we have

$$c^*_{\inf}(\xi) \geq \inf_{\mu>0} \frac{\lambda_0(\xi,\mu,a_0)}{\mu}. \tag{6.20}$$

By (6.19) and (6.20),

$$c^*_{\sup}(\xi) = c^*_{\inf}(\xi) = \inf_{\mu>0} \frac{\lambda_0(\xi,\mu,a_0)}{\mu}.$$

Hence $c^*(\xi)$ exists and

$$c^*(\xi) = \inf_{\mu>0} \frac{\lambda_0(\xi, \mu, a_0)}{\mu}.$$

□

Proof Theorem H. It can be proved by the arguments similar in [51, Theorem E]. □

Chapter 7

Traveling Wave Solutions of Nonlocal KPP Equations in Periodic Media

In this chapter, we explore the existence and uniqueness of traveling wave solutions of (2.4) connecting 0 and u^+ and prove Theorem I. The main results of this chapter have been submitted for publication (see [43]). Throughout this chapter, we assume (H1) and (H2).

7.1 Sub- and super-solutions

In this section, we construct some sub- and super-solutions of (2.4) to be used in the proof of Theorem I. Throughout this subsection, we assume (H1)-(H3) and put $a_0(t,x) = f(t,x,0)$.

For given $\xi \in S^{N-1}$, let $\mu^*(\xi)$ be such that

$$c^*(\xi) = \frac{\lambda_0(\xi, \mu^*(\xi), a_0)}{\mu^*(\xi)}.$$

Fix $\xi \in S^{N-1}$ and $c > c^*(\xi)$. Let $0 < \mu < \mu_1 < \min\{2\mu, \mu^*(\xi)\}$ be such that $c = \frac{\lambda_0(\xi,\mu,a_0)}{\mu}$ and $\frac{\lambda_0(\xi,\mu,a_0)}{\mu} > \frac{\lambda_0(\xi,\mu_1,a_0)}{\mu_1} > c^*(\xi)$. Put

$$\phi(\cdot,\cdot) = \phi(\mu,\cdot,\cdot), \;\; \phi_1(\cdot,\cdot) = \phi(\mu_1,\cdot,\cdot).$$

If no confusion occurs, we may write $\lambda_0(\mu,\xi,a_0)$ as $\lambda(\mu)$.

For given $d_1 > 0$, let

$$\underline{v}(t,x;z,d_1) = e^{-\mu(x\cdot\xi - ct)}\phi(t,x+z) - d_1 e^{-\mu_1(x\cdot\xi-ct)}\phi_1(t,x+z)$$

and

$$\underline{u}(t,x;z,d_1) = \max\{0, \underline{v}(t,x;z,d_1)\}. \tag{7.1}$$

We may write $\underline{u}(t,x;z)$ for $\underline{u}(t,x;z,d_1)$ for fixed $d_1 > 0$ if no confusion occurs.

Proposition 7.1. *For any $z \in \mathbb{R}^N$, $\underline{u}(t,x;z,d_1)$ is a sub-solution of (6.1) provided that d_1 is sufficiently large.*

Proof. It follows from the similar arguments as in [50, Propsotion 3.2]. □

For given $d_2 \geq 0$, let

$$\bar{v}(t,x;z,d_2) = e^{-\mu(x\cdot\xi - ct)}\phi(t,x+z) + d_2 e^{-\mu_1(x\cdot\xi-ct)}\phi_1(t,x+z)$$

and

$$\bar{u}(t,x;z,d_2) = \min\{\bar{v}(t,x;z,d_2), u^+(t,x+z)\}. \tag{7.2}$$

We may write $\bar{v}(t,x;z)$ and $\bar{u}(t,x;z)$ for $\bar{v}(t,x;z,d_2)$ and $\bar{u}(t,x;z,d_2)$, respectively, if no confusion occurs.

Proposition 7.2. *For any $d_2 \geq 0$ and $z \in \mathbb{R}^N$, $\bar{u}(t,x;z,d_2)$ is a super-solution of (6.1).*

Proof. It follows from the similar arguments as in [50, Proposition 3.5]. □

Proposition 7.3. *For $u_0(\cdot;z) \in X^+$ with $u_0(x;z) \leq u^+(0,x+z)$, if $\lim_{x\cdot\xi\to\infty} \frac{u_0(x;z)}{e^{-\mu x\cdot\xi}\phi(0,x+z)} = 1$ uniformly in $z \in \mathbb{R}^N$ and $\inf_{x\cdot\xi\leq O(1), z\in\mathbb{R}^N} u_0(x;z) > 0$, then*

$$\lim_{x\cdot\xi\to\infty} \frac{u(t,x+ct\xi;u_0(\cdot;z),z)}{e^{-\mu x\cdot\xi}\phi(t,x+ct\xi+z)} = 1 \tag{7.3}$$

uniformly in $t \geq 0$ and $z \in \mathbb{R}^N$, and

$$\inf_{x\cdot\xi\leq O(1), t\geq 0, z\in\mathbb{R}^N} u(t,x+ct\xi;u_0(\cdot;z),z) > 0. \tag{7.4}$$

Proof. Assume that $u_0 \in X^+$ satisfies the conditions in the proposition. We first prove (7.3). Observe that there are $d_1, d_2 > 0$ such that

$$\underline{u}(0,x,;z,d_1) \leq u_0(x;z) \leq \bar{u}(0,x;z,d_2) \quad \forall\ z \in \mathbb{R}^N.$$

By Propositions 7.1 and 7.2,

$$\underline{u}(t,x;z,d_1) \le u(t,x;u_0(\cdot;z),z) \le \bar{u}(t,x;z,d_2). \tag{7.5}$$

This implies that

$$\lim_{x\cdot\xi\to\infty} \frac{u(t,x+ct\xi;u_0(\cdot;z),z)}{e^{-\mu x\cdot\xi}\phi(t,x+ct\xi+z)} = 1$$

uniformly in $t \ge 0$ and $z \in \mathbb{R}^N$, i.e., (7.3) holds.

Next we prove (7.4). Without loss of generality, we may assume that $u_0(x) \le u_0(x-p_i\mathbf{e_i})$ for any $\mathbf{e_i}$ with $\mathbf{e_i}\cdot\xi > 0$. By (7.5), there are $M_- < M_+$ with $M_+ - M_- \ge p_1 + p_2 + \cdots + p_N$ and $\sigma > 0$ such that

$$u(t,x+ct\xi;u_0(\cdot;z),z) \ge \sigma \quad \forall\ t \ge 0,\ M_- \le x\cdot\xi \le M_+. \tag{7.6}$$

Then for any $\mathbf{e_i}$ with $\mathbf{e_i}\cdot\xi > 0$,

$$p_i\mathbf{e_i}\cdot\xi \le M_+ - M_-$$

and

$$u_0(x) \le u_0(x-p_i\mathbf{e_i}).$$

Observe that there is $\mathbf{e_{i_0}}$ such that $\mathbf{e_{i_0}}\cdot\xi > 0$. Then by Proposition 3.8, for any $k \in \mathbb{N}$,

$$\begin{aligned} u(t,x+ct\xi-kp_{i_0}\mathbf{e_{i_0}};u_0,z) &= u(t,x+ct\xi;u_0(\cdot-kp_{i_0}\mathbf{e_{i_0}}),z-kp_{i_0}\mathbf{e_{i_0}}) \\ &= u(t,x+ct\xi;u_0(\cdot-kp_{i_0}\mathbf{e_{i_0}}),z) \\ &\ge u(t,x+ct\xi;u_0(\cdot),z). \end{aligned}$$

This together with (7.6) implies that

$$u(t,x+ct\xi;u_0(\cdot;z),z) \ge \sigma \quad \forall\ t \ge 0,\ M_- - k\hat{p}_{i_0} \le x\cdot\xi \le M_+ - k\hat{p}_{i_0},\ z \in \mathbb{R}^N, \tag{7.7}$$

where $\hat{p}_{i_0} = p_{i_0}\mathbf{e_{i_0}} \cdot \xi (> 0)$. (7.6) and (7.7) together with $\hat{p}_{i_0} < M_+ - M_-$ imply that

$$u(t, x + ct\xi; u_0(\cdot; z), z) \geq \sigma \quad \forall\ t \geq 0, \quad x \cdot \xi \leq M_+,\ z \in \mathbb{R}^N.$$

(7.4) then follows. □

7.2 Traveling wave solutions

In this section, we investigate the existence of traveling wave solutions of (2.4) and prove Theorem I. Throughout this section, we assume (H1)-(H3).

Lemma 7.4. *Let*

$$u_n(x, z) = u(nT, x + cnT\xi; \bar{u}(0, \cdot; z - cnT\xi), z - cnT\xi).$$

Then

$$u_n(x, z) = u(nT, x; \bar{u}(0, \cdot + cnT\xi; z - cnT\xi), z)$$

and $u_n(x, z)$ *is non-increasing in* n.

Proof. First, by direct calculation,

$$u_n(x, z) = u(nT, x; \bar{u}(0, \cdot + cnT\xi; z - cnT\xi), z).$$

Next, observe that

$$\bar{u}(T, x + cT\xi; z - cnT\xi) = \bar{u}(0, x; z - c(n-1)T\xi) \quad \forall\ n \geq 1.$$

Hence

$u_n(x, z)$

$$
\begin{aligned}
&= u(nT, x + cnT\xi; \bar{u}(0, \cdot; z - cnT\xi), z - cnT\xi) \\
&= u((n-1)T, x + cnT\xi; u(T, \cdot; \bar{u}(0, \cdot; z - cnT\xi), z - cnT\xi), z - cnT\xi) \\
&= u((n-1)T, x + c(n-1)T\xi; u(T, \cdot + cT\xi; \bar{u}(0, \cdot; z - cnT\xi), z - cnT\xi); z - c(n-1)T\xi) \\
&\le u((n-1)T, x + c(n-1)T\xi; \bar{u}(T, \cdot + cT\xi; z - cnT\xi); z - c(n-1)T\xi) \quad \text{(byLemma6.4)} \\
&= u((n-1)T, x + c(n-1)T\xi; \bar{u}(0, \cdot; z - c(n-1)T\xi), z - c(n-1)T\xi) \\
&= u_{n-1}(x, z).
\end{aligned}
$$

The proposition is thus proved. □

Let

$$\Phi_0(x, z) = \lim_{n\to\infty} u_n(x, z).$$

Then $\Phi_0(x, z)$ is upper semi-continuous, $0 \le \Phi(x, z) \le u^+(0, x + z)$, and hence $\Phi(\cdot, z) \in \tilde{X}$.

The following lemma follows easily.

Lemma 7.5. *For each $z \in \mathbb{R}^N$, $u(t, x) = u(t, x; \Phi_0(\cdot, z), z)$ are entire solutions of (6.1).*

Proof of Theorem I. Let

$$\Phi(x, t, z + ct\xi) = u(t, x + ct\xi; \Phi_0(\cdot, z), z).$$

It suffices to prove that $\Phi(x, t, z)$ generates a traveling wave solution of (2.4).

First of all, $u(t, x; \Phi(\cdot, 0, z), z) = \Phi(x - ct\xi, t, z + ct\xi)$ follows directly from the definition of $\Phi(x, t, z)$.

Next, note that

$$
\begin{aligned}
\underline{u}(t, x; z) &= e^{-\mu(x\cdot\xi - ct)}\phi(t, x + z) - d_1 e^{-\mu_1(x\cdot\xi - ct)}\phi_1(x + z) \\
&\le u(t, x; \Phi(\cdot, 0, z), z) \\
&\le \bar{u}(t, x; z) \\
&= e^{-\mu(x\cdot\xi - ct)}\phi(t, x + z) + d_2 e^{-\mu_1(x\cdot\xi - ct)}\phi_1(x + z) \qquad (7.8)
\end{aligned}
$$

for $t \in \mathbb{R}$ and $x, z \in \mathbb{R}^N$. Note also that

$$\begin{aligned}
& \Phi(x,t,z) \\
= \ & u\Big(t, x+ct\xi; \Phi(0,\cdot,z-ct\xi), z-ct\xi\Big) \\
= \ & \lim_{n\to\infty} u\Big(t, x+ct\xi; u_n(\cdot, z-ct\xi), z-ct\xi\Big) \\
= \ & \lim_{n\to\infty} u\Big(t, x+ct\xi; u(nT, \cdot + cnT\xi; \bar{u}(0,\cdot; z-cnT\xi-ct\xi), z-cnT\xi-ct\xi), z-ct\xi\Big) \\
= \ & \lim_{n\to\infty} u\Big(t, x; u(nT, \cdot + ct\xi; \bar{u}(0,\cdot + cnT\xi; z-cnT\xi-ct\xi), z-ct\xi), z\Big) \\
= \ & \lim_{n\to\infty} u\Big(t, x; u(nT, \cdot; \bar{u}(0,\cdot + cnT\xi+ct\xi; z-cnT\xi-ct\xi), z), z\Big) \\
= \ & \lim_{n\to\infty} u\Big(t+nT, x; \bar{u}(0,\cdot + cnT\xi+ct\xi; z-cnT\xi-ct\xi), z\Big) \\
= \ & \lim_{n\to\infty} u\Big(t+nT, x+cnT\xi+ct\xi; \bar{u}(0,\cdot; z-cnT\xi-ct\xi), z-cnT\xi-ct\xi\Big). \qquad (7.9)
\end{aligned}$$

By (7.8),

$$\lim_{x\cdot\xi-ct\to\infty} \frac{\Phi(x-ct\xi, t, z+ct\xi)}{e^{-\mu(x\cdot\xi-ct)}\phi(t,x+z)} = 1,$$

which is equivalent to

$$\lim_{x\cdot\xi\to\infty} \frac{\Phi(x,t,z)}{e^{-\mu x\cdot\xi}\phi(t,x+z)} = 1, \qquad (7.10)$$

uniformly in $t \in \mathbb{R}$ and $z \in \mathbb{R}^N$.

By (7.9) and Proposition 7.3, there are $\sigma > 0$ and $M \in \mathbb{R}$ such that

$$u\Big(t+nT, x+cnT\xi+ct\xi; \bar{u}(0,\cdot; z-cnT\xi-ct\xi), z-cnT\xi-ct\xi\Big) \geq \sigma \quad \forall n \gg 1,\ x\cdot\xi \leq M.$$

It then follows from Lemma 6.2 that

$$\lim_{x\cdot\xi\to-\infty} (\Phi(t,x,z) - u^+(t,x+z) = 0 \qquad (7.11)$$

uniformly in $t \in \mathbb{R}$ and $z \in \mathbb{R}^N$.

By (7.9), we have

$$\begin{aligned}
&\Phi(x,T,z)\\
=\;& \lim_{n\to\infty} u\Big((n+1)T, x+c(n+1)T\xi; \bar{u}(0,\cdot;z-c(n+1)T\xi), z-c(n+1)T\xi\Big)\\
=\;& \lim_{n\to\infty} u\Big(nT, x+cnT\xi; \bar{u}(0,\cdot;z-cnT\xi), z-cnT\xi\Big)\\
=\;& \Phi(x,0,z) \qquad (7.12)
\end{aligned}$$

and

$$\begin{aligned}
&\Phi(x,t,z+p_i\mathbf{e_i})\\
=\;& \lim_{n\to\infty} u\Big(t+nT, x+cnT\xi+ct\xi; \bar{u}(0,\cdot;z+p_i\mathbf{e_i}-cnT\xi-ct\xi), z+p_i\mathbf{e_i}-cnT\xi-ct\xi\Big)\\
=\;& \lim_{n\to\infty} u\Big(t+nT, x+cnT\xi+ct\xi; \bar{u}(0,\cdot;z-cnT\xi-ct\xi), z-cnT\xi-ct\xi\Big)\\
=\;& \Phi(x,t,z). \qquad (7.13)
\end{aligned}$$

Moreover, for any $x, x^{'} \in \mathbb{R}^N$ with $x\cdot\xi = x^{'}\cdot\xi$,

$$\begin{aligned}
&\Phi(x,t,z-x)\\
=\;& \lim_{n\to\infty} u\Big(t+nT, x+cnT\xi+ct\xi; \bar{u}(0,\cdot;z-x-cnT\xi-ct\xi), z-x-cnT\xi-ct\xi\Big)\\
=\;& \lim_{n\to\infty} u\Big(t+nT, cnT\xi+ct\xi; \bar{u}(0,\cdot+x;z-x-cnT\xi-ct\xi), z-cnT\xi-ct\xi\Big)\\
=\;& \lim_{n\to\infty} u\Big(t+nT, cnT\xi+ct\xi; \bar{u}(0,\cdot+x^{'};z-x^{'}-cnT\xi-ct\xi), z-cnT\xi-ct\xi\Big)\\
=\;& \lim_{n\to\infty} u\Big(t+nT, x^{'}+cnT\xi+ct\xi; \bar{u}(0,\cdot;z-x^{'}-cnT\xi-ct\xi), z-x^{'}-cnT\xi-ct\xi\Big)\\
=\;& \Phi(x^{'},t,z-x^{'}). \qquad (7.14)
\end{aligned}$$

By (7.9)-(7.14), $\Phi(x,t,z)$ generates a traveling wave solution of (2.4) in the direction of ξ with speed c. □

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Appendices