

**On the Derivation Algebras of Parabolic Lie Algebras
with Applications to Zero Product Determined Algebras**

by

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A dissertation submitted to the Graduate Faculty of
Auburn University
in partial fulfillment of the
requirements for the Degree of
Doctor of Philosophy

Auburn, Alabama

August 2, 2014

Keywords: Reductive Lie algebra, parabolic subalgebra, derivation, zero product
determined algebra, direct sum decomposition

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Abstract

This dissertation builds upon and extends previous work completed by the author and his advisor in [5]. A Lie algebra \mathfrak{g} is said to be zero product determined if for each bilinear map $\varphi : \mathfrak{g} \times \mathfrak{g} \rightarrow V$ that satisfies $\varphi(x, y) = 0$ whenever $[x, y] = 0$ there is a linear map $f : [\mathfrak{g}, \mathfrak{g}] \rightarrow V$ such that $\varphi(x, y) = f([x, y])$ for all $x, y \in \mathfrak{g}$. A derivation D on a Lie algebra \mathfrak{g} is a linear map $D : \mathfrak{g} \rightarrow \mathfrak{g}$ satisfying $D([x, y]) = [D(x), y] + [x, D(y)]$ for all $x, y \in \mathfrak{g}$. $\text{Der } \mathfrak{g}$ denotes the space of all derivations on the Lie algebra \mathfrak{g} , which itself forms a Lie algebra. The study of derivations forms part of the classical theory of Lie algebras and is well understood, though some work has been done recently that generalizes some of the classical theory [9, 10, 14, 23, 24, 27, 30, 31, 34, 37]. In contrast, the theory of zero product determined algebras is new, motivated by applications to analysis, and supports a growing body of literature [1, 4, 5, 11, 33]. In this dissertation, we add to this body of knowledge, studying the two concepts of derivations and of zero product determined algebras individually and in relation to each other.

This dissertation contains two main results. Let \mathbb{K} denote an algebraically-closed, characteristic-zero field. Let \mathfrak{q} be a parabolic subalgebra of a reductive Lie algebra \mathfrak{g} over \mathbb{K} or \mathbb{R} . First we prove a direct sum decomposition of $\text{Der } \mathfrak{q}$. $\text{Der } \mathfrak{q}$ decomposes as the direct sum of ideals $\text{Der } \mathfrak{q} = \mathcal{L} \oplus \text{ad } \mathfrak{q}$, where \mathcal{L} consists of all linear maps on \mathfrak{q} that map into the center of \mathfrak{g} and map $[\mathfrak{q}, \mathfrak{q}]$ to 0. Second, we apply the decomposition, along with results of [5] and [33], to prove that \mathfrak{q} and $\text{Der } \mathfrak{q}$ are zero product determined in the case that \mathfrak{g} is a Lie algebra over \mathbb{K} .

We conclude by discussing several possible directions for future research and by applying the main results to providing tabular data for parabolic subalgebras of reductive Lie algebras of types A_5 , G_2 , and F_4 .

Acknowledgments

I'd like to thank the Department of Mathematics and Statistics at Auburn University for supporting me in my studies and research for the past six years. My time at Auburn has been wonderful and formative, and the people here have been nothing but welcoming and friendly.

Chief among these is my advisor, Dr. Huajun Huang. Dr. Huang, thank you for your patience with me when I often made the same mistake several times. Thank you for making the effort to contact me when I would disappear for weeks at a time, swallowed up in a miasma of depression. Thank you for having faith in me which I often lacked. I certainly would not have had the emotional will to continue if not for your guidance.

I'd like to thank Dr. Holmes and Dr. Tam, whose work has influenced mine, who have given me helpful input throughout the process of writing this dissertation, and from whom I learned my craft. Dr. Holmes, I hope that I make you proud when I say that yours is the primary influence on my mathematical writing style. Dr. Tam, you have made this dissertation possible by giving me the necessary mathematical tools. Thank you both.

I'd also like to thank my undergraduate professors: Dr. Elliott who introduced me to algebra, Dr. Grzegorzcyk who introduced me to geometry, and Dr. Garcia who introduced me to mathematical reasoning and proof. It was at CSU - Channel Islands that, under their expert tutelage, the elusive cognitive lubricant called *mathematical maturity* first began to ferment inside me.

It would be impossible to name everyone who has made this dissertation possible. I will close by thanking my family and my friends. I've had a great time these past six years, thanks to all of you. Now, anybody up for a round of Spades?

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Chapter 1

Introduction

Let \mathbb{K} be an algebraically-closed field of characteristic-zero. Let \mathfrak{g} be a reductive Lie algebra over \mathbb{K} or \mathbb{R} . Let \mathfrak{q} be a parabolic subalgebra of \mathfrak{g} . The purpose of this dissertation is, primarily, twofold: first, to construct a direct sum decomposition of the derivation algebra $\text{Der } \mathfrak{q}$; second, to use this direct sum decomposition to extend work done in [5] and [33] and show that $\text{Der } \mathfrak{q}$ is zero product determined.

An algebra \mathcal{A} with multiplication $*$ is said to be zero product determined if for each bilinear map $\varphi : \mathcal{A} \times \mathcal{A} \longrightarrow V$ that satisfies $\varphi(x, y) = 0$ whenever $x * y = 0$ there is a linear map $f : \mathcal{A}^2 \longrightarrow V$ such that $\varphi(x, y) = f(x * y)$ for all $x, y \in \mathcal{A}$. This is a relatively new concept, first appearing in [4] and expanded upon in [5, 11, 33] and others. In the initial paper, published in 2009, Brešar, Grašič, and Sánchez Ortega proved that the full matrix algebra over a commutative ring is zero product determined when $*$ is the usual matrix multiplication or when $*$ is the Jordan product $x * y = xy + yx$, and also that the general linear algebra \mathfrak{gl} over a field is zero product determined when $*$ is the Lie bracket $x * y = [x, y] = xy - yx$ [4]. Grašič expanded the pool of Lie algebras known to be zero product determined to include the classical algebras B_l , C_l , and D_l over an arbitrary field of characteristic not 2 [11]. In 2011, Wang et al. proved that the parabolic subalgebras of a simple Lie algebras over \mathbb{K} are zero product determined for \mathbb{K} algebraically-closed and characteristic-zero (in particular, the simple Lie algebras over \mathbb{K} are zero product determined) [33]. Our present research began as an attempt to extend this result to parabolic subalgebras of reductive Lie algebras over \mathbb{K} and over \mathbb{R} and to their derivation algebras.

A derivation D on a Lie algebra \mathfrak{g} is a linear map $D : \mathfrak{g} \rightarrow \mathfrak{g}$ satisfying $D([x, y]) = [D(x), y] + [x, D(y)]$ for all $x, y \in \mathfrak{g}$. Denote by $\text{Der } \mathfrak{g}$ the space of all derivations on the Lie algebra \mathfrak{g} , which itself forms a Lie algebra. The study of derivations forms part of the classical theory of Lie algebras. The classical theory of derivations can perhaps be said to begin with the well-known result that if \mathfrak{g} is a semisimple Lie algebra over a field of characteristic not equal to two, then any derivation of \mathfrak{g} is an inner derivation [13, 26], so in particular $\mathfrak{g} \cong \text{Der } \mathfrak{g}$ in case \mathfrak{g} is semisimple char $\mathbb{F} \neq 2$.

In 1955, Jacobson proved that a Lie algebra \mathfrak{g} is nilpotent if it has a derivation $f \in \text{Der } \mathfrak{g}$ that is nonsingular [14]. Dixmier and Lister provided an example in 1957 showing that the converse was not possible [9]. They constructed a particular Lie algebra L , showed that L is nilpotent, characterized the derivation algebra $\text{Der } L$ through explicit computation, and showed that every derivation of L had a non-trivial kernel. Of note is that Dixmier and Lister explicitly decompose $\text{Der } L$ in this particular case and arrive at results similar to our results for general parabolic subalgebras of reductive Lie algebras.

Tôgô, in 1961, proved a partial converse to our result in the special case when $\mathfrak{q} = \mathfrak{g}$ [27]. By 1972, Leger and Luks — working over an arbitrary field of characteristic not equal to two — were able to show that all derivations of a Borel subalgebra \mathfrak{b} of a semisimple Lie algebra \mathfrak{g} are inner derivations, analogous to the known result for the semisimple algebra \mathfrak{g} itself [19]. More generally, their result applies to the class of Lie algebras \mathfrak{g} that can be expressed as the semidirect product $\mathfrak{g} = \mathfrak{a} \rtimes \mathfrak{g}'$ where the subalgebra \mathfrak{g}' is nilpotent and the ideal \mathfrak{a} is abelian and acts diagonally on \mathfrak{g}' [19]. This wider class of Lie algebras includes Borel subalgebras but does not include parabolic subalgebras. Working independently, Tolpygo extended the result of Leger and Luks to apply to any parabolic subalgebra \mathfrak{q} between \mathfrak{b} and \mathfrak{g} , but only in case the scalar field is the complex numbers \mathbb{C} [29]. The balance of the work done in this era provides characterizations of derivations of special classes of Lie algebras [10, 24, 28].

The recent direction that work on derivations has taken has been to relax the definition of Lie algebra to include consideration of Lie algebras that draw scalars from commutative rings rather than from fields and to attempt to reproduce as much of the classical theory as can be, characterizing derivations of specific classes of such Lie algebras [23, 30, 31]. Zhang in 2008 takes a different approach, defining a new class of solvable Lie algebras over \mathbb{C} and characterizing their derivation algebras [37].

Other work has been in the direction of considering certain maps that are similar to but may fail to be derivations [6, 8, 34]. Unrelated to zero product determined algebras, except perhaps inspired by the concept, Wang et al. recently defined a *product zero derivation* of a Lie algebra \mathfrak{g} as a linear map $f : \mathfrak{g} \rightarrow \mathfrak{g}$ satisfying $[f(x), y] + [x, f(y)] = 0$ whenever $[x, y] = 0$ [34]. In the aforementioned paper, the authors go on to characterize the product zero derivations of parabolic subalgebras \mathfrak{q} of simple Lie algebras over an algebraically-closed, characteristic-zero field, ultimately showing all product zero derivations of \mathfrak{q} to be sums of inner derivations and scalar multiplication maps [34]. In papers appearing in 2011 and 2012, Chen et al. consider nonlinear maps satisfying derivability and nonlinear Lie triple derivations respectively [6, 8]. The authors characterize all such maps on parabolic subalgebras of a semisimple Lie algebra over \mathbb{C} as the sums of inner derivations and certain maps called quasi-derivations that may fail to be linear [6, 8].

This dissertation serves two purposes: to expand on the results characterizing derivations of classes of Lie algebras and to apply these results to further the study of zero product determined algebras. We prove, among other results, the following two theorems.

Theorem. *Let \mathfrak{q} be a parabolic subalgebra of a reductive Lie algebra \mathfrak{g} over an algebraically-closed, characteristic-zero field or over \mathbb{R} . The derivation Lie algebra $\text{Der } \mathfrak{q}$ decomposes as the direct sum of ideals*

$$\text{Der } \mathfrak{q} = \mathfrak{L} \oplus \text{ad } \mathfrak{q}$$

where \mathfrak{L} consists of all linear transformation mapping \mathfrak{q} into its center and mapping $[\mathfrak{q}, \mathfrak{q}]$ to 0.

Theorem. *Let \mathfrak{q} be a parabolic subalgebra of a reductive Lie algebra \mathfrak{g} over an algebraically-closed, characteristic-zero field. \mathfrak{q} and $\text{Der } \mathfrak{q}$ are zero product determined.*

We began this research with the goal of proving the latter theorem, as it is a natural extension of results found in [33]. As a tool to this end, we required an understanding of the structure of $\text{Der } \mathfrak{q}$, especially as it relates to the structure of \mathfrak{q} . Considering the vast body of literature on derivations, we assumed that results on the structure of $\text{Der } \mathfrak{q}$ for such a \mathfrak{q} would be readily available; however, a review of the literature found only partial results, as we summarized above. Without the necessary tools to proceed, developing those tools quickly overtook our work on zero product determined algebras and became a central part of our research in and of itself. The results are complete and satisfying, and enable us to pursue our original goals.

The method of proof of the former theorem relies on utilization of the grading on \mathfrak{q} afforded by the root system Φ . In order to motivate the methods employed, we offer the following example. The reader is encouraged to keep this example in mind during the general treatment in chapter 3.

Example 1. We consider the parabolic subalgebra \mathfrak{q} of $\mathfrak{g} = \mathfrak{gl}(\mathbb{C}^6)$ consisting of block-upper-triangular matrices in block sizes 3, 2, 1 (see figure 1.1). We write $\mathfrak{gl}(\mathbb{C}^6) = \mathfrak{g}_Z \oplus \mathfrak{g}_S$, where $\mathfrak{g}_Z = \mathbb{C}I$ and $\mathfrak{g}_S = \mathfrak{sl}(\mathbb{C}^6)$. We decompose \mathfrak{q} similarly: $\mathfrak{q} = \mathfrak{g}_Z \oplus \mathfrak{q}_S$, where $\mathfrak{q}_S = \mathfrak{q} \cap \mathfrak{g}_S$.

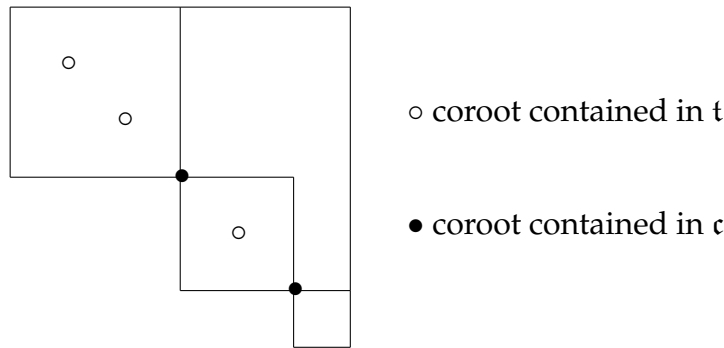


Figure 1.1: Decomposition of \mathfrak{q}_S

\mathfrak{g}_S has root space decomposition $\mathfrak{g}_S = \mathfrak{h} \dot{+} \sum_{i \neq j} \mathbb{C}e_{i,j}$ where \mathfrak{h} consists of traceless diagonal 6×6 matrices. It is well known that the coroots $h_i = e_{ii} - e_{i+1,i+1}$ form a basis of \mathfrak{h} . We further decompose \mathfrak{h} into $\mathfrak{t} \dot{+} \mathfrak{c}$, where $\mathfrak{t} = \text{Span}\{h_1, h_2, h_4\}$ and $\mathfrak{c} = \text{Span}\{h_3, h_5\}$ (see figure 1.1). It follows that $\mathfrak{t} = \mathfrak{h} \cap [\mathfrak{q}, \mathfrak{q}]$ and that \mathfrak{q} has the vector space direct sum decomposition

$$\mathfrak{q} = \mathfrak{g}_Z \dot{+} \mathfrak{c} \dot{+} [\mathfrak{q}, \mathfrak{q}].$$

In light of this decomposition, a linear transformation that sends \mathfrak{q} to \mathfrak{g}_Z and sends $[\mathfrak{q}, \mathfrak{q}]$ to 0 has the block matrix form illustrated by figure 1.2.

$$\begin{array}{c} \mathfrak{g}_Z \\ \mathfrak{c} \\ [\mathfrak{q}, \mathfrak{q}] \end{array} \begin{pmatrix} \mathfrak{g}_Z & \mathfrak{c} & [\mathfrak{q}, \mathfrak{q}] \\ * & * & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Figure 1.2: Block matrix form of derivations in \mathcal{L}

The claims of the two theorems — that $\text{Der } \mathfrak{q} = \mathcal{L} \oplus \text{ad } \mathfrak{q}$ and that $\text{Der } \mathfrak{q}$ is zero product determined — may then be explicitly verified via computation in this special case. The proofs of the theorems in general will rely on carrying out the same decomposition of \mathfrak{q} and the accompanying computations in abstract. \square

A brief outline of this dissertation: Chapter 2 provides the necessary background definitions and tools needed to understand the results in the sequel. Except where noted in section 2.8, this chapter does not contain original research and may be skimmed or even skipped by an experienced Lie algebraist. Chapter 3 proves the former theorem and several ancillary results, as mentioned, primarily relying on the root space decomposition. Chapter 4 formally introduces the notion of a zero product determined algebra, summarizes some of the known results, and extends those results. Chapter 5 contains a short discussion on possible directions in which to generalize the results of this dissertation for future research.

Before we begin, we shall make note of some conventions of terminology and notation. For the convenience of the reader, we shall use the term *proposition* for any result that is not original to this dissertation. The terms *lemma*, *theorem*, and *corollary* are used for results appearing for the first time in this dissertation. \mathbb{R} and \mathbb{C} will denote the field of real numbers and the field of complex numbers, respectively. If \mathbb{F} is a field, we will say that \mathbb{F} is *complex-like* to mean that \mathbb{F} is algebraically-closed and of characteristic-zero. Typically, \mathbb{K} will be used to denote complex-like fields and \mathbb{F} will be used to denote more general fields.

If V is a vector space, we denote the identity map on V by id_V . If V_1 and V_2 are vector spaces over the same field \mathbb{F} , we denote by $\text{Hom}_{\mathbb{F}}(V_1, V_2)$ the set of \mathbb{F} -linear maps from V_1 to V_2 , which is itself a vector spaces over \mathbb{F} . If V_1 and V_2 are both subspaces of a vector space V , intersect trivially, and together span V , we write $V = V_1 \dot{+} V_2$. If V is a vectorspace, W a subspace, and f a linear map $f : V \rightarrow V$, we say f stabilizes W , or W is f -invariant, to mean $f(W) \subseteq W$. We write $f|_W$ to denote the restriction of f to W ; namely, the linear map $W \rightarrow V$ defined by $f|_W(v) = f(v)$ for each $v \in W$. If V is a vector space whose scalar field is ambiguous or non-standard in some way, we write $V = V_{\mathbb{F}}$ to note that \mathbb{F} is the scalar field. For example, the notation \mathbb{C}^3 will denote the set of 3-entry column vectors with entries in \mathbb{C} viewed as a 3-dimensional complex vector space, and the notation $\mathbb{C}_{\mathbb{R}}^3$ will denote the same set viewed as a 6-dimensional vector space with real number scalars.

If M is a matrix, M^t denotes the transpose of M . If M is a matrix with complex entries, M^* denotes the conjugate transpose of M , ie $M^* = (\overline{M})^t$. The notation $\text{Tr } M$ denotes the trace of a matrix or linear transformation M (ie, the sum of the diagonal entries of M or a matrix representing M , respectively). In case $\text{Tr } M = 0$ we may say M is *traceless* for brevity. The notation $e_{i,j}$ is used to denote the matrix with 1 in the i -th row, j -th column entry and zeros elsewhere. The notation I_n (or simply I if n is understood) denotes the $n \times n$ identity matrix with 1-s along the main diagonal and zeros elsewhere.

Chapter 2

Background and Setting

This chapter reviews the basic facts of the classical theory of Lie algebras which are required for an understanding of the subsequent discussion. Proofs are included as space permits. Where a particular definition, theorem, or proof is not cited in the line of the text, the reader is referred to any of the standard texts on the subject, eg, [2, 13, 15, 17, 21, 25].

2.1 Lie algebras

Definition 2.1. A *Lie algebra* is a vector space \mathfrak{g} over a field \mathbb{F} together with a binary operation $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ satisfying:

1. $[\cdot, \cdot]$ is \mathbb{F} -bilinear, ie,

$$\forall x, y, z \in \mathfrak{g}, \forall a \in \mathbb{F}, \quad [x + ay, z] = [x, z] + a[y, z] \text{ and } [x, y + az] = [x, y] + a[x, z];$$

2. $[\cdot, \cdot]$ is alternating, ie,

$$\forall x \in \mathfrak{g}, \quad [x, x] = 0; \text{ and}$$

3. $[\cdot, \cdot]$ satisfies the Jacobi identity, ie,

$$\forall x, y, z \in \mathfrak{g}, \quad [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

Proposition 2.1. For all $x, y \in \mathfrak{g}$, $[x, y] = -[y, x]$.

Proof. Consider $[x + y, x + y]$. By condition 2, $[x + y, x + y] = 0$ and by conditions 1 and 2, $[x + y, x + y] = [x, x] + [x, y] + [y, x] + [y, y] = [x, y] + [y, x]$, so $0 = [x, y] + [y, x]$, or rather $[x, y] = -[y, x]$. □

Example 2. Real three-space \mathbb{R}^3 , together with the familiar cross product \times defined by

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \times \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} y_1 z_2 - y_2 z_1 \\ x_2 z_1 - x_1 z_2 \\ x_1 y_2 - x_2 y_1 \end{pmatrix}$$

is a Lie algebra. It is straightforward to verify that \times satisfies the three conditions required of $[\cdot, \cdot]$ in definition 2.1.

Example 3. Let n be some positive integer. A space of $n \times n$ matrices \mathcal{M} may be endowed with a bracket product by defining

$$\forall M, N \in \mathcal{M}, \quad [M, N] = MN - NM.$$

If \mathcal{M} is closed under taking linear combinations of bracket products of its members, then \mathcal{M} is a Lie algebra.

Example 4. Denote by $\mathfrak{so}(n)$ the space of all $n \times n$ matrices with entries in \mathbb{R} satisfying $M = -M^t$ (ie. M is skew-symmetric). $\mathfrak{so}(n)$ is closed under taking linear combinations of matrix bracket products $MN - NM$, so $\mathfrak{so}(n)$ is a Lie algebra under $[M, N]$ defined in example 3.

Example 5. A matrix M with complex entries is called *Hermitian* if $M = M^*$. M is called *skew-Hermitian* if $M = -M^*$. If M is Hermitian (res. skew-Hermitian), complex scalar multiple of M may fail to be Hermitian (res. skew-Hermitian). In fact, for Hermitian M , the scalar multiple iM is skew-Hermitian, and vice versa. Because of this, the vector space consisting of Hermitian (res. skew-Hermitian) matrices are vector spaces over \mathbb{R} , despite Hermitian (res. skew-Hermitian) matrices admitting complex entries.

Denote by $\mathfrak{su}(n)$ the space of all $n \times n$ skew-Hermitian traceless matrices. $\mathfrak{su}(n)$ is closed under the bracket $[M, N]$ defined in examples 3, and as such forms a Lie algebra

over \mathbb{R} . The space of Hermitian matrices is not closed under the bracket. In fact, if M and N are Hermitian, $[M, N]$ is skew-Hermitian.

Example 6. Let V be a vector space over a field \mathbb{F} . Denote by $\mathfrak{gl}(V)$ the space of all \mathbb{F} -linear maps from V to V . Denote by $\mathfrak{sl}(V)$ the subspace of $\mathfrak{gl}(V)$ consisting of traceless linear maps. Define the bracket product $[\cdot, \cdot]$ by

$$\forall f, g \in \mathfrak{gl}(V), \quad [f, g] = f \circ g - g \circ f.$$

Then, $\mathfrak{gl}(V)$ is a Lie algebra, and $\mathfrak{sl}(V)$ is a Lie algebra under the bracket restricted to $\mathfrak{sl}(V) \times \mathfrak{sl}(V)$. If the dimension of V is n , and if a basis for V is chosen, then $\mathfrak{gl}(V)$ and $\mathfrak{sl}(V)$ are concretely realized as the space of all $n \times n$ matrices with entries in \mathbb{F} and the space of all $n \times n$ traceless matrices with entries in \mathbb{F} , respectively, and the bracket defined here agrees with the bracket defined in example 3.

Example 7. The vector space $\mathbb{C}^{n \times n}$ consisting of $n \times n$ matrices with complex entries can be considered a Lie algebra over \mathbb{C} , as $\mathfrak{gl}(\mathbb{C}^n)$ with dimension n^2 , or as a Lie algebra over \mathbb{R} , as $\mathfrak{gl}(\mathbb{C}_{\mathbb{R}}^n)$ with dimension $2n^2$.

Definition 2.1 does not exclude the possibility of considering infinite-dimensional Lie algebras or Lie algebras over prime-characteristic fields, as is the case with — for example — [16] and [26], respectively. In addition, recent work in the study of Lie algebras has relaxed the definition to include the consideration of Lie algebras with scalars from a commutative ring rather than a field, such as in [2, 4, 11, 23, 30, 31]. This dissertation follows none of the aforementioned directions. Instead, all of the Lie algebras considered in this dissertation are finite-dimensional vector spaces, drawing scalars from a characteristic-zero field such as \mathbb{R} or \mathbb{C} .

If \mathfrak{g} is a Lie algebra and if A, B are subsets of \mathfrak{g} (written $A, B \subseteq \mathfrak{g}$) we use the notation $[A, B]$ to denote the \mathbb{F} -linear span of members of \mathfrak{g} of the form $[a, b]$ where $a \in A$ and

$b \in B$. Taking the \mathbb{F} -linear span is essential here, as the set $\{[a, b] \mid a \in A, b \in B\}$ often fails to be a linear subspace of \mathfrak{g} .

Definition 2.2. \mathfrak{h} is a *subalgebra* of \mathfrak{g} (written $\mathfrak{h} \leq \mathfrak{g}$) means that \mathfrak{h} is a linear subspace of \mathfrak{g} and that $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$. \mathfrak{a} is an *ideal* of \mathfrak{g} (written $\mathfrak{a} \trianglelefteq \mathfrak{g}$) means that \mathfrak{a} is a subalgebra of \mathfrak{g} and that $[\mathfrak{a}, \mathfrak{g}] \subseteq \mathfrak{a}$.

In light of proposition 2.1, $[\mathfrak{a}, \mathfrak{g}] = [\mathfrak{g}, \mathfrak{a}]$, so the condition that $[\mathfrak{a}, \mathfrak{g}] \subseteq \mathfrak{a}$ is equivalent to the condition that $[\mathfrak{g}, \mathfrak{a}] \subseteq \mathfrak{a}$.

Example 8. Let \mathfrak{g} be any Lie algebra. $\mathfrak{g} \trianglelefteq \mathfrak{g}$ trivially. Moreover, the bracket product of two ideals is an ideal, so $[\mathfrak{g}, \mathfrak{g}]$ is an ideal and is called the *derived algebra* of \mathfrak{g} .

Example 9. Consider $\mathfrak{gl}(\mathbb{R}^3)$. Notice that $\mathfrak{so}(3) \leq \mathfrak{gl}(\mathbb{R}^3)$, and $\mathfrak{sl}(\mathbb{R}^3) = [\mathfrak{gl}(\mathbb{R}^3), \mathfrak{gl}(\mathbb{R}^3)] \trianglelefteq \mathfrak{gl}(\mathbb{R}^3)$.

A subalgebra $\mathfrak{a} \leq \mathfrak{g}$ induces an equivalence relation on \mathfrak{g} by partitioning \mathfrak{g} into cosets $x + \mathfrak{a} = \{x + a \mid a \in \mathfrak{a}\}$ for each $x \in \mathfrak{g}$. The set of all such cosets has a natural structure as a Lie algebra if, and only if, \mathfrak{a} is an ideal.

Definition 2.3. Let \mathfrak{g} be a Lie algebra and let $\mathfrak{a} \trianglelefteq \mathfrak{g}$. The *quotient algebra* $\mathfrak{g}/\mathfrak{a}$ is the Lie algebra consisting of the set

$$\mathfrak{g}/\mathfrak{a} = \{x + \mathfrak{a} \mid x \in \mathfrak{g}\}$$

together with the bracket

$$[x + \mathfrak{a}, y + \mathfrak{a}] = [x, y] + \mathfrak{a}.$$

We omit the verification that the bracket on $\mathfrak{g}/\mathfrak{a}$ is well-defined, while we note that the proof relies on the fact that \mathfrak{a} is an ideal.

Definition 2.4. A Lie algebra \mathfrak{g} is *abelian* when $[\mathfrak{g}, \mathfrak{g}] = 0$.

The name is not accidental or arbitrary. In fact, if \mathcal{M} is a space of commuting matrices, then for any $M, N \in \mathcal{M}$ we have $[M, N] = MN - NM = MN - MN = 0$, so that

the term abelian used here agrees with the familiar usage of the term from group theory when used to mean *commutative*.

Definition 2.5. The *center* of \mathfrak{g} (written \mathfrak{g}_Z) is the set of all $z \in \mathfrak{g}$ such that $[z, \mathfrak{g}] = 0$.

The notation $[x, S]$ is undefined in the case we consider, where $S \subset \mathfrak{g}$ and $x \in \mathfrak{g}$, but we now fix this notation to mean $[\{x\}, S]$, ie, the linear span of the bracket products $[x, s]$ for fixed $x \in \mathfrak{g}$ and for all $s \in S$.

Proposition 2.2. \mathfrak{g}_Z is abelian and $\mathfrak{g}_Z \trianglelefteq \mathfrak{g}$.

Proof. We omit the verification that \mathfrak{g}_Z is a linear subspace of \mathfrak{g} . We have left to show that $[\mathfrak{g}_Z, \mathfrak{g}_Z] = 0$ and that $[\mathfrak{g}_Z, \mathfrak{g}] \subseteq \mathfrak{g}_Z$. Both follow from the observation that $[\mathfrak{g}_Z, \mathfrak{g}] = 0$. \square

Definition 2.6. A Lie algebra \mathfrak{g} is *simple* when \mathfrak{g} is non-abelian and the only ideals of \mathfrak{g} are 0 and \mathfrak{g} itself.

Example 10. Let $\dim V \geq 2$. $\mathfrak{gl}(V)$ is not simple because $(\mathfrak{gl}(V))_Z = \mathbb{F}\text{id}_V$ (where id_V is the identity map on V). $\mathfrak{sl}(V)$, however, is simple. It is somewhat non-trivial to prove this, though a proof may be found in any of the standard texts.

The simple Lie algebras were completely classified and enumerated, in case \mathbb{F} is complex-like, by Killing and Cartan as early as the 1890's [13]. The classification of all simple Lie algebras over \mathbb{R} appears in [17].

Definition 2.7. Let \mathfrak{g}_1 and \mathfrak{g}_2 be Lie algebras over \mathbb{F} . An \mathbb{F} -linear map $f : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ is called a *homomorphism* (or *endomorphism* in case $\mathfrak{g}_1 = \mathfrak{g}_2$) if

$$\forall x, y \in \mathfrak{g}_1, \quad f([x, y]) = [f(x), f(y)].$$

If in addition f is injective (ie, one-to-one) and surjective (ie, onto), then f is called an *isomorphism* (or *automorphism* in case $\mathfrak{g}_1 = \mathfrak{g}_2$) and \mathfrak{g}_1 and \mathfrak{g}_2 are said to be isomorphic, written $\mathfrak{g}_1 \cong \mathfrak{g}_2$.

Example 11. Let V be a vector space, and suppose that $T : V \rightarrow V$ is a change of basis, represented by an invertible matrix A in the sense that $T(v) = Av$. Let $\mathfrak{g} \leq \mathfrak{gl}(V)$. There is a change of basis $T' : \mathfrak{g} \rightarrow \mathfrak{g}$ corresponding to T , represented by matrix conjugation in that

$$T'(x) = A^{-1}xA.$$

Then, the map T' is an automorphism of \mathfrak{g} , since

$$\begin{aligned} T'([x, y]) &= A^{-1}(xy - yx)A \\ &= A^{-1}xyA - A^{-1}yxA \\ &= (A^{-1}xA)(A^{-1}yA) - (A^{-1}yA)(A^{-1}xA) \\ &= [T'(x), T'(y)]. \end{aligned}$$

Example 12. \mathbb{R}^3, \times is isomorphic to $\mathfrak{so}(3)$ by the isomorphism $\rho : \mathbb{R}^3 \rightarrow \mathfrak{so}(3)$ defined by

$$\rho \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix}.$$

Definition 2.8. Let $f : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ be a homomorphism. The *kernel* of f (written $\text{Ker } f$) is the set

$$\text{Ker } f = \{x \in \mathfrak{g}_1 \mid f(x) = 0\},$$

and the *image* of f (written $\text{Im } f$) is the set

$$\text{Im } f = \{f(x) \in \mathfrak{g}_2 \mid x \in \mathfrak{g}_1\}.$$

Proposition 2.3. Let $f : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ be a homomorphism. Then the kernel of f is an ideal of \mathfrak{g}_1 , and the image of f is a subalgebra of \mathfrak{g}_2 . □

The underlying vector space of a Lie algebra \mathfrak{g} may be decomposed as a direct sum of subspaces. For example $\mathfrak{g} = \mathfrak{h} \dot{+} \mathfrak{k}$ if \mathfrak{h} and \mathfrak{k} are subspaces of \mathfrak{g} , if $\mathfrak{h} \cap \mathfrak{k} = 0$, and if $\text{Span}(\mathfrak{h} \cup \mathfrak{k}) = \mathfrak{g}$, but an otherwise unqualified decomposition does not reflect any of the Lie algebra structure of \mathfrak{g} , meaning that the vector space decomposition is not necessarily compatible in any meaningful way with the bracket. We define two notions of direct sums of Lie algebras that are, to various degrees, compatible with the bracket.

Definition 2.9. Let $\mathfrak{g} = \mathfrak{a} \dot{+} \mathfrak{b}$. \mathfrak{g} is said to be the *Lie algebra direct sum* of \mathfrak{a} and \mathfrak{b} , written $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$ if both \mathfrak{a} and \mathfrak{b} are ideals of \mathfrak{g} .

The requirement that \mathfrak{a} and \mathfrak{b} be ideals of \mathfrak{g} guarantees that the bracket acts diagonally on either summand. In other words, $[\mathfrak{a}, \mathfrak{b}] = 0$ since $[\mathfrak{a}, \mathfrak{b}] \subseteq \mathfrak{a} \cap \mathfrak{b} = 0$. In this way, \mathfrak{g} is thought of as taking two distinct Lie algebras, \mathfrak{a} and \mathfrak{b} , and combining them together with the natural componentwise bracket rule

$$[a_1 + b_1, a_2 + b_2] = \underbrace{[a_1, a_2]}_{\in \mathfrak{a}} + \underbrace{[b_1, b_2]}_{\in \mathfrak{b}}$$

for $a_1, a_2 \in \mathfrak{a}, b_1, b_2 \in \mathfrak{b}$.

Definition 2.10. Let \mathfrak{g} be a Lie algebra. \mathfrak{g} is called *semisimple* if \mathfrak{g} is the direct sum of simple ideals. \mathfrak{g} is called *reductive* if $\mathfrak{g} = \mathfrak{g}_Z \oplus \mathfrak{g}_S$ for some semisimple ideal \mathfrak{g}_S .

Semisimple and reductive Lie algebras are natural generalization of simple Lie algebras, in the sense that much of the theory of simple Lie algebras can be extended to semisimple and reductive Lie algebras. We state without proof that the semisimple ideal \mathfrak{g}_S of a reductive \mathfrak{g} is maximal and unique up to isomorphism and that the simple summands of a semisimple \mathfrak{g} are unique up to isomorphism.

Example 13. $\mathfrak{sl}(\mathbb{C}^n)$ is semisimple because it is a sum of one simple ideal. $\mathfrak{gl}(\mathbb{C}^n) = \mathbb{C}I_n \oplus \mathfrak{sl}(\mathbb{C}^n)$ is neither simple nor semisimple but is reductive, with center consisting of scalar

matrices $\mathbb{C}I_n$ and maximal semisimple ideal $\mathfrak{sl}(\mathbb{C}^n)$ (Recall I_n denotes the $n \times n$ identity matrix).

Definition 2.11. Let $\mathfrak{g} = \mathfrak{a} \dot{+} \mathfrak{b}$. \mathfrak{g} is said to be the *Lie algebra semidirect sum* of \mathfrak{a} and \mathfrak{b} , written $\mathfrak{g} = \mathfrak{a} \rtimes \mathfrak{b}$, if \mathfrak{a} is an ideal of \mathfrak{g} and \mathfrak{b} is a subalgebra of \mathfrak{g} .

The notation $\mathfrak{g} = \mathfrak{a} \rtimes \mathfrak{b}$ is chosen to remind us that $\mathfrak{a} \trianglelefteq \mathfrak{g}$. Since \mathfrak{a} is an ideal, $[\mathfrak{b}, \mathfrak{a}] \subseteq \mathfrak{a}$. Explicitly, for each $x \in \mathfrak{b}$, the map $x \cdot : \mathfrak{a} \rightarrow \mathfrak{a}$ defined by $x \cdot a = [x, a]$ is a linear transformation on \mathfrak{a} . Because of this, we say that \mathfrak{b} acts on \mathfrak{a} and that \mathfrak{g} is an extension of \mathfrak{b} by \mathfrak{a} . Then the bracket rule on \mathfrak{g} may be described in terms of the brackets on the individual summands and the action of \mathfrak{b} on \mathfrak{a} as

$$[a_1 + b_1, a_2 + b_2] = \underbrace{[a_1, a_2] + b_1 \cdot a_2 - b_2 \cdot a_1}_{\in \mathfrak{a}} + \underbrace{[b_1, b_2]}_{\in \mathfrak{b}}.$$

2.2 Derivations and the adjoint map

Proposition 2.4. Let $\mathfrak{g} = \mathfrak{a} \rtimes \mathfrak{b}$. Define the map $\rho : \mathfrak{b} \rightarrow \mathfrak{gl}(\mathfrak{a})$ by $\rho(x) = x \cdot$ (ie, $\rho(x)(a) = [x, a]$) for $x \in \mathfrak{b}$. The map ρ is a homomorphism of Lie algebras.

Proof. We must show $\rho([x, y]) = [\rho(x), \rho(y)]$ for all $x, y \in \mathfrak{b}$. Let $a \in \mathfrak{a}$.

$$\begin{aligned} \rho([x, y])(a) &= [[x, y], a] \\ &= [x, [y, a]] - [y, [x, a]] && \text{by definition 2.1} \\ &= (\rho(x) \circ \rho(y) - \rho(y) \circ \rho(x))(a) \\ &= ([\rho(x), \rho(y)])(a). \end{aligned}$$

Since a was arbitrary, $\rho([x, y]) = [\rho(x), \rho(y)]$ for all $x, y \in \mathfrak{b}$. □

The map ρ is said to be a representation of \mathfrak{b} , and \mathfrak{a} is said to be a \mathfrak{b} -module.

Definition 2.12. Let \mathfrak{g} be a Lie algebra. A *representation* of \mathfrak{g} is a homomorphism $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ for some (finite-dimensional) vector space V . V , in this case, is said to be a \mathfrak{g} -*module*. When ρ is injective it is said to be a *faithful* representation.

Representations are a tool by which an abstract Lie algebra \mathfrak{g} may be studied more concretely by considering Lie algebras consisting of linear transformations. If a basis for V is chosen, the linear transformations themselves are then represented by matrices, further simplifying the study of \mathfrak{g} .

Proposition 2.5 (Ado's Theorem). *Let \mathbb{F} be a characteristic-zero field. Let \mathfrak{g} be a (finite-dimensional) Lie algebra over \mathbb{F} . Then, \mathfrak{g} admits a faithful representation. Explicitly, \mathfrak{g} is isomorphic to a space of matrices with entries in \mathbb{F} and bracket $[M, N] = MN - NM$ [2, Ch. I, §7.3].* □

Example 14. The map ρ of example 12 is a faithful representation of \mathbb{R}^3, \times onto $\mathfrak{so}(3)$. Since ρ is an isomorphism, it has an inverse $\rho^{-1} : \mathfrak{so}(3) \longrightarrow \mathbb{R}^3$; however, ρ^{-1} is not a representation because it does not map into a subspace of some $\mathfrak{gl}(V)$.

Definition 2.13. Let \mathfrak{g} be a Lie algebra. A linear map $D : \mathfrak{g} \longrightarrow \mathfrak{g}$ is called a *derivation* if

$$\forall x, y \in \mathfrak{g}, \quad D([x, y]) = [D(x), y] + [x, D(y)].$$

The definition of derivation is motivated by the familiar product rule of differentiation. In fact, the differential operator $\frac{d}{dx}$ is a derivation in a suitable context. We will not spend time developing this idea, other than to mention it. The interested reader is referred to [3] for an algebraic treatment or to [12] or [36] for a geometric point of view.

Definition 2.14. For a Lie algebra \mathfrak{g} , $\text{Der } \mathfrak{g}$ denotes the set of all derivations on \mathfrak{g} .

We note that if D_1 and D_2 are derivations, $D_1 \circ D_2$ need not necessarily be a derivation; however, $[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$ is a derivation. In light of this, we have $\text{Der } \mathfrak{g} \leq \mathfrak{gl}(\mathfrak{g})$.

Definition 2.15. Let \mathfrak{g} be a Lie algebra. For $x \in \mathfrak{g}$ the *adjoint* of x is the map $\text{ad } x : \mathfrak{g} \longrightarrow \mathfrak{g}$ defined by $\text{ad } x(y) = [x, y]$ for all $y \in \mathfrak{g}$. The *adjoint map* is the map $\text{ad} : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$.

Proposition 2.6. $\text{ad} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$ is a representation (in particular, a Lie algebra homomorphism). Moreover, for each $x \in \mathfrak{g}$, $\text{ad } x$ is a derivation on \mathfrak{g} .

The proposition follows from definition 2.1. In a sense, the definition of a Lie algebra \mathfrak{g} is intended to ensure that the action of multiplication by an element x (ie, the map $\text{ad } x$) is a derivation on \mathfrak{g} for all $x \in \mathfrak{g}$. It is for this reason that derivations take a primary role in the study of Lie algebras.

Proof of proposition 2.6. We omit the proofs that ad and $\text{ad } x$ are linear maps. Let $x, y, z \in \mathfrak{g}$ and consider $\text{ad } x$. We must to show that $\text{ad } x([y, z]) = [\text{ad } x(y), z] + [y, \text{ad } x(z)]$.

$$\begin{aligned} \text{ad } x([y, z]) &= [x, [y, z]] \\ &= -[z, [x, y]] - [y, [z, x]] && \text{by condition 3} \\ &= [[x, y], z] + [y, [x, z]] && \text{by condition 2} \\ &= [\text{ad } x(y), z] + [y, \text{ad } x(z)], \end{aligned}$$

so $\text{ad } x$ is a derivation. Next, we must show $\text{ad}[x, y] = [\text{ad } x, \text{ad } y]$.

$$\begin{aligned} \text{ad}[x, y](z) &= [[x, y], z] \\ &= -[[y, z], x] - [[z, x], y] && \text{by condition 3} \\ &= [x, [y, z]] - [y, [x, z]] && \text{by condition 2} \\ &= \text{ad } x(\text{ad } y(z)) - \text{ad } y(\text{ad } x(z)) \\ &= [\text{ad } x, \text{ad } y](z) \end{aligned}$$

so $\text{ad}[x, y] = [\text{ad } x, \text{ad } y]$, that is, ad is a homomorphism. □

We write $\text{ad } \mathfrak{g} = \text{Im } \text{ad}$ and note that $\text{ad } \mathfrak{g} \leq \text{Der } \mathfrak{g}$. Furthermore, notice $\text{Ker } \text{ad} = \mathfrak{g}_Z$.

Example 15. Consider the adjoint representation of $\mathfrak{gl}(\mathbb{C}^n)$. The kernel is CI_n and the image is isomorphic to $\mathfrak{gl}(\mathbb{C}^n)/CI_n \cong \mathfrak{sl}(\mathbb{C}^n)$.

In the example above, the traceless matrices $\mathfrak{sl}(\mathbb{C}^n)$ act as derivations on the space of matrices $\mathfrak{gl}(\mathbb{C}^n)$. A natural question that we will return to often in this dissertation is whether there are other derivations on $\mathfrak{gl}(\mathbb{C}^n)$, and if so, how we may characterize them.

Definition 2.16. Let \mathfrak{g} be a Lie algebra. An *inner derivation* of \mathfrak{g} is a member of $\text{ad } \mathfrak{g}$. Any member of $\text{Der } \mathfrak{g}$ not in $\text{ad } \mathfrak{g}$ is called an *outer derivation*.

Proposition 2.7. *An inner derivation maps \mathfrak{g} into $[\mathfrak{g}, \mathfrak{g}]$ and stabilizes ideals.*

Proof. Let D be an inner derivation, so $D = \text{ad } x$ for some $x \in \mathfrak{g}$. Let $y \in \mathfrak{g}$ be arbitrary and notice $D(y) = \text{ad } x(y) = [x, y] \in [\mathfrak{g}, \mathfrak{g}]$, verifying the first assertion. Next, let $\mathfrak{a} \trianglelefteq \mathfrak{g}$. $D(\mathfrak{a}) = \text{ad } x(\mathfrak{a}) = [x, \mathfrak{a}] \subseteq \mathfrak{a}$ by the definition of ideal. \square

In general, for a Lie algebra \mathfrak{g} we observe the chain of subspaces

$$\text{ad } \mathfrak{g} \leq \text{Der } \mathfrak{g} \leq \mathfrak{gl}(\mathfrak{g})$$

consisting of inner derivations, all derivations, and all linear maps respectively. The next theorem, a classical result in the theory of Lie algebras, completely characterizes the derivations of semisimple Lie algebras. Our work in chapter 3 of this dissertation can be understood as a generalization and extension of this classical result.

Proposition 2.8. *Let \mathfrak{g} be a semisimple Lie algebra over a field \mathbb{F} of characteristic not equal to two. The only derivations of \mathfrak{g} are inner derivations [13, 26].* \square

The proposition states that $\text{Der } \mathfrak{g} = \text{ad } \mathfrak{g}$ in case \mathfrak{g} is semisimple. A large portion of our work is to characterize outer derivations when \mathfrak{q} is a parabolic subalgebra (cf. section 2.6) of a reductive \mathfrak{g} .

Let \mathfrak{g} be semisimple and write as the sum of its simple ideals $\mathfrak{g} = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_k$. Each simple \mathfrak{g}_i is of course semisimple, so $\text{Der } \mathfrak{g}_i = \text{ad } \mathfrak{g}_i \cong \mathfrak{g}_i / (\mathfrak{g}_i)_Z = \mathfrak{g}_i$ by proposition 2.8 and since each $(\mathfrak{g}_i)_Z = 0$. Applying proposition 2.8 to the semisimple \mathfrak{g} gives

$$\text{Der } \mathfrak{g} = \text{ad } \mathfrak{g} \cong \mathfrak{g} / \mathfrak{g}_Z = \mathfrak{g} = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_k$$

so that we have

$$\text{Der}(\mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_k) \cong \text{Der}(\mathfrak{g}_1) \oplus \dots \oplus \text{Der}(\mathfrak{g}_k)$$

in case each \mathfrak{g}_i is simple. In this fashion, the direct sum structure of \mathfrak{g} is carried over to the derivation algebra $\text{Der } \mathfrak{g}$ when \mathfrak{g} is semisimple. This is not universally applicable to all direct sums of Lie algebras — it is true for semisimple Lie algebras because of propositions 2.7 and 2.8. An arbitrary derivation of a general Lie algebra does not necessarily stabilize ideals, and direct sum decomposition is not necessarily preserved. There are two ideals, however, that are stabilized by every derivation, related in the following proposition.

Proposition 2.9. *Let D be a derivation on an arbitrary Lie algebra \mathfrak{g} . D stabilizes $[\mathfrak{g}, \mathfrak{g}]$ and \mathfrak{g}_Z .*

Proof. Let $x, y \in \mathfrak{g}$.

$$D([x, y]) = [D(x), y] + [x, D(y)] \in [\mathfrak{g}, \mathfrak{g}],$$

so D stabilizes $[\mathfrak{g}, \mathfrak{g}]$ as desired. Next, let $z \in \mathfrak{g}_Z$. We need to show $D(z) \in \mathfrak{g}_Z$. Let $x \in \mathfrak{g}$ and consider $D([z, x])$.

$$0 = D(\underbrace{[z, x]}_{=0}) = [D(z), x] + \underbrace{[z, D(x)]}_{=0} = [D(z), x],$$

so $[D(z), x] = 0$ for all $x \in \mathfrak{g}$, meaning $D(z) \in \mathfrak{g}_Z$ as desired. □

2.3 Real Lie algebras and complexification

Let \mathbf{i} denote the imaginary unit. Let \mathfrak{g} be a Lie algebra over \mathbb{R} . \mathfrak{g} is an \mathbb{R} -vector space, but it is possible that vectors in \mathfrak{g} admit complex entries (cf. example 5). We would like to define $\hat{\mathfrak{g}}$ as the vector space $\mathfrak{g} + \mathbf{i}\mathfrak{g}$, but we are concerned about possible notational collisions between \mathbf{i} and entries of members of \mathfrak{g} . However, in light of Ado's Theorem (proposition 2.5), \mathfrak{g} is isomorphic to a real Lie algebra consisting of matrices with real entries, and we think of \mathfrak{g} in this way as we proceed in order to avoid this issue.

The vector space

$$\hat{\mathfrak{g}} = \mathfrak{g} + \mathbf{i}\mathfrak{g} = \{x + \mathbf{i}y \mid x, y \in \mathfrak{g}\}$$

of dimension $2 \dim \mathfrak{g}$ over \mathbb{R} may be thought of as a \mathbb{C} -vector space of dimension $\dim \mathfrak{g}$. We may define a bracket on $\hat{\mathfrak{g}}$ by

$$[x + \mathbf{i}y, u + \mathbf{i}v] = \underbrace{[x, u] - [y, v]}_{\in \mathfrak{g}} + \mathbf{i} \underbrace{([x, v] + [y, u])}_{\in \mathbf{i}\mathfrak{g}}.$$

We may verify that $\hat{\mathfrak{g}}$, together with the bracket defined above, satisfies definition 2.1 with $\mathbb{F} = \mathbb{C}$, making $\hat{\mathfrak{g}}$ a Lie algebra over \mathbb{C} .

Definition 2.17. The complex Lie algebra $\hat{\mathfrak{g}}$ is called the *complexification* of the real Lie algebra \mathfrak{g} . \mathfrak{g} is called a *real form* of $\hat{\mathfrak{g}}$.

It is possible for non-isomorphic real Lie algebras \mathfrak{g}_1 and \mathfrak{g}_2 to have isomorphic complexifications $\hat{\mathfrak{g}}_1 \cong \hat{\mathfrak{g}}_2 \cong \hat{\mathfrak{g}}$. In that case, both \mathfrak{g}_1 and \mathfrak{g}_2 are real forms of the complex Lie algebra $\hat{\mathfrak{g}}$.

Example 16. Since $\mathfrak{sl}(\mathbb{R}^n) + \mathbf{i}\mathfrak{sl}(\mathbb{R}^n) = \mathfrak{sl}(\mathbb{C}_\mathbb{R}^n) = \mathfrak{su}(n) + \mathbf{i}\mathfrak{su}(n)$, both $\mathfrak{sl}(\mathbb{R}^n)$ and $\mathfrak{su}(n)$ are real forms of $\mathfrak{sl}(\mathbb{C}^n)$.

Proposition 2.10. Let \mathfrak{g} be real, let $\hat{\mathfrak{g}} = \mathfrak{g} + \mathbf{i}\mathfrak{g}$ be the complexification of \mathfrak{g} . Then the center of $\hat{\mathfrak{g}}$ is the complexification of the center of \mathfrak{g} , namely $\hat{\mathfrak{g}}_Z = \widehat{\mathfrak{g}}_Z = \mathfrak{g}_Z + \mathbf{i}\mathfrak{g}_Z$.

Proof. Let $z \in \hat{\mathfrak{g}}_Z$. Write $z = x + \mathbf{i}y$ with $x, y \in \mathfrak{g}$. Now, for arbitrary $w = u + \mathbf{i}v \in \hat{\mathfrak{g}}$ with $u, v \in \mathfrak{g}$ we have

$$\begin{aligned} 0 = [z, w] &= [x + \mathbf{i}y, u + \mathbf{i}v] \\ &= [x, u] - [y, v] + \mathbf{i}([x, v] + [y, u]) \end{aligned}$$

and by direct sum decomposition $[x, u] = [y, v]$ and $[x, v] = -[y, u]$. Adding these equations gives

$$\forall u, v \in \mathfrak{g}, \quad [x, u + v] = [y, v - u] \quad (2.1)$$

Setting $v = u$ in equation 2.1 produces $[x, 2u] = 0$ for all $u \in \mathfrak{g}$, so $x \in \mathfrak{g}_Z$. Similarly, setting $u = -v$ in equation 2.1 produces $0 = [y, 2v]$ for all $v \in \mathfrak{g}$, so $y \in \mathfrak{g}_Z$. \square

Proposition 2.11. *Let \mathfrak{g} be a semisimple (res. reductive) Lie algebra over \mathbb{R} . The complexification $\hat{\mathfrak{g}}$ of \mathfrak{g} is semisimple (res. reductive) [17, Ch. VI, §9].* \square

Proposition 2.12. *Let D be a derivation of the real Lie algebra \mathfrak{g} . Then \hat{D} defined by $\hat{D}(x + \mathbf{i}y) = D(x) + \mathbf{i}D(y)$ is a derivation of $\hat{\mathfrak{g}}$. Moreover, \hat{D} stabilizes \mathfrak{g} .*

Proof. Let $z = x + \mathbf{i}y, w = u + \mathbf{i}v$ be arbitrary elements of $\hat{\mathfrak{g}}$.

$$\begin{aligned} \hat{D}([z, w]) &= \hat{D}([x + \mathbf{i}y, u + \mathbf{i}v]) \\ &= \hat{D}([x, u] - [y, v] + \mathbf{i}([x, v] + [y, u])) \\ &= D([x, u] - [y, v]) + \mathbf{i}D([x, v] + [y, u]) \\ &= D([x, u]) - D([y, v]) + \mathbf{i}(D([x, v]) + D([y, u])) \\ &= [D(x), u] + [x, D(u)] - [D(y), v] - [y, D(v)] \\ &\quad + \mathbf{i}([D(x), v] + [x, D(v)] + [D(y), u] + [y, D(u)]) \end{aligned}$$

$$\begin{aligned}
&= [D(x), u + \mathbf{i}v] + [x + \mathbf{i}y, D(u)] + \mathbf{i}[x + \mathbf{i}y, D(v)] + \mathbf{i}[D(y), u + \mathbf{i}v] \\
&= [D(x), w] + \mathbf{i}[D(y), w] + [z, D(u)] + \mathbf{i}[z, D(v)] \\
&= [D(x) + \mathbf{i}D(y), w] + [z, D(u) + \mathbf{i}D(v)] \\
&= [\hat{D}(z), w] + [z, \hat{D}(w)]
\end{aligned}$$

So \hat{D} is a derivation on $\hat{\mathfrak{g}}$.

\hat{D} stabilizes \mathfrak{g} by definition. Indeed, if $x \in \mathfrak{g}$, then $\hat{D}(x) = \hat{D}(x + \mathbf{i}0) = D(x) \in \mathfrak{g}$. \square

2.4 Root space decomposition

Let \mathbb{K} denote a complex-like field. Fix \mathfrak{g} as denoting a semisimple Lie algebra over \mathbb{K} . We will show how to decompose \mathfrak{g} into a (vector space) direct sum of certain subspaces that have desirable interactions with the bracket $[\cdot, \cdot]$, made explicit below. Our approach largely follows Humphreys, and the reader is referred to [13, Ch. II, §8] for proofs.

Definition 2.18. A *toral subalgebra* of \mathfrak{g} is a subalgebra $\mathfrak{h} \leq \mathfrak{g}$ that is abelian and for each $x \in \mathfrak{h}$, the map $\text{ad } x : \mathfrak{g} \longrightarrow \mathfrak{g}$ is diagonalizable. If \mathfrak{h} is a maximal toral subalgebra, it is called a *Cartan subalgebra* of \mathfrak{g} .

For a general Lie algebra \mathfrak{g} , a Cartan subalgebra is typically defined to be a self-normalizing nilpotent subalgebra, but this is more generality than we require. Within the class of semisimple Lie algebras, the general notion coincides with the simpler-to-state definition 2.18. Cartan subalgebras are unique in the sense made precise below.

Proposition 2.13. *The Cartan subalgebras of \mathfrak{g} are conjugate to one another, in the sense that any one may be transformed into any other by an appropriate automorphism of \mathfrak{g} .* \square

As a result of proposition 2.13, each Cartan subalgebra of \mathfrak{g} has the same dimension. We call this number the *rank* of \mathfrak{g} . Now, fix \mathfrak{h} as denoting a specific Cartan subalgebra of \mathfrak{g} . Consider the dual vector space \mathfrak{h}^* consisting of linear functionals $\alpha : \mathfrak{h} \longrightarrow \mathbb{C}$.

Definition 2.19. For a non-zero $\alpha \in \mathfrak{h}^*$, define the subspace \mathfrak{g}_α of \mathfrak{g} by

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid \forall h \in \mathfrak{h}, [h, x] = \alpha(h)x\}.$$

In case $\mathfrak{g}_\alpha \neq 0$, α is called a *root* and \mathfrak{g}_α is called a *root space*. Denote by Φ the set of roots. Φ is called a *root system*. The *rank* of Φ is the dimension of \mathfrak{h}^* .

Some basic facts about the root spaces \mathfrak{g}_α for $\alpha \in \Phi$:

Proposition 2.14. Let $\alpha, \beta \in \Phi$. $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ if $\alpha + \beta \in \Phi$ and $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$ if $\alpha + \beta \notin \Phi$. Moreover, each \mathfrak{g}_α is one-dimensional. \square

Let \mathbb{Z}^+ denote the set of positive integers and \mathbb{Z}^- denote the set of negative integers. Follows are some basic facts about the geometric properties of the root system Φ .

Proposition 2.15. Let r denote the rank of Φ . A set Δ of r roots (referred to as a *base* of Φ) may be selected so that each root $\beta \in \Phi$ can be written uniquely as either a \mathbb{Z}^+ -linear combination or a \mathbb{Z}^- -linear combination of roots in Δ . Moreover, $\Phi = -\Phi$ and if $\beta \in \Phi$ and $\beta = \sum_{i=1}^k \alpha_i$ with each (not-necessarily non-repeating) $\alpha_i \in \Delta$, we may rearrange the terms of the sum so the partial sums $\sum_{i=1}^j \alpha_{\sigma(i)}$ lie in Φ for each $j \leq k$ each (where σ denotes an appropriate permutation). \square

In light of propositions 2.14 and 2.15 we fix the following notation and terminology: a Δ as in proposition 2.15 is called a *base* of Φ , and roots in Δ are called *simple roots*; roots in Φ generated by positive integer combinations of simple roots are called *positive roots*, and Φ^+ denotes the set of positive roots; and similarly for *negative roots* and Φ^- . From each \mathfrak{g}_β for $\beta \in \Phi$ we arbitrarily choose an $x_\beta \in \mathfrak{g}_\beta$ so that $\mathfrak{g}_\beta = \mathbb{K}x_\beta$ by proposition 2.14.

Proposition 2.16 (Root space decomposition). *The semisimple Lie algebra \mathfrak{g} decomposes as the vector space direct sum*

$$\mathfrak{g} = \mathfrak{h} \dot{+} \sum_{\beta \in \Phi} \mathbb{K}x_\beta.$$

This decomposition is called the root space decomposition of \mathfrak{g} relative of \mathfrak{h} , and is essentially unique in that the root space decompositions of \mathfrak{g} relative to two Cartan subalgebras \mathfrak{h}_1 and \mathfrak{h}_2 differ only by an automorphism of \mathfrak{g} . \square

We may extend these notions to reductive Lie algebras over \mathbb{K} . When $\mathfrak{g} = \mathfrak{g}_Z \oplus \mathfrak{g}_S$ is reductive, then a Cartan subalgebra of \mathfrak{g} takes the form $\mathfrak{g}_Z \oplus \mathfrak{h}$, where \mathfrak{h} is a Cartan subalgebra of \mathfrak{g}_S . In this case, by *root system* Φ of the reductive \mathfrak{g} we mean the root system of the semisimple part \mathfrak{g}_S , and by *root space decomposition* of \mathfrak{g} we mean the vector space direct sum decomposition

$$\mathfrak{g} = \mathfrak{g}_Z \dot{+} \mathfrak{h} \dot{+} \sum_{\beta \in \Phi} \mathbb{K}x_\beta.$$

Example 17. Consider $\mathfrak{sl}(\mathbb{C}^3)$, concretely the Lie algebra of 3×3 traceless matrices with complex entries. One choice of Cartan subalgebra is the subalgebra \mathfrak{h} consisting of diagonal matrices. We take as basis for \mathfrak{h} the matrices $h_1 = e_{1,1} - e_{2,2}$ and $h_2 = e_{2,2} - e_{3,3}$. (Recall, $e_{i,j}$ denotes the matrix with a 1 in the i, j position and zeros elsewhere.)

Our next task is to find all $\beta \in \mathfrak{h}^*$ such that $\mathfrak{g}_\beta \neq 0$. Write $\beta = b_1 h_1^* + b_2 h_2^*$, where $h_i^*(h_j) = \delta_{i,j}$ is the dual functional to the vector h_i . h_1^* and h_2^* are not roots, as $\mathfrak{g}_{h_1^*} = \mathfrak{g}_{h_2^*} = 0$. However, since h_1^*, h_2^* together span \mathfrak{h}^* , any root β will be of the form $\beta = b_1 h_1^* + b_2 h_2^*$ with $b_1, b_2 \in \mathbb{C}$. In particular, straightforward computation show that $\mathfrak{g}_{2h_1^* - h_2^*} \neq 0$ and $\mathfrak{g}_{-h_1^* + 2h_2^*} \neq 0$, among others. We may write $\alpha_1 = 2h_1^* - h_2^*, \alpha_2 = -h_1^* + 2h_2^*$ and as we shall see below, $\Delta = \{\alpha_1, \alpha_2\}$ is a base of the root system Φ . We record the individual root spaces, our choice base Δ , and our choice of basis vector for each root space in table 2.1.

Root β	β in terms of Δ	Root Space \mathfrak{g}_β	Choice of x_β
$2h_1^* - h_2^*$	α_1	$\mathbb{C}e_{1,2}$	$x_{\alpha_1} = e_{1,2}$
$-h_1^* + 2h_2^*$	α_2	$\mathbb{C}e_{2,3}$	$x_{\alpha_2} = e_{2,3}$
$h_1^* + h_2^*$	$\alpha_1 + \alpha_2$	$\mathbb{C}e_{1,3}$	$x_{\alpha_1 + \alpha_2} = e_{1,3}$
$2h_1^* - h_2^*$	$-\alpha_1$	$\mathbb{C}e_{2,1}$	$x_{-\alpha_1} = e_{2,1}$
$-h_1^* + 2h_2^*$	$-\alpha_2$	$\mathbb{C}e_{3,2}$	$x_{-\alpha_2} = e_{3,2}$
$h_1^* + h_2^*$	$-\alpha_1 - \alpha_2$	$\mathbb{C}e_{3,1}$	$x_{-\alpha_1 - \alpha_2} = e_{3,1}$

Table 2.1: $\mathfrak{sl}(\mathbb{C}^3)$ root spaces

Because the root system Φ may be written $\Phi = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, -\alpha_1, -\alpha_2, -\alpha_1 - \alpha_2\}$, we see that $\Delta = \{\alpha_1, \alpha_2\}$ was a suitable choice of base. Since $\dim \mathfrak{h}^* = 2$, Φ lies in a two-dimensional plane. Figure 2.1 provides an illustration of Φ .

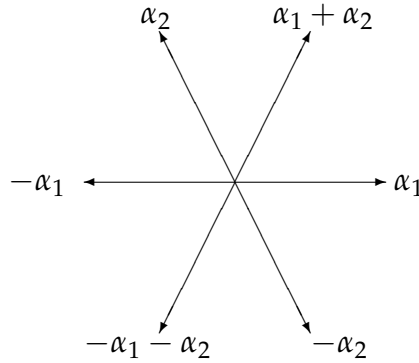


Figure 2.1: A_2 root system

Finally, $\mathfrak{sl}(\mathbb{C}^3)$ decomposes as the vector space direct sum of root spaces:

$$\mathfrak{sl}(\mathbb{C}^3) = \mathfrak{h} \dot{+} \sum_{\beta \in \Phi} \mathbb{C}x_{\beta}.$$

Refer to figure 2.2 for a graphical representation of the root space decomposition of $\mathfrak{sl}(\mathbb{C}^3)$.

$$\left[\begin{array}{c|c|c} \mathfrak{h} & \mathfrak{g}_{\alpha_1} & \mathfrak{g}_{\alpha_1+\alpha_2} \\ \hline \mathfrak{g}_{-\alpha_1} & \mathfrak{h} & \mathfrak{g}_{\alpha_2} \\ \hline \mathfrak{g}_{-\alpha_1-\alpha_2} & \mathfrak{g}_{-\alpha_2} & \mathfrak{h} \end{array} \right]$$

Figure 2.2: $\mathfrak{sl}(\mathbb{C}^3)$ root space decomposition

The root space decomposition of $\mathfrak{sl}(\mathbb{C}^3)$ — and more generally of $\mathfrak{sl}(\mathbb{C}^n)$ — is, in a sense, a triviality, since it merely write $\mathfrak{sl}(\mathbb{C}^3)$ in terms of the standard unit matrices $e_{i,j}$. Partly this is because we chose a well-behaved Cartan subalgebra, but partly this

situation is intentional. The root space decomposition of $\mathfrak{sl}(\mathbb{C}^n)$ relative to the Cartan subalgebra consisting of diagonal matrices can be thought of as the motivating example for root space decomposition in general. In this sense, the root space decomposition of a general reductive \mathfrak{g} provides an \mathfrak{sl} -esque basis for general \mathfrak{g} [13], allowing one to reduce questions in Lie algebra theory to questions in linear algebra and matrix theory. Root space decomposition also allows for induction on the height of roots as a proof technique (where, for example, the roots α_2 and $-\alpha_1 - \alpha_2$ have respective heights 1 and 2).

2.5 Restricted root space decomposition

In this section, let \mathfrak{g} denote a semisimple Lie algebra over \mathbb{R} . We will define the restricted root space decomposition of \mathfrak{g} , a coarser decomposition than the analogous root space decomposition for Lie algebras over complex-like fields. Except for some notational changes made for internal consistency, the development of these ideas here follows Knapp's, and the reader is referred to [17, Ch. VI, §2-4] for proofs.

Definition 2.20. Let $\theta : \mathfrak{g} \rightarrow \mathfrak{g}$ be an automorphism. θ is called a *Cartan involution* of \mathfrak{g} if $\theta \circ \theta = \text{id}_{\mathfrak{g}}$ and if the bilinear map

$$B_{\theta}(x, y) = -\text{Tr} \left((\text{ad } x)(\text{ad } \theta(y)) \right) \quad (2.2)$$

is positive definite.

For $x, y \in \mathfrak{g}$, notice that $\text{ad } x, \text{ad } \theta(y)$ are linear transformations $\mathfrak{g} \rightarrow \mathfrak{g}$, which we may think of as matrices. Observing this, we see that the B_{θ} , defined by equation 2.2, is well defined. The interested reader may note that, in general, the bilinear map $B(x, y) = \text{Tr}(\text{ad } x)(\text{ad } y)$ on $\mathfrak{g} \times \mathfrak{g}$ is called the *Killing form*, after Wilhelm Killing, and has many applications in the structure theory of Lie algebras [2, 13, 17]. We will not make use of the Killing form in the sequel, though Cartan involutions are necessary to arrive at the Cartan decomposition of \mathfrak{g} , described below.

Example 18. Recall $\mathfrak{sl}(\mathbb{R}^n)$, $\mathfrak{su}(n)$, and $\mathfrak{sl}(\mathbb{C}_{\mathbb{R}}^n)$ are Lie algebras over \mathbb{R} . The map $\theta(M) = -M^t$ on $\mathfrak{sl}(\mathbb{R}^n)$ is a Cartan involution. The identity map on $\mathfrak{su}(n)$ is a Cartan involution. Complex conjugation $\theta(M) = \overline{M}$ is a Cartan involution on $\mathfrak{sl}(\mathbb{C}_{\mathbb{R}}^n)$, as well as is the map $\theta(M) = -M^*$.

Proposition 2.17. *Every real Lie algebra \mathfrak{g} has a Cartan involution θ . The Cartan involution θ of \mathfrak{g} is essentially unique, in the sense that if θ' is another Cartan involution of \mathfrak{g} , then $\theta' = \varphi \circ \theta \circ \varphi^{-1}$ for some automorphism φ of \mathfrak{g} . \square*

Given a Cartan involution θ we may consider the eigenvalues of θ . Since $\theta \circ \theta = \text{id}_{\mathfrak{g}}$ the eigenvalues of θ are 1 and -1 . We write \mathfrak{k} for the eigenspace corresponding to 1 and \mathfrak{p} for the eigenspace corresponding to -1 , and observe the following results:

Proposition 2.18. *For θ a Cartan involution of \mathfrak{g} we have $\mathfrak{g} = \mathfrak{k} \dot{+} \mathfrak{p}$, with \mathfrak{k} and \mathfrak{p} as above. Furthermore, $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}$, and $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$. \square*

Definition 2.21. Notation as above, the decomposition $\mathfrak{g} = \mathfrak{k} \dot{+} \mathfrak{p}$ is called the *Cartan decomposition* of \mathfrak{g} .

Starting from a Cartan decomposition, we select a maximal abelian subspace $\mathfrak{a} \subseteq \mathfrak{p}$ and we set $\mathfrak{m} = \{x \in \mathfrak{k} \mid [x, \mathfrak{a}] = 0\} \subseteq \mathfrak{k}$. For each $\lambda \in \mathfrak{a}^*$ we write

$$\mathfrak{g}_{\lambda} = \{x \in \mathfrak{g} \mid \forall h \in \mathfrak{a}, [h, x] = \lambda(h)x\}$$

analogously to the complex-like case.

Definition 2.22. Let $\lambda \in \mathfrak{a}^*$ be non-zero. In case $\mathfrak{g}_{\lambda} \neq 0$, we call λ a *restricted root* of \mathfrak{g} and we call \mathfrak{g}_{λ} the *restricted root space* associated to λ . $\Phi = \{\lambda \in \mathfrak{a}^* \mid \lambda \text{ is a restricted root}\}$ is called the *restricted root system* of \mathfrak{g} relative to \mathfrak{a} .

Proposition 2.19 (Restricted root space decomposition). *\mathfrak{g} decomposes as the vector space direct sum*

$$\mathfrak{g} = \mathfrak{a} \dot{+} \mathfrak{m} \dot{+} \sum_{\lambda \in \Phi} \mathfrak{g}_{\lambda}.$$

Furthermore, the restricted root spaces $\mathfrak{g}_\alpha, \mathfrak{g}_\beta$ satisfy $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$. \square

In contrast to the complex-like case, the restricted root space \mathfrak{g}_λ need not be one-dimensional. The restricted root system exhibits some of the same geometric properties exhibited by the root system of a complex-like Lie algebra, which we discuss in the next two propositions.

Proposition 2.20. *The restricted root system Φ satisfies $\Phi = -\Phi$. Furthermore, a set of positive restricted roots Φ^+ may be selected with the properties that for each $\lambda \in \Phi$ exactly one of λ or $-\lambda$ lies in Φ^+ and for each $\alpha, \beta \in \Phi^+$, if $\alpha + \beta \in \Phi$ then $\alpha + \beta \in \Phi^+$.* \square

With Φ and Φ^+ fixed, by a *simple* restricted root we mean a positive restricted root that does not decompose as the sum of two or more other positive restricted roots. Write Δ for the set of simple restricted roots. We call Δ a base of Φ .

Proposition 2.21. *There are $\dim \mathfrak{a}$ simple restricted roots. Δ is linearly independent and spans Φ . $\Phi^+ \subseteq \text{Span}_{\mathbb{Z}^+}(\Delta)$ and $\Phi^- \subseteq \text{Span}_{\mathbb{Z}^-}(\Delta)$. If $\alpha, \beta \in \Delta$, then $\alpha - \beta \notin \Phi$.* \square

In case $\mathfrak{g} = \mathfrak{g}_Z \oplus \mathfrak{g}_S$ is reductive, then when we refer to the restricted root space decomposition or restricted root system of \mathfrak{g} , we mean respectively the restricted root space decomposition and restricted root system of \mathfrak{g}_S .

Example 19. Let $\mathfrak{g} = \mathfrak{sl}(\mathbb{R}^3)$. With respect to the Cartan involution $\theta(M) = -M^t$, the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is given by $\mathfrak{k} = \mathfrak{so}(3)$ and $\mathfrak{p} = \{M \in \mathfrak{sl}(\mathbb{R}^3) \mid M \text{ is symmetric}\}$. $\mathfrak{a} \subset \mathfrak{p}$ may be chosen as consisting of diagonal traceless matrices, and then $\mathfrak{m} = 0$ since skew-symmetric matrices have zeros along the diagonal. For each pair (i, j) with $i, j \leq 3$ and $i \neq j$ we have the restricted root $\lambda_{i,j}$ corresponding to the one-dimensional root space $\mathfrak{g}_{\lambda_{i,j}} = \mathbb{R}e_{i,j}$ giving the restricted root space decomposition

$$\mathfrak{sl}(\mathbb{R}^3) = \mathfrak{a} \dot{+} \sum_{\substack{i,j \leq 3 \\ i \neq j}} \mathbb{R}e_{i,j}$$

similar to the root space decomposition of $\mathfrak{sl}(\mathbb{C}^3)$ given in example 17.

Example 20. Consider $\mathfrak{g} = \mathfrak{sl}(\mathbb{C}_{\mathbb{R}}^3)$ as a Lie algebra over \mathbb{R} . The map θ defined by $\theta(M) = -M^*$ is a Cartan involution on \mathfrak{g} with corresponding Cartan decomposition $\mathfrak{sl}(\mathbb{C}_{\mathbb{R}}^3) = \mathfrak{su}(3) + \mathfrak{isu}(3)$ with $\mathfrak{k} = \mathfrak{su}(3)$ and $\mathfrak{p} = \mathfrak{isu}(3)$ (cf. example 5). Chose $\mathfrak{a} \subseteq \mathfrak{p}$ to consist of the diagonal matrices in \mathfrak{p} ; namely, \mathfrak{a} consists of traceless diagonal matrices with pure imaginary entries. Then \mathfrak{m} consists of the diagonal matrices in \mathfrak{k} — traceless diagonal matrices with real entries. For each pair (i, j) with $i, j \leq 3$ and $i \neq j$ we have a restricted root $\lambda_{i,j}$ where the corresponding restricted root space $\mathfrak{g}_{\lambda_{i,j}}$ is 2-dimensional and takes the form

$$\mathfrak{g}_{\lambda_{i,j}} = \{ a_1 e_{i,j} + a_2 i e_{i,j} \mid a_1, a_2 \in \mathbb{R} \}.$$

The restricted root space decomposition is then given by the vector space direct sum

$$\mathfrak{sl}(\mathbb{C}_{\mathbb{R}}^3) = \mathfrak{a} \dot{+} \mathfrak{m} \dot{+} \sum_{\substack{i,j \leq 3 \\ i \neq j}} \mathfrak{g}_{\lambda_{i,j}}.$$

2.6 Parabolic subalgebras

In this section, we will define Borel subalgebras and parabolic subalgebras of reductive Lie algebras over a complex-like field \mathbb{K} or over \mathbb{R} . We first consider parabolic subalgebras in the case that \mathfrak{g} is semisimple over \mathbb{K} and close by extending the definition to the larger class of Lie algebras.

Definition 2.23. Let \mathbb{K} be complex-like. Let \mathfrak{g} be semisimple over \mathbb{K} . A *Borel* subalgebra \mathfrak{b} of \mathfrak{g} is a subalgebra of the form

$$\mathfrak{b} = \mathfrak{h} \dot{+} \sum_{\beta \in \Phi^+} \mathfrak{g}_{\beta}$$

where \mathfrak{h} is a Cartan subalgebra of \mathfrak{g} and Φ is the root system of \mathfrak{g} relative to \mathfrak{h} .

Example 21. In case $\mathfrak{g} = \mathfrak{sl}(\mathbb{C}^3)$ (as a complex Lie algebra) and \mathfrak{h} consists of diagonal matrices, \mathfrak{b} is the subalgebra of \mathfrak{g} consisting of upper triangular matrices.

$$\begin{array}{cccc}
\begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix} & \begin{pmatrix} * & * & * \\ * & * & * \\ & & * \end{pmatrix} & \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix} & \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix} \\
\mathfrak{q}(\emptyset) = \mathfrak{b} & \mathfrak{q}(\{\alpha_1\}) & \mathfrak{q}(\{\alpha_2\}) & \mathfrak{q}(\{\alpha_1, \alpha_2\}) = \mathfrak{g}
\end{array}$$

Figure 2.3: Standard parabolic subalgebras of $\mathfrak{sl}(\mathbb{C}^3)$

Definition 2.24. Let \mathbb{K} be complex-like. Let \mathfrak{g} be semisimple over \mathbb{K} . With notation as above, a *standard parabolic subalgebra* of \mathfrak{g} relative to the Cartan subalgebra \mathfrak{h} is a subalgebra \mathfrak{q} of \mathfrak{g} satisfying $\mathfrak{b} \leq \mathfrak{q} \leq \mathfrak{g}$. A *parabolic subalgebra* of \mathfrak{g} is a subalgebra $\mathfrak{q} \leq \mathfrak{g}$ that is a standard parabolic subalgebra for some appropriate choice of \mathfrak{h} .

Example 22. Where $\mathfrak{g} = \mathfrak{sl}(\mathbb{C}^3)$ and \mathfrak{b} is as above, any standard parabolic subalgebras of \mathfrak{g} consists of block upper triangular matrices. Each parabolic subalgebra of \mathfrak{g} is simply one of the standard parabolic subalgebras conjugated by a change of basis of \mathbb{C}^3 .

Proposition 2.22. *The standard parabolic subalgebras of \mathfrak{g} relative to \mathfrak{b} are in one-to-one correspondence with subsets of Δ . Explicitly, for a subset $\Delta' \subseteq \Delta$, the corresponding parabolic subalgebra $\mathfrak{q} = \mathfrak{q}(\Delta')$ is*

$$\mathfrak{q} = \mathfrak{h} \dot{+} \sum_{\beta \in \Phi'} \mathfrak{g}_\beta$$

where

$$\Phi' = \Phi^+ \cup (\Phi \cap \text{Span } \Delta'),$$

ie, Φ' consists of all positive roots and the negative roots spanned by Δ' . □

Example 23. Since $\Delta = \{\alpha_1, \alpha_2\}$, $\mathfrak{g} = \mathfrak{sl}(\mathbb{C}^3)$ has four standard parabolic subalgebras, illustrated in figure 2.3. Additionally, figure 2.4 gives an illustration of Φ' when $\Delta' = \{\alpha_2\}$. That a particular root β is included in Φ' is indicated by placing \bullet at the head of β , while $\beta \notin \Phi'$ is indicated with \circ .

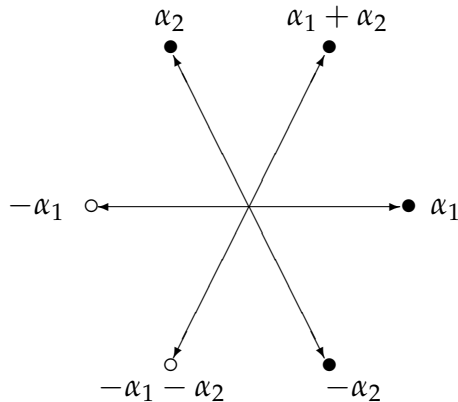


Figure 2.4: Φ' where $\Delta' = \{\alpha_2\}$

Since a parabolic subalgebra differs from a standard parabolic subalgebra only by an automorphism of \mathfrak{g} , proofs valid for any parabolic subalgebra $\mathfrak{q} \leq \mathfrak{g}$ need only consider the case where \mathfrak{q} is a standard parabolic subalgebra. We will often make use of this principle in the sequel.

In case $\mathfrak{g} = \mathfrak{g}_Z \oplus \mathfrak{g}_S$ is reductive over \mathbb{K} , a Borel subalgebra \mathfrak{b} of \mathfrak{g} is of the form $\mathfrak{b} = \mathfrak{g}_Z \oplus \mathfrak{b}_S$ where $\mathfrak{b}_S = \mathfrak{b} \cap \mathfrak{g}_S$ is a Borel subalgebra of \mathfrak{g}_S . A parabolic subalgebra \mathfrak{q} of \mathfrak{g} is of the form $\mathfrak{q} = \mathfrak{g}_Z \oplus \mathfrak{q}_S$ where $\mathfrak{q}_S = \mathfrak{q} \cap \mathfrak{g}_S$ is a parabolic subalgebra of \mathfrak{g}_S .

Having defined parabolic subalgebras in the complex-like semisimple and reductive cases, we now extend the definition to include real semisimple and reductive Lie algebras.

Definition 2.25. Let \mathfrak{g} be a reductive Lie algebra over \mathbb{R} . A *parabolic subalgebra* \mathfrak{q} of \mathfrak{g} is a subalgebra such that the complexification $\widehat{\mathfrak{q}}$ is a parabolic subalgebra of $\widehat{\mathfrak{g}}$.

Notice that because $(\widehat{\mathfrak{g}})_Z = \widehat{(\mathfrak{g}_Z)}$, a parabolic subalgebra \mathfrak{q} of a real reductive $\mathfrak{g} = \mathfrak{g}_Z \oplus \mathfrak{g}_S$ has the form $\mathfrak{q} = \mathfrak{g}_Z \oplus \mathfrak{q}_S$, where \mathfrak{q}_S is a parabolic subalgebra of \mathfrak{g} . In addition, parabolic subalgebras of a real Lie algebra exhibit a structure theory relating to the restricted root

space decomposition analogous to the structure theory of parabolic subalgebras of Lie algebras over a complex-like field.

Proposition 2.23. *Let \mathfrak{g} be reductive over \mathbb{R} . Let \mathfrak{q} be a parabolic subalgebra of \mathfrak{g} . We may choose a restricted root space decomposition*

$$\mathfrak{g} = \mathfrak{a} \dot{+} \mathfrak{m} \dot{+} \sum_{\lambda \in \Phi} \mathfrak{g}_\lambda$$

with restricted root system Φ and set of simple restricted roots Δ so that \mathfrak{q} has the form

$$\mathfrak{q} = \mathfrak{a} \dot{+} \mathfrak{m} \dot{+} \sum_{\lambda \in \Phi'} \mathfrak{g}_\lambda$$

where

$$\Phi' = \Phi^+ \cup (\Phi \cap \text{Span } \Delta'),$$

for an appropriate subset Δ' of Δ . □

2.7 Langland's decomposition

What follows is perhaps not part of the classical theory of Lie algebras, but can be found in chapter V, section 7 of [17] in case \mathfrak{g} is complex-like and in chapter VII, section 7 of [17] in case \mathfrak{g} is real.

Let \mathfrak{g} be semisimple over a complex-like field \mathbb{K} or over \mathbb{R} and let $\mathfrak{q} \leq \mathfrak{g}$ be a parabolic subalgebra. Without loss of generality, we may assume that \mathfrak{q} arises from a (restricted) root space decomposition of \mathfrak{g} .

In the complex-like case, we have the following situation:

$$\mathfrak{g} = \mathfrak{h} \dot{+} \sum_{\beta \in \Phi} \mathfrak{g}_\beta, \text{ where}$$

\mathfrak{h} is a Cartan subalgebra of \mathfrak{g} ,

Φ is the root system of \mathfrak{g} relative to \mathfrak{h} ,

Δ is a base of Φ ,

$\Delta' \subseteq \Delta$ is the subset of Δ corresponding to \mathfrak{q} , and

$$\Phi' = \Phi^+ \cup (\Phi \cap \text{Span } \Delta').$$

Then $\mathfrak{q} = \mathfrak{h} \dot{+} \sum_{\beta \in \Phi'} \mathfrak{g}_\beta$.

Considering the case where \mathfrak{g} is real, we have the analogous situation:

$$\mathfrak{g} = \mathfrak{a} \dot{+} \mathfrak{m} \dot{+} \sum_{\lambda \in \Phi} \mathfrak{g}_\lambda \text{ where,}$$

Φ is the restricted root system of \mathfrak{g} relative to \mathfrak{a} ,

Δ is a set of simple restricted roots of Φ ,

$\Delta' \subseteq \Delta$ is the subset of Δ corresponding to \mathfrak{q} , and

$$\Phi' = \Phi^+ \cup (\Phi \cap \text{Span } \Delta'),$$

so that $\mathfrak{q} = \mathfrak{a} \dot{+} \mathfrak{m} \dot{+} \sum_{\lambda \in \Phi'} \mathfrak{g}_\lambda$.

Φ' may be partitioned into two subsets, $\Phi' \cap -\Phi$ and $\Phi' \setminus -\Phi$. This partition of Φ' results in a vector space direct sum decomposition of \mathfrak{q} as

$$\mathfrak{q} = \mathfrak{l} \dot{+} \mathfrak{n}$$

where

$$\mathfrak{l} = \mathfrak{h} \dot{+} \sum_{\beta \in \Phi' \cap -\Phi} \mathfrak{g}_\beta$$

and

$$\mathfrak{n} = \sum_{\beta \in \Phi' \setminus -\Phi} \mathfrak{g}_\beta.$$

Proposition 2.24. *Notation as above, \mathfrak{n} is an ideal of \mathfrak{q} , \mathfrak{l} is a subalgebra of \mathfrak{q} , and \mathfrak{l} is reductive. [17, Ch. V, §7; Ch. VII, §7]* □

Definition 2.26. Notation as above, \mathfrak{l} is called the *Levi factor* of \mathfrak{q} and \mathfrak{n} is called the *nilradical* of \mathfrak{q} . The decomposition $\mathfrak{q} = \mathfrak{l} \dot{+} \mathfrak{n}$ is called *Langland's decomposition*. (Notice that since \mathfrak{n} is an ideal, \mathfrak{q} is the Lie algebra semidirect sum of the subalgebra \mathfrak{l} and the ideal \mathfrak{n} , and we may write $\mathfrak{q} = \mathfrak{l} \ltimes \mathfrak{n}$.)

Finally, we extend this terminology and these notions to the case there $\mathfrak{g} = \mathfrak{g}_Z \oplus \mathfrak{g}_S$ is reductive and $\mathfrak{q} = \mathfrak{g}_Z \oplus \mathfrak{q}_S$, by simply writing

$$\mathfrak{q} = \mathfrak{g}_Z \oplus (\mathfrak{l} \dot{+} \mathfrak{n})$$

where \mathfrak{l} is the Levi factor and \mathfrak{n} is the nilpotent radical of \mathfrak{q}_S . In such case, we say \mathfrak{l} (res. \mathfrak{n}) is the Levi factor (res. nilradical) of \mathfrak{q} and of \mathfrak{q}_S interchangeably.

Example 24. Let $\mathfrak{g} = \mathfrak{gl}(\mathbb{C}^6) = \mathbb{C}I_6 \oplus \mathfrak{sl}(\mathbb{C}^6)$, let \mathfrak{h} consist of traceless diagonal matrices. Then \mathfrak{h} has dimension 5. The root system Φ will embed into a five-dimensional euclidian space $\subseteq \mathfrak{h}^*$, and so we need a base Δ consisting of five simple roots.

Write $h_i = e_{i,i} - e_{i+1,i+1}$ for $1 \leq i \leq 5$. Then $\{h_i\}$ spans \mathfrak{h} and the dual functionals $\{h_i^*\}$ span \mathfrak{h}^* . By computing $[h_i, e_{j,j+1}]$ for each pair $(i, j) \in \{1, \dots, 5\}^2$ we find five simple roots $\Delta = \{\alpha_1, \dots, \alpha_5\}$, recorded in table 2.2.

Root α_i	α_i in terms of $\{h_i^*\}$	Root space \mathfrak{g}_{α_i}
α_1	$(2, -1, 0, 0, 0)$	$\mathbb{C}e_{1,2}$
α_2	$(-1, 2, -1, 0, 0)$	$\mathbb{C}e_{2,3}$
α_3	$(0, -1, 2, -1, 0)$	$\mathbb{C}e_{3,4}$
α_4	$(0, 0, -1, 2, -1)$	$\mathbb{C}e_{4,5}$
α_5	$(0, 0, 0, -1, 2)$	$\mathbb{C}e_{5,6}$

Table 2.2: Simple roots of $\mathfrak{sl}(\mathbb{C}^6)$

The root spaces of $\mathfrak{sl}(\mathbb{C}^6)$ are listed in table 2.3. We enumerate each (positive) root $\beta \in \Phi$ as a vector with respect to the basis $\Delta = \{\alpha_1, \dots, \alpha_5\}$ and also with respect to the basis $\{h_1^*, \dots, h_5^*\}$.

β in terms of Δ	Root space \mathfrak{g}_β	Root space $\mathfrak{g}_{-\beta}$	β in terms of $\{h_i^*\}$
(1, 0, 0, 0, 0)	$\mathbb{C}e_{1,2}$	$\mathbb{C}e_{2,1}$	(2, -1, 0, 0, 0)
(0, 1, 0, 0, 0)	$\mathbb{C}e_{2,3}$	$\mathbb{C}e_{3,2}$	(-1, 2, -1, 0, 0)
(0, 0, 1, 0, 0)	$\mathbb{C}e_{3,4}$	$\mathbb{C}e_{4,3}$	(0, -1, 2, -1, 0)
(0, 0, 0, 1, 0)	$\mathbb{C}e_{4,5}$	$\mathbb{C}e_{5,4}$	(0, 0, -1, 2, -1)
(0, 0, 0, 0, 1)	$\mathbb{C}e_{5,6}$	$\mathbb{C}e_{6,5}$	(0, 0, 0, -1, 2)
(1, 1, 0, 0, 0)	$\mathbb{C}e_{1,3}$	$\mathbb{C}e_{3,1}$	(1, 1, -1, 0, 0)
(0, 1, 1, 0, 0)	$\mathbb{C}e_{2,4}$	$\mathbb{C}e_{4,2}$	(-1, 1, 1, -1, 0)
(0, 0, 1, 1, 0)	$\mathbb{C}e_{3,5}$	$\mathbb{C}e_{5,3}$	(0, -1, 1, 1, -1)
(0, 0, 0, 1, 1)	$\mathbb{C}e_{4,6}$	$\mathbb{C}e_{6,4}$	(0, 0, -1, 1, 1)
(1, 1, 1, 0, 0)	$\mathbb{C}e_{1,4}$	$\mathbb{C}e_{4,1}$	(1, 0, 1, -1, 0)
(0, 1, 1, 1, 0)	$\mathbb{C}e_{2,5}$	$\mathbb{C}e_{5,2}$	(-1, 1, 0, 1, -1)
(0, 0, 1, 1, 1)	$\mathbb{C}e_{3,6}$	$\mathbb{C}e_{6,3}$	(0, -1, 1, 0, 1)
(1, 1, 1, 1, 0)	$\mathbb{C}e_{1,5}$	$\mathbb{C}e_{1,5}$	(1, 0, 0, 1, -1)
(0, 1, 1, 1, 1)	$\mathbb{C}e_{2,6}$	$\mathbb{C}e_{2,6}$	(-1, 1, 0, 0, 1)
(1, 1, 1, 1, 1)	$\mathbb{C}e_{1,6}$	$\mathbb{C}e_{6,1}$	(1, 0, 0, 0, 1)

Table 2.3: Root spaces of $\mathfrak{sl}(\mathbb{C}^6)$ relative to \mathfrak{h}

Take $\Delta' = \{\alpha_1, \alpha_2, \alpha_4\}$. The standard parabolic subalgebra \mathfrak{q} corresponding to Δ' consists of block upper triangular matrices corresponding to the partition $3 + 2 + 1$ of 6, as illustrated in figure 2.5.

$$\mathfrak{q} = \left(\begin{array}{ccc|cc|c} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ \hline & & & * & * & * \\ & & & * & * & * \\ \hline & & & & & * \end{array} \right)$$

Figure 2.5: The parabolic subalgebra $\mathfrak{q} \leq \mathfrak{gl}(\mathbb{C}^6)$ corresponding to $\Delta' = \{\alpha_1, \alpha_2, \alpha_4\}$

Consideration of the root system Φ shows that $\Phi \cap -\Phi = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 + \alpha_2), \pm\alpha_4\}$. $\Phi \setminus -\Phi$ consists of the remaining positive roots.

\mathfrak{l} (respectively \mathfrak{n}) consists of the block diagonal matrices (block strictly upper triangular matrices) in \mathfrak{q} that preserve the existing block structure, illustrated in figure 2.6.

$$\mathfrak{l} = \left(\begin{array}{ccc|cc} * & * & * & & \\ * & * & * & & \\ * & * & * & & \\ \hline & & & * & * \\ & & & * & * \\ \hline & & & & * \end{array} \right), \quad \mathfrak{n} = \left(\begin{array}{ccc|cc} & * & * & * & \\ & * & * & * & \\ & * & * & * & \\ \hline & & & & * \\ & & & & * \\ \hline & & & & \end{array} \right)$$

Figure 2.6: The Levi factor decomposition of $\mathfrak{q}_S \leq \mathfrak{sl}(\mathbb{C}^6)$

As a reductive Lie algebra, \mathfrak{l} decomposes as $\mathfrak{l} = \mathfrak{l}_Z \oplus \mathfrak{l}_S$. Write $h_i = e_{i,i} - e_{i+1,i+1}$ for $1 \leq i \leq 5$. Direct computation shows that center \mathfrak{l}_Z is two dimensional:

$$\mathfrak{l}_Z = \{ a(h_1 + 2h_2 + 3h_3 + 3h_4) + b(h_4 + 2h_5) \mid a, b \in \mathbb{C} \}.$$

Perhaps more naturally, we may describe \mathfrak{l}_Z in terms of matrices, illustrated in figure 2.7.

$$\mathfrak{l} = \left(\begin{array}{ccc|cc} a & & & & \\ & a & & & \\ & & a & & \\ \hline & & & b & \\ & & & & b \\ \hline & & & & -3a - 2b \end{array} \right)$$

Figure 2.7: A representative member of \mathfrak{l}_Z

A Cartan subalgebra of \mathfrak{l} is spanned by h_1, h_2 , and h_4 , and \mathfrak{l}_S is the Lie algebra direct sum of simple ideals isomorphic to $\mathfrak{sl}(\mathbb{C}^3)$ and $\mathfrak{sl}(\mathbb{C}^2)$ whereby $\mathfrak{l} \cong \mathbb{C}^2 \oplus \mathfrak{sl}(\mathbb{C}^3) \oplus \mathfrak{sl}(\mathbb{C}^2)$.

2.8 The center of a parabolic subalgebra

We conclude this chapter with a lemma — characterizing the center of parabolic subalgebras — that will be required later. We did not find the following result in the standard texts, but it is elementary. As such, we presume it is already well-known, and we include it here rather than in chapter 3.

Lemma 2.25. *Let $\mathfrak{q} = \mathfrak{g}_Z \oplus \mathfrak{q}_S$ be a parabolic subalgebra of the reductive Lie algebra $\mathfrak{g} = \mathfrak{g}_Z \oplus \mathfrak{g}_S$ over a complex-like field \mathbb{K} or over \mathbb{R} . The center of \mathfrak{q} is \mathfrak{g}_Z .*

Proof. We consider first the special case where $\mathfrak{g} = \mathfrak{h} \dot{+} \sum_{\beta \in \Phi}$ is semisimple over \mathbb{K} . We assume without loss of generality that \mathfrak{q} is a standard parabolic subalgebra and write $\mathfrak{q} = \mathfrak{h} \dot{+} \sum_{\beta \in \Phi'} \mathfrak{g}_\beta$.

Let $z \in \mathfrak{q}_Z$ so $z = z_{\mathfrak{h}} + \sum_{\beta \in \Phi'} z_{\mathfrak{g}_\beta}$. Then for any $x \in \mathfrak{q}$ we have

$$0 = [z, x] = [z_{\mathfrak{h}}, x] + \sum_{\beta \in \Phi'} [z_{\mathfrak{g}_\beta}, x]. \quad (2.3)$$

Specifically, for $0 \neq x \in \mathfrak{g}_\alpha$ with $\alpha \in \Phi'$, equation 2.3 becomes

$$0 = \underbrace{\alpha(z_{\mathfrak{h}})}_{\in \mathfrak{g}_\alpha} x + \sum_{\beta \in \Phi'} \underbrace{[z_{\mathfrak{g}_\beta}, x]}_{\in \mathfrak{g}_{\alpha+\beta}}$$

and by direct sum decomposition $0 = \alpha(z_{\mathfrak{h}})$ for all $\alpha \in \Phi'$. Since Φ' spans \mathfrak{h}^* it must be the case that $z_{\mathfrak{h}} = 0$, so $z = \sum_{\beta \in \Phi'} z_{\mathfrak{g}_\beta}$.

Now, let $0 \neq h \in \mathfrak{h}$ and apply equation 2.3. We see

$$0 = \sum_{\beta \in \Phi'} [z_{\mathfrak{g}_\beta}, h] = \sum_{\beta \in \Phi'} -\beta(h) z_{\mathfrak{g}_\beta}$$

which by direct sum decomposition yields $0 = \beta(h) z_{\mathfrak{g}_\beta}$ for all $\beta \in \Phi'$. Since h is arbitrary in \mathfrak{h} , $z_{\mathfrak{g}_\beta} = 0$ for all β , so $z = 0$.

Having established that $\mathfrak{q}_Z = 0$ when \mathfrak{g} is semisimple over \mathbb{K} , that $\mathfrak{q}_Z = \mathfrak{g}_Z$ when \mathfrak{g} is reductive over \mathbb{K} follows from the Lie algebra direct sum decomposition $\mathfrak{q} = \mathfrak{g}_Z \oplus \mathfrak{q}_S$. We now consider the case where \mathfrak{q} is a parabolic subalgebra of a real reductive \mathfrak{g} . We have $\hat{\mathfrak{q}}$ is a parabolic subalgebra of $\hat{\mathfrak{g}}$ by definition. Then

$$\mathfrak{g}_Z + \mathfrak{i}\mathfrak{g}_Z = \widehat{(\mathfrak{g}_Z)} = \underbrace{(\hat{\mathfrak{g}})_Z}_{\text{by above case}} = (\hat{\mathfrak{q}})_Z = \widehat{(\mathfrak{q}_Z)} = \mathfrak{q}_Z + \mathfrak{i}\mathfrak{q}_Z. \quad (2.4)$$

Finally, by Ado's Theorem (proposition 2.5), we may assume that \mathfrak{g} consists of real matrices, so that we may separate the real and imaginary part in equation 2.4, giving $\mathfrak{g}_Z = \mathfrak{q}_Z$, as desired. □

Chapter 3

Derivations of Parabolic Lie Algebras

We begin this chapter by stating the main result of this dissertation:

Theorem. *Let \mathfrak{q} be a parabolic subalgebra of a reductive Lie algebra \mathfrak{g} over \mathbb{R} or over a complex-like field. Let \mathcal{L} be the set of all linear transformations mapping \mathfrak{q} into \mathfrak{q}_Z that send $[\mathfrak{q}, \mathfrak{q}]$ to 0. Then \mathcal{L} is an ideal of $\text{Der } \mathfrak{q}$ and $\text{Der } \mathfrak{q}$ decomposes as the direct sum of ideals*

$$\text{Der } \mathfrak{q} = \mathcal{L} \oplus \text{ad } \mathfrak{q}.$$

Our main result is valid for \mathbb{R} or for any complex-like field (such as \mathbb{C}). However, the method of proof in the two cases is quite different. The proof of the complex-like case is highly technical: given a derivation D on \mathfrak{q} , we explicitly construct a linear map L and an element $x \in \mathfrak{q}$ such that $D = L + \text{ad } x$, after which we prove that our construction satisfies the stated properties. The real case, in contrast, is high-level and abstract, appealing to the complex case of the central theorem as applied to $\hat{\mathfrak{q}}$, the complexification of the real parabolic subalgebra \mathfrak{q} .

Lie algebras over complex-like fields support a more regular structural decomposition than real Lie algebras affords. In particular, Langland's decomposition — while possible in the former case — is completely unnecessary in order to prove theorem 3.1. Because of the less regular structure of real Lie algebras, Langland's decomposition becomes an important tool for the proof of theorem 3.4

3.1 The algebraically-closed, characteristic-zero case

Throughout this section we use the following notational conventions:

\mathbb{K} denotes a complex-like field;

$\mathfrak{g} = \mathfrak{g}_Z \oplus \mathfrak{g}_S$ denotes a reductive Lie algebra over \mathbb{K} , where

\mathfrak{g}_Z is the center of \mathfrak{g} , and

\mathfrak{g}_S is the maximal semisimple ideal of \mathfrak{g} ;

$\mathfrak{q} = \mathfrak{g}_Z \oplus \mathfrak{q}_S$ is a given parabolic subalgebra of \mathfrak{g} , where

$\mathfrak{q}_S = \mathfrak{q} \cap \mathfrak{g}_S$ is a parabolic subalgebra of \mathfrak{g}_S .

We choose a Cartan subalgebra \mathfrak{h} , a root system Φ , and a base Δ compatible with \mathfrak{q}_S in the sense that \mathfrak{q}_S is a standard parabolic subalgebra of \mathfrak{g}_S relative to $(\mathfrak{h}, \Phi, \Delta)$ and corresponds to a subset $\Delta' \subseteq \Delta$. Then

$$\mathfrak{q}_S = \mathfrak{h} \dot{+} \sum_{\alpha \in \Phi'} \mathbb{K}x_\alpha,$$

where

$$\Phi' = \Phi^+ \cup (\Phi \cap \text{Span } \Delta')$$

and where each x_α is chosen arbitrarily from the one-dimensional root space it spans.

Define \mathfrak{t} and \mathfrak{c} by

$$\mathfrak{t} = \mathfrak{h} \cap [\mathfrak{q}, \mathfrak{q}] \text{ and}$$

$$\mathfrak{c} = \text{Span} \{ [x_\alpha, x_{-\alpha}] \mid \alpha \in \Delta \setminus \Delta' \}.$$

Claim. \mathfrak{h} decomposes as $\mathfrak{h} = \mathfrak{c} \dot{+} \mathfrak{t}$.

Proof. Notice that $\mathfrak{h} = \text{Span} \{ [x_\alpha, x_{-\alpha}] \mid \alpha \in \Delta \}$ and that $\mathfrak{t} = \text{Span} \{ [x_\alpha, x_{-\alpha}] \mid \alpha \in \Delta' \}$.

From these observations, we see that $\mathfrak{c} \cap \mathfrak{t} = 0$ and that $\text{Span}(\mathfrak{c} \cup \mathfrak{t}) = \mathfrak{h}$. \square

Noting that $[\mathfrak{q}, \mathfrak{q}] = \mathfrak{t} + \sum_{\alpha \in \Phi'} \mathbb{K}x_\alpha$, we arrive at the desired vector space direct-sum decompositions of \mathfrak{q} :

$$\begin{aligned} \mathfrak{q} &= \mathfrak{g}_Z + \mathfrak{h} + \sum_{\alpha \in \Phi'} \mathbb{K}x_\alpha \\ &= \mathfrak{g}_Z + \overbrace{\mathfrak{c} + \mathfrak{t}}^{\mathfrak{h}} + \underbrace{\sum_{\alpha \in \Phi'} \mathbb{K}x_\alpha}_{[\mathfrak{q}, \mathfrak{q}]} \\ &= \mathfrak{g}_Z + \mathfrak{c} + [\mathfrak{q}, \mathfrak{q}]. \end{aligned}$$

We take a moment to note that alternatively \mathfrak{c} may have been chosen so that it coincides with the center of \mathfrak{l} in Langland's decomposition $\mathfrak{q} = \mathfrak{l} + \mathfrak{n}$. This approach is not required in order to prove the complex-like case, but it is taken in order to simplify the proof of theorem 3.4 in the real case.

For the remainder of the section, we assume all of the notational conventions described above without further mention, starting with a restatement of the central theorem in terms of the adopted notation.

Theorem 3.1. *For a parabolic subalgebra $\mathfrak{q} = \mathfrak{g}_Z \oplus \mathfrak{q}_S$ of a reductive Lie algebra $\mathfrak{g} = \mathfrak{g}_Z \oplus \mathfrak{g}_S$ over the complex-like field \mathbb{K} , the derivation algebra $\text{Der } \mathfrak{q}$ decomposes as the direct sum of ideals*

$$\text{Der } \mathfrak{q} = \mathfrak{L} \oplus \text{ad } \mathfrak{q},$$

where \mathfrak{L} consists of all \mathbb{K} -linear transformations on \mathfrak{q} mapping into \mathfrak{q}_Z and mapping $[\mathfrak{q}, \mathfrak{q}]$ to 0.

Explicitly, for any root system Φ with respect to which \mathfrak{q} is a standard parabolic subalgebra, \mathfrak{q} decomposes as $\mathfrak{q} = \mathfrak{g}_Z + \mathfrak{c} + [\mathfrak{q}, \mathfrak{q}]$ and the ideal \mathfrak{L} consists of all \mathbb{K} -linear transformations on \mathfrak{q} that map $\mathfrak{g}_Z + \mathfrak{c}$ into \mathfrak{g}_Z and map $[\mathfrak{q}, \mathfrak{q}]$ to 0, whereby

$$\text{Der } \mathfrak{q} \cong \text{Hom}_{\mathbb{K}}(\mathfrak{g}_Z + \mathfrak{c}, \mathfrak{g}_Z) \oplus \mathfrak{q}_S.$$

We must explain what we mean by $\text{Hom}_{\mathbb{K}}(\mathfrak{g}_Z \dot{+} \mathfrak{c}, \mathfrak{g}_Z)$ as a Lie algebra, since it is merely a space of linear maps and does not come equipped with a Lie bracket by default. For vector spaces V_1, V_2 , we consider the space $\text{Hom}_{\mathbb{K}}(V_2, V_1)$ an abelian Lie algebra. Then, $\text{Hom}_{\mathbb{K}}(V_1 \dot{+} V_2, V_1)$ may be realized as the Lie algebra semidirect sum

$$\text{Hom}_{\mathbb{K}}(V_1 \dot{+} V_2, V_1) = \mathfrak{gl}(V_1) \ltimes \text{Hom}_{\mathbb{K}}(V_2, V_1)$$

with the action of $\mathfrak{gl}(V_1)$ on $\text{Hom}_{\mathbb{K}}(V_2, V_1)$ defined by

$$f \cdot g = f \circ g \quad \forall f \in \mathfrak{gl}(V_1), g \in \text{Hom}_{\mathbb{K}}(V_2, V_1).$$

This definition is canonical in the sense that if we fix bases for V_1 and V_2 , then $\text{Hom}_{\mathbb{K}}(V_1 \dot{+} V_2, V_1)$ is identified with the subalgebra of $\mathfrak{gl}(V_1 \dot{+} V_2)$ consisting of block matrices of the form illustrated in figure 3.1 (compare to figure 1.2), and the Lie bracket defined by the action above coincides with the standard Lie bracket of matrices, ie, $[M, N] = MN - NM$.

$$\begin{array}{c} V_1 \quad V_2 \\ V_1 \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix} \\ V_2 \end{array}$$

Figure 3.1: Embedding of $\text{Hom}_{\mathbb{K}}(V_1 \dot{+} V_2, V_1)$ in $\mathfrak{gl}(V_1 \dot{+} V_2)$

Proof of theorem 3.1. For clarity, the proof of the theorem is organized into a progression of claims. The first three claims establish that an arbitrary derivation may be written as a sum of an inner derivation and a derivation mapping $\mathfrak{g}_Z \dot{+} \mathfrak{c}$ to \mathfrak{g}_Z and $[\mathfrak{q}, \mathfrak{q}]$ to 0. To this end, let D be an arbitrary derivation of \mathfrak{q} .

Claim 1. There is an $x \in \mathfrak{q}$ such that $D - \text{ad } x$ maps \mathfrak{c} to \mathfrak{g}_Z , annihilates \mathfrak{t} , and stabilizes each root space $\mathbb{K}x_\alpha$.

Let $h, k \in \mathfrak{h}$ be arbitrary and write $D(h) = z + h' + \sum_{\gamma} a_{\gamma}(h)x_{\gamma}$ and $D(k) = c + k' + \sum_{\gamma} a_{\gamma}(k)x_{\gamma}$ with $z, c \in \mathfrak{g}_Z$ and $h', k' \in \mathfrak{h}$ and $a_{\gamma}(h), a_{\gamma}(k) \in \mathbb{K}$. Recall $[h, k] = 0$ since

$h, k \in \mathfrak{h}$ and consider $D([h, k])$.

$$\begin{aligned}
0 &= D([h, k]) \\
&= [h, D(k)] - [k, D(h)] \\
&= \left[h, c + k' + \sum_{\gamma} a_{\gamma}(k) x_{\gamma} \right] - \left[k, z + h' + \sum_{\gamma} a_{\gamma}(h) x_{\gamma} \right] \\
&= \left[h, \sum_{\gamma} a_{\gamma}(k) x_{\gamma} \right] - \left[k, \sum_{\gamma} a_{\gamma}(h) x_{\gamma} \right] \\
&= \sum_{\gamma} a_{\gamma}(k) [h, x_{\gamma}] - \sum_{\gamma} a_{\gamma}(h) [k, x_{\gamma}] \\
&= \sum_{\gamma} (a_{\gamma}(k) \gamma(h) - a_{\gamma}(h) \gamma(k)) x_{\gamma}.
\end{aligned}$$

So

$$a_{\gamma}(k) \gamma(h) - a_{\gamma}(h) \gamma(k) = 0 \text{ for all } \gamma \in \Phi', h, k \in \mathfrak{h}. \quad (3.1)$$

Furthermore, for any pair h, k for which $\gamma(h) \neq 0$ and $\gamma(k) \neq 0$, we have that

$$\frac{a_{\gamma}(h)}{\gamma(h)} = \frac{a_{\gamma}(k)}{\gamma(k)}.$$

This observation, along with the fact that $\gamma(h) \neq 0$ for at least one $h \in \mathfrak{h}$, allows us to associate with each $\gamma \in \Phi'$ the numerical invariant

$$d_{\gamma} = \frac{a_{\gamma}(h)}{\gamma(h)}$$

independently of our choice of h . Notice that $a_{\gamma}(h) - d_{\gamma} \gamma(h) = 0$ by definition when $\gamma(h) \neq 0$. If $\gamma(h) = 0$, the same equality still holds, as equation 3.1 becomes $a_{\gamma}(h) \gamma(k) = 0$ for all $k \in \mathfrak{h}$. Since at least one $k \in \mathfrak{h}$ satisfies $\gamma(k) \neq 0$ we have $a_{\gamma}(h) = 0$ in case $\gamma(h) = 0$, giving

$$a_{\gamma}(h) - d_{\gamma} \gamma(h) = 0 \text{ for all } h \in \mathfrak{h}. \quad (3.2)$$

Now, set $x = \sum_{\gamma} -d_{\gamma}x_{\gamma}$. Write $D' = D - \text{ad } x$. We will show that D' maps \mathfrak{c} to \mathfrak{g}_Z , annihilates \mathfrak{t} , and stabilizes each root space $\mathbb{K}x_{\alpha}$.

We first show that D' maps \mathfrak{h} to $\mathfrak{g}_Z \dot{+} \mathfrak{h}$. Let $h \in \mathfrak{h}$ be arbitrary and again write $D(h) = z + h' + \sum_{\gamma} a_{\gamma}(h)x_{\gamma}$. We have that

$$\begin{aligned}
D'(h) &= D(h) - \text{ad } x(h) \\
&= z + h' + \sum_{\gamma} a_{\gamma}(h)x_{\gamma} - \sum_{\gamma} -d_{\gamma} \text{ad } x_{\gamma}(h) \\
&= z + h' + \sum_{\gamma} a_{\gamma}(h)x_{\gamma} - \sum_{\gamma} d_{\gamma} \text{ad } h(x_{\gamma}) \\
&= z + h' + \sum_{\gamma} a_{\gamma}(h)x_{\gamma} - \sum_{\gamma} d_{\gamma} \gamma(h)x_{\gamma} \\
&= z + h' + \sum_{\gamma} \underbrace{(a_{\gamma}(h) - d_{\gamma} \gamma(h))}_{=0 \text{ by 3.2}} x_{\gamma} \\
&= z + h',
\end{aligned}$$

affirming the assertion.

Having established that D' maps \mathfrak{h} into $\mathfrak{g}_Z \dot{+} \mathfrak{h}$, we have left to show that D' annihilates \mathfrak{t} and stabilizes each $\mathbb{K}x_{\alpha}$. Let $h \in \mathfrak{h}$ and $\alpha \in \Phi'$ be arbitrary, and write $D'(h) = z + h'$ and $D'(x_{\alpha}) = c + k + \sum_{\gamma} b_{\gamma}x_{\gamma}$ with $z, c \in \mathfrak{g}_Z$ and $h', k \in \mathfrak{h}$ and $b_{\gamma} \in \mathbb{K}$. Consider $D'([h, x_{\alpha}])$. On one hand,

$$\begin{aligned}
D'([h, x_{\alpha}]) &= D'(\alpha(h)x_{\alpha}) \\
&= \alpha(h)D'(x_{\alpha}) \\
&= \alpha(h)c + \alpha(h)k + \sum_{\gamma} \alpha(h)b_{\gamma}x_{\gamma}. \tag{\#}
\end{aligned}$$

On the other hand,

$$\begin{aligned}
D'([h, x_\alpha]) &= [D'(h), x_\alpha] + [h, D'(x_\alpha)] \\
&= [z + h', x_\alpha] + \left[h, c + k + \sum_{\gamma} b_{\gamma} x_{\gamma} \right] \\
&= [h', x_\alpha] + \sum_{\gamma} b_{\gamma} [h, x_{\gamma}] \\
&= \alpha(h') x_{\alpha} + \sum_{\gamma} \gamma(h) b_{\gamma} x_{\gamma} \\
&= (\alpha(h') + \alpha(h) b_{\alpha}) x_{\alpha} + \sum_{\gamma \neq \alpha} \gamma(h) b_{\gamma} x_{\gamma}. \tag{b}
\end{aligned}$$

By equating \sharp and \flat and by direct sum decomposition of \mathfrak{q} we obtain

$$\alpha(h)c = 0, \tag{3.3}$$

$$\alpha(h)k = 0, \tag{3.4}$$

$$\alpha(h)b_{\gamma} = \gamma(h)b_{\gamma} \quad \text{for } \gamma \neq \alpha, \text{ and} \tag{3.5}$$

$$\alpha(h)b_{\alpha} = \alpha(h') + \alpha(h)b_{\alpha}. \tag{3.6}$$

Since h is arbitrary, equations 3.3 and 3.4 give $c = 0$ and $k = 0$ respectively. Second, equations 3.5 give us $b_{\gamma}(\gamma - \alpha)(h) = 0$ for all $\gamma \neq \alpha$. If any one $b_{\gamma} \neq 0$, then we would have $\gamma = \alpha$, a contradiction, so each $b_{\gamma} = 0$, whence D' stabilizes each root space.

Next, equation 3.6 gives us $0 = \alpha(h')$. Since α is arbitrary in Φ' and Φ' contains a basis of \mathfrak{h}^* , $h' = 0$, so $D'(\mathfrak{h}) \subseteq \mathfrak{g}_Z$. Since derivations in general stabilize $[\mathfrak{q}, \mathfrak{q}]$, $D'(\mathfrak{t}) \subseteq \mathfrak{g}_Z \cap [\mathfrak{q}, \mathfrak{q}] = 0$, so D' annihilates \mathfrak{t} . The claim is verified.

Claim 2. There is an $h \in \mathfrak{h}$ whereby $D - \text{ad } x - \text{ad } h$ annihilates $[\mathfrak{q}, \mathfrak{q}]$.

We have the $D' = D - \text{ad } x$ maps \mathfrak{c} to \mathfrak{g}_Z , annihilate \mathfrak{t} , and stabilize each root space $\mathbb{K}x_{\alpha}$. For each $\gamma \in \Phi'$ write

$$D'(x_{\gamma}) = c_{\gamma} x_{\gamma}$$

with $c_\gamma \in \mathbb{K}$. Taking each $\alpha \in \Delta$, the scalars c_α define a linear functional

$$\tilde{c} : \mathfrak{h}^* \longrightarrow \mathbb{C}.$$

We first verify that for each $\gamma \in \Phi'$, $c_\gamma = \tilde{c}(\gamma)$.

We begin with $\gamma \in \Phi' \cap \Phi^+$. Let $\gamma = \alpha_1 + \dots + \alpha_k$ with each $\alpha_i \in \Delta$ and where each sequential partial sum $\alpha_1 + \dots + \alpha_i \in \Phi'$. Then

$$ax_\gamma = [\dots [[x_{\alpha_1}, x_{\alpha_2}], x_{\alpha_3}], \dots, x_{\alpha_k}]$$

for some $0 \neq a \in \mathbb{K}$. Apply D' to both sides. Since D' is a derivations, we have

$$\begin{aligned} c_\gamma ax_\gamma &= D' [\dots [[x_{\alpha_1}, x_{\alpha_2}], x_{\alpha_3}], \dots, x_{\alpha_k}] \\ &= [\dots [[D'(x_{\alpha_1}), x_{\alpha_2}], x_{\alpha_3}], \dots, x_{\alpha_k}] \\ &\quad + [\dots [[x_{\alpha_1}, D'(x_{\alpha_2})], x_{\alpha_3}], \dots, x_{\alpha_k}] \\ &\quad + [\dots [[x_{\alpha_1}, x_{\alpha_2}], D'(x_{\alpha_3})], \dots, x_{\alpha_k}] \\ &\quad + \dots + [\dots [[x_{\alpha_1}, x_{\alpha_2}], x_{\alpha_3}], \dots, D'(x_{\alpha_k})] \\ &= c_{\alpha_1} [\dots [[x_{\alpha_1}, x_{\alpha_2}], x_{\alpha_3}], \dots, x_{\alpha_k}] \\ &\quad + c_{\alpha_2} [\dots [[x_{\alpha_1}, x_{\alpha_2}], x_{\alpha_3}], \dots, x_{\alpha_k}] \\ &\quad + c_{\alpha_3} [\dots [[x_{\alpha_1}, x_{\alpha_2}], x_{\alpha_3}], \dots, x_{\alpha_k}] \\ &\quad + \dots + c_{\alpha_k} [\dots [[x_{\alpha_1}, x_{\alpha_2}], x_{\alpha_3}], \dots, x_{\alpha_k}] \\ &= c_{\alpha_1} ax_\gamma + \dots + c_{\alpha_k} ax_\gamma \\ &= \tilde{c}(\alpha_1 + \dots + \alpha_k) ax_\gamma \\ &= \tilde{c}(\gamma) ax_\gamma \end{aligned}$$

whereby $c_\gamma = \tilde{c}(\gamma)$ for all $\gamma \in \Phi' \cap \Phi^+$.

Next let $\gamma \in \Phi' \cap \Phi^-$. Consider $[x_\gamma, x_{-\gamma}] \in \mathfrak{t}$, and apply D' .

$$\begin{aligned}
0 &= D'([x_\alpha, x_{-\alpha}]) \\
&= [D'(x_\gamma), x_{-\gamma}] + [x_\gamma, D'(x_{-\gamma})] \\
&= c_\gamma[x_\gamma, x_{-\gamma}] + c_{-\gamma}[x_\gamma, x_{-\gamma}] \\
&= (c_\gamma + c_{-\gamma})[x_\gamma, x_{-\gamma}].
\end{aligned}$$

Since $[x_\gamma, x_{-\gamma}] \neq 0$, we have $c_\gamma + c_{-\gamma} = 0$ so

$$\begin{aligned}
c_\gamma &= -c_{-\gamma} \\
&= -\tilde{c}(-\gamma) && \text{(since } -\gamma \in \Phi' \cap \Phi^+) \\
&= \tilde{c}(\gamma)
\end{aligned}$$

as desired.

Next, we use the canonical isomorphism $\Psi : \mathfrak{h}^{**} \rightarrow \mathfrak{h}$ [22, Ch. VII, §4] to produce $\Psi(\tilde{c}) = h \in \mathfrak{h}$. Notice that for each $\gamma \in \Phi'$ we have the identity

$$\tilde{c}(\gamma) - \gamma(h) = 0 \tag{3.7}$$

by the definition of the canonical isomorphism.

The claim is that $D' - \text{ad } h$ annihilates $[\mathfrak{q}, \mathfrak{q}]$. Since $[h, \mathfrak{t}] = 0$, we need only check that $D' - \text{ad } h$ maps each x_γ to 0.

$$\begin{aligned}
(D' - \text{ad } h)(x_\gamma) &= \tilde{c}(\gamma)x_\gamma - \gamma(h)x_\gamma \\
&= \underbrace{(\tilde{c}(\gamma) - \gamma(h))}_{=0 \text{ by 3.7}} x_\gamma \\
&= 0
\end{aligned}$$

verifying the claim.

Claim 3. $D = L + \text{ad } p$ for some $p \in \mathfrak{q}$ and some derivation L which maps $\mathfrak{g}_Z + \mathfrak{c}$ to \mathfrak{g}_Z and maps $[\mathfrak{q}, \mathfrak{q}]$ to 0.

Set $p = x + h$ as above and set $L = D - \text{ad } p = D' - \text{ad } h$. Then $D = L + \text{ad } p$ as desired. We note that since L is the difference of two derivations, L is itself a derivation. We know from claim 2 that L annihilates $[\mathfrak{q}, \mathfrak{q}]$. We must check that L maps $\mathfrak{g}_Z + \mathfrak{c}$ to \mathfrak{g}_Z .

We have already seen that \mathfrak{g}_Z is the center of \mathfrak{q} , and more, that a derivation of \mathfrak{q} must stabilize the center of \mathfrak{q} . What is left to verify claim 3 is to check that L maps \mathfrak{c} into \mathfrak{g}_Z . Let $c \in \mathfrak{c}$ be arbitrary. We have

$$\begin{aligned} L(c) &= (D' - \text{ad } h)(c) \\ &= D'(c) - [h, c] \\ &= \underbrace{D'(c)}_{\in \mathfrak{g}_Z \text{ by claim 1}} \end{aligned}$$

verifying claim 3.

Since D was arbitrary, we now have that $\text{Der } \mathfrak{q}$ is spanned by $\text{ad } \mathfrak{q}$ and the subset of $\text{Der } \mathfrak{q}$ consisting of derivations that map $\mathfrak{g}_Z + \mathfrak{c}$ to \mathfrak{g}_Z and $[\mathfrak{q}, \mathfrak{q}]$ to 0. The next three claims establish facts about the relationship between these two sets.

Claim 4. $\mathfrak{L} \subseteq \text{Der } \mathfrak{q}$.

\mathfrak{L} is defined as the set of \mathbb{K} -linear endomorphisms of \mathfrak{q} mapping into the center of \mathfrak{q} and mapping $[\mathfrak{q}, \mathfrak{q}]$ to 0. We will show that any such linear map is indeed a derivation of \mathfrak{q} . Suppose $L : \mathfrak{q} \rightarrow \mathfrak{q}$ is any \mathbb{K} -linear map satisfying $L(\mathfrak{q}) \subseteq \mathfrak{g}_Z$ and $L([\mathfrak{q}, \mathfrak{q}]) = 0$. Then, for any $x, y \in \mathfrak{q}$ we have

$$\underbrace{[L(x), y]}_{=0} + \underbrace{[x, L(y)]}_{=0} = 0 = \underbrace{L([x, y])}_{=0}$$

so L is a derivation.

Claim 5. \mathfrak{L} is an ideal of $\text{Der } \mathfrak{q}$.

First we must show that \mathfrak{L} is closed under taking linear combinations of members.

Let $L_1, L_2 \in \mathfrak{L}$. We must show that $L_1 + kL_2 \in \mathfrak{L}$ (where $k \in \mathbb{K}$). Let $x \in \mathfrak{q}$. We have

$$(L_1 + kL_2)(x) = \underbrace{L_1(x)}_{\in \mathfrak{g}_Z} + k \underbrace{L_2(x)}_{\in \mathfrak{g}_Z} \in \mathfrak{g}_Z$$

so $L_1 + kL_2$ maps \mathfrak{q} into \mathfrak{g}_Z . Next, let $y \in [\mathfrak{q}, \mathfrak{q}]$. We have

$$(L_1 + kL_2)(y) = \underbrace{L_1(y)}_{=0} + k \underbrace{L_2(y)}_{=0} = 0$$

so $L_1 + kL_2$ sends $[\mathfrak{q}, \mathfrak{q}]$ to 0.

Second, let $L \in \mathfrak{L}$ and $D \in \text{Der } \mathfrak{q}$. We must show that $[D, L] = D \circ L - L \circ D \in \mathfrak{L}$, ie, that $[D, L]$ maps \mathfrak{q} into \mathfrak{g}_Z and maps $[\mathfrak{q}, \mathfrak{q}]$ to 0. Recall that $D(\mathfrak{g}_Z) \subseteq \mathfrak{g}_Z$ and $D([\mathfrak{q}, \mathfrak{q}]) \subseteq [\mathfrak{q}, \mathfrak{q}]$. Let $x \in \mathfrak{q}$. Consider $[D, L](x)$.

$$[D, L](x) = \underbrace{D(\overbrace{L(x)}^{\in \mathfrak{g}_Z})}_{\in \mathfrak{g}_Z} - \underbrace{L(\overbrace{D(x)}^{\in \mathfrak{q}})}_{\in \mathfrak{g}_Z} \in \mathfrak{g}_Z$$

so $[D, L]$ maps \mathfrak{q} into \mathfrak{g}_Z . Now, let $y \in [\mathfrak{q}, \mathfrak{q}]$ and consider $[D, L](y)$.

$$[D, L](y) = \underbrace{D(\overbrace{L(y)}^{=0})}_{=0} - \underbrace{L(\overbrace{D(y)}^{\in [\mathfrak{q}, \mathfrak{q}]})}_{=0} = 0$$

so $[D, L]$ maps $[\mathfrak{q}, \mathfrak{q}]$ to 0—ie, $[D, L] \in \mathfrak{L}$ —verifying the claim.

Claim 6. \mathfrak{L} and $\text{ad } \mathfrak{q}$ intersect trivially.

Suppose $D \in \mathfrak{L} \cap \text{ad } \mathfrak{q}$. Since $D \in \mathfrak{L}$, D maps \mathfrak{q} into \mathfrak{g}_Z . Since $D \in \text{ad } \mathfrak{q}$, D maps \mathfrak{q} into $[\mathfrak{q}, \mathfrak{q}]$. So, D maps \mathfrak{q} into $\mathfrak{g}_Z \cap [\mathfrak{q}, \mathfrak{q}] = 0$, whereby $D = 0$, completing the proof of the theorem. \square

As a simple application, we will use theorem 3.1 to derive a formula for the dimension of $\text{Der } \mathfrak{q}$ in terms of \mathfrak{g} and \mathfrak{q} .

Corollary 3.2. *For \mathfrak{q} a parabolic subalgebra of the reductive Lie algebra $\mathfrak{g} \cong \mathbb{C}^n \oplus \mathfrak{g}_S$ over \mathbb{K} , and with notation as above, the dimension of $\text{Der } \mathfrak{q}$ is given by*

$$\dim \text{Der } \mathfrak{q} = (n + |\Delta| - |\Delta'|) n + \dim \mathfrak{q}_S.$$

Proof. The corollary follows from the isomorphism $\text{Der } \mathfrak{q} \cong \text{Hom}_{\mathbb{K}}(\mathfrak{g}_Z + \mathfrak{c}, \mathfrak{g}_Z) \oplus \text{ad } \mathfrak{q}$, ie, the facts that for any vector spaces V_1, V_2 the dimension of $V_1 + V_2$ is the sum $\dim V_1 + \dim V_2$ and the dimension of $\text{Hom}_{\mathbb{K}}(V_1, V_2)$ is the product $(\dim V_1)(\dim V_2)$. Now, $\dim \mathfrak{c} = |\Delta| - |\Delta'|$, and $\dim \text{ad } \mathfrak{q} = \dim \mathfrak{q}_S$ because $\text{ad } \mathfrak{q} \cong \mathfrak{q}/\mathfrak{q}_Z = \mathfrak{q}_S$. Applying the mentioned general principles completes the proof. \square

We will note now that a similar dimension-counting result will not be possible — in general — in the real case, though formulas may be possible for classes of certain parabolic subalgebras of specific real Lie algebras. Notice that the statement and proof of corollary 3.2 rely heavily on the explicit description of the ideal \mathfrak{L} and knowledge of the dimension of the subspace $\mathfrak{c} \subseteq \mathfrak{q}$, which in turn rely on properties of the root space decomposition for Lie algebras over algebraically-close, characteristic-zero fields. Analogous properties fail to hold in general for the restricted root space decomposition of a real Lie algebra; however, the dimension of $\text{Der } \mathfrak{q}$ in the real case may be calculated on a example-by-example basis.

3.2 The real case

In this section, $\mathfrak{g} = \mathfrak{g}_Z \oplus \mathfrak{g}_S$ denotes a reductive real Lie algebra with center \mathfrak{g}_Z and maximal semisimple ideal \mathfrak{g}_S . $\mathfrak{q} = \mathfrak{g}_Z \oplus \mathfrak{q}_S$ is a parabolic subalgebra of \mathfrak{g} , where $\mathfrak{q}_S = \mathfrak{q} \cap \mathfrak{g}_S$ is a parabolic subalgebra of \mathfrak{g}_S .

We begin by proving the limited sense of the central theorem in the context of real Lie algebras. The proof will rely heavily on the complexification $\hat{\mathfrak{g}}$ of \mathfrak{g} , to which we will apply theorem 3.1. Afterwards, we consider the restricted root space decomposition of \mathfrak{g} and expand upon the central theorem.

Theorem 3.3. *For a parabolic subalgebra $\mathfrak{q} = \mathfrak{g}_Z \oplus \mathfrak{q}_S$ of a reductive Lie algebra $\mathfrak{g} = \mathfrak{g}_Z \oplus \mathfrak{g}_S$ over \mathbb{R} , the derivation algebra $\text{Der } \mathfrak{q}$ decomposes as the sum of ideals*

$$\text{Der } \mathfrak{q} = \mathcal{L} \oplus \text{ad } \mathfrak{q},$$

where \mathcal{L} consists of all \mathbb{R} -linear transformations on \mathfrak{q} mapping into \mathfrak{q}_Z and mapping $[\mathfrak{q}, \mathfrak{q}]$ to 0.

Proof. We may assume without loss of generality that \mathfrak{g} is realized as a set of real matrices by Ado's Theorem (proposition 2.5). We fix the following notation:

\mathbf{i} denotes the imaginary unit;

$\hat{\mathfrak{g}} = \mathfrak{g} + \mathbf{i}\mathfrak{g} = (\mathfrak{g}_Z + \mathbf{i}\mathfrak{g}_Z) \oplus (\mathfrak{g}_S + \mathbf{i}\mathfrak{g}_S)$ denotes the complexification of \mathfrak{g} ;

$\mathfrak{q} = \mathfrak{g}_Z \oplus \mathfrak{q}_S$ denotes a parabolic subalgebra of \mathfrak{g} ;

$\hat{\mathfrak{q}} = \mathfrak{q} + \mathbf{i}\mathfrak{q} = (\mathfrak{g}_Z + \mathbf{i}\mathfrak{g}_Z) \oplus (\mathfrak{q}_S + \mathbf{i}\mathfrak{q}_S)$ is a parabolic subalgebra of $\hat{\mathfrak{g}}$; and

$\widehat{\mathfrak{g}}_Z = \mathfrak{g}_Z + \mathbf{i}\mathfrak{g}_Z = \hat{\mathfrak{g}}_Z$ denotes the center of $\hat{\mathfrak{g}}$.

Given a derivation D of \mathfrak{q} , we have a corresponding derivation \hat{D} of $\hat{\mathfrak{q}}$ given by $\hat{D}(x + \mathbf{i}y) = D(x) + \mathbf{i}D(y)$. As a derivation of $\hat{\mathfrak{q}}$, \hat{D} decomposes as $\hat{D} = L + \text{ad}(x + \mathbf{i}y)$ with L

mapping \hat{q} into \hat{g}_Z and mapping $[\hat{q}, \hat{q}]$ to 0 and with $x, y \in \mathfrak{q}$. Note that L sends $[\mathfrak{q}, \mathfrak{q}]$ to 0, since $[\mathfrak{q}, \mathfrak{q}] \subseteq [\hat{q}, \hat{q}]$.

Let $u \in \mathfrak{q}$. \hat{D} stabilizes \mathfrak{q} , so we have

$$D(u) = \hat{D}(u) = L(u) + \text{ad}(x + \mathbf{i}y)(u) = L(u) + [x, u] + \mathbf{i}[y, u] \in \mathfrak{q}.$$

Now, $L(u) \in \hat{g}_Z = \mathfrak{g}_Z + \mathbf{i}\mathfrak{g}_Z$, so we may write $L(u) = v_1 + \mathbf{i}v_2$ with $v_1, v_2 \in \mathfrak{g}_Z$. Then

$$D(u) = v_1 + \mathbf{i}v_2 + [x, u] + \mathbf{i}[y, u] = (v_1 + [x, u]) + \mathbf{i}(v_2 + [y, u]) \in \mathfrak{q}$$

so $v_2 + [y, u] = 0$, and by the direct-sum decomposition $\mathfrak{q} = \mathfrak{g}_Z + [\mathfrak{q}, \mathfrak{q}]$, $v_2 = 0$ and $[y, u] = 0$. In particular, we have $L(u) = v_1$, so L maps \mathfrak{q} into \mathfrak{g}_Z . Furthermore, since u was arbitrary and $[y, u] = 0$, we have $y \in \mathfrak{g}_Z$. Since $y \in \mathfrak{g}_Z$, we have for any arbitrary $z = u + \mathbf{i}v \in \mathfrak{g}$

$$\begin{aligned} \text{ad}(x + \mathbf{i}y)(z) &= \text{ad } x(z) + \mathbf{i} \text{ad } y(z) \\ &= [x, u + \mathbf{i}v] + \mathbf{i}[y, u + \mathbf{i}v] \\ &= [x, u] - \underbrace{[y, v]}_{=0} + \mathbf{i}([x, v] + \underbrace{[y, u]}_{=0}) \\ &= [x, u] + \mathbf{i}[x, v] \\ &= [x, u + \mathbf{i}v] \\ &= \text{ad } x(z) \end{aligned}$$

thus we have

$$\text{ad}(x + \mathbf{i}y) = \text{ad } x.$$

We now have $D = L|_{\mathfrak{q}} + \text{ad } x$ with $x \in \mathfrak{q}$ and $L|_{\mathfrak{q}}$ an \mathbb{R} -linear transformation mapping \mathfrak{q} to \mathfrak{g}_Z and $[\mathfrak{q}, \mathfrak{q}]$ to 0, as desired. We have left to check that arbitrary \mathbb{R} -linear maps

sending \mathfrak{q} to \mathfrak{g}_Z and $[\mathfrak{q}, \mathfrak{q}]$ to 0 are derivations, that \mathfrak{L} as described is an ideal of \mathfrak{q} , and that \mathfrak{L} and $\text{ad } \mathfrak{q}$ intersect trivially, the proofs of which are identical to the proofs given of claims 4, 5, and 6 of theorem 3.1, respectively. \square

We will now examine the relationship between the direct sum decomposition of $\text{Der } \mathfrak{q}$ and the restricted root space decomposition of \mathfrak{g} . Given a parabolic subalgebra $\mathfrak{q} = \mathfrak{g}_Z \oplus \mathfrak{q}_S$ of a reductive real Lie algebra $\mathfrak{g} = \mathfrak{g}_Z \oplus \mathfrak{g}_S$, we may choose a restricted root space decomposition of \mathfrak{g} that is compatible with \mathfrak{q} in the sense that \mathfrak{q} is a standard parabolic subalgebra of \mathfrak{g} . We may then decompose \mathfrak{q} into the sum of $\mathfrak{q} = \mathfrak{g}_Z \dot{+} \mathfrak{c} \dot{+} [\mathfrak{q}, \mathfrak{q}]$ where \mathfrak{c} is an appropriately-chosen complimentary subspace, similar to the complex-like case. To achieve this decomposition, we rely on Langland's decomposition of \mathfrak{q}_S , described in chapter 2. We fix the following notation pertaining to the restricted root space decomposition of \mathfrak{g} :

$$\mathfrak{g} = \mathfrak{g}_Z \dot{+} \mathfrak{a} \dot{+} \mathfrak{m} \dot{+} \sum_{\alpha \in \Phi} \mathfrak{g}_\alpha;$$

Δ a base of Φ ;

$\Delta' \subseteq \Delta$ corresponding to \mathfrak{q}_S ;

$\Phi' = \Phi^+ \cup (\Phi \cap \text{Span } \Delta')$; and

$$\mathfrak{q} = \mathfrak{g}_Z \dot{+} \underbrace{\mathfrak{a} \dot{+} \mathfrak{m} \dot{+} \sum_{\gamma \in \Phi'} \mathfrak{g}_\gamma}_{\mathfrak{q}_S}.$$

Write Langland's decomposition of \mathfrak{q}_S :

$$\mathfrak{l} = \mathfrak{a} \dot{+} \mathfrak{m} \dot{+} \sum_{\gamma \in \Phi' \cap -\Phi'} \mathfrak{g}_\gamma \text{ and}$$

$$\mathfrak{n} = \sum_{\gamma \in \Phi' \setminus -\Phi'} \mathfrak{g}_\gamma$$

so that $\mathfrak{q}_S = \mathfrak{l} \dot{+} \mathfrak{n}$ with \mathfrak{l} reductive and \mathfrak{n} nilpotent. Write

\mathfrak{c} for the center of \mathfrak{l} and

\mathfrak{l}_S for the unique semisimple ideal of \mathfrak{l}

so that $\mathfrak{l} = \mathfrak{c} \dot{+} \mathfrak{l}_S$.

Claim. $[\mathfrak{q}, \mathfrak{q}] = \mathfrak{l}_S \dot{+} \mathfrak{n}$

Proof. Let $x, y \in \mathfrak{q}$. We must show $[x, y] \in \mathfrak{l}_S \dot{+} \mathfrak{n}$. Without loss of generality, we may assume $x, y \in \mathfrak{q}_S$, since their projections onto \mathfrak{g}_Z are lost upon applying bracket.

Write $x = x_{\mathfrak{l}} + x_{\mathfrak{n}}$ and $y = y_{\mathfrak{l}} + y_{\mathfrak{n}}$ with $x_{\mathfrak{l}}, y_{\mathfrak{l}} \in \mathfrak{l}$ and $x_{\mathfrak{n}}, y_{\mathfrak{n}} \in \mathfrak{n}$. Then

$$\begin{aligned} [x, y] &= [x_{\mathfrak{l}} + x_{\mathfrak{n}}, y_{\mathfrak{l}} + y_{\mathfrak{n}}] \\ &= \underbrace{[x_{\mathfrak{l}}, y_{\mathfrak{l}}]}_{\in \mathfrak{l}_S} + \underbrace{[x_{\mathfrak{l}}, y_{\mathfrak{n}}] + [x_{\mathfrak{n}}, y_{\mathfrak{l}}] + [x_{\mathfrak{n}}, y_{\mathfrak{n}}]}_{\in \mathfrak{n}} \in \mathfrak{l}_S \dot{+} \mathfrak{n} \end{aligned}$$

since \mathfrak{l} is reductive and \mathfrak{n} is an ideal. □

Thus we arrive at the desired decomposition,

$$\begin{aligned} \mathfrak{q} &= \mathfrak{g}_Z \dot{+} \underbrace{\mathfrak{l} \dot{+} \mathfrak{n}}_{\mathfrak{g}_S} \\ &= \mathfrak{g}_Z \dot{+} \mathfrak{c} \dot{+} \mathfrak{l}_S \dot{+} \mathfrak{n} \\ &= \mathfrak{g}_Z \dot{+} \mathfrak{c} \dot{+} [\mathfrak{q}, \mathfrak{q}]. \end{aligned}$$

Theorem 3.4. *For any root system Φ with respect to which \mathfrak{q} is a standard parabolic subalgebra, \mathfrak{q} decomposes as $\mathfrak{q} = \mathfrak{g}_Z \dot{+} \mathfrak{c} \dot{+} [\mathfrak{q}, \mathfrak{q}]$ and the ideal \mathfrak{L} of $\text{Der } \mathfrak{q}$ consists of all \mathbb{R} -linear transformation on \mathfrak{q} that map $\mathfrak{g}_Z \dot{+} \mathfrak{c}$ to \mathfrak{g}_Z and map $[\mathfrak{q}, \mathfrak{q}]$ to 0, whereby*

$$\text{Der } \mathfrak{q} \cong \text{Hom}_{\mathbb{R}}(\mathfrak{g}_Z \dot{+} \mathfrak{c}, \mathfrak{g}_Z) \oplus \text{ad } \mathfrak{q}.$$

Proof. The proof is essentially done. The majority is merely the description of the decomposition of \mathfrak{q} , already done above. We have left to show only that $\mathfrak{L} \cong \text{Hom}_{\mathbb{R}}(\mathfrak{g}_Z \dot{+} \mathfrak{c}, \mathfrak{g}_Z)$,

which is obvious in light of the decomposition $\mathfrak{q} = \mathfrak{g}_Z + \mathfrak{c} + [\mathfrak{q}, \mathfrak{q}]$. The sceptical reader is referred to figure 1.2, illustrating the block form of matrices in \mathfrak{L} . \square

Because of the coarseness of the restricted root space decomposition, the dimension of \mathfrak{c} is not readily available in the real case, in contrast to the complex-like case. $\dim \mathfrak{c}$ may be calculated if given a specific real Lie algebra \mathfrak{g} and a specific standard parabolic subalgebra \mathfrak{q} .

3.3 Corollaries

The following three corollaries represent extremal cases of the central theorem. Corollary 3.5 applies to arbitrary parabolic subalgebras of a semisimple Lie algebra (ie, the case $\mathfrak{g}_Z = 0$). Corollaries 3.6 and 3.7 apply specifically to minimal parabolic subalgebras (ie, Borel subalgebras) and maximal parabolic subalgebras (ie, the entire Lie algebra \mathfrak{g}), respectively.

Corollary 3.5. *For a parabolic subalgebra \mathfrak{q} of a semisimple Lie algebra \mathfrak{g} over the field \mathbb{F} (where \mathbb{F} is complex-like or \mathbb{R}), the derivation algebra $\text{Der } \mathfrak{q}$ satisfies*

$$\text{Der } \mathfrak{q} = \text{ad } \mathfrak{q}.$$

Proof. By the central theorem, $\text{Der } \mathfrak{q} = \mathfrak{L} \oplus \text{ad } \mathfrak{q}$, and since $\mathfrak{g}_Z = 0$, we have $\mathfrak{L} = 0$. \square

Corollary 3.5 was proven for Borel subalgebras of semisimple Lie algebras over an arbitrary field by Leger and Luks in [19]. Tolpygo found the same result for parabolic subalgebras of complex Lie algebras [29]. Our contribution is to include parabolic subalgebras of real Lie algebras.

Corollary 3.6. *For a Borel subalgebra $\mathfrak{b} = \mathfrak{g}_Z \dot{+} \mathfrak{g}_0 \dot{+} \sum_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$ of the reductive Lie algebra $\mathfrak{g} = \mathfrak{g}_Z \oplus \mathfrak{g}_S$ over the field \mathbb{F} (where \mathbb{F} is complex-like or \mathbb{R}), the derivation algebra $\text{Der } \mathfrak{b}$ satisfies*

$$\text{Der } \mathfrak{b} \cong \text{Hom}_{\mathbb{F}}(\mathfrak{g}_Z \dot{+} (\mathfrak{g}_0)_Z, \mathfrak{g}_Z) \oplus \text{ad } \mathfrak{b}.$$

Proof. Write $\mathfrak{b}_S = \mathfrak{g}_0 \dot{+} \sum_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$. Since $\sum_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$ is clearly the nilpotent radical of \mathfrak{b}_S , the Levi factor $\mathfrak{l} = \mathfrak{g}_0$. Applying the central theorem gives the result. \square

Farnsteiner proved corollary 3.6 over a complex-like field in [10]. As with corollary 3.5, our contribution is to extend this result to Borel subalgebras of real Lie algebras.

Corollary 3.7. *For a reductive Lie algebra $\mathfrak{g} = \mathfrak{g}_Z \oplus \mathfrak{g}_S$ over the field \mathbb{F} (where \mathbb{F} is complex-like or \mathbb{R}), the derivation algebra $\text{Der } \mathfrak{g}$ satisfies*

$$\text{Der } \mathfrak{g} \cong \mathfrak{gl}(\mathfrak{g}_Z) \oplus \text{ad } \mathfrak{g}.$$

Proof. \mathfrak{g}_S is its own Levi factor. Being semisimple, the center of \mathfrak{g}_S is trivial, so \mathfrak{L} consists of linear maps stabilizing \mathfrak{g}_Z and sending $\mathfrak{g}_S = [\mathfrak{g}, \mathfrak{g}]$ to 0, which is isomorphic to $\mathfrak{gl}(\mathfrak{g}_Z)$. \square

The final corollary provides a high-level abstract description of $\text{Der } \mathfrak{q}$ useful for dimension-counting arguments. It is also satisfying on a theoretical level, since it relies on simple constructions that can be carried out on any Lie algebra, suggests that the result here for reductive Lie algebras might be generalized to larger classes of Lie algebras.

Recall that $\mathfrak{q}/[\mathfrak{q}, \mathfrak{q}]$ is the minimal abelian quotient of \mathfrak{q} . Since $\mathfrak{q} = \mathfrak{g}_Z \dot{+} \mathfrak{c} \dot{+} [\mathfrak{q}, \mathfrak{q}]$, we have $\mathfrak{g}_Z \dot{+} \mathfrak{c} \cong \mathfrak{q}/[\mathfrak{q}, \mathfrak{q}]$. Also, $\mathfrak{g}_Z = \mathfrak{q}_Z$, and $\text{ad } \mathfrak{q} \cong \mathfrak{q}/\mathfrak{q}_Z$, thus:

Corollary 3.8. *For a parabolic subalgebra \mathfrak{q} of a reductive Lie algebra \mathfrak{g} over a complex-like field or over \mathbb{R} , we have*

$$\text{Der } \mathfrak{q} \cong \text{Hom}(\mathfrak{q}/[\mathfrak{q}, \mathfrak{q}], \mathfrak{q}_Z) \oplus (\mathfrak{q}/\mathfrak{q}_Z).$$

Proof. Above. \square

Chapter 4

Zero Product Determined Derivation Algebras

Let \mathfrak{g} be a Lie algebra over an arbitrary field \mathbb{F} . \mathfrak{g} is called *zero product determined* if for each \mathbb{F} -bilinear map $\varphi : \mathfrak{g} \times \mathfrak{g} \longrightarrow V$ into an \mathbb{F} -vector space V , the condition that

$$\varphi(x, y) = 0 \text{ whenever } [x, y] = 0 \tag{4.1}$$

implies the existence of an \mathbb{F} -linear map $f : \mathfrak{g} \longrightarrow V$ satisfying

$$\varphi(x, y) = f([x, y]) \text{ for all } x, y \in \mathfrak{g}. \tag{4.2}$$

We will drop the reference to the base field \mathbb{F} when the context is clear. However, the reader should remember that the condition that a Lie algebra is zero product determined is tied to the understood base field. Explicitly, a given Lie algebra \mathfrak{g} may consist of complex matrices and may be considered as either a real Lie algebra or a complex Lie algebra. It is conceivable that \mathfrak{g} may be zero product determined as a real Lie algebra, but not zero product determined as a complex Lie algebra, or vice versa.

A few remarks. First, some terminology. A bilinear map satisfying property 4.1 is said to *preserve zero products*. Second, property 4.2 can be thought of as a map factoring property. The bracket $[\cdot, \cdot]$ can be thought of as a bilinear map μ defined by

$$\mu : \begin{cases} \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g} \\ (x, y) \mapsto [x, y] \end{cases}$$

Then the definition can be understood in terms of function composition as saying that the bilinear map φ factors as the composition of a linear map f and the Lie bracket μ ; in symbols, \mathfrak{g} is zero product determined if and only if

$$(\varphi(x, y) = 0 \text{ whenever } \mu(x, y) = 0) \text{ implies } \exists f, \varphi = f \circ \mu.$$

In words, \mathfrak{g} is zero product determined if and only if every bilinear map φ that preserves zero products factors through the bracket map μ .

Third remark: the setting as described above is not entirely desirable, since it combines notions of bilinearity and linearity, making the study problematic in certain settings. An equivalent definition can be phrased entirely in terms of linear maps by considering tensor products of vector spaces [5]. One may replace the Cartesian product $\mathfrak{g} \times \mathfrak{g}$ with the tensor product $\mathfrak{g} \otimes \mathfrak{g}$ without ambiguity, since linear maps on the tensor product $\mathfrak{g} \otimes \mathfrak{g}$ are in one-to-one correspondence with bilinear maps on the Cartesian product $\mathfrak{g} \times \mathfrak{g}$ [22, Ch. IX, §8]. See figure 4.1 for a diagrammatic expression of the factorization $\varphi = f \circ \mu$ in the tensor-product setting.

Finally, the above definition (along with the reformulated definition in terms of tensor products) works equally well for associative algebras—such as the algebra $\mathbb{R}^{n \times n}$ of n by n real matrices with the usual matrix multiplication—and non-associative, non-Lie algebras if the bracket is replaced by an appropriate multiplication. In fact, the initial work on zero product determined algebras was done in the context of matrix algebras, considering the standard associative matrix product, the non-associative Lie bracket, and the non-associative Jordan product [4].

For the remainder of this chapter, let \mathbb{K} denote a complex-like field.

4.1 Zero product determined algebras

Definition 4.1. A \mathbb{K} -algebra is a pair (\mathcal{A}, μ) where \mathcal{A} is a \mathbb{K} -vector space and

$$\mu : \mathcal{A} \otimes_{\mathbb{K}} \mathcal{A} \longrightarrow \mathcal{A}$$

is an \mathbb{K} -linear map. The image $\text{Im } \mu$ is denoted \mathcal{A}^2 .

Definition 4.1 encompasses Lie algebras when μ is defined by $\mu(x \otimes y) = [x, y]$. (We note in the Lie algebra case that $\mathcal{A}^2 = [\mathcal{A}, \mathcal{A}]$.) The definition also includes associative algebras (eg, matrix algebras under the usual matrix product, Banach algebras) and other non-associative algebras, such as Jordan algebras. In the sequel, we will suppress the mention of the scalar field \mathbb{K} when there is no danger of ambiguity.

The definition of *zero product determined* as applied to algebras is originally due to Brešar, Grašič, and Sánchez Ortega; it was motivated by applications to analysis on Banach algebras [1, 4]. Definition 4.2, given below, is equivalent to that found in [4] but rephrased in terms of linear maps on tensor products rather than in terms of bilinear maps on Cartesian products [5]. A purely linear approach offers the advantage of considering kernels and images of linear maps, alleviating certain difficulties found in the bilinear approach. Consider that for a bilinear map φ , the image of φ is not necessarily a subspace of the target vector space, and the notion of a kernel of φ is non-existent.

Definition 4.2. An algebra (\mathcal{A}, μ) is called *zero product determined* if for each vector space V and for each linear map $\varphi : \mathcal{A} \otimes \mathcal{A} \longrightarrow V$, if

$$\varphi(a_1 \otimes a_2) = 0 \text{ whenever } \mu(a_1 \otimes a_2) = 0,$$

then there is a linear map $f : \mathcal{A}^2 \longrightarrow B$ whereby φ factors through μ as

$$\varphi = f \circ \mu.$$

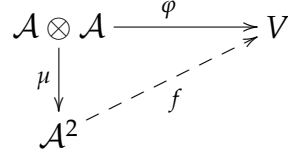


Figure 4.1: Tensor definition of zero product determined

The figure diagrammatically expresses the factorization $\varphi = f \circ \mu$. All maps depicted are linear. Existence of φ and μ is assumed a priori. f exists if φ preserves zero products.

We reference several results on zero product determined algebras that will be used in the sequel.

Proposition 4.1 (Theorem 2.3 in [5]). *An algebra (\mathcal{A}, μ) is zero product determined if and only if $\text{Ker } \mu$ is generated by elementary tensors.*

Some terminology will be helpful for understand the statement of the theorem. Elements of $\mathcal{A} \otimes \mathcal{A}$ are called *tensors*. An *elementary tensor* is a member of $\mathcal{A} \otimes \mathcal{A}$ of the form $a_1 \otimes a_2$ for $a_1, a_2 \in \mathcal{A}$. In general, an arbitrary tensor $t \in \mathcal{A} \otimes \mathcal{A}$ is a linear combination of elementary tensors, ie, $t = \sum_{i=1}^n a_1^{(i)} \otimes a_2^{(i)}$ for some positive integer n .

While $\mathcal{A} \otimes \mathcal{A}$ is generated by elementary tensors through taking linear combinations, an arbitrary subspace of $\mathcal{A} \otimes \mathcal{A}$ may fail to be generated by the elementary tensors it contains. In fact, there are non-trivial subspace of $\mathcal{A} \otimes \mathcal{A}$ that contain no elementary tensors other than the zero tensor $0 = 0 \otimes 0$.

Proposition 4.2 (Theorem 3.1 in [5]). *Let I be an arbitrary set and for each $i \in I$ let (\mathcal{A}_i, μ_i) be an algebra. Consider the algebra direct sum $(\mathcal{A}, \mu) = (\bigoplus_{i \in I} \mathcal{A}_i, \bigoplus_{i \in I} \mu_i)$. (\mathcal{A}, μ) is zero product determined if and only if each summand (\mathcal{A}_i, μ_i) is zero product determined.*

4.2 Parabolic subalgebras of reductive Lie algebras

For this section we adopt several of the notational conventions of [33] for clarity. Let $\mathfrak{g} = \mathfrak{g}_Z \oplus \mathfrak{g}_S$ be a finite-dimensional reductive Lie algebra over \mathbb{K} with \mathfrak{g}_Z abelian and \mathfrak{g}_S semisimple. Let \mathfrak{h} be a Cartan subalgebra of \mathfrak{g}_S and let Φ be the root system of \mathfrak{g}_S relative

to \mathfrak{h} . Let Δ be a base of Φ , and denote the positive roots relative to Δ by Φ^+ . Then \mathfrak{g}_S decomposes as $\mathfrak{g}_S = \mathfrak{h} \dot{+} \sum_{\beta \in \Phi} \mathfrak{g}_\beta$.

From each $\beta \in \Phi^+$ we select a non-zero vector $e_\beta \in \mathfrak{g}_\beta$. Then for each β there is a unique $e_{-\beta} \in \mathfrak{g}_{-\beta}$ with the property that $e_\beta, e_{-\beta}$, and $h_\beta = [e_\beta, e_{-\beta}]$ span a subalgebra of \mathfrak{g}_S isomorphic to $\mathfrak{sl}_2(\mathbb{K})$. Each \mathfrak{g}_β for $\beta \in \Phi$ is spanned by the vector e_β , and the vectors h_α for $\alpha \in \Delta$ form a basis of \mathfrak{h} . For each $\beta \in \Phi$ and each $h \in \mathfrak{h}$ we have $[h, e_\beta] = \beta(h)e_\beta$, and for each pair $\beta, \gamma \in \Phi$ we have $[e_\beta, e_\gamma] \in \mathfrak{g}_{\beta+\gamma}$, from which we define $N_{\beta, \gamma} \in \mathbb{K}$ by $[e_\beta, e_\gamma] = N_{\beta, \gamma} e_{\beta+\gamma}$.

Of any arbitrary parabolic subalgebra \mathfrak{q} of \mathfrak{g} , we assume without loss of generality that it is a standard parabolic subalgebra of \mathfrak{g} corresponding to some subset $\Delta' \subseteq \Delta$. More explicitly, the assumption is that that $\mathfrak{q} = \mathfrak{g}_Z \oplus \left(\mathfrak{h} \dot{+} \sum_{\beta \in \Phi'} \mathfrak{g}_\beta \right)$, where $\Phi' = \Phi^+ \cup (\Phi \cap \text{Span } \Delta')$.

In [33], the authors prove — with minor error — that the parabolic subalgebras of the finite-dimensional simple Lie algebras over \mathbb{K} are zero product determined. Specifically, lemma 2.2 of [33] contains the following unproven claim: *For any $\alpha, \gamma \in \Phi'$ where $\alpha + \gamma$ is a root, at least one of $\alpha + 2\gamma$ or $2\alpha + \gamma$ is not a root.* The root system of the simple Lie algebra G_2 provides a counterexample, as illustrated in figure 4.2.

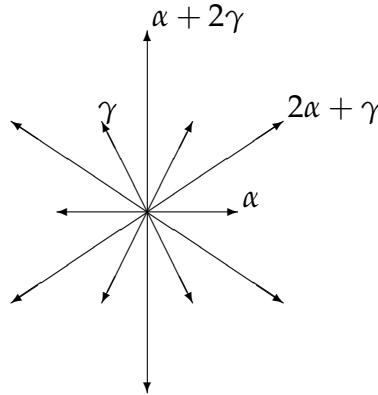


Figure 4.2: G_2 root system
Both $\alpha + 2\beta$ and $2\alpha + \beta$ are roots.

Lemma 2.2 of [33] does however provide a suitable base case for an induction argument, which we give below. We are given a parabolic subalgebra \mathfrak{q} and a bilinear map $\varphi : \mathfrak{q} \times \mathfrak{q} \longrightarrow V$ (V is some arbitrary vector space) of which we assume $\varphi(x, y) = 0$ whenever $[x, y] = 0$. For each $\gamma \in \Phi'$ we chose a $d_\gamma \in \mathfrak{h}$ so that $\gamma(d_\gamma) = 1$.

Following Wang et al, we define a linear map $f : [\mathfrak{q}, \mathfrak{q}] \longrightarrow V$ by

$$f(h_\alpha) = \varphi(e_\alpha, e_{-\alpha}) \text{ for each } \alpha \in \Delta,$$

$$f(e_\gamma) = \varphi(d_\gamma, e_\gamma) \text{ for each } \gamma \in \Gamma,$$

and extending linearly. The basic theorem of [33] is to show that $f([x, y]) = \varphi(x, y)$ for all $x, y \in \mathfrak{q}$. Lemma 2.2 of [33] gives a special case that Wang et al. use to facilitate the proof of the basic theorem.

Lemma 4.3 (Lemma 2.2 in [33]). *For $\alpha, \gamma \in \Phi'$, if $\alpha + \gamma \neq 0$, then*

$$f([e_\alpha, e_\gamma]) = \varphi(e_\alpha, e_\gamma).$$

We will need Wang et al.'s Lemma 2.1 for the proof of lemma 4.3. We state it now for use later: for the proof we refer the reader to [33].

Proposition 4.4 (Lemma 2.1 in [33]). *For all $h \in \mathfrak{h}$ and all $\beta \in \Phi'$, we have*

$$f([h, e_\beta]) = \varphi(h, e_\beta).$$

Proof of lemma 4.3. Let k be the largest integer such that $k\alpha + \gamma$ is a root. The proof of Lemma 2.2 in [33] shows that the proposition holds in case $k = 0, 1$. We proceed by assuming for induction that the proposition hold for all pairs of roots β, δ such that $\beta + \delta \neq 0$ and $\beta + \delta$ is not a root.

Pick $h \in \mathfrak{h}$ so that $\gamma(h) = 0$ and $\alpha(h) = 1$. Let $a_0 = 1$ and for each i from 1 to k let

$$a_i = \frac{1}{i} a_{i-1} N_{\alpha, (i-1)\alpha + \gamma}.$$

Notice that

$$ia_i - a_{i-1}N_{\alpha,(i-1)\alpha+\beta} = 0 \quad (4.3)$$

for each i from 1 to k . Then we have

$$\begin{aligned} \left[h - e_\alpha, \sum_{i=0}^k a_i e_{i\alpha+\gamma} \right] &= \sum_{i=0}^k a_i [h, e_{i\alpha+\gamma}] - \sum_{i=0}^k a_i [e_\alpha, e_{i\alpha+\gamma}] \\ &= \sum_{i=0}^k a_i (i\alpha + \gamma)(h) e_{i\alpha+\gamma} - \sum_{i=0}^k a_i N_{\alpha,i\alpha+\gamma} e^{(i+1)\alpha+\gamma} \\ &= \underbrace{\gamma(h)}_{=0} e_\gamma + \sum_{i=1}^k \underbrace{\left(ia_i - a_{i-1} N_{\alpha,(i-1)\alpha+\gamma} \right)}_{=0 \text{ by 4.3}} e_{i\alpha+\gamma} - a_k \underbrace{[e_\alpha, e_{k\alpha+\gamma}]}_{=0} \\ &= 0 \end{aligned}$$

and since φ preserves zero products by assumption we arrive at

$$\varphi \left(h - e_\alpha, \sum_{i=0}^k a_i e_{i\alpha+\gamma} \right) = 0. \quad (4.4)$$

By bilinearity of φ we have

$$0 = \varphi \left(h - e_\alpha, \sum_{i=0}^k a_i e_{i\alpha+\gamma} \right) = \sum_{i=0}^k a_i \varphi(h, e_{i\alpha+\gamma}) - \sum_{i=0}^k a_i \varphi(e_\alpha, e_{i\alpha+\gamma}),$$

and by proposition 4.4 and the definition of a_i and $N_{\alpha,(i-1)\alpha+\gamma}$

$$\begin{aligned} a_i \varphi(h, e_{i\alpha+\gamma}) &= a_i f([h, e_{i\alpha+\gamma}]) \\ &= a_i f((i\alpha + \gamma)(h) e_{i\alpha+\gamma}) \\ &= a_i f(i e_{i\alpha+\gamma}) \\ &= a_i \frac{i}{N_{\alpha,(i-1)\alpha+\gamma}} f([e_\alpha, e_{(i-1)\alpha+\gamma}]) \\ &= a_{i-1} f([e_\alpha, e_{(i-1)\alpha+\gamma}]). \end{aligned}$$

for each i from 1 to k . Then equation 4.4 becomes

$$\begin{aligned}
0 &= a_0 \underbrace{\varphi(h, e_\gamma)}_{=0} + \sum_{i=1}^k a_{i-1} f([e_\alpha, e_{(i-1)\alpha+\gamma}]) - \sum_{i=0}^k a_i \varphi(e_\alpha, e_{i\alpha+\gamma}) \\
&= \sum_{i=0}^{k-1} a_i f([e_\alpha, e_{i\alpha+\gamma}]) - \sum_{i=0}^{k-1} a_i \varphi(e_\alpha, e_{i\alpha+\gamma}) - \underbrace{a_k \varphi(e_\alpha, e_{k\alpha+\gamma})}_{=0} \\
&= \sum_{i=0}^{k-1} a_i (f([e_\alpha, e_{i\alpha+\gamma}]) - \varphi(e_\alpha, e_{i\alpha+\gamma})).
\end{aligned}$$

For $i \geq 1$, applying the inductive hypothesis to the pair $\alpha, i\alpha + \gamma$ gives $f([e_\alpha, e_{i\alpha+\gamma}]) - \varphi(e_\alpha, e_{i\alpha+\gamma}) = 0$, so the sum reduces to the $i = 0$ term:

$$0 = f([e_\alpha, e_\gamma]) - \varphi(e_\alpha, e_\gamma),$$

which is what we set out to show. □

We now state the Basic Theorem of Wang et al. for use later. The remainder of the proof is of course found in [33].

Proposition 4.5 (Basic Theorem in [33]). *A parabolic subalgebra \mathfrak{q} of a simple Lie algebra \mathfrak{g} over \mathbb{K} is zero product determined.*

The results of [5] allow us to generalize proposition 4.5 to reductive Lie algebras and their parabolic subalgebras.

Lemma 4.6. *If \mathfrak{g} is an abelian Lie algebra, then \mathfrak{g} is zero product determined.*

Proof. To say \mathfrak{g} is abelian is to say $\text{Ker } \mu = \mathfrak{g} \otimes \mathfrak{g}$, which is generated by elementary tensors $x \otimes y \quad \forall x, y \in \mathfrak{g}$. By proposition 4.1, \mathfrak{g} is zero product determined. □

Theorem 4.7. *Let $\mathfrak{q} = \mathfrak{g}_Z \oplus \mathfrak{g}_S$ be a parabolic subalgebra of a reductive Lie algebra $\mathfrak{g} = \mathfrak{g}_Z \oplus \mathfrak{g}_S$ over the field \mathbb{K} . Then \mathfrak{q} is zero product determined.*

Proof. \mathfrak{g}_Z is zero product determined by lemma 4.6, \mathfrak{q}_S is zero product determined by proposition 4.5. The direct sum $\mathfrak{q} = \mathfrak{g}_Z \oplus \mathfrak{q}_S$ is zero product determined by proposition 4.2. □

In particular, the Borel subalgebra $\mathfrak{b} = \mathfrak{g}_Z + \sum_{\beta \in \Phi^+} \mathfrak{g}_\beta$ and the reductive Lie algebra \mathfrak{g} are zero product determined.

4.3 Derivations of parabolic subalgebras

We now return to the original motivation of this dissertation. We apply the study of zero product determined algebras to the derivation algebra $\text{Der } \mathfrak{q}$ of a parabolic subalgebra $\mathfrak{q} = \mathfrak{g}_Z \oplus \mathfrak{q}_S$ of the reductive Lie algebra $\mathfrak{g} = \mathfrak{g}_Z \oplus \mathfrak{g}_S$ over the field \mathbb{K} .

Definition 4.3. Let $n, k \in \mathbb{Z}^{\geq 0}$. Denote by $L(n, k)$ the subalgebra of $\mathfrak{gl}(\mathbb{C}^{n+k})$ consisting of matrices whose $(n+i)$ -th rows are zero for $1 \leq i \leq k$.

$L(n, k)$ consists of complex matrices with block form

$$\begin{matrix} & n & k \\ n & \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix} \\ k & \begin{pmatrix} 0 & 0 \end{pmatrix} \end{matrix}.$$

As a Lie algebra, $L(n, k) \cong \mathfrak{m} \ltimes \mathfrak{n}$ where: $\mathfrak{m} \cong \mathfrak{gl}(\mathbb{C}^n)$; \mathfrak{n} is abelian, consisting of $n \times k$ matrices with trivial bracket; and the action of \mathfrak{m} on \mathfrak{n} is given by usual matrix multiplication à la $[x, y] = xy$ for all $x \in \mathfrak{m}, y \in \mathfrak{n}$.

Notice that $L(n, 0) \cong \mathfrak{gl}(\mathbb{C}^n)$, so $L(n, k)$ is zero product determined by 4.7 when $k = 0$.

Lemma 4.8. $L(n, k)$ is zero product determined.

Proof. Without loss of generality, we assume $k \geq 1$. Write $L = L(n, k)$, and define $\mu : L \otimes L \longrightarrow [L, L]$ by $\mu(x \otimes y) = [x, y]$. By the rank-nullity theorem, we have

$$\dim \text{Ker } \mu = n^4 + 2n^3k + n^2k^2 - n^2 - nk + 1.$$

We will exhibit a basis for $\text{Ker } \mu$ consisting of elementary tensors.

Notation as above, $L \cong \mathfrak{m} \ltimes \mathfrak{n}$. \mathfrak{m} is zero product determined by theorem 4.7. By proposition 4.1, $\text{Ker } \mu|_{\mathfrak{m} \otimes \mathfrak{m}}$ admits a basis consisting of $n^4 - n^2 + 1$ elementary tensors of the form $x \otimes y$ with $x, y \in \mathfrak{m}$ and $[x, y] = 0$.

Since \mathfrak{n} is abelian, \mathfrak{n} is zero product determined by lemma 4.6, and proposition 4.1 provides n^2k^2 more elementary tensors of the form $x \otimes y$ with $x, y \in \mathfrak{n}$ and $[x, y] = 0$. We require $2n^3k - nk$ more linearly independent elementary tensors in $\text{Ker } \mu$.

Consider the $2n^3k - 2n^2k$ tensors

$$T_{i,j,l,q} = e_{i,j} \otimes e_{l,n+q} \in \mathfrak{m} \otimes \mathfrak{n}$$

and

$$T^{i,j,l,q} = e_{l,n+q} \otimes e_{i,j} \in \mathfrak{n} \otimes \mathfrak{m}$$

for $i, j, l \leq n$ and $q \leq k$ with $j \neq l$. Additionally, we have $2n^2k - 2nk$ tensors

$$S_{i,j,q} = (e_{i,j} - e_{i,j+1}) \otimes (e_{j,n+q} + e_{j+1,n+q}) \in \mathfrak{m} \otimes \mathfrak{n}$$

and

$$S^{i,j,q} = (e_{j,n+q} + e_{j+1,n+q}) \otimes (e_{i,j} - e_{i,j+1}) \in \mathfrak{n} \otimes \mathfrak{m}$$

with $i \leq n, j \leq n - 1$, and $q \leq k$. Finally, we have nk tensors of the form

$$R(i, q) = (e_{i,i} + e_{i,n+q}) \otimes (e_{i,i} + e_{i,n+q}) \in (\mathfrak{m} \dot{+} \mathfrak{n}) \otimes (\mathfrak{m} \dot{+} \mathfrak{n})$$

for $i \leq n$ and $q \leq k$, giving the desired $2n^3k - nk$ elementary tensors in $\text{Ker } \mu$. We have left to show that these tensors are linearly independent.

Expanding $S_{i,j,q}$ we see that

$$S_{i,j,q} = \underbrace{e_{i,j} \otimes e_{j,n+q} - e_{i,j+1} \otimes e_{j+1,n+q}}_{\notin \text{Span}\{T_{i,j,l,q}\}} + \underbrace{e_{i,j} \otimes e_{j+1,n+q} - e_{i,j+1} \otimes e_{j,n+q}}_{\in \text{Span}\{T_{i,j,l,q}\}}$$

is not in the span of the $T_{i,j,l,q}$. A similar observation shows that $S^{i,j,q}$ is not in the span of the $T^{i,j,l,q}$ tensors.

Expanding $R(i, q)$ we have

$$R(i, q) = \underbrace{e_{i,i} \otimes e_{i,i}}_{\in \mathfrak{m} \otimes \mathfrak{m}} + \underbrace{e_{i,n+q} \otimes e_{i,n+q}}_{\in \mathfrak{n} \otimes \mathfrak{n}} + \underbrace{e_{i,i} \otimes e_{i,n+q} + e_{i,n+q} \otimes e_{i,i}}_{\in \mathfrak{m} \otimes \mathfrak{n} + \mathfrak{n} \otimes \mathfrak{m}}.$$

Since $e_{i,i} \otimes e_{i,i}$ and $e_{i,n+q} \otimes e_{i,n+q}$ are in $\mathfrak{m} \otimes \mathfrak{m}$ and $\mathfrak{n} \otimes \mathfrak{n}$, respectively, we may subtract them, and we have left to consider $R'(i, q) = e_{i,i} \otimes e_{i,n+q} + e_{i,n+q} \otimes e_{i,i}$. $R'(i, q)$ is not in the span of $\{T_{i,j,l,q}, T^{i,j,l,q}\}$ since individually $e_{i,i} \otimes e_{i,n+q}$ and $e_{i,n+q} \otimes e_{i,i}$ are not among the $\{T_{i,j,l,q}, T^{i,j,l,q}\}$. Now, consider $S_{i,i,q} + S^{i,i,q}$ if $i < n$ (in case $i = n$ we are done, since we require $j \leq n - 1$ in $S_{i,j,q}$). We have

$$S_{i,i,q} + S^{i,i,q} = \underbrace{e_{i,i} \otimes e_{i,n+q} + e_{i,n+q} \otimes e_{i,i}}_{=R'(i,q)} + T - (e_{i,i+1} \otimes e_{i+1,n+q} + e_{i+1,n+q} \otimes e_{i,i+1})$$

with $T \in \text{Span}\{T_{i,j,l,q}, T^{i,j,l,q}\}$, so we have

$$R'(i, q) = S_{i,i,q} + S^{i,i,q} - T + e_{i,i+1} \otimes e_{i+1,n+q} + e_{i+1,n+q} \otimes e_{i,i+1}.$$

Write $R''(i, q) = e_{i,i+1} \otimes e_{i+1,n+q} + e_{i+1,n+q} \otimes e_{i,i+1}$. If $i = n - 1$ we are done. If $i < n - 1$ we may reduce $R''(i, q)$ using the same method as above, and so by induction we are done.

Thus we have a basis for $\text{Ker } \mu$ consisting of elementary tensors, and $L(n, k)$ is zero product determined by proposition 4.1. \square

Recall from theorem 3.1 that \mathfrak{q} decomposes as

$$\mathfrak{q} = \mathfrak{g}_Z + \mathfrak{c} + [\mathfrak{q}, \mathfrak{q}]$$

and $\text{Der } \mathfrak{q}$ decomposes as

$$\text{Der } \mathfrak{q} = \mathfrak{L} \oplus \text{ad } \mathfrak{q} \cong \text{Hom}_{\mathbb{K}}(\mathfrak{g}_Z + \mathfrak{c}, \mathfrak{g}_Z) \oplus \mathfrak{q}_S.$$

Since \mathfrak{q}_S is known to be zero product determined by proposition 4.5, we direct our attention to $\mathfrak{L} \cong \text{Hom}_{\mathbb{K}}(\mathfrak{g}_Z + \mathfrak{c}, \mathfrak{g}_Z)$, which is zero product determined in light of lemma 4.8.

Recall that our study in chapter 3 on the direct sum decomposition of $\text{Der } \mathfrak{q}$ was originally motivated by the question of whether the derivation algebras of certain Lie algebras were zero product determined. We conclude this chapter with the answer to this original question in the affirmative.

Theorem 4.9. *Let \mathfrak{q} be a parabolic subalgebra of a reductive Lie algebra \mathfrak{g} over a complex-like field \mathbb{K} . The derivation algebra $\text{Der } \mathfrak{q}$ is zero product determined.*

Proof. We begin with the decomposition

$$\text{Der } \mathfrak{q} = \mathfrak{L} \oplus \text{ad } \mathfrak{q}$$

established by theorem 3.1. We have $\text{ad } \mathfrak{q} \cong \mathfrak{q}_S$, and \mathfrak{q}_S is zero product determined by proposition 4.5. Furthermore, \mathfrak{L} is zero product determined. To verify this, write $n = \dim \mathfrak{g}_Z$ and $k = \dim \mathfrak{c}$, and observe that $\mathfrak{L} \cong L(n, k)$. By lemma 4.8, \mathfrak{L} is zero product determined. By proposition 4.2, $\text{Der } \mathfrak{q}$, as the direct sum of zero product determined Lie algebras, is zero product determined. □

Chapter 5

Examples and Future Research

We close this dissertation by taking note of directions that future research could take and how such research would fit into the existing body of results, and by providing worked examples and tabular data for reductive Lie algebras of types A_5 , G_2 , and F_4 .

5.1 Examples

We provide an algorithmic method for computing the center \mathfrak{l}_Z of the Levi factor in the Langland's decomposition of a parabolic subalgebra corresponding to any given subset of the base Δ of the root system. We then enumerate all standard parabolic subalgebras and give the dimensions of \mathfrak{L} , \mathfrak{q}_S (which, recall, is isomorphic to $\text{ad } \mathfrak{q}$), and $\text{Der } \mathfrak{q}$ in tabular form.

5.1.1 Type A_5

Let $\mathfrak{g} \cong \mathbb{C}^n \oplus \mathfrak{g}_S$ where $\mathfrak{g}_S \cong \mathfrak{sl}(\mathbb{C}^6)$. \mathfrak{g}_S has the root space decomposition

$$\mathfrak{g}_S = \mathfrak{h} \dot{+} \sum_{i \neq j} \mathbb{C}e_{i,j}$$

where \mathfrak{h} consists of traceless diagonal 6×6 complex matrices. Chose $\Delta = \{\alpha_1, \dots, \alpha_5\}$ as a base where $\mathfrak{g}_{\alpha_i} = \mathbb{C}e_{i,i+1}$. Then $\Phi^+ = \left\{ \sum_{i=1}^5 a_i \alpha_i \mid a_i \in \{0, 1\} \right\}$ and $\Phi = \Phi^+ \cup -\Phi$. Write $x_i = e_{i,i+1}$, $y_i = e_{i+1,i}$, and $h_i = [x_i, y_i] = e_{i,i} - e_{i+1,i+1}$. For each i , let t_i be the coroot dual to α_i , so $\alpha_i(t_j) = \delta_{i,j}$. $\mathcal{T} = \{t_1, \dots, t_5\}$ is a basis for \mathfrak{h} . Partial multiplication table for \mathfrak{g} in terms of $\mathcal{H} = \{h_1, \dots, h_5\}$ and \mathcal{T} are provided in tables 5.1 and 5.2 respectively.

	x_1	x_2	x_3	x_4	x_5
h_1	$2x_1$	$-x_2$	0	0	0
h_2	$-x_1$	$2x_2$	$-x_3$	0	0
h_3	0	$-1x_2$	$2x_3$	$-x_4$	0
h_4	0	0	$-x_3$	$2x_4$	$-x_4$
h_5	0	0	0	$-x_4$	$2x_5$

Table 5.1: Partial multiplication table for $\mathfrak{sl}(\mathbb{C}^6)$ in terms of \mathcal{H}

	x_1	x_2	x_3	x_4	x_5
t_1	x_1	0	0	0	0
t_2	0	x_2	0	0	0
t_3	0	0	x_3	0	0
t_4	0	0	0	x_4	0
t_5	0	0	0	0	x_5

Table 5.2: Partial multiplication table for $\mathfrak{sl}(\mathbb{C}^6)$ in terms of \mathcal{T}

For any $\Delta' \subset \Delta$ with corresponding parabolic subalgebra $\mathfrak{q} = \mathfrak{g}_Z \dot{+} \mathfrak{h} \dot{+} \sum_{\beta \in \Phi'} \mathfrak{g}_\beta$, we make three observations. First, the derived algebra $[\mathfrak{q}, \mathfrak{q}]$ is determined by Δ' as

$$[\mathfrak{q}, \mathfrak{q}] = \text{Span} \{ h_i \mid \alpha_i \in \Delta' \} \dot{+} \sum_{\beta \in \Phi'} \mathfrak{g}_\beta.$$

Second, the center \mathfrak{l}_Z of the Levi factor \mathfrak{l} is given by

$$\mathfrak{l}_Z = \text{Span} \{ t_i \mid \alpha_i \in \Delta \setminus \Delta' \}.$$

Third, the matrix whose columns are the members of \mathcal{T} written as vectors in terms of the basis \mathcal{H} is the inverse of the transpose of the Cartan matrix of \mathfrak{g} . Figure 5.1 gives the Cartan matrix and the inverse transpose of \mathfrak{g} , and table 5.3 gives members of \mathcal{T} in terms of \mathcal{H} and as matrices.

Utilizing the three above observations allows one to explicitly compute a basis for the ideal \mathfrak{L} of $\text{Der } \mathfrak{g}$. Table 5.4 contains data on for all standard parabolic subalgebras of \mathfrak{g} with respect to \mathfrak{h} .

t_i	t_i in terms of \mathcal{H}	t_i as a diagonal matrix
t_1	$(5/6, 2/3, 1/2, 1/3, 1/6)$	$\text{diag}(5/6, -1/6, -1/6, -1/6, -1/6, -1/6)$
t_2	$(2/3, 4/3, 1, 2/3, 1/3)$	$\text{diag}(2/3, 2/3, -1/3, -1/3, -1/3, -1/3)$
t_3	$(1/2, 1, 3/2, 1, 1/2)$	$\text{diag}(1/2, 1/2, 1/2, -1/2, -1/2, -1/2)$
t_4	$(1/3, 2/3, 1, 4/3, 2/3)$	$\text{diag}(1/3, 1/3, 1/3, 1/3, -2/3, -2/3)$
t_5	$(1/6, 1/3, 1/2, 2/3, 5/6)$	$\text{diag}(1/6, 1/6, 1/6, 1/6, 1/6, -5/6)$

Table 5.3: \mathcal{T} in terms of \mathcal{H} and as matrices

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix} \quad (A^T)^{-1} = \begin{bmatrix} 5/6 & 2/3 & 1/2 & 1/3 & 1/6 \\ 2/3 & 4/3 & 1 & 2/3 & 1/3 \\ 1/2 & 1 & 3/2 & 1 & 1/2 \\ 1/3 & 2/3 & 1 & 4/3 & 2/3 \\ 1/6 & 1/3 & 1/2 & 2/3 & 5/6 \end{bmatrix}$$

Figure 5.1: Cartan matrix and transpose inverse for Type A_5

5.1.2 Type G_2

Let $\mathfrak{g} \cong \mathbb{C}^n \oplus \mathfrak{g}_S$ where \mathfrak{g}_S is simple of type G_2 . The same observations in the previous example apply to any parabolic subalgebra corresponding to a $\Delta' \subset \Delta$. In particular, $\mathfrak{l}_Z = \text{Span } t_i \alpha_i \in \Delta \setminus \Delta'$. Figure 5.2 gives the Cartan matrix for Type G_2 and gives the inverse transpose, whose columns are t_i in terms of the h_i . Table 5.5 gives data for all the standard parabolic subalgebras of \mathfrak{g} .

5.1.3 Type F_4

Let $\mathfrak{g} \cong \mathbb{C}^n \oplus \mathfrak{g}_S$ where \mathfrak{g}_S is simple of type F_4 . Again, to any parabolic subalgebra corresponding to a $\Delta' \subset \Delta$ $\mathfrak{l}_Z = \text{Span } t_i \alpha_i \in \Delta \setminus \Delta'$. Figure 5.3 gives the Cartan matrix for Type F_4 and gives the inverse transpose, whose columns are t_i in terms of the h_i . Table 5.6 gives data for all the standard parabolic subalgebras of \mathfrak{g} .

$$A = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \quad (A^T)^{-1} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

Figure 5.2: Cartan matrix and transpose inverse for Type G_2

Δ'	$\dim \mathfrak{l}_Z$	$\dim \mathfrak{L}$	$\dim \mathfrak{q}_S$	$\dim \text{Der } \mathfrak{q}$
\emptyset	5	$n^2 + 5n$	20	$n^2 + 5n + 20$
10000	4	$n^2 + 4n$	21	$n^2 + 4n + 21$
01000	4	$n^2 + 4n$	21	$n^2 + 4n + 21$
00100	4	$n^2 + 4n$	21	$n^2 + 4n + 21$
00010	4	$n^2 + 4n$	21	$n^2 + 4n + 21$
00001	4	$n^2 + 4n$	21	$n^2 + 4n + 21$
11000	3	$n^2 + 3n$	23	$n^2 + 3n + 23$
10100	3	$n^2 + 3n$	22	$n^2 + 3n + 22$
10010	3	$n^2 + 3n$	22	$n^2 + 3n + 22$
10001	3	$n^2 + 3n$	22	$n^2 + 3n + 22$
01100	3	$n^2 + 3n$	23	$n^2 + 3n + 23$
01010	3	$n^2 + 3n$	22	$n^2 + 3n + 22$
01001	3	$n^2 + 3n$	22	$n^2 + 3n + 22$
00110	3	$n^2 + 3n$	23	$n^2 + 3n + 23$
00101	3	$n^2 + 3n$	22	$n^2 + 3n + 22$
00011	3	$n^2 + 3n$	23	$n^2 + 3n + 23$
11100	2	$n^2 + 2n$	26	$n^2 + 2n + 26$
11010	2	$n^2 + 2n$	24	$n^2 + 2n + 24$
11001	2	$n^2 + 2n$	24	$n^2 + 2n + 24$
10110	2	$n^2 + 2n$	24	$n^2 + 2n + 24$
10101	2	$n^2 + 2n$	23	$n^2 + 2n + 23$
10011	2	$n^2 + 2n$	24	$n^2 + 2n + 24$
01110	2	$n^2 + 2n$	26	$n^2 + 2n + 26$
01101	2	$n^2 + 2n$	24	$n^2 + 2n + 24$
01011	2	$n^2 + 2n$	24	$n^2 + 2n + 24$
00111	2	$n^2 + 2n$	26	$n^2 + 2n + 26$
11110	1	$n^2 + n$	30	$n^2 + n + 30$
11101	1	$n^2 + n$	27	$n^2 + n + 27$
11011	1	$n^2 + n$	26	$n^2 + n + 26$
10111	1	$n^2 + n$	27	$n^2 + n + 27$
01111	1	$n^2 + n$	30	$n^2 + n + 30$
Δ	0	n^2	35	$n^2 + 35$

Table 5.4: Parabolic subalgebras of type A_5

Δ'	\mathfrak{l}_Z	$\dim \mathfrak{L}$	$\dim \mathfrak{q}_S$	$\dim \text{Der } \mathfrak{q}$
\emptyset	\mathfrak{h}	$n^2 + 2n$	8	$n^2 + 2n + 8$
$\{\alpha_1\}$	$\text{Span}\{h_1 + 2h_2\}$	$n^2 + n$	9	$n^2 + n + 9$
$\{\alpha_2\}$	$\text{Span}\{2h_1 + 3h_2\}$	$n^2 + n$	9	$n^2 + n + 9$
Δ	0	n^2	14	$n^2 + 14$

Table 5.5: Parabolic subalgebras of type G_2

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \quad (A^T)^{-1} = \begin{bmatrix} 2 & 3 & 2 & 1 \\ 3 & 6 & 4 & 2 \\ 4 & 8 & 6 & 3 \\ 2 & 4 & 3 & 2 \end{bmatrix}$$

Figure 5.3: Cartan matrix and transpose inverse for Type F_4

Δ'	$\dim \mathfrak{t}_Z$	$\dim \mathfrak{L}$	$\dim \mathfrak{q}_S$	$\dim \text{Der } \mathfrak{q}$
\emptyset	4	$n^2 + 4n$	28	$n^2 + 4n + 28$
1000	3	$n^2 + 3n$	29	$n^2 + 3n + 29$
0100	3	$n^2 + 3n$	29	$n^2 + 3n + 29$
0010	3	$n^2 + 3n$	29	$n^2 + 3n + 29$
0001	3	$n^2 + 3n$	29	$n^2 + 3n + 29$
1100	2	$n^2 + 2n$	31	$n^2 + 2n + 31$
1010	2	$n^2 + 2n$	30	$n^2 + 2n + 30$
1001	2	$n^2 + 2n$	30	$n^2 + 2n + 30$
0110	2	$n^2 + 2n$	32	$n^2 + 2n + 32$
0101	2	$n^2 + 2n$	30	$n^2 + 2n + 30$
0011	2	$n^2 + 2n$	31	$n^2 + 2n + 31$
1110	1	$n^2 + n$	37	$n^2 + n + 37$
1101	1	$n^2 + n$	32	$n^2 + n + 32$
1011	1	$n^2 + n$	32	$n^2 + n + 32$
0111	1	$n^2 + n$	37	$n^2 + n + 37$
Δ	0	n^2	54	$n^2 + 54$

Table 5.6: Parabolic subalgebras of type F_4

5.2 Directions for future research

The theorems of chapter 4 apply primarily to reductive Lie algebras over complex-like fields. An immediate extension would be to prove these results for reductive Lie algebras over \mathbb{R} . Our results in chapter 4 extend work carried out by Wang et al. in [33], where the authors prove that the parabolic subalgebras of a simple Lie algebra over a complex-like field are zero product determined. The arguments employed by Wang et al. do not appear to be easily extended to the real case, as the computational method employed relies on the fact that root spaces are one-dimensional, where restricted root spaces can be arbitrarily large in dimension. An abstract argument, however, similar to our proof of theorem 3.3 from theorem 3.1 might produce the desired result.

Along the same lines, we may extend the class of Lie algebras under consideration by including Lie algebras over prime-characteristic fields [19, 26] or over general commutative rings [23, 30, 31]. Alternatively, we may consider infinite-dimensional Lie algebras. Kac-Moody algebras are infinite-dimensional generalizations of the (finite-dimensional) semisimple Lie algebras, and they share many of the properties of semisimple Lie algebras especially as they relate to root space decomposition [16]. Farnsteiner in 1988 investigated the derivations of Borel subalgebras of Kac-Moody algebras, and perhaps similar techniques can be employed to extend these results to parabolic subalgebras [10].

The methods we employ in our investigation have several noteworthy precedents in the literature. Recall, for instance, the discussion in chapter 1 of the work of Jacobson in 1955 in [14] and the related work by Dixmier and Lister in 1957 in [9]. Dixmier and Lister show that a converse to a result proved by Jacobson is not possible by constructing an example of a nilpotent Lie algebra and explicitly describing its derivation algebra. Dixmier and Lister employ methods in their concrete example that mirror the abstract methods we use in this dissertation. The interesting point is this: parabolic subalgebras are never nilpotent. This suggests that perhaps the methods employed here may be extended to a much wider classes of Lie algebras.

In contrast, we may consider the methods of Leger and Luks in [19] and the methods of Tolpygo in [29]. In these papers, the authors prove special cases of our results, though they use completely different methods. Leger's and Luks's results imply that all derivations of a Borel subalgebra of a simple Lie algebra are inner (over any field with characteristic not 2) [19], and similarly Tolpygo's results (applicable specifically over the complex field) imply that all derivations of a parabolic subalgebra of a semisimple Lie algebra are inner [29]. In fact, these results are more general and stated in the language of cohomology: The authors prove that all cohomology group $H^n(\mathfrak{g}, \mathfrak{g})$ are trivial for their respective classes of Lie algebras \mathfrak{g} under consideration [19, 29].

Very briefly, cohomology groups are computable invariants of a Lie algebra that provide information about the Lie algebra under consideration. (For context, the reader is reminded that the familiar Calculus derivative f' of a real-valued function f is a computable invariant that provides information about f .) For instance, the first cohomology group $H^1(\mathfrak{g}; \mathfrak{g})$ of a Lie algebra \mathfrak{g} satisfies the isomorphism

$$H^1(\mathfrak{g}; \mathfrak{g}) \cong \text{Der } \mathfrak{g} / \text{ad } \mathfrak{g}.$$

From this isomorphism, it follows that $H^1(\mathfrak{g}; \mathfrak{g}) = 0$ implies that all derivations of \mathfrak{g} are inner. A further application of cohomology is to extensions of a Lie algebra \mathfrak{b} by an ideal \mathfrak{a} . We have the isomorphism

$$H^2(\mathfrak{b}; \mathfrak{a}) \cong \text{Ext}(\mathfrak{b}; \mathfrak{a})$$

meaning that the second cohomology group $H^2(\mathfrak{b}; \mathfrak{a})$ parametrizes the set of all possible extensions of \mathfrak{b} by \mathfrak{a} (cf. definition 2.11). If we happen to know that $H^2(\mathfrak{b}; \mathfrak{a}) = 0$, then we know that the only extension of \mathfrak{b} by \mathfrak{a} is the trivial extension defined by the action $\forall b \in \mathfrak{b}, \forall a \in \mathfrak{a}, b \cdot a = 0$. Such an action results in a component-wise bracket rule, so the extensions is in fact the Lie algebra direct sum $\mathfrak{b} \oplus \mathfrak{a}$.

In light of these two isomorphisms, the language and methods of cohomology provide a strong framework for discovering structural properties of $\text{Der } \mathfrak{g}$ as they relate to properties of \mathfrak{g} . Our results on derivations apply to reductive Lie algebras, trivial extensions of semisimple Lie algebras by an abelian Lie algebra. Cohomology might be employed to study the derivations of general extensions of Lie algebras. Our results on direct sums of zero product determined algebras might likewise be generalized to extensions of algebras. Our present results, in turn, can enrich the study of cohomology by providing further examples of classes of Lie algebras that exhibit non-trivial first cohomology groups.

Another direction for extending our results is to generalize the notion of derivation itself. A straightforward way of doing this is to drop the requirement that a derivation be linear, an approach studied in by Chen and Wang in [6] and [7]. The authors use the term *nonlinear map satisfying derivability* for a not-necessarily-linear map $D : \mathfrak{g} \rightarrow \mathfrak{g}$ satisfying

$$\forall x, y \in \mathfrak{g} \quad D([x, y]) = [D(x), y] + [x, D(y)]. \quad (5.1)$$

Alternatively, one can generalize the notion of a derivation by relaxing the product rule (equation 5.1), studying linear maps $D : \mathfrak{g} \rightarrow \mathfrak{g}$ satisfying the weaker condition

$$\forall x, y, z \in \mathfrak{g} \quad D\left([x, y], z\right) = \left[[D(x), y], z\right] + \left[[x, D(y)], z\right] + \left[[x, y], D(z)\right]. \quad (5.2)$$

Such a map is called a *Lie triple derivation*, and these maps are studied in [20], [35], and [32] among others. The two approaches can be combined, studying maps D that are not necessarily linear and satisfy condition 5.2 rather than condition 5.1. Chen and Wang take this approach in [8], naming such a map a *nonlinear Lie triple derivation*. Figure 5.4 illustrates the logical connection between the various types of derivation-like maps considered.

We offer a brief summary of the results in [6] and [8]. In [6], Chen and Wang study non-linear maps satisfying derivability on parabolic subalgebras of simple Lie algebras

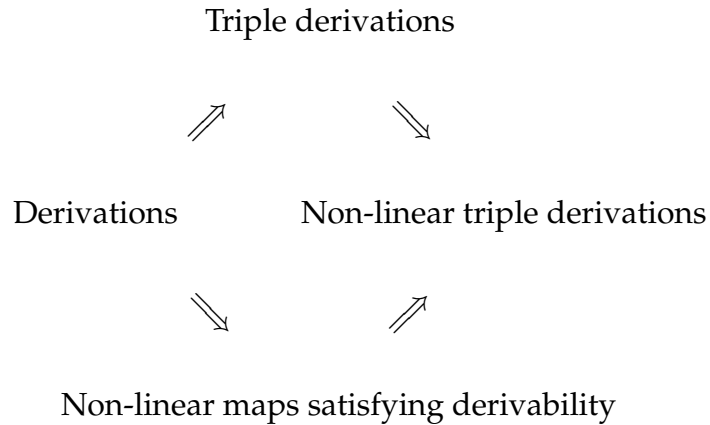


Figure 5.4: Logical relations among various types of derivation-like maps

over a complex-like field. The authors show that any such map is the sum of an inner derivation and what the authors call an additive quasi-derivation (a map that, notably, fails to be homogeneous). As an aside, this gives an alternate proof that $\text{Der}(\mathfrak{q}) = \text{ad}(\mathfrak{q})$ when \mathfrak{q} is a parabolic subalgebra of a simple Lie algebra. In [8], the same authors study non-linear Lie triple derivations in the same setting, parabolic subalgebras of a simple Lie algebra \mathfrak{g} over a complex-like field. What is worth noting, though, is that their result is exactly the same: a non-linear Lie triple derivation is the sum of an inner derivation and an additive quasi-derivation. In other words, the weaker assumption of requiring condition 5.2 rather than condition 5.1 resulted in no new maps — the non-linear Lie triple derivations and the non-linear maps satisfying derivability on a parabolic subalgebra exactly coincide in case \mathfrak{g} is simple.

The results of Chen and Wang motivate the following questions: Are there non-linear triple derivations that are not non-linear maps satisfying derivability? If so, what classes of Lie algebras must we consider in order to differentiate between the two types of maps? It would be interesting to extend these results to parabolic subalgebras of reductive Lie algebras for a number of reasons. Considering derivations of reductive algebras has provided examples of derivations that are non inner — a sort of non-triviality result about

outer derivations. In parallel, considering non-linear maps satisfying derivability and non-linear triple derivations of parabolic subalgebras of reductive algebras could perhaps lead to examples of non-linear triple derivations that are not non-linear maps satisfying derivability.

A final vehicle for future research that we will discuss deals with the abstract form of the decomposition of $\text{Der } \mathfrak{q}$ established by theorems 3.1 and 3.3. If we denote by \mathfrak{g} a parabolic subalgebra, we have that the derivation algebra $\text{Der } \mathfrak{g}$ decomposes as

$$\text{Der } \mathfrak{g} \cong \text{Hom}(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}], \mathfrak{g}_Z) \oplus \text{ad } \mathfrak{g}. \quad (5.3)$$

by corollary 3.8. The constructions $\text{ad } \mathfrak{g}$, \mathfrak{g}_Z , $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$, and $\text{Hom}(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}], \mathfrak{g}_Z)$ can be carried out for any Lie algebra \mathfrak{g} , motivating the following question: for which Lie algebras \mathfrak{g} does isomorphism 5.3 hold?

We remind the reader that a Lie algebra \mathfrak{g} is called *complete* if $\mathfrak{g}_Z = 0$ and \mathfrak{g} has only inner derivations. Analogously, we propose the following definition: a Lie algebra \mathfrak{g} is *almost complete* if isomorphism 5.3 holds. We see that the almost complete Lie algebras are an intermediate class between the complete Lie algebras and general Lie algebras, and as an area for future investigation we may wish to characterize almost complete Lie algebras in order to refine the classification of Lie algebras in general.

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