Monotonic Covering Properties

by

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Abstract

Monotonic versions of classical topological properties have been of some interest for several years. That adding monotonic to a covering property makes it stronger is well-known, but exactly how much stronger is still, in many cases, unknown. Here we trace the progression of results regarding monotonic covering properties, and then look at monotonic versions of metacompactness and meta-Lindelöfness, providing a useful property these spaces exhibit, and from that obtain several original results, many of which answer open questions. We strengthen a theorem of G. Gruenhage, (that monotonically compact, T_2 spaces are metrizable) by showing that compact, T_2, monotonically metacompact spaces are metrizable. We also show that every monotonically metacompact space is hereditarily metacompact, and show by way of a counterexample that Bennett, Hart, and Lutzer's theorem that every regular, developable, metacompact space is monotonically metacompact cannot have the condition "developable" weakened to "quasi-developable", or replaced by "stratifiable". In examining several different spaces for their relationship to this property, we also show that a monotonically Lindelöf space need not be monotonically metacompact.

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There is no such thing as a 'self-made' man. We are all made up of thousands of others. Everyone who has ever done a kind deed for us...has entered into the make-up of our character and of our thoughts, as well as our success.

–George Matthew Adams

For this achievement of mine, I owe thanks to a great number of people, without whom I would never have been able to succeed:

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Chapter 1

Introduction

General topology is a complex and intricate field of mathematics. It is often the case that we not only look at a property and its interactions and relationships to other known properties, but it is also common to consider some special properties that are modifiers of other properties. These "meta"- properties are quite prevalent throughout topology - there are many well-known results involving properties that are *locally*, *hereditarily*, or *relatively* some common property. Here we take a look at one of these "meta"-properties, *monotonically*, and its relationship to some common topological covering properties.

The idea of a covering property being monotonic has its roots in the definition of a property that has nothing to do with open covers. Monotonic normality was first examined in Borges' paper[7], in 1966, and Zenor officially named the property in 1970 in his joint paper with Heath and Lutzer [19]. After that there was quite an explosion of material regarding this new property. Shortly after, the style of this definition was adapted and applied to other kinds of properties, including those of primary interest here, covering properties.

K.P. Hart, in his online review of Junilla and Kunzi's *Ortho-bases and Monotonic Properties*, described a process for obtaining a monotonic version of any well-known covering property: "by requiring that there is an operator, R, that assigns to every open cover a refinement of the right kind in such a way that $R(\mathcal{O})$ refines $R(\mathcal{U})$ whenever $\mathcal{O}$ refines $\mathcal{U}$ [18]."

Using this process, just about any covering property can be "upgraded" into a monotonic property, thus obviously acquiring a new property. The resulting monotonic version of the property turns out to be stronger than the original covering property, often much stronger.

For example, Gruenhage [15] showed that every monotonically compact Hausdorff space must be a compact metric space.

Most of the topological covering properties have monotonic versions that have been examined over the years. Monotone paracompactness was one of the first considered in 1993 by Gartside and Moody, [11], although they did not define it by monotonizing the standard definition of paracompactness. It turns out that the utilization of several different, although equivalent, characterizations of paracompactness has given rise to several different, unequal monotone properties. Indeed, we give an example which shows that monotonizing he standard definition of paracompactness yields a different property than that of Gartside and Moody.

Monotonically compact and monotonically Lindelöf were both initially considered around 2005, and over the years there have been several results published regarding those properties as well. The properties of monotonically metacompact and monotonically meta-Lindelöf have been some of the most recently examined of the monotonic covering properties, and are the two that receive the greatest consideration here.

The monotonically metacompact property was first introduced by Popvasillev in 2009, as an extension of the monotonically compact property [30]. He showed that neither of the ordinal spaces ω_1 nor $\omega_1 + 1$ are monotonically (countably) metacompact. Bennett, Hart, and Lutzer later showed that any metric space as well as any metacompact Moore space is monotonically metacompact, and discovered several other results relating to monotone met-compactness in LOTS and GO-spaces; in particular, they proved that every monotonically metacompact compact LOTS is metrizable [4]. More recently Liang-Xue Peng and Hui Li have extended some results for monotonically compact and monotonically Lindelöf spaces to monotonically metacompact spaces [26].

The definition of monotone metacompactness studied here is that of Popvassilev[1], and will be formally defined in the next chapter. Our work extends several of the results mentioned above, most notably we show that every compact, monotonically metacompact, Hausdorff space is metrizable, which is an extention of a theorem by Gruenhage, and answers a question asked by Popvassilev [30] and by Bennett, Hart, and Lutzer [4]. Additionally we show that every T_3, monotonically metacompact space having caliber ω_1 is hereditarily Lindelöf, which is an extention of G. Gruenhage's theorem [14]. We also show that every monotonically metacompact space must be hereditarily metacompact, which we use to answer another question of Bennet, Hart, and Lutzer. Finally we examine several spaces which have appeared in the literature regarding other monotonic properties. We show whether these spaces are monotonically metacompact or monotonically meta-Lindelöf or not, and consequently show that a space can be monotonically Lindelöf without being monotonically metacompact, among other results.

1.1 Definitions

Most of the advanced definitions used throughout will be provided as needed. A general understanding of basic topology is assumed, essentially any information provided in a first-year graduate level topology course, and definitions and techniques most commonly introduced there will be omitted. I will, for the reader's benefit, put some of the most frequently used "basic" definitions below, in addition to any special notations I use. For a more in depth examination of any of these definitions, the reader is encouraged to review [9].

By a *space* we will mean a topological space.

By a *refinement* of an open cover, we will mean a collection of open sets every element of which is contained in some element of the original open cover. In other words we will say

[1] In the literature, there is another definition of monotone countable metacompactness, which is given by Good, Knight, and Stares [13]. Their definition utilizes decreasing sequences of closed sets, and has been shown that their definition is the class of β-spaces of Hodel [36]. Consequently, as explained in [30] and [4], this definition and the defnition of monotonically (countably) metacompact of Popvassilev are entirely different, and neither implies the other.

$\mathcal{V}$ refines $\mathcal{U}$, and write $\mathcal{V} \prec \mathcal{U}$, if for every $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $V \subset U$. Unless otherwise stated, we will not assume that a refinement of an open cover also covers the space.

If X is a space and $\mathcal{U}$ is a collection of its subets, then for $S \subset X$, we will say the *star of S with respect to $\mathcal{U}$* is $st(S,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : S \cap U \neq \emptyset\}$.

An open cover $\mathcal{V}$ is a *star-refinement* of $\mathcal{U}$ if for every $V \in \mathcal{V}$ there exists some $U \in \mathcal{U}$ such that $st(V, \mathcal{V}) \subset U$.

A collection $\mathcal{U}$ of open subsets of a space X is *locally finite* if any point in the space has a neighborhood that intersects only finitely many members of the collection, and is *point-finite* if any point in the space is contained in only finitely many elements of the collection. Obviously every locally finite collection of sets is point-finite.

For a topological space X, and a propery P, we will say that X is *hereditarily P* if every subset of X also has property P.

Although the reader is assumed to be familiar with the concepts of compactness and Lindelöfness, there are several other covering properties whose monotonic versions will be explored here. A list of all of the important definitions, and their relationships to each other is given:

Definition 1. *A space X is **compact** if every open cover $\mathcal{U}$ of the space X has a finite subcover.*

Definition 2. *A space X is **Lindelöf** if every open cover $\mathcal{U}$ of the space X has a countable subcover.*

It is easily seen that every compact space is Lindelöf.

Definition 3. *A space X is **paracompact** if for every open cover $\mathcal{U}$ of X, there is an open refinement, $\mathcal{V}$, covering the space, such that $\mathcal{V}$ is locally finite.*

It is known that every compact space and every T_3, Lindelöf space is paracompact.

Definition 4. *A space X is **metacompact (meta-Lindelöf)** if for every open cover $\mathcal{U}$ of X, there exists an open refinement $\mathcal{V}$ covering the space, such that $\mathcal{V}$ is point-finite (point-countable).*

Paracompact spaces are metacompact, and metacompact spaces are meta-Lindelöf.

Figure 1.1: Relationships between weakenings of compactness.

Metrizability is another property that is very prominent in set theoretic topology that the reader is assumed to be familiar with, but some of the important, though less well known weakenings of metrizability will also be examined regarding their relationship to monotonic covering properties. The following is a list of these properties and their known relationships:

Definition 5. *A topological space (X, τ) is a **metric space** or **metrizable** if there is a metric $d : X \times X \to [0, \infty)$ such that the topology induced by d is τ.*

Every metric space is known to be paracompact. For a metric space to be compact or Lindelöf, however, requires additional properties.

Definition 6. *A space X is **protometrizable** if it is paracompact and has an orthobase.*

An *orthobase* for a space X is a base $\mathcal{B}$ for the topology on X such that for any collection $\mathcal{F} \subset \mathcal{B}$, either $\bigcap \mathcal{F}$ is open, or $\bigcap \mathcal{F} = \{x\}$, and $\mathcal{F}$ is a local base at x.

Every metrizable space is protometrizable.

Definition 7. *A space X is **stratifiable** if for every open set U, one can assign a countable collection of open sets $\mathcal{U}_n = \{U_n\}_{n \in \omega}$, such that:*

i. $\overline{U_n} \subset U$;

ii. $\bigcup_{n\in\omega} U_n = U$;

iii. $U_n \subset V_n$ *whenever* $U \subset V$.

Every metric space is stratifiable.

Definition 8. *A space X is* ***quasi-developable*** *if there exists a sequence $\{\mathcal{U}_1, \mathcal{U}_2, \dots\}$ of collections of open subsets of X such that for each $x \in X$ and each open set O containing x, there exists $n \in \omega$ such that for some $U \in \mathcal{U}_n$, $x \in U$, and each element of $\mathcal{U}_n$ that contains x is contained in O. In other words, $st(x, \mathcal{U}_n) \subset O$.*

Definition 9. *A space X is* ***developable*** *if it is quasi-developable, and each collection $\mathcal{U}_i$ in the defining sequence is an open cover of X. Regular developable spaces are called* ***Moore spaces***.

It is known every metric space is a Moore space, and it is obvious that every developable space is quasi-developable.

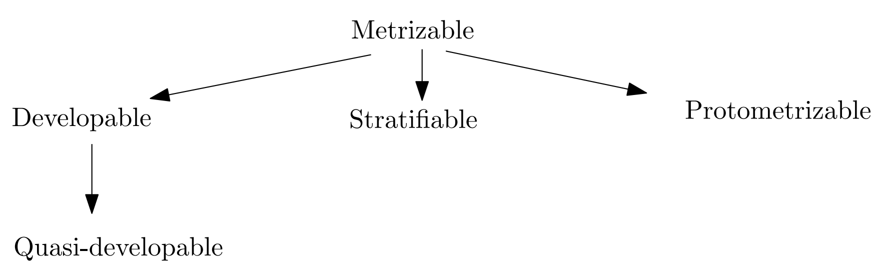

Figure 1.2: Relationships between weakenings of metrizability.

Separability is another important property in general topology, and we frequently consider weakenings of it. The relationships between these properties is given in the following figure. These implications are not reversible in ZFC.

Definition 10. *A space X is* ***separable*** *if it contains a countable dense subset.*

Definition 11. *A space X has* ***caliber*** ω_1 *if for any uncountable collection of nonempty open sets* $\mathcal{U} = \{U_\alpha\}_{\alpha\in\omega_1}$ *there exists an uncountable* $A \subset \omega_1$ *such that* $\bigcap_{\alpha\in A} U_\alpha \neq \emptyset$. .

Definition 12. *A space X has* ***property K*** *if every uncountable collection of open sets has an uncountable linked subcollection.*

A collection of sets is *linked* if the intersection of any two sets from the collection is non-empty.

Definition 13. *A space X satisfies the* ***countable chain condition*** *(**CCC**) if every pair-wise disjoint collection of non-empty open subsets of the space is countable.*

Separable $\rightarrow$ caliber ω_1 $\rightarrow$ property K $\rightarrow$ CCC

Figure 1.3: Relationships between weakenings of separability.

In the literature, the monotonic covering properties are often considered within the framework of special topological spaces - two of the most common are linearly ordered topological spaces, and generalized order spaces:

Definition 14. *A space X is a* ***linearly ordered topological space*** *(**LOTS**) if it is totally ordered, and the topology on X is the order topology generated by taking all sets of the form:*

$(-\infty, b) = \{x : x < b\}$,

$(a, \infty) = \{x : a < x\}$, *and*

$(a, b) = \{x : a < x < b\}$

as a basis.

Definition 15. *A space X is a* ***GO-space*** *if X has a linear ordering such that the topology on X is T_2 and has a base of order-convex sets. Equivalently, X is homeomorphic to a subspace of a LOTS.*

In several of our results we make good use of some "counting arguments" from partition calculus. Partition calculus is an area of set theory devoted to the study of Ramsey Theory, and often considers the combinatorics of infinite sets. For more information on partition calculus, we refer the reader to [20]. Two theorems we apply several times are:

Theorem 1.1.1. ***Ramsey's Theorem:*** - $\omega \to (\omega)^r_n$: *Assume $r, n \in \omega$. If the r-element subsets of a countably infinite set A are divided into n-many different pots, (Pot I, Pot II, ... , Pot n), then there is an infinite subset $A' \subset A$, such that all the r-element subsets of A' are in the same pot.*

Theorem 1.1.2. ***Erdos' Theorem*** - $\omega_1 \to (\omega, \omega_1)^2$: *If the unordered pairs $\{a, b\}$ of elements an uncountable set A are divided into two pots, say Pot I and Pot II, then there is either an infinite subset of A, all pairs of which are from Pot I, or there is an uncountable subset of A, all pairs of which are from Pot II.*

Chapter 2

Main Results

Our initial interest in spaces having monotone covering properties started with the examination of monotonically metacompact spaces, and the attempt to answer an open question posed both by Bennett, Hart, and Lutzer [4], as well as by Popvassilev [30]: whether every montonically metacompact, compact, Hausdorff space is metrizable. In exploring that question, it became useful to consider the related property of monotonically meta-Lindelöf, and we found several results related to that property as well.

Definition 16. *A space X is* ***monotonically (countably) metacompact*** *if there is a function r that assigns to each (countable) open cover $\mathcal{U}$ of a space X, a point-finite open refinement $r(\mathcal{U})$ covering X such that if $\mathcal{V}$ is a (countable) open cover of X and $\mathcal{V}$ refines $\mathcal{U}$, then $r(\mathcal{V})$ refines $r(\mathcal{U})$. The function r is called a* ***monotone (countable) metacompactness operator****.*

Definition 17. *A space X is* ***monotonically meta-Lindelöf*** *if there is a function r that assigns to each open cover $\mathcal{U}$ of a space X, a point-countable open refinement $r(\mathcal{U})$ covering X such that if $\mathcal{V}$ is an open cover of X and $\mathcal{V}$ refines $\mathcal{U}$, then $r(\mathcal{V})$ refines $r(\mathcal{U})$.*

In many ways these two properties behave much like the original covering properties. It is easy to see that closed subspaces of monotonically metacompact or monotonically meta-Lindelöf spaces are monotonically metacompact and monotonically meta-Lindelöf, respectively. Similarly it is trivially seen that every monotonically compact space is monotonically metacompact, and that all the implications in Figure 2.1 hold. It is also known that these implications are not reversible.

Initially we attempted to address the open question by applying the definition of monotonically metacompact directly. While it is possible to show spaces are not monotonically

$$\begin{tikzcd}
\text{Monotonically Compact} \arrow[r] \arrow[dr] & \text{Monotonically Paracompact} \arrow[r] & \text{Monotonically Metacompact} \arrow[d] \\
 & \text{Monotonically Lindelöf} \arrow[r] & \text{Monotonically Meta-Lindelöf}
\end{tikzcd}$$

Figure 2.1: Relationships between monotone compactness-type properties.

metacompact by using only the definition, it can be very cumbersome, as you can see in Example 3.1.1, and we found it significantly easier to apply a property that these spaces exhibit involving a special neighborhood-pair assignment, described in Lemma 2.0.3. Almost all of the results here are found by applying that lemma, which was inspired by a similar characterization described in [15].

For a space X let $\mathcal{P}_X$ be the collection of all triples $p = (x^p, U_0^p, U_1^p)$ where U_0^p, U_1^p are open in X, and $x^p \in U_0^p \subset \overline{U_0^p} \subset U_1^p$.

Lemma 2.0.3. *Suppose X is monotonically (countably) metacompact. Then to each $p \in \mathcal{P}_X$ one can assign an open V^p satisfying:*

i. $x^p \in V^p \subset U_1^p$;

ii. *Whenever (countable) $\mathcal{Q} \subset \mathcal{P}_X$, then either $\bigcap_{q\in\mathcal{Q}} V^q = \emptyset$, or there exists a $\mathcal{Q}' \subset \mathcal{Q}$, with $\mathcal{Q}'$ finite, such that for any $q \in \mathcal{Q}$ there exists $q' \in \mathcal{Q}'$ such that either $V^q \subset U_1^{q'}$ or $V^q \cap U_0^{q'} = \emptyset$.*

Proof. Let X be monotonically (countably) metacompact, r the monotonically (countably) metacompact operator, and $\mathcal{P}_X$ defined as above. Notice that for every $p \in \mathcal{P}_X$, $\mathcal{U}^p = \{U_1^p,\ X \setminus \overline{U_0^p}\}$ is an open cover of X. Also note that $r(\mathcal{U}^p)$ is a point-finite refinement of $\mathcal{U}^p$. Let V^p be any $V \in r(\mathcal{U}^p)$ such that $x^p \in V$. Obviously V^p is open and $x^p \in V^p \subset U_1^p$.

Now let $\mathcal{Q} \subset \mathcal{P}_X$, ($|\mathcal{Q}| = \omega$), and we will assume that $\bigcap_{q\in\mathcal{Q}} V^q \neq \emptyset$.

Let $\mathcal{U} = \bigcup\{\mathcal{U}^q : q \in \mathcal{Q}\}$. $\mathcal{U}$ is a (countable) open cover of X, and for all $q \in \mathcal{Q}$, $\mathcal{U}^q \prec \mathcal{U}$. Thus $r(\mathcal{U}^q) \prec r(\mathcal{U})$ for all $q \in \mathcal{Q}$.

Let $t \in \bigcap_{q \in \mathcal{Q}} V^q$, and let $\mathcal{U}' = \{U \in r(\mathcal{U}) : t \in U\}$. Since $r(\mathcal{U})$ refines $\mathcal{U}$, for each $U \in \mathcal{U}'$ there is some $q(U) \in \mathcal{Q}$ such that $\{U\} \prec \mathcal{U}^{q(U)}$. Let $\mathcal{Q}' = \{q(U) : U \in \mathcal{U}'\}$.

We show that this $\mathcal{Q}'$ satisfies the conclusion of condition (ii). Suppose $q \in \mathcal{Q}$. We have $\{V^q\} \prec r(\mathcal{U}^q) \prec r(\mathcal{U})$, and $t \in V^q$, so there is some $U \in \mathcal{U}'$ such that $V^q \subset U$. Then from $\{V^q\} \prec \{U\} \prec \mathcal{U}^{q(U)}$, we have $V^q \subset U_1^{q(U)}$ or $V^q \cap U_0^{q(U)} = \emptyset$, as desired. □

We also have a monotonically meta-Lindelöf version of Lemma 2.0.3:

Lemma 2.0.4. *Suppose X is monotonically meta-Lindelöf. Then to each $p \in \mathcal{P}_X$ one can assign an open V^p satisfying:*

i. $x^p \in V^p \subset U_1^p$;

ii. Whenever $\mathcal{Q} \subset \mathcal{P}_X$, then either $\bigcap_{q \in \mathcal{Q}} V^q = \emptyset$, or there exists a $\mathcal{Q}' \subset \mathcal{Q}$, with $\mathcal{Q}'$ countable, such that for any $q \in \mathcal{Q}$ there exists $q' \in \mathcal{Q}'$ such that either $V^q \subset U_1^{q'}$ or $V^q \cap U_0^{q'} = \emptyset$.

Proof. The proof of Lemma 2.0.4 is identical to that of Lemma 2.0.3, one just needs to substitute "countable" for "finite" in the proof, and ignore the parenthetical comments. In all other ways, the proofs are the same. □

There is a "weaker" version of these lemmas that are implied by monotone (countable) metacompactness, and monotone meta-Lindelöfness, respectively. The proof of these lemmas follow in the same manner as in Theorem 2.3 in [15]. This version of the property is sometimes easier to work with, since one need not worry about dealing with $\mathcal{P}_X$ and the triples. However, when using it to show that a spaces is not monotonically (countably) metacompact, it requires that there be several points in a space that exhibit the space not being monotonically metacompact, which may not be possible to find. (Many spaces we considered had only one non-isolated point, thus rendering the following lemma useless.)

Lemma 2.0.5. *Let X be a monotonically countably metacompact T_3-space, and $Y \subset X$. If for each $y \in Y$, U_y is some open neighborhood containing y, then there exists an open neighborhood V_y of y with $V_y \subset U_y$ such that if $Y' \subset Y$, and $\bigcap_{y \in Y'} V_y \neq \emptyset$, then there is a finite $Y'' \subset Y'$ such that $Y' \subset \bigcup_{y \in Y''} U_y$.*

Proof. Let U'_y be an open neighborhood of y such that $\overline{U'_y} \subset U_y$. Then $p(y) = (y, U'_y, U_y) \in \mathcal{P}_X$. Set $V_y = V^{p(y)} \cap U'_y$, where $V^{p(y)}$ is as in Lemma 2.0.1; also suppose $Y' \subset Y$, and $\bigcap_{y \in Y'} V_y \neq \emptyset$. Suppose that no finite $Y'' \subset Y'$ satisfying the conclusion of the lemma exists. Then we can find $y_0, y_1, ... \in Y'$ such that $y_n \notin \bigcup_{i<n} U_{y_i}$. Let $\mathcal{Q} = \{(y_n, U'_{y_n}, U_{y_n}) : n \in \omega\}$. Then there must be a finite subset $\mathcal{Q}'$ of $\mathcal{Q}$ satisfying the conclusion of Lemma 2.0.3. Let $n \in \omega$ be such that $(y_i, U'_{y_i}, U_{y_i}) \in \mathcal{Q}'$ implies that $i < n$. By Lemma 2.0.3, there must be $i < n$ such that either $V^{p(y_n)} \subset U_{y_i}$ or $V^{p(y_n)} \cap U'_{y_i} = \emptyset$. Now, $y_n \in V^{p(y_n)} \setminus U_{y_i}$, so $V^{p(y_n)} \not\subset U_{y_i}$. But $V^{p(y_n)} \cap U'_{y_i} \supset V_{y_n} \cap V_{y_i}$ and $V_{y_n} \cap V_{y_i} \neq \emptyset$. So $V^{p(y_n)} \cap U'_{y_i} \neq \emptyset$, a contradiction, and hence the lemma holds. □

This lemma also has a monotonically meta-Lindelöf version, whose proof is essentially identical to the one given above, just substituting "countable" in place of "finite".

Lemma 2.0.6. *Let X be a monotonically meta-Lindelöf, T_3-space, and $Y \subset X$. If for each $y \in Y$, U_y is some open neighborhood containing y, then there exists an open neighborhood V_y of y with $V_y \subset U_y$ such that if $Y' \subset Y$, and $\bigcap_{y \in Y'} V_y \neq \emptyset$, then there is a countable $Y'' \subset Y'$ such that $Y' \subset \bigcup_{y \in Y''} U_y$.*

Proof. Same as above. □

For our first major result, (Theorem 2.0.10), we utilize the following property: A space X has **caliber** ω_1 if for any uncountable collection of nonempty open sets $\mathcal{U} = \{U_\alpha\}_{\alpha \in \omega_1}$ there exists an uncountable $A \subset \omega_1$ such that $\bigcap_{\alpha \in A} U_\alpha \neq \emptyset$. Another way to think of this property is that for the space X, there is no uncountable, point-countable collection of open sets in X.

The following lemma will be useful in proving later results, and is a good example that makes use of Lemma 2.0.5.

Lemma 2.0.7. *If the T_3-space X is monotonically (countably) metacompact, and has caliber ω_1, then X is hereditarily Lindelöf.*[1]

Proof. If X is not hereditarily Lindelöf, it contains a right-separated subspace $\{x_\alpha : \alpha < \omega_1\}$ []. That is, for each $\alpha \in \omega_1$, there exists an open neighborhood U_{x_α} of x_α such that $U_{x_\alpha} \cap \{x_\beta : \beta > \alpha\} = \emptyset$. Now let V_{x_α} be as in Lemma 2.0.5. Since X has caliber ω_1, there must be an uncountable $A \subset \omega_1$ such that $\bigcap\{V_{x_\alpha} : \alpha \in A\} \neq \emptyset$. From Lemma 2.0.5, there must be a finite $A' \subset A$ such that $\{x_\alpha : \alpha \in A\} \subset \bigcup_{\beta \in A'} U_{x_\beta}$, which is impossible. Hence X is hereditarily Lindelöf. □

The next two lemmas and subsequent theorem together answer in the affirmative the question posed by both Popvassilev in [30] and Bennett, Hart, and Lutzer, in [4], "Is every compact, monotonically (countably) metacompact, Hausdorff space metrizable?" The proofs make good use of Lemma 2.0.3. This proof originally appeared in [8]. It should be noted that proof of Lemma 2.0.9 is similar to the proof that monotonically compact Hausdorff spaces having property (K) are metrizable, which was proved in [15].

Lemma 2.0.8. *Let X be compact T_2 and monotonically countably metacompact. Then X has caliber ω_1.*

Proof. Assume the hypotheses and suppose X does not have caliber ω_1. Then there exists a collection of nonempty open sets $\hat{\mathcal{U}} = \{U_\alpha : \alpha \in \omega_1\}$ such that any uncountable subcollection of $\hat{\mathcal{U}}$ has empty intersection. In other words, $\hat{\mathcal{U}}$ is a point-countable collection.

[1] The proof of this lemma is essentially identical to the proof of Theorem 2.4 of [15].

Pick $x_0 \in U_0$, and let $\alpha_0 = 0$. Since $\hat{\mathcal{U}}$ is point countable, there exists an $\alpha_1 \in \omega_1$ such that $x_0 \notin U_{\alpha_1}$. Pick $x_{\alpha_1} \in U_{\alpha_1}$. Suppose that x_{α_γ} and U_{α_γ} have been defined for each $\gamma < \delta$, where $\delta < \omega_1$, such that:

(i) $x_{\alpha_\gamma} \in U_{\alpha_\gamma}$

(ii) $\gamma < \gamma' < \delta \Rightarrow x_{\alpha_\gamma} \notin U_{\alpha_{\gamma'}}$

By point-countability, there exists an α_δ such that $U_{\alpha_\delta} \cap \{x_{\alpha_\gamma} : \gamma < \delta\} = \emptyset$. Now choose $x_{\alpha_\delta} \in U_{\alpha_\delta}$. In this manner, we get an uncountable collection $A \subset \omega_1$ and $x_\alpha \in U_\alpha$ for every $\alpha \in A$, but $x_\alpha \notin U_\beta$ for any $\beta > \alpha$, where $\beta \in A$.

By regularity, for each $\alpha \in A$ we can find a U'_α such that $x_\alpha \in U'_\alpha \subset \overline{U'_\alpha} \subset U_\alpha$. Let $\mathcal{U}_\alpha = \{U_\alpha,\ X \setminus \overline{U'_\alpha}\}$, for each $\alpha \in A$. Then $\mathcal{U}_\alpha$ is an open cover of X for each α, and $r(\mathcal{U}_\alpha) \prec \mathcal{U}_\alpha$.

Since X is compact, for each $\alpha \in A$ there exists a finite $\mathcal{V}_\alpha \subseteq \{V \in r(\mathcal{U}_\alpha) : V \subset X \setminus \overline{U'_\alpha}\}$ such that $X \setminus U_\alpha \subset \bigcup \mathcal{V}_\alpha$. Also notice that $x_\alpha \notin \bigcup \mathcal{V}_\alpha$.

Since we have uncountably many finite collections $\mathcal{V}_\alpha$, there exists $n \in \omega$ and an $A' \subset A$, with $|A'| = \omega_1$, such that $|\mathcal{V}_\alpha| = n$ for all $\alpha \in A'$. Denote $\mathcal{V}_\alpha = \{V_{\alpha,1},\ V_{\alpha,2},\ \ldots\ V_{\alpha,n}\}$ for all $\alpha \in A'$.

Let $A'' \subset A'$ have order type ω. Note that for $s < t$ in A'', we have $x_s \notin U_t$, hence $x_s \in \bigcup \mathcal{V}_t$. Put $\{s,t\}$ in Pot i if $x_s \in V_{t,i}$. From Ramsey's Theorem, there is an infinite $B \subset A''$ and an $m \leq n$ such that $s < t \in B$ implies that $\{s,t\} \in$ Pot m, and thus $x_s \in V_{t,m}$. If $s_0 = \min B$ then $x_{s_0} \in V_{t,m}$ for all $t \in B \setminus \{s_0\}$.

Let $\mathcal{U} = \bigcup\{\mathcal{U}_t : t \in B \setminus \{s_0\}\}$. If $x_{s_0} \in V' \in r(\mathcal{U})$, then there is some $t \in B \setminus \{s_0\}$ such that $V' \subset U_t$ or $V' \subset (X \setminus \overline{U'_t})$. Since $x_{s_0} \notin U_t$, the latter must hold and so $x_t \notin V'$. Thus, for every $V' \in \{V \in r(\mathcal{U}) : x_{s_0} \in V\}$ there exists a $k(V') \in B \setminus \{s_0\}$ such that $x_{k(V')} \notin V'$.

Let $d \in B$ such that $d > k(V')$, for every $V' \in \{V \in r(\mathcal{U}) : x_{s_0} \in V\}$. We have $x_{s_0} \in V_{d,m} \in r(\mathcal{U}_d) \prec r(\mathcal{U})$, so there exists a $V' \in r(\mathcal{U})$ such that $V_{d,m} \subset V'$. But $k(V') < d$ implies $x_{k(V')} \in V_{d,m} \subset V'$, which is a contradiction. $\square$

Lemma 2.0.9. *Let X be compact T_2 and monotonically countably metacompact. If X has caliber ω_1, then X is metrizable.*

Proof. Assume that X is compact, T_2, monotonically countably metacompact with operator r, and has caliber ω_1.

By Lemma 2.0.7, X is hereditarily Lindelöf, and hence perfectly normal. Suppose X is not metrizable. Choose $x_0, y_0 \in X$ such that $x_0 \neq y_0$, and let U_0 be an open neighborhood of x_0 with $y_0 \notin \overline{U}_0$. Suppose $\alpha < \omega_1$ and $x_\beta,\ y_\beta,$ and U_β have been chosen for each $\beta < \alpha$. There cannot exist a countable collection of open sets in X that separates points in the T_0 sense, for otherwise, by perfect normality, we would also then have a collection that separates points in the T_1 sense, making X metrizable (see, e.g., Theorem 7.6 of [16]). Therefore there are points x_α and y_α, $x_\alpha \neq y_\alpha$, such that if $\beta < \alpha$, then $U_\beta \cap \{x_\alpha, y_\alpha\} = \emptyset$, or $\{x_\alpha, y_\alpha\} \subset U_\beta$. Now let U_α be an open neighborhood of x_α with $y_\alpha \notin \overline{U}_\alpha$. Thus we have defined $x_\alpha,\ y_\alpha$, and U_α for each $\alpha \in \omega_1$.

For each $\alpha \in \omega_1$, let U'_α be open such that $x_\alpha \in U'_\alpha \subset \overline{U'_\alpha} \subset U_\alpha$. Let $p_\alpha = (x_\alpha, U'_\alpha, U_\alpha)$, and let V^{p_α} be as in Lemma 2.0.3.

Since X has caliber ω_1, there is an uncountable $A \subset \omega_1$ such that $\bigcap\{V^{p_\alpha} \cap U'_\alpha : \alpha \in A\} \neq \emptyset$. For $\beta < \alpha \in A$, we will put $\{\alpha, \beta\}$ in Pot I if $V^{p_\alpha} \not\subset U_\beta$; otherwise put $\{\alpha, \beta\}$ in Pot II.

We claim that there can be no infinite subset A' of A which is homogeneous for Pot I. If there were, we may assume A' has order type ω. Then from Lemma 2.0.3, there must be a finite $A'' \subset A'$ such that for any $\alpha \in A'$ there is some $\beta \in A''$ such that $V^{p_\alpha} \subset U_\beta$ (since $\bigcap\{V^{p_\alpha} \cap U'_\alpha : \alpha \in A\} \neq \emptyset$, the alternative $V^{p_\alpha} \cap U'_\beta = \emptyset$ never holds). But then choosing $\alpha \in A'$ with $\alpha > \beta$ for every $\beta \in A''$ yields a contradiction.

Hence by Erdos' theorem $\omega_1 \to (\omega, \omega_1)^2$, there is an uncountable $A'' \subset A'$ that is homogenous for Pot II. In other words, $\beta < \alpha \in A''$ implies that $V^{p_\alpha} \subset U_\beta$.

Applying hereditarily Lindelöf, we must have a $\gamma < \omega_1$ so that for every $\mu,\ \nu \in \omega_1 \smallsetminus \gamma$,

$$\overline{\bigcup_{\alpha \in A'' \smallsetminus \mu} \{x_\alpha, y_\alpha\}} = \overline{\bigcup_{\alpha \in A'' \smallsetminus \nu} \{x_\alpha,\ y_\alpha\}}$$

If β is the least element of $A'' \smallsetminus \gamma$, then $y_\beta \in \overline{\bigcup_{\alpha \in A'' \smallsetminus \beta} \{x_\alpha, y_\alpha\}} = \overline{\bigcup_{\alpha \in A'' \smallsetminus (\beta+1)} \{x_\alpha,\ y_\alpha\}}$. For each $\alpha \in A'' \smallsetminus (\beta + 1)$, we have that $U_\beta \supset V^{p_\alpha} \ni x_\alpha$ and thus $\{x_\alpha, y_\alpha\} \subset U_\beta$. Thus $U_\beta \supset \bigcup_{\alpha \in A'' \smallsetminus (\beta+1)} \{x_\alpha, y_\alpha\}$ and so $\overline{U}_\beta \supset \overline{\bigcup_{\alpha \in A'' \smallsetminus (\beta+1)} \{x_\alpha, y_\alpha\}}$, and consequently $y_\beta \in \overline{U}_\beta$, a contradiction. □

Theorem 2.0.10. *Let X be compact T_2 and monotonically countably metacompact. Then X is metrizable.*

Proof. This theorem is immediate from the above two lemmas. □

2.1 Just from the Lemmas

Although both Lemma 2.0.3 and Lemma 2.0.5 are quite powerful and quite useful when it comes to proving things involving monotonically metacompact property (Lemmas 2.0.4 and 2.0.6 for monotonically meta-Lindelöf spaces), it is known that the "weaker" lemmas, 2.0.5 and 2.0.6, are not equivalent to the original properties of monotonically metacompact and monotonically meta-Lindelöf, respectively.

If we let X be the one-point compactification of an uncountable discrete space, and we define $V_y = \{y\}$ if y is isolated, and $V_y = U_y$ if y is the single non-isolated point, we can easily see that this space satisfies the conclusion of Lemma 2.0.5, but it is not monotonically countably metacompact, since it is compact but not metrizable.

Similarly, in Example 3.8.1 we show that the one-point Lindelöfication of a discrete space of size greater than or equal to ω_2 is not monotonically meta-Lindelöf, but this space, with a similar definition for the V_y's as that described above, will satisfy the conclusion of Lemma 2.0.6.

Unfortunately, it is currently unknown whether there are any spaces that are not monotonically metacompact or monotonically meta-Lindelöf, but will satisfy the conclusion of Lemma 2.0.3 or Lemma 2.0.4, respectively.

Considering the above, it is quite natural to see which of the above results could be obtained just from the lemmas themselves. It actually turns out that most of the results can be obtained by just applying lemmas. What follows looks at just that.

In G. Gruenhage's "Monotonically Compact and Monotonically Lindelöf spaces", one finds the "original" lemma relating a monotone covering propery to the neighborhood-pair assignment. In proving that monotonically compact Hausdorff spaces are metrizable [14], the author applied both the definition of monotonically compact directly, as well as the characterization given by the lemma. We looked at the proof, and found that only the lemma was necessary to achieve the result. While this was later found to be a known result, our result was found independently.

In a similar manner, we then examined the above proof of Theorem 2.0.10, and found that only the lemma was necessary for that result.

Theorem 2.1.1. *Let X be compact, Hausdorff, and satisfy the conclusion of Lemma 2.0.3. Then X is metrizable.*

Proof. Assume X is not metrizable. Then from Lemma 2.0.9 (which only employed the definition of monotonically metacompact in order to apply Lemma 2.0.3) X can not have caliber ω_1. So there exists an uncountable, point-countable collection $\mathcal{U} = \{U_\alpha\}_{\alpha \in \omega_1}$, of open sets in X. Without loss of generality, we can assume that there exists $p_\alpha \in U_\alpha$ such that for all $\beta > \alpha$, $p_\alpha \notin U_\beta$.

Since X is regular, we can find open U_α^0 and U_α^1 such that $p_\alpha \in U_\alpha^0 \subset \overline{U_\alpha^0} \subset U_\alpha^1 \subset \overline{U_\alpha^1} \subset U_\alpha$. Notice that the collections $\{U_\alpha^0\}_{\alpha\in\omega_1}$ and $\{U_\alpha^1\}_{\alpha\in\omega_1}$ also witness X not having caliber ω_1. Now the triple $(x, X \setminus \overline{U_\alpha^1}, X \setminus \overline{U_\alpha^0}) \in \mathcal{P}_X$ for all $x \notin \overline{U_\alpha^1}$. Thus, by Lemma 2.0.3 we get a $V(x,\alpha) \subset X \setminus \overline{U_\alpha^0}$, satisfying the conclusion of the Lemma.

If we consider the collection $\{V(x,\alpha) : x \in (X \setminus U_\alpha^1)\}$, then since X is compact, there must be a finite subcollection covering $X \setminus U_\alpha$. Call this finite subcolletion $\mathcal{V}_\alpha = \{V(x_{\alpha,i},\alpha) | i = 1, ..., k_\alpha\}$. Now there exists $A \subset \omega_1$, $|A| = \omega_1$, and $k \in \omega$, such that $k_\alpha = |\mathcal{V}_\alpha| = k$ for all $\alpha \in A$. From here, we will simply induct on the value of k, to get a contradiction and finish the proof. However, we will first prove the following claim that will allow us to more efficiently complete the induction:

Claim 1. *Let F be a proper subset of $\{1, 2, \ldots, k\}$. Suppose F has the following property:*

$(*)$*: There is an uncountable subset A' of A such that for any $\beta < \alpha \in A'$, we have* $p_\beta \notin \bigcup_{i\in F} V(x_{\alpha,i},\alpha)$.

Then there is some F' with $F \subsetneq F' \subseteq \{1, 2, \ldots, k\}$ such that F' satisfies $(*)$.

Proof of Claim 1. Assume $F \subset \{1, 2, \ldots, k\}$ and that F satisfies $(*)$ above.

If $\beta_0 = \min A'$, then $p_{\beta_0} \notin U_\alpha$ for all $\alpha \in (A' \setminus \{\beta_0\})$. So $p_{\beta_0} \in (X \setminus U_\alpha)$ for all $\alpha \in (A' \setminus \{\beta_0\})$.

From $(*)$ we have that $p_{\beta_0} \notin V(x_{\alpha,i},\alpha)$ for any $i \in F$. So for all $\alpha \in (A' \setminus \{\beta_0\})$ there exists $j_\alpha \in (\{1, 2, ..., k\} \setminus F)$ such that $p_{\beta_0} \in V(x_{\alpha,j_\alpha},\alpha)$.

Now, since $|A' \setminus \{\beta_0\}| = \omega_1$ then uncountably often $j_\alpha = j$, so let $\hat{A} = \{\alpha \in (A' \setminus \{\beta_0\}) : p_{\beta_0} \in V(x_{\alpha,j},\alpha)\}$.

Notice then that $\bigcap_{\alpha\in\hat{A}} V(x_{\alpha,j},\alpha) \neq \emptyset$.

Now, for $\beta < \alpha \in \hat{A}$, put $\{\alpha,\beta\}$ in Pot I if $p_\beta \in V(x_{\alpha,j},\alpha)$ and in Pot II otherwise. By $\omega_1 \to (\omega,\omega_1)^2$ we have that either there exists an infinite $B \subset \hat{A}$ such that B is homogeneous for Pot I, or there exists an uncountable $B \subset \hat{A}$ such that B is homogenous for Pot II.

Assume that B is infinite homogeneous for Pot I. WLOG, B has order type ω. Then $\beta < \alpha \in B$ implies that $p_\beta \in V(x_{\alpha,j}, \alpha)$.

If we now apply Lemma 2.0.3 to $\mathcal{Q} = \{(x_{\alpha,j}, X \setminus \overline{U_\alpha^1}, X \setminus \overline{U_\alpha^0}) : \alpha \in B\}$, then since $\bigcap_{\alpha \in B} V(x_{\alpha,j}, \alpha) \neq \emptyset$, we get a finite $B' \subset B$ such that for each $\alpha \in B$ there exists $\alpha' \in B'$ with either $V(x_{\alpha,j}, \alpha) \subset (X \setminus \overline{U_{\alpha'}^0})$ or $V(x_{\alpha,j}, \alpha) \cap (X \setminus \overline{U_{\alpha'}^1}) = \emptyset$.

Choose $\alpha \in B$ with $\alpha > \max(B')$.

For no $\alpha' \in B'$ will $V(x_{\alpha,j}, \alpha) \cap (X \setminus \overline{U_{\alpha'}^1}) = \emptyset$. If there were such an α', then $V(x_{\alpha,j}, \alpha) \subset U_{\alpha'}$, which is a contradiction, since $p_{\beta_0} \in V(x_{\alpha,j}, \alpha)$ but $p_{\beta_0} \notin U_{\alpha'}$.

So we must have $V(x_{\alpha,j}, \alpha) \subset (X \setminus \overline{U_{\alpha'}^0})$. But this is also a contradiction, since $p'_\alpha \in V(x_{\alpha,j}, \alpha) \cap U_{\alpha'}^0$.

So there is no infinite subset of $\hat{A}$ that is homogeneous for Pot I. But then we must have that there exists an uncountable $B \subset \hat{A}$ such that B is homogenous for Pot II.

Thus, if we let $F' = F \cup \{j\}$, then B is an uncountable subset of A such that for all $\beta < \alpha \in B$ we have $p_\beta \notin \bigcup_{i \in F'} V(x_{\alpha,j}, \alpha)$. And thus F' satisfies $(*)$. This proves Claim 1.

We now complete the proof of the theorem.

It is easy to see that if $F_0 = \emptyset$, then F trivially satisfies $(*)$: since F_0 is empty, then $\bigcup_{i \in F_0} V(x_{\alpha,i}, \alpha) = \emptyset$, and hence, A itself is an uncountable set such that $p_\beta \notin \bigcup_{i \in F_0} V(x_{\alpha,i}, \alpha)$.

Now, from the claim there exists $F_1 \supsetneq F_0$ such that F_1 satisfies $(*)$. If $F_1 \neq \{1, 2, ..., k\}$, then apply the claim again to F_1 to get a superset F_2. We can continue in this way until we get $F_n = \{1, 2, \ldots, k\}$. But, if $F_n = \{1, 2, \ldots, k\}$ satisfies $(*)$, we get our desired contradiction: for then we have an uncountable $A' \subset A$ such that $\forall \beta < \alpha \in A'$, we have $p_\beta \notin \bigcup_{i \in F_n} V(x_{\alpha,i}, \alpha)$. But this then implies that $p_\beta \in U_\alpha$, since $\bigcup_{i=1}^{k} V(x_{\alpha,i}, \alpha)$ covers $X \setminus U_\alpha$. But this is a contradiction to our initial assumption on $p_\beta \notin U_\alpha^1$ for all $\alpha > \beta$.

Hence X must have caliber ω_1, and is therefore metrizable. $\square$

In [4], Bennett, Hart, and Lutzer showed that every monotonically (countably) metacompact, monotonically normal space must be paracompact, but left it open if such spaces must be hereditarily paracompact. We answer that question in the affirmative below. First, we will adapt a characterization given by Gruenhage, Michael, and Tanaka involving hereditarily meta-Lindelöf spaces.

If $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$ is a well-ordered cover of a space X, then define $\alpha(x)$ to be the first α such that $x \in U_\alpha$. Also denote $\widetilde{U}_\alpha = \{x \in X : \alpha(x) = \alpha\}$. Clearly $\widetilde{U}_\alpha = U_\alpha \setminus \bigcup_{\beta<\alpha} U_\beta$, and thus $\widetilde{U}_\alpha \subset U_\alpha$.

Lemma 2.1.2. *(Gruenhage, Michael, and Tanaka) [17] The following are equivalent:*

a. X is hereditarily meta-Lindelöf;

b. Every well-ordered open cover $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$ of X has a point-countable open refinement $\mathcal{V}$ such that for any $x \in X$ there exists $V \in \mathcal{V}$ such that $x \in V \subset U_{\alpha(x)}$;

c. Every well-ordered open cover $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$ of X has a point-countable open refinement $\mathcal{V} = \{V_\alpha\}_{\alpha\in A}$ such that $\widetilde{U}_\alpha \subset V_\alpha \subset U_\alpha$ for all α.

We show that a similar characterization exists for hereditarily metacompact spaces:

Lemma 2.1.3. *The following are equivalent:*

a. X is hereditarily metacompact;

b. Every well-ordered open cover $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$ of X has a point-finite open refinement $\mathcal{V}$ such that for any $x \in X$ there exists $V \in \mathcal{V}$ such that $x \in V \subset U_{\alpha(x)}$;

c. Every well-ordered open cover $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$ of X has a point-finite open refinement $\mathcal{V} = \{V_\alpha\}_{\alpha\in A}$ such that $\widetilde{U}_\alpha \subset V_\alpha \subset U_\alpha$ for all α.

Proof. (a) $\rightarrow$ (b): Assume X is hereditarily metacompact, and λ is an ordinal, indexing the well-ordered open cover $\mathcal{U}$ of X. We will induct on λ, the size of the indexed open cover.

For $\lambda = 1$, this is trivial: if $\mathcal{V} = \mathcal{U}$, then $\mathcal{V}$ satisfies (b).

If λ is a successor ordinal, ($\lambda = \alpha + 1$ for some α), then the well-ordered open cover $\{U_\beta\}_{\beta<\alpha}$ of $\bigcup_{\beta<\alpha} U_\beta$ must have a point-finite open refinement $\mathcal{V}'$ that satisfies (b) in the lemma. It is easy to see that $\mathcal{V} = \mathcal{V}' \cup \{U_\alpha\}$ is a refinement of $\mathcal{U}$ that also must satisfy the conditions of (b).

Now, if λ is a limit ordinal, let $\mathcal{W}$ be a point-finite open refinement of the open cover $\mathcal{U}$ of X. For each $W \in \mathcal{W}$ choose $\alpha(W) < \lambda$ such that $W \subset U_{\alpha(W)}$. For each $\alpha < \lambda$, the well-ordered open cover $\{U_\beta\}_{\beta<\alpha}$ of $\bigcup_{\beta<\alpha} U_\beta$ must have a point-finite open refinement $\mathcal{V}_\alpha$ that satisfies (b).

Define $\mathcal{V} = \{W \cap V : W \in \mathcal{W},\ V \in \mathcal{V}_{\alpha(W)+1}\}$. Since $\mathcal{W}$ and $\mathcal{V}_\alpha$ are point-finite open covers, then so is $\mathcal{V}$. If we suppose $x \in X$ and pick $W \in \mathcal{W}$ so that $x \in W$, then $x \in U_{\alpha(W)} \subset \bigcup\{U_\beta : \beta < (\alpha(W) + 1)\}$. Thus, there exists $V \in \mathcal{V}_{\alpha(W)+1}$ with $x \in V \subset U_{\alpha(x)}$. Therefore, $W \cap V \in \mathcal{V}$, and $x \in W \cap V \subset U_{\alpha(x)}$. So $\mathcal{V}$ is a refinement of $\mathcal{U}$ and $\mathcal{V}$ satisfies the conditions of (b).

(b) $\rightarrow$ (c): Let $\mathcal{U}$ be a well-ordered open cover of X, indexed by A. Choose an open cover $\mathcal{V}$ of X as in (b). For each $\alpha \in A$, let $V_\alpha = \bigcup\{V \in \mathcal{V} : V \subset U_\alpha,\ \textit{but } V \not\subset U_\beta \textit{ for any } \beta < \alpha\}$. Then it is easy to verify that the well ordered cover $\{V_\alpha : \alpha \in A\}$ satisfies (c).

(c) $\rightarrow$ (a): Suppose that $Y \subset X$ and that X satisfies (c). Let $\mathcal{W} = \{W_\alpha\}_{\alpha<\lambda}$ be a well-ordered open cover of Y, each W_α open with respect to the subspace topology. Now, for each $\alpha < \lambda$, there exists a U_α open in X, such that $U_\alpha \cap Y = W_\alpha$. If we let $U_\lambda = X$, then $\mathcal{U} = \{U_\alpha\}_{\alpha\leq\lambda}$ is a well-ordered open cover of X. Since X satisfies (c), we then can let $\mathcal{V} = \{V_\alpha\}_{\alpha\leq\lambda}$ be a point finite open refintement of $\mathcal{U}$ with $\widetilde{U}_\alpha \subset V_\alpha \subset U_\alpha$ for all $\alpha \leq \lambda$. Note that each $y \in Y$ is an element of $\widetilde{U}_\alpha$ for some $\alpha < \lambda$. Thus $\{(V_\alpha \cap Y : \alpha < \lambda\}$ must be a point-finite open refinement of $\mathcal{W}$ covering Y. Hence Y is metacompact, and so X is hereditarily metacompact. □

The following lemma shows that any space satisfying just the conclusion of Lemma 2.0.5 will satisfy condition (c.) of the above characterization, and thus we get that every monotonically metacompact space is hereditarily metacompact. So the "weak" lemma has considerable strength after all!

Lemma 2.1.4. *Let X satisfy the conclusion of Lemma 2.0.5. Then X is hereditarily metacompact.*

Proof. Let X satisfy Lemma 2.0.5, and let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be a well-ordered open cover of X indexed by A. For all $x \in X$ let $\alpha(x)$ be the least α such that $x \in U_\alpha$. So for each x, we have that $x \in U_{\alpha(x)} \setminus \bigcup_{\beta < \alpha(x)} U_\beta$. Define $\widetilde{U}_\alpha = \{x : \alpha(x) = \alpha\}$ and note that $\widetilde{U}_\alpha = U_\alpha \setminus \bigcup_{\beta < \alpha} U_\beta$.

By Lemma 2.0.5, for all $x \in \widetilde{U}_\alpha$, there exists V_x such that $x \in V_x \subset U_\alpha$, such that the V_x's satisfy the conclusion of 2.0.5. Let $V_\alpha = \bigcup_{x \in \widetilde{U}_\alpha} V_x$.

Claim: $\{V_\alpha\}_{\alpha \in A}$ is point-finite.

Assume not. Then there exists a $y \in X$ and an infinite $A' \subset A$, such that $y \in V_\alpha$ for all $\alpha \in A'$. Thus for all $\alpha \in A'$, $y \in \bigcup_{x \in \widetilde{U}_\alpha} V_x$. So, since y is in each of those unions, then for each $\alpha \in A'$ there exists $x_\alpha \in \widetilde{U}_\alpha$ with $y \in V_{x_\alpha}$. WLOG, A' has order type ω. Applying the lemma to the set $Y' = \{x_\alpha : \alpha \in A'\}$ will yield the contradiction. Since $y \in V_{x_\alpha}$ for all $\alpha \in A'$, then $\bigcap_{\alpha \in A'} V_{x_\alpha} \neq \emptyset$. So, from the lemma, there must exist a finite $A'' \subset A'$ such that $Y' \subset \bigcup_{\alpha \in A''} U_\alpha$. Now, if we let $\gamma \in A'$ such that $\gamma > \max(A'')$, then $x_\gamma \in \bigcup_{\alpha \in A''} U_\alpha$, but for $\alpha \in A''$, $U_\alpha \cap \widetilde{U}_\gamma = \emptyset$, and hence $x_\gamma \notin \bigcup_{\alpha \in A''} V_\alpha$, a contradiction.

Thus $\{V_\alpha\}_{\alpha \in A}$ is point-finite.

□

Lemma 2.1.3 and Lemma 2.1.4 together give us:

Corollary 2.1.5. *Every monotonically countably metacompact space is hereditarily metacompact.*

A similar argument to Lemma 2.1.4, but exchanging point-finite for point-countable, will show that any space satisfying the conclusion of Lemma 2.0.6 will satisfy condition (*c*.) of Theorem 2.1.2, and thus must be hereditarily meta-Lindelöf, yielding the following corollary:

Corollary 2.1.6. *Every monotonically meta-Lindelöf space is hereditarily meta-Lindelöf.*

It is known that every hereditarily metacompact, monotonically normal space is hereditarily paracompact. This comes from the combination of the known fact that no stationary subset of a regular uncountable cardinal is metacompact (or even meta-Lindelöf), combined with the result of Balogh and Rudin, that a monotonically normal space is hereditarily paracompact if and only if it does not contain a copy of such a stationary subset [2]. From this, we have that since a monotonically meta-Lindelöf space is hereditarily meta-Lindelöf, then it cannot contain a copy of a stationary subset of a regular uncountable cardinal. Thus we get the following corollary, which answers a question of Bennett, Hart, and Lutzer [4]:

Corollary 2.1.7. *Any monotonically normal, monotonically meta-Lindelöf space is hereditarily paracompact.*

The above result was shown by Li and Peng in [27], our result was found independently, and utilizes the hereditarily meta-Lindelöf result of Corollary 2.1.7.

The property of a monotonic covering property implying a space has the property hereditarily is not universal, however. Popvassilev and Porter have shown that while a space that is monotonically paracompact under the definition of Gartside and Moody must be hereditarily paracompact, they have also gone on to show that montonically paracompact in the locally finite sense need not imply hereditarily paracompact [29].

Chapter 3

Examples

In this section we look at some monotonic covering properties of some specific spaces. Some of our results about these spaces answer open questions, while others are examples for pedagogical purposes. Still others have been examined due to their prominence in the literature with regards to other monotonic properties. First we will start with a few spaces that were examined initially in order to gain an understanding of how the properties of monotonically metacompact and monotonically meta-Lindelöf really "worked".

3.1 Alexandroff Duplicate of the Interval

Example 3.1.1 is actually a corollary of Theorem 2.0.10. Since the space is compact and Hausdorff, but it is not metrizable, it therefore cannot be monotonically metacompact. The direct proof, however, shows that proving a space is not monotonicaly metacompact directly from the definition can be quite cumbersome.

The Alexandroff Duplicate consists of two copies of the unit interval, $[0,1]$, called X^0 and X^1. All points in X^1 are isolated, and for points in X^0, a basic open neighborhood of x^0 is of the form $(a^0,b^0) \cup ((a^1,b^1) \setminus \{x^1\})$, where $x^0 \in (a^0,b^0)$.

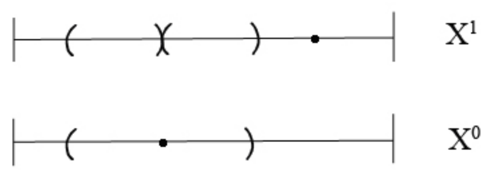

Figure 3.1: The Alexandroff Duplicate of $[0,1]$

Example 3.1.1. *The Alexandoff Duplicate of* $[0,1]$ *is not Monotonically Metacompact.*

Proof. Let X be the Alexandroff Duplicate. For each $x \in [0,1]$ let $\mathcal{U}_x = \{\{x^1\}, X \setminus \{x^1\}\}$. Then for each $x \in [0,1]$, $\mathcal{U}_x$ is an open cover of X. Assume that X is monotonically metacompact, and let r be the monotoniclly metacompact operator. Now, since $r(\mathcal{U}_x)$ is point finite, then for any $x \in [0,1]$ there are only finitely many $V \in r(\mathcal{U}_x)$ such that $x^0 \in V$. Therefore, there are rationals a_x and b_x such that $x^0 \in (a^0_x, b^0_x) \cup ((a^1_x, b^1_x) \setminus \{x^1\}) \subseteq \bigcap\{V \in r(\mathcal{U}_x) : x^0 \in V\}$. Let $O_x = (a^0_x, b^0_x) \cup ((a^1_x, b^1_x) \setminus \{x^1\}$.

Now, the preceding can be done for all $x \in [0,1]$ and the corresponding $\mathcal{U}_x$, and since there are only countably many rationals in $[0,1]$, there must be an uncountable $A \subset [0,1]$ such that $(a^0_x, b^0_x) = (a^0, b^0)$ for all $x \in A$. In other words, $O_x = (a^0, b^0) \cup ((a^1, b^1) \setminus \{x^1\})$ for each $x \in A$.

Let $\mathcal{U} = \bigcup\{\mathcal{U}_x : x \in A\}$. For every $U \in r(\mathcal{U})$, since $\{U\} \prec \mathcal{U}$, there exists $x(U) \in A$ such that $\{U\} \prec \mathcal{U}_{x(U)}$. Let $q \in (a^0, b^0)$ and pick $x \in A$ such that $x \notin \{x(U) : q \in U \in r(\mathcal{U})\}$. Now $r(\mathcal{U}_x) \prec r(\mathcal{U})$, so $\{O_x\} \prec r(\mathcal{U}_x) \prec r(\mathcal{U})$. Therefore there exists $U \in r(\mathcal{U})$ such that $O_x \subseteq U$. But $x(U)^1 \notin U$, while $x(U)^1 \in O_x$, a contradiction.

Thus X is not monotonically metacompact. □

3.2 The Double Arrow Space

This next example is a space that is somewhat similar to the Alexandroff Duplicate. Like the previous example, the Alexandroff Double Arrow space is also a compact, Hausdorff space. It is also known to be hereditarily Lindelöf, hereditarily separable, but not metrizable, so just like the previous example, as a corollary to Theorem 2.0.10, it is not monotonically metacompact. The proof here helps illustrate, however, how Lemma 2.0.3 can be used to show a space is not monotonically metacompact.

The Double Arrow space is the space $X = [0,1] \times \{0,1\}$. We will call the "top" unit interval X^1 and the "bottom" unit interval X^0. The topology on X is given by: for a point $x \in X^1$, a basic open set is of the form $((x,a) \times \{0\}) \cup ([x,a) \times \{1\})$, where $a \in [0,1]$,

$x < a < 1$. If $x \in X^0$, then a basic open set is of the form $((b, x] \times \{0\}) \cup ((b, x) \times \{1\})$ where $0 < b < x$. Another way to describe this space is that it is the top and bottom edges of the unit square with the lexicographic order.

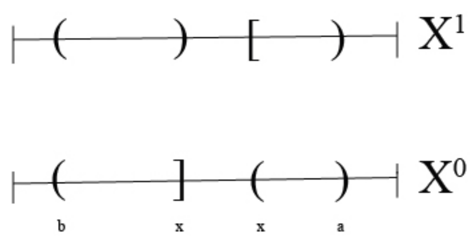

Figure 3.2: The Double Arrow space

Example 3.2.1. *The Double Arrow Space is not monotonically metacompact.*

Proof. Let X be the Double Arrow space. Let $a \in (0, 1)$. For each $x < a$, $x \in (0, 1)$, let $U_x = ((x, a) \times \{0\}) \cup ([x, a) \times \{1\})$. Then U_x is a basic open set containing the point $(x, 1)$. Since X is regular, we can find another basic open set U_x^0 such that $x \in U_x^0 \subset \overline{U}_x^0 \subset U_x$. So $p_x = (x, U_x^0, U_x) \in \mathcal{P}_X$. Suppose X is monotonically metacompact. Then from Lemma 2.0.3, for each p_x we have V_{p_x} satisfying the lemma. Now, without loss of generality, we can assume that V_{p_x} is a basic open set of the form $((x, b_x) \times \{0\}) \cup ([x, b_x) \times \{1\})$, where $x < b_x < a$, and that the right endpoint, b_x, is rational. Now, since there are uncountably many V_{p_x}'s, there are uncountably many that have the same rational right endpoint $b_x = b$. Similarly since there are uncountably many V_{p_x} with right endpoint b, but only countably many rational numbers, there must exist a rational q such that for uncountably many of those p_x, $x < q < b$. Let A be the set of all x such that $V_{p_x} = ((x, b) \times \{0\}) \cup ([x, b) \times \{1\})$, and $x < q < b$. We will pass to an infinite subset A' of A, such that $\inf(A') \notin A'$. Now, the point $(q, 1) \in V_{p_x}$ for all $p_x \in A'$. Hence $\bigcap_{x \in A'} V_{p_x} \neq \emptyset$. From Lemma 2.0.3, this means there must exist a finite $A'' \subset A'$, so that for any $x \in A'$, there exists $x' \in A''$, such that either $V_{p_x} \subset U_{p_{x'}}$, or $V_{p_x} \cap U_{x'}^0 = \emptyset$. The latter is impossible, so the former must be the case. Yet, since $\inf(A') \notin A'$, then choosing $x \in A'$ such that $x < \min(A'')$ yields a contradiction, for $V_{p_x} \not\subset U_y$ whenever $x < y$. Hence X is not monotonically metacompact.

□

3.3 Ceder and McAuley Spaces

One of the more examined problems involving monotonic metacompactness has been the attempt to discover which properties, when combined with metacompactness, will give monotonically metacompact. It was shown in [4] that every metacompact Moore space is monotonically metacompact. But which properties, weaker than developable, will give the same result? In [4] it was asked which stratifiable spaces were monotonically metacompact, and in particular, if either the Ceder space or McAuley space held this property. These two spaces are quite similar, and a description of these spaces is taken from [1].

The McAuley space is defined as $X = X_0 \cup X_1$ where $X_0 = \{(x, 0) \ : \ x \in \mathbb{R}\}$, and $X_1 = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$. A basic open neighborhood for a point $x = (x_0, 0) \in X_0$ is of the form $M\left(x, \dfrac{1}{n}\right) = \{x\} \cup \left\{(x', y') \in X \mid y' < \dfrac{1}{n}|x' - x| < \dfrac{1}{n^2}\right\}$, for $n \in \mathbb{N}$. The points $x = (x', y') \in X_1$ have the usual Euclidean open balls as basic neighborhoods.

The Ceder space uses the same underlying set for X, and as in the McAuley space, basic open neighborhoods of points above the x-axis are Euclidean open balls. A basic open neighborhood for a point $x = (x_0, 0) \in X_0$ is anything of the form $C\left(x, \dfrac{1}{n}\right) = \{x\} \cup \left\{(x', y') \in X \ : \ y' < n - \sqrt{n^2 - (x' - x)^2}, |x' - x| < \frac{1}{n}\right\}$, for $n \in \mathbb{N}$.

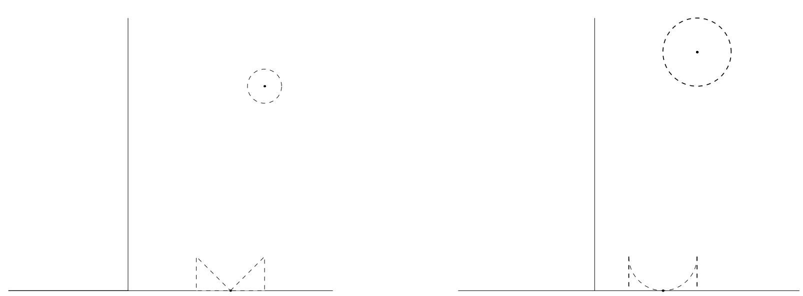

Figure 3.3: The McAuley bowtie space and Ceder space

Li and Peng [26] examined this space, and were able to show that is was paracompact, and hence metacompact, utilizing a direct argument, but were not able to determine if it was monotonically metacompact. We show that the McAuley space is not monotonically meta-Lindelöf, hence not monotonically metacompact either. A similar argument will show that the Ceder space is neither monotonically metacompact nor monotonically meta-Lindelöf as well, thus showing that it is not enough for a regular, metacompat space to be stratifiable in order for it to be monotonically metacompact.

Example 3.3.1. *The McAuley Space is not monotonically metacompact.*

Proof. Let X be the McAuley space. For each $x \in X$, choose a basic open neighborhood U_x^1 with $x \in U_x^1$. Since X is regular, we can find some other basic open neighborhood U_x^0 of x, such that $x \in U_x^0 \subset \overline{U_x^0} \subset U_x^1$. Thus $p = (x, U_x^0, U_x^1) \in P_X$.

Assume that X is monotonically metacompact. Then by Lemma 2.0.4, for each $p = (x, U_x^0, U_x^1) \in P_X$, there is an open $V^x \subset X$ satisfying the conclusions of the lemma. Without loss of generality, we may assume each V^x is also a basic neighborhood. For each $x \in X_0$, we may also consider the set $V^x \cap X_0 = (a_x, b_x) \times \{0\}$ to be an interval in X_0 with rational endpoints, such that $a_x < x < b_x$.

Now, since X_0 is uncountable, then for some $a, b \in \mathbb{Q}$ and for some uncountable $A \subset X_0$, we must have $V \cap X_0 = (a, b) \times \{0\}$ for all $x \in A$. Then if $\mathcal{Q} = \{(x, U_x^0, U_x^1) : x \in A\}$ we have from Lemma 2.0.4 that since $\bigcap_{x \in A} V^x \neq \emptyset$, and $V^x \cap U_{x'}^0 \neq \emptyset$ for any $x, x' \in A$, then there must be a countable $A' \subset A$ such that for any $x \in A$ there exists $y \in A'$ such that $V^x \subset U_y^1$. Yet if $x \neq y \in A$, then $V^x \not\subset U_y^1$, since there will be some point $(y, z) \in X_1$ such that $(y, z) \in V^x$, but for no $z > 0$ is the point $(y, z) \in U_y^1$, a contradiction. Thus X is not monotonically meta-Lindelöf.

□

The Ceder space, as described above, is quite similar to the McAuley space, and the proof that the Ceder space is not monotonically metacompact is almost identical to the above proof, it is just the shape of the basic open sets that have changed.

3.4 The Sequential Fan Space

This next example was motivated by a result of Levy and Matveev [25] in which they show that under the Continuum Hypothesis the sequential fan space is monotonically Lindelöf. It is currently unknown whether the space is monotonically Lindelöf in ZFC. We show here, however, that the space is not monotonically (countably) metacompact. In particular, this means that even though regular Lindelöf spaces are (countably) metacompact, monotonically Lindelöf does not imply monotonically (countably) metacompact.

The sequential fan space can be defined as follows: $X = (\omega \times \omega) \cup \{\infty\}$, where all points in $\omega \times \omega$ are isolated and an open neighborhood of ∞ is of the form $B(\infty, f) = \{(m, n) \in (\omega \times \omega) \mid f(m) \leq n\}$ where $f \in \omega^\omega$.

To show this space is not monotonically metacompact, we will first show two lemmas useful in dealing with the functions in this space.

Given two functions $f, g \in \omega^\omega$, we will say $f \leq^* g$ if for all but finitely many $n \in \omega$, we have $f(n) \leq g(n)$.

We will also say that a collection $\mathcal{F} \subset \omega^\omega$ of functions is $\leq^*$-*unbounded* if for any $g \in \omega^\omega$ there exists $f \in \mathcal{F}$ such that $g \leq^* f$.

Lemma 3.4.1. *If $\mathcal{F} \subset \omega^\omega$ is $\leq^*$-unbounded and $\mathcal{F} = \bigcup_{n\in\omega} \mathcal{F}_n$, then there exists a $n_0 \in \omega$ such that $\mathcal{F}_{n_0}$ is $\leq^*$-unbounded.*

Proof. Assume not. Then for each $n \in \omega$, there exists $g_n \in \omega^\omega$ such that for every $f \in \mathcal{F}_n$ we have $f(i) < g_n(i)$ for all sufficiently large $i \in \omega$.

Define a function $g^* \in \omega^\omega$ by $g^*(k) = \sum_{i=0}^{k} g_i(k)$.

Then $g_n \leq^* g^*$ for all $n \in \omega$, and it follows that $g \leq^* g^*$ for all $g \in \mathcal{F}$, which is a contradiction. $\square$

Armed with the above lemma, we are now able to show that the sequential fan space is not monotonically metacompact.

Example 3.4.2. *The sequential fan space is not monotonically countably metacompact.*

Proof. Assume that X is monotonically countably metacompact, and note that if $B(\infty, g) \subset B(\infty, f) \Leftrightarrow f(n) \leq g(n)$ for all $n \in \omega$.

Notice that for each $f \in \omega^\omega$, $p_f = (\infty, B(\infty, f), B(\infty, f)) \in P_X$. Since X is monotonically countably metacompact, then by Lemma 2.0.3, there is a $g_f \in \omega^\omega$ such that $\infty \in B(\infty, g_f) \subset B(\infty, f)$, and for each $\mathcal{F} \subset \omega^\omega$ there exists a finite $\mathcal{F}' \subset \mathcal{F}$ such that for any $f \in \mathcal{F}$ there is a $f' \in \mathcal{F}'$ with $f'(n) \leq g_f(n)$ for all $n \in \omega$.

For each $f \in \omega^\omega$, put $f \in \mathcal{F}_n$ if $g_f(0) = n$. From Lemma 3.4.1, there exists an $n_0 \in \omega$ such that $\mathcal{F}_{n_0}$ is $\leq^*$-unbounded in ω^ω. Now, for each $f \in \mathcal{F}_{n_0}$ put $f \in \mathcal{F}_{n_0,n}$ if $g_f(1) = n$. Again, from Lemma 3.4.1, there exists a $n_1 \in \omega$ such that $\mathcal{F}_{n_0,n_1}$ is $\leq^*$-unbounded. Continue on in this way: if $\mathcal{F}_\sigma$ has been defined, where $\sigma \in \omega^{<\omega}$, let $k = |\sigma|$ and partition $\mathcal{F}_\sigma$ by placing $f \in \mathcal{F}_\sigma$ into $\mathcal{F}_{\sigma^\frown n}$ if $f(k) = n$. Then there exists a $n_k \in \omega$ such that $\mathcal{F}_{\sigma^\frown n_k}$ is $\leq^*$-unbounded. We get a $g^* \in \omega^\omega$ such that for each $f \in \mathcal{F}_{g^*\restriction n}$, $g_{f\restriction n} = g^*_{\restriction n}$, and $\mathcal{F}_{g^*\restriction n}$ is $\leq^*$-unbounded in ω^ω for each $n \in \omega$.

There exists a $f_0 \in \omega^\omega$ such that $f_0 \not\leq^* g^*$, or in other words, $f_0(i) > g^*(i)$ for infinitely many $i \in \omega$. Let $i_0 \in \omega$ be the first such that $f_0(i_0) > g^*(i_0)$.

Now $\mathcal{F}_{g^*\restriction(i_0+1)}$ is $\leq^*$-unbounded in ω^ω, so there exists a $f_1 \in \mathcal{F}_{g^*\restriction(i_0+1)}$ such that $f_1(i) > g^*(i)$ for infinitely many $i \in \omega$. So $g_{f_1}(i_0) = g^*(i_0) < f_0(i_0)$, where $B(\infty, g_{f_1}) \not\subset B(\infty, f_0)$. Again let $i_1 \in \omega$ be greater than i_0 such that $f_1(i_1) > g^*(i_1)$. We will continue on in this way.

In general, assume that $f_j \in \mathcal{F}_{g^*\restriction_{(i_{j-1}+1)}}$ has been chosen for all $j \leq k \in \omega$, such that $g^*(i) < f_j(i)$ for infinitely many $i \in \omega$.

Let i_k be the first greater than i_{k-1} such that $f_k(i_k) > g^*(i_k)$. But then since $\mathcal{F}_{g^*\restriction_{(i_k+1)}}$ is $\leq^*$-unbounded in ω^ω, we have that there exists a $f_{k+1} \in \mathcal{F}_{g^*\restriction_{(i_k+1)}}$ such that $f_{k+1}(i) > g^*(i)$ for infinitely many $i \in \omega$, while $f_{k+1}(n) < g^*(n)$ for all $n \leq k$.

So, proceeding thus for all $k \in \omega$, we get a collection $\mathcal{Q} = \{f_0, f_1, f_2, ..., f_k, ...\}$ of elements of ω^ω such that $g_{f_k} \not\geq f_i$ if $i < k$.

By Lemma 2.0.3, there exists a $Q' \subset \omega$, $|Q'| < \omega$, such that for any $k \in Q$, there exists a $j \in Q'$ such that $B(\infty, gf_k) \subset B(\infty, f_j)$, whence $g_{f_k} \geq f_j$. So let $m = \max\{k \in \omega : f_k \in Q'\}$. But $f_{m+1} \in Q$ is guaranteed by construction to have $g_{f_{(m+1)}} \not\geq f_k$ for all $k \in Q'$, which is a contradiction. Therefore the sequential fan is not monotonically metacompact. $\square$

3.5 Single Ultrafilter Space

Another space that was examined by Levy and Matveev [25] with regards to the monotone Lindelöf property was the single ultrafilter space. Similar to the sequential fan space, under the Continuum Hypothesis it was found that there exists a single ultrafilter space that is monotonically Lindelöf. It is not currently known if every single ulrafilter space is monotonically Lindelöf, however, nor if the single ultrafilter space is monotonically Lindelöf in ZFC. We examined this space, and found that no single ultrafilter space is monotonically (countably) metacompact.

Recall, that a collection of subsets $\mathcal{F}$ of a topological space X is a *filter* if for any two elements $F_a, F_b \in \mathcal{F}$, then $F_a \cap F_b \in \mathcal{F}$, and if for any $F \in \mathcal{F}$, and any $G \subset X$ with $F \subset G$ then we must have $G \in \mathcal{F}$. A filter on a space X is an *ultrafilter* if there is no strictly finer filter containing it. An equivalent condition for an ultrafilter is it is a filter such that for any subset B of the space, either B or its complement $X \setminus B$ is an element of the filter. An ultrafilter is call a *free ultrafilter* if $\bigcap \mathcal{F} = \emptyset$.

The single ultrafilter space, or the single ultrafilter space on ω, is the space: $X = \omega \cup \{\mathcal{F}\}$ where $\mathcal{F}$ is a free ultrafilter on ω. In X, the points of ω are isolated, and a basic neighborhood of $\mathcal{F}$ is any member F of the ultrafilter, together with the point $\mathcal{F}$.

We will need to utilize two useful lemmas in order to show that this space is not monotonically (countably) metacompact.

For two infinite sets A and B, we will say that $A \subset^* B$ if $A \setminus B$ is finite.

Lemma 3.5.1. *If $\mathcal{F}$ is a free ultrafilter on ω, and if $\mathcal{F} = \bigcup_{n\in\omega} \mathcal{F}_n$, then $\exists n \in \omega$ such that $\nexists A \subset \omega$ with A infinite and $A \subset^* F$ for all $F \in \mathcal{F}_n$.*

Proof. Assume not. Then for all $n \in \omega$ there exists $A_n \subset \omega$ such that $A_n \subset^* F$ for all $F \in \mathcal{F}_n$. By shrinking the A_n's if necessary, we may assume that they are pairwise disjoint. Now, for each $F \in \mathcal{F}_n$, $F \cap A_n \neq \emptyset$. Thus, if $A = \bigcup_{n\in\omega} A_n$, we have that for all $F \in \mathcal{F}$, $F \cap A \neq \emptyset$. So, $A \in \mathcal{F}$.

For each $n \in \omega$, divide A_n into A_n^0 and A_n^1, where $A_n^0 \cap A_n^1 = \emptyset$ and $A_n^0 \cup A_n^1 \subset A_n$. Then $A_n^0 \cap F \neq \emptyset$, and $A_n^1 \cap F \neq \emptyset$ for all $F \in \mathcal{F}_n$.

Now, if we consider $\bigcup_{n\in\omega} A_n^0$ and $\bigcup_{n\in\omega} A_n^1$, notice that for all $F \in \mathcal{F}$, $F \cap \bigcup_{n\in\omega} A_n^0 \neq \emptyset$, and $F \cap \bigcup_{n\in\omega} A_n^1 \neq \emptyset$. So both $\bigcup_{n\in\omega} A_n^0$ and $\bigcup_{n\in\omega} A_n^1$ are in $\mathcal{F}$, yet $\bigcup_{n\in\omega} A_n^0 \cap \bigcup_{n\in\omega} A_n^1 = \emptyset$, which is impossible. Hence the lemma holds. $\square$

A set P is called the *pseudointersection* of a filter $\mathcal{F}$ if $P \setminus F$ is finite for every $F \in \mathcal{F}$. Notice that a collection of sets $\mathcal{F}$ that satisfies $(*)$ in the following lemma implies that $\mathcal{F}$ has no infinite pseudointersection.

Lemma 3.5.2. *Suppose that $\mathcal{F}$ is a collection of sets satisfying:*

$(*)$ *Whenever $\mathcal{F} = \bigcup_{n\in\omega} \mathcal{F}_n$, then there exists $n \in \omega$ such that there does not exist $A \subset \omega$, $|A| = \omega$, with $A \subset^* F$ for all $F \in \mathcal{F}_n$.*

Then whenever $\mathcal{F} = \bigcup_{n\in\omega} \mathcal{F}_n$, $\exists n \in \omega$ *such that* $\mathcal{F}_n$ *satisfies* $(*)$*.*

Proof. Suppose that $\mathcal{F}$ satisfies $(*)$, and $\mathcal{F} = \bigcup_{n\in\omega} \mathcal{F}_n$, but that for all $n \in \omega$, $\mathcal{F}_n$ does not satisfy $(*)$. Then for each $n \in \omega$, $\mathcal{F}_n = \bigcup_{m\in\omega} \mathcal{F}_{n,m}$, and for each $m \in \omega$, there exists $A_m \subset \omega$ with $|A_m| = \omega$, such that $A_m \subset^* F$ for every $F \in \mathcal{F}_{n,m}$. So, $\mathcal{F} = \bigcup_{n\in\omega} \bigcup_{m\in\omega} \mathcal{F}_{n,m}$, and every $\mathcal{F}_{n,m}$ has infinite pseudointersection, which violates $\mathcal{F}$ satisfying $(*)$. $\square$

Example 3.5.3. *The single ultrafilter space is not monotonically metacompact.*

Proof. Assume that X is monotonically metacompact, with monotone metacompactness operator r.

Now for each $F \in \mathcal{F}$, $p_F = (\mathcal{F}, F \cup \{\mathcal{F}\}, F \cup \{\mathcal{F}\}) \in P_X$. Since X is monotonically metacompact, then by Lemma 2.0.3, then there is a $V^{p_F} \in \mathcal{F}$ such that $V^{p_F} \subset F$, and for each $Q \subset P_X$, there exists a finite $Q' \subset Q$ such that for any $p_F \in Q$ there is a $p_{F'} \in Q'$ such that $V^{p_F} \subset F'$.

For each $n \in \omega$, let $\mathcal{F}_n = \{F \in \mathcal{F} \mid n \in V^{p_F}\}$. Then $\mathcal{F} = \bigcup_{n\in\omega} \mathcal{F}_n$.
By Lemma 3.5.1, there exists $n_0 \in \omega$ such that $\mathcal{F}_{n_0}$ satisfies $(*)$ from Lemma 3.5.2. Now, for each $n \in (\omega \setminus n_0)$, let $\mathcal{F}_{n_0,n} = \{F \in \mathcal{F}_{n_0} \mid n \in V^{p_F}\}$. Applying Lemma 3.5.2, we then have that $\exists n_1 \in (\omega \setminus n_0)$ such that $\mathcal{F}_{n_0,n_1}$ satisfies $(*)$. In general, if we have constructed $\mathcal{F}_\sigma$, for $\sigma \in \omega^{<\omega}$, such that $\mathcal{F}_\sigma$ satisfies $(*)$ from Lemma 3.5.2, we can let $\mathcal{F}_{\sigma^\frown n} = \{F \in \mathcal{F}_\sigma \mid n \in V^F\}$, and then from Lemma 3.5.2, we get that there exists $n_i \in (\omega \setminus S)$, where S is the range of σ and $i = |\sigma|$, such that $\mathcal{F}_{\sigma^\frown n_i}$ satisfies $(*)$.

Thus we get a sequence $s = (n_0, n_1, \ldots)$, such that $\mathcal{F}_{s\restriction n}$ has no infinite pseudointersection for any $n \in \omega$, and $ran(s_{\restriction n}) \subset V^F$, for any $F \in \mathcal{F}_{s_{\restriction n}}$.

If we let $G = \{n_0, n_1, ...,\}$, then there exists $F_0 \in \mathcal{F}$ such that $G \not\subset^* F_0$. In other words, $G \setminus F_0$ is infinite. Now let $i_0 \in \omega$ such that $n_{i_0} \in (G \setminus F_0)$. Then there is $F_1 \in \mathcal{F}_{n_0,\ldots,n_{i_0}}$ such that $G \setminus (F_0 \cup F_1)$ is infinite. We will continue on in this way:

In general, assume that $F_0, F_1, ..., F_k$ have all been found such that $G \setminus (\bigcup_0^k F_j)$ is infinite. Let $n_{i_k} \in G \setminus (\bigcup_0^k F_j)$. Then there exists $F_{k+1} \in \mathcal{F}_{n_0, n_1, ..., n_{i_k}}$ such that $G \setminus (\bigcup_{j=0}^{k+1} F_j)$ is infinite. Proceeding this way for all $k \in \omega$, we get a collection $\mathcal{Q} = \{F_0, F_1, ..., F_k, ...\}$ of elements of $\mathcal{F}$ such that $V^{F_k} \not\subset F_i$ if $i < k$. (This is because $n_i \in V^{F_k} \setminus F_i$.)

If $Q = \{F_i : i \in \omega\}$, then by Lemma 2.0.3, there exists a finite $Q' \subset Q$, such that for any $F_k \in Q$, there exists $F_j \in Q'$ such that $V^{F_k} \subset F_j$. So, let $m = \max\{k \in \omega : F_k \in Q'\}$. We have that F_{m+1} is guaranteed by the construction to have $V^{p_{F(m+1)}} \not\subset F_k$ for all $p_{F_k} \in Q'$, which is a contradiction. Therefore the single ultrafilter space is not monotonically metacompact.

□

3.6 A Stratifiable, Monotonically Paracompact, Not Protometrizable Space

The next example was motivated by a question of Gartside and Moody [11], who after defining monotone paracompactness using star-finite refinements, asked if it would have been equivalent for their definition to use the standard locally finite refinements instead.

Definition 18. *A space is* ***monotonically paracompact (using star-refinements)*** *if there exists a function $m : \mathcal{C} \to \mathcal{C}$,(where $\mathcal{C}$ is the set of open covers of X) such that:*

(MP1) for every $\mathcal{U} \in \mathcal{C}$, $m(\mathcal{U})$ star-refines $\mathcal{U}$

(MP2) if $\mathcal{U}$ refines $\mathcal{V}$ then $m(\mathcal{U})$ refines $\mathcal{V}$.

Gartside and Moody [11] were able to show that for T_1 spaces their version of monotonically paracompact was equivalent to a space being protometrizable, but made no progress on the locally finite verison of monotonically paracompact.

Definition 19. *A space X is **monotonically paracompact (in the locally finite sense)** if there exists a function r assigning to each open cover $\mathcal{U}$ of X a locally finite open refinement $r(\mathcal{U})$ such that for any other open cover $\mathcal{V}$, whenever $\mathcal{V}$ refines $\mathcal{U}$ then we must have $r(\mathcal{V})$ refines $r(\mathcal{U})$.*

Stares [32] observed that utilizing different equivalent definitions of paracompactness can give rise to definitions of monotonically paracompact that are not equivalent and showed it to be the case for the following two characterizations of paracompactness:

Theorem 3.6.1. *(Michael) [28] A space X is paracompact if and only if for each open coeer $\mathcal{U}$ of X there is a semineighborhood D of the diagonal in X^2 such that $\{D[x] : x \in X\}$ refines $\mathcal{U}$.*

Theorem 3.6.2. *(Kelley) [22] A space X is paracompact if and only if for each open cover $\mathcal{U}$ of X there is a neighborhood D of the diagonal in X^2 such that $\{D[x] : x \in X\}$ refines $\mathcal{U}$.*

When "monotonized", these two equivalent and similarly worded characterizations of paracompactness yield different classes of spaces. If we let $\mathcal{C}$ denote the collection of open covers of a space X, $\mathcal{S}$ denote the collection of semi-neighborhoods of the diagonal in X^2, and $\mathcal{N}$ the colleciton of neighborhoods of the diagonal in X^2, then the monotonized versions of the above characterizations are defined as follows:

Definition 20 (Stares)**.** *A space X is **monotonically semineighborhood refining** (MSNR) if there is a map $r : \mathcal{C} \to \mathcal{S}$ such that:*

a. $\{r(\mathcal{U})[x] : x \in X\}$ refines $\mathcal{U}$ for each $\mathcal{U} \in \mathcal{C}$,

b. $r(\mathcal{U}) \prec r(\mathcal{V})$ whenever $\mathcal{U} \prec \mathcal{V}$.

Definition 21 (Stares)**.** *A space X is **monotonically neighborhood refining** (MNR) if there is a map $r : \mathcal{C} \to \mathcal{N}$ such that:*

a. $\{r(\mathcal{U})[x] : x \in X\}$ *refines* $\mathcal{U}$ *for each* $\mathcal{U} \in \mathcal{C}$,

b. $r(\mathcal{U}) \prec r(\mathcal{V})$ *whenever* $\mathcal{U} \prec \mathcal{V}$.

Stares showed that X being MNR is equivalent to the space being protometrizable, and hence MNR is equivalent to monotonically paracompact (under Gartside and Moody's definition). He also showed that if X is MNR it is obviously MSNR. However, he goes on to show that the McAuley space (see Example 3.3.1 above), is a space that is not protometrizable, (it is stratifiable but not metrizable), so cannot be MNR, yet he showed this space is MSNR. Considering this in the light of our above findings for the McAuley space, this gives a monotonized characterization of paracompactness that does not imply monotone meta-Lindelöfness.

Although Stares showed that different, equivalent definitions for paracompactnesss can give rise to different classes of spaces when monotonized, he was not able to determine whether the monotonized version of the standard definition of paracompactness, using locally finite refinements, and the defnition of monotonically paracompact given by Gartside and Moody give rise to different monotonic properties.

We show here an example of a space that is monotonically paracompact under the locally finite definition, and stratifiable, but not protometrizable and hence not monotonically paracompact by Gartside and Moody's definition.[1]

Example 3.6.3. *Let* $X = (\omega_1 \times \omega) \cup \{\infty\}$, *where points of* $\omega_1 \times \omega$ *are isolated, and a basic open neighborhood of* ∞ *is of the form* $B(\alpha, n) = \{(\beta, m) : \alpha \leq \beta, n \leq m\} \cup \{\infty\}$. *Then* X *is stratifiable and monotonically paracompact in the locally finite sense, but not protometrizable.*

X is stratifiable. Let U be an open subset in X. If $\infty \in U$, let $U_n = U$ for all $n \in \omega$. If $\infty \notin U$, then let $U_n = U \cap \{(b, m) : m \leq n\}$.

That condition (i) of Definition 7 holds is obvious, since all sets are clopen in X.

[1] Popvassilev and Porter independently obtained a similar example, although their example is not stratifiable.[29]

Condition (ii) is just as obvious: if $\infty \in U$, it is trivial, and if $\infty \notin U$, then clearly $\bigcup_{n\in\omega} U_n = U$.

For condition (iii), let U and V be open subsets of X, with $U \subset V$. If $\infty \in V$, then $U_n \subset V_n$ for all $n \in \omega$, since $U_n \subseteq U$ and $V_n = V$ for all $n \in \omega$. If $\infty \notin V$, then condition (iii) is just as easily realized, since $U_n = U \cap \{(b,m) : m \leq n\} \subseteq V \cap \{(b,m) : m \leq n\} = V_n$ for all $n \in \omega$.

Thus X is stratifiable. $\square$

X is monotonically paracompact. Let $\mathcal{U}$ be an open cover of X.
Define $\mathcal{U}_\infty = \{U \in \mathcal{U} : \infty \in U\}$, and $\mathcal{U}_\infty^B = \{(B(\alpha,n) : B(\alpha,n) \subset U,\ for\ some\ U \in \mathcal{U}_\infty\}$. For any basic open neighborhood $B(\alpha,n)$ of ∞, we will often wish to refer its minimum point, (α,n).

Now, we are only interested in the maximal elements of $\mathcal{U}_\infty^B$. (A basic open set V is maximal in $\mathcal{U}_\infty^B$ if $V \not\subseteq U$ for every $U \in \mathcal{U}_\infty^B$). Let $\widehat{\mathcal{U}}_\infty^B$ be the collection of all maximal elements of $\mathcal{U}_\infty^B$.

Notice that if $B(\beta_1,m_1), B(\beta_2,m_2) \in \widehat{\mathcal{U}}_\infty^B$ then $\beta_1 \neq \beta_2$ and $m_1 \neq m_2$. In fact, we must have either $\beta_1 < \beta_2$ and $m_1 > m_2$, or $\beta_1 > \beta_2$ and $m_1 < m_2$.

Claim: $\widehat{\mathcal{U}}_\infty^B$ is finite.

Suppose that $\{(\alpha_1,n_1),(\alpha_2,n_2),...,(\alpha_i,n_i),...\}$ is an infinite collection of distinct minimal points from elements of $\widehat{\mathcal{U}}_\infty^B$. Consider the collection of all unordered pairs from that infinite collection: $\{\{(\alpha_i,n_i),(\alpha_j,n_j)\} : i,j \in \omega\}$.

We will put $\{(\alpha_i,n_i),(\alpha_j,n_j)\}$ in Pot I if $i < j$ and $\alpha_i > \alpha_j$, and in Pot II otherwise. By Ramsey's Theorem there exists an infinite $A \subset \omega$, such that the pairs $\{\{(\alpha_k,n_k),(\alpha_t,n_t)\} : k,t \in A\}$ are all in the same pot.

If Pot I is homogenous for A, then we must have $\forall i < k \in A$, $\alpha_i > \alpha_k$, and we get an infinite decreasing sequence of elements in ω_1, which is impossible.

If Pot II is homogeneous for A, and if i is the least element of A, then $\alpha_i \leq \alpha_k$ for all $k \in A$, we must have $n_i \geq n_k$ for all $k \in A$, and thus we have infinitely many elements of ω less than some specific element $n_i \in \omega$, also a contradiction.

Thus $\widehat{\mathcal{U}}_\infty^B$ contains only finitely many elements. Define $r(\mathcal{U}) = \widehat{\mathcal{U}}_\infty^B \cup \{\{(\alpha, n)\} : (\alpha, n) \notin \bigcup \widehat{\mathcal{U}}_\infty^B\}$.

Claim: r is the desired monotone paracompactness operator.

i.) $r(\mathcal{U})$ refines $\mathcal{U}$:

This is obvious since each element of $r(\mathcal{U})$ is either a singleton, or is a basic open neighborhood of ∞, chosen because it was contained in some element of $\mathcal{U}$.

ii.) $r(\mathcal{U})$ is locally-finite.

If some point of $(\omega_1 \times \omega)$ were contained in infinitely many elements of $r(\mathcal{U})$, then the point ∞ would also be contained in infinitely many elements of $r(\mathcal{U})$ as well. But there are only finitely many elements in $\widehat{\mathcal{U}}_\infty^B$. So any isolated point has a neighborhood, namely the point itself, which meets only finitely many members of $r(\mathcal{U})$. It remains to show that $r(\mathcal{U})$ is locally finite at ∞. But any member of $\widehat{\mathcal{U}}_\infty^B$ is a neighborhood of ∞ which meets only other members of $\widehat{\mathcal{U}}_\infty^B$, which is finite.

iii.) If $\mathcal{V}$ is any open cover of X with $\mathcal{U} \prec \mathcal{V}$, then $r(\mathcal{U}) \prec r(\mathcal{V})$.

Let $U' \in r(\mathcal{U})$. Then there exists $U \in \mathcal{U}$ such that $U' \subset U$. Now, since $\mathcal{U} \prec \mathcal{V}$, then there must exist $V \in \mathcal{V}$ such that $U \subset V$, and hence $U' \subset V$. Now, U' is a basic open subset of X. If U' is a singleton, then we're done. If $\infty \in U'$, then since U' is a basic open subset contained in V, we must have that $U' \in \mathcal{V}_\infty^B$. If $U' \in \widehat{\mathcal{V}}_\infty^B$ then we're done. If $U' \notin \widehat{\mathcal{V}}_\infty^B$ then it means that U' is not maximal in $\mathcal{V}_\infty^B$. Hence there exists a $V' \in \widehat{\mathcal{V}}_\infty^B$ such that $U' \subset V'$. But then $V' \in r(\mathcal{V})$, and so $\mathcal{U} \prec \mathcal{V}$.

So r is a monotone paracompactness operator, and X is monotonically paracompact. $\square$

X *is not protometrizable.* Assume that X is protometrizable. Then X is paracompact and has an orthobase. It should first be noted that the collection of basic open neighborhoods described above is not an orthobase for X. Consider the collection $\mathcal{F} = \{B(\alpha_1, n) : n \in \omega\}$ for some fixed $a_1 \in \omega_1$. Then $\bigcap \mathcal{F} = \{\infty\}$, but $\mathcal{F}$ is not a base at ∞ since no member of $\mathcal{F}$ is a subset of the open neighborhood $B(\alpha_2, n)$ of ∞ for any $\alpha_2 > \alpha_1$. Now, we claim that no other collection of open subsets of X can be an orthobase. So assume X has an orthobase $\mathcal{B}$. First, start by choosing $B_0 \in \mathcal{B}$ so that $\infty \in B_0$. Next, let $B_0(\alpha_0, n_0)$ be any basic open set of ∞, such that $B_0(\alpha_0, n_0) \subset B_0$. Now, since $\mathcal{B}$ is an orthobase, there exists a $B_1 \in \mathcal{B}$ such that $\infty \in B_1 \subset B_0(\alpha_0, n_0)$. But now we can let $B_1(\alpha_1, n_1)$, with $\alpha_1 > \alpha_0$ and $n_1 > n_0$, be a basic open set of ∞, such that $B_1(\alpha_1, n_1) \subset B_1$.

In general, assuming we have $B_k \in \mathcal{B}$, and $B_k(\alpha_k, n_k)$ selected for all $i \leq k$ such that $\infty \in B_{k-1}(\alpha_{k-1}, n_{k-1}) \subset B_k \subset B_k(\alpha_k, n_k)$ then since $\mathcal{B}$ is an orthobase, there must be a $B_{k+1} \in \mathcal{B}$ such that $B_{k+1} \subset B_k(\alpha_k, n_k)$. Then we can select a basic open set $B_{k+1}(\alpha_{k+1}, n_{k+1})$, with $\alpha_{k+1} > \alpha_k$ and $n_{k+1} > n_k$, such that $\infty \in B_{k+1}(\alpha_{k+1}, n_{k+1}) \subset B_{k+1}$. We do this for all $k \in \omega$.

In this way we get a countable collection, $\mathcal{F} = \{B_k \in \mathcal{B} : k \in \omega\}$ of open subsets such that $\bigcap \mathcal{F} = \bigcap\limits_{k \in \omega} B_k(\alpha_k, n_k) = \{\infty\}$. But the sequence α_k must be bounded in ω_1 and hence there exists $\beta \in \omega_1$ such that for all $k \in \omega$, $\beta > \alpha_k$. Thus, for all $B_k \in \mathcal{F}$, $B(\beta, n) \not\subset B_k$ for any specific $n \in \omega$. So $\bigcap\limits_{k \in \omega} B_k$ is not open, and $\{B_k : k \in \omega\}$ is not a base at ∞. So $\mathcal{B}$ is not an orthobase. $\square$

Corollary 3.6.4. *The locally finite version of monotonically paracompact is not equivalent to the version studied by Gartside and Moody [11].*

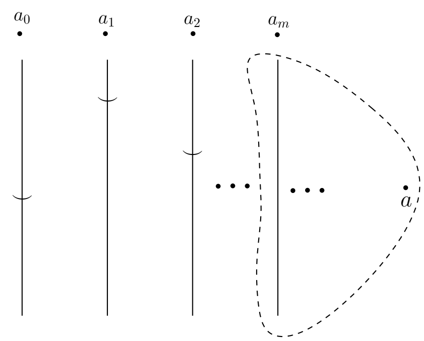

Figure 3.4: The Song space

3.7 The Song Space

In [23] it was asked "Is it consistent that every countable, monotonically Lindelöf space is metrizable?". Song [31] answered this in the negative by constructing a space that was Hausdorff, countable, and monotonically Lindelöf, but not metrizable.[2] We show that this space is monotonically metacompact.

The space defined by Song is:

Let $A = \{a_n : n \in \omega\}$ and $B = \{b_m : m \in \omega\}$ and let $Y = \{(a_n, b_m) : n, m \in \omega\}$. Then define $X = Y \cup A \cup \{a\}$ where $a \notin Y \cup A$.

Finally, define a topology on X by: every point of Y is isolated. A basic open neighborhood of a point $a_n \in A$ is of the form $U_{a_n}(m) = \{a_n\} \cup \{(a_n, b_j) : j > m\}$, for some $m \in \omega$. A basic open neighborhood of a is of the form $U_a(k) = \{a\} \cup \{(a_n, b_m) : n > k,\ m \in \omega\}$ for some $k \in \omega$. [3]

Example 3.7.1. *The Song Space is monotonically metacompact.*

Proof. For any open cover $\mathcal{V}$ of X, let $p(\mathcal{V}) = \min\{k : \{U_a(k)\} \prec \mathcal{V}\}$. For each $n \in \omega$ define $q(n, \mathcal{V}) = \min\{m : \{U_{a_n}(m)\} \prec \mathcal{V}\}$.

Finally, we will define $r(\mathcal{V}) = (A \times B) \cup \{U_a(p(\mathcal{V}))\} \cup \{U_{a_n}(q(n, \mathcal{V})) : n \in \omega\}$.

Claim: r is a monotone metacompactness operator.

[2] Song's space is not regular, so the question remains if there is a regular example of a space exhibiting these properties.

[3] In Song's original definition of the space, a basic neighborhood of a is of the form $U_a(F) = \{a\} \cup \bigcup\{(a_n, b_m) : a_n \in A \smallsetminus F,\ m \in \omega\}$ where F is any finite subset of A. The definition we give is equivalent.

It is easy to see that if $\mathcal{V}$ is an open cover of X, that $r(\mathcal{V})$ covers X, and that $r(\mathcal{V})$ is point-finite open refinement of $\mathcal{V}$. This last is because the points a and a_n, for each $n \in \omega$, are in exactly one element of $r(\mathcal{V})$, and any point in $(A \times B)$ is in at most three open sets in $r(\mathcal{V})$.

So assume that $\mathcal{W}$ and $\mathcal{V}$ are open covers of X with $\mathcal{W} \prec \mathcal{V}$, and let $W' \in r(\mathcal{W})$. Obviously, if $W' = \{(a_i, b_j)\}$ for some $i, j \in \omega$, then $W' \in \mathcal{V}$. If $W' = U_a(p(\mathcal{W}))$ or $W' = U_{a_n}(q(n, \mathcal{W}))$, then first note that $\mathcal{W} \prec \mathcal{V}$ implies $p(\mathcal{W}) \geq p(\mathcal{V})$ and $q(n, \mathcal{W}) \geq q(n, \mathcal{V})$. So there exists $V' = U_a(p(\mathcal{V})) \in r(V)$ or $V' = U_{a_n}(q(n, \mathcal{V})) \in r(\mathcal{V})$ such that $W' \subset V'$. Thus $r(\mathcal{W}) \prec r(\mathcal{V})$, and X is monotonically metacompact.

$\square$

3.8 One Point Lindelöfication

This example is another space examined due to its presence in the literature relating to the monotone Lindelöf property. In [25] Levy and Matveev show that the one point Lindelöfication of the discrete space of cardinality ω_1 is monotonically Lindelöf. They also show the one point Lindelöfication of a discrete space of cardinality $\geq \omega_2$ is not monotonically Lindelöf. [25]. We strengthen the result by showing that the one point Lindelöfication of the discrete space of cardinality $\geq \omega_2$ is not monotonically meta-Lindelöf, and hence not monotonically metacompact..

Let X be the one point Lindelöfication of an uncountable discrete space of size greater than or equal to ω_2. For simplicity, we will consider $X = \omega_2 \cup \{\infty\}$, where points of ω_2 are isolated, and a basic open set of ∞ contains all but countably many points of ω_2.

Example 3.8.1. *The one point Lindelöfication X of an uncountable discrete space of size greater than or equal to ω_2, is not monotonically meta-Lindelöf.*

Proof. Let X be the one point Lindelöfication of an uncountable discrete space of cardinality ω_2, and assume that X is monotonically meta-Lindelöf.

Now, for each $x \in \omega_2$ let $U_x = X \setminus \{x\}$ be a basic open neighborhood of ∞. Notice, that for all $x \in \omega_2$ the triple $(\infty, U_x, U_x) \in \mathcal{P}_X$. Hence from Lemma 2.0.4, for each $x \in \omega_2$, we get an open V_x, such that $\infty \in V_x \subset U_x$. Also note that $\bigcap_{x \in \omega_2} U_x \neq \emptyset$.

Now with this space it is actually more useful to consider the complements of the above open sets: thus since for each $x \in \omega_2$, $V_x = X \setminus C_x$, where C_x is a countable subset of ω_2, we will consider the collection $\{C_x : x \in \omega_2\}$ and note that for all $x \in \omega_2$, $x \in C_x$.

Rephrasing the conclusion of Lemma 2.0.4 in terms of this space and the complements of these open sets yields the following:

For every $A \subset \omega_2$ there exists a countable $B \subset A$ such that for each $x \in A$ there exists $b \in B$ such that $b \in C_x$.

Now, from $A3.5$ in "Cardinal Funcitons in Topology" by Juhasz[4], if we let $F : \omega_2 \to \mathcal{P}(\omega_2)$ be the function assigning to each $x \in \omega_2$ its respective C_x, then we get that there exists $M \subset \omega_2$, $|M| = \omega_2$, such that for any $x \neq y \in M$, $x \notin C_y$, and $y \notin C_x$.

We will apply the adapted conclusion of the monotone meta-Lindelöf lemma to the collection M to arrive at our contradiciton.

From the lemma, there must exist a countable $M' \subset M$, such that for any $x \in M$, there exists a $y \in M'$ with $y \in C_x$. Let $x \in M \setminus M'$. Then for every $y \in M'$, we have $y \in C_x$. But this contradicts $x \neq y$ implying that $x \notin C_y$ and $y \notin C_x$.

Therefore X is not monotonically meta-Lindelöf. $\square$

[4]Juhasz's theorem states: *If* $|X| = \alpha$ *and* $\beta < \alpha$ *and* $F : X \to \mathcal{P}(X)$ *satisfies* $\forall x\ inX,\ x \notin F(x)$ *and* $\forall x \in X,\ |F(x)| < \beta$ *then* $\exists M \subset X,\ |M| = \alpha$, *such that* M *is free.* We will utilize a related theorem, which modifies the Juhasz's as follows:*If* $|X| = \alpha$ *and* $\beta < \alpha$ *and* $F : X \to \mathcal{P}(X)$ *satisfies* $\forall x\ inX,\ |F(x)| < \beta$ *then* $\exists M \subset X,\ |M| = \alpha$, *such that for any* $x \neq y \in M,\ x \notin F(y)$ *and* $y \notin F(x)$.

3.9 Quasi-developable Spaces

Bennett, Hart, and Lutzer [4] also asked whether a metacompact, quasi-developable space must be monotonically metacompact. The following example, given by Bennett in [3], combined with Lemma 2.1.4 answers this question in the negative.

Let X be $[0,1] \times [0,1]$. Let T be an uncountable, dense subset of $[0,1]$ such that the only compact (with respect to the usual topology on $[0,1]$) subsets of T are countable. If $t \in T$ and ϵ is a positive real number, let $D(t,\epsilon) = \{z \in X : |z - (t,\epsilon)| < \epsilon\} \cup \{(t,0)\}$. In other words, $D(t,\epsilon)$ is a disc of radius ϵ that is tangent to the x-axis at $(t,0)$, together with the point $(t,0)$ itself. Let a basis for the topology on X consist of all sets of the following form:

i. $D(t,\epsilon)$ if $t \in T$ and $\epsilon > 0$

ii. $R \cap X$ where R is any open (in the usual topology) set in $\mathbb{R}^2$.

Example 3.9.1. *(Bennett) [3] X is a separable, Lindelöf, $\aleph_1$-compact quasi-developable regular space that is not a Moore space, does not have a point countable base, and is not hereditarily metacompact.*

Since this space is Lindelöf, it is paracompact. But the subspace $T \cup \{(x,y) : 0 < y \leq 1\}$ is not metacompact. It is actually easy to see it is not meta-Lindelöf, and hence X is neither hereditarily metacompact nor hereditarily meta-Lindelöf. Now from Lemma 2.1.4 we can give the following corollary:

Corollary 3.9.2. *There is a separable, Lindelöf, quasi-developable regular space that is not monotonically meta-Lindelöf.*

Chapter 4

Summary and Outlook

The monotonically metacompact and monotonically meta-Lindelöf covering properties have, in recent years, been examined fairly assiduously. Despite this fact, there is still quite a bit that is not known regarding these properties.

We have shown that every compact, Hausdorff, monotonically countably metacompact space is metrizable. This is an extension of both the result of Bennett, Hart, and Lutzer that every compact, monotonically metacompact LOTS is metrizable, and the related result of Gruenahge involving the stronger covering property, monotonically compact, that every monotonically compact Hausdorff space is metrizable, and answers the question posed by Popvassilev and Bennett, Hart, and Lutzer.

In proving this theorem, we have introduced two useful properties (Lemma 2.0.3 and Lemma 2.0.5) that monotonically metacompact spaces possess, and shown that the first property alone is enough to get metrizability in compact, Hausdorff spaces. We have shown that these properties, and hence monotone metacompactness as well, imply a space is hereditarily metacompact. We also extended Lemma 2.0.3 and Lemma 2.0.5 to versions implied by the monotonically meta-Lindelöf property, and showed that every monotonically meta-Lindelöf space is hereditarily Lindelöf.

We show that Lemma 2.0.5 is not equivalent to either the monotone metacompactness property or Lemma 2.0.3. However, we put forth this question:

Question 4.0.1. *Is there a space satisfying the conclusion of Lemma 2.0.3, that is not monotonically metacompact?*

In [4] it was shown that every metrizable space, as well as every metacompact Moore space was monotonically metacompact, and asked which other properties, when paired with metacompactness, would be monotonically metacompact. By means of counterexamples, we show that neither quasi-developable, nor stratifiable can replace the Moore property, and thus answer a couple of questions posed by Bennet, Hart, and Lutzer in [4]. However, we were unable to answer one of their questions fully, so we restate here:

Question 4.0.2. *Must a hereditarily metacompact, quasi-developable space be metacompact?*

In response to another question put forth by Bennett, Hart and Lutzer, and utilizing the hereditarily meta-Lindelöf result previously mentioned, we independently showed a result initially published by Li and Peng, that every monotonically normal, monotonically meta-metaöf space is hereditarily paracompact.

Finally, in Example 3.6.3 we took a look at the some of the many "monotonically paracompact properties", and by means of an example, showed that monotonically paracompact in the locally finite sense is not equivalent to monotonically paracompact in the sense of Gartside and Moody. This was a result also discovered independantly by Popvassilev and Porter, although our example was a space with the added property of being stratifiable. Since monotonically paracompact in the locally finite sense is the most natural definition of monotonically paracompact, yet is now known to be different from the version studied by Gartside and Moody, it would definitely be worth investigating this property further.

We looked at the monotone metacompactness and monotone meta-Lindelófness covering properties with regard to several spaces over the course of this paper, and in a couple of examples showed that a monotonically Lindelöf space need not be montotonically metacompact, however we are still unable to characterize these properties completely with regard to countable spaces, even those having only a single non-isolated point, hence we give the following question:

Question 4.0.3. *Must every countable space that is monotonically metacompact be metrizble?*

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