# **Bounded Complete Embedding Graphs**

by

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Abstract

In the study of graph embeddings, there is particular interest in small embeddings and in bounds on the minimum number of vertices that must be added in order to achieve an embedding of a particular design. We introduce two new terms, defining a natural approach to the bounding question. A simple graph $G$ is a *bounded complete embedding graph* if and only if there is some positive integer $b$ such that, for every $n \in \mathbb{P}$ such that a complete $G$-design of order $n$ exists, every complete $G$-design of order $n$ can be embedded in a complete $G$-design of order $n + x$, for some positive integer $x$ such that $x \leq b$. A simple graph $G$ is a *bounded embedding graph* if and only if there is some positive integer $c$ such that, for every positive integer $n$, every partial $G$-design of order $n$ can be embedded in a complete $G$-design of order $n + x$, for some positive integer $x$ such that $x \leq c$.

By definition, every bounded embedding graph is a bounded complete embedding graph; we show that the converse of this fact is false, and that all bounded complete embedding graphs are bipartite. We identify results in the literature that provide, as immediate corollaries, the following results: that $k$-stars are bounded embedding graphs, that 3-paths are bounded embedding graphs, and that even cycles are bounded complete embedding graphs.

We establish that paths and complete bipartite graphs are bounded complete embedding graphs. We show that all simple bipartite graphs $G$ that have $2^t$ edges (for some positive integer $t$), have all vertices of even degree, and admit a $\beta^+$-labeling, are bounded complete embedding graphs.

We show that the graph $\mathcal{C}_{2k}^p$, consisting of $p$ vertex-disjoint $2k$-cycles, is a bounded complete embedding graph if $p = 2^t$, for some positive integer $t$, or if $2 \leq p \leq 128$ and $2 \leq k \leq 128$. These results on the graph $\mathcal{C}_{2k}^p$ and the supporting constructions comprise a major portion of our work. We produce two constructions of $\mathcal{C}_{2k}^p$-designs on complete bipartite graphs; we apply these constructions to obtain the designs on complete bipartite graphs that are necessary to build embeddings. We rely on existing graph labeling results that establish, for all values of $p$ and $k$, the existence of a complete $\mathcal{C}_{2k}^p$-design of order $4kp+1$; we also present our own independently achieved designs for some values of $p$ and $k$, and we compare our designs to those created by graph labelings. Furthermore, we establish the spectrum of complete $\mathcal{C}_{2k}^p$-designs when $4kp = 2^t$ (for some positive integer $t$), and we exhibit the additional designs necessary to establish the spectrum of complete $\mathcal{C}_{2k}^p$-designs for $p = 2$ and $k \in \{3, 5\}$.

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## List of Notation

| | |
|---|---|
| $\mathbb{Z}$ | the set of integers |
| $\mathbb{N}$ | the set of non-negative integers |
| $\mathbb{P}$ | the set of positive integers |
| $\mathbb{Z}_m$ | the set of integers modulo $m$ |
| $[\![a, b]\!]$ | the set of integers $i$ satisfying $a \leq i \leq b$ |
| $\mathrm{abs}(x)$ | the absolute value of the real number $x$ |
| $\lvert A \rvert$ | the number of elements in the set $A$ |
| $C_k$ | the cycle on $k$ edges |
| $K_n$ | the complete graph on $n$ vertices |
| $K_{r,s}$ | the complete bipartite graph with partite sets of sizes $r$ and $s$ |
| $P_k$ | the path on $k$ edges |
| $S_k$ | the $k$-star, $K_{1,k}$ |
| $V(G)$ | the vertex set of the graph $G$ |
| $E(G)$ | the edge set of the graph $G$ |
| $v(G)$ | the number of vertices in the graph $G$ |
| $e(G)$ | the number of edges in the graph $G$ |
| $\deg_G(v)$ | the degree of the vertex $v$ in the graph $G$ |
| $G[X]$ | the subgraph of the graph $G$ induced by the set $X$ |
| $\mathrm{Spec}(G)$ | the spectrum of the graph $G$ |
| $\mathrm{SSpec}(G)$ | the superspectrum of the graph $G$ |

# Chapter 1

# Introduction

Graph designs, also known as graph decompositions, are a major topic in both design theory and graph theory. For a particular graph $G$, the typical first question about $G$-designs is that of their existence; subsequent questions typically involve embedding smaller $G$-designs in larger ones or building $G$-designs with desirable properties. Embeddings of graph designs are widely studied, and they are the focus of this dissertation. In particular, for a given graph $G$, we wish to bound the increase in size between the smaller $G$-design and the larger $G$-design as strictly as possible: to bound it by a value that is constant with respect to the size of the smaller $G$-design. We introduce two terms to describe graphs for which this bound exists; these terms are *bounded complete embedding graphs* and *bounded embedding graphs*. Our primary goal is the identification of bounded complete embedding graphs; we are successful in identifying several infinite families.

## 1.1 Notation, Conventions, and Graph Terminology

We denote by $\mathbb{Z}$, the set of integers; by $\mathbb{N}$, the set of nonnegative integers; and by $\mathbb{P}$, the set of positive integers. For any positive integer $m$, we denote by $\mathbb{Z}_m$ the set of integers modulo $m$. For integers $a$ and $b$, we denote by $[\![a, b]\!]$ the set of integers $x$ satisfying $a \leq x \leq b$, with the understanding that $[\![a, b]\!] = \varnothing$ if $a > b$. We now introduce the essential terminology and notation of graphs; we refer the reader to the text by Bondy and Murty [7] for any graph theory terms we do not define.

**Definition 1.1.** A ***graph*** $G$ is a triple consisting of a nonempty set $V(G)$ of ***vertices***, a set $E(G)$ (disjoint from $V(G)$) of ***edges***, and an incidence relation $\psi_G$ that assigns to each edge of $G$ two (not necessarily distinct) elements of $V(G)$, called its ***ends***.

We denote by $v(G)$ the number of vertices in $G$ and by $e(G)$ the number of edges in $G$.

Two vertices are said to be ***adjacent*** if and only if they are the ends of an edge; two edges are said to be ***adjacent*** if and only if they have at least one common end. An edge and a vertex are said to be ***incident*** to each other if and only if the vertex is an end of the edge. A ***loop*** is an edge whose two ends are the same vertex. Two edges $e$ and $f$ that are not loops are said to be ***parallel*** if and only if the ends of $e$ are the ends of $f$. ■

**Definition 1.2.** A graph is said to be ***simple*** if and only if it has no loops and no parallel edges. ■

Note that edges in a simple graph are uniquely determined by their endpoints, so we may specify a simple graph $G$ simply by specifying $V(G)$ and a set of two-element subsets of $V(G)$ that we call $E(G)$.

**Definition 1.3.** The ***complement*** of the simple graph $G$ is the simple graph $\overline{G}$ defined by $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{\{u,v\} \subseteq V(G) \mid \{u,v\} \notin E(G)\}$. ■

**Definition 1.4.** The ***degree*** of vertex $v$ in the graph $G$, denoted $\deg(v)$ or $\deg_G(v)$, is the number of edges of $G$ incident with $v$, with each loop at $v$ counted twice. ■

**Definition 1.5.** A graph $G$ is said to be ***regular*** if and only if all vertices in $G$ have the same degree. We say that $G$ is ***k-regular*** if and only if $\deg_G(v) = k$ for all $v \in V(G)$. ■

**Definition 1.6.** A simple graph is ***complete*** if and only if every vertex in $G$ is adjacent to every other vertex in $G$. For $n \in \mathbb{P}$, we denote the complete graph on $n$ vertices by $K_n$. ■

**Definition 1.7.** A graph is ***empty*** if and only if it has no edges. ■

**Definition 1.8.** A graph is said to be ***bipartite*** if and only if its vertex set can be partitioned into two sets $A$ and $B$ such that every edge has one end in $A$ and one end in $B$. Such a partition $[A,B]$ is called a ***bipartition*** of the graph, and the sets $A$ and $B$ are called its ***parts***. ■

**Definition 1.9.** A bipartite graph $G$ on bipartition $[A, B]$ is a ***complete bipartite graph*** if and only if $G$ is simple and every vertex of $A$ is adjacent to every vertex of $B$.

For $r, s \in \mathbb{P}$, the complete bipartite graph on parts of sizes $r$ and $s$ is denoted $K_{r,s}$. ■

**Definition 1.10.** A graph is a ***k-star***, denoted $S_k$, if and only if it is a complete bipartite graph with one part of size one and one part of size $k$. ■

**Definition 1.11.** A ***path*** is a simple graph whose vertices can be arranged in a linear sequence so that two vertices are adjacent if and only if they are consecutive in the sequence. The ***length*** of a path is its number of edges; a path of length $k$ is called a ***k-path*** and denoted $P_k$. ■

**Definition 1.12.** A ***cycle*** is a simple graph whose vertices can be arranged in a cyclic sequence so that two vertices are adjacent if and only if they are consecutive in the sequence. The ***length*** of a cycle is its number of edges; a cycle of length $k$ is called a ***k-cycle*** and denoted $C_k$. ■

**Definition 1.13.** A ***subgraph*** of a graph $G$ is a graph $H$ such that $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$, and $\psi_H$ agrees with $\psi_G$ on $E(H)$. ■

**Definition 1.14.** Let $G$ be a graph, and let $W \subseteq V(G)$. The ***subgraph of G induced by W***, denoted $G[W]$, is the subgraph of $G$ having vertex set $W$ and edge set consisting of all edges in $G$ with both ends in $W$. Subgraphs induced by sets of vertices are commonly called *vertex-induced subgraphs* or simply *induced subgraphs.* ■

**Definition 1.15.** Let $G$ be a graph, and let $F \subseteq E(G)$. The ***subgraph of G induced by F***, denoted $G[F]$, is the subgraph of $G$ having edge set $F$ and vertex set consisting of all vertices in $G$ that are incident to at least one edge in $F$. Subgraphs induced by sets of edges are sometimes called *edge-induced subgraphs.* ■

**Definition 1.16.** A graph $G$ is ***connected*** if and only if, for any partition of $V(G)$ into two nonempty sets $X$ and $Y$, there is at least one edge of $G$ with one end in $X$ and one end in $Y$. ■

**Definition 1.17.** The ***components*** of a graph $G$ are its maximal connected subgraphs. ■

Since we are primarily concerned with simple graphs, we will henceforth use the word *graph* to mean a simple graph; when we wish to emphasize that we have restricted a result to the simple graph case, we may temporarily use the full phrase *simple graph* for purposes of such emphasis. On the limited occasions when we wish to allow loops or parallel edges, we will use the term *multigraph*. We will use the symbol $\uplus$ to denote the vertex-disjoint union of graphs, meaning that each given graph is taken as one component of the new graph.

## 1.2 Graph Designs and Decompositions

In this section, we define and discuss a single idea with two names: that of graph decompositions or graph designs. We begin with definitions of the two terms.

**Definition 1.18.** Let $G$ and $H$ be graphs. A ***G-decomposition of H*** is a system $\mathscr{G}$ of subgraphs of $H$, each isomorphic to $G$, which are pairwise edge-disjoint and whose union is the graph $H$. ■

**Definition 1.19.** Let $G$ and $H$ be graphs, and let $n = v(H)$. A ***G-design on H*** is a collection $\mathscr{B}$ of subgraphs of $H$, each isomorphic to $G$, whose edge sets partition $E(H)$. Elements of $\mathscr{B}$ are called ***G-blocks***. The collection $\mathscr{B}$ is also called a ***partial G-design of order n***. $\overline{H}$ is called the ***leave*** of the partial $G$-design. If $H = K_n$, then $\mathscr{B}$ is called a (***complete***) ***G-design of order n***. ■

We observe that, for any graph $H$ with at least one edge and any graph $G$, a $G$-design on $H$ and a $G$-decomposition of $H$ are equivalent. If, however, the graph $H$ has no edges, the terms are not equivalent: the empty collection *is not* a graph decomposition of $H$, since the union of its elements is the empty set and thus not a graph, but the empty collection *is* a graph design on $H$. Furthermore, if $G$ has at least one edge and $H$ has no edges, the *only* $G$-design on $H$ is the empty collection. In particular, the empty collection is a $G$-design on $K_1$; we will henceforth refer to this design as the trivial complete $G$-design of order 1.

In general, for a graph $G$, it is natural to ask for which positive integers $n$ a complete $G$-design of order $n$ exists. This question is fundamental to the study of graph designs, and is still open for numerous infinite families of graphs. The following term is common in the literature on the question of existence.

**Definition 1.20.** The ***spectrum*** of a graph $G$, denoted Spec($G$), is

$$\operatorname{Spec}(G) = \Big\{ n \in \mathbb{P} \,\Big|\, \text{there is a complete } G\text{-design of order } n \Big\}. \qquad \blacksquare$$

For any graph $G$, there are three "obvious" necessary conditions (on the positive integer $n$) for the existence of a complete $G$-design of order $n$. These conditions dictate the size of $n$ and confine it to certain residue classes. We will refer to these conditions often; for convenience, we name them as we discuss them.

The first obvious necessary condition is that $K_n$ must have enough vertices to admit a complete $G$-design. In particular, unless $n = 1$, $K_n$ must have at least as many vertices as $G$, so that $K_n$ has subgraphs isomorphic to $G$.

(SSC–1) $\qquad n = 1 \quad \text{or} \quad n \geq v(G)$ .

The second obvious necessary condition is that it must be possible to partition $E(K_n)$ into sets of the correct size. In particular, since the edge sets of the $G$-blocks partition $E(K_n)$ into sets of size $e(G)$, the number of edges in $K_n$ must be a multiple of $e(G)$; that is, $e(G) \,|\, e(K_n)$. Since $e(K_n) = n(n-1)/2$, we obtain the condition below.

(SSC–2) $\qquad 2e(G) \,|\, n(n-1)$

The third obvious necessary condition is slightly less obvious than the previous two. Suppose there is a $G$-design $\mathscr{B}$ of order $n$, and consider $v \in V(K_n)$. Clearly, $v$ belongs to some (possibly all) of the $G$-blocks in the design, and every edge incident with $v$ belongs to exactly one $G$-block in $\mathscr{B}$; thus

$$\deg_{K_n}(v) = \sum_{B \in \mathscr{B}} \deg_B(v). \tag{1.1}$$

For each $B \in \mathscr{B}$, either $v$ is not a vertex of $B$, in which case $\deg_B(v) = 0$, or $v$ is a vertex of $B$. If $v \in V(B)$, then, since $B$ is isomorphic to $G$, there is some $u_B \in V(G)$ so that $\deg_B(v) = \deg_G(u_B)$. Thus

$$n - 1 = \deg_{K_n}(v) = \sum_{B \in \mathscr{B}} \deg_B(v) = \sum_{B \in \mathscr{B}} \deg_G(u_B) \tag{1.2}$$

The greatest common divisor of all the degrees of the vertices of $G$ must divide the rightmost sum above; hence it must also divide $n-1$. We take this condition of divisibilty as the third condition. Clearly, this condition takes its strongest form when the graph $G$ is regular.

(SSC–3) $\gcd \left\{ \deg_G(v) \,\middle|\, v \in V(G) \right\} \Big| \, (n-1)$

Since all positive integers $n$ for which a $G$-design of order $n$ exists (that is, all $n$ in the spectrum of $G$) must satisfy SSC–1, SSC–2, and SSC–3, we will refer collectively to these conditions as the Superspectral Conditions. Since many graphs still have unknown spectra, it will be useful to have a name for the set of positive integers $n$ that satisfy the Superspectral Conditions.

**Definition 1.21.** The ***superspectrum*** of a graph $G$, denoted SSpec($G$), is the set of all positive integers $n$ that satisfy SSC–1, SSC–2, and SSC–3. ■

The existence, for any graph $G$, of complete $G$-designs of infinitely many orders was established in a 1976 paper by Richard M. Wilson, as a special case of a more general result [35]. In this paper, Wilson establishes necessary and "asymptotically sufficient" conditions for the existence of digraph decompositions of complete digraphs. Wilson highlights the special case of symmetric-digraph-decompositions of complete digraphs, restating the conditions as they apply to this special case. Decompositions of complete digraphs by symmetric digraphs are easily seen to be equivalent to $G$-decompositions of complete graphs. We state Wilson's result for this special case below, using the language of complete $G$-designs; we will refer to this result as *Wilson's Theorem*.

**Theorem 1.22** (Wilson)**.** *Let $G$ be a graph. For all sufficiently large integers $n$ satisfying SSC–2 and SSC–3, a complete $G$-design of order $n$ exists.*

Wilson's Theorem says, in essence, that there are at most finitely many elements of the superspectrum of $G$ that are not elements of the spectrum of $G$. Wilson's work in the paper goes one step further, providing a description of the spectrum of $G$ in terms of residue classes. He establishes that, for any graph $G$, there is a positive integer $S_G$ such that the spectrum of $G$ may be expressed as a subset of the union of certain residue classes modulo $S_G$, and that only finitely many positive elements of each residue class are missing from the spectrum. Since the spectrum may be expressed in this way, it must have infinitely many elements. We make the additional observation that one particular congruence class is always present in the spectrum, as we describe in the following lemma.

**Lemma 1.23.** *Let $G$ be a graph. Then we have the following.*

(*i*) *All positive integers $n$ such that $n \equiv 1 \pmod{2e(G)}$ satisfy* SSC–2 *and* SSC–3.

(*ii*) *If* $2e(G)+1 \geq v(G)$, *then* $\left\{ n \in \mathbb{P} \,\middle|\, n \equiv 1 \pmod{2e(G)} \right\} \subseteq \mathrm{SSpec}(G)$.

(*iii*) *There is some positive integer $N(G)$ such that*

$$\left\{ n \in \mathbb{P} \;\middle|\; n \geq N(G) \;\; \textit{and} \;\; n \equiv 1 \pmod{2e(G)} \right\} \subseteq \mathrm{Spec}(G)\,.$$

*Proof.* Let $G$ be a graph. Let $n \in \mathbb{P}$ such that $n \equiv 1 \pmod{2e(G)}$; then $2e(G) \,\big|\, (n-1)$, so $2e(G) \,\big|\, n(n-1)$; hence SSC–2 is satisfied. Now we recall an elementary fact from graph theory, relating the sum of the vertex degrees of the graph $G$ to its number of edges:

$$\sum_{v \in V(G)} \deg_G(v) = 2e(G). \tag{1.3}$$

Since the sum of the degrees of the vertices $G$ must be divisible by the greatest common divisor of those degrees, we must have that $\gcd\left\{ \deg_G(v) \,\middle|\, v \in V(G) \right\} \Big| \, 2e(G)$. Then, since

$2e(G) \,\Big|\, (n-1)$ (as observed above), $\gcd\big\{ \deg_G(v) \,\big|\, v \in V(G) \big\} \,\Big|\, (n-1)$, so SSC–3 is also satisfied; hence we have proved item (i).

If $2e(G) + 1 \geq v(G)$, then every positive integer $n$ satisfying $n \equiv 1 \big(\text{mod } 2e(G)\big)$ also satisfies SSC–1, so $\Big\{ n \in \mathbb{P} \,\Big|\, n \equiv 1 \big(\text{mod } 2e(G)\big) \Big\} \subseteq \text{SSpec}(G)$; thus item (ii) holds.

Wilson's Theorem guarantees that all sufficiently large integers satisfying SSC–2 and SSC–3 are elements of the spectrum of $G$; by item (i), we have that every positive integer $n$ such that $n \equiv 1 \big(\text{mod } 2e(G)\big)$ satisfies SSC–2 and SSC–3. Hence there is some $N(G)$ such that $\Big\{ \; n \in \mathbb{P} \;\; \Big| \;\; n \geq N(G) \;\text{ and }\; n \equiv 1 \,(\text{mod } 2e(G)) \Big\} \subseteq \text{Spec}(G)$, so item (iii) is verified. □

We note that any graph $G$ that has no isolated vertices satisfies the condition in item (ii) of Lemma 1.23, namely that $2e(G) + 1 \geq v(G)$. Since we rarely consider graphs with isolated vertices as candidate graphs for building graph designs, the conclusion of item (ii) holds for almost all graphs that are of interest in the study of graph designs.

### 1.3 Embeddings of Graph Designs

We now venture beyond the question of existence for graph designs, and discuss when it is possible to build a graph design with a smaller design residing within it; this idea of building one design around another is called *embedding* the smaller design.

**Definition 1.24.** Let $G$ be a graph; let $n \in \mathbb{P}$; let $\mathscr{B}$ be a partial $G$-design of order $n$. Let $r \in \text{Spec}(G)$, such that $r \geq n$. An ***embedding of $\mathscr{B}$ of order $r$*** is a complete $G$-design $\mathscr{D}$ of order $r$ such that $\mathscr{B} \subseteq \mathscr{D}$. If such a design $\mathscr{D}$ exists, we say that $\mathscr{B}$ *can be embedded in a* (*complete*) *$G$-design of order $r$*. ■

We observe that every complete $G$-design has a trivial embedding, namely in itself. The above definition also allows the possibility of a partial $G$-design being embedded in a complete $G$-design of the same order; such embeddings are commonly called *completions* of partial designs. Clearly, a completion of a partial $G$-design of order $n$ is only possible if

$n \in \mathrm{Spec}(G)$; with rare exception, this condition is far from sufficient. We primarily concern ourselves with the creation of embeddings of order strictly greater than that of the original design; henceforth, we will refer to such embeddings as *nontrivial* embeddings. It follows from Wilson's Theorem that nontrivial embeddings of $G$-designs always exist, for any graph $G$, for both partial and complete $G$-designs. We state these facts as Theorems 1.25 and 1.26 and offer short proofs.

**Theorem 1.25.** *Let $G$ be a graph. Then, for each $n \in \mathrm{Spec}(G)$, there is some $v(n) \in \mathbb{P}$ such that $v(n) > n$ and there is an embedding of every complete $G$-design of order $n$ in a complete $G$-design of order $v(n)$.*

*Proof.* Let $G$ be a graph, let $n \in \mathrm{Spec}(G)$, and let $\mathscr{B}$ be a complete $G$-design on $K_n$. Wilson's Theorem guarantees that $\mathrm{Spec}(K_n)$ is infinite, so let $v(n) \in \mathrm{Spec}(K_n)$ such that $v(n) > n$, and let $\mathscr{C}$ be a $K_n$-design of order $v(n)$.

For each $C \in \mathscr{C}$, let $f_C$ denote the isomorphism of $K_n$ with $C$. We observe that, for any $C \in \mathscr{C}$ and any $B \in \mathscr{B}$, since $B$ is a subgraph of $K_n$ isomorphic to $G$, the graph $f_C(B)$ is a subgraph of $C$ (and thus of $K_{v(n)}$) that is isomorphic to $G$. So the set

$$\mathscr{D} = \left\{ f_C(B) \;\middle|\; C \in \mathscr{C} \text{ and } B \in \mathscr{B} \right\}$$

is a $G$-design on $K_{v(n)}$. Observe that $\mathscr{D}$ is an embedding of $\mathscr{B}$ of order $v(n)$. $\square$

**Theorem 1.26.** *Let $G$ be a graph, let $n \in \mathbb{P}$, and let $\mathscr{B}$ be a partial $G$-design of order $n$. Then there is some $v \in \mathbb{P}$ such that $v > n$ and there is an embedding of $\mathscr{B}$ in a complete $G$-design of order $v$.*

*Proof.* Let $G$ be a graph, let $n \in \mathbb{P}$, and let $\mathscr{B}$ be a partial $G$-design of order $n$. If $\mathscr{B}$ is complete, let $v = v(n)$ as in Theorem 1.25. Otherwise, observe that $\mathscr{B}$ is a $G$-design on some proper subgraph $H$ of $K_n$; Wilson's Theorem guarantees that $\mathrm{Spec}(H)$ is infinite, so let $v \in \mathrm{Spec}(H)$ such that $v > n$, and let $\mathscr{C}$ be an $H$-design of order $v$. We now proceed as in the proof of Theorem 1.25 to combine the designs $\mathscr{C}$ and $\mathscr{B}$ to form an embedding of $\mathscr{B}$ of order $v$. □

With the existence of embeddings thus established, it is natural to ask what orders of embeddings can be achieved for a particular design, and, in particular, how close the order of the embedding can be to the order of the original design. Many of the existing results on embeddings of graph designs focus on finding embeddings for which the difference between the order, $r$, of the embedding and the order, $n$, of the original design is small, and on finding bounds on this difference. One famous bounding result concerns embeddings of partial $K_3$-designs, which are more commonly known as partial *Steiner triple systems*. In 1975, C. C. Lindner showed [24] that every partial Steiner triple system of order $n$ can be embedded in a complete Steiner triple system of order $6n + 3$. He conjectured a smaller bound (see [24], p. 351, and [25], p. 59): that every partial Steiner triple system of order $n$ can be embedded in a complete Steiner triple system of order $v$, for any $v \equiv 1$ or $3 \pmod 6$ such that $v \geq 2n+1$; this bound was finally proved by D. Bryant and D. Horsley over thirty years later [8].

The ultimate question of smallness for embeddings is to determine the size of the smallest possible embedding; this is typically expressed as determining, for a partial or complete design of order $n$, the minimum number, $x$, for which an embedding of order $n + x$ is possible. Many results on the existence of small embeddings are stated as bounds on this minimum number.

## 1.4 Bounded Complete Embedding Graphs and Bounded Embedding Graphs

In our study of small embeddings, we ask for a particularly restrictive kind of smallness: we ask that the bound on this minimum number of added vertices be constant for a particular graph, that is, independent of the size, $n$, of the original design for which an embedding is sought. We formulate two versions of this problem; one concerns only the embedding of complete $G$-designs, while the other addresses the more general problem of embedding partial

$G$-designs. In both versions, we restrict ourselves to nontrivial embeddings by insisting that the number of vertices added be positive.

**Definition 1.27.** A graph $G$ is a ***bounded complete embedding graph*** (***BCE graph***) if and only if there is some positive integer $b$ such that, for every $n \in \text{Spec}(G)$, every complete $G$-design of order $n$ can be embedded in a $G$-design of order $n+x$, for some $x \in [\![1, b]\!]$. ■

**Definition 1.28.** A graph $G$ is a ***bounded embedding graph*** (***BE graph***) if and only if there is some positive integer $c$ such that, for every $n \in \mathbb{P}$, every partial $G$-design of order $n$ can be embedded in a $G$-design of order $n+x$, for some $x \in [\![1, c]\!]$. ■

With these definitions in hand, our questions are now quite simple to state: we wish to know which graphs are bounded complete embedding graphs and which graphs are bounded embedding graphs. We note that, by definition, complete $G$-designs are a special case of partial $G$-designs; we therefore have the following.

**Remark 1.29.** Let $G$ be a bounded embedding graph. Then $G$ is also a bounded complete embedding graph. ■

Due to this fact, we amend our earlier statement: we wish to know which graphs are bounded complete embedding graphs, and which of those graphs are bounded embedding graphs. We begin to answer these questions with some elementary examples.

**Example 1.30.** The graph $K_2$ is both a bounded complete embedding graph and a bounded embedding graph. For any $n \in \mathbb{P}$, a unique complete $K_2$-design of order $n$ exists and can be embedded in a complete $K_2$-design of order $n+1$. Thus $K_2$ is a bounded complete embedding graph. Furthermore, for any $n \in \mathbb{P}$, any partial $K_2$-design of order $n$ can be embedded in a complete $K_2$-design of order $n+1$. Thus $K_2$ is a bounded embedding graph. ■

**Example 1.31.** Consider the path $P_2$. Let $n \in \text{Spec}(P_2)$ and suppose $n$ is even. Then a complete $P_2$-design of order $n$ can be embedded in a complete $P_2$-design of order $n+1$, as follows. Suppose we have a complete $P_2$-design $\mathscr{B}$ on $K_n$ with $V(K_n) = \left\{v_i \,\middle|\, i \in [\![1, n]\!]\right\}$

and we add vertex $z$ to create a $K_{n+1}$. Since $n$ is even, we have $n = 2t$ for some $t \in \mathbb{P}$. Since all new edges are incident with $z$, let $\mathscr{C} = \left\{ v_{2i-1}\, z\, v_{2i} \;\middle|\; i \in [\![1, t]\!] \right\}$. Then $\mathscr{B} \cup \mathscr{C}$ is an embedding of $\mathscr{B}$ in a complete $P_2$-design on $K_{n+1}$.

Let $n \in \mathrm{Spec}(P_2)$ and suppose $n$ is odd. Then any complete $P_2$-design of order $n$ can be embedded in a complete $P_2$-design of order $n+3$, as follows. Suppose we have a complete $P_2$-design $\mathscr{B}$ on $K_n$ with $V(K_n) = \left\{ v_i \;\middle|\; i \in [\![1, n]\!] \right\}$ and we add vertices $z_1$, $z_2$, and $z_3$ to create a $K_{n+3}$. Since $n$ is odd, we have $n = 2s+1$ for some $s \in \mathbb{N}$. The subgraph of $K_{n+3}$ induced by the vertex set $\{v_n, z_1, z_2, z_3\}$ is a $K_4$-subgraph of $K_{n+3}$, which admits the $P_2$-design $\mathscr{C} = \{ z_1\, v_n\, z_2,\; z_1\, z_2\, z_3,\; z_1\, z_3\, v_n \}$ shown in Figure 1.1. For each $j \in \{1, 2, 3\}$, define $\mathscr{D}_j = \left\{ v_{2i-1}\, z_j\, v_{2i} \;\middle|\; i \in [\![1, s]\!] \right\}$. Then $\mathscr{B} \cup \mathscr{C} \cup \mathscr{D}_1 \cup \mathscr{D}_2 \cup \mathscr{D}_3$ is an embedding of $\mathscr{B}$ in a complete $P_2$-design on $K_{n+3}$.

We have shown that, for any $n \in \mathrm{Spec}(P_2)$, every complete $G$-design of order $n$ can be embedded in a complete $G$-design of order $n+x$ for some $x \in [\![1, 3]\!]$; thus $P_2$ is a bounded complete embedding graph. ■

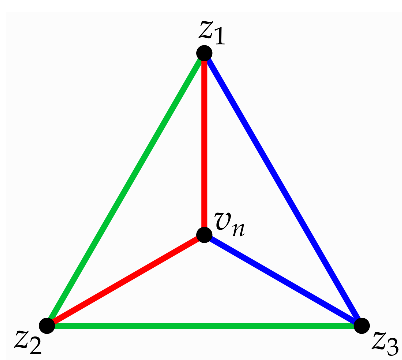



Figure 1.1: A $P_2$-design on the $K_4$-subgraph on vertices $v_n$, $z_1$, $z_2$, and $z_3$

**Example 1.32.** Consider the path $P_2$ again. Let $n \in \mathbb{P}$ and suppose $\mathscr{B}$ is a partial $P_2$-design of order $n$. Form a maximal partial design $\mathscr{B}^*$ by adding $P_2$-blocks from the leave of $\mathscr{B}$ until no $P_2$-subgraphs remain in the leave of $\mathscr{B}^*$. If the leave of $\mathscr{B}^*$ has no edges, it is a complete $G$-design of order $n$, so it can be embedded in a complete $G$-design of order $n+x$ for some $x \in [\![1, 3]\!]$ as shown in the previous example. Suppose, then, that the leave of $\mathscr{B}^*$ has at least one edge. Note that no two edges in the leave of $\mathscr{B}^*$ can be adjacent, since two adjacent edges form a $P_2$-subgraph. Hence the leave consists of some positive number, $s$, of isolated edges and some nonnegative number, $t$, of isolated vertices. Let the set of edges in the leave be $\left\{e_i \,\middle|\, i \in [\![1,s]\!]\right\}$, and let $V(K_n) = \left\{u_i \,\middle|\, i \in [\![1,s]\!]\right\} \cup \left\{v_i \,\middle|\, i \in [\![1,s]\!]\right\} \cup \left\{w_j \,\middle|\, j \in [\![1,t]\!]\right\}$, so that, for all $i \in [\![1,s]\!]$, the ends of edge $e_i$ are $u_i$ and $v_i$. We now consider cases according to the parity of $s$ and $t$.

Case 1: $s$ and $t$ are both even. We add one vertex, $z$, to form a $K_{n+1}$. We pair the isolated edges; each pair corresponds to a bowtie subgraph of $K_{n+1}$, all of whose edges are unused by $P_2$-blocks in $\mathscr{B}^*$. The bowtie graph admits a $P_2$-design, as shown in Figure 1.2. Let

$$\mathscr{C} = \left\{ u_{i-1}\, z\, v_i,\ u_{i-1}\, v_{i-1}\, z,\ z\, u_i\, v_i \,\middle|\, i \in [\![1,s]\!] \text{ and } i \text{ is even} \right\}.$$

We pair the isolated vertices, forming paths centered at $z$; let

$$\mathscr{D} = \left\{ w_{j-1}\, z\, w_j \,\middle|\, j \in [\![1,t]\!] \text{ and } j \text{ is even} \right\}.$$

Then $\mathscr{B}^* \cup \mathscr{C} \cup \mathscr{D}$ is an embedding of $\mathscr{B}$ in a complete $P_2$-design on $K_{n+1}$.

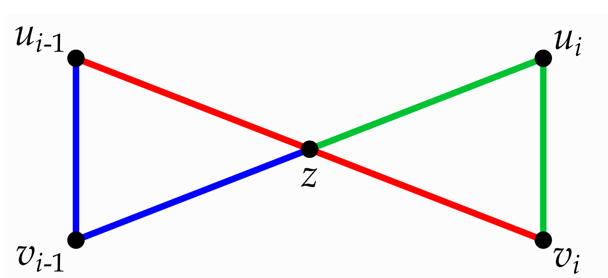



Figure 1.2: A $P_2$-design on the bowtie subgraph on vertices $u_{i-1}$, $v_{i-1}$, $u_i$, $v_i$, and $z$

Case 2: $s$ is even and $t$ is odd. We add three vertices, $z_1$, $z_2$, and $z_3$, to form a $K_{n+3}$. We pair the isolated edges as in the previous case, forming bowtie graphs centered at $z_1$. Let $\mathscr{C}_1 = \left\{ u_{i-1}\, z_1\, v_i,\ u_{i-1}\, v_{i-1}\, z_1,\ z_1\, u_i\, v_i \,\middle|\, i \in [\![1,s]\!] \text{ and } i \text{ is even} \right\}$. We then form paths from $u_i$ to $v_i$ centered at $z_2$ and at $z_3$; we let $\mathscr{C}_2 = \left\{ u_i\, z_2\, v_i \;\middle|\; i \in [\![1, s]\!] \right\}$ and $\mathscr{C}_3 = \left\{ u_i\, z_3\, v_i \;\middle|\; i \in [\![1, s]\!] \right\}$. We then pair all but one of the isolated vertices, omitting $w_t$, and form three paths for each pair: one centered at $z_1$, one centered at $z_2$, and one centered at $z_3$. We let $\mathscr{D}_1 = \left\{ w_{j-1}\, z_q\, w_j \;\middle|\; q \in \{1, 2, 3\} \text{ and } j \in [\![1, t]\!] \text{ and } j \text{ is even} \right\}$. The subgraph of $K_{n+3}$ induced by the vertex set $\{w_t, z_1, z_2, z_3\}$ is a $K_4$-subgraph of $K_{n+3}$, which admits the $P_2$-design $\mathscr{D}_2 = \{z_1\, w_t\, z_2,\ z_1\, z_2\, z_3,\ z_1\, z_3\, w_t\}$. Now we have a $P_2$-design on $K_{n+3}$: the design $\mathscr{B}^* \cup \mathscr{C}_1 \cup \mathscr{C}_2 \cup \mathscr{C}_3 \cup \mathscr{D}_1 \cup \mathscr{D}_2$ is an embedding of $\mathscr{B}$ in a complete $P_2$-design on $K_{n+3}$.

CASE 3: $s$ is odd. We add two vertices, $z_1$ and $z_2$, to form a $K_{n+2}$. We pair all but one of the isolated edges, omitting $e_s$, and form bowtie graphs centered at $z_1$, as in the previous case. Let $\mathscr{C}_1 = \left\{ u_{i-1}\, z_1\, v_i,\ u_{i-1}\, v_{i-1}\, z_1,\ z_1\, u_i\, v_i \;\middle|\; i \in [\![1, s]\!] \text{ and } i \text{ is even} \right\}$. We then form paths from $u_i$ to $v_i$ centered at $z_2$ and at $z_3$ for all $i < s$; we let $\mathscr{C}_2 = \left\{ u_i\, z_2\, v_i \;\middle|\; i \in [\![1, s-1]\!] \right\}$ and $\mathscr{C}_3 = \left\{ u_i\, z_3\, v_i \;\middle|\; i \in [\![1, s-1]\!] \right\}$. The subgraph of $K_{n+3}$ induced by the vertex set $\{u_s, v_s, z_1, z_2\}$ is a $K_4$-subgraph of $K_{n+3}$, which admits the $P_2$-design $\mathscr{C}_4 = \{z_1\, u_s\, v_s,\ v_s\, z_1\, z_2,\ u_s\, z_2\, v_s\}$. For each isolated vertex $w_j$, we form a path from $z_1$ to $z_2$ centered at $w_j$; we let $\mathscr{D} = \left\{ z_1\, w_j\, z_2 \;\middle|\; j \in [\![1, t]\!] \right\}$. Then the design $\mathscr{B}^* \cup \mathscr{C}_1 \cup \mathscr{C}_2 \cup \mathscr{C}_3 \cup \mathscr{C}_4 \cup \mathscr{D}$ is an embedding of $\mathscr{B}$ in a complete $P_2$-design on $K_{n+2}$.

We have shown that, for any $n \in \mathbb{P}$, every partial $G$-design of order $n$ can be embedded in a complete $G$-design of order $n+x$ for some $x \in [\![1, 3]\!]$; therefore $P_2$ is a bounded embedding graph. ■

The examples we have seen thus far are of graphs that are both bounded complete embedding graphs and bounded embedding graphs; this is not, however, the general pattern concerning the two questions we have asked, as the converse of Remark 1.29 is false. We offer $G = C_4$ as a counterexample.

**Example 1.33.** In Section 1.5, we see that all even cycles are bounded complete embedding graphs; this result is stated as Corollary 1.43. Thus $C_4$ is a bounded complete embedding graph.

We claim that $C_4$ is not a bounded embedding graph. To show this, we will exhibit, for each $n \in \mathrm{Spec}(C_4)$, a partial $C_4$-design $\mathscr{C}_n$ of order $2n$ such that any embedding of $\mathscr{C}_n$ in a complete $G$-design of order $2n+t$ requires that $\binom{t}{2} \geq n$.

Let $n \in \mathrm{Spec}(C_4)$, and let $\mathscr{B}$ be a complete $G$-design of order $n$. Let $V(K_n) = [\![1, n]\!]$, and let $V(K_{2n}) = \left\{ u_i, v_i \;\middle|\; i \in [\![1, n]\!] \right\}$. For each $C_4$-block $B = (a, b, c, d) \in \mathscr{B}$, we define a corresponding set of four $C_4$-subgraphs of $K_{2n}$:

$$\mathscr{C}(B) = \left\{ (u_a, u_b, u_c, u_d),\ (v_a, v_b, v_c, v_d),\ (u_a, v_b, u_c, v_d),\ (v_a, u_b, v_c, u_d) \right\}.$$

These cycles are shown in Figure 1.3.

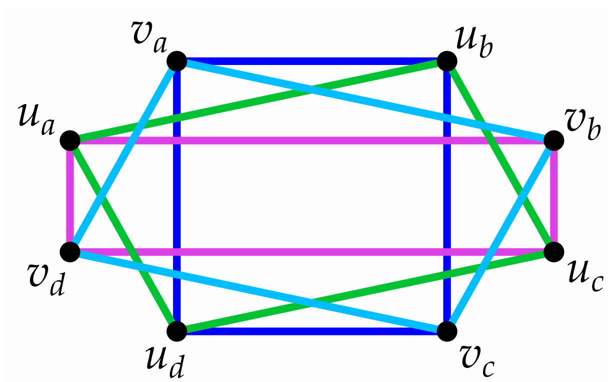



Figure 1.3: Four $C_4$-blocks corresponding to $(a, b, c, d) \in \mathscr{B}$

Let $\mathscr{C}_n = \bigcup_{B \in \mathscr{B}} \mathscr{C}(B)$. Then $\mathscr{C}_n$ is a partial $C_4$-design on $K_{2n}$, and the leave of $\mathscr{C}_n$ is the subgraph induced by the perfect matching $\left\{ \{u_i, v_i\} \;\middle|\; i \in [\![1, n]\!] \right\}$. Note that this subgraph consists of $n$ isolated edges.

By Theorem 1.26, there is some $t \in \mathbb{P}$ such that there is an embedding of $\mathscr{C}_n$ in a complete $C_4$-design of order $2n+t$. Let $V(K_{2n+t}) = \left\{ u_i, v_i \;\middle|\; i \in [\![1, n]\!] \right\} \cup \left\{ w_r \;\middle|\; r \in [\![1, t]\!] \right\}$, and let $\mathscr{D}$ be an embedding of $\mathscr{C}_n$ in a complete $C_4$-design on $K_{2n+t}$. For each $i \in [\![1, n]\!]$, there is a $C_4$-block $D_i \in \mathscr{D} - \mathscr{C}_n$ such that $\{u_i, v_i\} \in E(D_i)$. Fix $i \in [\![1, n]\!]$, and note that $V(D_i)$ cannot contain $u_j$ for any $j \in [\![1, n]\!]$ such that $j \neq i$, since this would require either $\{u_i, u_j\} \in E(D_i)$ or $\{v_i, u_j\} \in E(D_i)$, but the edges $\{u_i, u_j\}$ and $\{v_i, u_j\}$ are already in some block of $\mathscr{C}_n$. Similarly, $V(D_i)$ cannot contain $v_j$ for any $j \in [\![1, n]\!]$ such that $j \neq i$. Hence $V(D_i) = \{u_i, v_i, w_r, w_s\}$ for some distinct $r, s \in [\![1, t]\!]$, and therefore $\{w_r, w_s\} \in E(D_i)$. Since each block $D_i$ has exactly one edge in the $K_t$-subgraph on vertex set $W = \left\{ w_r \;\middle|\; r \in [\![1, t]\!] \right\}$, and since no two distinct blocks can contain the same edge, there must be at least as many edges in the $K_t$-subgraph on $W$ as there are blocks $D_i$. Thus $e(K_t) \geq \left| [\![1, n]\!] \right|$, that is, $\binom{t}{2} \geq n$.

We now see that $C_4$ cannot be a bounded embedding graph. For any positive integer $c$, let $N(c)$ be the smallest element of $\mathrm{Spec}(C_4)$ that is strictly greater than $\binom{c}{2}$; note that the partial $C_4$-design $\mathscr{C}_{N(c)}$ has order $2N(c)$. If $t \in \mathbb{P}$ such that $\mathscr{C}_{N(c)}$ can be embedded in a complete $C_4$-design of order $2N(c) + t$, then $\binom{t}{2} \geq 2N(c) > \binom{c}{2}$, so $t > c$. Hence $\mathscr{C}_{N(c)}$ cannot be embedded in a complete $C_4$-design of order $2N(c) + x$ for any $x \in [\![1, c]\!]$. ■

We now establish two parallel results regarding bounded complete embedding graphs and bounded embedding graphs; these results allow us to ignore finitely many admissible design orders in the process of showing that a graph is a bounded complete embedding graph or a bounded embedding graph.

**Theorem 1.34.** *Let $G$ be a graph. Suppose that there exist $N, b \in \mathbb{P}$ such that, for all $n \in \mathrm{Spec}(G)$ such that $n \geq N$, every complete $G$-design of order $n$ can be embedded in a complete $G$-design of order $n+x$, for some integer $x \in [\![1, b]\!]$. Then $G$ is a bounded complete embedding graph.*

*Proof.* Let $G$ be a graph, and suppose that there exist $N, b \in \mathbb{P}$ such that, for all $n \in \mathrm{Spec}(G)$ such that $n \geq N$, every complete $G$-design of order $n$ can be embedded in a complete $G$-design of order $n + x$, for some integer $x \in [\![1, b]\!]$. Let $\alpha$ denote the number of elements in the set $S = \left\{ n \in \mathrm{Spec}(G) \;\middle|\; n < N \right\}$, and note that $\alpha \leq N - 1$; denote the elements of $S$ by

$n_1, n_2, \ldots n_\alpha$. By Theorem 1.25, there exist positive integers $v(n_i)$ for $1 \leq i \leq \alpha$ such that every complete $G$-design of order $n_i$ can be embedded in a complete $G$-design of order $v(n_i)$. Let

$$\mu = \max\Big\{v(n_1) - n_1,\, v(n_2) - n_2,\, \ldots,\, v(n_\alpha) - n_\alpha,\, b\Big\}.$$

Then, for all $n \in \mathrm{Spec}(G)$, every complete $G$-design of order $n$ can be embedded in a complete $G$-design of order $n + x$ for some positive integer $x \leq \mu$; thus $G$ is a bounded complete embedding graph. □

**Theorem 1.35.** *Let $G$ be a graph. Suppose that there exist $N, c \in \mathbb{P}$ such that, for all $n \in \mathbb{P}$ such that $n \geq N$, every partial $G$-design of order $n$ can be embedded in a complete $G$-design of order $n + x$, for some integer $x \in [\![1,\, c]\!]$. Then $G$ is a bounded embedding graph.*

*Proof.* Let $G$ be a graph, and suppose that there exist $N, c \in \mathbb{P}$ such that, for all $n \in \mathbb{P}$ such that $n \geq N$, every partial $G$-design of order $n$ can be embedded in a complete $G$-design of order $n + x$, for some integer $x \in [\![1,\, c]\!]$. By Theorem 1.26, there exist positive integers $v(n)$ for $1 \leq n \leq N - 1$ such that every partial $G$-design of order $n$ can be embedded in a complete $G$-design of order at most $v(n)$. Let

$$\mu = \max\Big\{v(1) - 1,\, v(2) - 2,\, \ldots,\, v(N-1) - N + 1,\, c\Big\}.$$

Then, for all $n \in \mathbb{P}$, every partial $G$-design of order $n$ can be embedded in a complete $G$-design of order $n + x$ for some positive integer $x \leq \mu$; thus $G$ is a bounded embedding graph. □

In the next section, we describe the known significant results on bounded embedding graphs and bounded complete embedding graphs. These results appear in the literature without mention of the terms bounded embedding graph or bounded complete embedding graph; one result even appears without any mention of embeddings. The value of these results to our investigation is easily recognized, however, with only a few observations.

## 1.5 Existing Significant Results

We are aware of only two significant results on the identification of bounded embedding graphs. The first is due to D. Roberts and D. G. Hoffman [17].

**Theorem 1.36** (Roberts and Hoffman, 2014)**.** *Let $k \in \mathbb{P}$. If $k$ is odd, then a partial $S_k$-design of order $n$ can be embedded in a complete $S_k$-design of order $n + x$ for some positive integer $x \leq 7k - 4$. If $k$ is even, then a partial $S_k$-design of order $n$ can be embedded in a complete $S_k$-design of order $n + x$ for some positive integer $x \leq 8k - 4$.*

This result is precisely what is required to show that $k$-stars are bounded embedding graphs, so Roberts and Hoffman have achieved the following result.

**Corollary 1.37.** *For each $k \in \mathbb{P}$, the $k$-star is a bounded embedding graph.*

A result by T. R. Whitt and C. A. Rodger [34] identifies another bounded embedding graph.

**Theorem 1.38** (Whitt and Rodger)**.** *Let $n, r \in \mathbb{P}$ such that $r \geq n + 2$. A partial $P_3$-design of order $n$ can be embedded in a $P_3$-design of order $r$ if and only if $r \equiv 0$ or $1 \pmod 3$ and $r \geq 4$.*

This result guarantees that, for any $n \in \mathbb{P}$, any partial $P_3$-design of order $n$ can be embedded in a complete design of order $n+x$ for some $x \in \{2, 3\}$. Hence Whitt and Rodger have achieved the following result.

**Corollary 1.39.** *The graph $P_3$ is a bounded embedding graph.*

Whitt and Rodger also show that a partial $P_3$-design of order $n$ may be embedded in a design of order $n + 1$ under certain conditions, which include a lower bound on the number of edges in the leave; see [34].

In addition to the above results on bounded embedding graphs, the literature affords a single result for bounded complete embedding graphs. A result by C. A. Rodger [27] is the critical ingredient in showing that the graph $C_{2k}$ is a bounded complete embedding graph for all positive integers $k \geq 2$.

**Theorem 1.40** (Rodger, 1990)**.** *Let $k \in \mathbb{P}$ such that $k \geq 2$. For each integer $a \in [\![2k+1,\, 6k]\!]$, if there exists a $C_{2k}$-design on $K_a$, then there exists a $C_{2k}$-design on $K_{a+4kx}$ for all positive integers $x$.*

An examination of the proof of this result reveals that the design constructions are actually accomplished by building embeddings of smaller designs. In particular, Rodger builds, for each $a \in \mathrm{Spec}(C_{2k})$ such that $2k+1 \leq a \leq 6k$ and each positive integer $x$, an embedding of an arbitrary complete $C_{2k}$-design of order $a+4k(x-1)$ in a complete $C_{2k}$-design of order $a+4kx$. So Rodger's construction in fact establishes the following result.

**Corollary 1.41.** *Let $k \in \mathbb{P}$ such that $k \geq 2$. For each integer $a \in \mathrm{Spec}(C_{2k})$ such that $2k+1 \leq a \leq 6k$ and each $x \in \mathbb{N}$, there is an embedding of every complete $C_{2k}$-design of order $n = a + 4kx$ in a complete $C_{2k}$-design of order $n + 4k = a + 4k(x+1)$.*

Note that the above corollary does not address the the (trivial) complete $C_{2k}$-design of order 1; since this design can clearly be embedded in any larger $C_{2k}$-design, we must show that all other orders in the spectrum of $C_{2k}$ are addressed in the above corollary. In Theorem 1.42, we state the spectrum of $C_{2k}$, which appears in a 2002 paper by M. Šajna [31]. Šajna's paper is the culmination of work by numerous authors in several papers published over a 38-year period. This work was begun by Kotzig [20] and Rosa [28] and completed by Alspach and Gavlas [2] and Šajna [31]. (See also [5], [3], [27], [4], [23], and [30].)

**Theorem 1.42** (Šajna, 2002)**.** *Let $k, n \in \mathbb{P}$, and suppose $k \geq 2$. There is a complete $C_{2k}$-design of order $n$ if and only if the following conditions are satisfied.*

(*i*) $n = 1$ *or* $n \geq 2k$

(*ii*) $4k \mid n(n-1)$

(*iii*) $2 \mid (n-1)$

From Theorem 1.42, we can see how to describe $\mathrm{Spec}(C_{2k})$ in terms of congruence classes modulo $4k$. For any $n \in \mathrm{Spec}(C_{2k})$, we see from condition (iii) that $n$ is odd, and from conditions (i) and (iii) that $n \notin [\![2,\, 2k]\!]$. Consider the prime factorization of the product $4k$; let $s$ denote the number of distinct odd prime factors of $4k$. If $s = 0$, then $4k = 2^\alpha$ for some integer $\alpha \geq 3$, so by condition (ii) and the fact that $n$ is odd, we must have $2^\alpha \big| (n-1)$, so $n \equiv 1 \pmod{4k}$ for all $n \in \mathrm{Spec}(C_{2k})$. If instead $s > 0$, then there exist $s$ distinct odd primes $q_1, q_2, \ldots, q_s$ and $s+1$ positive integers $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_s$ such that

$$4k = 2^{\alpha_0} q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_s^{\alpha_s} \,. \tag{1.4}$$

Then, by condition (ii) and the fact that $n$ is odd, we must have that $2^{\alpha_0} \,\Big|\, (n-1)$ and that, for each $i \in [\![1, s]\!]$, $q_i^{\alpha_i} \,\Big|\, n(n-1)$, so either $q_i^{\alpha_i} \,\Big|\, n$ or $q_i^{\alpha_i} \,\Big|\, (n-1)$. Hence every element of $\mathrm{Spec}(C_{2k})$ is a solution to the system of $(s+1)$ congruences in (1.5) below.

$$\begin{cases} n \equiv 1 \left(\mathrm{mod}\ 2^{\alpha_0}\right) & \\ n(n-1) \equiv 0 \left(\mathrm{mod}\ q_i^{\alpha_i}\right) & \text{for each } i \in [\![1,\, s]\!] \end{cases} \tag{1.5}$$

There are exactly $2^s$ congruence classes modulo $4k$ that are solutions to this system. Each one of these classes has exactly one representative among the $4k$ consecutive integers in the set $[\![2k+1,\, 6k]\!]$; hence the integers $n = a + 4kx$ described in Corollary 1.41 are precisely the elements of $\mathrm{Spec}(C_{2k})$, except $n = 1$. We thus have the following corollary to Rodger's result.

**Corollary 1.43.** *Let $k \in \mathbb{P}$, and suppose $k \geq 2$. Then $C_{2k}$ is a bounded complete embedding graph.*

In subsequent chapters, we identify additional families of bounded complete embedding graphs, including paths, complete bipartite graphs, and certain graphs whose components are even cycles. Before we proceed with these identifications, we pause to discuss two new topics and to state a few known results from these topics that we will use later.

Chapter 2

# Known Results on Graph Labelings and Cyclic Designs

In this chapter, we divert ourselves from direct discussion of bounded complete embedding graphs to explore results from the literature that will be helpful to us later. These results all pertain to two closely related topics, namely cyclic methods for generating designs and graph labelings. We begin with the numerous definitions that are required for entry into these topics, and close the chapter with those results that are most important to us. We add, for emphasis, that all results mentioned in this chapter are the work of others.

## 2.1 Cyclic Designs on Complete Graphs and Complete Bipartite Graphs

In the process of building design embeddings, we build many designs on complete graphs and complete bipartite graphs. In this section, we present terminology and notation for the construction of certain designs, including the *difference* of an edge in $K_n$, the *length* of an edge in $K_{n,n}$, the process of *clicking* edges and subgraphs of $K_n$ and $K_{n,n}$, and *cyclic designs* on $K_n$ and $K_{n,n}$.

**Definition 2.1.** Let $n$ be a positive integer, and let $V(K_n) = [\![0, n-1]\!]$. Let $i, j \in V(K_n)$, with $i \neq j$. The ***difference*** of the edge $e = \{i, j\}$, denoted $\mathrm{diff}(e)$ or $\mathrm{diff}(i, j)$, is given by

$$\mathrm{diff}(i,j) = \min\left\{ \left|j-i\right|,\ n - \left|j-i\right| \right\}.$$

The edge $e$ is called a ***wrap-around edge*** if and only if $\mathrm{diff}(e) = n - \left|j-i\right|$. ■

If we arrange the vertices of $K_n$ in order as the vertices of a regular $n$-gon, then the difference of the edge $\{i, j\}$ is the length of the shorter path from $i$ to $j$ along the perimeter of the regular $n$-gon. The possible differences in $K_n$ are the integers from 1 to $\lfloor n/2 \rfloor$.

**Notation 2.2.** We denote by $\mathcal{D}_n$ the set of available differences in $K_n$:

$$\mathcal{D}_n = \left\{ i \in \mathbb{Z} \,\middle|\, 1 \leq i \leq \lfloor n/2 \rfloor \right\} = \left[\!\left[ 1, \lfloor n/2 \rfloor \right]\!\right]. \qquad \blacksquare$$

The *length* of an edge in $K_{n,n}$ is analogous to the difference of an edge in $K_n$; we use different terms for these two ideas for added clarity in distinguishing designs on $K_n$ from designs on $K_{n,n}$.

**Definition 2.3.** Let $n$ be a positive integer, and let $V(K_{n,n}) = [\![0, n-1]\!] \times \{0,1\}$. Let $i, j \in [\![0, n-1]\!]$. The ***length*** of the edge $e = \{(i,0), (j,1)\}$, denoted $\operatorname{Lnth}(e)$ or $\operatorname{Lnth}(i,j)$, is given by

$$\operatorname{Lnth}(i,j) = \begin{cases} j - i, & \text{if } i \leq j \\ n + j - i, & \text{if } i > j \end{cases}.$$

The edge $e$ is called a ***wrap-around edge*** if and only if $i > j$. $\blacksquare$

Note that, in $K_n$, the edges $\{i, j\}$ and $\{i+1, j+1\}$ (with addition computed modulo $n$) have the same difference, and that, in $K_{n,n}$, the edges $\{(i,0), (j,1)\}$ and $\{(i+1,0), (j+1,1)\}$ (with addition computed modulo $n$) have the same length. The process of *clicking* a subgraph provides a means to use these facts to advantage in the construction of designs.

**Definition 2.4.** The process of ***clicking*** an edge $\{i, j\}$ of $K_n$ is the increase of each vertex label by one to obtain the edge $\{i+1, j+1\}$, with vertex labels computed modulo $n$. The process of ***clicking*** an edge $\{(i,0), (j,1)\}$ of $K_{n,n}$ is the increase of the first coordinate of each vertex label by one to obtain the edge $\{(i+1,0), (j+1,1)\}$, with vertex labels computed modulo $n$. The process of ***clicking*** a subgraph, $G$, of $K_n$ or $K_{n,n}$ is simultaneously clicking all the edges of $G$. $\blacksquare$

Note that clicking, when applied to the entire graph $K_n$ or $K_{n,n}$, is an automorphism of $K_n$ or $K_{n,n}$, respectively. We use the clicking process to build designs by clicking a $G$-block to obtain other $G$-blocks for the design. In some cases, we build a design by clicking a single $G$-block $B$ repeatedly to obtain all the other blocks in the design. In such designs, the original block is usually called a *base block* for the design.

**Definition 2.5.** Let $n$ be a positive integer, let $K \in \{K_n, K_{n,n}\}$, and let $G$ be a subgraph of $K$. Let $\mathscr{B}$ be a $G$-design on $K$, and let $s = |\mathscr{B}|$. The design $\mathscr{B}$ is called a ***cyclic*** design if and only if the clicking automorphism is a permutation of $\mathscr{B}$. The design $\mathscr{B}$ is called a ***purely cyclic*** design if and only if the clicking automorphism is an $s$-cycle of $\mathscr{B}$. ■

Much theory has been developed in the pursuit of cyclic designs and especially purely cyclic designs. In the next section, we describe a family of graph labelings that were developed as tools for building such designs.

### 2.2 Graph Labelings and Cyclic Designs

In his 1967 paper, A. Rosa developed four types of *labelings*, or *valuations*, of a graph that are helpful in building cyclic designs [29]. Since then, several authors have defined variations on these labelings. We refer the reader to the 2009 survey by S. El-Zanati and C. Vanden Eynden [13] for further reading; we will focus on the variations called *ordered labelings*, which were developed by El-Zanati and Vanden Eynden in a series of several papers (see, for example, [11], [12], and [6]). In this section, we give definitions for Rosa's four original labelings and for ordered labelings. We also state several results from the literature that establish connections between the existence of certain types of labelings of a graph $G$ and the existence of certain $G$-designs.

**Definition 2.6.** A ***labeling***, or ***valuation***, of a graph $G$ is an injection $f : V(G) \to X$, where $X \subseteq \mathbb{N}$. For any vertex $v \in V(G)$, the number $f(v)$ is called the ***value*** of the vertex $v$; for any edge $e = \{u, v\} \in E(G)$, the number $f^*(u, v)$, defined by $f^*(u, v) = |f(u) - f(v)|$, is called the (***induced***) ***value*** of the edge $e$. We denote by $\mathscr{L}(V)$ and $\mathscr{L}(E)$, respectively, the sets of values of the vertices and edges of $G$, that is,

$$\mathscr{L}(V) = \{\, f(v) \mid v \in V(G) \,\} \quad \text{and} \quad \mathscr{L}(E) = \{\, f^*(u, v) \mid \{u, v\} \in E(G) \,\}. \qquad ■$$

### 2.2.1 Restrictions on Graph Labelings

We give several conditions on labelings, which we then use to define the various labeling types. We name these conditions for convenient reference; note that these names are not standard in the literature, and that the conditions are often stated in other equivalent forms. There are six conditions total, which provide weak and strong options for each of three possible restrictions on a labeling. For brevity in the statements below, we let $m = e(G)$.

(I) The first restriction concerns the codomain, $X$, which is the set of available values for the vertices. Both conditions assign $X$ to be a subset of $\mathbb{Z}_{2m+1}$; the weak condition assigns the entire set, while the strong condition assigns a proper subset.

***Weak Codomain Condition*** (WCDC): $X = [\![0, 2m]\!]$

***Strong Codomain Condition*** (SCDC): $X = [\![0, m]\!]$

(II) The second restriction concerns the set $\mathscr{L}(E)$ of induced values on the edges. Both conditions require that no two edges have the same induced value.

***Weak Edge Value Condition*** (WEVC):

For each $i \in [\![1, m]\!]$, there is a unique edge $\{u, v\} \in E(G)$ such that

$$\min\Big\{f^*(u,v),\ (2m+1) - f^*(u,v)\Big\} = i\,.$$

***Strong Edge Value Condition*** (SEVC):

For each $i \in [\![1, m]\!]$, there is a unique edge $\{u, v\} \in E(G)$ such that $f^*(u, v) = i$.

(III) The third restriction is only defined when $G$ is bipartite: we may require that the values of the vertices be ordered in some way with respect to the bipartition of the vertex set. Suppose $G$ is bipartite on bipartition $[A, B]$.

***Weak Ordering Condition*** (WOC):

For each edge $\{a, b\} \in E(G)$ (with $a \in A$ and $b \in B$), $f(a) < f(b)$.

***Strong Ordering Condition*** (SOC):

There exists $\lambda \in \mathbb{Z}$ such that $f(a) \leq \lambda$ for all $a \in A$ and $f(b) > \lambda$ for all $b \in B$.

As previously mentioned, Rosa's four original labeling types were created to facilitate the building of cyclic designs on complete graphs; some of the labeling types we present are also useful in producing cyclic designs on complete bipartite graphs with both parts of the same size. We observe that, when vertex values are chosen from $\mathbb{Z}_{2m+1}$ (as required by both of the codomain conditions above), we may interpret a labeling of the graph $G$ as the identification of a $G$-subgraph of $K_{2m+1}$: the value of a vertex in $G$ gives the corresponding vertex of $K_{2m+1}$ in the $G$-subgraph. Under this interpretation, the difference of an edge $\{u, v\}$ of $G$ is the difference of the corresponding edge $\{f(u), f(v)\}$ in the $G$-subgraph of $K_{2m+1}$, which is

$$\begin{aligned}\operatorname{diff}\Big(f(u),\, f(v)\Big) &= \min\Big\{\big|f(u) - f(v)\big|,\, (2m+1) - \big|f(u) - f(v)\big|\Big\} \\ &= \min\Big\{f^*(u,v),\, (2m+1) - f^*(u,v)\Big\}.\end{aligned}$$

With this perspective, we see that the Weak Edge Value Condition requires that the $G$-subgraph of $K_{2m+1}$ have exactly one edge of each difference $d \in \mathcal{D}_{2m+1} = [\![1, m]\!]$. The Strong Edge Value Condition also requires this, with the extra restriction that wrap-around edges are forbidden.

### 2.2.2 Nine Graph Labeling Types

We now present the definitions of the labeling types. We begin with the three labeling types that do not include an ordering condition; these are three of Rosa's four original labeling types. Note that graphs that are not bipartite may admit these labelings.

**Definition 2.7.** Let $G$ be a graph, and let $m = e(G)$. A labeling, or valuation, of $G$ is called a ***$\rho$-labeling***, or ***$\rho$-valuation***, of $G$ if and only if the Weak Codomain Condition and the Weak Edge Value Condition hold. ■

**Definition 2.8.** Let $G$ be a graph, and let $m = e(G)$. A labeling, or valuation, of $G$ is called a ***σ-labeling***, or ***σ-valuation***, of $G$ if and only if the Weak Codomain Condition and the Strong Edge Value Condition hold. ■

**Definition 2.9.** Let $G$ be a graph, and let $m = e(G)$. A labeling, or valuation, of $G$ is called a ***β-labeling***, or ***β-valuation***, of $G$ if and only if the Strong Codomain Condition and the Strong Edge Value Condition hold. ■

We may impose either the Weak Ordering Condition or the Strong Ordering Condition on any of the three labeling types we have just defined, creating six additional labeling types. Since the ordering conditions are only defined on bipartite graphs, these six new labeling types are also defined only on bipartite graphs.

**Definition 2.10.** Let $G$ be a bipartite graph on bipartition $[A, B]$, and let $m = e(G)$. A labeling, or valuation, of $G$ is called an ***ordered labeling***, or ***ordered valuation***, of $G$ if and only if the Weak Ordering Condition holds. ■

**Notation 2.11.** We denote an ordered labeling by adding a superscripted plus sign (+) to the name of the labeling type:

- a $\rho^+$-labeling is an ordered $\rho$-labeling;
- a $\sigma^+$-labeling is an ordered $\sigma$-labeling; and
- a $\beta^+$-labeling is an ordered $\beta$-labeling. ■

**Definition 2.12.** Let $G$ be a bipartite graph on bipartition $[A, B]$, and let $m = e(G)$. A labeling, or valuation, of $G$ is called a ***uniformly ordered labeling***, or ***uniformly ordered valuation***, of $G$ if and only if the Strong Ordering Condition holds. ■

**Notation 2.13.** We denote a uniformly ordered labeling by adding a superscripted double plus sign (++) to the name of the labeling type:

- a $\rho^{++}$-labeling is a uniformly ordered $\rho$-labeling;

- a $\sigma^{++}$-labeling is a uniformly ordered $\sigma$-labeling; and
- a $\beta^{++}$-labeling is a uniformly ordered $\beta$-labeling. ■

Rosa's fourth labeling type is the $\alpha$-labeling, which is identical to the uniformly ordered $\beta$-labeling; such labelings are typically called $\alpha$-labelings in the literature, but occasionally the name *bipartite labeling* is used. We summarize the nine labeling types and their defining conditions in Table 2.1.

Table 2.1: The nine labeling types and their defining conditions

| Labeling Type | Codomain | Edge Values | Ordering |
|---|---|---|---|
| $\rho$ | WCDC | WEVC | — |
| $\rho^{+}$ | WCDC | WEVC | WOC |
| $\rho^{++}$ | WCDC | WEVC | SOC |
| $\sigma$ | WCDC | SEVC | — |
| $\sigma^{+}$ | WCDC | SEVC | WOC |
| $\sigma^{++}$ | WCDC | SEVC | SOC |
| $\beta$ | SCDC | SEVC | — |
| $\beta^{+}$ | SCDC | SEVC | WOC |
| $\alpha\ (\beta^{++})$ | SCDC | SEVC | SOC |

### 2.2.3 Obtaining Cyclic Designs from Graph Labelings

In his 1967 paper, Rosa proved two results that connect the existence of graph labelings to the existence of cyclic designs on certain complete graphs [29]. In a later paper with C. Huang, Rosa connects $\alpha$-labelings to the existence of cyclic designs on certain complete bipartite graphs [18]. We begin with these important results.

**Theorem 2.14** (Rosa, 1967)**.** *Let $G$ be a graph, and let $m = e(G)$. The graph $G$ admits a $\rho$-labeling if and only if there is a purely cyclic $G$-design on $K_{2m+1}$.*

**Theorem 2.15** (Rosa, 1967)**.** *Let $G$ be a bipartite graph on bipartition $[A, B]$; let $m = e(G)$. If $G$ admits an $\alpha$-labeling, then there exists a cyclic $G$-design on $K_{2mx+1}$ for all $x \in \mathbb{P}$.*

**Theorem 2.16** (Huang and Rosa, 1978)**.** *Let $G$ be a bipartite graph on bipartition $[A, B]$; let $m = e(G)$. If $G$ admits an $\alpha$-labeling, then there exists a purely cyclic $G$-design on $K_{m,m}$.*

Since the $\rho$-labeling is the least restrictive labeling, all of the labelings we have defined are also $\rho$-labelings; hence, if a graph $G$ (having $m$ edges) admits a labeling of any of these nine types, then there is a purely cyclic $G$-design on $K_{2m+1}$, by Theorem 2.14.

Note that the $\alpha$-labeling is the only one of Rosa's original labelings that includes an ordering condition. El-Zanati, Vanden Eynden, and their co-authors have shown that the designs in Theorems 2.15 and 2.16 may be obtained from less restrictive labelings; note, in particular, that the Strong Ordering Condition may be relaxed to the Weak Ordering Condition in both cases. The following results are proved in [12] and [11].

**Theorem 2.17** (El-Zanati, Vanden Eynden, and Punnim, 2001)**.** *Let $G$ be a bipartite graph on bipartition $[A, B]$, and let $m = e(G)$. If $G$ admits an ordered $\rho$-labeling, then there exists a cyclic $G$-design on $K_{2mx+1}$ for all $x \in \mathbb{P}$.*

**Theorem 2.18** (El-Zanati, Kenig, and Vanden Eynden, 2000)**.** *Let $G$ be a bipartite graph on bipartition $[A, B]$, and let $m = e(G)$. If $G$ admits an ordered $\beta$-labeling, then there exists a purely cyclic $G$-design on $K_{m,m}$.*

### 2.3 Existence Results for Labelings of Certain Graphs

In this section, we present two known results on the existence of certain types of labelings of various graphs. The first result is a list of all graphs that are currently known to admit $\beta^+$-labelings; before we provide this list, we pause to define a few families of graphs, so that all items in the list are meaningful to the reader.

**Definition 2.19.** The ***base*** of a graph $G$ is the subgraph of $G$ obtained by deleting all vertices in $G$ of degree one. A ***caterpillar*** is a tree whose base is a path. ■

**Definition 2.20.** The ***comet*** $S_{k,m}$ is the graph obtained from the $k$-star $S_k$ by replacing each edge by a path with $m$ edges. ■

**Definition 2.21.** The ***cube*** $Q_n$ is the graph whose vertex set is the set of binary $n$-tuples, where two $n$-tuples are adjacent if and only if they differ in exactly one coordinate. ■

**Example 2.22.** A caterpillar and the comet $S_{4,3}$ are shown in Figure 2.1. The cube $Q_4$ is shown in Figure 2.2. ■

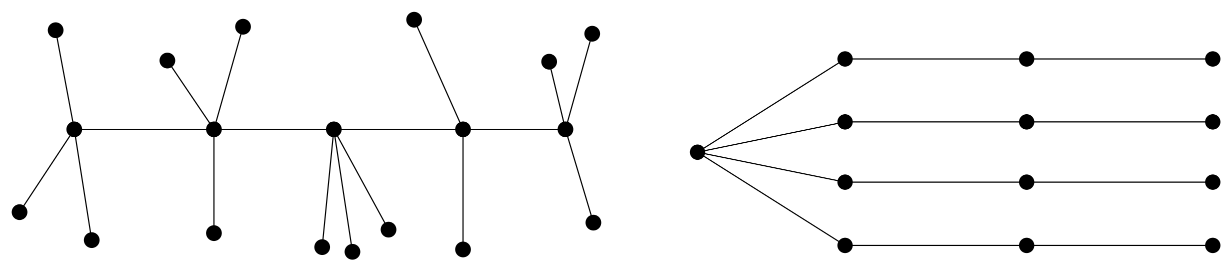


Figure 2.1: A caterpillar graph (left) and the comet $S_{4,3}$ (right)

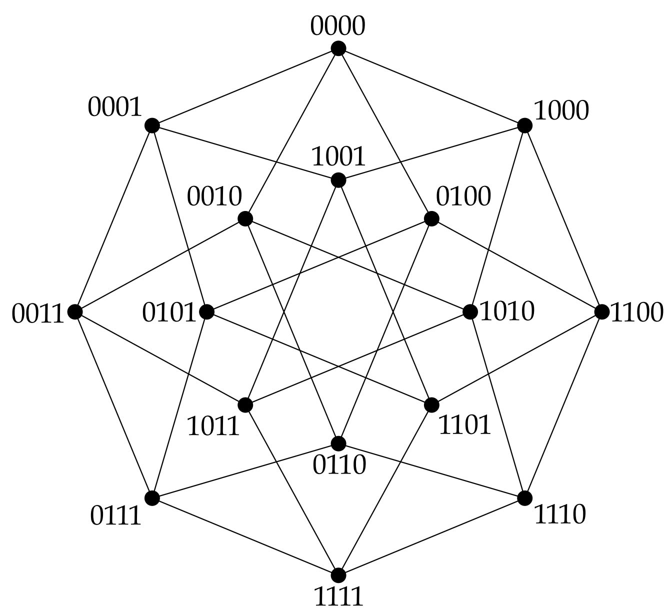



Figure 2.2: The cube $Q_4$

The list of graphs that admit $\beta^+$-labelings is found in the survey by El-Zanati and Vanden Eynden [13]; these results are the work of several authors. The *parity condition* mentioned in item (8) of Theorem 2.23 below is explained in [29] and [13].

**Theorem 2.23.** *The following graphs admit $\beta^+$-labelings.*

(1) *trees with at most 20 edges* [15]

(2) *the comets $S_{k,2}$, for all $k \in \mathbb{P}$* [11]

*The following graphs admit $\alpha$-labelings. Note that all $\alpha$-labelings are $\beta^+$-labelings.*

(3) *caterpillars* [29]

(4) *complete bipartite graphs* [29]

(5) *the cubes $Q_n$, for all $n \in \mathbb{P}$* [22]

(6) *the $4k$-cycles, for all $k \in \mathbb{P}$* [29]

(7) *the graphs $\biguplus_{i=1}^{r} C_4$, for all $r \in \mathbb{P}$ such that $r \neq 3$* [1]

(8) *all 2-regular bipartite graphs that have at most three components and satisfy the parity condition, except $C_4 \uplus C_4 \uplus C_4$* [14]

By Theorem 2.15, we have the following corollary to item (4) of Theorem 2.23 above.

**Corollary 2.24.** *Let $r, s \in \mathbb{P}$. Then there is a $K_{r,s}$-design on $K_{2rsx+1}$ for all $x \in \mathbb{P}$.*

The second result is a $\rho^+$-labeling result on 2-regular bipartite graphs by A. Blinco and S. El-Zanati [6]. In Chapters 4 and 5, we will be particularly interested in the 2-regular bipartite graphs in which all components have the same number of vertices.

**Theorem 2.25** (Blinco and El-Zanati, 2004)**.** *Let $G$ be a 2-regular bipartite graph. Then $G$ admits a $\rho^+$-labeling.*

**Corollary 2.26.** *Let $G$ be a 2-regular bipartite graph, and let $q = e(G)$. Then there is a $G$-design on $K_{2qx+1}$ for all $x \in \mathbb{P}$.*

Equipped with the language of graph labelings and the designs provided by the labeling results we have presented, we now return to the identification of bounded complete embedding graphs.

# Chapter 3

## A Few Bounded Complete Embedding Graphs

In this chapter, we present a few infinite families of bounded complete embedding graphs and some other results relevant to the identification problem. We begin the chapter with the fundamental result that all bounded complete embedding graphs are bipartite. In Section 3.1, we present two constructions for building embeddings. Construction I is used in Section 3.2; Construction II is used for all subsequent embedding constructions and is thus one of our most significant tools for bounded complete embedding graph results. In Sections 3.2 and 3.3, we show that paths and complete bipartite graphs, respectively, are bounded complete embedding graphs. We close the chapter with a result for bipartite graphs satisfying certain conditions; this last result is primarily informed by the graph labeling results in Chapter 2.

**Theorem 3.1.** *Let $G$ be a bounded complete embedding graph. Then $G$ is bipartite.*

*Proof.* Let $G$ be a graph, and suppose that $G$ is not bipartite. We will show that $G$ is not a bounded complete embedding graph.

Let $n \in \text{Spec}(G)$, and let $\mathscr{B}$ be a $G$-design of order $n$. By Theorem 1.25, let $t_n \in \mathbb{P}$ such that we can embed $\mathscr{B}$ in a $G$-design $\mathscr{D}$ of order $r = n + t_n$. Let $\mathscr{C}$ denote the set of $G$-blocks that are elements of $\mathscr{D}$ but not $\mathscr{B}$. Let $V(K_r) = [\![1, n + t_n]\!]$, with vertex names assigned so that $\mathscr{B}$ is a $G$-design on the $K_n$-subgraph of $K_r$ induced by the vertex set $A = [\![1, n]\!]$; let $H_1$ denote this subgraph of $K_r$. Let $B = [\![n + 1, n + t_n]\!]$, and let $H_2$ denote the $K_{t_n}$ subgraph of $K_r$ induced by $B$. Let $H_3$ denote the $K_{n,t_n}$-subgraph of $K_r$ on bipartition $[A, B]$. Note that $H_1$, $H_2$, and $H_3$ partition the set $E(K_r)$. Furthermore, note that edges of blocks in $\mathscr{C}$ can only be edges of $H_2$ and $H_3$, because $\mathscr{B}$ is a complete $G$-design on $H_1$.

Claim: Let $C \in \mathscr{C}$. If $C$ has an edge in $H_3$, then it must have at least one edge in $H_2$.

PROOF OF CLAIM: Let $C \in \mathscr{C}$ and suppose $C$ has an edge in $H_3$. Since $H_3$ is bipartite, $H_3$ has no $G$-subgraphs, because $G$ is not bipartite. Thus, since $C$ is a $G$-block, $C$ is not a subgraph of $H_3$, so it must have at least one edge that is not in $H_3$. Since $C$ cannot have an edge in $H_1$, $C$ must have at least one edge in $H_2$. ◊

Let $\mathscr{C}^*$ denote the set of $G$-blocks in $\mathscr{C}$ that have an edge in $H_3$. Since each of these blocks has at least one edge in $H_2$, we have $e(H_2) \geq \left|\mathscr{C}^*\right|$, and each such block has at most $e(G)-1$ edges in $H_3$. Since $e(H_3) = e(K_{n,t_n}) = nt_n$, we have that

$$\frac{t_n(t_n-1)}{2} = e(H_2) \geq \left|\mathscr{C}^*\right| \geq \frac{nt_n}{e(G)-1}\,. \tag{3.1}$$

Thus

$$t_n \geq \frac{2n}{e(G)-1} + 1\,. \tag{3.2}$$

Now, for any positive integer $b$, let $N(b)$ be the smallest element of $\mathrm{Spec}(G)$ that is strictly greater than $b \cdot \big(e(G)-1\big)$, and let $\mathscr{F}$ be a complete $G$-design of order $N(b)$. Then, for any $t \in \mathbb{P}$ such that $\mathscr{F}$ can be embedded in a complete $G$-design of order $n+t$, we have that

$$t \geq \frac{2N(b)}{e(G)-1} + 1 > 2b + 1\,, \tag{3.3}$$

so no complete $G$-design of order $N(b)$ can be embedded in a complete $G$-design of order $N(b)+x$ for any $x \in [\![1, b]\!]$. Therefore $G$ is not a bounded complete embedding graph. □

We have already seen that even cycles are bounded complete embedding graphs; since odd cycles are not bipartite, the above theorem guarantees that we need not consider them. Before we proceed to other families of bounded complete embedding graphs, we develop two essential tools, namely, the two embedding constructions we use to establish our results in this chapter and the next.

## 3.1 Constructions for Embeddings

In order to show that a particular graph is a bounded complete embedding graph, we must build embeddings of certain designs; in this section, we present two constructions for building these embeddings. The two constructions are quite similar; they are slight variations on the same idea. Many other constructions of a similar nature are possible, but these two constructions are, in many ways, the simplest of the possible variations. We have obtained all of our results on bounded complete embedding graphs using only these two constructions.

**Lemma 3.2** (Construction I)**.** *Let $G$ be a bipartite graph, and let $n \in \mathrm{Spec}(G)$ such that $n > 1$. Suppose that there is some positive integer $t$ such that there exist $G$-designs on $K_t$ and $K_{t,n}$. Then every $G$-design of order $n$ can be embedded in a $G$-design of order $n + t$.*

*Proof.* Let $G$ be a bipartite graph, and let $n \in \mathrm{Spec}(G)$ such that $n > 1$. Suppose that there is some positive integer $t$ such that there exist $G$-designs on $K_t$ and $K_{t,n}$.

We first identify three edge-disjoint subgraphs $H_1$, $H_2$, and $H_3$ of $K_{n+t}$, as follows. Partition $V(K_{n+t})$ into a set $A$ of $t$ vertices and a set $B$ of $n$ vertices. Let $H_1$ denote the $K_n$-subgraph induced by $B$. Let $H_2$ denote the $K_t$-subgraph induced by $A$. The remaining edges are the edges of a $K_{t,n}$-subgraph on bipartition $[A, B]$; let $H_3$ denote this subgraph. Note that these three subgraphs induce a partition of the edge set of $K_{n+t}$.

Since $n \in \mathrm{Spec}(G)$, let $\mathscr{B}$ be a $G$-design on $H_1$. By assumption, there is a $G$-design $\mathscr{D}$ on $H_2$, and there is a $G$-design $\mathscr{F}$ on $H_3$. Then $\mathscr{B} \cup \mathscr{D} \cup \mathscr{F}$ is an embedding of $\mathscr{B}$ in a complete $G$-design on $K_{n+t}$, as desired. □

**Lemma 3.3** (Construction II)**.** *Let $G$ be a bipartite graph, and let $n \in \mathrm{Spec}(G)$ such that $n > 1$. Suppose that there is some positive integer $t$ such that there exist $G$-designs on $K_{t+1}$ and $K_{t,n-1}$. Then every $G$-design of order $n$ can be embedded in a $G$-design of order $n + t$.*

*Proof.* Let $G$ be a bipartite graph, and let $n \in \mathrm{Spec}(G)$ such that $n > 1$. Suppose that there is some positive integer $t$ such that there exist $G$-designs on $K_{t+1}$ and $K_{t,n-1}$.

We first identify three edge-disjoint subgraphs $H_1$, $H_2$, and $H_3$ of $K_{n+t}$. Partition $V(K_{n+t})$ into a set $A$ of $t$ vertices and a set $B$ of $n$ vertices. Let $H_1$ denote the $K_n$-subgraph induced by $B$. Fix a vertex $b \in B$; let $H_2$ denote the $K_{t+1}$-subgraph induced by $A \cup \{b\}$. The remaining edges are the edges of a $K_{t,n-1}$-subgraph on bipartition $\big[A, (B - \{b\})\big]$; let $H_3$ denote this subgraph. Note that these three subgraphs induce a partition of the edge set of $K_{n+t}$.

Since $n \in \text{Spec}(G)$, let $\mathscr{B}$ be a $G$-design on $H_1$. By assumption, there is a $G$-design $\mathscr{D}$ on $H_2$, and there is a $G$-design $\mathscr{F}$ on $H_3$. Then $\mathscr{B} \cup \mathscr{D} \cup \mathscr{F}$ is an embedding of $\mathscr{B}$ in a complete $G$-design on $K_{n+t}$, as desired. □

### 3.2 Paths

In this section, we prove that paths are bounded complete embedding graphs. We appeal to Construction I (Lemma 3.2) in order to build the embeddings that we need; the constructions of these embeddings differ slightly according to the parity of the length, $r$, of the path. We begin by establishing the existence of several $P_r$-designs that are needed for these constructions.

We need a $P_r$-design on $K_t$ for an appropriate value of $t$; if $r$ is even, we use $t = 2r$; if $r$ is odd, we use $t = r + 1$. We appeal to the spectral result on $P_r$-designs on $K_n$ in order to establish the existence of the needed designs. This result is due to M. Tarsi [33]; we modify Tarsi's original statement to use the language of graph designs.

**Theorem 3.4** (Tarsi, 1983)**.** *Let $m$, $v$, and $\lambda$ be positive integers. There is a $P_m$-design on $\lambda K_v$ if and only if the following conditions are satisfied.*

$$(i) \qquad v = 1 \ \text{or} \ v \geq m + 1$$

$$(ii) \qquad \lambda v(v-1) \equiv 0 \pmod{2m}$$

We now apply Theorem 3.4 to obtain the desired $P_r$-designs. We address the design for paths of even length first.

**Lemma 3.5.** *Let $r$ be an even positive integer. There is a $P_r$-design on $K_{2r}$.*

*Proof.* Let $r$ be an even positive integer. We wish to apply Theorem 3.4, with $m = r$, $v = 2r$, and $\lambda = 1$; in order to do so, we must verify that the two conditions in the theorem hold for these values of $m$, $v$, and $\lambda$.

Condition (i): $2r \geq r + 1$.

Since $r$ is a positive integer, this is true.

Condition (ii): $(2r)(2r - 1) \equiv 0 \pmod{2r}$.

This is clearly true.

We have verified that both conditions hold; hence there is a $P_r$-design on $K_{2r}$. □

**Lemma 3.6.** *Let $r$ be an odd positive integer. There is a $P_r$-design on $K_{r+1}$.*

*Proof.* Let $r$ be an odd positive integer. We wish to apply Theorem 3.4, with $m = r$, $v = r + 1$, and $\lambda = 1$; in order to do so, we must verify that the two conditions in the theorem hold for these values of $m$, $v$, and $\lambda$.

Condition (i): $r + 1 \geq r + 1$.

This is obviously true.

Condition (ii): $(r + 1)(r) \equiv 0 \pmod{2r}$.

Since $r$ is odd, $r + 1$ is even; thus $(r + 1)(r)$ is a multiple of $2r$, so the condition holds.

We have verified that both conditions hold; hence there is a $P_r$-design on $K_{r+1}$. □

The embedding constructions require several path designs on complete bipartite graphs; as with the previous designs, the requirements differ slightly for paths of even length and paths of odd length. In order to establish the existence of the needed designs, we will appeal to the spectral result on path designs by C. Parker [26]. This theorem provides the exact spectrum of $P_k$-designs on $K_{a,b}$ in eight cases, according to the parity of $k$, $a$, and $b$.

**Theorem 3.7** (Parker, 1998)**.** *Let $k$, $a$, and $b$ be positive integers.*

(1) *Suppose $k$, $a$, and $b$ are all even. Then there is a $P_k$-design on $K_{a,b}$ if and only if $k \leq 2a$ and $k \leq 2b$, with strict inequality in at least one of these inequalities.*

(2) *Suppose $k$ and $a$ are even and $b$ is odd. Then there is a $P_k$-design on $K_{a,b}$ if and only if $k \leq 2a - 2$ and $k \leq 2b$.*

(3) *Suppose $k$ and $b$ are even and $a$ is odd. Then there is a $P_k$-design on $K_{a,b}$ if and only if $k \leq 2a$ and $k \leq 2b - 2$.*

(4) *Suppose $k$ is even and $a$ and $b$ are odd. Then there is no $P_k$-design on $K_{a,b}$.*

(5) *Suppose $k$ is odd and $a$ and $b$ are both even. Then there is a $P_k$-design on $K_{a,b}$ if and only if $k \leq 2a - 1$ and $k \leq 2b - 1$.*

(6) *Suppose $k$ and $b$ are odd and $a$ is even. Then there is a $P_k$-design on $K_{a,b}$ if and only if $k \leq 2a - 1$ and $k \leq b$.*

(7) *Suppose $k$ and $a$ are odd and $b$ is even. Then there is a $P_k$-design on $K_{a,b}$ if and only if $k \leq a$ and $k \leq 2b - 1$.*

(8) *Suppose $k$, $a$, and $b$ are all odd. Then there is a $P_k$-design on $K_{a,b}$ if and only if $k \leq a$ and $k \leq b$.*

**Lemma 3.8.** *Let $r$ be an even positive integer, and let $n \in \mathrm{Spec}(P_r)$ such that $n > 1$. There is a $P_r$-design on $K_{2r,n}$.*

*Proof.* Let $r$ be an even positive integer, and let $n \in \mathrm{Spec}(P_r)$ such that $n > 1$. We apply Theorem 3.7, with $k = r$, $a = n$, and $b = 2r$. Note that $k = r$ and $b = 2r$ are both even. If $n$ is even, we will appeal to case (1) of this theorem; if $n$ is odd, we will appeal to case (3).

Suppose first that $n$ is even. Since $n \geq r + 1$ by Theorem 3.4, we have that

$$k = r < r + 1 \leq n < 2n = 2a\,.$$

Since $r$ is a positive integer, $k = r < 2(2r) = 2b$. We have thus satisfied the conditions of Theorem 3.7, case (1), so there is a $P_r$-design on $K_{2r,\,n}$.

Now suppose $n$ is odd. As argued for $n$ even, $k = r \le 2n = 2a$. Since $r$ is a positive integer, $k = r < 2(2r) - 2 = 2b - 2$. We have thus satisfied the conditions of Theorem 3.7, case (3), so there is a $P_r$-design on $K_{2r,\,n}$. □

**Lemma 3.9.** *Let $r$ be an odd positive integer, and let $n \in \mathrm{Spec}(P_r)$ such that $n > 1$. There is a $P_r$-design on $K_{r+1,\,n}$.*

*Proof.* Let $r$ be an odd positive integer, and let $n \in \mathrm{Spec}(P_r)$ such that $n > 1$. We apply Theorem 3.7, with $k = r$, $a = n$, and $b = r + 1$. Note that $b = r + 1$ is even. If $n$ is even, we will appeal to case (5) of this theorem; if $n$ is odd, we will appeal to case (7).

Suppose first that $n$ is even. Since $n \ge r + 1$ by Theorem 3.4, we have that

$$k = r < r + 1 \le n < 2n - 1 = 2a - 1\,.$$

Since $r$ is a positive integer, $k = r < 2r + 1 = 2(r + 1) - 1 = 2b - 1$. We have thus satisfied the conditions of Theorem 3.7, case (5), so there is a $P_r$-design on $K_{r+1,\,n}$.

Now suppose $n$ is odd. Since $n \ge r + 1$ by Theorem 3.4, we have that

$$k = r < r + 1 \le n = a\,.$$

As argued for $n$ even, $k < 2b - 1$. We have thus satisfied the conditions of Theorem 3.7, case (7), so there is a $P_r$-design on $K_{r+1,\,n}$. □

We are now ready to prove the main result of this section: that paths are bounded complete embedding graphs. The proof is straightforward: we apply Construction I to build, for each $n \in \mathrm{Spec}(P_r)$, an embedding of any $P_r$-design of order $n$ in a $P_r$-design of order $n + t$, where $t = 2r$ for all even values of $r$ and $t = r + 1$ for all odd values of $r$.

**Theorem 3.10.** *Let $r$ be a positive integer. Then $P_r$ is a bounded complete embedding graph.*

*Proof.* We approach the proof in two parts, in order to handle separately the slightly different constructions required for paths of even and odd lengths.

Part I: Paths of Even Length

Let $r$ be a positive even integer, and consider $P_r$-designs. Clearly, the trivial complete $P_r$-design of order 1 can clearly be embedded in any larger $P_r$-design; in particular, it can be embedded in the $P_r$-design of order $2r$ guaranteed by Lemma 3.5.

By Lemma 3.5, there is a $P_r$-design on $K_{2r}$; by Lemma 3.8, there is a $P_r$-design on $K_{2r,\,n}$ for all $n \in \text{Spec}(P_r)$ such that $n > 1$. The conditions of Construction I (Lemma 3.2) with $t = 2r$ are thus satisfied for all $n \in \text{Spec}(P_r)$ such that $n > 1$.

Hence for all even positive integers $r$, every $P_r$-design of order $n$ can be embedded in a $P_r$-design of order $n + x$ for some $x \in [\![1,\, 2r]\!]$. Therefore $P_r$ is a bounded complete embedding graph for even $r$. □

Part II: Paths of Odd Length

Now let $r$ be a positive odd integer, and consider $P_r$-designs. Clearly, the trivial complete $P_r$-design of order 1 can clearly be embedded in any larger $P_r$-design; in particular, the $P_r$-design of order 1 can be embedded in the $P_r$-design of order $r + 1$ guaranteed by Lemma 3.6.

By Lemma 3.6, there is a $P_r$-design on $K_{r+1}$; by Lemma 3.9, there is a $P_r$-design on $K_{r+1,\,n}$ for all $n \in \text{Spec}(P_r)$ such that $n > 1$. The conditions of Construction I (Lemma 3.2) with $t = r + 1$ are thus satisfied for all $n \in \text{Spec}(P_r)$ such that $n > 1$.

Hence for all odd positive integers $r$, every $P_r$-design of order $n$ can be embedded in a $P_r$-design of order $n+x$ for some $x \in [\![1,\, (r+1)]\!]$. Therefore $P_r$ is a bounded complete embedding graph for odd $r$. □

### 3.3 Complete Bipartite Graphs

In this section, we prove that all complete bipartite graphs are bounded complete embedding graphs. We begin by analyzing the superspectral conditions as they apply to the complete bipartite graph $K_{r,s}$.

**Remark 3.11.** Let $r$, $s$, and $n$ be positive integers, and suppose there is a $K_{r,s}$-design on $K_n$. Note that $v(K_{r,s}) = r + s$, that $e(K_{r,s}) = rs$, and that all vertices in $K_{r,s}$ have degree either $r$ or $s$. Applying conditions SSC–1, SSC–2, and SSC–3, we have that

(1) $n = 1$ or $n \geq r + s$,

(2) $2rs \,\big|\, n(n-1)$, and

(3) $\gcd(r,s) \,\big|\, (n-1)$.

These conditions define the superspectrum of $K_{r,s}$. ■

We will appeal to Construction II (Lemma 3.3) to build the embeddings that we need. One of the designs we will need for Construction II is a $K_{r,s}$-design on $K_{2rs+1}$, whose existence is guaranteed by Corollary 2.24. The other designs required for Construction II are $K_{r,s}$-designs on $K_{2rs,n-1}$ for each $n \in \mathrm{Spec}(K_{r,s})$ except $n = 1$. We appeal to a theorem of D. G. Hoffman and M. Liatti [16] to provide these designs.

**Theorem 3.12** (Hoffman and Liatti, 1995)**.** *Let $a$, $b$, $c$, and $d$ be positive integers. Let $g = \gcd(a,b)$; let $e$ and $f$ be integers satisfying $ae - bf = g$, and let $h = ae + bf$.*

*For each integer $x$, let*

$$\alpha(x) = \left\lceil \frac{xf}{a} \right\rceil, \qquad \beta(x) = \left\lfloor \frac{xe}{b} \right\rfloor, \qquad \text{and} \qquad \gamma(x) = \frac{x}{ab}.$$

*Then there is a $K_{a,b}$-design on $K_{c,d}$ if and only if the following conditions are satisfied.*

$$
\begin{array}{ll}
(i) & ab \mid cd \\
(ii) & g \mid c \ \ and \ \ \alpha(c) \le \beta(c) \\
(iii) & g \mid d \ \ and \ \ \alpha(d) \le \beta(d) \\
(iv) & c \cdot \alpha(d) + d \cdot \alpha(c) \le h \cdot \gamma(cd) \le c \cdot \beta(d) + d \cdot \beta(c)
\end{array}
$$

We now apply Theorem 3.12 to obtain $K_{r,s}$-designs on certain complete bipartite graphs. The application of this theorem is done as broadly as possible; note that no conditions are placed on the positive integers $r$ and $s$. We restrict $n$ by the condition $n \ge rs + 1$; this restriction is sufficient, but not necessary, for condition (ii) to hold.

**Lemma 3.13.** *Let $r$ and $s$ be positive integers. Let $n \in \mathrm{SSpec}(K_{r,s})$, and suppose $n \ge rs+1$. Then there exists a $K_{r,s}$-design on $K_{2rs,\,n-1}$.*

*Proof.* Let $r$ and $s$ be positive integers. Let $n \in \mathrm{SSpec}(K_{r,s})$, and suppose $n \ge rs + 1$. Since $n \in \mathrm{SSpec}(K_{r,s})$, $n$ must satisfy the three conditions stated in Remark 3.11. We apply Theorem 3.12, with $a = r$, $b = s$, $c = n - 1$, and $d = 2rs$. As in this theorem, let $g = \gcd(r, s)$, and let $e$ and $f$ be integers such that $re - sf = g$. Let $h$, $\alpha(x)$, $\beta(x)$, and $\gamma(x)$ be defined as in the theorem. We need to verify that the four conditions in Theorem 3.12 hold for the chosen values of $a$, $b$, $c$, and $d$.

Condition (i): $rs \mid (n-1)2rs$ This is clearly true.

Condition (ii): $\gcd(r, s) \mid (n-1)$ and $\alpha(n-1) \le \beta(n-1)$

The condition $\gcd(r, s) \mid (n-1)$ is one of the conditions in Remark 3.11, so it holds.

Now consider the condition $\alpha(n-1) \le \beta(n-1)$. We observe that $g \ge 1$, because $g = \gcd(r, s)$; we recall that $n \ge rs + 1$ by assumption. Combining these facts, we obtain

$$rs \le n-1 \le (n-1)g. \tag{3.4}$$

Thus

$$rs + (n-1)sf \;\leq\; (n-1)g + (n-1)sf. \tag{3.5}$$

Dividing by $rs$ and applying the fact that $sf + g = re$, we obtain

$$1 + \frac{(n-1)f}{r} \leq \frac{(n-1)e}{s}. \tag{3.6}$$

Thus

$$\left\lceil \frac{(n-1)f}{r} \right\rceil \;\leq\; \left\lfloor 1 + \frac{(n-1)f}{r} \right\rfloor \;\leq\; \left\lfloor \frac{(n-1)e}{s} \right\rfloor, \tag{3.7}$$

that is, the condition $\alpha(n-1) \leq \beta(n-1)$ holds.

Condition (iii): $\gcd(r,s) \,\big|\, 2rs$ and $\alpha(2rs) \leq \beta(2rs)$

The condition $\gcd(r,s) \,\big|\, 2rs$ clearly holds, as $\gcd(r,s)$ divides both $r$ and $s$.

Furthermore,

$$\begin{aligned} \alpha(2rs) \leq \beta(2rs) \quad &\Longleftrightarrow \quad \left\lceil \frac{(2rs)f}{r} \right\rceil \;\leq\; \left\lfloor \frac{(2rs)e}{s} \right\rfloor \\ &\Longleftrightarrow \quad \lceil 2sf \rceil \;\leq\; \lfloor 2re \rfloor \\ &\Longleftrightarrow \quad 2sf \;\leq\; 2re \\ &\Longleftrightarrow \quad sf \;\leq\; re \\ &\Longleftrightarrow \quad sf \;\leq\; sf + g \end{aligned} \tag{3.8}$$

The last inequality is clearly true, since $g = \gcd(r,s)$ is a positive integer; hence the condition $\alpha(2rs) \leq \beta(2rs)$ holds.

Condition (iv): $(n-1)\cdot\alpha(2rs) + 2rs\cdot\alpha(n-1) \leq h\cdot\gamma\big((n-1)2rs\big) \leq (n-1)\cdot\beta(2rs) + 2rs\cdot\beta(n-1)$

By condition (ii), we have that $\left\lceil \frac{(n-1)f}{r} \right\rceil \leq \left\lfloor \frac{(n-1)e}{s} \right\rfloor$, so

$$\frac{(n-1)f}{r} \leq \left\lceil \frac{(n-1)f}{r} \right\rceil \leq \left\lfloor \frac{(n-1)e}{s} \right\rfloor \leq \frac{(n-1)e}{s}.$$

In particular,

$$\left\lceil \frac{(n-1)f}{r} \right\rceil \leq \frac{(n-1)e}{s} \tag{3.9}$$

and

$$\frac{(n-1)f}{r} \leq \left\lfloor \frac{(n-1)e}{s} \right\rfloor. \tag{3.10}$$

From inequality (3.9), we obtain

$$\frac{(n-1)f}{r} + \left\lceil \frac{(n-1)f}{r} \right\rceil \leq \frac{(n-1)f}{r} + \frac{(n-1)e}{s}. \tag{3.11}$$

Note that the right-hand side of inequality (3.11) is equal to $(re+sf)(n-1)/rs$.

Multiplying through by $2rs$, we obtain

$$(n-1) \cdot \frac{2rsf}{r} + 2rs \cdot \left\lceil \frac{(n-1)f}{r} \right\rceil \leq (re+sf)\frac{(n-1)2rs}{rs}. \tag{3.12}$$

Note that $(2rsf)/r$ is an integer, so

$$\frac{2rsf}{r} = \left\lceil \frac{2rsf}{r} \right\rceil = \alpha(2rs); \tag{3.13}$$

thus inequality (3.12) can be written as

$$(n-1) \cdot \alpha(2rs) + 2rs \cdot \alpha(n-1) \leq h \cdot \gamma\big((n-1)2rs\big), \tag{3.14}$$

which is the left-hand inequality to be verified in condition (iv).

From inequality (3.10), we obtain

$$\frac{(n-1)e}{s} + \frac{(n-1)f}{r} \leq \frac{(n-1)e}{s} + \left\lfloor \frac{(n-1)e}{s} \right\rfloor. \tag{3.15}$$

Note that the left-hand side of inequality (3.15) is equal to $(re+sf)(n-1)/rs$.

Multiplying through by $2rs$, we obtain

$$(re + sf)\frac{(n-1)2rs}{rs} \leq (n-1) \cdot \frac{2rse}{s} + 2rs \cdot \left\lfloor \frac{(n-1)e}{s} \right\rfloor . \qquad (3.16)$$

Note that $(2rse)/s$ is an integer, so

$$\frac{2rse}{s} = \left\lfloor \frac{2rse}{s} \right\rfloor = \beta(2rs); \qquad (3.17)$$

thus inequality (3.16) can be written as

$$h \cdot \gamma\big((n-1)2rs\big) \leq (n-1) \cdot \beta(2rs) + 2rs \cdot \beta(n-1), \qquad (3.18)$$

which is the right-hand inequality to be verified in condition (iv).

Since we have verified all four conditions, there is indeed a $K_{r,s}$-design on $K_{2rs,\, n-1}$ for all $n \in \mathrm{SSpec}(K_{r,s})$ such that $n \geq rs+1$. □

We are now ready to prove the main result of this section: that complete bipartite graphs are bounded complete embedding graphs. We use Construction II to build embeddings of $K_{r,s}$-designs of all large orders in the spectrum of $K_{r,s}$. We then appeal to Theorem 1.34, which guarantees that these embeddings are sufficient.

**Theorem 3.14.** *Let $r$ and $s$ be positive integers. The complete bipartite graph $K_{r,s}$ is a bounded complete embedding graph.*

*Proof.* Let $r$ and $s$ be positive integers; let $n \in \mathrm{Spec}(K_{r,s})$, and suppose that $n \geq rs+1$. By Corollary 2.24, there is a $K_{r,s}$-design on $K_{2rs+1}$. By Lemma 3.13, there is a $K_{r,s}$-design on $K_{2rs,\, n-1}$. The conditions of Construction II (Lemma 3.3) with $t = 2rs$ are thus satisfied for $n$. Hence every $K_{r,s}$-design of order $n \geq rs+1$ can be embedded in a $K_{r,s}$-design of order $n + 2rs$. Therefore, by Theorem 1.34 (with $N = rs+1$ and $b = 2rs$), $K_{r,s}$ is a bounded complete embedding graph. □

## 3.4 A Special Class of Bipartite Graphs

We now turn our attention to a special class of bipartite graphs, which we will show are bounded complete embedding graphs. For any bipartite graph $G$ that admits a $\beta^+$-labeling, certain designs that are useful for Construction II are guaranteed by the existence of the $\beta^+$-labeling. For some such graphs, these designs are sufficient to build embeddings for a bounded complete embedding graph result. We see the conditions that describe these graphs in the following lemma.

**Lemma 3.15.** *Let $s \in \mathbb{P}$, and let $G$ be a graph such that $e(G) = 2^s$ and $\deg(v)$ is even for all $v \in V(G)$, and suppose that $2e(G) + 1 \geq v(G)$. Then*

$$\mathrm{SSpec}(G) = \left\{ n \in \mathbb{P} \;\middle|\; n \equiv 1 \pmod{2^{s+1}} \right\}.$$

*Proof.* Let $s \in \mathbb{P}$, and let $G$ be a graph such that $e(G) = 2^s$ and $\deg(v)$ is even for all $v \in V(G)$, and suppose that $2e(G) + 1 \geq v(G)$. By Lemma 1.23, we have that

$$\left\{ n \in \mathbb{P} \;\middle|\; n \equiv 1 \pmod{2^{s+1}} \right\} \subseteq \mathrm{SSpec}(G).$$

Now suppose that $n \in \mathrm{SSpec}(G)$ and $n > 1$. Since $n$ must satisfy SSC–2, we have that $2^{s+1} \big| n(n-1)$. Since $n$ must also satisfy SSC–3, we have $\gcd\left\{ \deg(v) \;\middle|\; v \in V(G) \right\} \Big| (n-1)$; then, since $\gcd\left\{ \deg(v) \;\middle|\; v \in V(G) \right\}$ is divisible by 2 by assumption, we have that $2 \big| (n-1)$. Thus $(n-1)$ is even and $n$ is odd, so $2^{s+1} \big| (n-1)$; thus $n \equiv 1 \pmod{2^{s+1}}$. Hence

$$\mathrm{SSpec}(G) \subseteq \left\{ n \in \mathbb{P} \;\middle|\; n \equiv 1 \pmod{2^{s+1}} \right\},$$

so $\mathrm{SSpec}(G) = \left\{ n \in \mathbb{P} \;\middle|\; n \equiv 1 \pmod{2^{s+1}} \right\}$, as desired. □

We observe that, for any bipartite graph $G$, a $G$-design on a complete bipartite graph of a certain size may be used to build $G$-designs on certain larger complete bipartite graphs.

**Lemma 3.16.** *Let $G$ be a graph, let $m \in \mathbb{P}$, and suppose that there is a $G$-design on $K_{m,m}$. Then there is a $G$-design on $K_{pm,qm}$ for every $p, q \in \mathbb{P}$.*

*Proof.* Let $G$ be a graph, let $m \in \mathbb{P}$, and suppose there is a $G$-design on $K_{m,m}$. Let $U = [\![1, m]\!] \times \{0\}$ and $V = [\![1, m]\!] \times \{1\}$, and let $K_{m,m}$ be on bipartition $[U, V]$. Let $p, q \in \mathbb{P}$, and let $K_{pm,qm}$ be on bipartition $[X, Y]$, where

$$X = [\![1, p]\!] \times [\![1, m]\!] \times \{0\} \ \text{ and } \ Y = [\![1, q]\!] \times [\![1, m]\!] \times \{1\}.$$

Furthermore, for each $i \in [\![1, p]\!]$, let $X_i = \{i\} \times [\![1, m]\!] \times \{0\}$, and for each $j \in [\![1, q]\!]$, let $Y_j = \{j\} \times [\![1, m]\!] \times \{1\}$. Then, for each $(i, j) \in [\![1, p]\!] \times [\![1, q]\!]$, the subgraph $H(i, j)$ of $K_{pm,qm}$ induced by the vertex set $X_i \cup Y_j$ is a $K_{m,m}$-subgraph, so it admits a $G$-design $\mathscr{B}(i, j)$ by assumption. Hence the design

$$\mathscr{D} = \bigcup_{\substack{1 \le i \le p \\ 1 \le j \le q}} \mathscr{B}(i, j)$$

is a $G$-design on $K_{pm,qm}$, as desired. □

We can now establish that any bipartite graph satisfying our selected conditions is a bounded complete embedding graph. The most important of these conditions is that the graph admit a $\beta^+$-labeling. We recall for the reader that the labeling types are defined in subsection 2.2.2, and that a list of graphs that admit $\beta^+$-labelings is given in Theorem 2.23.

**Theorem 3.17.** *Let $G$ be a bipartite graph that satisfies the following conditions:*

*(i)* $e(G) = 2^s$ *for some* $s \in \mathbb{P}$,

*(ii)* $\deg(v)$ *is even, for all* $v \in V(G)$, *and*

*(iii)* *$G$ admits a $\beta^+$-labeling.*

*Then $G$ is a bounded complete embedding graph.*
*Furthermore, the spectrum of $G$ is the set* $\left\{ n \in \mathbb{P} \mid n \equiv 1 \pmod{2^{s+1}} \right\}$.

*Proof.* Let $G$ be a graph that satisfies conditions (i) – (iii) as given in the statement of the theorem, and let $m = e(G) = 2^s$. Since $G$ admits a $\beta^+$-labeling, $G$ has at most $(m+1)$ vertices. Since $m = e(G) \geq 2$, we have $2e(G)+1 = 2m+1 > m+1 \geq v(G)$, so we have satisfied the hypotheses of Lemma 3.15; hence the superspectrum of $G$ consists only of those positive integers $n$ satisfying the congruence $n \equiv 1 \,(\mathrm{mod}\ 2^{s+1})$. Since $G$ admits a $\beta^+$-labeling, and since every $\beta^+$-labeling is a $\rho^+$-labeling, we have from Theorem 2.17 that there is a complete $G$-design of order $2m+1 = 2^{s+1}+1$. Furthermore, we have from Theorem 2.18 that there is a $G$-design on $K_{m,m}$. If $n \in \mathrm{SSpec}(G)$ and $n > 1$, then $n-1 = x2^{s+1} = 2xm$ for some $x \in \mathbb{P}$; thus, by Lemma 3.16 with $p = 2x$ and $q = 2$, there is a $G$-design on $K_{n-1,\,2m} = K_{2xm,\,2m}$. We have therefore satisfied the conditions of Construction II, with $t = 2m = 2e(G)$, for every $n \in \mathrm{Spec}(G)$ except $n = 1$, so $G$ is a bounded complete embedding graph.

We now consider the spectrum of $G$. The above construction establishes the existence of a complete $G$-design of order $n$ for all $n \in \mathrm{SSpec}(G)$ such that $n > 2m+1$. The existence of a complete $G$-design of order $2m+1$ is already established by Theorem 2.17, and the existence of a complete $G$ design of order 1 is trivial, so the spectrum of $G$ is precisely its superspectrum; that is, $\mathrm{Spec}(G) = \left\{\, n \in \mathbb{P} \;\middle|\; n \equiv 1 \left(\mathrm{mod}\ 2^{k+1}\right) \right\}$. □

We note that the restrictions on the number of edges in $G$ and on the parity of the degrees of vertices in $G$ are necessary, as relaxing either condition causes the superspectrum of $G$ to expand, so that the collection of $G$-designs guaranteed by the $\beta^+$-labeling is no longer sufficient. From Theorem 2.23, we see that Theorem 3.17 provides two new infinite classes of bounded complete embedding graphs; further results on graphs that admit $\beta^+$-labelings may add additional families.

**Corollary 3.18.** *Let $t \in \mathbb{P}$, and let $n = 2^t$. Then the cube $Q_n$ is a bounded complete embedding graph.*

*Proof.* Let $t \in \mathbb{P}$, and let $n = 2^t$. Then the cube $Q_n$ admits a $\beta^+$-labeling by Theorem 2.23; furthermore, $v(Q_n) = 2^n$, and $Q_n$ is $2^t$-regular, so $e(Q_n) = 2^{n+t-1}$. Hence, by Theorem 3.17, the cube $Q_n$ is a bounded complete embedding graph. □

**Corollary 3.19.** *Let $a$ and $b$ be distinct positive integers, each at least two, such that $2a + 2b = 2^t$ for some $t \in \mathbb{P}$. Then $C_{2a} \uplus C_{2b}$ is a bounded complete embedding graph.*

*Proof.* Let $a$ and $b$ be distinct positive integers, each at least two, such that $2a + 2b = 2^t$ for some $t \in \mathbb{P}$. Then $C_{2a} \uplus C_{2b}$ admits a $\beta^+$-labeling by Theorem 2.23; furthermore, $C_{2a} \uplus C_{2b}$ is 2-regular, and $e(C_{2a} \uplus C_{2b}) = 2a + 2b = 2^t$ by assumption. Hence, by Theorem 3.17, the graph $C_{2a} \uplus C_{2b}$ is a bounded complete embedding graph. □

**Corollary 3.20.** *Let $a, b, c \in \mathbb{P}$, each at least two, such that $2a + 2b + 2c = 2^t$ for some $t \in \mathbb{P}$. Then $C_{2a} \uplus C_{2b} \uplus C_{2c}$ is a bounded complete embedding graph.*

*Proof.* Let $a$, $b$, and $c$ be positive integers, each at least two, such that $2a + 2b + 2c = 2^t$ for some $t \in \mathbb{P}$. Then $C_{2a} \uplus C_{2b} \uplus C_{2c}$ admits a $\beta^+$-labeling by Theorem 2.23; furthermore, $C_{2a} \uplus C_{2b} \uplus C_{2c}$ is 2-regular, and $e(C_{2a} \uplus C_{2b} \uplus C_{2c}) = 2a + 2b + 2c = 2^t$ by assumption. Hence, by Theorem 3.17, the graph $C_{2a} \uplus C_{2b} \uplus C_{2c}$ is a bounded complete embedding graph.

□

We note that item (8) of Theorem 2.23 addresses all 2-regular bipartite graphs with at most three components. All of the graphs in Corollaries 3.19 and 3.20 have cycles of non-uniform length, but the graphs $C_{2a} \uplus C_{2a}$, where $a$ is a positive integer power of two, also satisfy the conditions of Theorem 3.17. The 2-regular bipartite graphs with components of uniform size are the subject of Chapters 4 and 5, so we omit the statement of the relevant corollary to Theorem 3.17 for such graphs in favor of stating a more general result later.

# Chapter 4

## Cohorts of Even Cycles, Part One

We have so far seen several families of bounded complete embedding graphs, including three of the most commonly studied families of bipartite graphs; we now turn our attention to a more obscure family of bipartite graphs. Each component of a 2-regular bipartite graph is an even cycle; these cycles may be of the same length or of different lengths. In this chapter and the next, we consider specifically those 2-regular bipartite graphs having all cycles of the same length. Some authors have used the term *uniform 2-regular graph* to refer to 2-regular graphs having all cycles of the same length. In keeping with this usage, we could call our graphs *uniform 2-regular bipartite graphs*, but we find this term somewhat cumbersome, and so have coined our own. We have given a name, in general, to any graph whose components are all isomorphic to one another.

**Definition 4.1.** Let $p \in \mathbb{Z}$ with $p \geq 2$, and let $H$ be a graph. The ***p-cohort of the graph H*** is the graph $G$ with exactly $p$ components, each of which is isomorphic to $H$. We use the shortened phrase $(p, H)$-cohort to denote these graphs. ■

Using this terminology, the graphs we wish to consider are the $\big(p,\, C_{2k}\big)$-cohorts, for all integers $p \geq 2$ and $k \geq 2$. We refer to these graphs informally as *cohorts of even cycles*; for convenience, we denote the $\big(p,\, C_{2k}\big)$-cohort by $\mathcal{C}_{2k}^{p}$. We begin our discussion of these graphs with a statement of the superspectral conditions in terms of the parameters $p$ and $k$.

**Remark 4.2.** Let $p$, $k$, and $n$ be positive integers such that $p \geq 2$ and $k \geq 2$, and suppose there is a $\mathcal{C}_{2k}^{p}$-design on $K_n$. Note that $v\big(\mathcal{C}_{2k}^{p}\big) = 2kp = e\big(\mathcal{C}_{2k}^{p}\big)$, and that all vertices in $\mathcal{C}_{2k}^{p}$ have degree two. Applying conditions SSC–1, SSC–2, and SSC–3, we have that

(1) $n = 1$ or $n \geq 2kp$,

(2) $4kp \,\big|\, n(n-1)$, and

(3) $2 \,\big|\, (n-1)$.

These conditions define the superspectrum of $\mathcal{C}_{2k}^{p}$. ■

We believe that $\mathcal{C}_{2k}^{p}$ is a bounded complete embedding graph for all values of $p$ and $k$. In this chapter, we establish this conjecture as fact for certain values of $p$ and $k$; all other cases remain open. We achieve our results using Construction II (Lemma 3.3), with $t = 4kp$, to build the required embeddings. For each $p$ and $k$, a $\mathcal{C}_{2k}^{p}$-design on $K_{4kp+1}$ is required to build the embeddings; the existence of these designs is guaranteed by Corollary 2.26. In this chapter, we construct the required $\mathcal{C}_{2k}^{p}$-designs on $K_{4kp,n-1}$ for all but finitely many $n \in \mathrm{SSpec}\big(\mathcal{C}_{2k}^{p}\big)$ and use Theorem 1.34 to establish the bounded complete embedding graph results. We conclude the chapter with a few results on the spectrum of $\mathcal{C}_{2k}^{p}$.

### 4.1 Constructions for Designs on Complete Bipartite Graphs

In our first construction, we combine a $C_{2k}$-design on a small complete bipartite graph and a *p-matching decomposition* of another small complete bipartite graph in a product-like way in order to create a $\mathcal{C}_{2k}^{p}$-design on $K_{4kp,\,n-1}$. Before we can describe this construction in more detail, we need two definitions and two important theorems.

**Definition 4.3.** Let $G$ be a graph. A ***matching*** in $G$ is a set $M \subseteq E(G)$ such that no two elements of $M$ are incident to the same vertex. ■

**Definition 4.4.** Let $G$ be a graph, and let $r \in \mathbb{P}$. An ***r-matching decomposition*** of $G$ is a partition of $E(G)$ into matchings having $r$ edges each. ■

Observe that, for a graph $G$ and a matching $M$ in $G$, the graph $G[M]$ consists precisely of the edges in $M$ and the vertices that are ends of edges in $M$; if $r$ denotes the number of edges in $M$, then, in the language of cohorts, the graph $G[M]$ is the $(r, K_2)$-cohort. Thus the existence of an $r$-matching-decomposition of a graph $G$ is equivalent to the existence of an $(r, K_2)$-cohort-design on $G$; note that $r \mid e(G)$ is a necessary condition for this structure.

Now we can state the first of the two important theorems we need; this result, by D. de Werra, guarantees that we can obtain a matching decomposition of a bipartite graph under certain conditions [9]. We note that de Werra's result is more general in several ways; we state only the special case that suits our needs.

**Theorem 4.5** (de Werra, 1972)**.** *Let $B$ be a simple bipartite graph, and let $\Delta$ denote the maximum degree of $B$. For any positive integer $m \geq \Delta$, there is a decomposition of $B$ into $m$ matchings of uniform size if and only if* $m \mid e(B)$.

We apply the above theorem to obtain an $r$-matching decomposition of a particular complete bipartite graph.

**Lemma 4.6.** *Let $r, s \in \mathbb{P}$ such that $r \leq s$. Then there exists an $r$-matching decomposition of $K_{r,s}$.*

*Proof.* Let $r, s \in \mathbb{P}$ such that $r \leq s$. Then $K_{r,s}$ is a bipartite graph with maximum degree $\Delta = s$; since $e(K_{r,s}) = rs$ is divisible by $s$, the graph $K_{r,s}$ has a decomposition into $s$ matchings of uniform size by Theorem 4.5. Clearly, all matchings have size $\dfrac{rs}{s} = r$, so this decomposition is an $r$-matching decomposition of $K_{r,s}$. □

Our second important theorem is the following theorem of Sotteau [32], which gives necessary and sufficient conditions for a $C_{2k}$-design on a complete bipartite graph.

**Theorem 4.7** (Sotteau, 1981)**.** *Let $k, a, b, \in \mathbb{P}$, and suppose $k \geq 2$. There is a $C_{2k}$-design on $K_{a,b}$ if and only if the following conditions are satisfied.*

(i) $a \geq k$ and $b \geq k$

(ii) $a$ and $b$ are even

(iii) $2k \mid ab$

We now present the construction of a $\mathcal{C}_{2k}^{p}$-design on $K_{4kp,\,n-1}$; we have dubbed it the Dovetail Construction due to the assembly of a $C_{2k}$-design and a $p$-matching decomposition to produce the desired $\mathcal{C}_{2k}^{p}$-design.

**Theorem 4.8** (Dovetail Construction)**.** *Let $k, p \in \mathbb{P}$ such that $k \geq 2$ and $p \geq 2$, and let $n \in \operatorname{SSpec}\big(\mathcal{C}_{2k}^{p}\big)$ such that $n > 1$. If there exists a divisor $z$ of $n-1$ such that $p \leq z \leq \dfrac{n-1}{k}$ and $\dfrac{n-1}{z}$ is even, then there exists a $\mathcal{C}_{2k}^{p}$-design on $K_{4kp,\,n-1}$.*

*Proof.* Let $k, p \in \mathbb{P}$ such that $k \geq 2$ and $p \geq 2$, and let $n \in \operatorname{SSpec}\big(\mathcal{C}_{2k}^{p}\big)$. Suppose that there exists a divisor $z$ of $n-1$ such that $p \leq z \leq \dfrac{n-1}{k}$ and $\dfrac{n-1}{z}$ is even. Then there is some $w \in \mathbb{P}$ such that $w = \dfrac{n-1}{z}$. Let $G = K_{4kp,\,n-1}$ be on bipartition $[A, B]$, where

$$A = \Big\{(a_j, x) \;\Big|\; j \in [\![1, 4k]\!],\ x \in [\![1, p]\!]\Big\} \text{ and} \tag{4.1}$$

$$B = \Big\{(b_i, y) \;\Big|\; i \in [\![1, w]\!],\ y \in [\![1, z]\!]\Big\}. \tag{4.2}$$

For each $s \in [\![1, p]\!]$, define $A_s = \Big\{(a_j, s) \;\Big|\; j \in [\![1, 4k]\!]\Big\}$; for each $r \in [\![1, z]\!]$, define $B_r = \Big\{(b_i, r) \;\Big|\; i \in [\![1, w]\!]\Big\}$. We observe that $\big\{A_s \;\big|\; 1 \leq s \leq p\big\}$ is a partition of $A$, and $\big\{B_r \;\big|\; 1 \leq r \leq z\big\}$ is a partition of $B$. Furthermore, for each $(s, r) \in [\![1, p]\!] \times [\![1, z]\!]$, the graph $G[A_s \cup B_r]$ is a $K_{4k,w}$-subgraph of $G$.

Claim: A $C_{2k}$-design on $K_{4k,\,w}$ exists.

Proof of Claim: We apply Sotteau's Theorem (4.7) with $a = 4k$ and $b = w$. Clearly, $4k \geq k$, $4k$ is even, and $2k \,\big|\, 4kw$. Since our assumptions about $z$ guarantee that $w \geq k$ and $w$ is even, the conditions of Sotteau's Theorem are satisfied. Hence a $C_{2k}$-design on $K_{4k,\,w}$ exists. ◇

We now build a $\mathcal{C}_{2k}^{p}$-design on $K_{4kp,\,n-1}$. Let $\Big([\![1, p]\!] \times \{0\}\Big) \cup \Big([\![1, z]\!] \times \{1\}\Big)$ be the vertex set of $K_{p,\,z}$, so that no two vertices having the same second coordinate are adjacent. Since $z \geq p$, the graph $K_{p,\,z}$ admits a $p$-matching decomposition $\mathscr{M}$ by Lemma 4.6. Let $H = K_{4k,\,w}$

be on bipartition $[X, Y]$, where $X = \{x_j \mid j \in [\![1, 4k]\!]\}$ and $Y = \{y_i \mid i \in [\![1, w]\!]\}$. Since a $C_{2k}$-design on $K_{4k,\,w}$ exists, let $\mathscr{D}$ be such a design on $H$. For an arbitrary $C_{2k}$-block $D \in \mathscr{D}$, note that the vertices of $D$ must alternate between vertices in $X$ and vertices in $Y$; so $D$ has the form

$$D = \left(x_{j_1},\, y_{i_1},\, x_{j_2},\, y_{i_2},\, \ldots,\, x_{j_k},\, y_{i_k}\right). \tag{4.3}$$

For any $r \in [\![1, z]\!]$ and $s \in [\![1, p]\!]$, the map defined by $x_j \mapsto (a_j, s)$ and $y_i \mapsto (b_i, r)$ is an isomorphism from $H$ to $G[A_s \cup B_r]$. Under this isomorphism, the cycle $D$ maps to the cycle

$$D_{s,\,r} = \Big((a_{j_1}, s),\; (b_{i_1}, r),\; (a_{j_2}, s),\; (b_{i_2}, r),\; \ldots,\; (a_{j_k}, s),\; (b_{i_k}, r)\Big). \tag{4.4}$$

For each matching $M \in \mathscr{M}$ and each $C_{2k}$-block $D \in \mathscr{D}$, we define the corresponding $\mathcal{C}_{2k}^{p}$-subgraph of $G = K_{4kp,\,n-1}$, denoted $\mathcal{C}(D, M)$, to be the union of the $p$ distinct, vertex-disjoint $2k$-cycles in the set

$$\Big\{D_{s,r} \;\Big|\; \big\{(s, 0),\, (r, 1)\big\} \in M\Big\}. \tag{4.5}$$

Let $\mathscr{C} = \Big\{\mathcal{C}(D, M) \;\Big|\; D \in \mathscr{D},\; M \in \mathscr{M}\Big\}$. Then $\mathscr{C}$ is the desired $\mathcal{C}_{2k}^{p}$-design on $K_{4kp,\,n-1}$. □

In the special case that $p \big| (n-1)$ in the above theorem, $p$ can serve as the required divisor $z$. We state this fact as the following corollary; for convenience, we refer to this result as the Dovetail Corollary.

**Corollary 4.9** (Dovetail Corollary). *Let $k, p \in \mathbb{P}$ such that $k \geq 2$ and $p \geq 2$, and let $n \in \mathrm{SSpec}\big(\mathcal{C}_{2k}^{p}\big)$ such that $n > 1$. If $n \equiv 1 \pmod{p}$, then there exists a $\mathcal{C}_{2k}^{p}$-design on $K_{4kp,\,n-1}$.*

*Proof.* Let $k, p \in \mathbb{P}$ such that $k \geq 2$ and $p \geq 2$, and let $n \in \mathrm{SSpec}\big(\mathcal{C}_{2k}^{p}\big)$ such that $n > 1$. Suppose that $n \equiv 1 \pmod{p}$; then $p \big| (n-1)$. It therefore suffices to show that $p$ satisfies the conditions imposed on the divisor $z$ in Theorem 4.8.

Since $n \in \mathrm{SSpec}\big(\mathcal{C}_{2k}^{p}\big)$ and $n > 1$, we have $n \geq 2kp$ by SSC–1. Then, since $k, p \geq 2$,

$$\frac{n-1}{p} \geq \frac{2kp-1}{p} = 2k - \frac{1}{p} > k\,. \tag{4.6}$$

Then $\dfrac{n-1}{k} \geq p$, so we have $p \leq p \leq \dfrac{n-1}{k}$, as required.

Since $n \in \mathrm{SSpec}\big(\mathcal{C}_{2k}^{p}\big)$ and every vertex of $\mathcal{C}_{2k}^{p}$ has degree 2, $(n-1)$ is even by SSC–3; thus $n$ is not divisible by 2. Furthermore, $(n-1)$ is divisible by $p$ by assumption; thus $n$ is not divisible by $p$. By SSC–2, we have that $4kp \big| n(n-1)$, so $4p \,\Big|\, n(n-1)$. Since neither 2 nor $p$ divides $n$, we must have that $(n-1)$ is divisible by $4p$; then $\dfrac{n-1}{p}$ is divisible by 4 and is thus even.

We have verified that $z = p$ satisfies the conditions of Theorem 4.8; hence a $\mathcal{C}_{2k}^{p}$-design on $K_{4kp,\,n-1}$ exists. □

Once we obtain a $\mathcal{C}_{2k}^{p}$-design on $K_{4kp,\,N-1}$ for a particular $N \in \mathrm{SSpec}\big(\mathcal{C}_{2k}^{p}\big)$, the following construction provides $\mathcal{C}_{2k}^{p}$-designs on $K_{4kp,\,n-1}$ for all $n \in \mathrm{SSpec}\big(\mathcal{C}_{2k}^{p}\big)$ such that $n > N$ and $n \equiv N \pmod{4kp}$. We will refer to this construction as the $4kp$-Increment Construction.

**Theorem 4.10** ($4kp$-Increment Construction)**.** *Let $k, p, m \in \mathbb{P}$ such that $k \geq 2$ and $p \geq 2$. If there is a $\mathcal{C}_{2k}^{p}$-design on $K_{4kp,\,m}$, then there is a $\mathcal{C}_{2k}^{p}$-design on $K_{4kp,\,m+4kpq}$ for all $q \in \mathbb{P}$.*

*Proof.* Let $k, p, m, q \in \mathbb{P}$ such that $k \geq 2$ and $p \geq 2$. Suppose there is a $\mathcal{C}_{2k}^{p}$-design on $K_{4kp,\,m}$. Note that $4kp+1 \in \mathrm{SSpec}\big(\mathcal{C}_{2k}^{p}\big)$ and that, since $4kp+1 \equiv 1 \pmod{p}$, a $\mathcal{C}_{2k}^{p}$-design on $K_{4kp,\,4kp}$ exists by Corollary 4.9. Let $G = K_{4kp,\,m+4kpq}$ be on bipartition $[X, Y]$, where $X = \big\{x_j \bigm| j \in [\![1, 4kp]\!]\big\}$ and $Y = \big\{y_i \bigm| i \in [\![1, m+4kpq]\!]\big\}$. Let $Y_0 = \big\{y_i \bigm| i \in [\![1, m]\!]\big\}$, and, for all $r \in [\![1, q]\!]$, let $Y_r = \big\{y_i \bigm| i \in [\![m+1+4kp(r-1), m+4kpr]\!]\big\}$. Then $G[X, Y_0]$ is a $K_{4kp,\,m}$-subgraph of $G$; let $\mathscr{C}$ be a $\mathcal{C}_{2k}^{p}$-design on $G[X, Y_0]$. Furthermore, $G[X, Y_r]$ is a $K_{4kp,\,4kp}$-subgraph of $G$ for all $r \in [\![1, q]\!]$; let $\mathscr{B}_r$ be a $\mathcal{C}_{2k}^{p}$-design on $G[X, Y_r]$ for each $r \in [\![1, q]\!]$. Then $\mathscr{D} = \mathscr{C} \cup \mathscr{B}_1 \cup \cdots \cup \mathscr{B}_q$ is the desired $\mathcal{C}_{2k}^{p}$-design on $G = K_{4kp,\,m+4kpq}$. □

The Dovetail and $4kp$-Increment Constructions allow us to create the necessary designs for our embedding results, provided we have enough information about the superspectrum of the graph $\mathcal{C}_{2k}^{p}$.

## 4.2 Superspectra of Cohorts of Even Cycles

In this section, we describe what is known about the superspectra of cohorts of even cycles. We have already translated the superspectral conditions into statements in terms of the parameters $k$ and $p$ in Remark 4.2; we now analyze those statements to determine what modular congruences they generate. We then give a few results on families of graphs $\mathcal{C}_{2k}^{p}$ for which the prime factorizations of $k$ and $p$ have the same form.

**Remark 4.11.** Let $k, p \in \mathbb{P}$ such that $k \geq 2$ and $p \geq 2$. Consider the prime factorization of the product $4kp$; let $s$ denote the number of distinct odd prime factors of $4kp$.

If $s = 0$, then $4kp = 2^{\alpha}$ for some integer $\alpha \geq 4$; since $n$ is odd by SSC–3, we have $4kp \,\Big|\, (n-1)$. Thus, by Remark 4.2 and Lemma 1.23,

$$\mathrm{SSpec}\big(\mathcal{C}_{2k}^{p}\big) = \Big\{ n \in \mathbb{P} \;\Big|\; n \equiv 1 \pmod{4kp} \Big\} = \Big\{ n \in \mathbb{P} \;\Big|\; n \equiv 1 \pmod{2^{\alpha}} \Big\}. \tag{4.7}$$

If $s > 0$, then there exist $s$ distinct odd primes $q_1, q_2, \ldots, q_s$ and $s+1$ positive integers $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_s$ such that

$$4kp = 2^{\alpha_0} q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_s^{\alpha_s}. \tag{4.8}$$

For any $n \in \mathrm{SSpec}\big(\mathcal{C}_{2k}^{p}\big)$ such that $n > 1$, since $n$ and $(n-1)$ are relatively prime, condition (3) of Remark 4.2 guarantees that $2^{\alpha_0} \,\Big|\, (n-1)$. Furthermore, for each integer $i \in [\![1, s]\!]$, if $q_i^{\alpha_i} \,\Big|\, n(n-1)$, then either $q_i^{\alpha_i} \,\Big|\, n$ or $q_i^{\alpha_i} \,\Big|\, (n-1)$. Condition (2) of Remark 4.2 therefore generates, for each partition $\{I,\ J\}$ of $[\![1, s]\!]$, a system of congruences of the form

$$n \equiv 0 \left( \mathrm{mod}\ \prod_{i \in I} q_i^{\alpha_i} \right) \quad \text{and} \quad n \equiv 1 \left( \mathrm{mod}\ 2^{\alpha_0} \prod_{j \in J} q_j^{\alpha_j} \right). \tag{4.9}$$

There are clearly $2^s$ such systems. The solution of each of these systems is a single congruence class modulo $4kp$; since distinct systems generate distinct classes, the superspectrum of $\mathcal{C}_{2k}^{p}$ consists of $2^s$ distinct congruence classes modulo $4kp$. We note that any elements of these congruence classes that are in the set $[\![2, 2kp-1]\!]$ are excluded from the superspectrum by condition (1) of Remark 4.2. ■

If the product $4kp$ has exactly one odd prime factor, then we can give a more specific description of the superspectrum of $\mathcal{C}_{2k}^{p}$ in terms of congruence classes modulo $4kp$.

**Lemma 4.12.** *Let $q$ be an odd prime, and let $\alpha, \beta, \gamma, \varepsilon \in \mathbb{N}$ such that $\alpha$ and $\gamma$ are not both zero and $\beta$ and $\varepsilon$ are not both zero. Let $k = 2^{\alpha} q^{\gamma}$ and let $p = 2^{\beta} q^{\varepsilon}$. Since $q^{\gamma+\varepsilon}$ and $2^{\alpha+\beta+2}$ are relatively prime, let $x$ and $y$ be the unique positive integers such that $yq^{\gamma+\varepsilon} - x2^{\alpha+\beta+2} = 1$. Then the superspectrum of the graph $\mathcal{C}_{2k}^{p}$ is*

$$\left\{ n \in \mathbb{P} \;\middle|\; n \equiv 1 \text{ or } yq^{\gamma+\varepsilon} \pmod{4kp} \text{ and } n \notin [\![2, 2kp-1]\!] \right\}.$$

*Note that $4kp = 2^{\alpha+\beta+2} q^{\gamma+\varepsilon}$ for these values of $k$ and $p$.*

*Proof.* Let $q$ be an odd prime, and let $\alpha, \beta, \gamma, \varepsilon \in \mathbb{N}$ such that $\alpha$ and $\gamma$ are not both zero and $\beta$ and $\varepsilon$ are not both zero. Let $k = 2^{\alpha} q^{\gamma}$ and let $p = 2^{\beta} q^{\varepsilon}$. Since $q^{\gamma+\varepsilon}$ and $2^{\alpha+\beta+2}$ are relatively prime, let $x$ and $y$ be the unique positive integers such that $yq^{\gamma+\varepsilon} - x2^{\alpha+\beta+2} = 1$. Let $n \in \mathrm{SSpec}(\mathcal{C}_{2k}^{p})$; by SSC–1, we must have either $n = 1$ or $n \geq 2kp$. By SSC–3, $(n-1)$ is even; hence $n$ is odd. Now suppose $n > 1$; by SSC–2, we have $4kp \,\big|\, n(n-1)$; that is, $2^{\alpha+\beta+2} q^{\gamma+\varepsilon} \,\big|\, n(n-1)$. Since $n$ and $(n-1)$ are relatively prime, this condition holds if and only if either

$$2^{\alpha+\beta+2} q^{\gamma+\varepsilon} \,\Big|\, (n-1) \tag{4.10}$$

or

$$2^{\alpha+\beta+2} \,\Big|\, (n-1) \quad \text{and} \quad q^{\gamma+\varepsilon} \,\Big|\, n\,. \tag{4.11}$$

If the divisibility condition (4.10) holds, then $n \equiv 1 \pmod{4kp}$. If, on the other hand, the divisibility conditions in (4.11) hold, then $n \equiv 1 \left(\mathrm{mod}\ 2^{\alpha+\beta+2}\right)$ and $n \equiv 0 \left(\mathrm{mod}\ q^{\gamma+\varepsilon}\right)$. Clearly, the unique solution to this pair of congruences is $n \equiv yq^{\gamma+\varepsilon} \pmod{4kp}$. So we must have

$$n \equiv 1 \text{ or } yq^{\gamma+\varepsilon} \pmod{4kp} \ . \tag{4.12}$$

The superspectrum of $\mathcal{C}_{2k}^{p}$ is precisely the set of positive integers $n$ that satisfy congruence (4.12) but do not satisfy the inequality $2 \leq n \leq 2kp - 1$.

Thus $\mathrm{SSpec}\big(\mathcal{C}_{2k}^{p}\big) = \Big\{ n \in \mathbb{P} \ \Big| \ n \equiv 1 \text{ or } yq^{\gamma+\varepsilon} \pmod{4kp} \text{ and } n \notin [\![2, 2kp-1]\!] \Big\}$. □

Using our knowledge of the superspectrum from Remark 4.11 and the techniques of Lemma 4.12, we have computed the exact superspectrum of $\mathcal{C}_{2k}^{p}$ for all pairs $(k, p)$ such that $p \in [\![2, 128]\!]$ and $k \in [\![2, 128]\!]$; we refer the reader to Appendix A for further information about these superspectra, including the Python code we used to compute the superspectrum of $\mathcal{C}_{2k}^{p}$ for specific values of $k$ and $p$.

### 4.3 Bounded Complete Embedding Results on Cohorts of Even Cycles

We can now establish that certain cohorts of even cycles are bounded complete embedding graphs. In order to illustrate the general approach to the proof, we begin with two specific graphs in this family, namely $\mathcal{C}_{14}^{3}$ and $\mathcal{C}_{34}^{5}$.

**Theorem 4.13.** *$\mathcal{C}_{14}^{3}$ is a bounded complete embedding graph.*

*Proof.* Consider the graph $\mathcal{C}_{14}^{3}$, which is $\mathcal{C}_{2k}^{p}$ with $p = 3$ and $k = 7$. We use Construction II, with $t = 4kp = 84$, to build embeddings of complete $\mathcal{C}_{14}^{3}$-designs. Note that there is a $\mathcal{C}_{14}^{3}$-design on $K_{85}$ by Corollary 2.26. Furthermore, it is easily verified (see Appendix A) that the superspectrum of $\mathcal{C}_{14}^{3}$ is

$$\mathrm{SSpec}\big(\mathcal{C}_{14}^{3}\big) = \Big\{ n \in \mathbb{P} \ \Big| \ n \equiv 1,\ 21,\ 49, \text{ or } 57 \pmod{84} \text{ and } n \neq 21 \Big\}.$$

In order to apply Construction II, we also need $\mathcal{C}_{14}^3$-designs on complete bipartite graphs of certain sizes; we rely on the Dovetail Construction and the $4kp$-Increment Construction to provide these designs.

Table 4.1 shows divisors that have been verified for the application of the Dovetail Construction, separated according to the modular congruence classes in the superspectrum. The number in the MIN DIVISOR column is the smallest positive integer satisfying the conditions imposed on the divisor $z$ in the Dovetail Construction. Stars in the $n$ and $n-1$ columns in a particular row indicate that the information in the MIN DIVISOR column is valid for all values of $n$ in the modular class that are greater than one.

Table 4.1: Dovetail divisors by modular class for $p = 3$, $k = 7$

| MODULAR CLASS | $n$ | $n-1$ | MIN DIVISOR |
|---|---|---|---|
| $n \equiv 1 \pmod{84}$ | $\star$ | $\star$ | $p$ |
| $n \equiv 21 \pmod{84}$ | 105 | 104 | 4 |
| $n \equiv 49 \pmod{84}$ | $\star$ | $\star$ | $p$ |
| $n \equiv 57 \pmod{84}$ | 57 | 56 | 4 |

From Table 4.1, we see that we may apply the Dovetail Construction to the following values of $n$: $n = 57$, $n = 105$, and all $n \in \mathrm{SSpec}(\mathcal{C}_{14}^3)$ such that $n > 1$ and $n \equiv 1 \pmod{84}$ or $n \equiv 49 \pmod{84}$. We obtain, for each such $n$, a $\mathcal{C}_{14}^3$-design on $K_{84,\,n-1}$ from the Dovetail Construction. For each $n \in \mathrm{SSpec}(\mathcal{C}_{14}^3)$ such that $n \equiv 21 \pmod{84}$ and $n > 105$ or such that $n \equiv 57 \pmod{84}$ and $n > 57$, we apply the $4kp$-Increment construction to obtain a $\mathcal{C}_{14}^3$-design on $K_{84,\,n-1}$. Hence a $\mathcal{C}_{14}^3$-design on $K_{84,\,n-1}$ exists for every $n \in \mathrm{SSpec}(\mathcal{C}_{14}^3)$ except $n = 1$, and thus such a design exists for every $n \in \mathrm{Spec}(\mathcal{C}_{14}^3)$ except $n = 1$.

We have thus satisfied the conditions of Construction II, with $t = 4kp = 84$, for all $n \in \mathrm{Spec}(\mathcal{C}_{14}^3)$ except $n = 1$; thus for all such $n$, every complete $\mathcal{C}_{14}^3$-design of order $n$ can be embedded in a complete $\mathcal{C}_{14}^3$-design of order $n + 84$. We note that the complete $\mathcal{C}_{14}^3$-design of order 1 can be embedded in the complete $\mathcal{C}_{14}^3$-design of order 85; hence we have shown that $\mathcal{C}_{14}^3$ is a bounded complete embedding graph. □

**Theorem 4.14.** *$\mathcal{C}_{34}^5$ is a bounded complete embedding graph.*

*Proof.* Consider the graph $\mathcal{C}_{34}^5$, which is $\mathcal{C}_{2k}^p$ with $p = 5$ and $k = 17$. We use Construction II, with $t = 4kp = 340$, to build embeddings of complete $\mathcal{C}_{34}^5$-designs. Note that there is a $\mathcal{C}_{34}^5$-design on $K_{341}$ by Corollary 2.26. Furthermore, it is easily verified (see Appendix A) that the superspectrum of $\mathcal{C}_{34}^5$ is

$$\mathrm{SSpec}\big(\mathcal{C}_{34}^5\big) = \Big\{ n \in \mathbb{P} \;\Big|\; n \equiv 1,\, 85,\, 205, \text{ or } 221 \pmod{340} \text{ and } n \neq 85 \Big\}.$$

In order to apply Construction II, we also need $\mathcal{C}_{34}^5$-designs on complete bipartite graphs of certain sizes; we rely on the Dovetail Construction and the $4kp$-Increment Construction to provide these designs.

Table 4.2 shows divisors that have been verified for the application of the Dovetail Construction, separated according to the modular congruence classes in the superspectrum. The number in the MIN DIVISOR column is the smallest positive integer satisfying the conditions imposed on the divisor $z$ in the Dovetail Construction; if NONE appears in this column, then no divisor of $(n - 1)$ satisfying these conditions exists. Stars in the $n$ and $n - 1$ columns in a particular row indicate that the information in the MIN DIVISOR column is valid for all values of $n$ in the modular class that are greater than one.

From Table 4.2, we see that we may apply the Dovetail Construction to the following values of $n$: $n = 205$, $n = 1105$, and all $n \in \mathrm{SSpec}\big(\mathcal{C}_{34}^5\big)$ such that $n > 1$ and $n \equiv 1 \pmod{340}$ or $n \equiv 221 \pmod{340}$. We obtain, for each such $n$, a $\mathcal{C}_{34}^5$-design on $K_{340,\,n-1}$ from the Dovetail Construction. For each $n \in \mathrm{SSpec}\big(\mathcal{C}_{34}^5\big)$ such that $n \equiv 85 \pmod{340}$ and $n > 1105$ or such that $n \equiv 205 \pmod{340}$ and $n > 205$, we apply the $4kp$-Increment construction to obtain a $\mathcal{C}_{34}^5$-design on $K_{340,\,n-1}$. Hence a $\mathcal{C}_{34}^5$-design on $K_{340,\,n-1}$ exists for every $n \in \mathrm{SSpec}\big(\mathcal{C}_{34}^5\big)$

Table 4.2: Dovetail divisors by modular class for $p = 5$, $k = 17$

| MODULAR CLASS | $n$ | $n-1$ | MIN DIVISOR |
|---|---|---|---|
| $n \equiv 1 \pmod{340}$ | $\star$ | $\star$ | $p$ |
| $n \equiv 85 \pmod{340}$ | 425 | 424 | NONE |
| | 765 | 764 | NONE |
| | 1105 | 1104 | 6 |
| $n \equiv 205 \pmod{340}$ | 205 | 204 | 6 |
| $n \equiv 221 \pmod{340}$ | $\star$ | $\star$ | $p$ |

except $n = 1$, 425, and 765, and thus such a design exists for every $n \in \mathrm{Spec}(\mathcal{C}_{34}^5)$ except $n = 1$, 425, and 765.

We have thus satisfied the conditions of Construction II, with $t = 4kp = 340$, for all $n \in \mathrm{Spec}(\mathcal{C}_{34}^5)$ except $n = 1$, 425, and 765; thus for all such $n$, every complete $\mathcal{C}_{34}^5$-design of order $n$ can be embedded in a complete $\mathcal{C}_{34}^5$-design of order $n + 340$. Then, by Theorem 1.34 (with $N = 766$ and $b = 340$), $\mathcal{C}_{34}^5$ is a bounded complete embedding graph. □

Observe that, in the proofs of the previous two theorems, once the Dovetail Construction is used to obtain a design for one value in a modular congruence class, we may apply the $4kp$-Increment Construction to obtain designs for all larger values in the class. Furthermore, we may invoke Theorem 1.34 with an appropriate value of $N$ as necessary, so we may neglect to build a $\mathcal{C}_{2k}^p$-design on $K_{4kp,\,n-1}$ for finitely many values of $n$ in $\mathrm{SSpec}(\mathcal{C}_{2k}^p)$.

**Theorem 4.15.** *Let $k, p \in \mathbb{P}$ such that $2 \leq k \leq 128$ and $2 \leq p \leq 128$. Then there is some positive integer $N(k,p)$ such that a $\mathcal{C}_{2k}^p$-design on $K_{4kp,\,n-1}$ exists for every $n \in \mathrm{SSpec}(\mathcal{C}_{2k}^p)$ such that $n \geq N(k,p)$.*

*Proof.* Let $k, p \in \mathbb{P}$ such that $k \geq 2$ and $p \geq 2$. Let $M$ denote the number of distinct modular congruence classes in $\mathrm{SSpec}(\mathcal{C}_{2k}^p)$, and let $\{n_i \mid i \in [\![1, M]\!]\}$ be the set of canonical representatives of those congruence classes.

Claim: It suffices to show that, for each $i \in [\![1, M]\!]$, there is some $N_i \in \mathrm{SSpec}(\mathcal{C}_{2k}^p)$ such that $N_i \equiv n_i \pmod{4kp}$ and there is a divisor $z_i$ of $N_i - 1$ satisfying the conditions of the Dovetail Construction.

Proof of Claim: Suppose that, for each $i \in [\![1, M]\!]$, there is some $N_i \in \mathrm{SSpec}(\mathcal{C}_{2k}^p)$ such that $N_i \equiv n_i \pmod{4kp}$ and there is a divisor $z_i$ of $N_i - 1$ satisfying the conditions of the Dovetail Construction. Then we may apply the Dovetail Construction to obtain a $\mathcal{C}_{2k}^p$-design on $K_{4kp,\,n-1}$ for each $n \in \{N_i \mid i \in [\![1, M]\!]\}$. Then, for each $i \in [\![1, M]\!]$, we may apply the $4kp$-Increment Construction to obtain a $\mathcal{C}_{2k}^p$-design on $K_{4kp,\,n-1}$ for all $n \in \mathrm{SSpec}(\mathcal{C}_{2k}^p)$ such that $n \equiv n_i \pmod{4kp}$ and $n > N_i$.

Let $N(k,p) = \max\{N_i \mid 1 \leq i \leq M\}$. Then there is indeed a $\mathcal{C}_{2k}^p$-design on $K_{4kp,\,n-1}$ for every $n \in \mathrm{SSpec}(\mathcal{C}_{2k}^p)$ such that $n \geq N(k,p)$, as desired. ◇

We have verified, for all $k, p \in \mathbb{P}$ such that $2 \leq k \leq 128$ and $2 \leq p \leq 128$, that, for each $i \in [\![1, M]\!]$, there is some $N_i \in \mathrm{SSpec}(\mathcal{C}_{2k}^p)$ such that $N_i \equiv n_i \pmod{4kp}$ and there is a divisor $z_i$ of $N_i - 1$ satisfying the conditions of the Dovetail Construction. This verification is a simple matter of computation; we refer the reader to Appendix B for a complete listing of the source code we implemented to complete these computations. □

**Theorem 4.16.** *Let $k, p \in \mathbb{P}$ such that $2 \leq k \leq 128$ and $2 \leq p \leq 128$. Then $\mathcal{C}_{2k}^p$ is a bounded complete embedding graph.*

*Proof.* Let $k, p \in \mathbb{P}$ such that $2 \leq k \leq 128$ and $2 \leq p \leq 128$. By Corollary 2.26, there is a $\mathcal{C}_{2k}^p$-design on $K_{4kp+1}$. By Theorem 4.15, there is some positive integer $N(k,p)$ such that a $\mathcal{C}_{2k}^p$-design on $K_{4kp,\,n-1}$ exists for every $n \in \mathrm{SSpec}(\mathcal{C}_{2k}^p)$ such that $n \geq N(k,p)$; hence a $\mathcal{C}_{2k}^p$-design on $K_{4kp,\,n-1}$ exists for every $n \in \mathrm{Spec}(\mathcal{C}_{2k}^p)$ such that $n \geq N(k,p)$. We have thus satisfied the conditions of Construction II, with $t = 4kp$, for all $n \in \mathrm{Spec}(\mathcal{C}_{2k}^p)$ such that $n \geq N(k,p)$; hence for all such $n$, every complete $\mathcal{C}_{2k}^p$-design of order $n$ can be embedded in a complete $\mathcal{C}_{2k}^p$-design of order $n + 4kp$. Therefore, by Theorem 1.34 (with $N = N(k,p)$ and $b = 4kp$), $\mathcal{C}_{2k}^p$ is a bounded complete embedding graph. □

Corollary 4.9 provides an additional result in a special case: if $p$ is a positive power of two, then $p$ must divide $(n-1)$ for all $n \in \mathrm{SSpec}\big(\mathcal{C}_{2k}^p\big)$, so we may apply the corollary to obtain all desired designs on the graphs $K_{4kp,\,n-1}$.

**Theorem 4.17.** *Let $k \in \mathbb{P}$ such that $k \geq 2$; let $\beta \in \mathbb{P}$, and let $p = 2^\beta$. Then there is a $\mathcal{C}_{2k}^p$-design on $K_{4kp,\,n-1}$ for all $n \in \mathrm{SSpec}\big(\mathcal{C}_{2k}^p\big)$ except $n = 1$.*

*Proof.* Let $k \in \mathbb{P}$ such that $k \geq 2$; let $\beta \in \mathbb{P}$, and let $p = 2^\beta$. Let $n \in \mathrm{SSpec}\big(\mathcal{C}_{2k}^p\big)$ such that $n \neq 1$. Since $n-1$ must be even and $n$ must be odd by SSC–3, and since $4kp \big| n(n-1)$ by SSC–2, we must have that $p\big|(n-1)$; hence we may apply Corollary 4.9 to obtain a $\mathcal{C}_{2k}^p$-design on $K_{4kp,\,n-1}$. □

**Theorem 4.18.** *Let $k \in \mathbb{P}$ such that $k \geq 2$; let $\beta \in \mathbb{P}$, and let $p = 2^\beta$. Then $\mathcal{C}_{2k}^p$ is a bounded complete embedding graph.*

*Proof.* Let $k \in \mathbb{P}$ such that $k \geq 2$; let $\beta \in \mathbb{P}$, and let $p = 2^\beta$. By Corollary 2.26, there is a $\mathcal{C}_{2k}^p$-design on $K_{4kp+1}$. By Theorem 4.17, there is a $\mathcal{C}_{2k}^p$-design on $K_{4kp,\,n-1}$ for all $n \in \mathrm{SSpec}\big(\mathcal{C}_{2k}^p\big)$ except $n = 1$, and thus there is a $\mathcal{C}_{2k}^p$-design on $K_{4kp,\,n-1}$ for all $n \in \mathrm{Spec}\big(\mathcal{C}_{2k}^p\big)$ except $n = 1$. We have thus satisfied the conditions of Construction II, with $t = 4kp$, for all $n \in \mathrm{Spec}\big(\mathcal{C}_{2k}^p\big)$ such that $n \geq 2$; hence for all such $n$, every complete $\mathcal{C}_{2k}^p$-design of order $n$ can be embedded in a complete $\mathcal{C}_{2k}^p$-design of order $n + 4kp$. Furthermore, the complete $\mathcal{C}_{2k}^p$-design of order 1 can be embedded in the $\mathcal{C}_{2k}^p$-design of order $4kp + 1$. Therefore $\mathcal{C}_{2k}^p$ is a bounded complete embedding graph. □

### 4.3.1 A Special Design

In this section, we exhibit a construction for a $\mathcal{C}_{10}^3$-design on $K_{60,\,44}$. This design was constructed during our investigations into the reach and power of the Dovetail Construction. We note that this design is not strictly necessary in order to complete the bounded complete embedding graph result on $\mathcal{C}_{10}^3$; we have included it here because the construction technique may have other applications.

For the case $p = 3$, $k = 5$, we are able to use the Dovetail and $4kp$-Increment Constructions to obtain $\mathcal{C}_{2k}^{p}$-designs on $K_{4kp,\,n-1}$ for all $n \in \mathrm{SSpec}\big(\mathcal{C}_{2k}^{p}\big)$, except $n = 1$ and $n = 45$. For $n = 45$, no divisor exists to allow use of the Dovetail Construction, so we have built a $\mathcal{C}_{10}^{3}$-design on $K_{60,\,44}$ separately.

**Remark 4.19.** It is easily verified that there is a $K_{30,\,22}$-design on $K_{60,\,44}$; this may be accomplished by an argument similar to that given in the proof of Lemma 3.16, or by applying Theorem 3.12. It therefore suffices to produce a $\mathcal{C}_{10}^{3}$-design on $K_{30,\,22}$. ■

For the remainder of our discussion, we consider the graph $\mathcal{G} = K_{10,\,22}$ on bipartition $[U, W]$, where $W = [\![1, 22]\!]$ and $U = \big\{1^\star,\, 2^\star,\, 3^\star,\, 4^\star,\, 5^\star,\, 6^\star,\, 7^\star,\, 8^\star,\, 9^\star,\, 10^\star\big\}$, and the graph $\mathcal{H} = K_{30,\,22}$ on bipartition $[U_1 \cup U_2 \cup U_3,\, W]$, where $W$ is as previously defined and, for each $i \in \{1, 2, 3\}$, $U_i = [\![100i + 1, 100i + 10]\!]$. Also, for each $i \in \{1, 2, 3\}$, we let $\mathcal{H}_i = \mathcal{H}[U_i \cup W]$, the subgraph of $\mathcal{H}$ induced by the vertex set $U_i \cup W$. Note that $\mathcal{H}_i$ is isomorphic to $\mathcal{G}$ for all $i \in \{1, 2, 3\}$.

**Definition 4.20.** Let $A$ and $B$ be subgraphs of $\mathcal{G}$; we say that $A$ and $B$ are ***W-separated*** if and only if $V(A) \cap V(B) \cap W = \varnothing$, that is, if and only if $A$ and $B$ have no vertex in $W$ in common. Let $\mathscr{A}$ be a collection of subgraphs of $\mathcal{G}$; we say that $\mathscr{A}$ is ***W-separated*** if and only if each vertex in $W$ is a vertex of at most one subgraph in $\mathscr{A}$. ■

**Remark 4.21.** Let $\mathscr{B}$ be a $C_{10}$-design on $\mathcal{G}$, and suppose that the collection $\{A, B, C\} \subseteq \mathscr{B}$ is $W$-separated. If we copy $\mathscr{B}$ onto $\mathcal{H}_1$, $\mathcal{H}_2$, and $\mathcal{H}_3$, then we may obtain three $\mathcal{C}_{10}^{3}$-subgraphs of $\mathcal{H}$ from the copies of $A$, $B$, and $C$ in these designs as follows. For each $i \in \{1, 2, 3\}$, let $(A, i)$, $(B, i)$, and $(C, i)$ denote the copies of $A$, $B$, and $C$, respectively, in the copy of $\mathscr{B}$ on $\mathcal{H}_i$. The three $\mathcal{C}_{10}^{3}$-subgraphs are $H_{ABC} = (A, 1) \cup (B, 2) \cup (C, 3)$, $H_{BCA} = (B, 1) \cup (C, 2) \cup (A, 3)$, and $H_{CAB} = (C, 1) \cup (A, 2) \cup (B, 3)$. ■

**Remark 4.22.** Let $\mathscr{B}$ be a $C_{10}$-design on $\mathcal{G}$, and suppose that $\mathscr{B}$ can be partitioned into seven sets, of which six have three elements each and the seventh has four elements, so that each set in the partition is a $W$-separated collection. Label the cycles in $\mathscr{B}$ so that the sets in the partition are $\{B_1, B_2, B_3\}$, $\{B_4, B_5, B_6\}$, $\{B_7, B_8, B_9\}$, $\{B_{10}, B_{11}, B_{12}\}$, $\{B_{13}, B_{14}, B_{15}\}$, $\{B_{16}, B_{17}, B_{18}\}$, and $\{B_{19}, B_{20}, B_{21}, B_{22}\}$. We obtain a $\mathcal{C}_{10}^3$-design on $\mathcal{H}$ as follows. As in Remark 4.21, we copy the design $\mathscr{B}$ onto $\mathcal{H}_1$, $\mathcal{H}_2$, and $\mathcal{H}_3$; we denote these copies by $\mathscr{B}_1$, $\mathscr{B}_2$, and $\mathscr{B}_3$, respectively. For each $B_j \in \mathscr{B}$ and each $i \in \{1, 2, 3\}$, let $(B_j, i)$ denote the copy of cycle $B_j$ in $\mathscr{B}_i$. Then the set $\mathscr{C}$ consisting of the $\mathcal{C}_{10}^3$-subgraphs of $\mathcal{H}$ listed in Table 4.3 is a $\mathcal{C}_{10}^3$-design on $\mathcal{H}$. ■

Table 4.3: The $\mathcal{C}_{10}^3$-subgraphs of $\mathcal{H}$ obtained from $\mathscr{B}_1$, $\mathscr{B}_2$, and $\mathscr{B}_3$.

| Subset | $\mathcal{C}_{10}^3$-subgraphs of $\mathcal{H}$ |
|---|---|
| $\{B_1, B_2, B_3\}$ | $H_{1,2,3} = (B_1, 1) \cup (B_2, 2) \cup (B_3, 3)$ |
| | $H_{2,3,1} = (B_2, 1) \cup (B_3, 2) \cup (B_1, 3)$ |
| | $H_{3,1,2} = (B_3, 1) \cup (B_1, 2) \cup (B_2, 3)$ |
| $\{B_4, B_5, B_6\}$ | $H_{4,5,6} = (B_4, 1) \cup (B_5, 2) \cup (B_6, 3)$ |
| | $H_{5,6,4} = (B_5, 1) \cup (B_6, 2) \cup (B_4, 3)$ |
| | $H_{6,4,5} = (B_6, 1) \cup (B_4, 2) \cup (B_5, 3)$ |
| $\{B_7, B_8, B_9\}$ | $H_{7,8,9} = (B_7, 1) \cup (B_8, 2) \cup (B_9, 3)$ |
| | $H_{8,9,7} = (B_8, 1) \cup (B_9, 2) \cup (B_7, 3)$ |
| | $H_{9,7,8} = (B_9, 1) \cup (B_7, 2) \cup (B_8, 3)$ |
| $\{B_{10}, B_{11}, B_{12}\}$ | $H_{10,11,12} = (B_{10}, 1) \cup (B_{11}, 2) \cup (B_{12}, 3)$ |
| | $H_{11,12,10} = (B_{11}, 1) \cup (B_{12}, 2) \cup (B_{10}, 3)$ |
| | $H_{12,10,11} = (B_{12}, 1) \cup (B_{10}, 2) \cup (B_{11}, 3)$ |
| $\{B_{13}, B_{14}, B_{15}\}$ | $H_{13,14,15} = (B_{13}, 1) \cup (B_{14}, 2) \cup (B_{15}, 3)$ |
| | $H_{14,15,13} = (B_{14}, 1) \cup (B_{15}, 2) \cup (B_{13}, 3)$ |
| | $H_{15,13,14} = (B_{15}, 1) \cup (B_{13}, 2) \cup (B_{14}, 3)$ |
| $\{B_{16}, B_{17}, B_{18}\}$ | $H_{16,17,18} = (B_{16}, 1) \cup (B_{17}, 2) \cup (B_{18}, 3)$ |
| | $H_{17,18,16} = (B_{17}, 1) \cup (B_{18}, 2) \cup (B_{16}, 3)$ |
| | $H_{18,16,17} = (B_{18}, 1) \cup (B_{16}, 2) \cup (B_{17}, 3)$ |
| $\{B_{19}, B_{20}, B_{21}, B_{22}\}$ | $H_{19,20,21} = (B_{19}, 1) \cup (B_{20}, 2) \cup (B_{21}, 3)$ |
| | $H_{20,21,22} = (B_{20}, 1) \cup (B_{21}, 2) \cup (B_{22}, 3)$ |
| | $H_{21,22,19} = (B_{21}, 1) \cup (B_{22}, 2) \cup (B_{19}, 3)$ |
| | $H_{22,19,20} = (B_{22}, 1) \cup (B_{19}, 2) \cup (B_{20}, 3)$ |

**Definition 4.23.** For any $C_{10}$-design $\mathscr{B}$ on $\mathcal{G}$, the ***W-separation graph*** of $\mathscr{B}$ is the graph $\mathrm{WS}(\mathscr{B})$ with vertex set $\mathscr{B}$ and edge set $\Big\{\{A, B\} \;\Big|\; A \text{ and } B \text{ are } W\text{-separated}\Big\}$. ■

By Remark 4.22, it suffices to exhibit a $C_{10}$-design on $\mathcal{G}$ that has a partition into $W$-separated collections, six of size 3 and one of size 4. A $C_{10}$-design $\mathscr{B}$ on $\mathcal{G}$ has such a partition if and only if there exists a spanning subgraph of $\mathrm{WS}(\mathscr{B})$ whose seven components are six $K_3$'s and one $K_4$. In what follows, we describe the construction of a $C_{10}$-design on $\mathcal{G}$ with the desired property. We begin with an observation about partitions of the set $U$ that are induced by a $C_{10}$-design on $\mathcal{G}$; this observation informs the first stage of our construction.

**Remark 4.24.** Suppose $\mathscr{B}$ is a $C_{10}$-design on $\mathcal{G}$. Then each vertex in $W$ is a vertex of exactly five cycles in $\mathscr{B}$; furthermore, if $w \in W$ is a vertex of the cycle $B \in \mathscr{B}$, then $w$ is adjacent to exactly two vertices of $U$ in the cycle $B$. Let $U(B, w)$ denote the set of vertices in $U$ that are adjacent to $w$ in cycle $B$. For each vertex $w \in W$, the collection

$$U(\mathscr{B}, w) = \Big\{U(B, w) \;\Big|\; B \in \mathscr{B} \text{ and } w \in V(B)\Big\}$$

is a partition of $U$ into five subsets, each of size two. ■

We begin our construction of a $C_{10}$-design $\mathscr{B}$ on $\mathcal{G}$ by assigning partitions of $U$ (each consisting of five 2-element subsets of $U$) to the vertices in $W$. We denote the partition of $U$ assigned to the vertex $w$ by $\mathscr{P}_w$. The partition assignment is given in Table 4.4. Our assignment scheme uses nine partitions of $U$, among which each 2-element subset of $U$ occurs exactly once; each of these partitions is repeated either twice or thrice in the assignment scheme. We note that many other assignment schemes are clearly possible; we leave the identification of which characteristics produce an admissible scheme to future work.

We build the design $\mathscr{B}$ from the assigned partitions, so that $U(\mathscr{B}, w) = \mathscr{P}_w$ for each $w \in W$. Our process for obtaining cycles from these partitions begins with small paths. Each pair $\big(w, S\big)$ consisting of a vertex $w \in W$ and a set $S \in \mathscr{P}_w$ defines a unique $P_2$-subgraph of $\mathcal{G}$ centered at $w$. For example, note that $S = \{2^\star, 8^\star\}$ is one of the sets in the partition

Table 4.4: The assignment of partitions of $U$ to the elements of $W = [\![1, 22]\!]$

| Vertex $w$ | Partition of $U$ assigned to $w$ | | | | |
|---|---|---|---|---|---|
| 1 | $\{1^\star, 2^\star\}$ | $\{3^\star, 5^\star\}$ | $\{4^\star, 9^\star\}$ | $\{6^\star, 10^\star\}$ | $\{7^\star, 8^\star\}$ |
| 2 | $\{1^\star, 3^\star\}$ | $\{2^\star, 10^\star\}$ | $\{4^\star, 6^\star\}$ | $\{5^\star, 7^\star\}$ | $\{8^\star, 9^\star\}$ |
| 3 | $\{1^\star, 4^\star\}$ | $\{2^\star, 7^\star\}$ | $\{3^\star, 8^\star\}$ | $\{5^\star, 10^\star\}$ | $\{6^\star, 9^\star\}$ |
| 4 | $\{1^\star, 5^\star\}$ | $\{2^\star, 4^\star\}$ | $\{3^\star, 6^\star\}$ | $\{7^\star, 9^\star\}$ | $\{8^\star, 10^\star\}$ |
| 5 | $\{1^\star, 6^\star\}$ | $\{2^\star, 8^\star\}$ | $\{3^\star, 4^\star\}$ | $\{5^\star, 9^\star\}$ | $\{7^\star, 10^\star\}$ |
| 6 | $\{1^\star, 7^\star\}$ | $\{2^\star, 3^\star\}$ | $\{4^\star, 5^\star\}$ | $\{6^\star, 8^\star\}$ | $\{9^\star, 10^\star\}$ |
| 7 | $\{1^\star, 8^\star\}$ | $\{2^\star, 9^\star\}$ | $\{3^\star, 7^\star\}$ | $\{4^\star, 10^\star\}$ | $\{5^\star, 6^\star\}$ |
| 8 | $\{1^\star, 9^\star\}$ | $\{2^\star, 6^\star\}$ | $\{3^\star, 10^\star\}$ | $\{4^\star, 7^\star\}$ | $\{5^\star, 8^\star\}$ |
| 9 | $\{1^\star, 10^\star\}$ | $\{2^\star, 5^\star\}$ | $\{3^\star, 9^\star\}$ | $\{4^\star, 8^\star\}$ | $\{6^\star, 7^\star\}$ |
| 10 | $\{1^\star, 5^\star\}$ | $\{2^\star, 4^\star\}$ | $\{3^\star, 6^\star\}$ | $\{7^\star, 9^\star\}$ | $\{8^\star, 10^\star\}$ |
| 11 | $\{1^\star, 7^\star\}$ | $\{2^\star, 3^\star\}$ | $\{4^\star, 5^\star\}$ | $\{6^\star, 8^\star\}$ | $\{9^\star, 10^\star\}$ |
| 12 | $\{1^\star, 9^\star\}$ | $\{2^\star, 6^\star\}$ | $\{3^\star, 10^\star\}$ | $\{4^\star, 7^\star\}$ | $\{5^\star, 8^\star\}$ |
| 13 | $\{1^\star, 2^\star\}$ | $\{3^\star, 5^\star\}$ | $\{4^\star, 9^\star\}$ | $\{6^\star, 10^\star\}$ | $\{7^\star, 8^\star\}$ |
| 14 | $\{1^\star, 4^\star\}$ | $\{2^\star, 7^\star\}$ | $\{3^\star, 8^\star\}$ | $\{5^\star, 10^\star\}$ | $\{6^\star, 9^\star\}$ |
| 15 | $\{1^\star, 8^\star\}$ | $\{2^\star, 9^\star\}$ | $\{3^\star, 7^\star\}$ | $\{4^\star, 10^\star\}$ | $\{5^\star, 6^\star\}$ |
| 16 | $\{1^\star, 3^\star\}$ | $\{2^\star, 10^\star\}$ | $\{4^\star, 6^\star\}$ | $\{5^\star, 7^\star\}$ | $\{8^\star, 9^\star\}$ |
| 17 | $\{1^\star, 6^\star\}$ | $\{2^\star, 8^\star\}$ | $\{3^\star, 4^\star\}$ | $\{5^\star, 9^\star\}$ | $\{7^\star, 10^\star\}$ |
| 18 | $\{1^\star, 2^\star\}$ | $\{3^\star, 5^\star\}$ | $\{4^\star, 9^\star\}$ | $\{6^\star, 10^\star\}$ | $\{7^\star, 8^\star\}$ |
| 19 | $\{1^\star, 8^\star\}$ | $\{2^\star, 9^\star\}$ | $\{3^\star, 7^\star\}$ | $\{4^\star, 10^\star\}$ | $\{5^\star, 6^\star\}$ |
| 20 | $\{1^\star, 4^\star\}$ | $\{2^\star, 7^\star\}$ | $\{3^\star, 8^\star\}$ | $\{5^\star, 10^\star\}$ | $\{6^\star, 9^\star\}$ |
| 21 | $\{1^\star, 10^\star\}$ | $\{2^\star, 5^\star\}$ | $\{3^\star, 9^\star\}$ | $\{4^\star, 8^\star\}$ | $\{6^\star, 7^\star\}$ |
| 22 | $\{1^\star, 7^\star\}$ | $\{2^\star, 3^\star\}$ | $\{4^\star, 5^\star\}$ | $\{6^\star, 8^\star\}$ | $\{9^\star, 10^\star\}$ |

assigned to the vertex $5 \in W$; the pair $\left(5,\ \{2^\star, 8^\star\}\right)$ defines the $P_2$-subgraph of $\mathcal{G}$ shown in Figure 4.1.

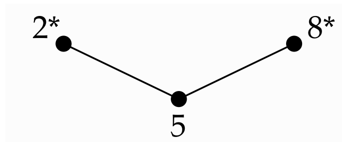



Figure 4.1: The $P_2$-subgraph of $\mathcal{G}$ defined by the pair $\left(5,\ \{2^\star, 8^\star\}\right)$

**Remark 4.25.** For each $t \in \{1, 2, 3, 4, 5\}$, let $w_t \in W$, and let $S_t \in \mathscr{P}_{w_t}$. The collection $\left\{ (w_t,\ S_t) \;\middle|\; 1 \le t \le 5 \right\}$ corresponds to a $C_{10}$-subgraph of $\mathcal{G} = K_{10,22}$ if and only if the following conditions hold.

(i) The five vertices $w_1$, $w_2$, $w_3$, $w_4$, and $w_5$ are distinct.

(ii) The five subsets $S_1$, $S_2$, $S_3$, $S_4$, and $S_5$ of $U$ are distinct.

(iii) The set $\mathcal{S} = \bigcup_{t=1}^{5} S_t$ has exactly five elements.

Observe that, in order for these conditions to hold, each element of $\mathcal{S}$ must be an element of exactly two of the subsets $S_t$, $1 \le t \le 5$. If such a collection corresponds to a $C_{10}$-subgraph of $\mathcal{G}$, then that subgraph has vertex set $\{w_1, w_2, w_3, w_4, w_5\} \cup \mathcal{S}$, and, for each $t \in \{1, 2, 3, 4, 5\}$, the vertex $w_t$ is adjacent to the two vertices in $S_t$. ■

As an illustrative example, consider the pairs $\left(1,\ \{7^\star, 8^\star\}\right)$, $\left(3,\ \{6^\star, 9^\star\}\right)$, $\left(5,\ \{2^\star, 8^\star\}\right)$, $\left(19,\ \{2^\star, 9^\star\}\right)$, and $\left(21,\ \{6^\star, 7^\star\}\right)$. The $P_2$-subgraphs of $\mathcal{G}$ defined by these pairs are shown in Figure 4.2. The collection consisting of these five pairs satisfies the conditions in Remark 4.25; this collection corresponds to the $C_{10}$-subgraph of $\mathcal{G}$ shown in Figure 4.3.

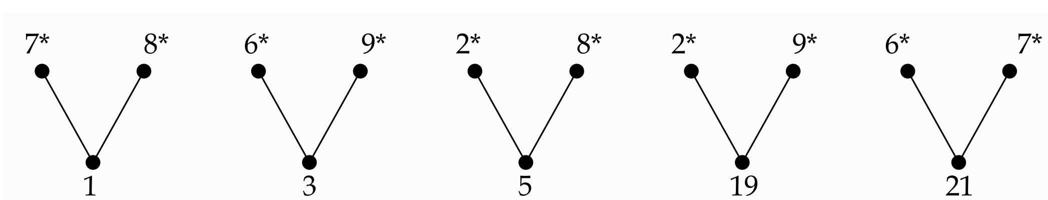



Figure 4.2: Five $P_2$-subgraphs of $\mathcal{G}$

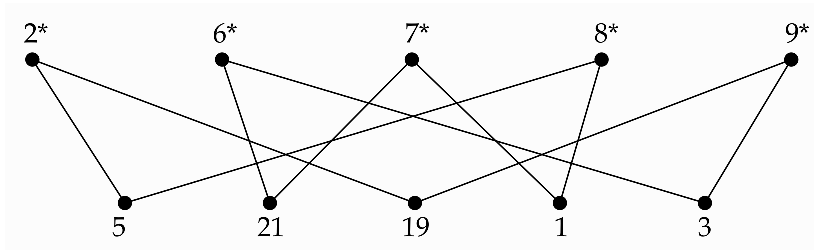



Figure 4.3: A $C_{10}$-subgraph of $\mathcal{G}$

In order to create a $C_{10}$-design from such collections, we must identify twenty-two pairwise-disjoint collections of the form $\left\{ (w_t,\, S_t) \;\middle|\; 1 \le t \le 5 \right\}$, each of which corresponds to a $C_{10}$-subgraph of $\mathcal{G}$; these subgraphs are the blocks of the design. We obtain a $C_{10}$-design that can be partitioned into $W$-separated collections of size 3 and 4 by careful selection and adjustment. The cycles in $\mathscr{B}$, the $C_{10}$-design on $\mathcal{G}$, are listed in Table 4.5. The $W$-separation graph for this design is shown in Figure 4.4; the desired spanning subgraph, consisting of six $K_3$'s and one $K_4$, is shown in multiple colors in this figure. The cycles of the design $\mathscr{B}$ are shown in Figures 4.5 – 4.11; each figure shows the cycles in a $W$-separated collection corresponding to a component of the spanning subgraph shown in Figure 4.4.

Table 4.5: The cycles of $\mathscr{B}$, the $C_{10}$-design on $\mathcal{G} = K_{10,\,22}$

| | |
|---|---|
| $A = \big(1, 1^\star, 2, 3^\star, 4, 6^\star, 14, 9^\star, 15, 2^\star\big)$ | $L = \big(9, 2^\star, 14, 7^\star, 5, 10^\star, 1, 6^\star, 15, 5^\star\big)$ |
| $B = \big(13, 3^\star, 20, 8^\star, 10, 10^\star, 19, 4^\star, 22, 5^\star\big)$ | $M = \big(20, 1^\star, 19, 8^\star, 14, 3^\star, 18, 5^\star, 6, 4^\star\big)$ |
| $C = \big(14, 1^\star, 15, 8^\star, 12, 5^\star, 5, 9^\star, 1, 4^\star\big)$ | $N = \big(7, 2^\star, 2, 10^\star, 4, 8^\star, 3, 3^\star, 9, 9^\star\big)$ |
| $D = \big(8, 1^\star, 21, 10^\star, 13, 6^\star, 6, 8^\star, 16, 9^\star\big)$ | $O = \big(18, 1^\star, 9, 10^\star, 14, 5^\star, 8, 8^\star, 17, 2^\star\big)$ |
| $E = \big(6, 2^\star, 10, 4^\star, 9, 8^\star, 18, 7^\star, 7, 3^\star\big)$ | $P = \big(5, 2^\star, 19, 9^\star, 3, 6^\star, 21, 7^\star, 1, 8^\star\big)$ |
| $F = \big(4, 1^\star, 22, 7^\star, 20, 2^\star, 8, 6^\star, 7, 5^\star\big)$ | $Q = \big(11, 2^\star, 16, 10^\star, 6, 9^\star, 4, 7^\star, 19, 3^\star\big)$ |
| $G = \big(3, 10^\star, 12, 3^\star, 21, 9^\star, 10, 7^\star, 16, 5^\star\big)$ | $R = \big(13, 1^\star, 12, 9^\star, 11, 10^\star, 20, 5^\star, 21, 2^\star\big)$ |
| $H = \big(11, 1^\star, 3, 4^\star, 5, 3^\star, 1, 5^\star, 2, 7^\star\big)$ | $S = \big(6, 1^\star, 7, 8^\star, 22, 6^\star, 18, 10^\star, 17, 7^\star\big)$ |
| $I = \big(22, 2^\star, 4, 4^\star, 21, 8^\star, 13, 7^\star, 15, 3^\star\big)$ | $T = \big(16, 4^\star, 15, 10^\star, 22, 9^\star, 17, 5^\star, 19, 6^\star\big)$ |
| $J = \big(10, 1^\star, 17, 6^\star, 9, 7^\star, 12, 4^\star, 11, 5^\star\big)$ | $U = \big(12, 2^\star, 3, 7^\star, 8, 4^\star, 18, 9^\star, 20, 6^\star\big)$ |
| $K = \big(16, 1^\star, 5, 6^\star, 2, 4^\star, 7, 10^\star, 8, 3^\star\big)$ | $V = \big(17, 3^\star, 10, 6^\star, 11, 8^\star, 2, 9^\star, 13, 4^\star\big)$ |

We obtain the $\mathcal{C}_{10}^{3}$-design $\mathscr{B}'$ on $\mathcal{H} = K_{30,22}$ from $\mathscr{B}$ as described in Remark 4.22. The $\mathcal{C}_{10}^{3}$-blocks of this design are shown in Figures 4.12 – 4.33. In these figures, individual cycles in these blocks are colored to match the colors used in Figures 4.5 – 4.11.

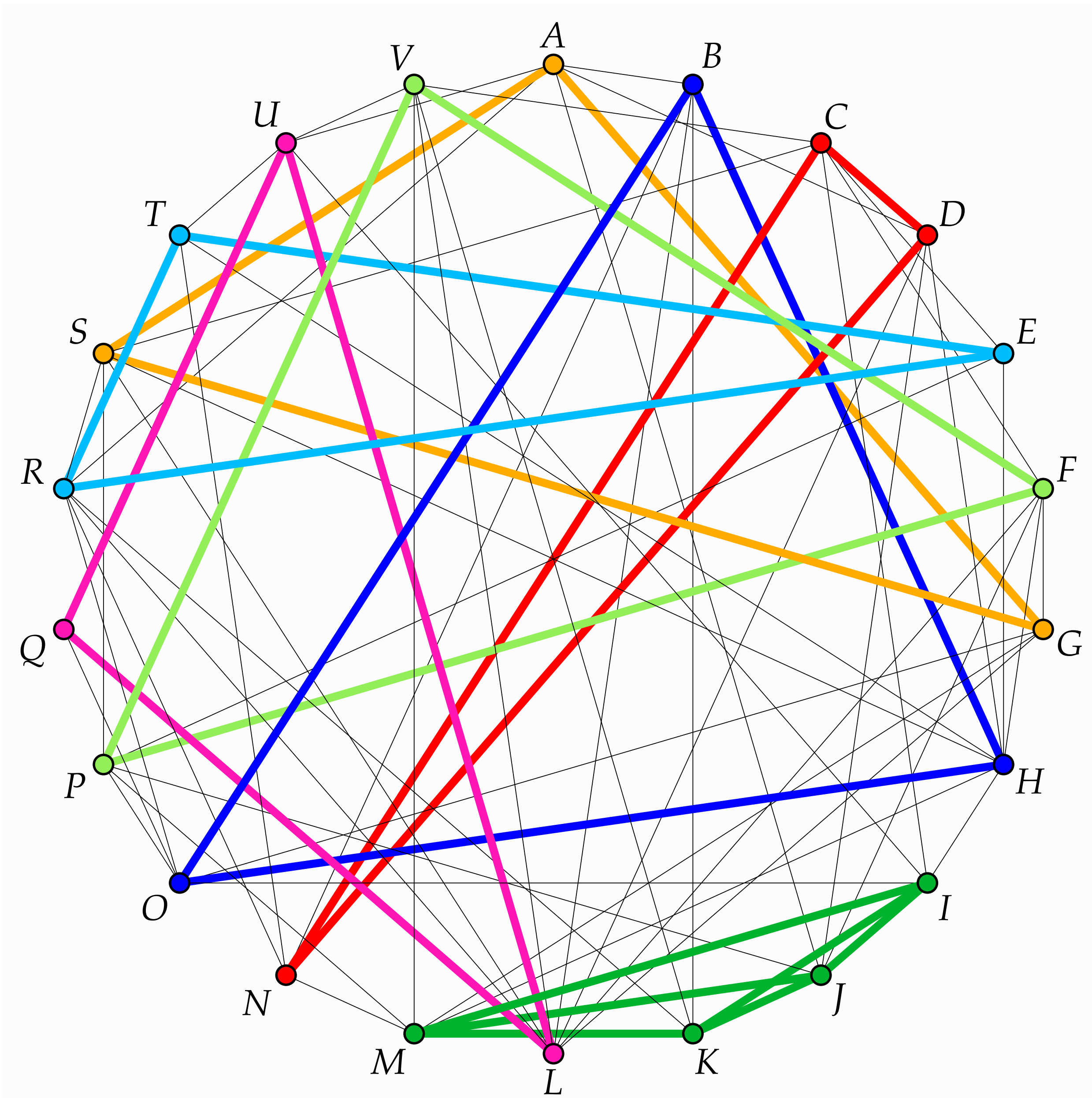



Figure 4.4: The $W$-separation graph for the $C_{10}$-design $\mathscr{B}$ on $\mathcal{G}$

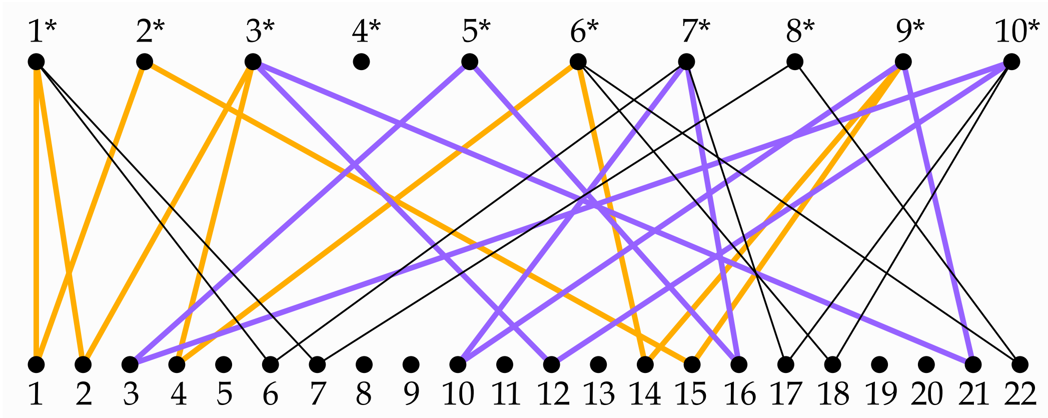



Figure 4.5: Cycles $A$ (apricot), $G$ (violet), and $S$ (black) are $W$-separated.

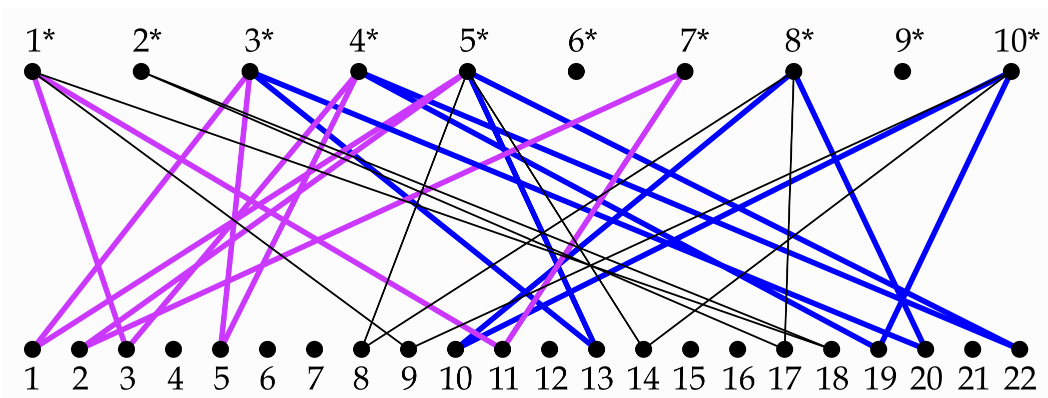



Figure 4.6: Cycles $B$ (cobalt), $H$ (lilac), and $O$ (black) are $W$-separated.

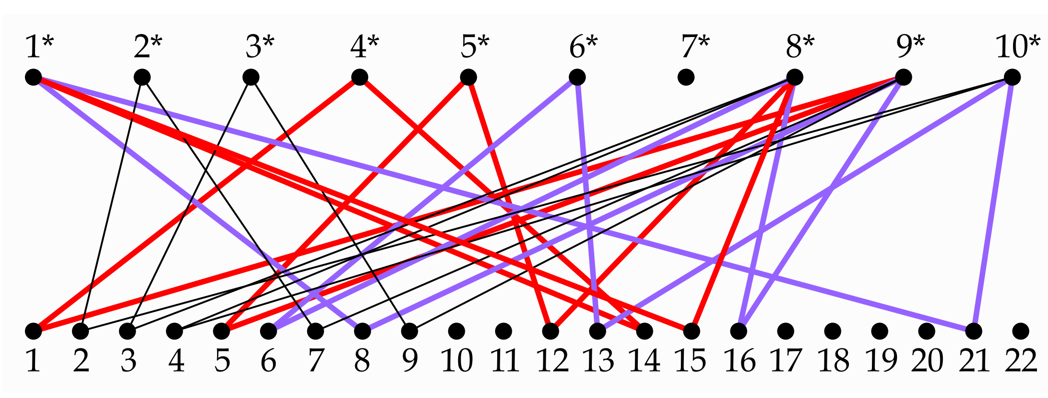



Figure 4.7: Cycles $C$ (red), $D$ (violet), and $N$(black) are $W$-separated.

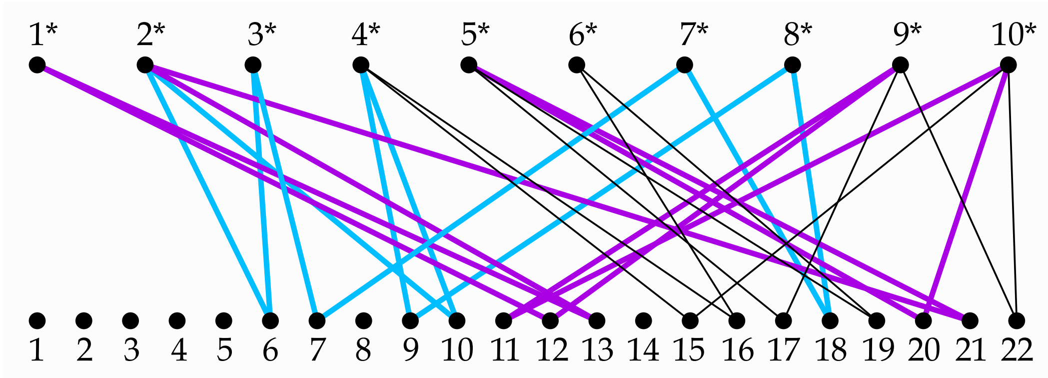



Figure 4.8: Cycles $E$ (sky blue), $R$ (plum), and $T$ (black) are $W$-separated.

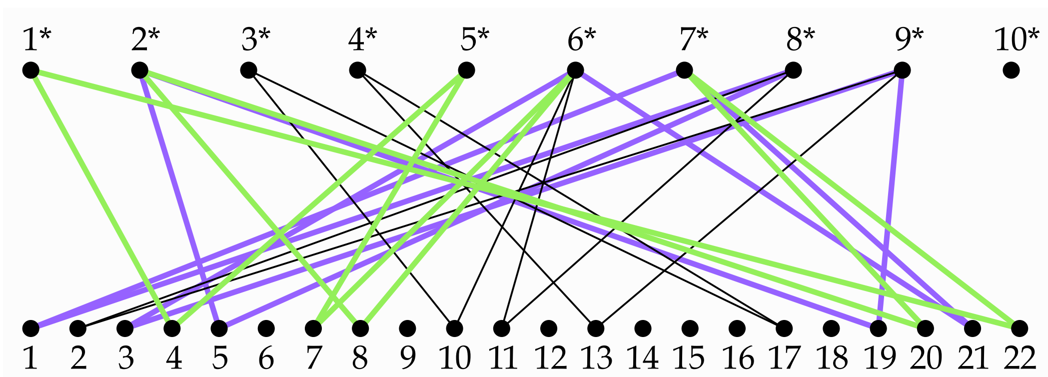



Figure 4.9: Cycles $F$ (lime), $P$ (violet), and $V$ (black) are $W$-separated.

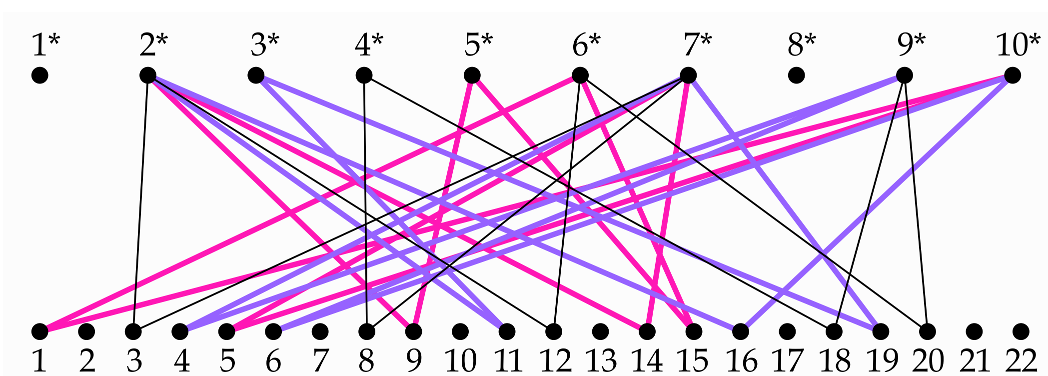



Figure 4.10: Cycles $L$ (pink), $Q$ (violet), and $U$ (black) are $W$-separated.

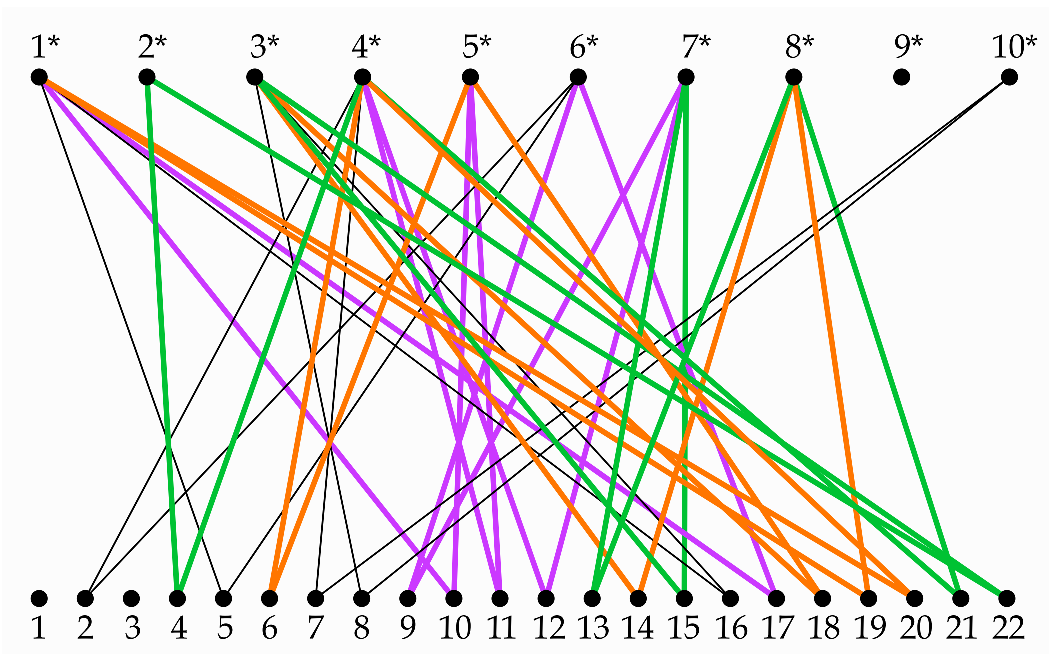
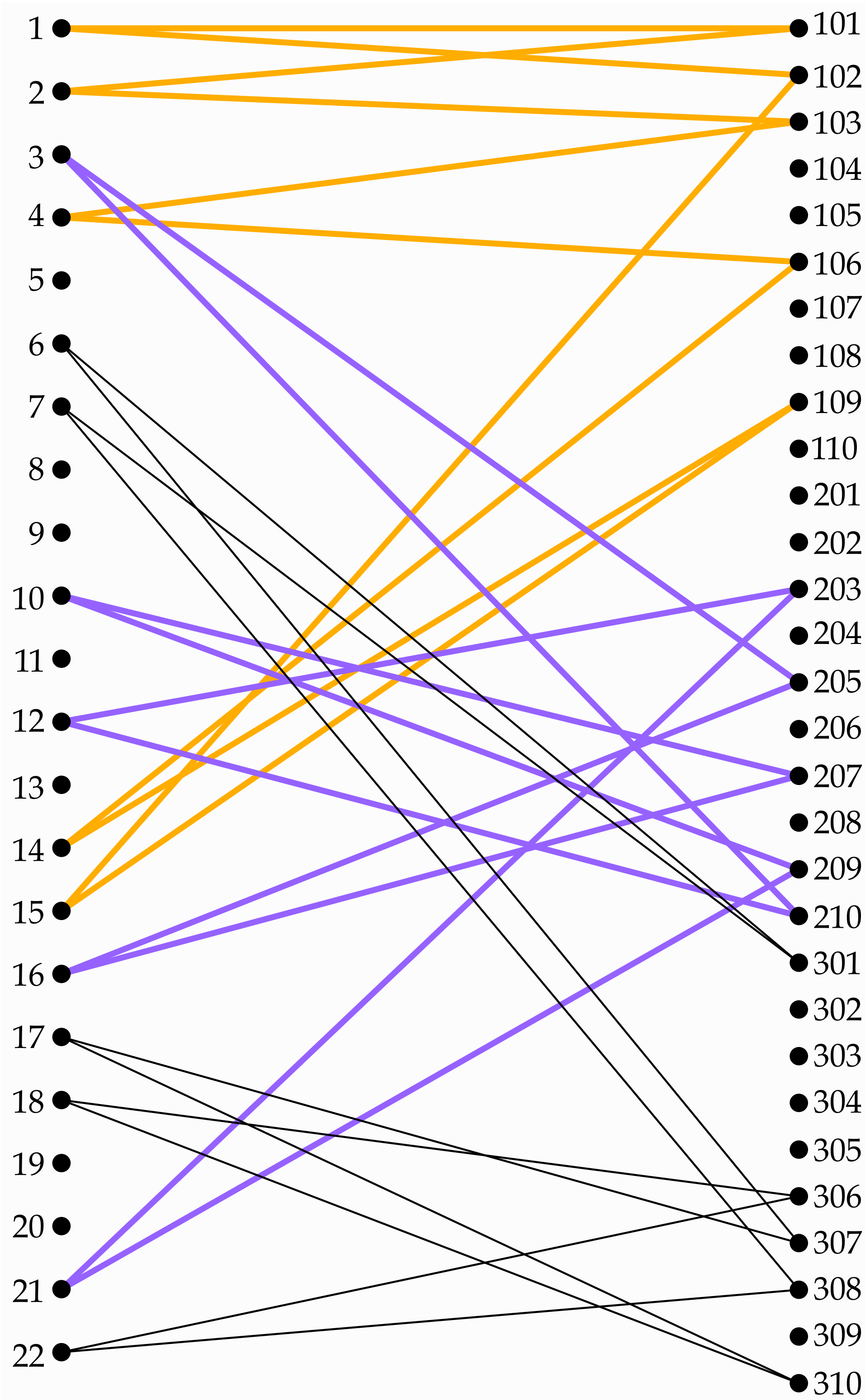
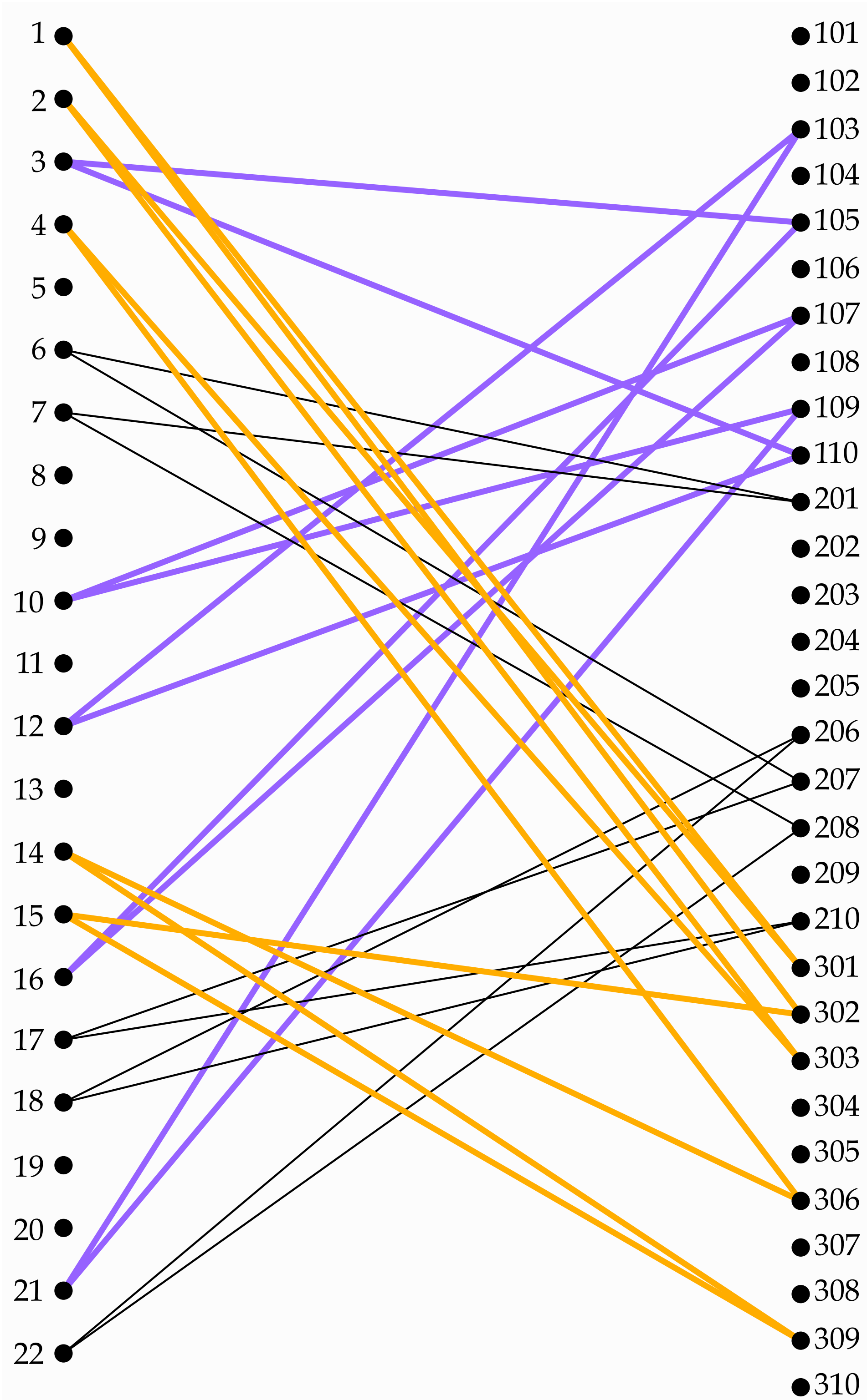



Figure 4.11: Cycles $I$ (jade), $J$ (lilac), $K$ (black), and $M$ (orange) are $W$-separated.



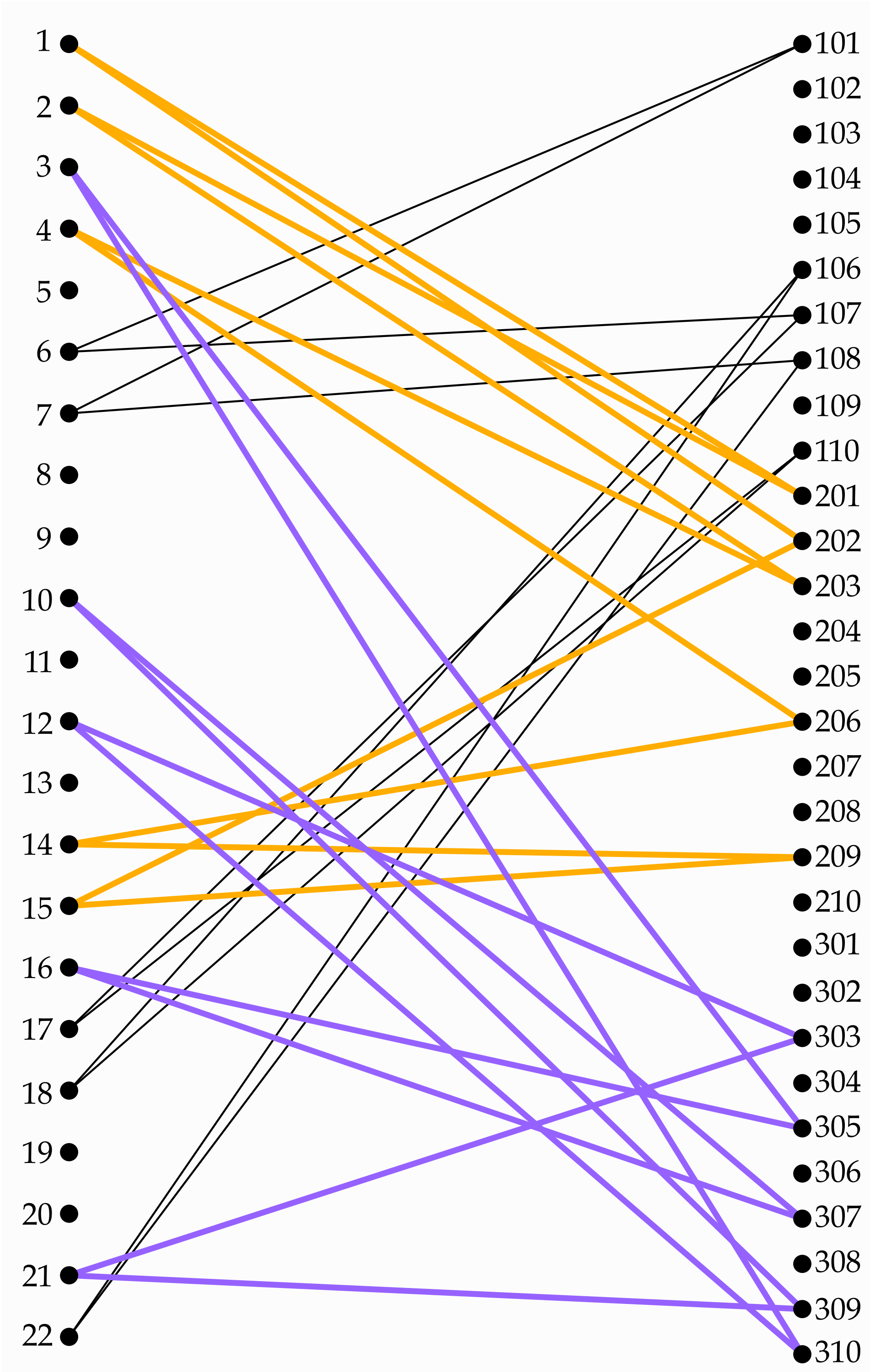

$(A, 1)$ in apricot; $(G, 2)$ in violet; $(S, 3)$ in black

Figure 4.12: The $\mathcal{C}_{10}^{3}$-block $(A, 1) \cup (G, 2) \cup (S, 3)$.



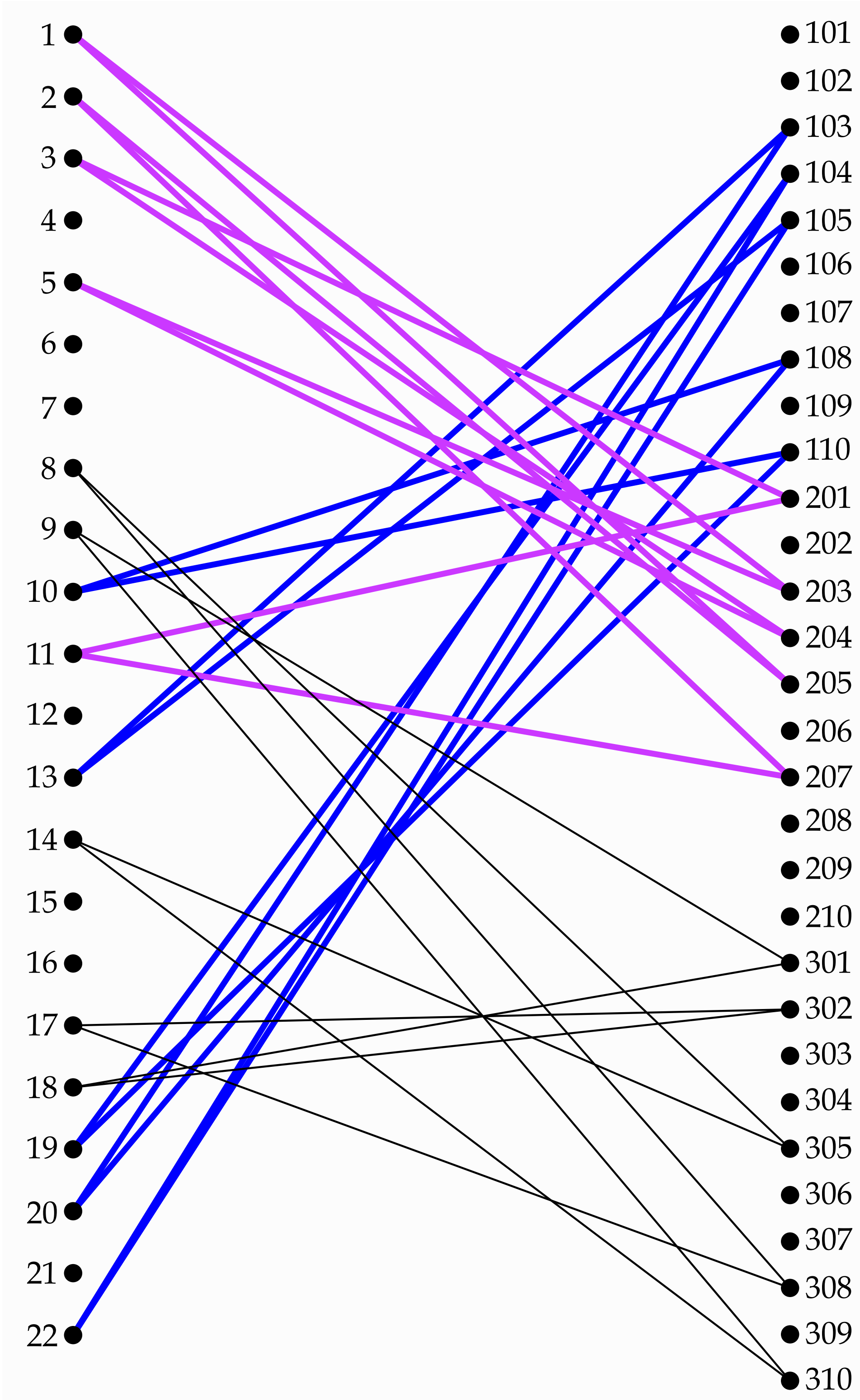

$(A, 3)$ in apricot; $(G, 1)$ in violet; $(S, 2)$ in black

Figure 4.13: The $\mathcal{C}_{10}^3$-block $(G, 1) \cup (S, 2) \cup (A, 3)$.



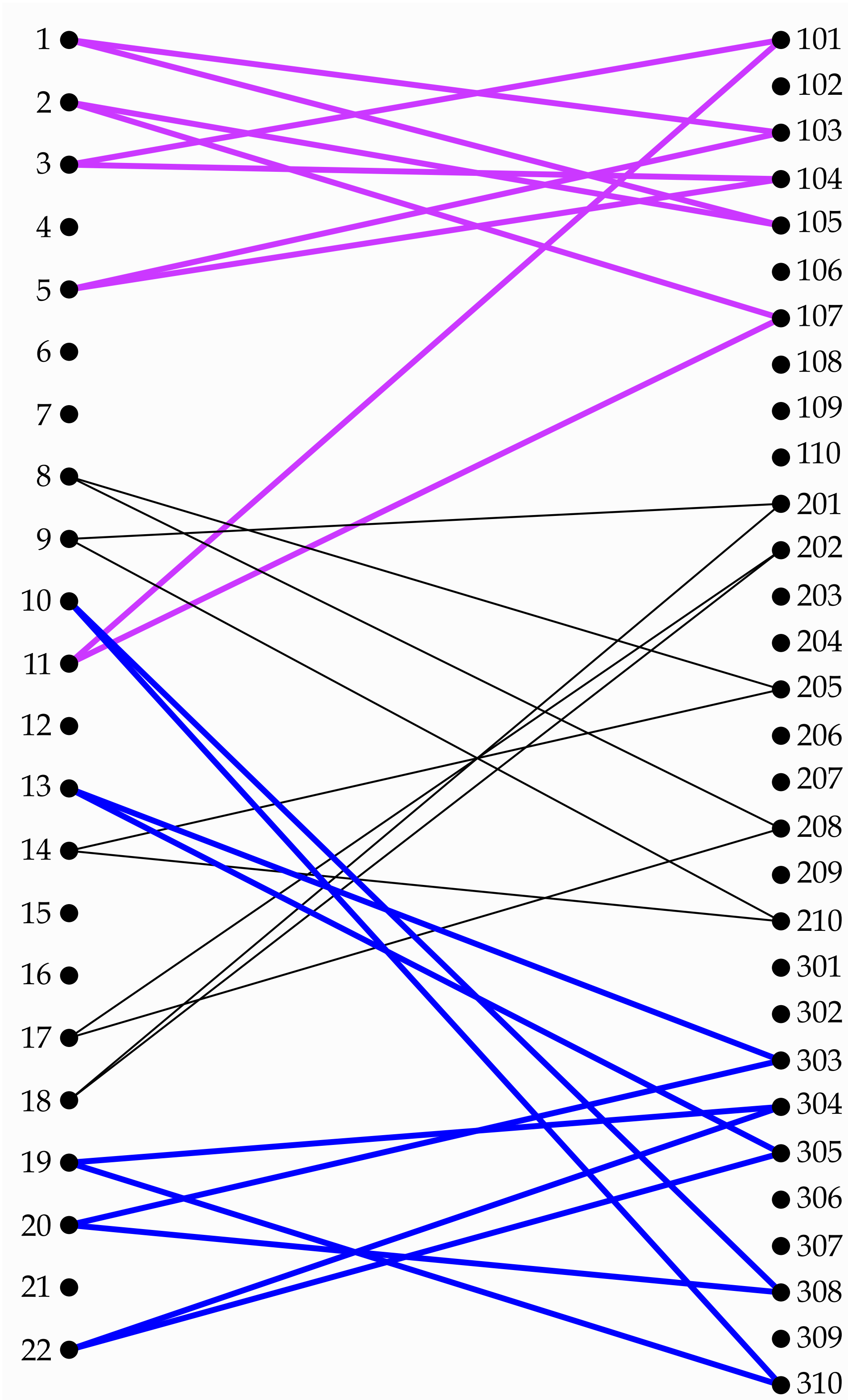

$(A, 2)$ in apricot; $(G, 3)$ in violet; $(S, 1)$ in black

Figure 4.14: The $\mathcal{C}_{10}^{3}$-block $(S, 1) \cup (A, 2) \cup (G, 3)$.



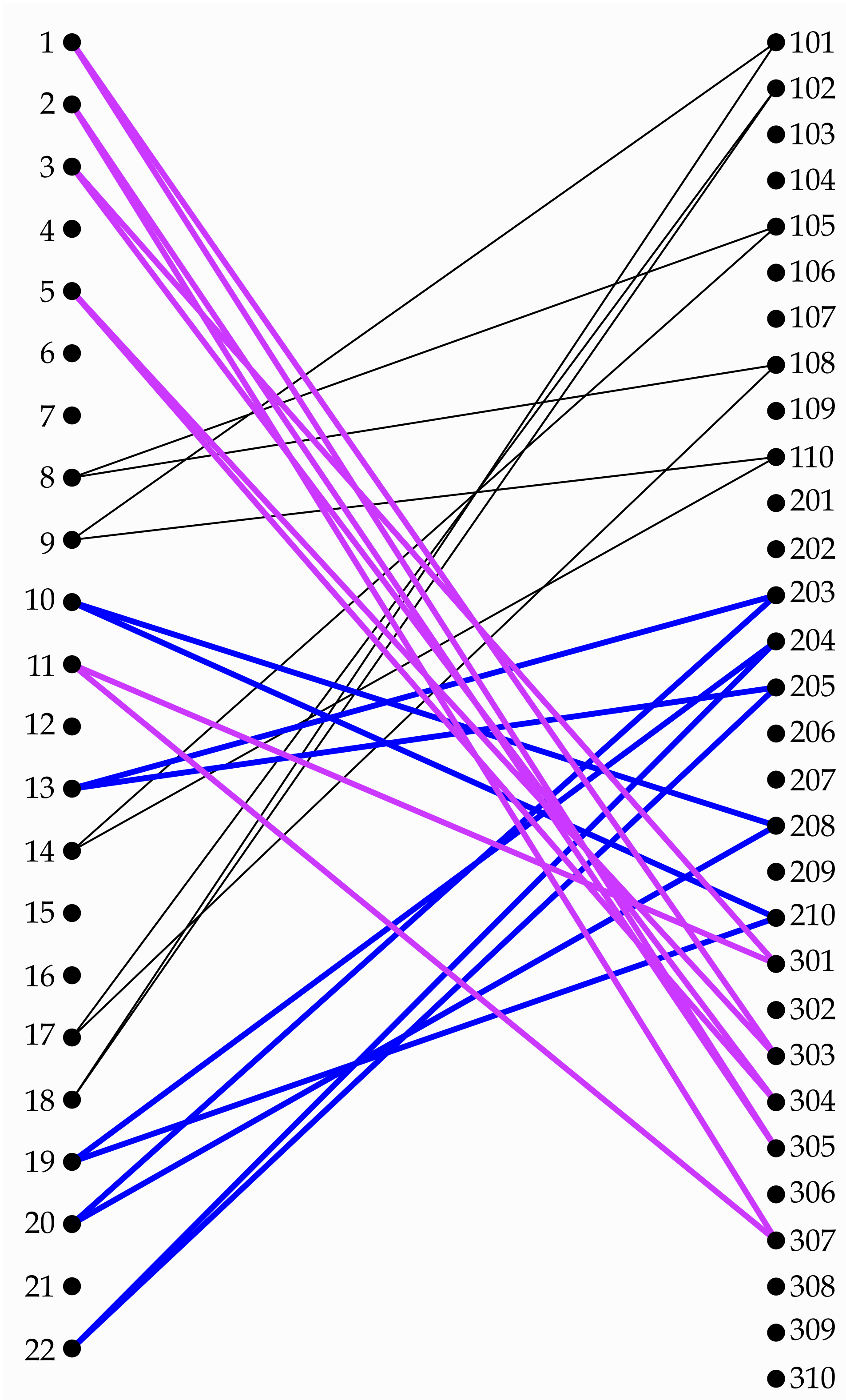

$(B,1)$ in cobalt; $(H,2)$ in lilac; $(O,3)$ in black

Figure 4.15: The $\mathcal{C}_{10}^{3}$-block $(B,1)\cup(H,2)\cup(O,3)$.



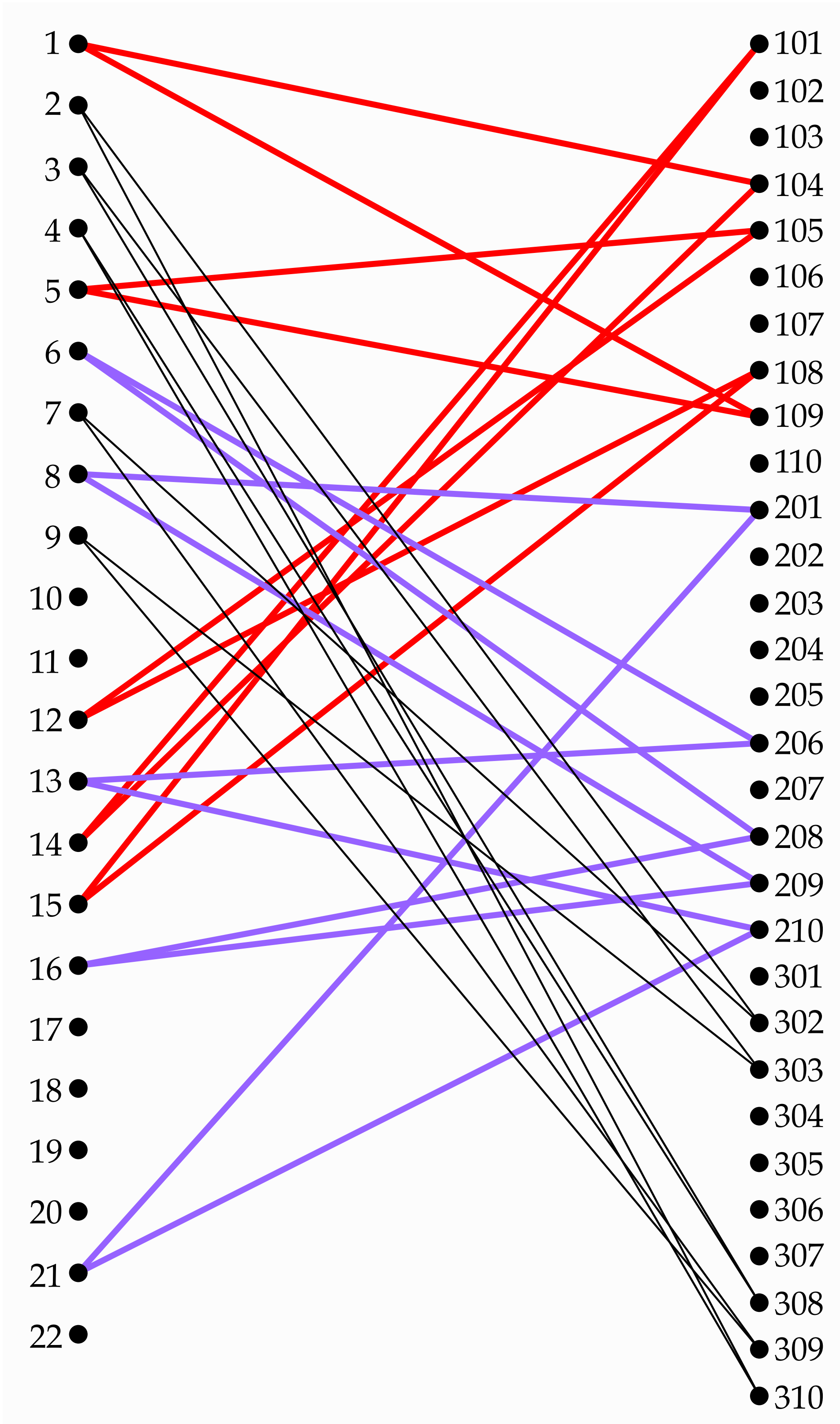

$(B, 3)$ in cobalt; $(H, 1)$ in lilac; $(O, 2)$ in black

Figure 4.16: The $\mathcal{C}_{10}^{3}$-block $(H, 1) \cup (O, 2) \cup (B, 3)$.



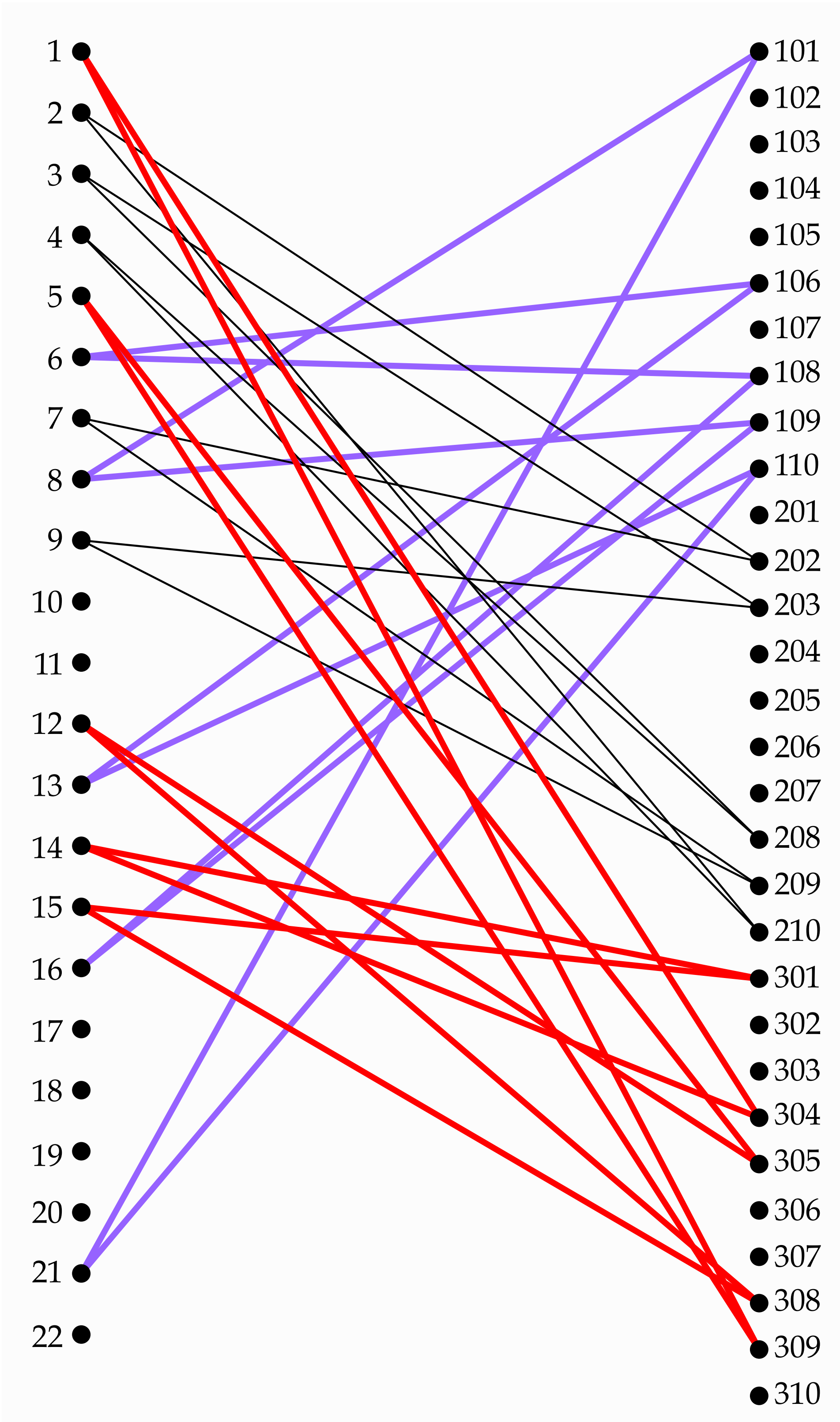

$(B, 2)$ in cobalt; $(H, 3)$ in lilac; $(O, 1)$ in black

Figure 4.17: The $\mathcal{C}_{10}^3$-block $(O, 1) \cup (B, 2) \cup (H, 3)$.



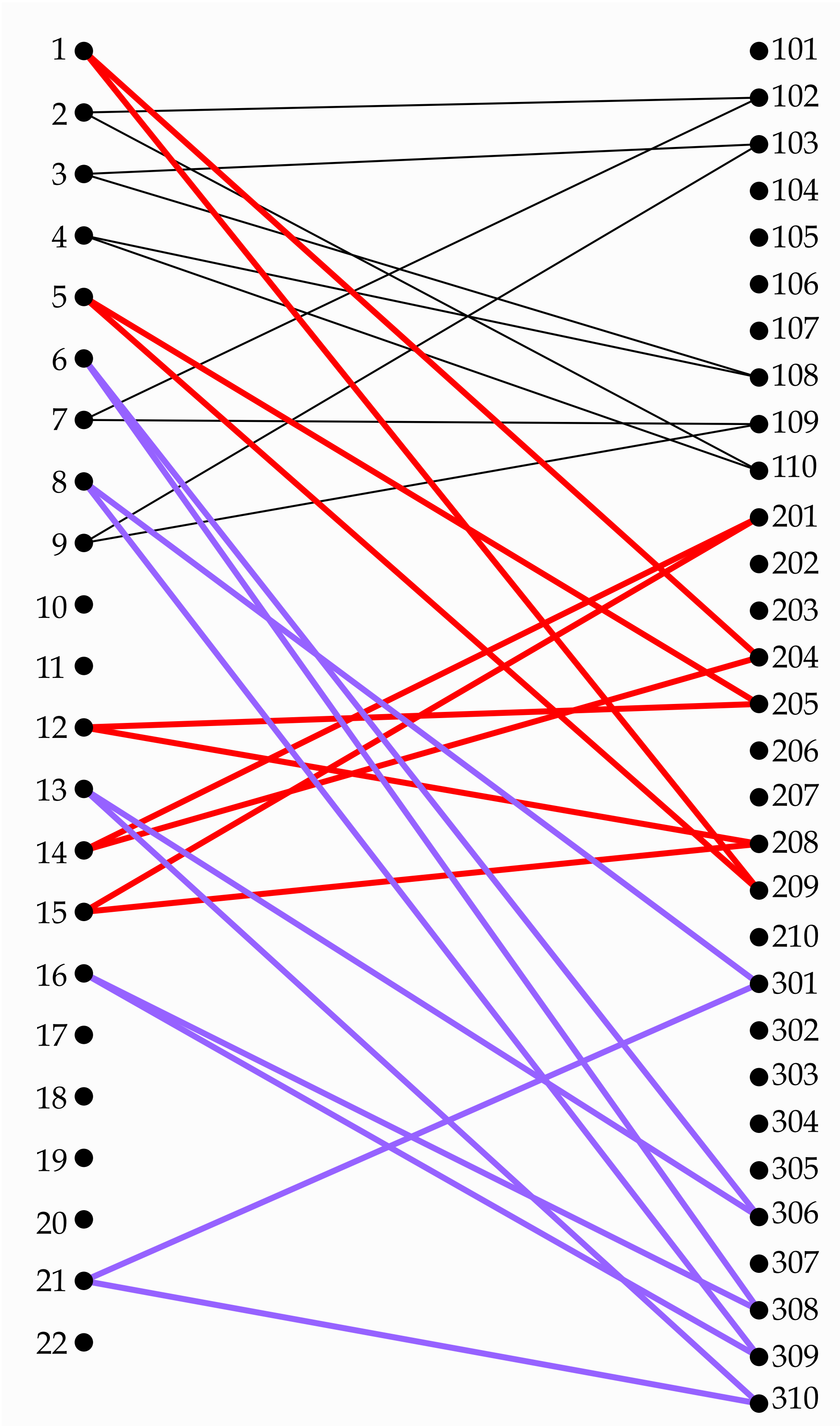

$(C, 1)$ in red; $(D, 2)$ in violet; $(N, 3)$ in black

Figure 4.18: The $\mathcal{C}_{10}^3$-block $(C, 1) \cup (D, 2) \cup (N, 3)$.



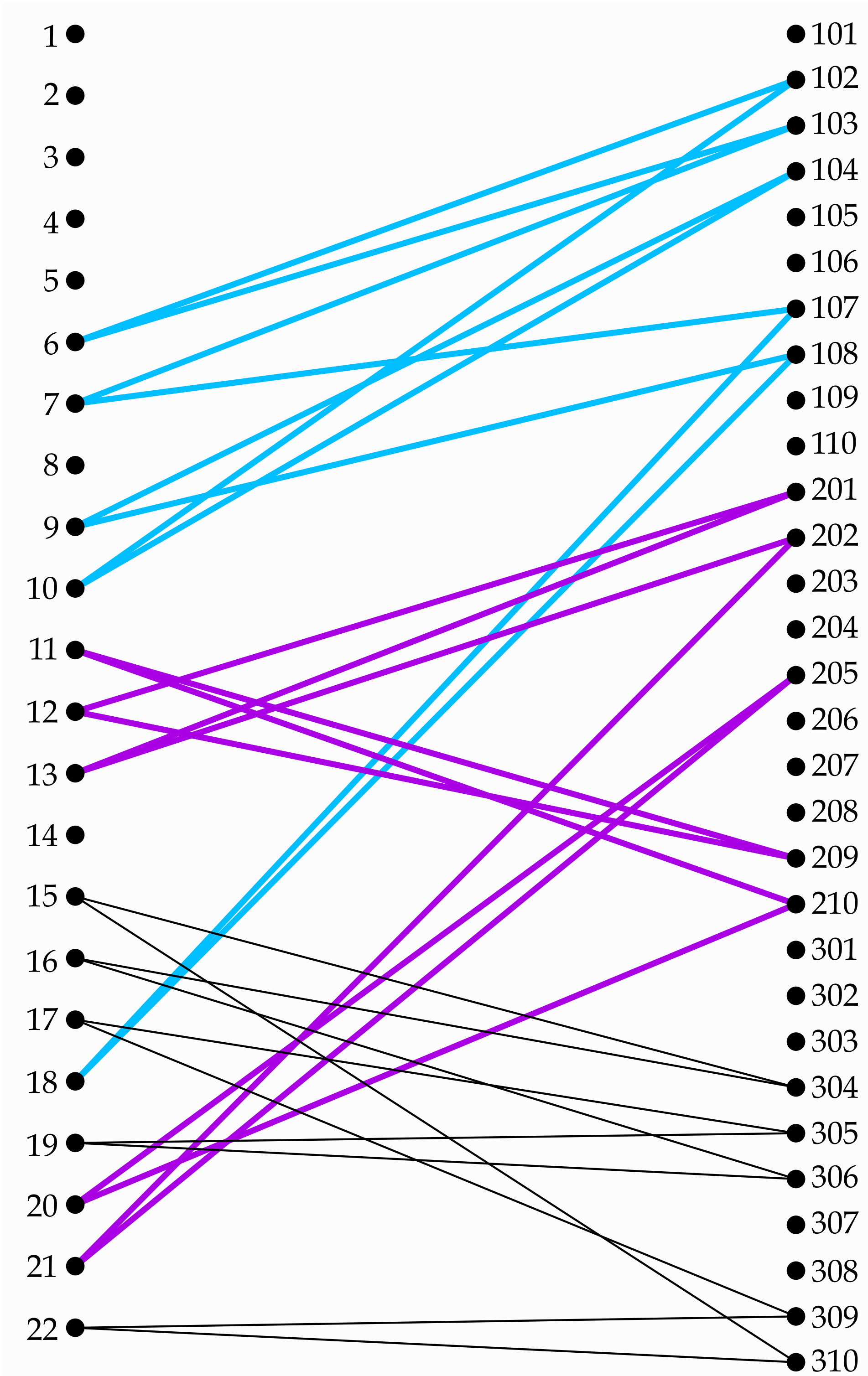

$(C,3)$ in red; $(D,1)$ in violet; $(N,2)$ in black

Figure 4.19: The $\mathcal{C}_{10}^3$-block $(D,1)\cup(N,2)\cup(C,3)$.



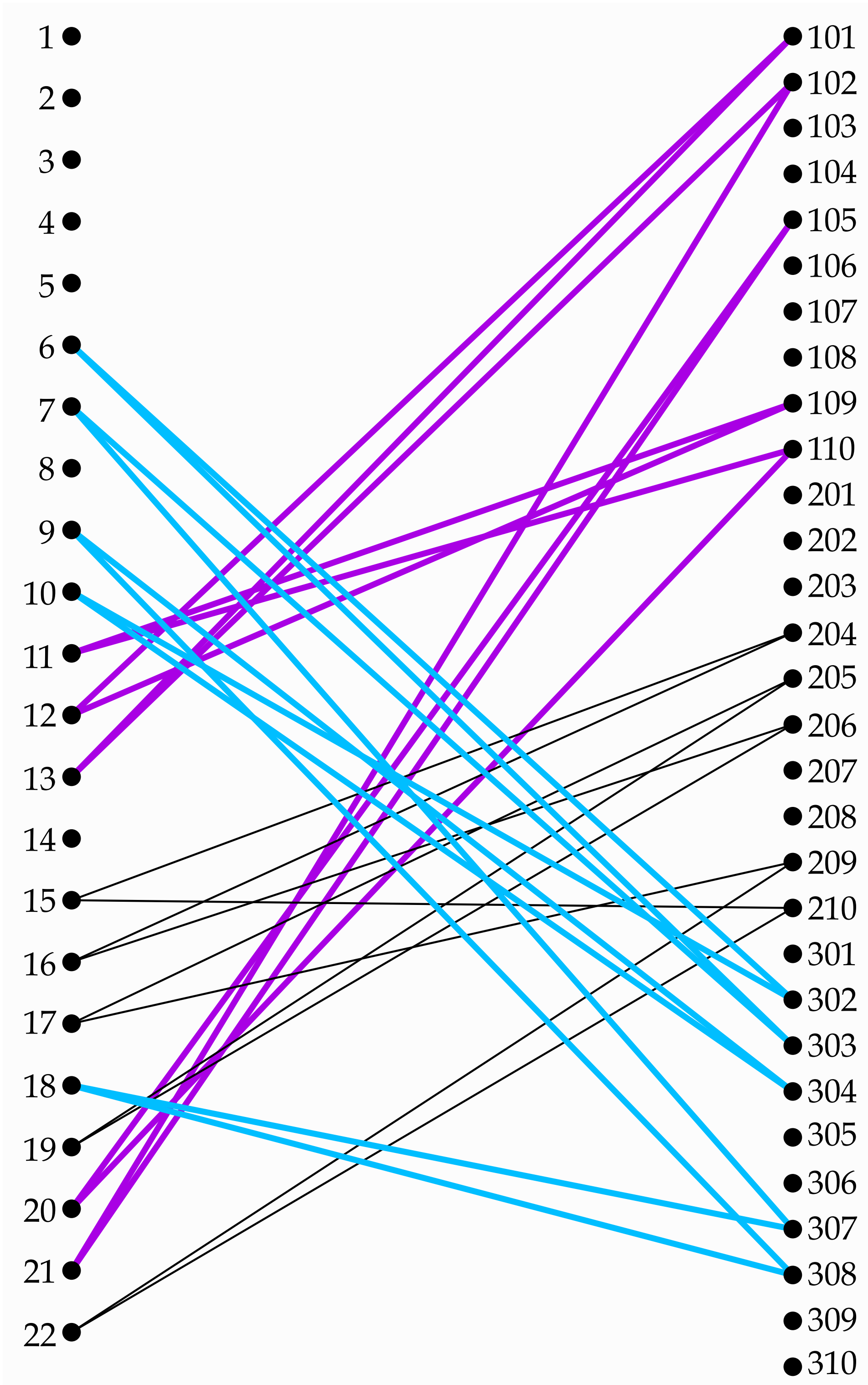

$(C, 2)$ in red; $(D, 3)$ in violet; $(N, 1)$ in black

Figure 4.20: The $\mathcal{C}_{10}^{3}$-block $(N, 1) \cup (C, 2) \cup (D, 3)$.



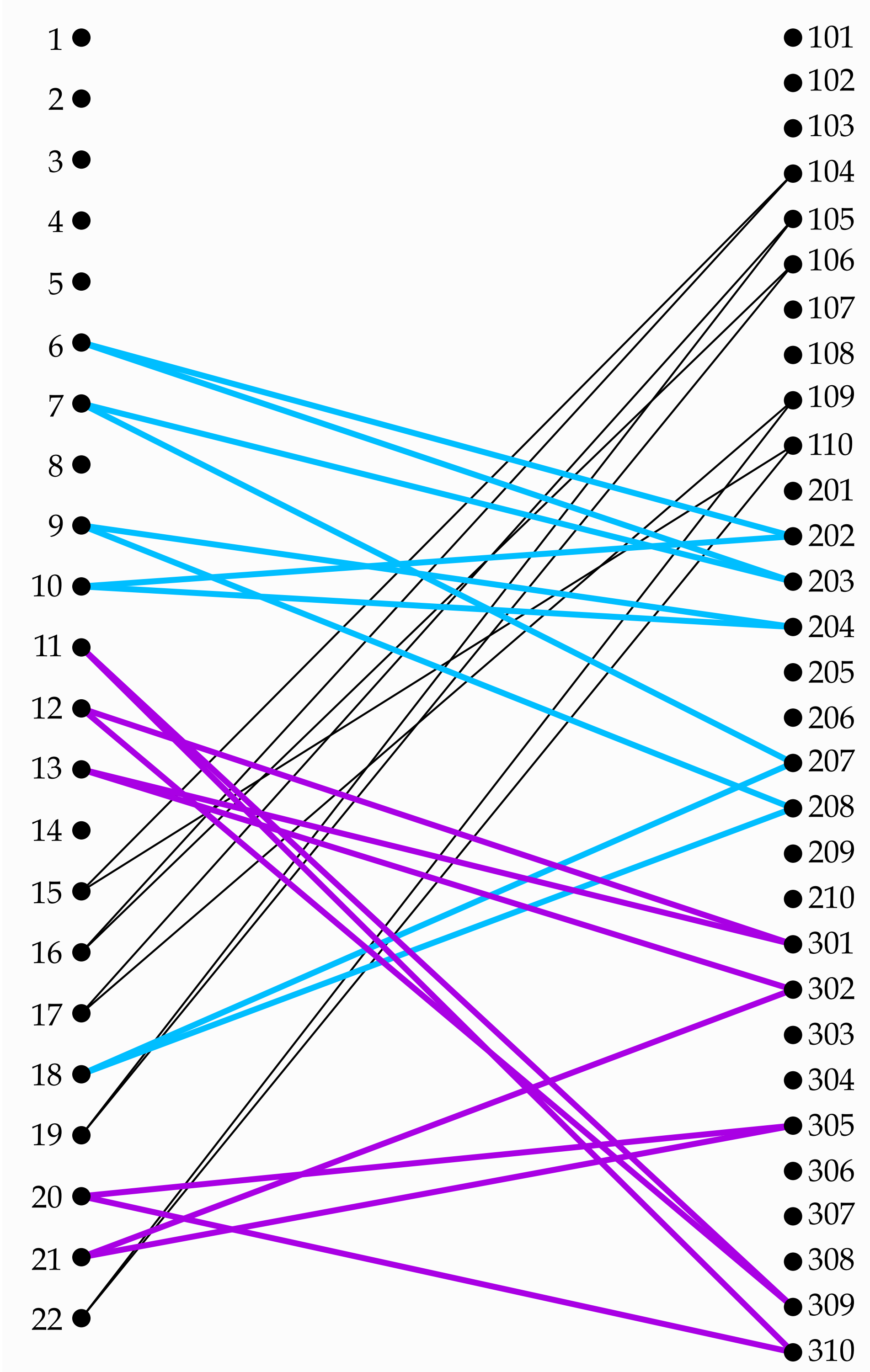

$(E, 1)$ in sky blue; $(R, 2)$ in plum; $(T, 3)$ in black

Figure 4.21: The $\mathcal{C}_{10}^3$-block $(E, 1) \cup (R, 2) \cup (T, 3)$.



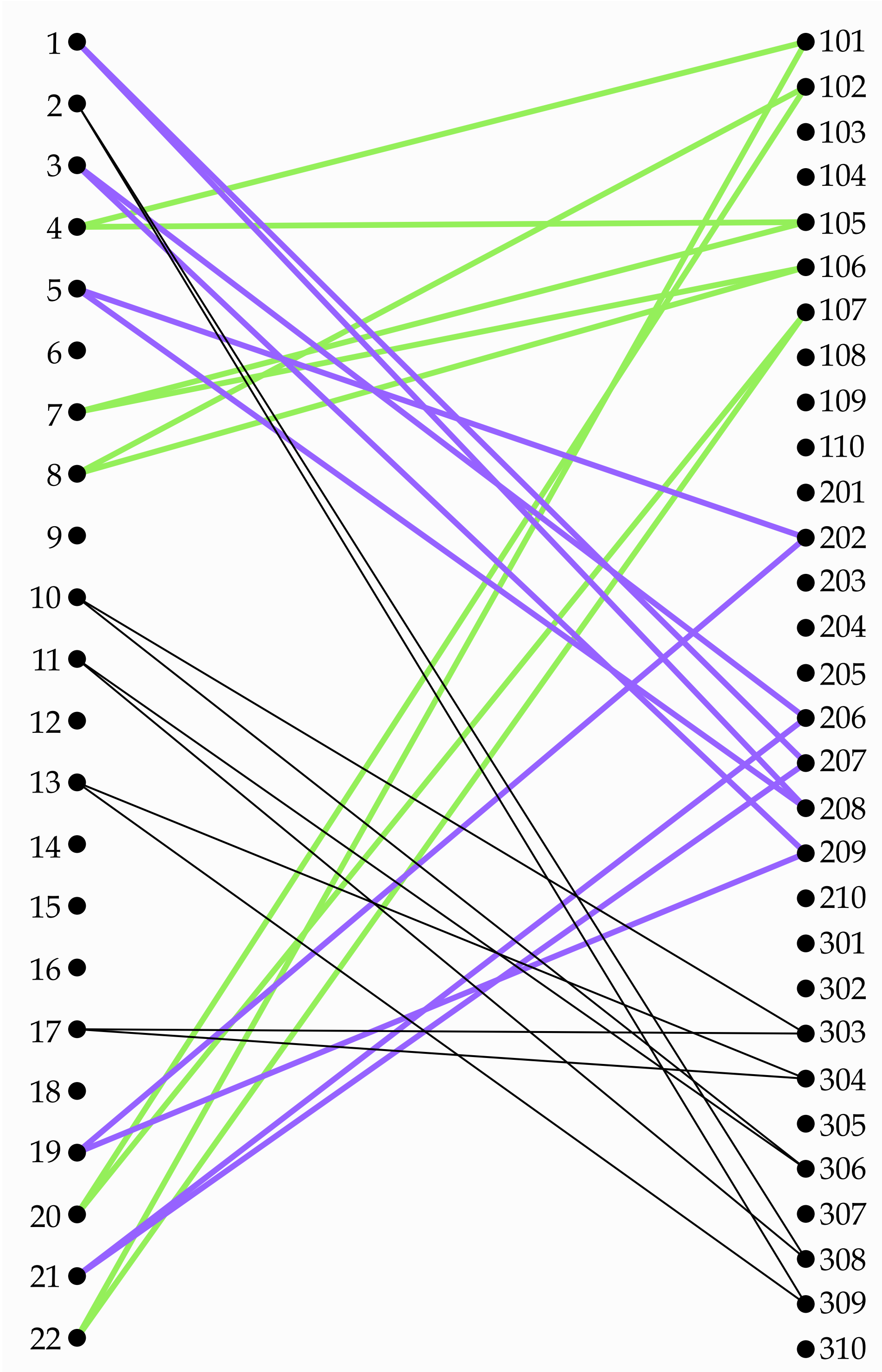

$(E, 3)$ in sky blue; $(R, 1)$ in plum; $(T, 2)$ in black

Figure 4.22: The $\mathcal{C}_{10}^{3}$-block $(R, 1) \cup (T, 2) \cup (E, 3)$.



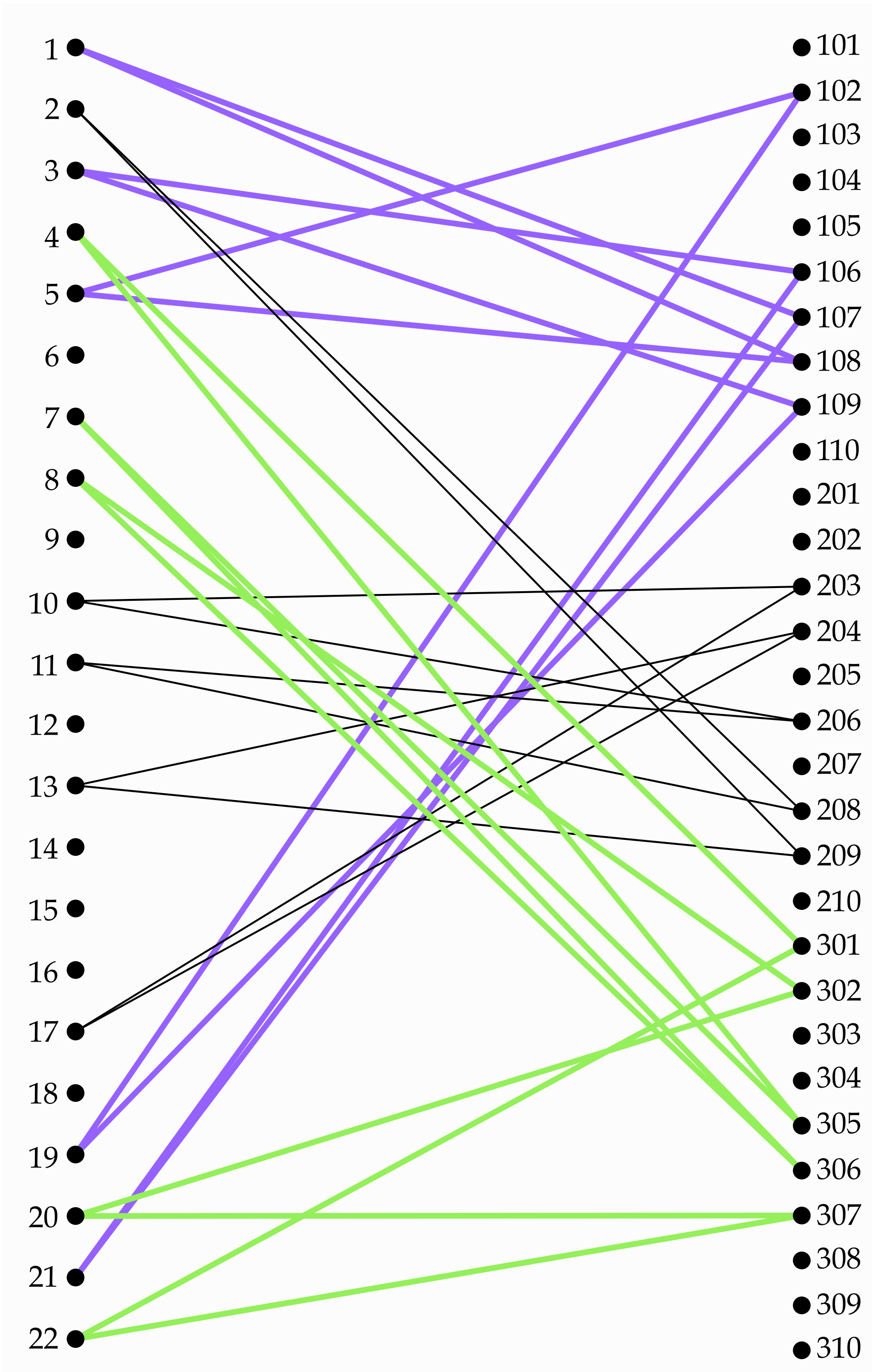

$(E, 2)$ in sky blue; $(R, 3)$ in plum; $(T, 1)$ in black

Figure 4.23: The $\mathcal{C}_{10}^{3}$-block $(T, 1) \cup (E, 2) \cup (R, 3)$.



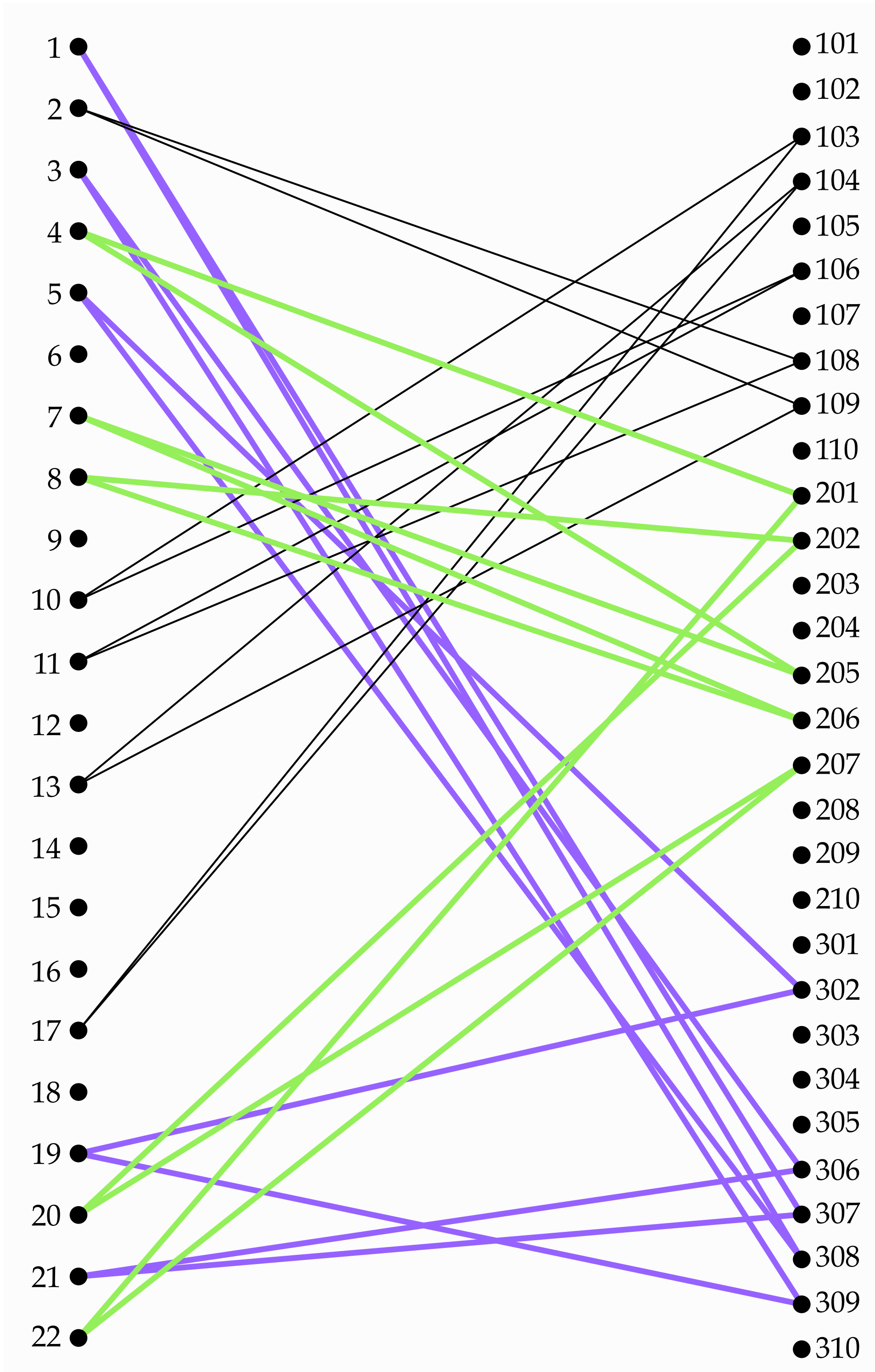

$(F, 1)$ in lime; $(P, 2)$ in violet; $(V, 3)$ in black

Figure 4.24: The $\mathcal{C}_{10}^{3}$-block $(F, 1) \cup (P, 2) \cup (V, 3)$.



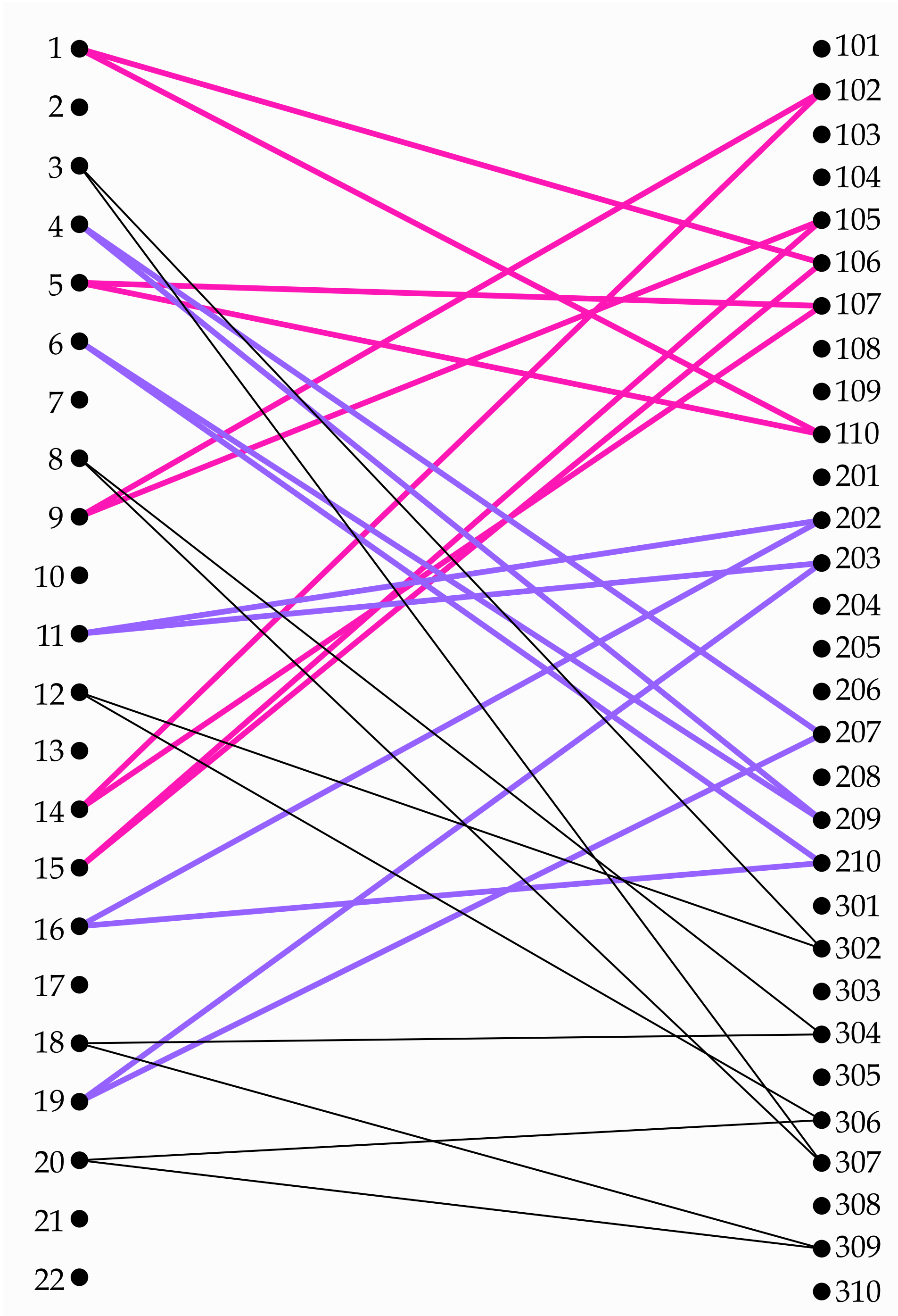

$(F, 3)$ in lime; $(P, 1)$ in violet; $(V, 2)$ in black

Figure 4.25: The $\mathcal{C}_{10}^3$-block $(P, 1) \cup (V, 2) \cup (F, 3)$.



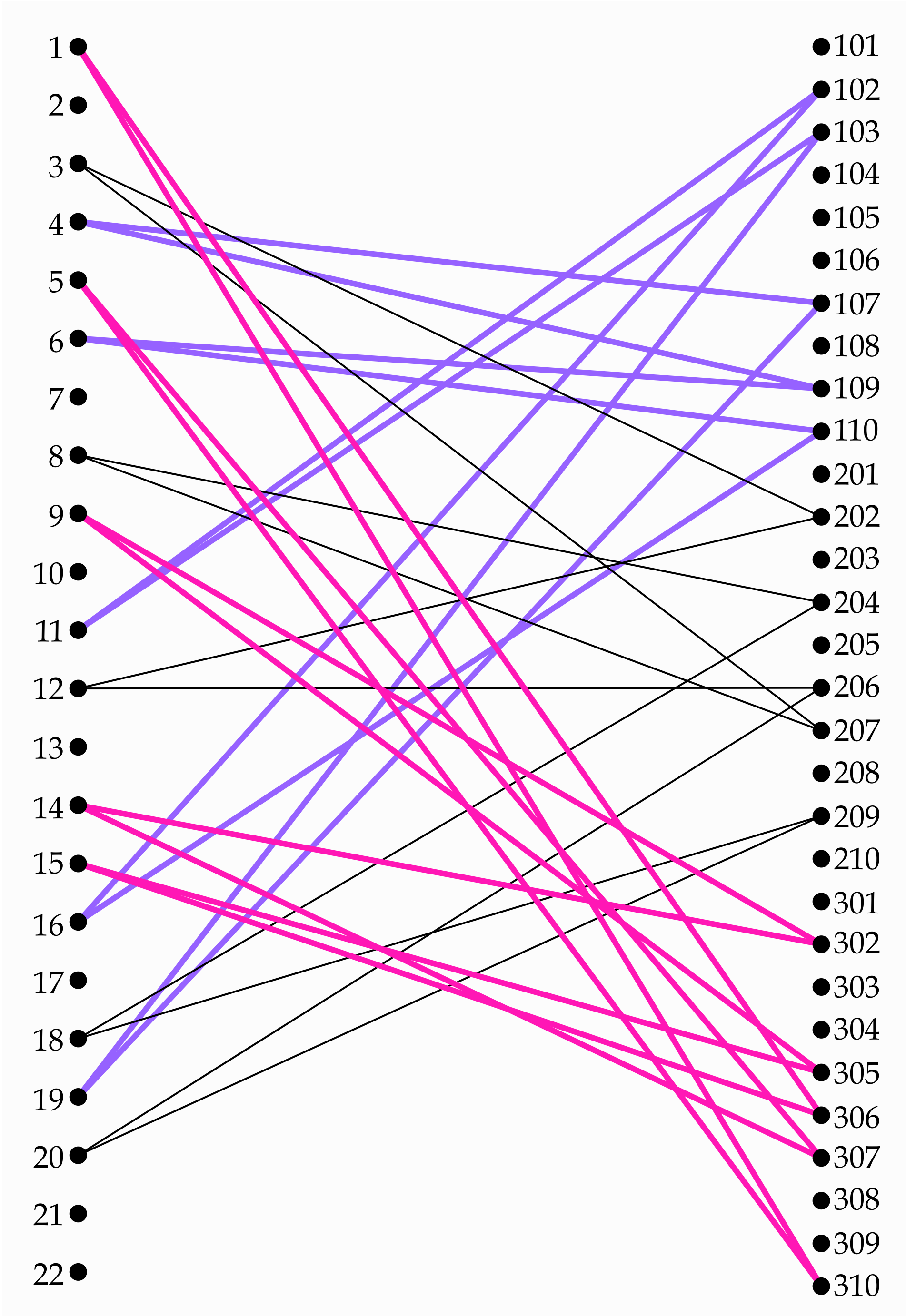

$(F, 2)$ in lime; $(P, 3)$ in violet; $(V, 1)$ in black

Figure 4.26: The $\mathcal{C}_{10}^3$-block $(V, 1) \cup (F, 2) \cup (P, 3)$.



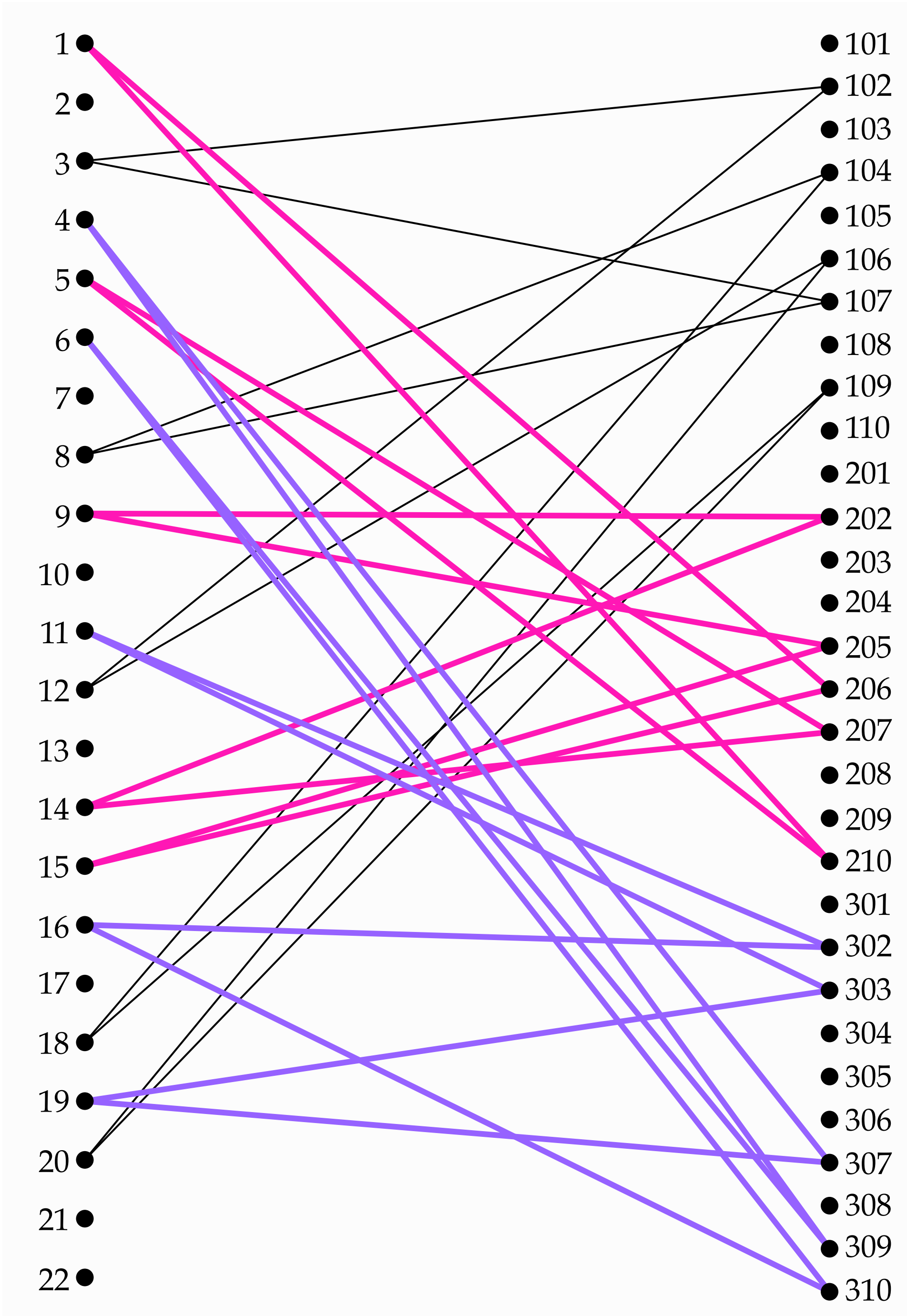

$(L, 1)$ in pink; $(Q, 2)$ in violet; $(U, 3)$ in black

Figure 4.27: The $\mathcal{C}_{10}^{3}$-block $(L, 1) \cup (Q, 2) \cup (U, 3)$.



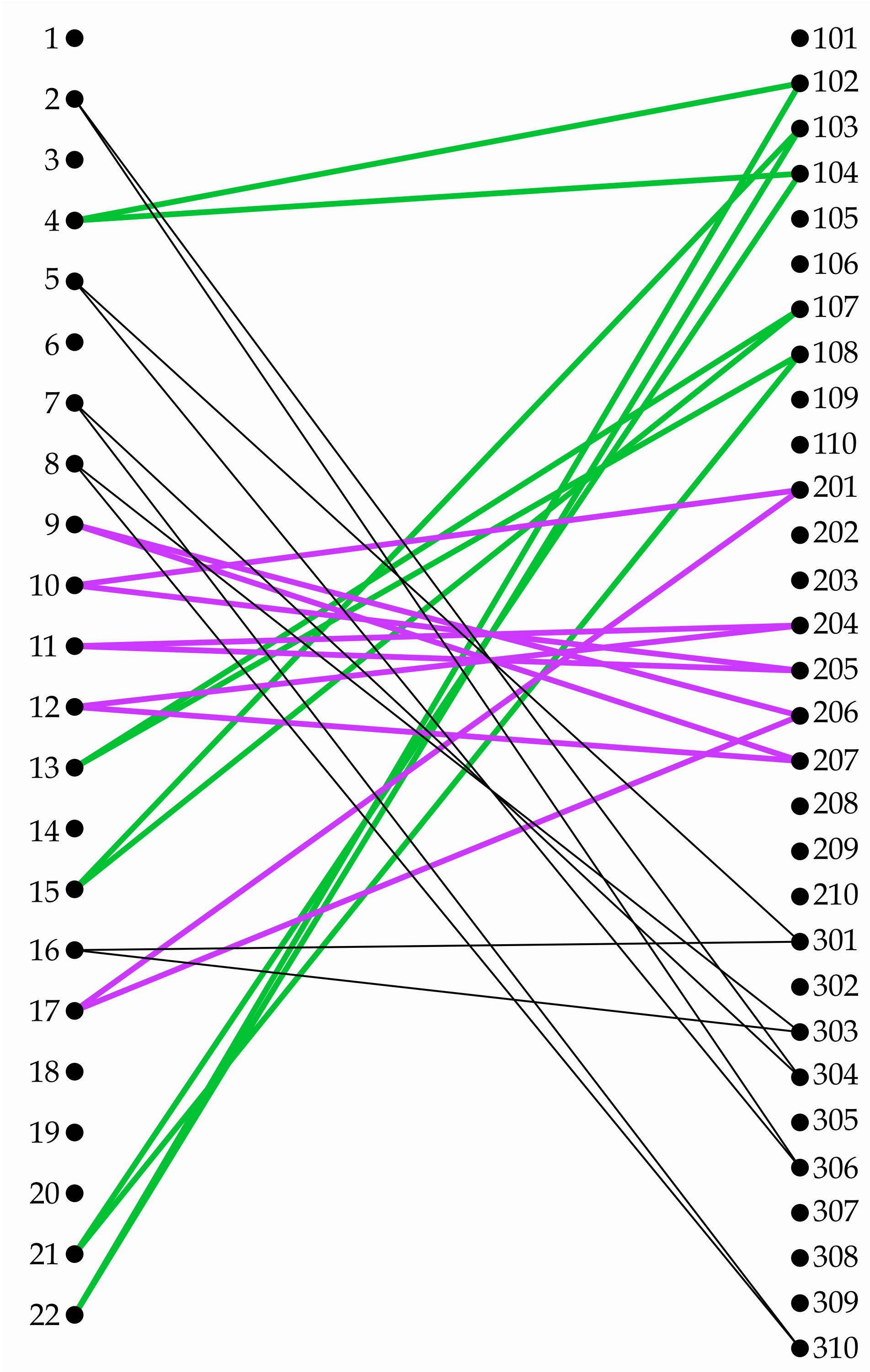

$(L, 3)$ in pink; $(Q, 1)$ in violet; $(U, 2)$ in black

Figure 4.28: The $\mathcal{C}_{10}^3$-block $(Q, 1) \cup (U, 2) \cup (L, 3)$.



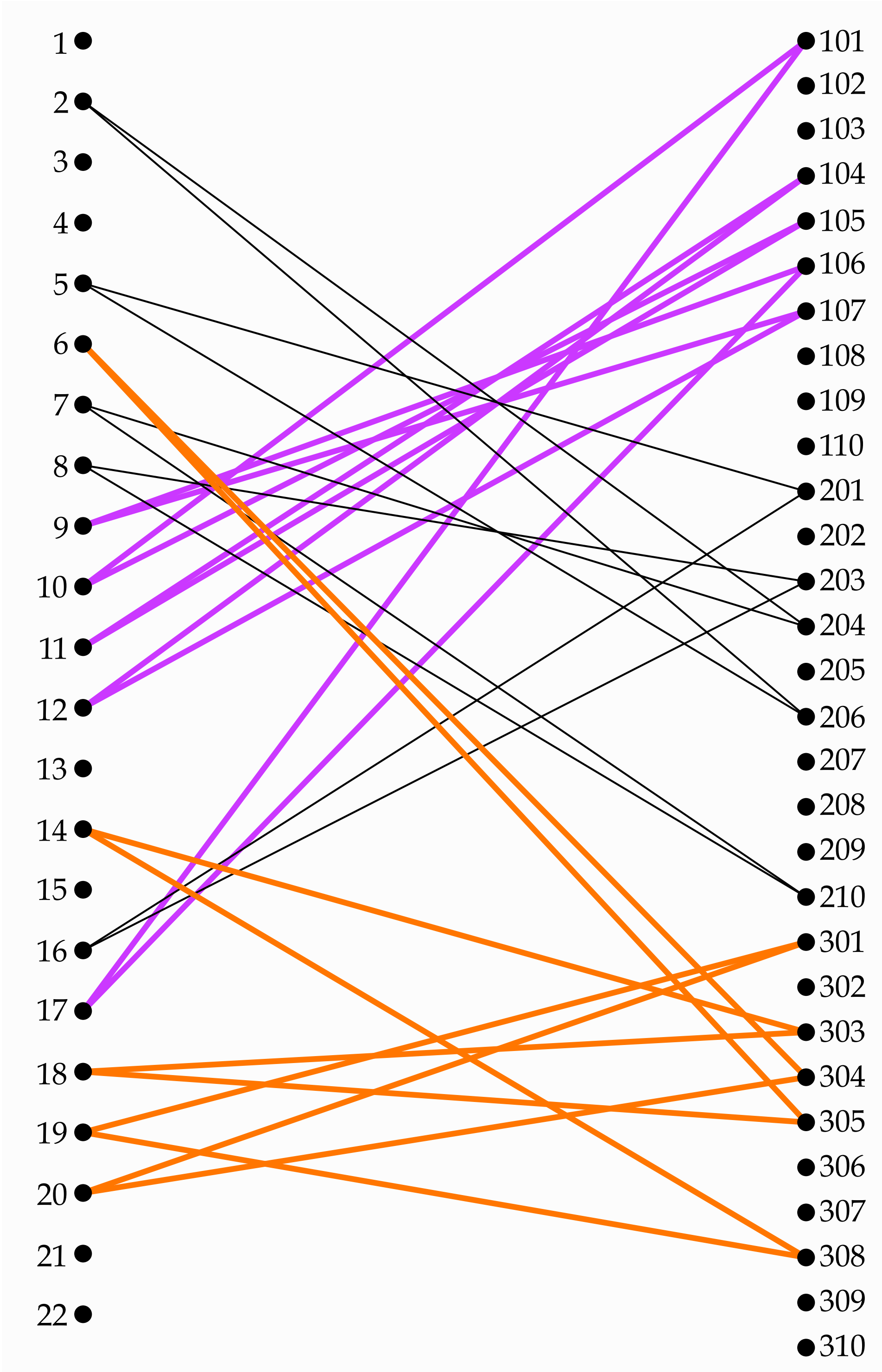

$(L, 2)$ in pink; $(Q, 3)$ in violet; $(U, 1)$ in black

Figure 4.29: The $\mathcal{C}_{10}^3$-block $(U, 1) \cup (L, 2) \cup (Q, 3)$.



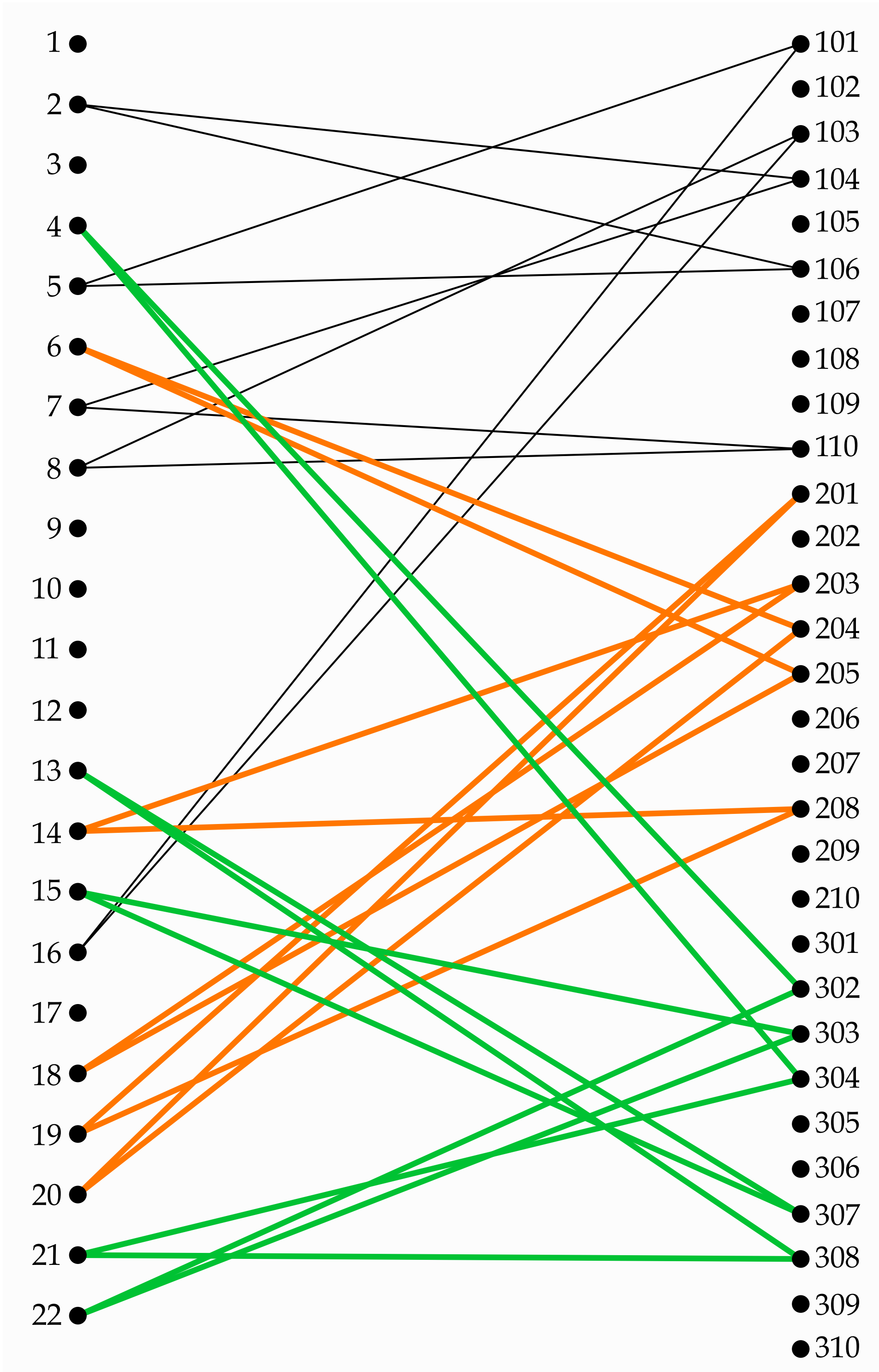

$(I, 1)$ in jade; $(J, 2)$ in lilac; $(K, 3)$ in black

Figure 4.30: The $\mathcal{C}_{10}^3$-block $(I, 1) \cup (J, 2) \cup (K, 3)$.



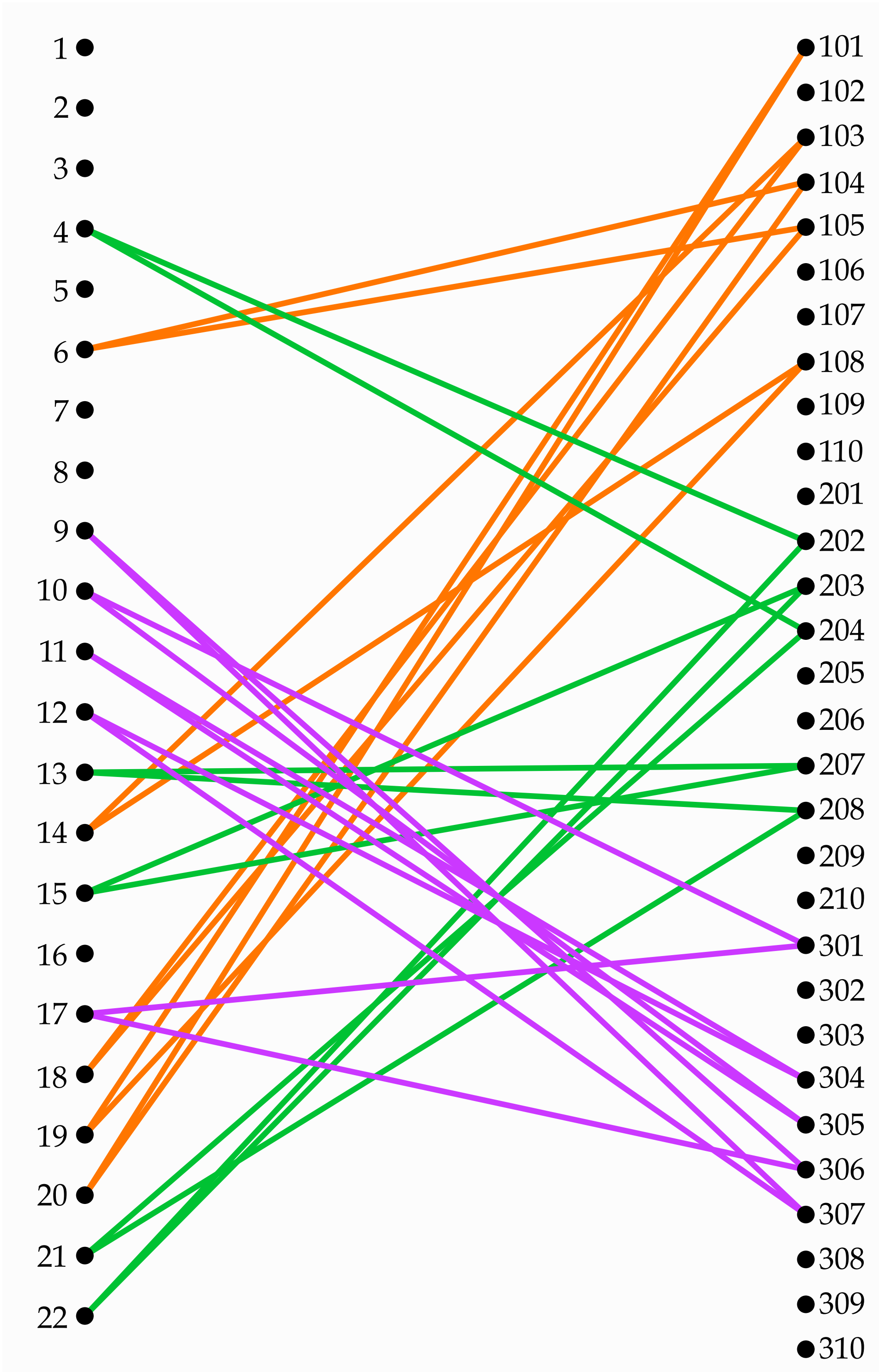

$(J, 1)$ in lilac; $(K, 2)$ in black; $(M, 3)$ in orange

Figure 4.31: The $\mathcal{C}_{10}^3$-block $(J, 1) \cup (K, 2) \cup (M, 3)$.




$(I,3)$ in jade; $(K,1)$ in black; $(M,2)$ in orange

Figure 4.32: The $\mathcal{C}_{10}^{3}$-block $(K,1) \cup (M,2) \cup (I,3)$.




$(I, 2)$ in jade; $(J, 3)$ in black; $(M, 1)$ in orange

Figure 4.33: The $\mathcal{C}_{10}^{3}$-block $(M, 1) \cup (I, 2) \cup (J, 3)$.

## 4.4 Spectral Results on Cohorts of Even Cycles

In this section, we obtain a few results on the spectra of cohorts of even cycles. Designs we have built in previous sections provide most of what is needed to determine the spectrum of some of these graphs.

**Theorem 4.26.** *Let $k, p \in \mathbb{P}$, and suppose there is some $\alpha \in \mathbb{P}$ such that $4kp = 2^\alpha$. Then the spectrum of $\mathcal{C}^p_{2k}$ is $\left\{n \in \mathbb{P} \,\middle|\, n \equiv 1 \pmod{4kp}\right\}$.*

*Proof.* Let $k, p \in \mathbb{P}$, and suppose there is some $\alpha \in \mathbb{P}$ such that $4kp = 2^\alpha$. Recall from Remark 4.11 that, in this case,

$$\mathrm{SSpec}\left(\mathcal{C}^p_{2k}\right) = \left\{n \in \mathbb{P} \,\middle|\, n \equiv 1 \pmod{4kp}\right\} = \left\{n \in \mathbb{P} \,\middle|\, n \equiv 1 \pmod{2^\alpha}\right\}. \tag{4.13}$$

We also recall that, in the proof of Theorem 4.18, we established that, if a complete $\mathcal{C}^p_{2k}$-design of order $n$ exists, it can be embedded in a complete $\mathcal{C}^p_{2k}$-design of order $n+4kp$. Note that the trivial complete $\mathcal{C}^p_{2k}$-design of order 1 clearly exists. Since, by Corollary 2.26, there is a complete $\mathcal{C}^p_{2k}$-design of order $4kp+1$, inductively applying the fact that we can embed any complete $\mathcal{C}^p_{2k}$-design of order $n$ in a complete design of order $n+4kp$ provides a complete $\mathcal{C}^p_{2k}$-design of order $n$ for each $n \in \mathrm{SSpec}\left(\mathcal{C}^p_{2k}\right)$. Therefore the spectrum of $\mathcal{C}^p_{2k}$ is identical to its superspectrum: $\mathrm{Spec}\left(\mathcal{C}^p_{2k}\right) = \left\{n \in \mathbb{P} \,\middle|\, n \equiv 1 \pmod{4kp}\right\}$. □

### 4.4.1 The Spectrum of the $(2, C_6)$-Cohort

We next address the spectrum of $\mathcal{C}^2_6$. We note that, by Corollary 2.26, there is a complete $\mathcal{C}^2_6$-design of order $24x+1$ for all $x \in \mathbb{P}$; in particular, a complete $\mathcal{C}^2_6$-design of order 25 exists. We exhibit a complete $\mathcal{C}^2_6$-design of order 33.

**Lemma 4.27.** *There is a $\mathcal{C}^2_6$-design of order 33.*

*Proof.* Let $V(K_{33}) = [\![0, 32]\!]$. Note that a $\mathcal{C}^2_6$-design on $K_{33}$ must have exactly 44 $\mathcal{C}^2_6$-blocks, as $e(K_{33}) = 528$ and $e\left(\mathcal{C}^2_6\right) = 12$. We exhibit two $\mathcal{C}^2_6$-blocks, $B$ and $A$, from which we obtain the remaining blocks in the design by clicking. We thus obtain a cyclic (but not purely cyclic) $\mathcal{C}_6^2$-design on $K_{33}$. Let $B = \mathfrak{C}_1 \uplus \mathfrak{C}_2$, where $\mathfrak{C}_1 = (0, 7, 20, 29, 14, 10)$ and $\mathfrak{C}_2 = (32, 13, 25, 27, 16, 15)$; the block $B$ is shown in Figure 4.34. Observe that $B$ has exactly one edge of each difference in the set $\{1, 2, 4, 7, 9, 10, 11, 12, 13, 14, 15, 16\}$. We click $B = B_0$ 32 times to obtain blocks $B_i$ for all $i \in [\![1, 32]\!]$.

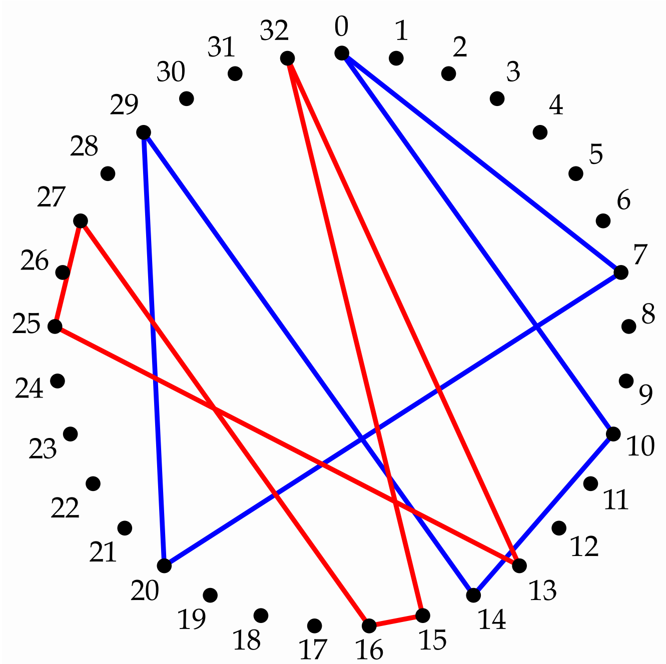



Figure 4.34: The block $B$, consisting of cycles $\mathfrak{C}_1$ (blue) and $\mathfrak{C}_2$ (red)

Let $A = \mathfrak{C}_3 \uplus \mathfrak{C}_4$, where $\mathfrak{C}_3 = (1, 4, 12, 15, 23, 26)$ and $\mathfrak{C}_4 = (0, 5, 11, 16, 22, 27)$; the block $A$ is shown in Figure 4.35. Observe that $A$ has exactly three edges of each difference in the set $\{3, 5, 6, 8\}$. We click $A = A_0$ 10 times to obtain blocks $A_j$ for all $j \in [\![1, 10]\!]$. Note that consecutive edges in the cycle $A$ have distinct differences, so that no edge of $K_{33}$ occurs in more than one block $A_j$.

We have produced 44 $\mathcal{C}_6^2$-blocks that are edge-disjoint; hence the collection

$$\mathscr{B} = \left\{ B_i \;\middle|\; i \in [\![0, 32]\!] \right\} \cup \left\{ A_j \;\middle|\; j \in [\![0, 10]\!] \right\}$$

is a cyclic $\mathcal{C}_6^2$-design on $K_{33}$, as desired. □

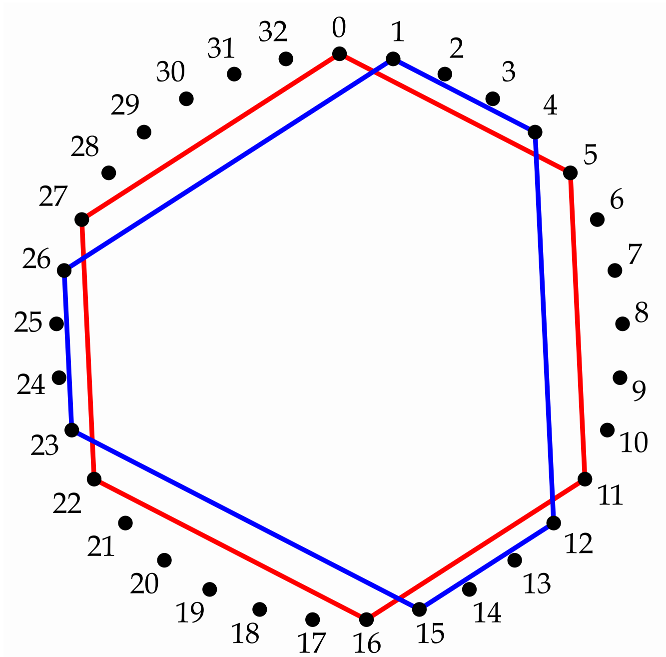



Figure 4.35: The block $A$, consisting of cycles $\mathfrak{C}_3$ (blue) and $\mathfrak{C}_4$ (red)

**Theorem 4.28.** *The spectrum of* $\mathcal{C}_6^2$ *is* $\left\{n \in \mathbb{P} \;\middle|\; n \equiv 1 \text{ or } 9 \pmod{24} \text{ and } n \neq 9\right\}$.

*Proof.* Note that $\mathcal{C}_6^2$ is the graph $\mathcal{C}_{2k}^p$ for $p = 2$ and $k = 3$, and that the trivial complete $\mathcal{C}_6^2$-design of order 1 exists. We have computed that

$$\mathrm{SSpec}\big(\mathcal{C}_6^2\big) = \left\{n \in \mathbb{P} \;\middle|\; n \equiv 1 \text{ or } 9 \pmod{24} \text{ and } n \neq 9\right\}. \tag{4.14}$$

Since $p = 2$ is a power of two, we have from the proof of Theorem 4.18 that, if a complete $\mathcal{C}_6^2$-design of order $n$ exists, it can be embedded in a complete $\mathcal{C}_6^2$-design of order $n+24$. We have shown that complete $\mathcal{C}_6^2$-designs of orders 25 and 33 exist; inductively applying the fact that we can embed any complete $\mathcal{C}_6^2$-design of order $n$ in a complete design of order $n+24$ provides a complete $\mathcal{C}_6^2$-design of order $n$ for each $n \in \mathrm{SSpec}\big(\mathcal{C}_6^2\big)$. Therefore the spectrum of $\mathcal{C}_6^2$ is its superspectrum: $\mathrm{Spec}\big(\mathcal{C}_6^2\big) = \left\{n \in \mathbb{P} \;\middle|\; n \equiv 1 \text{ or } 9 \pmod{24} \text{ and } n \neq 9\right\}$. □

### 4.4.2 The Spectrum of the $(2,\, C_{10})$-Cohort

We now consider $\mathcal{C}_{10}^2$. We note that, by Corollary 2.26, there is a complete $\mathcal{C}_{10}^2$-design of order $40x + 1$ for all $x \in \mathbb{P}$; in particular, a complete $\mathcal{C}_{10}^2$-design of order 41 exists. We exhibit a complete $\mathcal{C}_{10}^2$-design of order 25.

**Lemma 4.29.** *There is a $\mathcal{C}_{10}^2$-design of order 25.*

*Proof.* Let $V(K_{25}) = \mathbb{Z}_5 \times \mathbb{Z}_5$; for convenience, we shorten the usual notation of an ordered pair in diagrams and tables, denoting the pair $(i, j)$ by the two-digit string $ij$. Note that a $\mathcal{C}_{10}^2$-design of order 25 must have exactly fifteen $\mathcal{C}_{10}^2$-blocks, as $e(K_{25}) = 300$ and $e\big(\mathcal{C}_{10}^2\big) = 20$.

We apply a technique that is similar to cyclic difference methods in order to build this design: we arrange the vertices of $K_{25}$ in a 5-by-5 rectangular array, so that the vertex $(i, j)$ may be found in row $i$ and column $j$. For each of three base blocks, we perform an operation similar to clicking: we increase all first coordinates by one, computing modulo five; this operation is a cyclic shift on the rows, so we call it *shifting* for convenience. Note that shifting a subgraph of $K_{25}$ four times generates five subgraphs (including the original), which, in general, may or may not be distinct and may or may not be edge-disjoint; we take care to choose our base blocks so that the shifting operation, performed four times, generates a collection of five subgraphs that are indeed all distinct and pairwise edge-disjoint.

In order to choose our base blocks wisely, we consider a property of edges that is similar to the difference of an edge in cyclic difference methods. Applying the shifting operation to an edge with ends $(i_1, j_1)$ and $(i_2, j_2)$, we obtain the edge with ends $(i_1 + 1, j_1)$ and $(i_2 + 1, j_2)$. We observe that the shifting operation therefore preserves the second coordinates of both ends of the edge, and the difference (modulo 5) between the first coordinates of the ends. So we may identify edge classes by these three pieces of information. For $r, s, t \in [\![0, 4]\!]$ such that $r \leq s$, we denote by the ordered triple $[r, s, t]$ the set of edges with ends $(i, r)$ and $(i + t, s)$, with addition computed modulo 5. We give as illustrative examples the sets $[0, 0, 2]$ and $[1, 3, 1]$, which are shown in Figure 4.36.

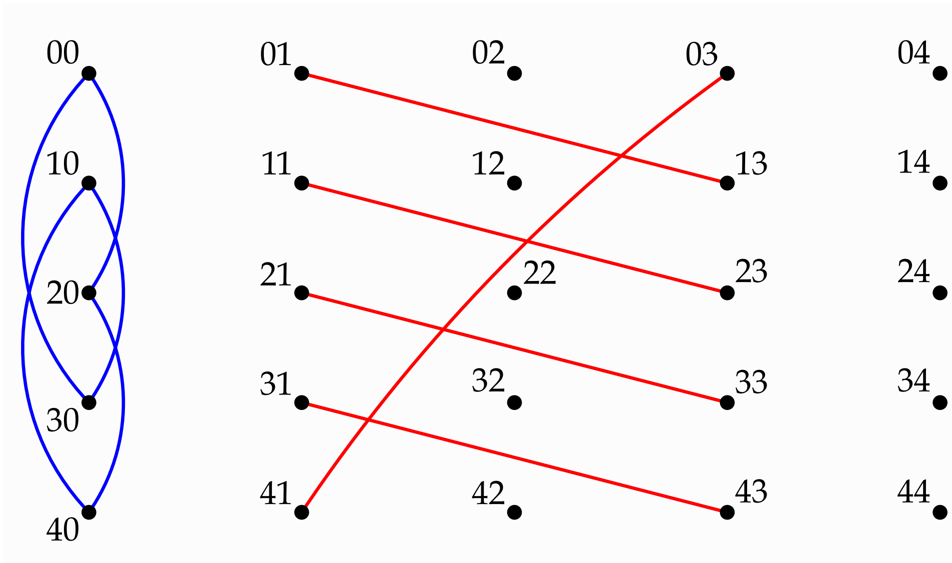



Figure 4.36: The sets $[0, 0, 2]$ (edges in cobalt) and $[1, 3, 1]$ (edges in red)

We observe that distinct triples define distinct, disjoint sets, with two exceptions: for any $r \in [\![0, 4]\!]$, the triples $[r, r, 1]$ and $[r, r, 4]$ define the same set, and the triples $[r, r, 2]$ and $[r, r, 3]$ define the same set. We therefore omit the triples $[r, r, 3]$ and $[r, r, 4]$ from the remainder of our discussion. The sixty triples that define distinct sets are given in Table 4.6.

Table 4.6: The sixty triples $[r, s, t]$ that define distinct sets of edges in $K_{25}$

| | | | | | | |
|---|---|---|---|---|---|---|
| $[0, 0, 1]$ | $[0, 1, 0]$ | $[0, 3, 0]$ | $[1, 2, 0]$ | $[1, 4, 0]$ | $[2, 3, 0]$ | $[3, 4, 0]$ |
| $[0, 0, 2]$ | $[0, 1, 1]$ | $[0, 3, 1]$ | $[1, 2, 1]$ | $[1, 4, 1]$ | $[2, 3, 1]$ | $[3, 4, 1]$ |
| $[1, 1, 1]$ | $[0, 1, 2]$ | $[0, 3, 2]$ | $[1, 2, 2]$ | $[1, 4, 2]$ | $[2, 3, 2]$ | $[3, 4, 2]$ |
| $[1, 1, 2]$ | $[0, 1, 3]$ | $[0, 3, 3]$ | $[1, 2, 3]$ | $[1, 4, 3]$ | $[2, 3, 3]$ | $[3, 4, 3]$ |
| $[2, 2, 1]$ | $[0, 1, 4]$ | $[0, 3, 4]$ | $[1, 2, 4]$ | $[1, 4, 4]$ | $[2, 3, 4]$ | $[3, 4, 4]$ |
| $[2, 2, 2]$ | $[0, 2, 0]$ | $[0, 4, 0]$ | $[1, 3, 0]$ | | $[2, 4, 0]$ | |
| $[3, 3, 1]$ | $[0, 2, 1]$ | $[0, 4, 1]$ | $[1, 3, 1]$ | | $[2, 4, 1]$ | |
| $[3, 3, 2]$ | $[0, 2, 2]$ | $[0, 4, 2]$ | $[1, 3, 2]$ | | $[2, 4, 2]$ | |
| $[4, 4, 1]$ | $[0, 2, 3]$ | $[0, 4, 3]$ | $[1, 3, 3]$ | | $[2, 4, 3]$ | |
| $[4, 4, 2]$ | $[0, 2, 4]$ | $[0, 4, 4]$ | $[1, 3, 4]$ | | $[2, 4, 4]$ | |

Now we exhibit our three base blocks, $A$, $B$, and $C$. Block $A$ is shown in Figure 4.37, with its cycles in different colors for clarity. The list below gives block $A$ and the four additional blocks $A_1$, $A_2$, $A_3$, and $A_4$ obtained by shifting $A$ four times.

$$A = \big(00, 11, 02, 13, 04, 34, 01, 32, 03, 30\big) \uplus \big(10, 20, 33, 21, 22, 23, 31, 24, 14, 42\big)$$

$$A_1 = \big(10, 21, 12, 23, 14, 44, 11, 42, 13, 40\big) \uplus \big(20, 30, 43, 31, 32, 33, 41, 34, 24, 02\big)$$

$$A_2 = \big(20, 31, 22, 33, 24, 04, 21, 02, 23, 00\big) \uplus \big(30, 40, 03, 41, 42, 43, 01, 44, 34, 12\big)$$

$$A_3 = \big(30, 41, 32, 43, 34, 14, 31, 12, 33, 10\big) \uplus \big(40, 00, 13, 01, 02, 03, 11, 04, 44, 22\big)$$

$$A_4 = \big(40, 01, 42, 03, 44, 24, 41, 22, 43, 20\big) \uplus \big(00, 10, 23, 11, 12, 13, 21, 14, 04, 32\big)$$

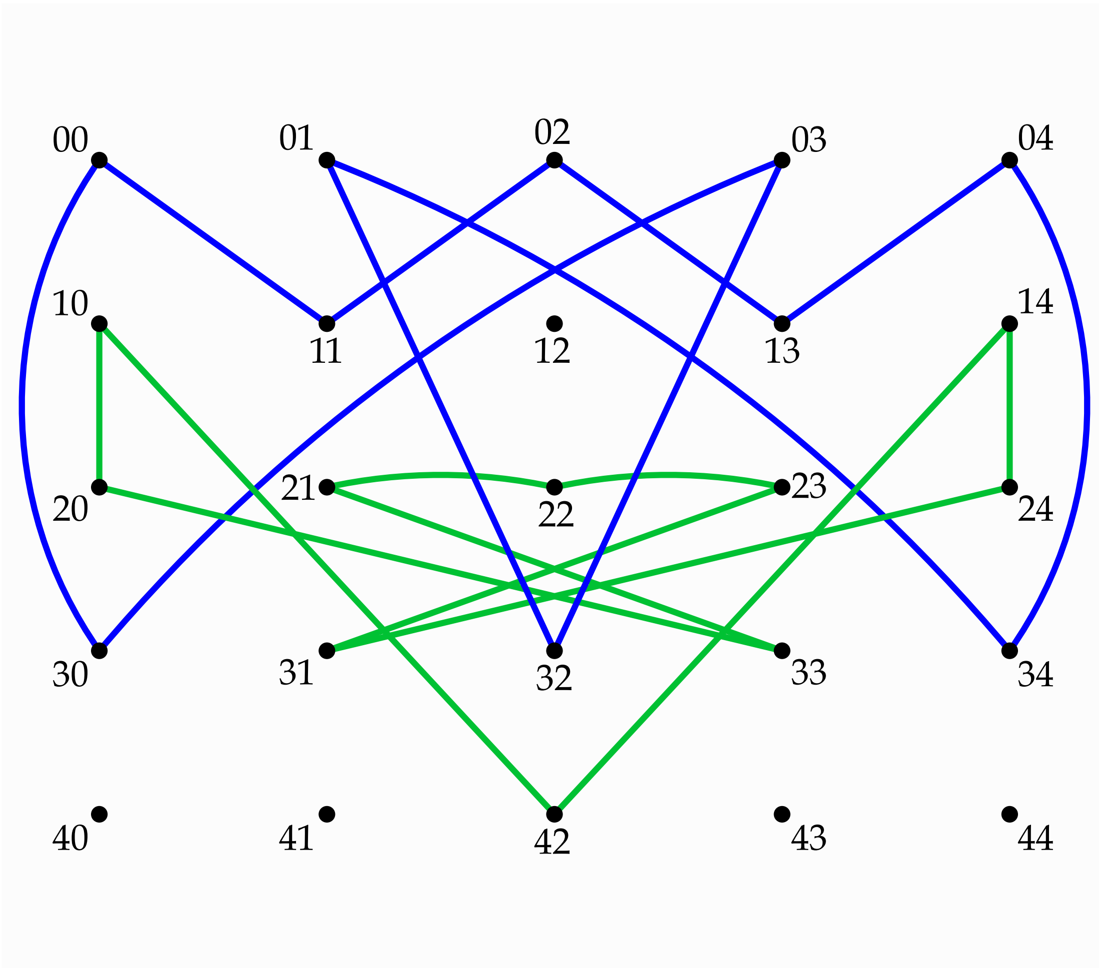



Figure 4.37: Base block $A$ for the $\mathcal{C}_{10}^2$-design on $K_{25}$

Block $B$ is shown in Figure 4.38, with its cycles in different colors for clarity. The list below gives block $B$ and the four additional blocks $B_1$, $B_2$, $B_3$, and $B_4$ obtained by shifting $B$ four times.

$$B = \big(00, 33, 02, 31, 04, 30, 21, 32, 23, 34\big) \uplus \big(10, 11, 41, 13, 43, 44, 42, 22, 12, 24\big)$$

$$B_1 = \big(10, 43, 12, 41, 14, 40, 31, 42, 33, 44\big) \uplus \big(20, 21, 01, 23, 03, 04, 02, 32, 22, 34\big)$$

$$B_2 = \big(20, 03, 22, 01, 24, 00, 41, 02, 43, 04\big) \uplus \big(30, 31, 11, 33, 13, 14, 12, 42, 32, 44\big)$$

$$B_3 = \big(30, 13, 32, 11, 34, 10, 01, 12, 03, 14\big) \uplus \big(40, 41, 21, 43, 23, 24, 22, 02, 42, 04\big)$$

$$B_4 = \big(40, 23, 42, 21, 44, 20, 11, 22, 13, 24\big) \uplus \big(00, 01, 31, 03, 33, 34, 32, 12, 02, 14\big)$$

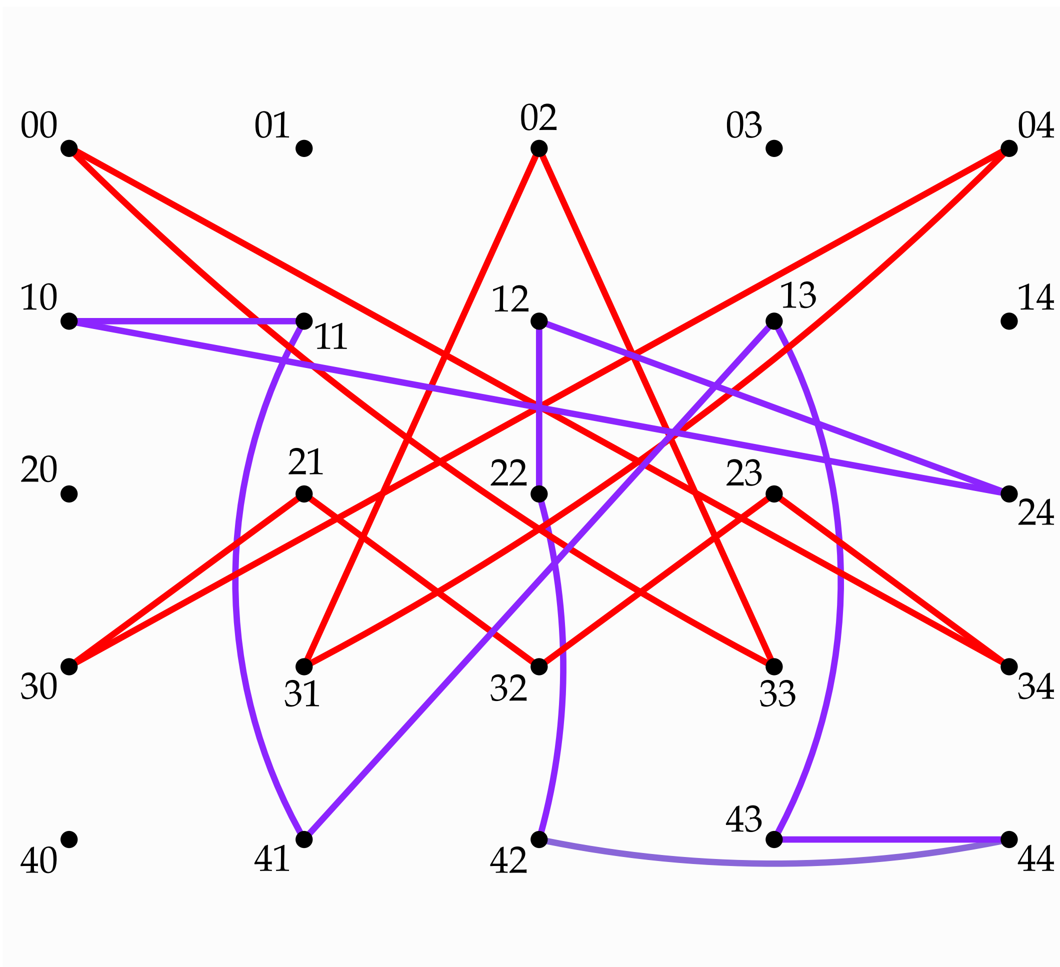



Figure 4.38: Base block $B$ for the $\mathcal{C}_{10}^2$-design on $K_{25}$

Block $C$ is shown in Figure 4.39, with its cycles in different colors for clarity. The list below gives block $C$ and the four additional blocks $C_1$, $C_2$, $C_3$, and $C_4$ obtained by shifting $C$ four times.

$$C = (00, 02, 40, 32, 24, 42, 20, 23, 41, 44) \uplus (10, 03, 13, 34, 30, 11, 01, 14, 33, 31)$$
$$C_1 = (10, 12, 00, 42, 34, 02, 30, 33, 01, 04) \uplus (20, 13, 23, 44, 40, 21, 11, 24, 43, 41)$$
$$C_2 = (20, 22, 10, 02, 44, 12, 40, 43, 11, 14) \uplus (30, 23, 33, 04, 00, 31, 21, 34, 03, 01)$$
$$C_3 = (30, 32, 20, 12, 04, 22, 00, 03, 21, 24) \uplus (40, 33, 43, 14, 10, 41, 31, 44, 13, 11)$$
$$C_4 = (40, 42, 30, 22, 14, 32, 10, 13, 31, 34) \uplus (00, 43, 03, 24, 20, 01, 41, 04, 23, 21)$$

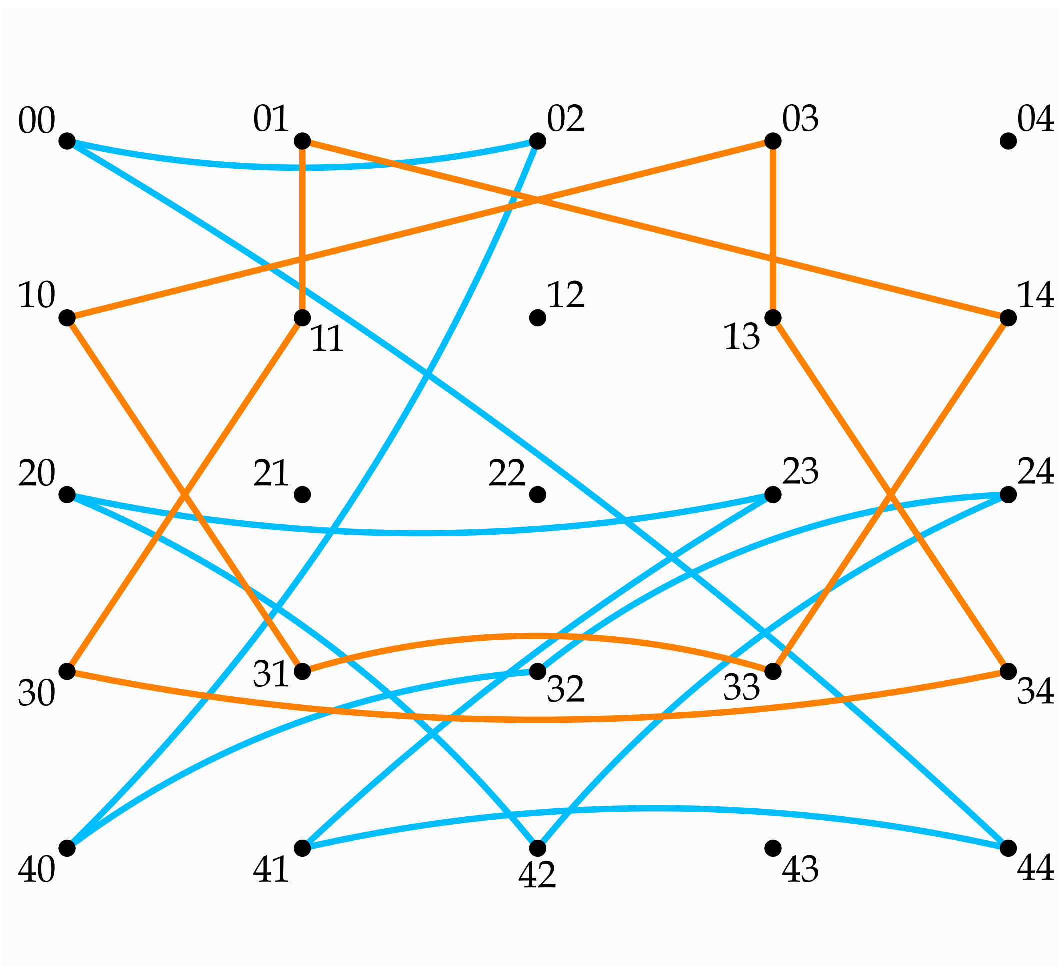



Figure 4.39: Base block $C$ for the $\mathcal{C}_{10}^2$-design on $K_{25}$

Each of the sixty edge types listed in Table 4.6 is represented by exactly one edge in exactly one of the three base blocks. This condition guarantees that the 15 $\mathcal{C}_{10}^2$-blocks we have produced are edge-disjoint. Table 4.7 gives the distribution of these types over the base blocks; for each triple $[r, s, t]$, we give the block, cycle, and edge where it occurs. We denote the block and cycle in the form X.x, where X $\in \{A, B, C\}$ and x $\in \{\text{i}, \text{ii}\}$; for each base block, the cycle that contains vertex 00 is cycle (i), and the other is cycle (ii).

Table 4.7: The distribution of the edge classes $[r, s, t]$ over the base blocks $A$, $B$, and $C$

| | | |
|---|---|---|
| $[0,0,1]$: A.ii $\{10,20\}$ | $[0,3,0]$: C.i $\{20,23\}$ | $[1,4,0]$: C.i $\{41,44\}$ |
| $[0,0,2]$: A.i $\{30,00\}$ | $[0,3,1]$: A.ii $\{20,33\}$ | $[1,4,1]$: C.ii $\{01,14\}$ |
| $[1,1,1]$: C.ii $\{01,11\}$ | $[0,3,2]$: A.i $\{30,03\}$ | $[1,4,2]$: B.i $\{31,04\}$ |
| $[1,1,2]$: B.ii $\{41,11\}$ | $[0,3,3]$: B.i $\{00,33\}$ | $[1,4,3]$: A.i $\{01,34\}$ |
| $[2,2,1]$: B.ii $\{12,22\}$ | $[0,3,4]$: C.ii $\{10,03\}$ | $[1,4,4]$: A.ii $\{31,24\}$ |
| $[2,2,2]$: B.ii $\{22,42\}$ | $[0,4,0]$: C.ii $\{30,34\}$ | $[2,3,0]$: A.ii $\{22,23\}$ |
| $[3,3,1]$: C.ii $\{03,13\}$ | $[0,4,1]$: B.ii $\{10,24\}$ | $[2,3,1]$: A.i $\{02,13\}$ |
| $[3,3,2]$: B.ii $\{43,13\}$ | $[0,4,2]$: B.i $\{30,04\}$ | $[2,3,2]$: A.i $\{32,03\}$ |
| $[4,4,1]$: A.ii $\{14,24\}$ | $[0,4,3]$: B.i $\{00,34\}$ | $[2,3,3]$: B.i $\{02,33\}$ |
| $[4,4,2]$: A.i $\{34,04\}$ | $[0,4,4]$: C.i $\{00,44\}$ | $[2,3,4]$: B.i $\{32,23\}$ |
| $[0,1,0]$: B.ii $\{10,11\}$ | $[1,2,0]$: A.ii $\{21,22\}$ | $[2,4,0]$: B.ii $\{42,44\}$ |
| $[0,1,1]$: A.i $\{00,11\}$ | $[1,2,1]$: B.i $\{21,32\}$ | $[2,4,1]$: B.ii $\{12,24\}$ |
| $[0,1,2]$: C.ii $\{10,31\}$ | $[1,2,2]$: B.i $\{31,02\}$ | $[2,4,2]$: A.ii $\{42,14\}$ |
| $[0,1,3]$: C.ii $\{30,11\}$ | $[1,2,3]$: A.i $\{01,32\}$ | $[2,4,3]$: C.i $\{42,24\}$ |
| $[0,1,4]$: B.i $\{30,21\}$ | $[1,2,4]$: A.i $\{11,02\}$ | $[2,4,4]$: C.i $\{32,24\}$ |
| $[0,2,0]$: C.i $\{00,02\}$ | $[1,3,0]$: C.ii $\{31,33\}$ | $[3,4,0]$: B.ii $\{43,44\}$ |
| $[0,2,1]$: C.i $\{40,02\}$ | $[1,3,1]$: A.ii $\{21,33\}$ | $[3,4,1]$: B.i $\{23,34\}$ |
| $[0,2,2]$: C.i $\{20,42\}$ | $[1,3,2]$: B.ii $\{41,13\}$ | $[3,4,2]$: C.ii $\{13,34\}$ |
| $[0,2,3]$: A.ii $\{10,42\}$ | $[1,3,3]$: C.i $\{41,23\}$ | $[3,4,3]$: C.ii $\{33,14\}$ |
| $[0,2,4]$: C.i $\{40,32\}$ | $[1,3,4]$: A.ii $\{31,23\}$ | $[3,4,4]$: A.i $\{13,04\}$ |

Since we have produced fifteen edge-disjoint $\mathcal{C}_{10}^2$-blocks, the collection

$$\mathscr{B} = \Big\{A, A_1, A_2, A_3, A_4, B, B_1, B_2, B_3, B_4, C, C_1, C_2, C_3, C_4\Big\}$$

is a $\mathcal{C}_{10}^2$-design on $K_{25}$, as desired. □

**Theorem 4.30.** *The spectrum of* $\mathcal{C}_{10}^2$ *is* $\Big\{n \in \mathbb{P} \;\Big|\; n \equiv 1 \text{ or } 25 \pmod{40}\Big\}$.

*Proof.* Note that $\mathcal{C}_{10}^2$ is the graph $\mathcal{C}_{2k}^p$ for $p = 2$ and $k = 5$, and that the trivial complete $\mathcal{C}_{10}^2$-design of order 1 exists. We have computed that

$$\text{SSpec}\big(\mathcal{C}_{10}^2\big) = \Big\{n \in \mathbb{P} \;\Big|\; n \equiv 1 \text{ or } 25 \pmod{40}\Big\}. \tag{4.15}$$

Since $p = 2$ is a power of two, we have from the proof of Theorem 4.18 that, if a complete $\mathcal{C}_{10}^2$-design of order $n$ exists, it can be embedded in a complete $\mathcal{C}_{10}^2$-design of order $n + 40$. We have shown that complete $\mathcal{C}_{10}^2$-designs of orders 25 and 41 exist; inductively applying the fact that we can embed any complete $\mathcal{C}_{10}^2$-design of order $n$ in a complete design of order $n + 40$ provides a complete $\mathcal{C}_{10}^2$-design of order $n$ for each $n \in \text{SSpec}\big(\mathcal{C}_{10}^2\big)$. Therefore the spectrum of $\mathcal{C}_{10}^2$ is its superspectrum: $\text{Spec}\big(\mathcal{C}_{10}^2\big) = \Big\{n \in \mathbb{P} \;\Big|\; n \equiv 1 \text{ or } 25 \pmod{40}\Big\}$. □

We observe that results similar to Theorems 4.28 and 4.30 are potentially possible for many pairs of values of $p$ and $k$ with the exhibition of a small number of designs. We now turn our attention back to the topic of Chapter 2 and explore in-depth the use of graph labelings to create $\mathcal{C}_{2k}^p$-designs on the graphs $K_{4kp+1}$.

Chapter 5

# Cohorts of Even Cycles, Part Two

In this chapter, we discuss $\mathcal{C}_{2k}^{p}$-designs on $K_{4kp+1}$. The existence of such designs is guaranteed by the labeling results presented as Theorem 2.25 and Corollary 2.26 in Chapter 2. Prior to becoming aware of these extensive results in graph labelings, we independently achieved many results on the existence of $\mathcal{C}_{2k}^{p}$-designs on $K_{4kp+1}$ for certain values of $k$ and $p$. In the first section of this chapter, we present further details of the labeling results, in particular the constructions used to achieve them, for comparison purposes with our own designs. In subsequent sections, we present our own constructions and some commentary on comparisons between the constructions. Both approaches to building $\mathcal{C}_{2k}^{p}$-designs on $K_{4kp+1}$ separate into three cases, namely: (1) $k$ is even; (2) $k$ is odd and $p$ is even; and (3) $k$ is odd and $p$ is odd.

Since our discussion of $\mathcal{C}_{2k}^{p}$-designs on $K_{4kp+1}$ exclusively concerns cyclic difference methods, we describe base blocks by listing their cycles, and we accompany each cycle by a description of the pattern in which differences are to be used to form the cycle. Since each vertex in a complete graph is incident to exactly two edges of each difference, there are two directions in which we can achieve difference $d$ on an edge originating at vertex $v$; we establish the following notation for the direction in which a difference is to be achieved.

**Notation 5.1.** In the statement of a difference pattern, a difference written without brackets indicates that the difference is to be achieved by moving clockwise from the current vertex (*i.e.,* in the direction of increasing vertex names), while a difference written inside brackets indicates that the difference is to be achieved by moving counterclockwise from the current vertex (*i.e.,* in the direction of decreasing vertex names). ■

**Example 5.2.** We may produce a cycle of length six in $K_{61}$ using the difference pattern

$$20 \quad [5] \quad 25 \quad 16 \quad [4] \quad 9\,.$$

If we begin at vertex 0, we obtain the cycle $(0, 20, 15, 40, 56, 52)$. This cycle is shown in Figure 5.1. ■

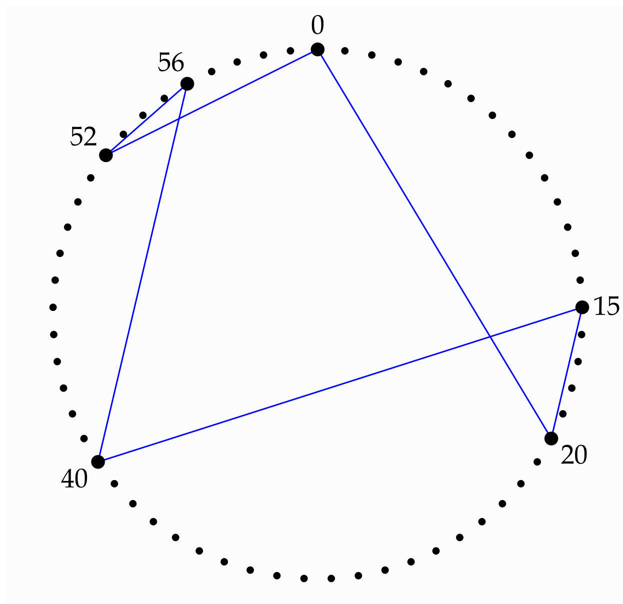



Figure 5.1: The 6-cycle in Example 5.2

With this notation in hand, we begin our discussion with a detailed analysis of one of the graph labeling results presented at the end of Chapter 2.

## 5.1 Constructions by Blinco and El-Zanati

In this section, we present the constructions behind Theorem 2.25 (due to A. Blinco and S. El-Zanati, 2004), with supporting results from earlier papers and several illustrative examples. All results presented in this section are the work of others; the examples are of our own choosing, for purposes of comparison in later sections. We begin by repeating the statement of Theorem 2.25 for convenient reference.

**Theorem** (2.25)**.** *Let $G$ be a 2-regular bipartite graph. Then $G$ admits a $\rho^+$-labeling.*

In Blinco and El-Zanati's paper [6], the above theorem is stated as a corollary to a more powerful result, which produces a $\rho^+$-labeling on the disjoint union of one or more graphs that admit $\alpha$-labelings and a single graph that admits a $\rho^{++}$-labeling.

**Theorem 5.3** (Blinco and El-Zanati, 2004)**.** *Let $G_1$ be a bipartite graph with $n_1$ edges. Suppose $G_1$ has a $\rho^{++}$-labeling in which no vertex is labeled $2n_1$. Let $G_2, G_3, \ldots, G_t$ be graphs with $\alpha$-labelings. Then the graph $G_1 \uplus G_2 \uplus \cdots \uplus G_t$ has a $\rho^+$-labeling.*

The proof of Theorem 2.25 by Blinco and El-Zanati relies on the above theorem and several other results from the literature. We state here only those results that are useful in the proof for the particular graphs we wish to consider. The most significant of these is by S. El-Zanati, C. Vanden Eynden, and N. Punnim [12].

**Theorem 5.4** (El-Zanati, Vanden Eynden, and Punnim, 2001)**.** *The disjoint union of graphs with $\alpha$-labelings has a $\sigma^+$-labeling.*

An additional result in the same paper provides a $\rho^{++}$-labeling of the cycle $C_{2k}$.

**Theorem 5.5** (El-Zanati, Vanden Eynden, and Punnim, 2001)**.** *Let $V(C_{2k}) = [\![1, 2k]\!]$, so that vertices $u$ and $v$ are adjacent whenever $v \equiv u + 1 \pmod{2k}$. Define a labeling $H$ on the vertices of $C_{2k}$ by*

$$H(v) = \begin{cases} (v-1)/2, & \text{if } v \text{ is odd,} \\ 2k - (v/2), & \text{if } v \text{ is even and } 0 < v < k-1, \\ 2k - 1 - (v/2), & \text{if } v \text{ is even and } k-1 \le v < 2k, \\ 2k+1, & \text{if } v = 2k. \end{cases} \tag{5.1}$$

*Then $H$ is a $\rho^{++}$-labeling of $C_{2k}$.*

We note that the critical value of $H$ in Theorem 5.5 is $\lambda = k - 1$. A 1975 paper by A. Kotzig [21] provides an $\alpha$-labeling of $C_{2k} \uplus C_{2k}$; we state the result here and exhibit the labeling when it is needed.

**Theorem 5.6** (Kotzig, 1975). *The graph $C_s \uplus C_s$ admits an $\alpha$-labeling if and only if $s$ is even and $s > 2$.*

Blinco and El-Zanati's proof of Theorem 2.25 for all 2-regular bipartite graphs is accomplished in two cases. We consider only the cohorts of even cycles; once restricted to these graphs, the construction by Blinco and El-Zanati has three natural cases, as mentioned at the beginning of this chapter: (1) $k$ is even; (2) $k$ is odd and $p$ is even; and (3) $k$ is odd and $p$ is odd.

### 5.1.1 The Blinco–El-Zanati Construction for Even $k$

We first consider the graph $\mathcal{C}_{2k}^p$ in the case that $k$ is even. In this case, the cycle length is a multiple of four. We observe that, since $k$ is even, $C_{2k}$ admits an $\alpha$-labeling; this result is commonly cited from a 1965 paper by Kotzig [20]. In our discussion, we employ two $\alpha$-labelings of $C_{2k}$, which we call $L_3$ and $L_1$; the labeling $L_3$ appears in the paper by Kotzig. Let $V(C_{2k}) = [\![1, 2k]\!]$, so that vertices $i$ and $j$ are adjacent whenever $j \equiv i + 1 \pmod{2k}$. We define the labeling $L_3$ by

$$L_3(v) = \begin{cases} \dfrac{v-1}{2}\,, & \text{if } v \text{ is odd,} \\ 2k + 1 - \dfrac{v}{2}\,, & \text{if } v \text{ is even and } 2 \le v \le k, \\ 2k - \dfrac{v}{2}\,, & \text{if } v \text{ is even and } k + 2 \le v \le 2k. \end{cases} \tag{5.2}$$

The critical value of $L_3$ is $\lambda = k - 1$. We define the labeling $L_1$ by

$$L_1(v) = \begin{cases} k - \dfrac{v-1}{2}\,, & \text{if } v \text{ is odd and } 1 \le v \le k-1\,, \\ k - \dfrac{v+1}{2}\,, & \text{if } v \text{ is odd and } k+1 \le v \le 2k-1\,, \\ k + \dfrac{v}{2}\,, & \text{if } v \text{ is even.} \end{cases} \tag{5.3}$$

The critical value of $L_1$ is $\lambda = k$.

In order to produce a $\rho^+$-labeling of $\mathcal{C}_{2k}^p$, the Blinco–El-Zanati Construction relies on the construction given in the proof of Theorem 5.4 by El-Zanati, Vanden Eynden, and Punnim. We give below the entire definition of this $\rho^+$-labeling, $h$, as it is given in [12]; we have made changes to the text only to ensure consistent notation within this document.

> For $i \in [\![1, t]\!]$, let $G_i$ be a graph with $n_i$ edges having an $\alpha$-labeling $h_i$ with critical value $\lambda_i$ and vertex bipartition $[A_i, B_i]$, where $A_i = \{v \in V(G_i) \mid h_i(v) \leq \lambda_i\}$. Define integers $\alpha_i$ and $\beta_i$ for each $i \in [\![1, t]\!]$ by
>
> $$\alpha_i = \begin{cases} \dfrac{i-1}{2} + \displaystyle\sum_{\substack{j \text{ odd}, \\ j < i}} \lambda_j \,, & \text{if } i \text{ is odd,} \\ \displaystyle\sum_{j=1}^{t} n_j \;+ \sum_{\substack{j \text{ odd}, \\ j \leq i}} n_j \;+ \sum_{\substack{j \text{ even}, \\ j < i}} \lambda_j \,, & \text{if } i \text{ is even,} \end{cases} \tag{5.4}$$
>
> and
>
> $$\beta_i = \alpha_i + \sum_{j > i} n_j \,. \tag{5.5}$$
>
> Assuming the graphs $G_i$ are vertex-disjoint, we define $h$ on their union, $G$, by
>
> $$h(v) = \begin{cases} h_i(v) + \alpha_i \,, & \text{if } v \in A_i \,, \\ h_i(v) + \beta_i \,, & \text{if } v \in B_i \,. \end{cases} \tag{5.6}$$

We note that the result by El-Zanati, Vanden Eynden, and Punnim actually provides a $\sigma^+$-labeling.

We wish to describe the labeling $h$ as specifically as possible for the graph $G = \mathcal{C}_{2k}^p$ in the case in which $k$ is even. Let $m \in \mathbb{P}$ such that $k = 2m$; let $t = p$; let $G_i = C_{2k}$ and $n_i = 2k$ for all $i \in [\![1, p]\!]$. For each $i \in [\![1, p]\!]$, we may choose $h_i$ to be any $\alpha$-labeling of $C_{2k}$; we describe the result of choosing $h_i = L_3$ for all $i \in [\![1, p]\!]$. Let $V(G_i) = [\![1, 2k]\!] \times \{i\}$, so that $V(G) = [\![1, 2k]\!] \times [\![1, p]\!]$; then we have, for all $i \in [\![1, p]\!]$, that $\lambda_i = k - 1$, and that

$$A_i = \left\{ (a, i) \;\middle|\; a \in [\![1, 2k]\!] \text{ and } a \text{ is odd} \right\}, \text{ and} \tag{5.7}$$

$$B_i = \left\{ (b, i) \;\middle|\; b \in [\![1, 2k]\!] \text{ and } b \text{ is even} \right\}. \tag{5.8}$$

We can therefore simplify the sums in the definitions of $\alpha_i$ and $\beta_i$ as follows.

$$(i \text{ odd}) \sum_{\substack{j \text{ odd}, \\ j < i}} \lambda_j = \sum_{\substack{j \text{ odd}, \\ j < i}} (k-1) = (k-1) \left( \frac{i-1}{2} \right) = (2m-1) \left( \frac{i-1}{2} \right) \tag{5.9}$$

$$\sum_{j=1}^{t} n_j = \sum_{j=1}^{p} 2k = 2kp = 4mp \tag{5.10}$$

$$(i \text{ even}) \sum_{\substack{j \text{ odd}, \\ j \leq i}} n_j = \sum_{\substack{j \text{ odd}, \\ j \leq i}} 2k = 2k \left( \frac{i}{2} \right) = ki = 2mi \tag{5.11}$$

$$(i \text{ even}) \sum_{\substack{j \text{ even}, \\ j < i}} \lambda_j = \sum_{\substack{j \text{ even}, \\ j < i}} (k-1) = (k-1) \left( \frac{i}{2} - 1 \right) = (2m-1) \left( \frac{i}{2} - 1 \right) \tag{5.12}$$

$$\sum_{j > i} n_j = \sum_{i < j \leq p} 2k = 2k(p-i) = 4m(p-i) \tag{5.13}$$

We obtain the following definitions of $\alpha_i$, $\beta_i$, and $h$.

$$\alpha_i = \begin{cases} m(i-1), & \text{if } i \text{ is odd}, \\ m\big(4p + 3i - 2\big) + 1 - \dfrac{i}{2}, & \text{if } i \text{ is even}. \end{cases} \tag{5.14}$$

$$\beta_i = \begin{cases} m\big(4p - 3i - 1\big), & \text{if } i \text{ is odd}, \\ m\big(8p - i - 2\big) + 1 - \dfrac{i}{2}, & \text{if } i \text{ is even}. \end{cases} \tag{5.15}$$

$$h(v, i) = \begin{cases} L_3(v) + \alpha_i\,, & \text{if } v \text{ is odd}, \\ L_3(v) + \beta_i\,, & \text{if } v \text{ is even}, \end{cases} \qquad \text{for all } (v, i) \in V(G). \tag{5.16}$$

We may combine formulas (5.2), (5.14), (5.15), and (5.16) to obtain a direct formula for $h(v,i)$, without reference to $\alpha_i$ or $\beta_i$. For added clarity, we give the formula statement in two parts.

If $i$ is odd, we have

$$h(v,i) = \begin{cases} m(i-1) + \dfrac{v-1}{2}\,, & \text{if } v \text{ is odd,} \\ m\bigl(4p+3-3i\bigr) + 1 - \dfrac{v}{2}\,, & \text{if } v \text{ is even and } 2 \le v \le 2m\,, \\ m\bigl(4p+3-3i\bigr) - \dfrac{v}{2}\,, & \text{if } v \text{ is even and } 2m+2 \le v \le 4m\,. \end{cases} \tag{5.17}$$

If $i$ is even, we have

$$h(v,i) = \begin{cases} m\bigl(4p+3i-2\bigr) + \dfrac{v+1-i}{2}\,, & \text{if } v \text{ is odd,} \\ m\bigl(8p-i+2\bigr) + 2 - \dfrac{v+i}{2}\,, & \text{if } v \text{ is even and } 2 \le v \le 2m\,, \\ m\bigl(8p-i+2\bigr) + 1 - \dfrac{v+i}{2}\,, & \text{if } v \text{ is even and } 2m+2 \le v \le 4m\,. \end{cases} \tag{5.18}$$

We note that several alternative labelings can be achieved by applying the construction by El-Zanati, Vanden Eynden, and Punnim with different choices of $\alpha$-labeling for some or all $C_{2k}$-subgraphs of $G = \mathcal{C}_{2k}^{p}$; we could even apply Kotzig's $\alpha$-labeling of $C_{2k} \uplus C_{2k}$ to some pairs of cycles to obtain further different results. We give two examples of $\sigma^{+}$-labelings, both achieved by choosing the labeling $L_3$ for all cycles in $G = \mathcal{C}_{2k}^{p}$, as we have described in detail above; these examples are $\sigma^{+}$-labelings of $\mathcal{C}_{12}^{4}$ and $\mathcal{C}_{16}^{5}$.

**Example 5.7.** Consider $\mathcal{C}_{12}^{4}$; this is the case $p = 4$ and $k = 6$. We exhibit a $\sigma^{+}$-labeling generated by the Blinco–El-Zanati Construction as the base block for a purely cyclic design of order $4kp + 1 = 97$. This base block is shown in Figure 5.2; distinct cycles are shown in different colors for enhanced visibility. The cycles and their corresponding difference patterns are listed in Table 5.1. ■

**Example 5.8.** Consider $\mathcal{C}_{16}^{5}$; this is the case $p = 5$ and $k = 8$. We exhibit a $\sigma^+$-labeling generated by the Blinco–El-Zanati Construction as the base block for a purely cyclic design of order $4kp + 1 = 161$. This base block is shown in Figure 5.3; distinct cycles are shown in different colors for enhanced visibility. The cycles and their corresponding difference patterns are listed in Table 5.2. ■

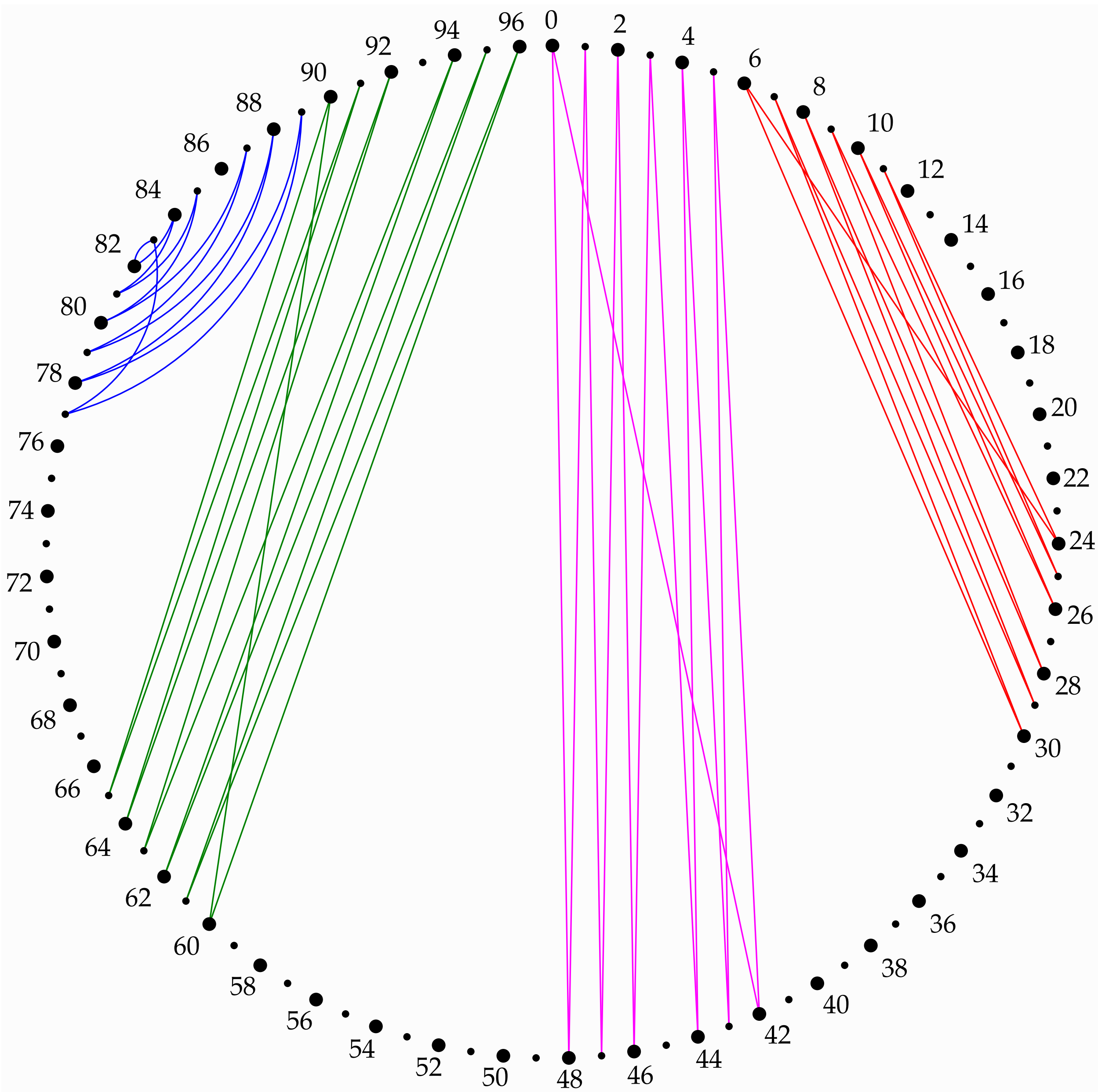



Figure 5.2: A Blinco–El-Zanati $\sigma^+$-labeling of $\mathcal{C}_{12}^{4}$

Table 5.1: Cycle list for the $\sigma^+$-labeling of $\mathcal{C}_{12}^4$ in Figure 5.2

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **Graph** $G_1 = C_{12}$**: Cycle** $\mathfrak{C}_1$ | | | | (fuchsia) | | | | | | | |
| $\mathfrak{C}_1 = \big(0, 48, 1, 47, 2, 46, 3, 44, 4, 43, 5, 42\big)$ | | | | | | | | | | | |
| 48 | [47] | 46 | [45] | 44 | [43] | 41 | [40] | 39 | [38] | 37 | [42] |
| **Graph** $G_2 = C_{12}$**: Cycle** $\mathfrak{C}_2$ | | | | (green) | | | | | | | |
| $\mathfrak{C}_2 = \big(60, 96, 61, 95, 62, 94, 63, 92, 64, 91, 65, 90\big)$ | | | | | | | | | | | |
| 36 | [35] | 34 | [33] | 32 | [31] | 29 | [28] | 27 | [26] | 25 | [30] |
| **Graph** $G_3 = C_{12}$**: Cycle** $\mathfrak{C}_3$ | | | | (red) | | | | | | | |
| $\mathfrak{C}_3 = \big(6, 30, 7, 29, 8, 28, 9, 26, 10, 25, 11, 24\big)$ | | | | | | | | | | | |
| 24 | [23] | 22 | [21] | 20 | [19] | 17 | [16] | 15 | [14] | 13 | [18] |
| **Graph** $G_4 = C_{12}$**: Cycle** $\mathfrak{C}_4$ | | | | (cobalt) | | | | | | | |
| $\mathfrak{C}_4 = \big(77, 89, 78, 88, 79, 87, 80, 85, 81, 84, 82, 83\big)$ | | | | | | | | | | | |
| 12 | [11] | 10 | [9] | 8 | [7] | 5 | [4] | 3 | [2] | 1 | [6] |

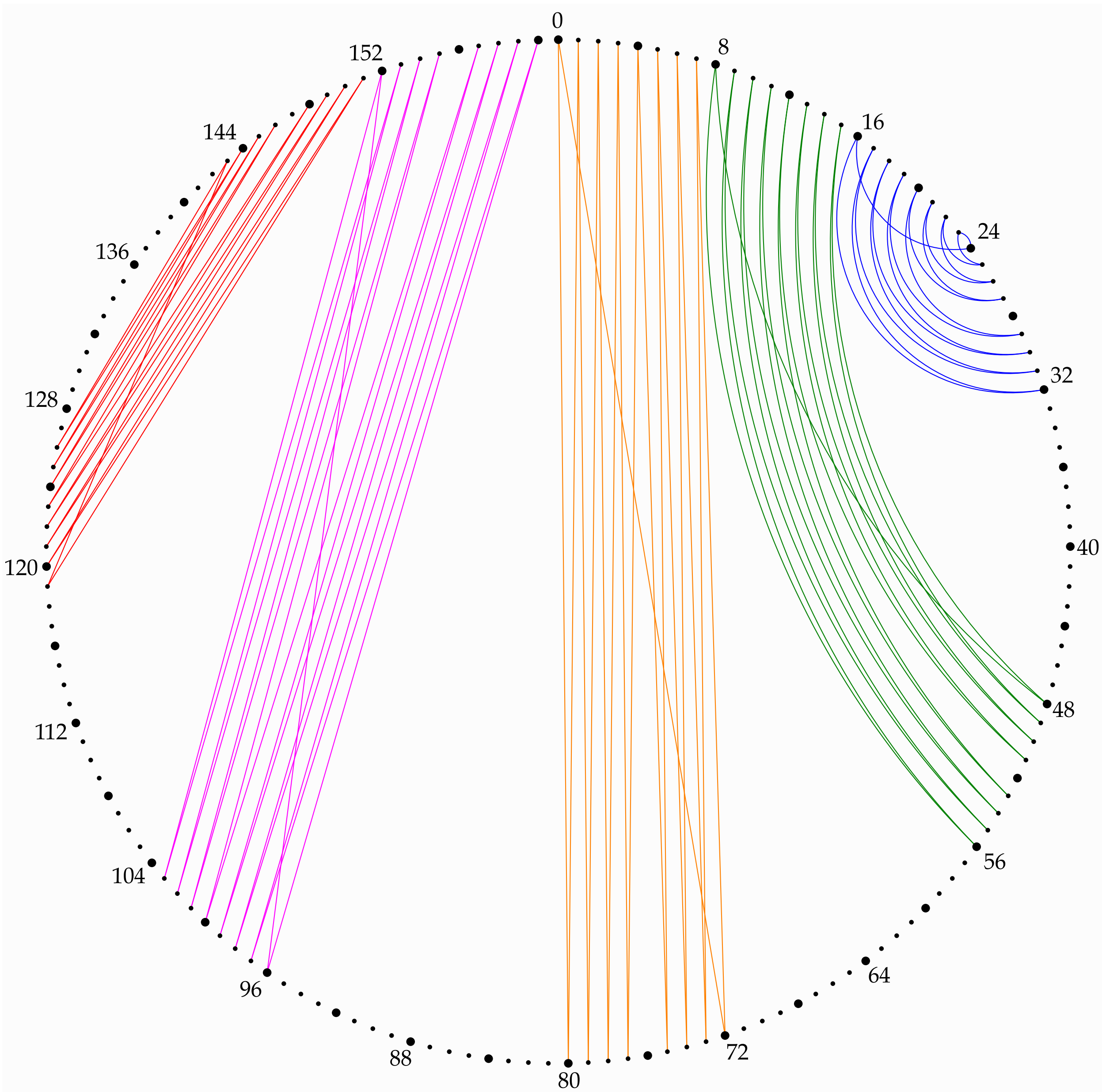



Figure 5.3: A Blinco–El-Zanati $\sigma^{+}$-labeling of $\mathcal{C}_{16}^{5}$

Table 5.2: Cycle list for the $\sigma^+$-labeling of $\mathcal{C}_{16}^5$ in Figure 5.3

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Graph** $G_1 = C_{16}$**: Cycle** $\mathfrak{C}_1$ (orange) | | | | | | | | | | | | | | | |
| $\mathfrak{C}_1 = (0, 80, 1, 79, 2, 78, 3, 77, 4, 75, 5, 74, 6, 73, 7, 72)$ | | | | | | | | | | | | | | | |
| 80 | [79] | 78 | [77] | 76 | [75] | 74 | [73] | 71 | [70] | 69 | [68] | 67 | [66] | 65 | [72] |
| **Graph** $G_2 = C_{16}$**: Cycle** $\mathfrak{C}_2$ (fuchsia) | | | | | | | | | | | | | | | |
| $\mathfrak{C}_2 = (96, 160, 97, 159, 98, 158, 99, 157, 100, 155, 101, 154, 102, 153, 103, 152)$ | | | | | | | | | | | | | | | |
| 64 | [63] | 62 | [61] | 60 | [59] | 58 | [57] | 55 | [54] | 53 | [52] | 51 | [50] | 49 | [56] |
| **Graph** $G_3 = C_{16}$**: Cycle** $\mathfrak{C}_3$ (forest) | | | | | | | | | | | | | | | |
| $\mathfrak{C}_3 = (8, 56, 9, 55, 10, 54, 11, 53, 12, 51, 13, 50, 14, 49, 15, 48)$ | | | | | | | | | | | | | | | |
| 48 | [47] | 46 | [45] | 44 | [43] | 42 | [41] | 39 | [38] | 37 | [36] | 35 | [34] | 33 | [40] |
| **Graph** $G_4 = C_{16}$**: Cycle** $\mathfrak{C}_4$ (red) | | | | | | | | | | | | | | | |
| $\mathfrak{C}_4 = (119, 151, 120, 150, 121, 149, 122, 148, 123, 146, 124, 145, 125, 144, 126, 143)$ | | | | | | | | | | | | | | | |
| 32 | [31] | 30 | [29] | 28 | [27] | 26 | [25] | 23 | [22] | 21 | [20] | 19 | [18] | 17 | [24] |
| **Graph** $G_5 = C_{16}$**: Cycle** $\mathfrak{C}_5$ (cobalt) | | | | | | | | | | | | | | | |
| $\mathfrak{C}_5 = (16, 32, 17, 31, 18, 30, 19, 29, 20, 27, 21, 26, 22, 25, 23, 24)$ | | | | | | | | | | | | | | | |
| 16 | [15] | 14 | [13] | 12 | [11] | 10 | [9] | 7 | [6] | 5 | [4] | 3 | [2] | 1 | [8] |

### 5.1.2 The Blinco–El-Zanati Construction for Odd $k$ and Even $p$

We next consider the graph $\mathcal{C}_{2k}^{p}$ in the case that $k$ is odd and $p$ is even. In this case, the cycle length is congruent to two modulo four; we let $q \in \mathbb{P}$ such that $p = 2q$. We observe that, since $k$ is odd, $C_2$ does not admit an $\alpha$-labeling (because it fails the *parity condition*; see [29] and [13]), but $C_{2k} \uplus C_{2k}$ does admit an $\alpha$-labeling, by Kotzig's result (Theorem 5.6). Let $V(C_{2k} \uplus C_{2k}) = [\![1, 4k]\!]$ so that vertices $i$ and $j$ in $[\![1, 2k]\!]$ are adjacent whenever $j \equiv i + 1 \pmod{2k}$ and vertices $i$ and $j$ in $[\![2k + 1, 4k]\!]$ are adjacent whenever $j \equiv i + 1 \pmod{2k}$. Kotzig's $\alpha$-labeling, $L$, of $C_{2k} \uplus C_{2k}$ is given by

$$L(v) = \begin{cases} 0\,, & \text{if } v = 1\,, \\ k + \dfrac{v-1}{2}\,, & \text{if } v \text{ is odd and } 3 \le v \le k\,, \\ k - \dfrac{v-1}{2}\,, & \text{if } v \text{ is odd and } k+2 \le v \le 2k-1\,, \\ \\ 4k\,, & \text{if } v = 2\,, \\ 3k + 2 - \dfrac{v}{2}\,, & \text{if } v \text{ is even and } 4 \le v \le k-1\,, \\ 3k - 1 + \dfrac{v}{2}\,, & \text{if } v \text{ is even and } k+1 \le v \le 2k\,, \\ \\ 2k\,, & \text{if } v = 2k+1\,, \\ 2k - \dfrac{v-1}{2}\,, & \text{if } v \text{ is odd and } 2k+3 \le v \le 3k\,, \\ \dfrac{v-1}{2}\,, & \text{if } v \text{ is odd and } 3k+2 \le v \le 4k-1\,, \\ \\ 2k+1\,, & \text{if } v = 2k+2\,, \\ 2k - 1 + \dfrac{v}{2}\,, & \text{if } v \text{ is even and } 2k+4 \le v \le 3k-1\,, \\ 4k + 2 - \dfrac{v}{2}\,, & \text{if } v \text{ is even and } 3k+1 \le v \le 4k\,. \end{cases} \tag{5.19}$$

The critical value of $L$ is $\lambda = 2k$.

In order to produce a $\rho^+$-labeling of $\mathcal{C}_{2k}^p$, the Blinco–El-Zanati Construction relies again on the construction given in the proof of Theorem 5.4 by El-Zanati, Vanden Eynden, and Punnim, which actually produces a $\sigma^+$-labeling, as previously noted. This $\sigma^+$-labeling was given in formulas (5.4), (5.5), and (5.6).

We wish to describe the labeling $h$ as specifically as possible for the graph $G = \mathcal{C}_{2k}^p$ in the case that $k$ is odd and $p$ is even. We let $t = q$, and, for all $i \in [\![1, q]\!]$, we let $G_i = C_{2k} \uplus C_{2k}$, so $n_i = 4k$, and $h_i = L$, with $\lambda_i = 2k$. Let $V(G_i) = [\![1, 4k]\!] \times \{i\}$, so that $V(G) = [\![1, 4k]\!] \times [\![1, q]\!]$; then we have, for all $i \in [\![1, q]\!]$, that

$$A_i = \left\{(a, i) \;\middle|\; a \in [\![1, 4k]\!] \text{ and } a \text{ is odd}\right\} \text{ and} \tag{5.20}$$

$$B_i = \left\{(b, i) \;\middle|\; b \in [\![1, 4k]\!] \text{ and } b \text{ is even}\right\}. \tag{5.21}$$

We can therefore simplify the sums in the definitions of $\alpha_i$ and $\beta_i$ as follows.

$$(i \text{ odd}) \sum_{\substack{j \text{ odd},\\ j<i}} \lambda_j = \sum_{\substack{j \text{ odd},\\ j<i}} 2k = 2k\left(\frac{i-1}{2}\right) = k(i-1) \tag{5.22}$$

$$\sum_{j=1}^{t} n_j = \sum_{j=1}^{q} 4k = 4kq = 2kp \tag{5.23}$$

$$(i \text{ even}) \sum_{\substack{j \text{ odd},\\ j\le i}} n_j = \sum_{\substack{j \text{ odd},\\ j\le i}} 4k = 4k\left(\frac{i}{2}\right) = 2ki \tag{5.24}$$

$$(i \text{ even}) \sum_{\substack{j \text{ even},\\ j<i}} \lambda_j = \sum_{\substack{j \text{ even},\\ j<i}} 2k = 2k\left(\frac{i}{2}-1\right) = k(i-2) \tag{5.25}$$

$$\sum_{j>i} n_j = \sum_{j=i+1}^{q} 4k = 4k(q-i) = 2k(p-2i) \tag{5.26}$$

We obtain the following definitions of $\alpha_i$, $\beta_i$, and $h$.

$$\alpha_i = \begin{cases} k(i-1) + \left(\dfrac{i-1}{2}\right), & \text{if } i \text{ is odd,} \\ k\big(2p+3i-2\big), & \text{if } i \text{ is even.} \end{cases} \tag{5.27}$$

$$\beta_i = \begin{cases} k\big(2p-3i-1\big) + \dfrac{i-1}{2}, & \text{if } i \text{ is odd,} \\ k\big(4p-i-2\big), & \text{if } i \text{ is even.} \end{cases} \tag{5.28}$$

$$h(v,i) = \begin{cases} L(v) + \alpha_i, & \text{if } v \text{ is odd,} \\ L(v) + \beta_i, & \text{if } v \text{ is even,} \end{cases} \qquad \text{for all } (v,i) \in V(G) \tag{5.29}$$

We may combine formulas (5.19), (5.27), (5.28), and (5.29) to obtain a direct formula for $h(v,i)$, without reference to $\alpha_i$ or $\beta_i$. For added clarity, we split the formula statement into two parts, by the parity of $i$, with four subdivisions each.

If $i$ is odd, then

$$h(v,i) = \begin{cases} k(i-1) + \dfrac{i-1}{2}, & \text{if } v = 1, \\ ki - 1 + \dfrac{v+i}{2}, & \text{if } v \text{ is odd and } 3 \le v \le k, \\ ki + \dfrac{i-v}{2}, & \text{if } v \text{ is odd and } k+2 \le v \le 2k-1, \end{cases} \tag{5.30}$$

$$h(v,i) = \begin{cases} k(2p-3i+3) + \dfrac{i-1}{2}, & \text{if } v = 2, \\ k(2p-3i+2) + \dfrac{i-v+3}{2}, & \text{if } v \text{ is even and } 4 \le v \le k-1, \\ k(2p-3i+2) + \dfrac{v+i-3}{2}, & \text{if } v \text{ is even and } k+1 \le v \le 2k, \end{cases} \tag{5.31}$$

$$h(v,i) = \begin{cases} k(i+1) + \dfrac{i-1}{2}, & \text{if } v = 2k+1, \\ k(i+1) + \dfrac{i-v}{2}, & \text{if } v \text{ is odd and } 2k+3 \le v \le 3k, \\ k(i-1) - 1 + \dfrac{v+i}{2}, & \text{if } v \text{ is odd and } 3k+2 \le v \le 4k-1, \end{cases} \tag{5.32}$$

$$h(v,i)=\begin{cases} k(2p-3i+1)+\dfrac{i+1}{2}\,, & \text{if } v=2k+2\,,\\ k(2p-3i+1)+\dfrac{v+i-3}{2}\,, & \text{if } v \text{ is even and } 2k+4\le v\le 3k-1\,,\\ k(2p-3i+3)+\dfrac{i-v+3}{2}\,, & \text{if } v \text{ is even and } 3k+1\le v\le 4k\,. \end{cases} \tag{5.33}$$

If $i$ is even, then

$$h(v,i)=\begin{cases} k\big(2p+3i-2\big)\,, & \text{if } v=1\,,\\ k\big(2p+3i-1\big)+\dfrac{v-1}{2}\,, & \text{if } v \text{ is odd and } 3\le v\le k\,,\\ k\big(2p+3i-1\big)-\dfrac{v-1}{2}\,, & \text{if } v \text{ is odd and } k+2\le v\le 2k-1\,, \end{cases} \tag{5.34}$$

$$h(v,i)=\begin{cases} k\big(4p-i+2\big)\,, & \text{if } v=2\,,\\ k\big(4p-i+1\big)+2-\dfrac{v}{2}\,, & \text{if } v \text{ is even and } 4\le v\le k-1\,,\\ k\big(4p-i+1\big)-1+\dfrac{v}{2}\,, & \text{if } v \text{ is even and } k+1\le v\le 2k\,, \end{cases} \tag{5.35}$$

$$h(v,i)=\begin{cases} k\big(2p+3i\big)\,, & \text{if } v=2k+1\,,\\ k\big(2p+3i\big)-\dfrac{v-1}{2}\,, & \text{if } v \text{ is odd and } 2k+3\le v\le 3k\,,\\ k\big(2p+3i-2\big)+\dfrac{v-1}{2}\,, & \text{if } v \text{ is odd and } 3k+2\le v\le 4k-1\,, \end{cases} \tag{5.36}$$

$$h(v,i)=\begin{cases} k\big(4p-i\big)+1\,, & \text{if } v=2k+2\,,\\ k\big(4p-i\big)-1+\dfrac{v}{2}\,, & \text{if } v \text{ is even and } 2k+4\le v\le 3k-1\,,\\ k\big(4p-i+2\big)+2-\dfrac{v}{2}\,, & \text{if } v \text{ is even and } 3k+1\le v\le 4k\,. \end{cases} \tag{5.37}$$

We give two examples of this labeling construction: $\sigma^+$-labelings of $\mathcal{C}_{10}^6$ and $\mathcal{C}_{10}^8$. Observe that, in both examples, cycles are entwined in pairs by the labeling; this is caused by the need to use $C_{2k} \uplus C_{2k}$-subgraphs of $\mathcal{C}_{2k}^p$ in order to obtain the required $\alpha$-labelings.

**Example 5.9.** Consider $\mathcal{C}_{10}^{6}$; this is the case $p = 6$ and $k = 5$. We exhibit a $\sigma^{+}$-labeling generated by the Blinco–El-Zanati Construction as the base block for a purely cyclic design of order $4kp + 1 = 121$. This base block is shown in Figure 5.4; distinct cycles are shown in different colors for enhanced visibility. The cycles and their corresponding difference patterns are listed in Table 5.3. ■

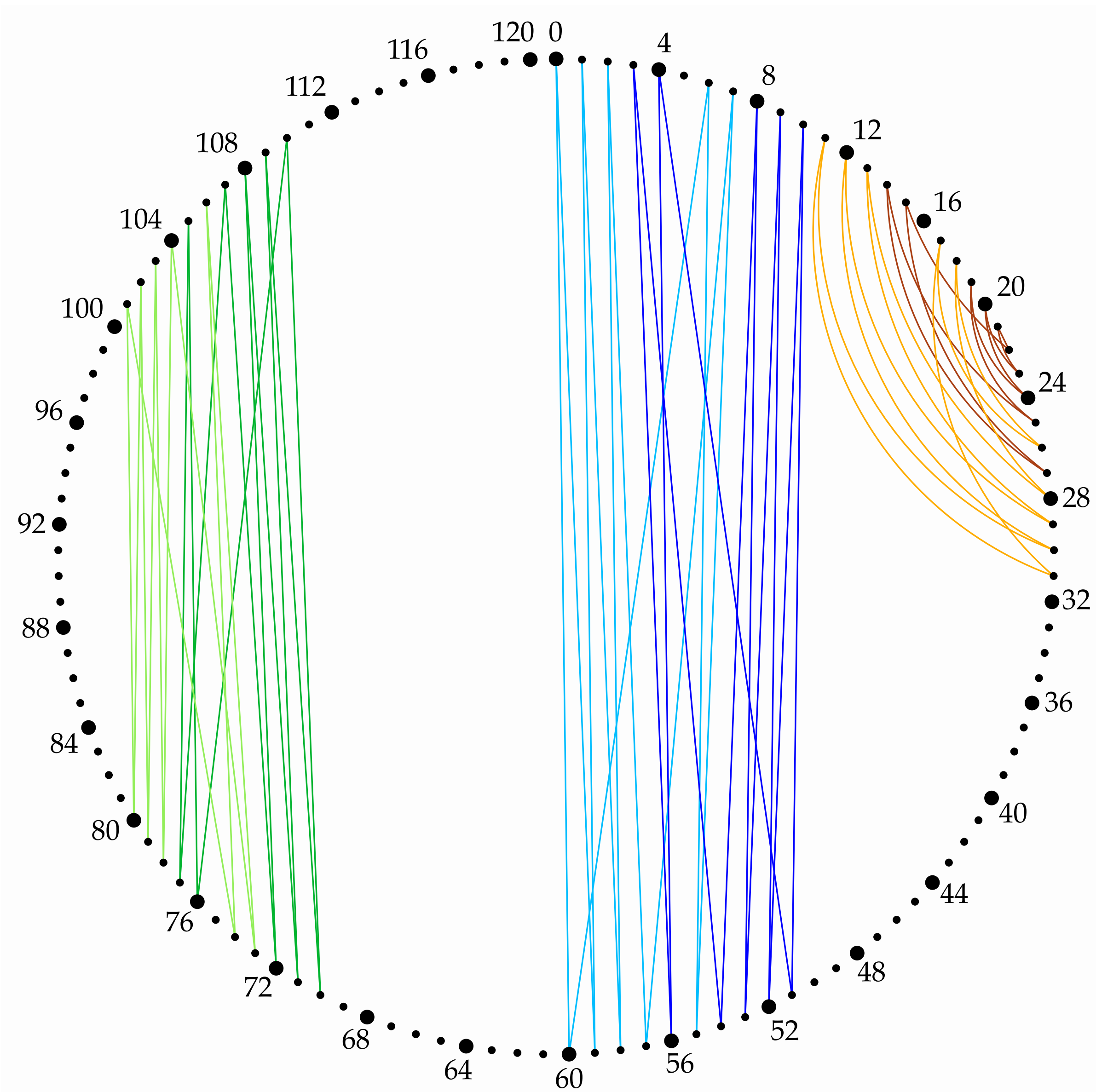



Figure 5.4: A Blinco–El-Zanati $\sigma^{+}$-labeling of $\mathcal{C}_{10}^{6}$

Table 5.3: Cycle list for the $\sigma^+$-labeling of $\mathcal{C}_{10}^{6}$ in Figure 5.4

| **Graph** $G_1 = C_{10} \uplus C_{10}$**: Cycles** $\mathfrak{C}_1$ **and** $\mathfrak{C}_2$ | (sky and cobalt) |
|---|---|
| $\mathfrak{C}_1 = (0, 60, 6, 55, 7, 57, 2, 58, 1, 59)$ | |
| 60 [54] 49 [48] 50 [55] 56 [57] 58 [59] | |
| $\mathfrak{C}_2 = (10, 51, 4, 56, 3, 54, 8, 53, 9, 52)$ | |
| 41 [47] 52 [53] 51 [46] 45 [44] 43 [42] | |
| **Graph** $G_2 = C_{10} \uplus C_{10}$**: Cycles** $\mathfrak{C}_3$ **and** $\mathfrak{C}_4$ | (jade and lime) |
| $\mathfrak{C}_3 = (70, 110, 76, 105, 77, 107, 72, 108, 71, 109)$ | |
| 40 [34] 29 [28] 30 [35] 36 [37] 38 [39] | |
| $\mathfrak{C}_4 = (80, 101, 74, 106, 73, 104, 78, 103, 79, 102)$ | |
| 21 [27] 32 [33] 31 [26] 25 [24] 23 [22] | |
| **Graph** $G_3 = C_{10} \uplus C_{10}$**: Cycles** $\mathfrak{C}_5$ **and** $\mathfrak{C}_6$ | (apricot and sienna) |
| $\mathfrak{C}_5 = (11, 31, 17, 26, 18, 28, 13, 29, 12, 30)$ | |
| 20 [14] 9 [8] 10 [15] 16 [17] 18 [19] | |
| $\mathfrak{C}_6 = (21, 22, 15, 27, 14, 25, 19, 24, 20, 23)$ | |
| 1 [7] 12 [13] 11 [6] 5 [4] 3 [2] | |

**Example 5.10.** Consider $\mathcal{C}_{10}^{8}$; this is the case $p = 8$ and $k = 5$. We exhibit a $\sigma^+$-labeling generated by the Blinco–El-Zanati Construction as the base block for a purely cyclic design of order $4kp + 1 = 161$. This base block is shown in Figure 5.5; distinct cycles are shown in different colors for enhanced visibility. The cycles and their corresponding difference patterns are listed in Table 5.4. ■

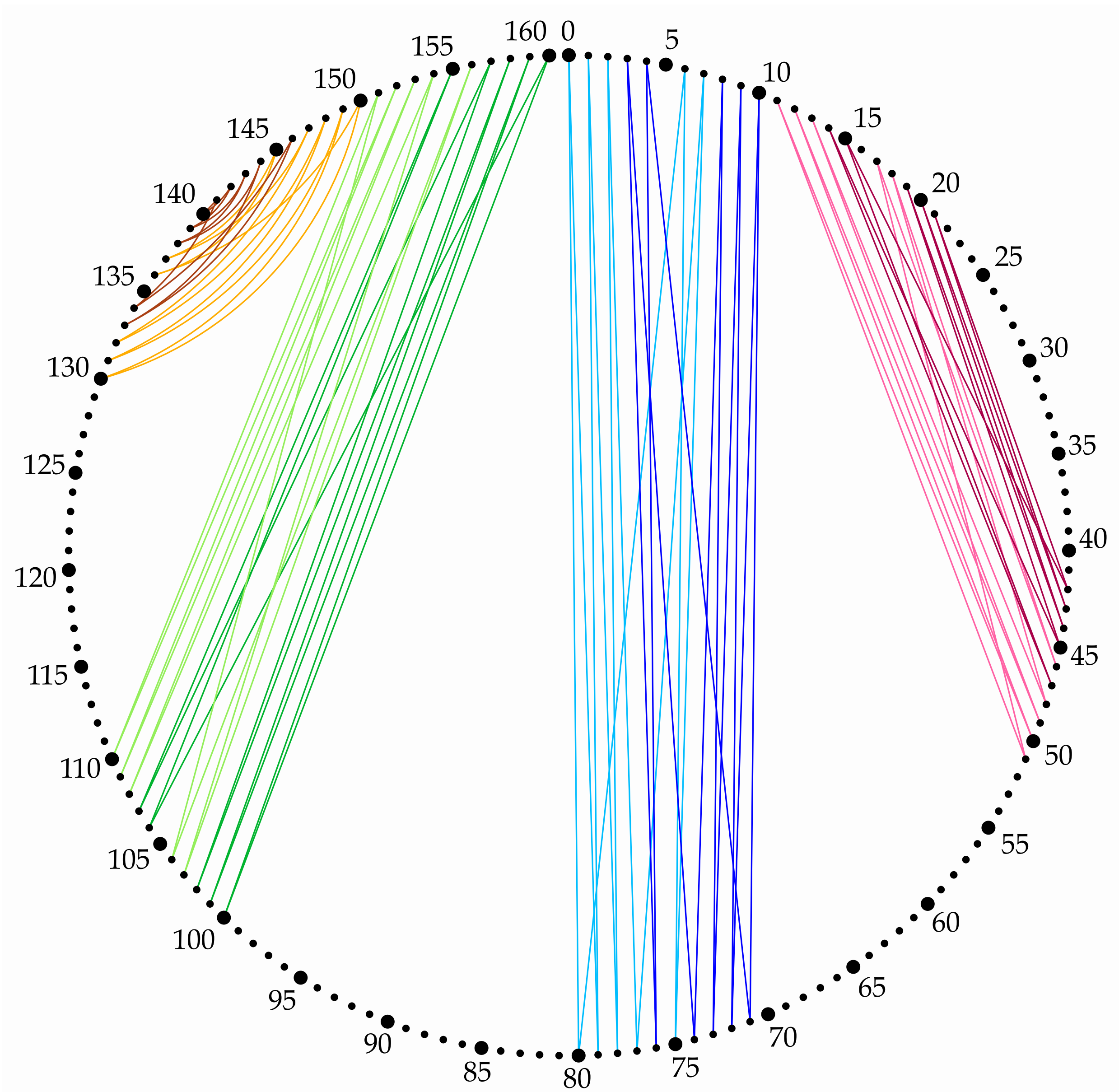



Figure 5.5: A Blinco–El-Zanati $\sigma^+$-labeling of $\mathcal{C}_{10}^{8}$

Table 5.4: Cycle list for the $\sigma^+$-labeling of $\mathcal{C}_{10}^{8}$ in Figure 5.5

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| **Graph** $G_1 = C_{10} \uplus C_{10}$: **Cycles** $\mathfrak{C}_1$ **and** $\mathfrak{C}_2$ (sky and cobalt) | | | | | | | | | | |
| $\mathfrak{C}_1 = \big(0, 80, 6, 75, 7, 77, 2, 78, 1, 79\big)$ | | | | | | | | | | |
| | 80 | [74] | 69 | [68] | 70 | [75] | 76 | [77] | 78 | [79] |
| $\mathfrak{C}_2 = \big(10, 71, 4, 76, 3, 74, 8, 73, 9, 72\big)$ | | | | | | | | | | |
| | 61 | [67] | 72 | [73] | 71 | [66] | 65 | [64] | 63 | [62] |
| **Graph** $G_2 = C_{10} \uplus C_{10}$: **Cycles** $\mathfrak{C}_3$ **and** $\mathfrak{C}_4$ (jade and lime) | | | | | | | | | | |
| $\mathfrak{C}_3 = \big(100, 160, 106, 155, 107, 157, 102, 158, 101, 159\big)$ | | | | | | | | | | |
| | 60 | [54] | 49 | [48] | 50 | [55] | 56 | [57] | 58 | [59] |
| $\mathfrak{C}_4 = \big(110, 151, 104, 156, 103, 154, 108, 153, 109\ 152\big)$ | | | | | | | | | | |
| | 41 | [47] | 52 | [53] | 51 | [46] | 45 | [44] | 43 | [42] |
| **Graph** $G_3 = C_{10} \uplus C_{10}$: **Cycles** $\mathfrak{C}_5$ **and** $\mathfrak{C}_6$ (rose and wine) | | | | | | | | | | |
| $\mathfrak{C}_5 = \big(11, 51, 17, 46, 18, 48, 13, 49, 12, 50\big)$ | | | | | | | | | | |
| | 40 | [34] | 29 | [28] | 30 | [35] | 36 | [37] | 38 | [39] |
| $\mathfrak{C}_6 = \big(21, 42, 15, 47, 14, 45, 19, 44, 20, 43\big)$ | | | | | | | | | | |
| | 21 | [27] | 32 | [33] | 31 | [26] | 25 | [24] | 23 | [22] |
| **Graph** $G_4 = C_{10} \uplus C_{10}$: **Cycles** $\mathfrak{C}_7$ **and** $\mathfrak{C}_8$ (apricot and sienna) | | | | | | | | | | |
| $\mathfrak{C}_7 = \big(130, 150, 136, 145, 137, 147, 132, 148, 131, 149\big)$ | | | | | | | | | | |
| | 20 | [14] | 9 | [8] | 10 | [15] | 16 | [17] | 18 | [19] |
| $\mathfrak{C}_8 = \big(140, 141, 134, 146, 133, 144, 138, 143, 139, 142\big)$ | | | | | | | | | | |
| | 1 | [7] | 12 | [13] | 11 | [6] | 5 | [4] | 3 | [2] |

### 5.1.3 The Blinco–El-Zanati Construction for Odd $k$ and Odd $p$

We last consider the graph $\mathcal{C}_{2k}^{p}$ in the case that $k$ and $p$ are both odd. In this case, the cycle length is congruent to two modulo four; we let $q \in \mathbb{P}$ such that $p = 2q + 1$. We use the $\rho^{++}$-labeling of $C_{2k}$ by El-Zanati, Vanden Eynden, and Punnim (see Theorem 5.5) and the $\alpha$-labeling of $C_{2k} \uplus C_{2k}$ by Kotzig (see Theorem 5.6 and formula 5.19) in our description of this construction.

In order to produce a $\rho^{+}$-labeling of $\mathcal{C}_{2k}^{p}$, the Blinco–El-Zanati Construction relies on the construction given in the proof of Theorem 5.3 by Blinco and El-Zanati. This construction is identical to the construction by El-Zanati, Vanden Eynden, and Punnim given in [12], except in its weaker hypotheses. We have reproduced below the entire definition of the $\rho^{+}$-labeling, $h$, produced in the proof of Theorem 5.3, as it is given in [6]; we have made changes to the text only to ensure consistent notation in this document.

> Let $G_1$ be a bipartite graph with $n_1$ edges and a $\rho^{++}$-labeling $h_1$ such that no vertex of $G_1$ is labeled $2n_1$. Let $[A_1, B_1]$ be the bipartition of $V(G_1)$ induced by $h_1$. Let $\lambda_1$ be the critical value of $h_1$. For $i \in [\![2, t]\!]$, let $G_i$ be a graph with $n_i$ edges having an $\alpha$-labeling $h_i$ with critical value $\lambda_i$ and vertex bipartition $[A_i, B_i]$, where $A_i = \{v \in V(G_i) \mid h_i(v) \leq \lambda_i\}$. Define integers $c_i$ and $d_i$ for each $i \in [\![1, t]\!]$ by

$$c_i = \begin{cases} \dfrac{i-1}{2} + \displaystyle\sum_{\substack{j \text{ odd}, \\ j < i}} \lambda_j \,, & \text{if } i \text{ is odd}, \\ \displaystyle\sum_{j=1}^{t} n_j + \sum_{\substack{j \text{ odd}, \\ j \leq i}} n_j + \sum_{\substack{j \text{ even}, \\ j < i}} \lambda_j \,, & \text{if } i \text{ is even}, \end{cases} \tag{5.38}$$

and

$$d_i = c_i + \sum_{j > i} n_j \,. \tag{5.39}$$

Assuming the graphs $G_i$ are vertex-disjoint, we define $h$ on their union, $G$, by

$$h(v) = \begin{cases} h_i(v) + c_i\,, & \text{if } v \in A_i\,, \\ h_i(v) + d_i\,, & \text{if } v \in B_i\,. \end{cases} \tag{5.40}$$

We wish to describe the labeling $h$ as specifically as possible for the graph $G = \mathcal{C}_{2k}^p$ in the case that both $k$ and $p$ are odd. For this choice of $G$, we must let $G_1 = C_{2k}$, so that $n_1 = 2k$, and let $h_1$ be the labeling $H$ given in Theorem 5.5, so that $\lambda_1 = k - 1$. We also let $t = q + 1$, and, for all $i \in [\![2, q+1]\!]$, we let $G_i = C_{2k} \uplus C_{2k}$, so $n_i = 4k$, and $h_i = L$, with $\lambda_i = 2k$. Furthermore, let $V(G_1) = [\![1, 2k]\!] \times \{1\}$ and $V(G_i) = [\![1, 4k]\!] \times \{i\}$ for all $i \in [\![2, q+1]\!]$, so that $V(G) = \Big([\![1, 2k]\!] \times \{1\}\Big) \cup \Big([\![1, 4k]\!] \times [\![2, q+1]\!]\Big)$. Then we have, for all $i \in [\![1, q+1]\!]$, that

$$A_i = \Big\{(a, i) \;\Big|\; a \in V(G_i) \text{ and } a \text{ is odd}\Big\}\,, \quad \text{and} \tag{5.41}$$

$$B_i = \Big\{(b, i) \;\Big|\; b \in V(G_i) \text{ and } b \text{ is even}\Big\}\,. \tag{5.42}$$

We can therefore simplify the sums in the definitions of $c_i$ and $d_i$ as follows.

$$(i > 1,\ \text{odd}) \sum_{\substack{j \text{ odd},\\ j<i}} \lambda_j = (k-1) + \sum_{\substack{j \text{ odd},\\ 1<j<i}} 2k = (k-1) + k(i-3) \tag{5.43}$$

$$\sum_{j=1}^{t} n_j = 2k + \sum_{j=2}^{q+1} 4k = 2k + 4kq = 2kp \tag{5.44}$$

$$(i \text{ even}) \sum_{\substack{j \text{ odd},\\ j\le i}} n_j = 2k + \sum_{\substack{j \text{ odd},\\ 1<j\le i}} 4k = 2k(i-1) \tag{5.45}$$

$$(i \text{ even}) \sum_{\substack{j \text{ even},\\ j<i}} \lambda_j = \sum_{\substack{j \text{ even},\\ 1<j<i}} 2k = k(i-2) \tag{5.46}$$

$$\sum_{j>i} n_j = \sum_{j=i+1}^{q+1} 4k = 2k(p+1-2i) \tag{5.47}$$

We obtain the following definitions of $c_i$, $d_i$, and $h$.

$$c_i = \begin{cases} 0\,, & \text{if } i = 1, \\ k(i-2) + \dfrac{i-3}{2}\,, & \text{if } i \text{ is odd and } i > 1, \\ k\big(2p+3i-4\big)\,, & \text{if } i \text{ is even.} \end{cases} \tag{5.48}$$

$$d_i = \begin{cases} 2k(p-1)\,, & \text{if } i = 1, \\ k(2p-3i) + \dfrac{i-3}{2}\,, & \text{if } i \text{ is odd and } i > 1, \\ k\big(4p-i-2\big)\,, & \text{if } i \text{ is even.} \end{cases} \tag{5.49}$$

$$h(v,1) = \begin{cases} H(v)\,, & \text{if } v \text{ is odd,} \\ H(v) + 2k(p-1)\,, & \text{if } v \text{ is even.} \end{cases} \tag{5.50}$$

$$h(v,i) = \begin{cases} L(v) + c_i\,, & \text{if } v \text{ is odd,} \\ L(v) + d_i\,, & \text{if } v \text{ is even,} \end{cases} \qquad \text{for all } i \in [\![2, q+1]\!]\,. \tag{5.51}$$

We may combine formulas (5.1), (5.19), (5.48), (5.49), (5.50), and (5.51) to obtain a direct formula for $h(v,i)$, without reference to $c_i$ or $d_i$. We give the formula statement in three parts: $i = 1$; $i > 1$ and $i$ is odd; and $i > 1$ and $i$ is even.

$$h(v,1) = \begin{cases} (v-1)/2, & \text{if } v \text{ is odd,} \\ 2kp - (v/2), & \text{if } v \text{ is even and } 2 \le v \le k-3, \\ 2kp - 1 - (v/2), & \text{if } v \text{ is even and } k-1 \le v \le 2k-2, \\ 2kp+1, & \text{if } v = 2k. \end{cases} \tag{5.52}$$

If $i$ is odd and $i > 1$, then

$$h(v,i) = \begin{cases} k(i-2) + \dfrac{i-3}{2}\,, & \text{if } v = 1\,, \\ k(i-1) - 2 + \dfrac{v+i}{2}\,, & \text{if } v \text{ is odd and } 3 \le v \le k\,, \\ k(i-1) - 1 + \dfrac{i-v}{2}\,, & \text{if } v \text{ is odd and } k+2 \le v \le 2k-1\,, \end{cases} \tag{5.53}$$

$$h(v,i) = \begin{cases} k(2p-3i+4) + \dfrac{i-3}{2}\,, & \text{if } v = 2\,, \\ k(2p-3i+3) + \dfrac{i-v+1}{2}\,, & \text{if } v \text{ is even and } 4 \le v \le k-1\,, \\ k(2p-3i+3) + \dfrac{v+i-5}{2}\,, & \text{if } v \text{ is even and } k+1 \le v \le 2k\,, \end{cases} \tag{5.54}$$

$$h(v,i) = \begin{cases} ki + \dfrac{i-3}{2}\,, & \text{if } v = 2k+1\,, \\ ki - 1 + \dfrac{i-v}{2}\,, & \text{if } v \text{ is odd and } 2k+3 \le v \le 3k\,, \\ k(i-2) - 2 + \dfrac{v+i}{2}\,, & \text{if } v \text{ is odd and } 3k+2 \le v \le 4k-1\,, \end{cases} \tag{5.55}$$

$$h(v,i) = \begin{cases} k(2p-3i+2) + \dfrac{i-1}{2}\,, & \text{if } v = 2k+2\,, \\ k(2p-3i+2) + \dfrac{v+i-5}{2}\,, & \text{if } v \text{ is even and } 2k+4 \le v \le 3k-1\,, \\ k(2p-3i+4) + \dfrac{i-v+1}{2}\,, & \text{if } v \text{ is even and } 3k+1 \le v \le 4k\,. \end{cases} \tag{5.56}$$

If $i$ is even, then

$$h(v,i) = \begin{cases} k\big(2p+3i-4\big)\,, & \text{if } v = 1\,, \\ k\big(2p+3i-3\big) + \dfrac{v-1}{2}\,, & \text{if } v \text{ is odd and } 3 \le v \le k\,, \\ k\big(2p+3i-3\big) - \dfrac{v-1}{2}\,, & \text{if } v \text{ is odd and } k+2 \le v \le 2k-1\,, \end{cases} \tag{5.57}$$

$$h(v,i)=\begin{cases} k\big(4p-i+2\big)\,, & \text{if } v=2\,,\\ k\big(4p-i+1\big)+2-\dfrac{v}{2}\,, & \text{if } v \text{ is even and } 4\le v\le k-1\,,\\ k\big(4p-i+1\big)-1+\dfrac{v}{2}\,, & \text{if } v \text{ is even and } k+1\le v\le 2k\,, \end{cases} \tag{5.58}$$

$$h(v,i)=\begin{cases} k\big(2p+3i-2\big)\,, & \text{if } v=2k+1\,,\\ k\big(2p+3i-2\big)-\dfrac{v-1}{2}\,, & \text{if } v \text{ is odd and } 2k+3\le v\le 3k\,,\\ k\big(2p+3i-4\big)+\dfrac{v-1}{2}\,, & \text{if } v \text{ is odd and } 3k+2\le v\le 4k-1\,, \end{cases} \tag{5.59}$$

$$h(v,i)=\begin{cases} k\big(4p-i\big)+1\,, & \text{if } v=2k+2\,,\\ k\big(4p-i\big)-1+\dfrac{v}{2}\,, & \text{if } v \text{ is even and } 2k+4\le v\le 3k-1\,,\\ k\big(4p-i+2\big)+2-\dfrac{v}{2}\,, & \text{if } v \text{ is even and } 3k+1\le v\le 4k\,. \end{cases} \tag{5.60}$$

We give two examples of this labeling construction: $\rho^+$-labelings of $\mathcal{C}_6^5$ and $\mathcal{C}_{14}^5$. Observe that, in both examples, cycles are entwined in pairs by the labeling; this is caused by the need to use $C_{2k} \uplus C_{2k}$-subgraphs of $\mathcal{C}_{2k}^p$ in order to obtain the required $\alpha$-labeling.

**Example 5.11.** Consider $\mathcal{C}_6^5$; this is the case $p=5$ and $k=3$. We exhibit a $\rho^+$-labeling generated by the Blinco–El-Zanati Construction as the base block for a purely cyclic design of order $4kp+1=61$. This base block is shown in Figure 5.6; distinct cycles are shown in different colors for enhanced visibility. The cycles and their corresponding difference patterns are listed in Table 5.5. ■

**Example 5.12.** Consider $\mathcal{C}_{14}^5$; this is the case $p=5$ and $k=7$. We exhibit a $\rho^+$-labeling generated by the Blinco–El-Zanati Construction as the base block for a purely cyclic design of order $4kp+1=141$. This base block is shown in Figure 5.7; distinct cycles are shown in different colors for enhanced visibility. The cycles and their corresponding difference patterns are listed in Table 5.6. ■

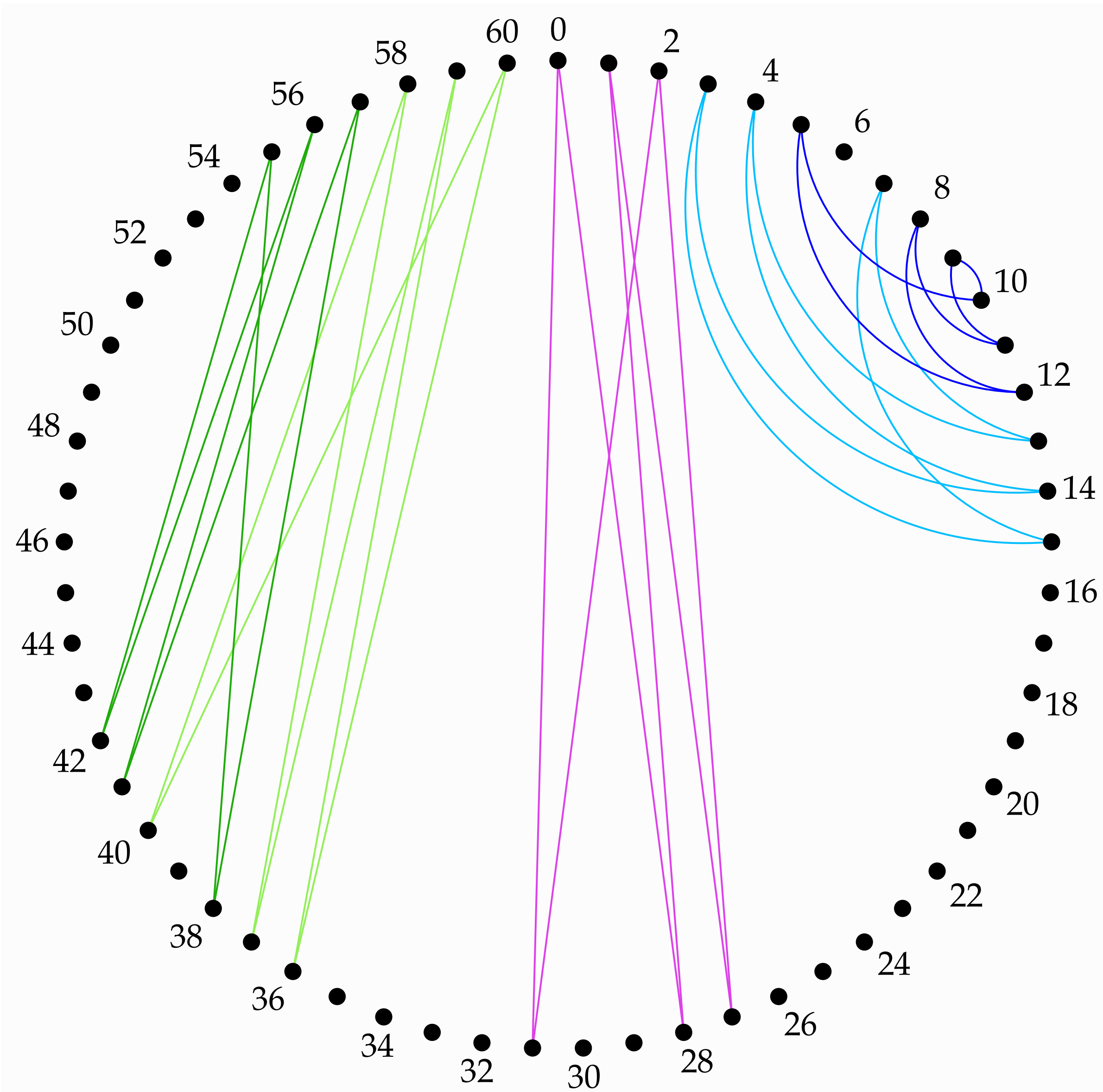



Figure 5.6: A Blinco–El-Zanati $\rho^+$-labeling of $\mathcal{C}_6^5$

Table 5.5: Cycle list for the $\rho^+$-labeling of $\mathcal{C}_6^5$ in Figure 5.4

| | | | | | | |
|---|---|---|---|---|---|---|
| **Graph** $G_1 = C_6$**: Cycle** $\mathfrak{C}_1$ (lilac) | | | | | | |
| $\mathfrak{C}_1 = (0, 28, 1, 27, 2, 31)$ | 28 | [27] | 26 | [25] | 29 | 30 |
| **Graph** $G_2 = C_6 \uplus C_6$**: Cycles** $\mathfrak{C}_2$ **and** $\mathfrak{C}_3$ (lime and jade) | | | | | | |
| $\mathfrak{C}_2 = (36, 60, 40, 58, 37, 59)$ | 24 | [20] | 18 | [21] | 22 | [23] |
| $\mathfrak{C}_3 = (42, 55, 38, 57, 41, 56)$ | 13 | [17] | 19 | [16] | 15 | [14] |
| **Graph** $G_3 = C_6 \uplus C_6$**: Cycles** $\mathfrak{C}_4$ **and** $\mathfrak{C}_5$ (sky and cobalt) | | | | | | |
| $\mathfrak{C}_4 = (3, 15, 7, 13, 4, 14)$ | 12 | [8] | 6 | [9] | 10 | [11] |
| $\mathfrak{C}_5 = (9, 10, 5, 12, 8, 11)$ | 1 | [5] | 7 | [4] | 3 | [2] |

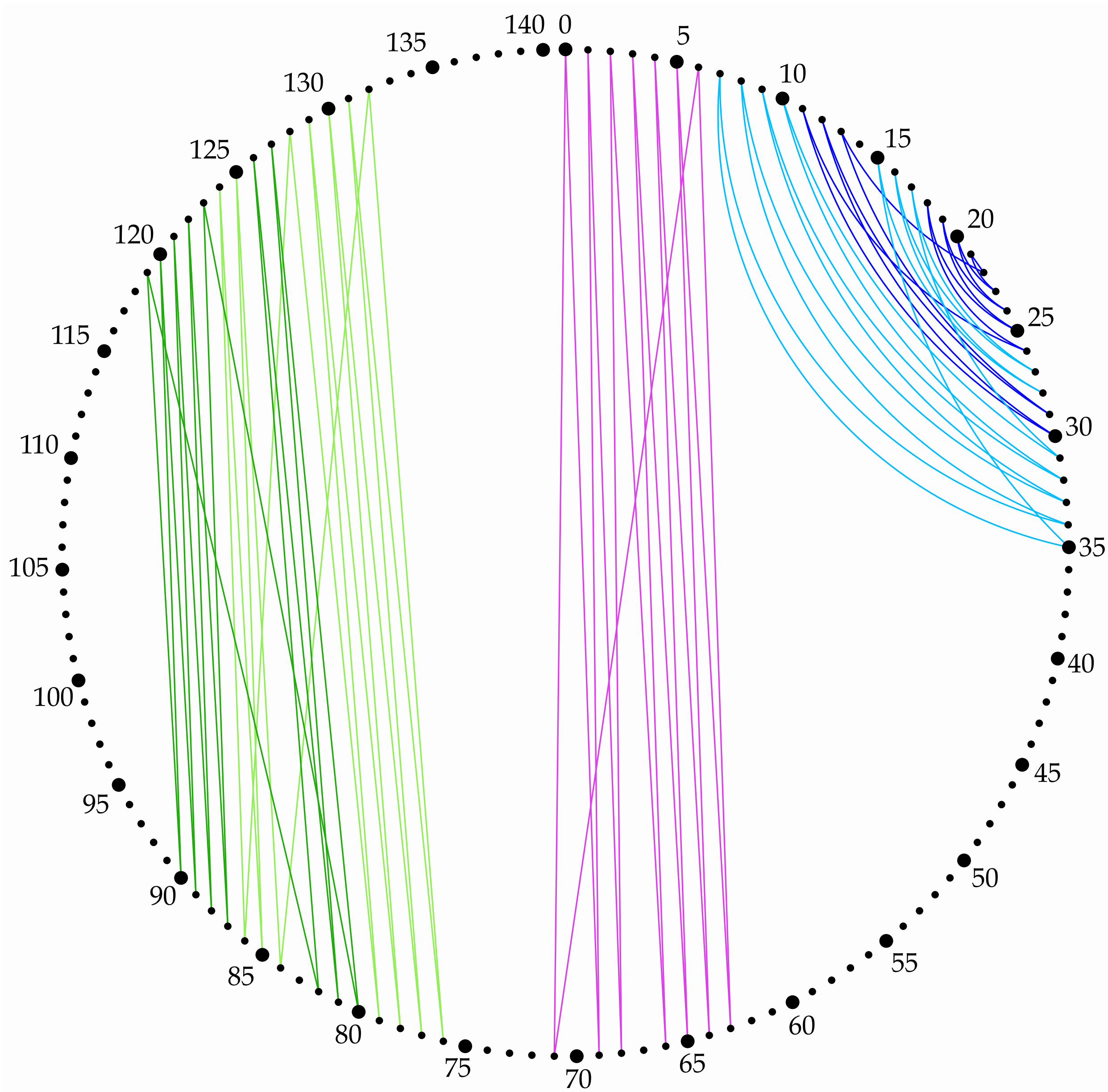



Figure 5.7: A Blinco–El-Zanati $\rho^+$-labeling of $\mathcal{C}_{14}^5$

Table 5.6: Cycle list for the $\rho^+$-labeling of $\mathcal{C}_{14}^5$ in Figure 5.7

**Graph** $G_1 = C_{14}$**: Cycle** $\mathfrak{C}_1$ (lilac)

$\mathfrak{C}_1 = \big(0, 69, 1, 68, 2, 66, 3, 65, 4, 64, 5, 63, 6, 71\big)$

69 [68] 67 [66] 64 [63] 62 [61] 60 [59] 58 [57] 65 70

**Graph** $G_2 = C_{14} \uplus C_{14}$**: Cycles** $\mathfrak{C}_2$ **and** $\mathfrak{C}_3$ (lime and jade)

$\mathfrak{C}_2 = \big(76, 132, 84, 125, 85, 124, 86, 128, 79, 129, 78, 130, 77, 131\big)$

56 [48] 41 [40] 39 [38] 42 [49] 50 [51] 52 [53] 54 [55]

$\mathfrak{C}_3 = \big(90, 119, 82, 126, 81, 127, 80, 123, 87, 122, 88, 121, 89, 120\big)$

29 [37] 44 [45] 46 [47] 43 [36] 35 [34] 33 [32] 31 [30]

**Graph** $G_3 = C_{14} \uplus C_{14}$**: Cycles** $\mathfrak{C}_4$ **and** $\mathfrak{C}_5$ (sky and cobalt)

$\mathfrak{C}_4 = \big(7, 35, 15, 28, 16, 27, 17, 31, 10, 32, 9, 33, 8, 34\big)$

28 [20] 13 [12] 11 [10] 14 [21] 22 [23] 24 [25] 26 [27]

$\mathfrak{C}_5 = \big(21, 22, 13, 29, 12, 30, 11, 26, 18, 25, 19, 24, 20, 23\big)$

1 [9] 16 [17] 18 [19] 15 [8] 7 [6] 5 [4] 3 [2]

## 5.2 Complete Designs of Order $4kp+1$ for Even $k$

In this and the two subsequent sections, we present our own constructions of base blocks for purely cyclic $\mathcal{C}_{2k}^{p}$-designs on the graphs $K_{4kp+1}$. These base blocks do, of course, induce $\rho$-labelings of $\mathcal{C}_{2k}^{p}$, since the existence of such a base block is logically equivalent to the existence of a $\rho$-labeling of $\mathcal{C}_{2k}^{p}$. We split our discussion into the same three cases we used to describe the constructions by Blinco and El-Zanati: (1) $k$ is even; (2) $k$ is odd and $p$ is even; and (3) $k$ is odd and $p$ is odd. Each case occupies a separate section; in this section, we address the case that $k$ is even. We conclude each section with comparative analysis of the constructions presented, namely, the construction by Blinco and El-Zanati and our own.

In each section, we describe our construction of a base block for a purely cyclic $\mathcal{C}_{2k}^{p}$-design on $K_{4kp+1}$ in three stages. We begin by partitioning the set of differences in $K_{4kp+1}$; recall that this set of differences is $\mathcal{D}_{4kp+1} = [\![1, 2kp]\!]$. We then choose a pattern for each set of differences in the partition, and realize each pattern as a $C_{2k}$-subgraph of $K_{4kp+1}$. We form the $\mathcal{C}_{2k}^{p}$-base block, $B$, as the disjoint union of these cycles.

### 5.2.1 Our Construction for Even $k$

Since $k$ is even, there is some $m \in \mathbb{P}$ such that $k = 2m$. In order to form the $\mathcal{C}_{2k}^{p}$ base block, we partition the set of differences into $p$ subsets $\mathcal{S}_1, \ldots, \mathcal{S}_p$ of size $2k = 4m$; each subset $\mathcal{S}_r$ is used to form a cycle $\mathfrak{C}_r$, so that we obtain $p$ vertex-disjoint cycles, as required. For each integer $r \in [\![1, p]\!]$, the set $\mathcal{S}_r$ is given by

$$\mathcal{S}_r = \left[\!\!\left[ 2(r-1)k + 1,\ 2rk \right]\!\!\right]. \tag{5.61}$$

In order to form the cycles, we use the differences in increasing order in alternating directions, except that we use the difference $(2r-1)k$ last, to close the cycle. Using the differences in this way creates a zig-zag pattern between two sets of consecutive vertices, with a one-vertex skip in the center of one of the two sets. We use this difference pattern for each set $\mathcal{S}_r$ to generate the corresponding cycle $\mathfrak{C}_r$; the base block, $B$, is the disjoint union of these cycles. Further details of cycle formation depend on the parity of the index of the cycle, so we separate our discussion into two cases.

For each odd integer $r \in [\![1, p]\!]$, we form cycle $\mathfrak{C}_r$ from $\mathcal{S}_r$ as follows. We use the difference pattern $\left\{ d_{2i-1} \;\; [d_{2i}] \right\}_{i=1}^{k}$, where the differences are given by

$$d_j = \begin{cases} 2k(r-1) + j\,, & \text{if } 1 \le j \le k-1\,, \\ 2k(r-1) + j + 1\,, & \text{if } k \le j \le 2k-1\,, \\ 2k(r-1) + k\,, & \text{if } j = 2k\,. \end{cases} \tag{5.62}$$

For each integer $j \in [\![1, 2k]\!]$, we denote the $j$th term of cycle $\mathfrak{C}_r$ by $x_j$.

If $j = 2i$ is even, then

$$x_j = x_{2i} = m(r-1) + i\,. \tag{5.63}$$

If $j = 2i - 1$ is odd, then

$$x_j = x_{2i-1} = \begin{cases} m\big(8p - 3r + 3\big) - i + 2\,, & \text{if } 1 \le i \le m\,, \\ m\big(8p - 3r + 3\big) - i + 1\,, & \text{if } m+1 \le i \le k\,. \end{cases} \tag{5.64}$$

Figure 5.8 shows a schematic diagram of the cycle $\mathfrak{C}_r$ we have just defined (for odd $r$). In this diagram, odd-indexed vertices appear on the left, and even-indexed vertices appear on the right. In order to simplify the labels, we use $a = 8p - 3r + 3$ in the diagram. We note that, in the case $r = 1$, we have $a = 8p$, so the top vertex on the left is vertex 0, and the top vertex on the right is vertex 1.

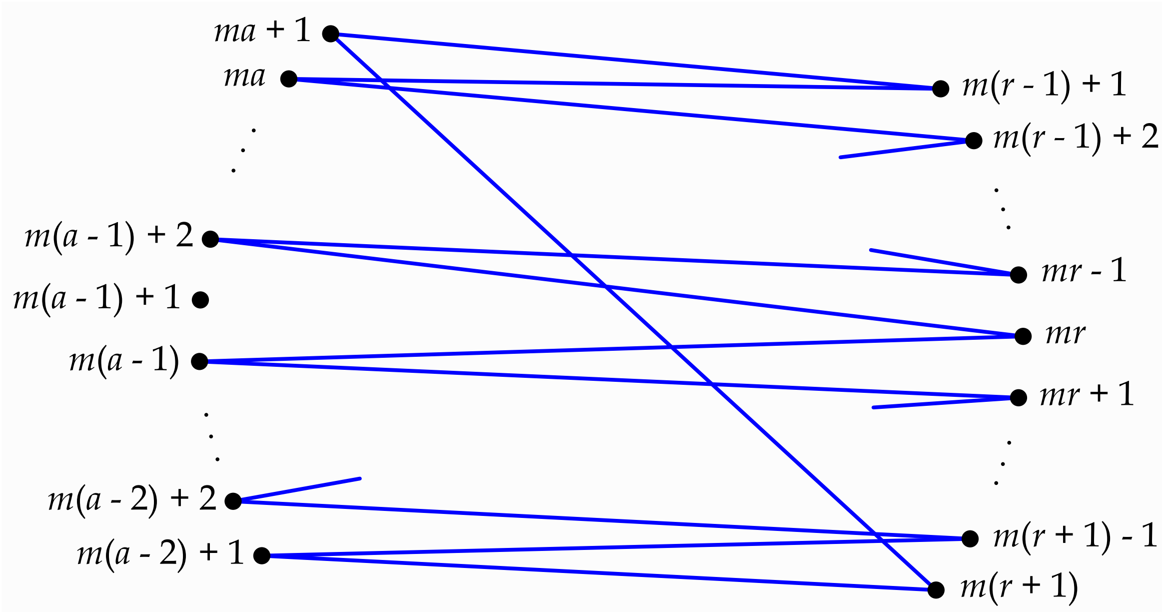



Figure 5.8: Schematic diagram of cycle $\mathfrak{C}_r$, for $r$ odd

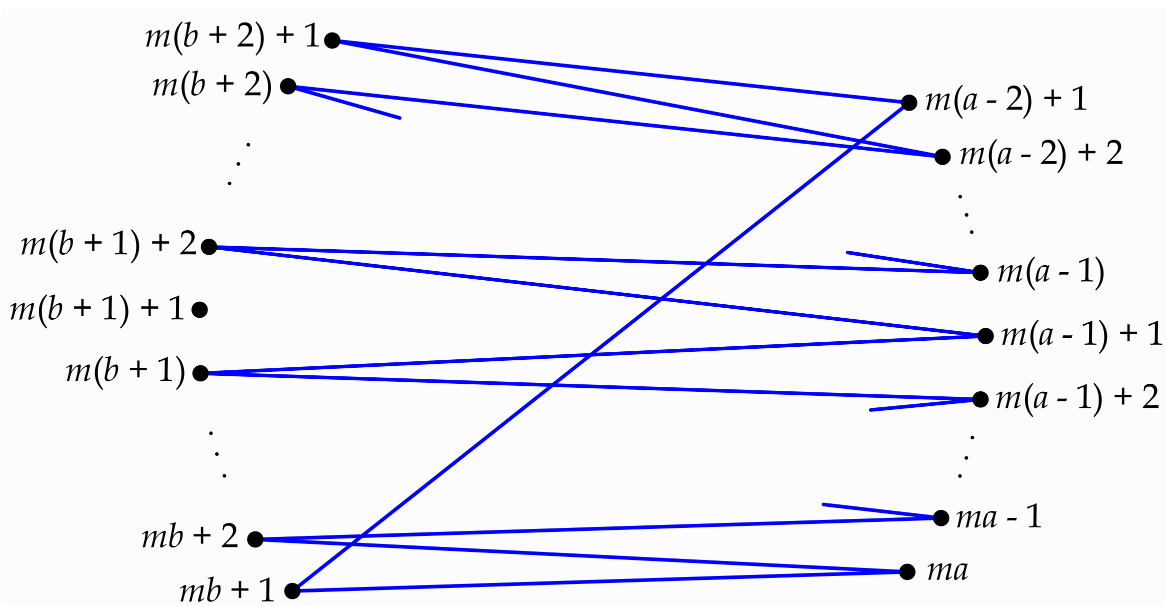



Figure 5.9: Schematic diagram of cycle $\mathfrak{C}_r$, for $r$ even

For each even integer $r \in [\![1, p]\!]$, we form cycle $\mathfrak{C}_r$ from $\mathcal{S}_r$ as follows. We use the difference pattern $\left\{ [d_{2i-1}] \;\; d_{2i} \right\}_{i=1}^{k}$, where the differences are given by

$$d_j = \begin{cases} 2k(r-1) + j\,, & \text{if } 1 \le j \le k-1 \\ 2k(r-1) + j + 1\,, & \text{if } k \le j \le 2k-1 \\ 2k(r-1) + k\,, & \text{if } j = 2k \end{cases} . \tag{5.65}$$

For each integer $j \in [\![1, 2k]\!]$, we denote the $j$th term of cycle $\mathfrak{C}_r$ by $x_j$.
If $j = 2i$ is even, then

$$x_j = x_{2i} = m(2p - r + 2) - i + 1\,. \tag{5.66}$$

If $j = 2i - 1$ is odd, then

$$x_j = x_{2i-1} = \begin{cases} m\big(2p + 3r - 2\big) + i\,, & \text{if } 1 \le i \le m\,, \\ m\big(2p + 3r - 2\big) + i + 1\,, & \text{if } m + 1 \le i \le k\,. \end{cases} \tag{5.67}$$

Figure 5.9 shows a schematic diagram of the cycle $\mathfrak{C}_r$ we have just defined (for even $r$). In this diagram, odd-indexed vertices appear on the left, and even-indexed vertices appear on the right. In order to simplify the labels, we use $a = 2p - r + 2$ and $b = 2p + 3r - 2$ in the diagram.

We form the base block $B$ by defining $B = \biguplus_{r=1}^{p} \mathfrak{C}_r$. This completes the construction.

**Theorem 5.13.** *The subgraph $B$ of $K_{4kp+1}$ generated by the above construction is a base block for a purely cyclic $\mathcal{C}_{2k}^{p}$-design on $K_{4kp+1}$, and hence exhibits a $\rho$-labeling of $\mathcal{C}_{2k}^{p}$. There is therefore a $\mathcal{C}_{2k}^{p}$-design on $K_{4kp+1}$ for each pair of integers $p$ and $k$ such that $p \ge 2$, $k \ge 2$, and $k$ is even.*

*Proof.* It is clear from the construction that each difference in $\mathcal{D}_{4kp+1}$ occurs on exactly one edge in the subgraph $B$, and that each cycle in $B$ has length $2k$. It remains to verify that the cycles $\mathfrak{C}_1, \ldots, \mathfrak{C}_p$ in $B$ are pairwise vertex-disjoint.

For $r \in [\![1, p]\!]$, define

$$\alpha(r) = \begin{cases} \frac{1}{2}(r+1), & \text{if } r \text{ is odd,} \\ p + 1 - \frac{r}{2}, & \text{if } r \text{ is even,} \end{cases} \tag{5.68}$$

and

$$\beta(r) = \begin{cases} 4p - \frac{3}{2}(r-1), & \text{if } r \text{ is odd,} \\ p + \frac{3}{2}r, & \text{if } r \text{ is even.} \end{cases} \tag{5.69}$$

Then, for $r \in [\![1, p]\!]$, cycle $\mathfrak{C}_r$ alternates between vertices in the sets

$$U_r = \left\{ u \in \mathbb{Z} \;\middle|\; \big(\alpha(r) - 1\big)k + 1 \le u \le \alpha(r)k \right\} \tag{5.70}$$

and

$$V_r = \left\{ v \in \mathbb{Z} \;\middle|\; \begin{array}{c} \big(\beta(r) - 1\big)k + 1 \le v \le \beta(r)k + 1, \\ v \ne \left(\beta(r) - \frac{1}{2}\right)k + 1 \end{array} \right\}. \tag{5.71}$$

We observe that, if $r$ is odd, then

$$1 \;\le\; \alpha(r) \;\le\; \left\lceil \frac{p}{2} \right\rceil \tag{5.72}$$

and

$$\frac{5}{2}p + \frac{3}{2} + \frac{3}{2}\left( \left\lceil \frac{p-1}{2} \right\rceil - \left\lfloor \frac{p-1}{2} \right\rfloor \right) \;\le\; \beta(r) \;\le\; 4p\,. \tag{5.73}$$

If $r$ is even, then

$$\left\lceil \frac{p}{2} \right\rceil + 1 \;\le\; \alpha(r) \;\le\; p\,, \tag{5.74}$$

and

$$p+3 \;\leq\; \beta(r) \;\leq\; \frac{5}{2}p - \frac{3}{2}\left(\left\lceil \frac{p}{2} \right\rceil - \left\lfloor \frac{p}{2} \right\rfloor\right). \tag{5.75}$$

Each set $U_r$ is a set of $k$ consecutive integers, which contains a unique multiple of $k$, namely $\alpha(r)k$. Each set $V_r$ is a set of $k$ integers among $k+1$ consecutive integers, and each set $V_r$ contains a unique multiple of $k$, namely $\beta(r)k$. In order to show that two sets $U_r$ and $U_{r^*}$ are disjoint, it suffices to show that $\alpha(r) \neq \alpha(r^*)$. In order to show that two sets $V_r$ and $V_{r^*}$ are disjoint, it suffices to show that $\left|\beta(r) - \beta(r^*)\right| > 1$. In order to show that two sets $U_r$ and $V_{r^*}$ are disjoint, it suffices to show that $\left|\alpha(r) - \beta(r^*)\right| > 1$.

CLAIM 1: If $r, r^* \in [\![1, p]\!]$ such that $r \neq r^*$, then $U_r$ and $U_{r^*}$ are disjoint.

PROOF OF CLAIM 1: Suppose that $r, r^* \in [\![1, p]\!]$ such that $r \neq r^*$. If $r$ and $r^*$ have the same parity, then $\alpha(r) \neq \alpha(r^*)$ by formula (5.68), since $r \neq r^*$. If $r$ and $r^*$ have different parity, then the inequalities (5.72) and (5.74) guarantee that $\alpha(r) \neq \alpha(r^*)$. Hence $U_r$ and $U_{r^*}$ are disjoint. ◇

CLAIM 2: If $r, r^* \in [\![1, p]\!]$ such that $r \neq r^*$, then $V_r$ and $V_{r^*}$ are disjoint.

PROOF OF CLAIM 2: Suppose that $r, r^* \in [\![1, p]\!]$ such that $r \neq r^*$. If $r$ and $r^*$ have the same parity, then $\left|\beta(r) - \beta(r^*)\right| \geq 3$ by formula (5.69), since $r \neq r^*$. If $r$ and $r^*$ have different parity, then suppose, without loss of generality, that $r$ is odd and $r^*$ is even; then, by inequalities (5.73) and (5.75), we have that

$$\beta(r^*) \leq \frac{5}{2}p - \frac{3}{2}\left(\left\lceil \frac{p}{2} \right\rceil - \left\lfloor \frac{p}{2} \right\rfloor\right)$$

and

$$\frac{5}{2}p + \frac{3}{2} + \frac{3}{2}\left(\left\lceil \frac{p-1}{2} \right\rceil - \left\lfloor \frac{p-1}{2} \right\rfloor\right) \leq \beta(r).$$

Since

$$\frac{5}{2}p + \frac{3}{2} + \frac{3}{2}\left(\left\lceil \frac{p-1}{2} \right\rceil - \left\lfloor \frac{p-1}{2} \right\rfloor\right) - \left(\frac{5}{2}p - \frac{3}{2}\left(\left\lceil \frac{p}{2} \right\rceil - \left\lfloor \frac{p}{2} \right\rfloor\right)\right) = 3\,,$$

we have that $\left|\beta(r) - \beta(r^*)\right| \geq 3$. Hence $V_r$ and $V_{r^*}$ are disjoint. ◇

CLAIM 3: If $r, r^* \in [\![1, p]\!]$ such that $r \neq r^*$, then $U_r$ and $V_{r^*}$ are disjoint, and $U_{r^*}$ and $V_r$ are disjoint.

PROOF OF CLAIM 3: Suppose that $r, r^* \in [\![1, p]\!]$ such that $r \neq r^*$. The inequalities (5.72) through (5.75) guarantee that $\alpha(r), \alpha(r^*) \leq p$ and $\beta(r), \beta(r^*) \geq p + 3$, so $\left|\alpha(r) - \beta(r^*)\right| \geq 3$ and $\left|\alpha(r^*) - \beta(r)\right| \geq 3$; hence $U_r$ and $V_{r^*}$ are disjoint, and $U_{r^*}$ and $V_r$ are disjoint. ◇

We have shown that, for $r, r^* \in [\![1, p]\!]$ such that $r \neq r^*$, the sets $U_r$, $U_{r^*}$, $V_r$, and $V_{r^*}$ are pairwise disjoint; therefore cycles $\mathfrak{C}_r$ and $\mathfrak{C}_{r^*}$ are vertex-disjoint, as desired. □

We give examples of our construction that parallel the examples given in subsection 5.1.1 in order to facilitate comparisons between the two constructions. We make such comparisons after the examples.

**Example 5.14.** We consider $\mathcal{C}_{12}^{4}$; for this graph, we have $p = 4$ and $k = 6$, so $4kp + 1 = 97$. The cycles in the base block and the difference patterns that generate them are listed in Table 5.7; the base block itself is shown in Figure 5.10. ■

**Example 5.15.** We consider $\mathcal{C}_{16}^{5}$; for this graph, we have $p = 5$ and $k = 8$, so $4kp + 1 = 161$. The cycles in the base block and the difference patterns that generate them are listed in Table 5.8; the base block itself is shown in Figure 5.11. ■

Table 5.7: Cycle list for the $\mathcal{C}^4_{12}$ base block in Figure 5.10

| **Cycle $\mathfrak{C}_1$** (cobalt) | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | [2] | 3 | [4] | 5 | [7] | 8 | [9] | 10 | [11] | 12 | [6] |
| $\mathfrak{C}_1 = \big(0, 1, 96, 2, 95, 3, 93, 4, 92, 5, 91, 6\big)$ | | | | | | | | | | | |
| **Cycle $\mathfrak{C}_2$** (red) | | | | | | | | | | | |
| [13] | 14 | [15] | 16 | [17] | 19 | [20] | 21 | [22] | 23 | [24] | 18 |
| $\mathfrak{C}_2 = \big(37, 24, 38, 23, 39, 22, 41, 21, 42, 20, 43, 19\big)$ | | | | | | | | | | | |
| **Cycle $\mathfrak{C}_3$** (forest) | | | | | | | | | | | |
| 25 | [26] | 27 | [28] | 29 | [31] | 32 | [33] | 34 | [35] | 36 | [30] |
| $\mathfrak{C}_3 = \big(79, 7, 78, 8, 77, 9, 75, 10, 74, 11, 73, 12\big)$ | | | | | | | | | | | |
| **Cycle $\mathfrak{C}_4$** (fuchsia) | | | | | | | | | | | |
| [37] | 38 | [39] | 40 | [41] | 43 | [44] | 45 | [46] | 47 | [48] | 42 |
| $\mathfrak{C}_4 = \big(55, 18, 56, 17, 57, 16, 59, 15, 60, 14, 61, 13\big)$ | | | | | | | | | | | |

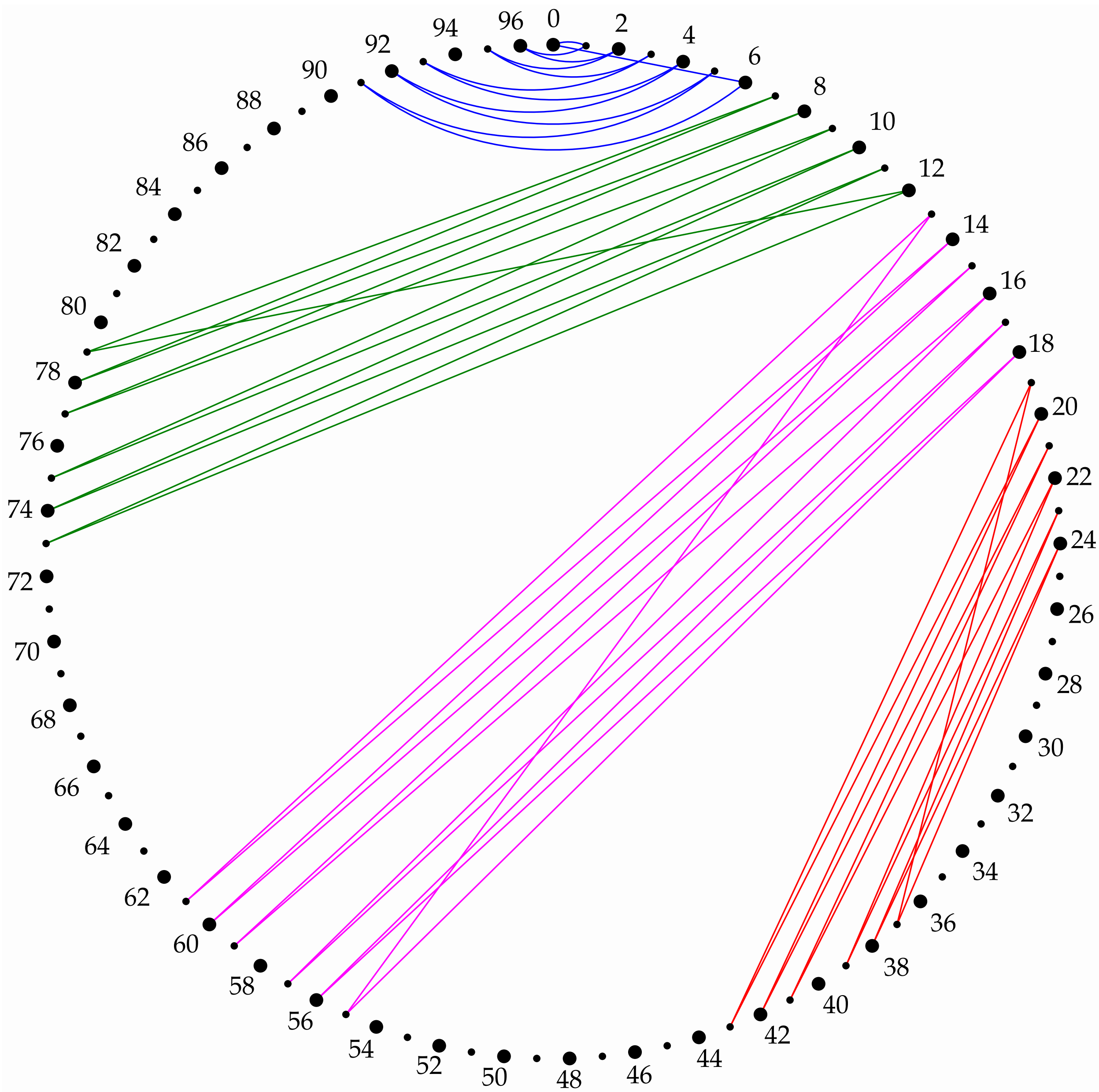



Figure 5.10: The $\mathcal{C}_{12}^4$ base block from Example 5.14

Table 5.8: Cycle list for the $\mathcal{C}_{16}^{5}$ base block in Figure 5.11

| **Cycle** $\mathfrak{C}_1$ (cobalt) |
|---|
| 1 [2] 3 [4] 5 [6] 7 [9] 10 [11] 12 [13] 14 [15] 16 [8] |
| $\mathfrak{C}_1 = \big(0, 1, 160, 2, 159, 3, 158, 4, 156, 5, 155, 6, 154, 7, 153, 8\big)$ |
| **Cycle** $\mathfrak{C}_2$ (red) |
| [17] 18 [19] 20 [21] 22 [23] 25 [26] 27 [28] 29 [30] 31 [32] 24 |
| $\mathfrak{C}_2 = \big(57, 40, 58, 39, 59, 38, 60, 37, 62, 36, 63, 35, 64, 34, 65, 33\big)$ |
| **Cycle** $\mathfrak{C}_3$ (forest) |
| 33 [34] 35 [36] 37 [38] 39 [41] 42 [43] 44 [45] 46 [47] 48 [40] |
| $\mathfrak{C}_3 = \big(137, 9, 136, 10, 135, 11, 134, 12, 132, 13, 131, 14, 130, 15, 129, 16\big)$ |
| **Cycle** $\mathfrak{C}_4$ (fuchsia) |
| [49] 50 [51] 52 [53] 54 [55] 57 [58] 59 [60] 61 [62] 63 [64] 56 |
| $\mathfrak{C}_4 = \big(81, 32, 82, 31, 83, 30, 84, 29, 86, 28, 87, 27, 88, 26, 89, 25\big)$ |
| **Cycle** $\mathfrak{C}_5$ (orange) |
| 65 [66] 67 [68] 69 [70] 71 [73] 74 [75] 76 [77] 78 [79] 80 [72] |
| $\mathfrak{C}_5 = \big(113, 17, 112, 18, 111, 19, 110, 20, 108, 21, 107, 22, 106, 23, 105, 24\big)$ |

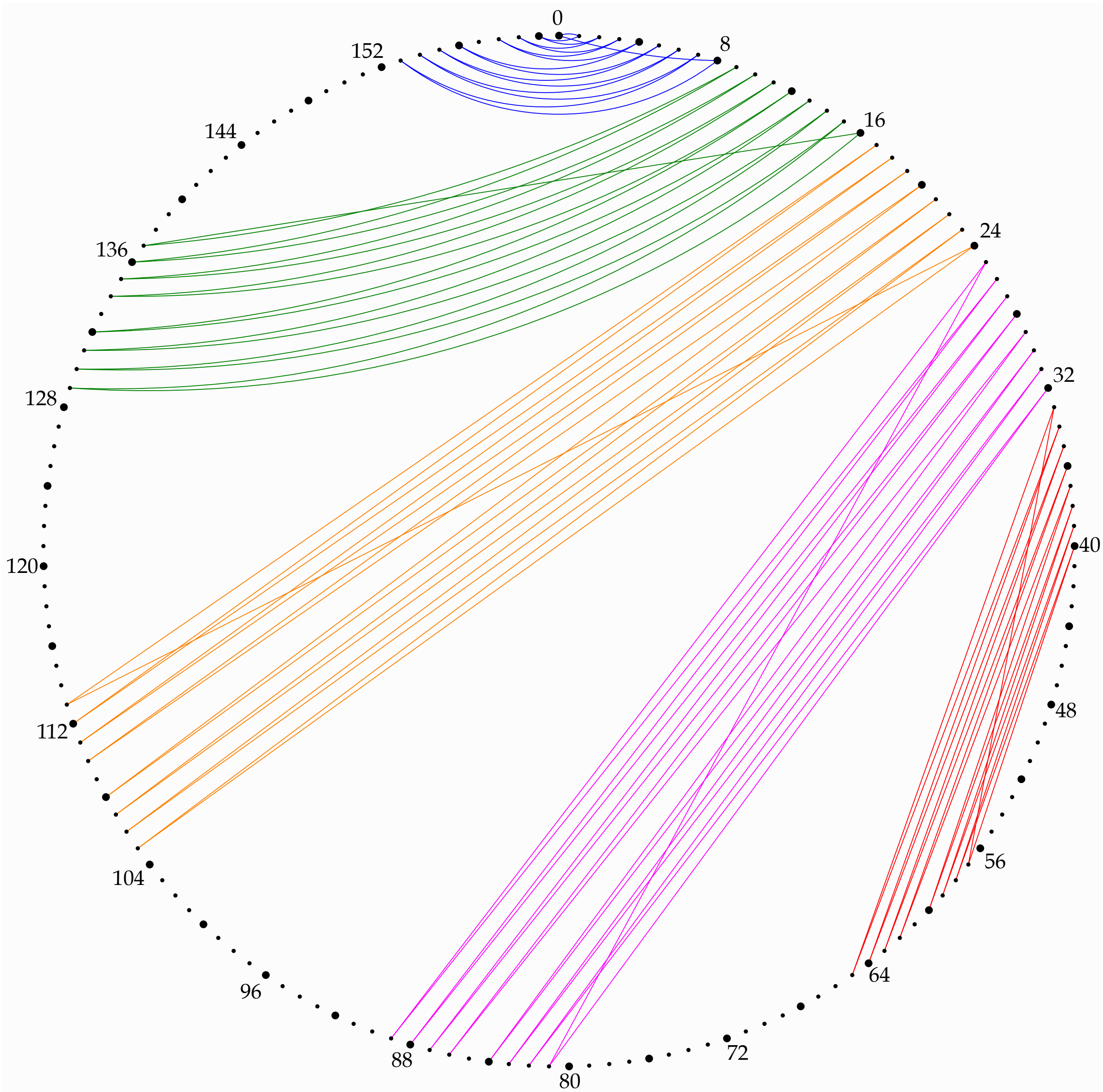



Figure 5.11: The $\mathcal{C}_{16}^{5}$ base block from Example 5.15

### 5.2.2 Comparative Analysis for Even $k$

We observe that our construction of a $\mathcal{C}_{2k}^p$ base block in the case that $k$ is even is highly similar to the construction by Blinco and El-Zanati. We have reproduced (at a reduced size) in Figure 5.12 the images of the base blocks from Examples 5.7, 5.14, 5.8, and 5.15, to facilitate direct visual comparison.

**Remark 5.16.** We can obtain a $\sigma^{++}$-labeling both from the $\sigma^+$-labeling constructed by Blinco and El-Zanati and from our base block, which induces a $\sigma$-labeling. This is achieved in a simple way: by adding a constant to all vertex labels, which has the effect of rotating the base blocks.

Recall that, in the construction by Blinco and El-Zanati, we take $G_i = C_{2k}$ for each $i \in [\![1, p]\!]$, and we take $G_i$ to have vertex set $[\![1, 2k]\!] \times \{i\}$ with bipartition $[A_i, B_i]$, where $A_i = \{(a, i) \in V(G_i) \mid a \text{ is odd}\}$ and $B_i = \{(b, i) \in V(G_i) \mid b \text{ is even}\}$. $G$ is then assumed to have bipartition $[A, B]$, where $A$ and $B$ are obtained from the bipartitions $[A_i, B_i]$ of the graphs $G_i$ by

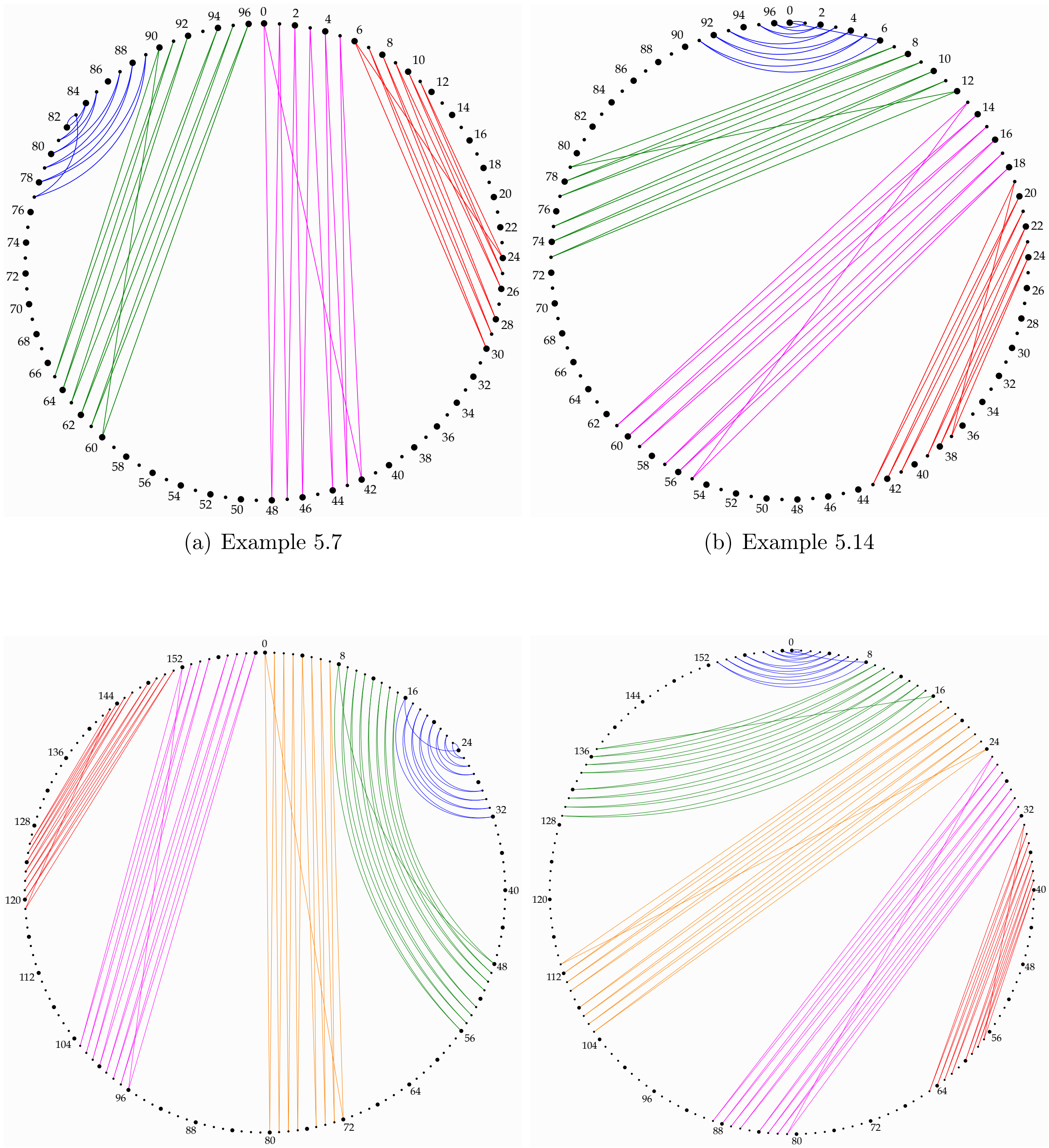
$$A = \bigcup_{i=1}^{p} A_i \quad \text{and} \quad B = \bigcup_{i=1}^{p} B_i \,.$$

We also consider two additional bipartitions of $G$: the bipartition $[B, A]$, with $A$ and $B$ defined as above, and the $[A^*, B^*]$, where

$$A^* = \bigcup_{\substack{i \text{ odd,} \\ 1 \le i \le p}} A_i \;\cup \bigcup_{\substack{i \text{ even,} \\ 1 \le i \le p}} B_i \quad \text{and} \quad B^* = \bigcup_{\substack{i \text{ odd,} \\ 1 \le i \le p}} B_i \;\cup \bigcup_{\substack{i \text{ even,} \\ 1 \le i \le p}} A_i \,.$$

The construction by Blinco and El-Zanati produces a $\sigma^+$-labeling of $G = \mathcal{C}_{2k}^p$ on bipartition $[A, B]$. If we use instead bipartition $[A^*, B^*]$ and add $(k+1) \cdot \lfloor p/2 \rfloor$ to each vertex label, with computations done modulo $(4kp+1)$, we obtain a $\sigma^{++}$-labeling of $\mathcal{C}_{2k}^p$ with critical value $\lambda = kp - 1 + \lfloor p/2 \rfloor$.

Our base block, as given, induces a $\sigma$-labeling of $G = \mathcal{C}_{2k}^p$ that is not a $\sigma^+$-labeling for any of the three bipartitions we have defined. If we take $G$ to have bipartition $[B, A]$, then



(a) Example 5.7

(b) Example 5.14

(c) Example 5.8

(d) Example 5.15

Figure 5.12: Small reproductions of $\mathcal{C}_{12}^4$ and $\mathcal{C}_{16}^5$ base blocks from Examples 5.7, 5.14, 5.8, and 5.15

by subtracting 1 from each vertex label, again computing modulo $(4kp+1)$, we obtain a $\sigma^{++}$-labeling of $\mathcal{C}_{2k}^{p}$ having critical value $\lambda = kp - 1$. ■

For $\mathcal{C}_{12}^{4}$, we see that the red and fuchsia cycles are configured identically in the two base blocks, so that an appropriate rotation of one base block aligns these cycles with their counterparts in the other base block; this is not the case with the cobalt and forest cycles. We can transform one block into another visually by first reflecting the half of $K_{97}$ containing the cobalt and forest cycles over an appropriate line, then rotating the cobalt and forest cycles into correct positions, and then rotating the entire block. We can accomplish these changes within the construction by Blinco and El-Zanati by trading the $\alpha$-labeling $L_3$ of $C_{2k}$ for the labeling $L_1$ on the forest and cobalt cycles and inverting the order in which the four cycles occur. The inversion of the cycle order is required because the cycle order is fuchsia, forest, red, cobalt, in the base block by Blinco and El-Zanati, while the order is cobalt, red, forest, fuchsia, in our base block. We describe how to achieve this in general for the case in which $k$ and $p$ are both even, thereby producing the labeling induced by our base block from the construction by Blinco and El-Zanati.

**Remark 5.17.** Let $p$ and $k$ be positive even integers, each at least two. Let $m, q \in \mathbb{P}$ such that $k = 2m$ and $p = 2q$. Let $\psi$ denote the $\rho$-labeling of $\mathcal{C}_{2k}^{p} = \mathcal{C}_{4m}^{2q}$ induced by the base block described in Theorem 5.13. Let $h$ be as given in equation (5.6) in the construction for Theorem 5.4, and apply this construction to $G = \mathcal{C}_{2k}^{p}$ with $G_i = C_{2k}$ for all $i \in [\![1, p]\!]$, setting $h_i = L_3$ for all odd $i \in [\![1, p]\!]$ and $h_i = L_1$ for all even $i \in [\![1, p]\!]$. Then

$$\psi(u, j) = \begin{cases} h(u, p-j) + mq + 1\,, & \text{if } j \text{ is odd,} \\ h(2k+1-u, p-j) + mq + 1\,, & \text{if } j \text{ is even.} \end{cases} \tag{5.76}$$

for all $(u, j) \in V\big(\mathcal{C}_{2k}^{p}\big)$, with computations done modulo $(4kp+1)$. ■

For $\mathcal{C}_{16}^{5}$, we see that, in order to transform one block into another, we must reflect the entire half of $K_{161}$ containing the red and fuchsia cycles over an appropriate line, rotate the red and fuchsia cycles into correct positions, and then rotate the entire block. We can accomplish these changes within the construction by Blinco and El-Zanati by trading the $\alpha$-labeling $L_3$ of $C_{2k}$ for the labeling $L_1$ on the red and fuchsia cycles and inverting the order in which the five cycles occur. The inversion of the cycle order is required because the cycle order is orange, fuchsia, forest, red, cobalt in the base block by Blinco and El-Zanati, while the order is cobalt, red, forest, fuchsia, orange, in our base block. We describe how to achieve this in general for the case in which $k$ is even and $p$ is odd, thereby producing the labeling induced by our base block from the construction by Blinco and El-Zanati.

**Remark 5.18.** Let $p$ and $k$ be positive integers, each at least two, such that $k$ is even and $p$ is odd. Let $m, q \in \mathbb{P}$ such that $k = 2m$ and $p = 2q + 1$. Let $\psi$ denote the $\rho^+$-labeling of $\mathcal{C}_{2k}^p = \mathcal{C}_{4m}^{2q}$ induced by the base block described in Theorem 5.13. Let $h$ be as given in equation (5.6) in the construction for Theorem 5.4, and apply this construction to $G = \mathcal{C}_{2k}^p$ with $G_i = C_{2k}$ for all $i \in [\![1, p]\!]$, setting $h_i = L_3$ for all odd $i \in [\![1, p]\!]$ and $h_i = L_1$ for all even $i \in [\![1, p]\!]$. Then

$$\psi(u, j) = \begin{cases} 2m(q+1) - h(2k+1-u, p-j)\,, & \text{if } j \text{ is odd,} \\ 2m(q+1) - h(u, p-j)\,, & \text{if } j \text{ is even.} \end{cases} \tag{5.77}$$

for all $(u, j) \in V\big(\mathcal{C}_{2k}^p\big)$, with computations done modulo $(4kp + 1)$. ■

We conclude that the construction by Blinco and El-Zanati and our construction are essentially the same in the case that $k$ is even, and that both constructions would benefit from the small modifications that are necessary to produce the $\sigma^{++}$-labeling directly.

## 5.3 Complete Designs of Order $4kp + 1$ for Odd $k$ and Even $p$

In this section, we present base block constructions for $\mathcal{C}_{2k}^p$ in the case that $k$ is odd and $p$ is even. There are two variations on the constructions presented in this section. The first variation is limited in two significant ways: first, we achieve $\mathcal{C}_{2k}^p$ base blocks only for

$p \in \{2, 4, 6, 8\}$; second, the general description of the construction applies only to values of $k$ such that $k > p+4$, and the smaller cases must be adapted individually. We altered these constructions in a slight but significant way to produce the second variation, which gives a single construction for the entire case that $k$ is odd and $p$ is even.

### 5.3.1 Our Construction for Odd $k$ and Even $p$, Variation I

Since $p$ is even, there is some $q \in \mathbb{P}$ such that $p = 2q$. In order to form the $\mathcal{C}_{2k}^p$ base block, we partition the set of differences into $p$ subsets $\mathcal{S}_1, \ldots, \mathcal{S}_p$ of size $2k$; each subset $\mathcal{S}_r$ is used to form a cycle $\mathfrak{C}_r$, so that we obtain $p$ vertex-disjoint cycles, as required. For small values of $k$, namely those such that $k < p + 4$, the general construction must be adapted slightly; we describe the general construction for $k > p + 4$ first. We emphasize that this construction applies only to $p \in \{2, 4, 6, 8\}$.

We first reserve the differences in the set $D = [\![1,\, p]\!] \cup [\![k,\, k+p-1]\!]$ for use in closing the cycles. We then assign the remaining differences, in consecutive pairs, to the sets in the partition in a descending, alternating pattern that assigns larger differences to sets with smaller indices, so that the pairs $\{k-2,\, k-1\}$ and $\{k+p,\, k+p+1\}$ are assigned to different sets. The sets $\mathcal{S}_{p-1}$ and $\mathcal{S}_p$ and their corresponding cycles are special in this construction; if $p \geq 4$, then there are at least two other sets; we define these other sets first. Recall that $p = 2q$. For all $z \in [\![1, q-1]\!]$, we define sets $\mathcal{S}_{2z-1}$ and $\mathcal{S}_{2z}$ by

$$\begin{aligned} \mathcal{S}_{2z-1} = \Big\{2z-1,\, k+2z-2\Big\} &\cup \Big\{4(q+1-z)k + 4z - 4i \,\Big|\, i \in [\![1, k-1]\!]\Big\} \\ &\cup \Big\{4(q+1-z)k + 4z - 1 - 4i \,\Big|\, i \in [\![1, k-1]\!]\Big\} \end{aligned} \tag{5.78}$$

and

$$\begin{aligned} \mathcal{S}_{2z} = \Big\{2z,\, k+2z-1\Big\} &\cup \Big\{4(q+1-z)k + 4z - 2 - 4i \,\Big|\, i \in [\![1, k-1]\!]\Big\} \\ &\cup \Big\{4(q+1-z)k + 4z - 3 - 4i \,\Big|\, i \in [\![1, k-1]\!]\Big\}. \end{aligned} \tag{5.79}$$

For all $z \in [\![1, q-1]\!]$, we use the differences in set $\mathcal{S}_{2z-1}$ in the pattern $\Big\{ d_{2i-1} \;\; [d_{2i}] \Big\}_{i=1}^{k}$, where

$$d_{2i-1} = \begin{cases} 4(q+1-z)k + 4z - 4i\,, & \text{if } 1 \leq i \leq k-1\,, \\ 2z-1\,, & \text{if } i = k\,, \end{cases} \tag{5.80}$$

and

$$d_{2i} = \begin{cases} 4(q+1-z)k + 4z - 1 - 4i\,, & \text{if } 1 \leq i \leq k-1\,, \\ k + 2z - 2\,, & \text{if } i = k\,. \end{cases} \tag{5.81}$$

We form cycle $\mathfrak{C}_{2z-1}$ from set $\mathcal{S}_{2z-1}$ as follows. For each integer $j \in [\![1, 2k]\!]$, we denote the $j$th term of cycle $\mathfrak{C}_{2z-1}$ by $x_j$.

If $j = 2i-1$ is odd, then

$$x_j = x_{2i-1} = (z-1)(k+z) - z + i\,. \tag{5.82}$$

If $j = 2i$ is even, then

$$x_j = x_{2i} = \begin{cases} 2kp - 3k(z-1) + z(z+2) - 3i\,, & \text{if } 1 \leq i \leq k-1\,, \\ z(k+z) - 1\,, & \text{if } i = k\,. \end{cases} \tag{5.83}$$

For all $z \in [\![1, q-1]\!]$, we use the differences in set $\mathcal{S}_{2z}$ in the pattern $\Big\{ [d_{2i-1}] \;\; d_{2i} \Big\}_{i=1}^{k}$, where

$$d_{2i-1} = \begin{cases} 4(q+1-z)k + 4z - 2 - 4i\,, & \text{if } 1 \leq i \leq k-1\,, \\ 2z\,, & \text{if } i = k\,, \end{cases} \tag{5.84}$$

and

$$d_{2i} = \begin{cases} 4(q+1-z)k + 4z - 3 - 4i\,, & \text{if } 1 \le i \le k-1\,, \\ k + 2z - 1\,, & \text{if } i = k\,. \end{cases} \tag{5.85}$$

We form cycle $\mathfrak{C}_{2z}$ from set $\mathcal{S}_{2z}$ as follows. For each integer $j \in [\![1, 2k]\!]$, we denote the $j$th term of cycle $\mathfrak{C}_{2z}$ by $x_j$.

If $j = 2i-1$ is odd, then

$$x_j = x_{2i-1} = 4kp - k(z-1) - \tfrac{3}{2}z(z-1) - i + 1\,. \tag{5.86}$$

If $j = 2i$ is even, then

$$x_j = x_{2i} = \begin{cases} 2kp + 3k(z-1) - \tfrac{1}{2}z(3z+5) + 3(i+1)\,, & \text{if } 1 \le i \le k-1\,, \\ 4kp - zk - \tfrac{1}{2}z(3z+1) + 1\,, & \text{if } i = k\,. \end{cases} \tag{5.87}$$

The last two sets of differences, $\mathcal{S}_{p-1}$ and $\mathcal{S}_p$, and their corresponding difference patterns and cycles differ slightly in two cases, according to whether $k \equiv 3 \pmod 4$ or $k \equiv 1 \pmod 4$. We address the subtle differences in these two cases by defining two new parameters, $\eta$ and $\theta$, for use in the formulas. We define

$$\eta(k) = \begin{cases} 3a + \left\lceil \dfrac{p+1}{4} \right\rceil, & \text{if } k \equiv 3 \pmod 4\,, \\ 3b + \left\lfloor \dfrac{p-1}{4} \right\rfloor, & \text{if } k \equiv 1 \pmod 4\,, \end{cases} \tag{5.88}$$

and

$$\theta(k) = \begin{cases} 3a + \left\lceil \dfrac{p+3}{4} \right\rceil, & \text{if } k \equiv 3 \pmod 4\,, \\ 3b + \left\lfloor \dfrac{p+1}{4} \right\rfloor, & \text{if } k \equiv 1 \pmod 4\,. \end{cases} \tag{5.89}$$

We now define the sets $\mathcal{S}_{p-1}$ and $\mathcal{S}_p$, their difference patterns, and the cycles $\mathfrak{C}_{p-1}$ and $\mathfrak{C}_p$ for all odd values of $k$ such that $k > p+4$. We define sets $\mathcal{S}_{p-1}$ and $\mathcal{S}_p$ by

$$\mathcal{S}_{p-1} = \Big\{p-1,\ k+p-2\Big\} \cup \left\{\ 4k+2p-4i,\ 4k+2p-1-4i \ \middle|\ i \in [\![1,\ \eta(k)+1]\!]\right\} \cup \left\{\ 4k+p-4i,\ 4k+p-1-4i \ \middle|\ i \in [\![\eta(k)+2,\ k-1]\!]\right\} \tag{5.90}$$

and

$$\mathcal{S}_{p} = \Big\{p,\ k+p-1\Big\} \cup \left\{\ 4k+2p-2-4i,\ 4k+2p-3-4i \ \middle|\ i \in [\![1,\ \eta(k)+1]\!]\right\} \cup \left\{\ 4k+p-2-4i,\ 4k+p-3-4i \ \middle|\ i \in [\![\eta(k)+2,\ k-1]\!]\right\}. \tag{5.91}$$

We use the differences in set $\mathcal{S}_{p-1}$ in the pattern $\left\{\ d_{2i-1}\ \ [d_{2i}]\ \right\}_{i=1}^{k}$, where

$$d_{2i-1} = \begin{cases} 4k+2p-4i\,, & \text{if } 1 \le i \le \eta(k)+1\,, \\ 4k+p-4i\,, & \text{if } \eta(k)+2 \le i \le k-1\,, \\ p-1\,, & \text{if } i = k\,, \end{cases} \tag{5.92}$$

and

$$d_{2i} = \begin{cases} 4k+2p-1-4i\,, & \text{if } 1 \le i \le \eta(k)+1\,, \\ 4k+p-1-4i\,, & \text{if } \eta(k)+2 \le i \le k-1\,, \\ k+p-2\,, & \text{if } i = k\,. \end{cases} \tag{5.93}$$

We form cycle $\mathfrak{C}_{p-1}$ from set $\mathcal{S}_{p-1}$ as follows. For each integer $j \in [\![1, 2k]\!]$, we denote the $j$th term of cycle $\mathfrak{C}_{p-1}$ by $x_j$.

If $j = 2i-1$ is odd, then

$$x_j = x_{2i-1} = (q-1)k + \tfrac{1}{6}(q^3-q) - 1 + i\,. \tag{5.94}$$

If $j = 2i$ is even, then

$$x_j = x_{2i} = \begin{cases} (q+3)k + \frac{1}{6}(q^3 + 23q) - 1 - 3i\,, & \text{if } 1 \le i \le \eta(k) + 1\,, \\ (q+3)k + \frac{1}{6}(q^3 + 11q) - 1 - 3i\,, & \text{if } \eta(k) + 2 \le i \le k - 1\,, \\ qk + \frac{1}{6}(q^3 + 11q) - 2\,, & \text{if } i = k\,. \end{cases} \tag{5.95}$$

We use the differences in set $\mathcal{S}_p$ in the pattern $\Big\{ \, [d_{2i-1}] \;\; d_{2i} \, \Big\}_{i=1}^{k}$, where

$$d_{2i-1} = \begin{cases} 4k + 2p - 2 - 4i\,, & \text{if } 1 \le i \le \theta(k)\,, \\ 4k + p - 2 - 4i\,, & \text{if } \theta(k) + 1 \le i \le k - 1\,, \\ p\,, & \text{if } i = k\,, \end{cases} \tag{5.96}$$

and

$$d_{2i} = \begin{cases} 4k + 2p - 3 - 4i\,, & \text{if } 1 \le i \le \theta(k)\,, \\ 4k + p - 3 - 4i\,, & \text{if } \theta(k) + 1 \le i \le k - 1\,, \\ k + p - 1\,, & \text{if } i = k\,. \end{cases} \tag{5.97}$$

We form cycle $\mathfrak{C}_p$ from set $\mathcal{S}_p$ as follows. For each integer $j \in [\![1, 2k]\!]$, we denote the $j$th term of cycle $\mathfrak{C}_p$ by $x_j$.

If $j = 2i - 1$ is odd, then

$$x_j = x_{2i-1} = (7q+1)k + \tfrac{1}{2}(3q - 3q^2) + 1 - i\,. \tag{5.98}$$

If $j = 2i$ is even, then

$$x_j = x_{2i} = \begin{cases} (7q-3)k - \frac{1}{2}(5q + 3q^2) + 3 + 3i\,, & \text{if } 1 \le i \le \theta(k)\,, \\ (7q-3)k - \frac{1}{2}(q + 3q^2) + 3 + 3i\,, & \text{if } \theta(k) + 1 \le i \le k - 1\,, \\ 7qk - \frac{1}{2}(q + 3q^2) + 1\,, & \text{if } i = k\,. \end{cases} \tag{5.99}$$

We form the base block $B$ by defining $B = \biguplus_{r=1}^{p} \mathfrak{C}_r$. This completes the construction.

**Theorem 5.19.** *The subgraph $B$ of $K_{4kp+1}$ generated by the above construction is a base block for a purely cyclic $\mathcal{C}_{2k}^p$-design on $K_{4kp+1}$, and hence exhibits a $\rho$-labeling of $\mathcal{C}_{2k}^p$. There is therefore a $\mathcal{C}_{2k}^p$-design on $K_{4kp+1}$ for each pair of integers $p$ and $k$ such that $p \in \{2, 4, 6, 8\}$, $k$ is odd, and $k > p + 4$.*

*Proof.* It is clear from the construction that each difference in $\mathcal{D}_{4kp+1}$ occurs on exactly one edge in the subgraph $B$, and that each cycle in $B$ has length $2k$. It remains to verify that the cycles $\mathfrak{C}_1, \ldots, \mathfrak{C}_p$ in $B$ are pairwise vertex-disjoint. We will exhibit, separately for each value of $p$, sets $V_r$ such that, for each $r \in [\![1, p]\!]$, the vertices of $\mathfrak{C}_r$ are all elements of the set $V_r$. For $p = 2$, the sets are $V_1 = [\![0,\, 4k]\!]$ and $V_2 = [\![4k + 2,\, 8k]\!]$.
For $p = 4$, the sets are

$$V_1 = [\![0,\, k]\!] \cup [\![5k + 6,\, 8k]\!]\,,$$

$$V_2 = [\![8k + 2,\, 11k - 7]\!] \cup \{11k - 4\} \cup [\![15k - 1,\, 16k]\!]\,,$$

$$V_3 = [\![k + 1,\, 2k + 3]\!] \cup [\![2k + 7,\, 5k + 5]\!]\,, \text{ and}$$

$$V_4 = \{11k - 5\} \cup [\![11k - 2,\, 15k - 3]\!]\,.$$

For $p = 6$, the sets are

$$V_1 = [\![0,\, k]\!] \cup [\![9k + 6,\, 12k]\!]\,,$$

$$V_2 = [\![12k + 2,\, 15k - 7]\!] \cup \{15k - 4\} \cup [\![23k - 1,\, 24k]\!]\,,$$

$$V_3 = [\![k + 1,\, 2k + 3]\!] \cup \{6k + 11\} \cup [\![6k + 14,\, 9k + 5]\!]\,,$$

$$V_4 = \{15k - 5\} \cup [\![15k - 2,\, 18k - 17]\!] \cup \{18k - 14, 18k - 11\} \cup [\![22k - 6,\, 23k - 3]\!]\,,$$

$$V_5 = [\![2k + 4,\, 6k + 9]\!] \cup \{6k + 12\}\,, \text{ and}$$

$$V_6 = \{18k-15, 18k-12\} \cup [\![18k-9,\ 22k-9]\!] .$$

For $p = 8$, the sets are

$$V_1 = [\![0,\ k]\!] \cup [\![13k+6,\ 16k]\!] ,$$

$$V_2 = [\![16k+2,\ 19k-7]\!] \cup \{19k-4\} \cup [\![31k-1,\ 32k]\!] ,$$

$$V_3 = [\![k+1,\ 2k+3]\!] \cup \{10k+11\} \cup [\![10k+14,\ 13k+5]\!] ,$$

$$V_4 = \{19k-5\} \cup [\![19k-2,\ 22k-17]\!] \cup \{22k-14, 22k-11\} \cup [\![30k-6,\ 31k-3]\!] ,$$

$$V_5 = [\![2k+4,\ 3k+8]\!] \cup \{7k+18, 7k+21\} \cup [\![7k+24,\ 10k+9]\!] \cup \{10k+12\}$$

$$V_6 = \{22k-15, 22k-12\} \cup [\![22k-9,\ 25k-30]\!] \cup \{25k-27, 25k-24, 25k-21\} \cup [\![29k-14,\ 30k-9]\!]$$

$$V_7 = [\![3k+10,\ 7k+16]\!] \cup \{7k+19, 7k+22\} , \text{ and}$$

$$V_8 = \{25k-28, 25k-25, 25k-22\} \cup [\![25k-19,\ 29k-18]\!] .$$

It is clear that the sets $V_r$ are pairwise disjoint; hence the cycles $\mathfrak{C}_r$ are pairwise vertex disjoint, as desired. □

**Example 5.20.** The smallest example for $p = 2$ corresponds to $k = 7$, as $k$ must be greater than 6. With these values of $p$ and $k$, we have $4kp + 1 = 57$. We exhibit the base block for $\mathcal{C}_{14}^2$. The cycles in the base block and the difference patterns that generate them are listed in Table 5.9; the base block itself is shown in Figure 5.13. ■

**Example 5.21.** The smallest example for $p = 4$ corresponds to $k = 9$, as $k$ must be greater than 8. With these values of $p$ and $k$, we have $4kp + 1 = 145$. We exhibit the base block for $\mathcal{C}_{18}^4$. The cycles in the base block and the difference patterns that generate them are listed in Table 5.10; the base block itself is shown in Figure 5.14. ■

**Example 5.22.** The smallest example for $p = 6$ corresponds to $k = 11$, as $k$ must be greater than 10. With these values of $p$ and $k$, we have $4kp+1 = 265$. We exhibit the base block for $\mathcal{C}_{22}^{6}$. The cycles in the base block and the difference patterns that generate them are listed in Table 5.11; the base block itself is shown in Figure 5.15. ■

**Example 5.23.** The smallest example for $p = 8$ corresponds to $k = 13$, as $k$ must be greater than 12. With these values of $p$ and $k$, we have $4kp+1 = 417$. We exhibit the base block for $\mathcal{C}_{26}^{8}$. The cycles in the base block and the difference patterns that generate them are listed in Tables 5.12 and 5.13; the base block itself is shown in Figures 5.16 and 5.17. ■

Table 5.9: Cycle list for the $\mathcal{C}_{14}^{2}$ base block in Figure 5.13

| **Cycle $\mathfrak{C}_1$** | (cobalt) | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 28 | [27] | 24 | [23] | 20 | [19] | 16 | [15] | 12 | [11] | 6 | [5] | 1 | [7] |
| $\mathfrak{C}_1 = (0, 28, 1, 25, 2, 22, 3, 19, 4, 16, 5, 11, 6, 7)$ | | | | | | | | | | | | | |
| **Cycle $\mathfrak{C}_2$** | (red) | | | | | | | | | | | | |
| [26] | 25 | [22] | 21 | [18] | 17 | [14] | 13 | [10] | 9 | [4] | 3 | [2] | 8 |
| $\mathfrak{C}_2 = (56, 30, 55, 33, 54, 36, 53, 39, 52, 42, 51, 47, 50, 48)$ | | | | | | | | | | | | | |

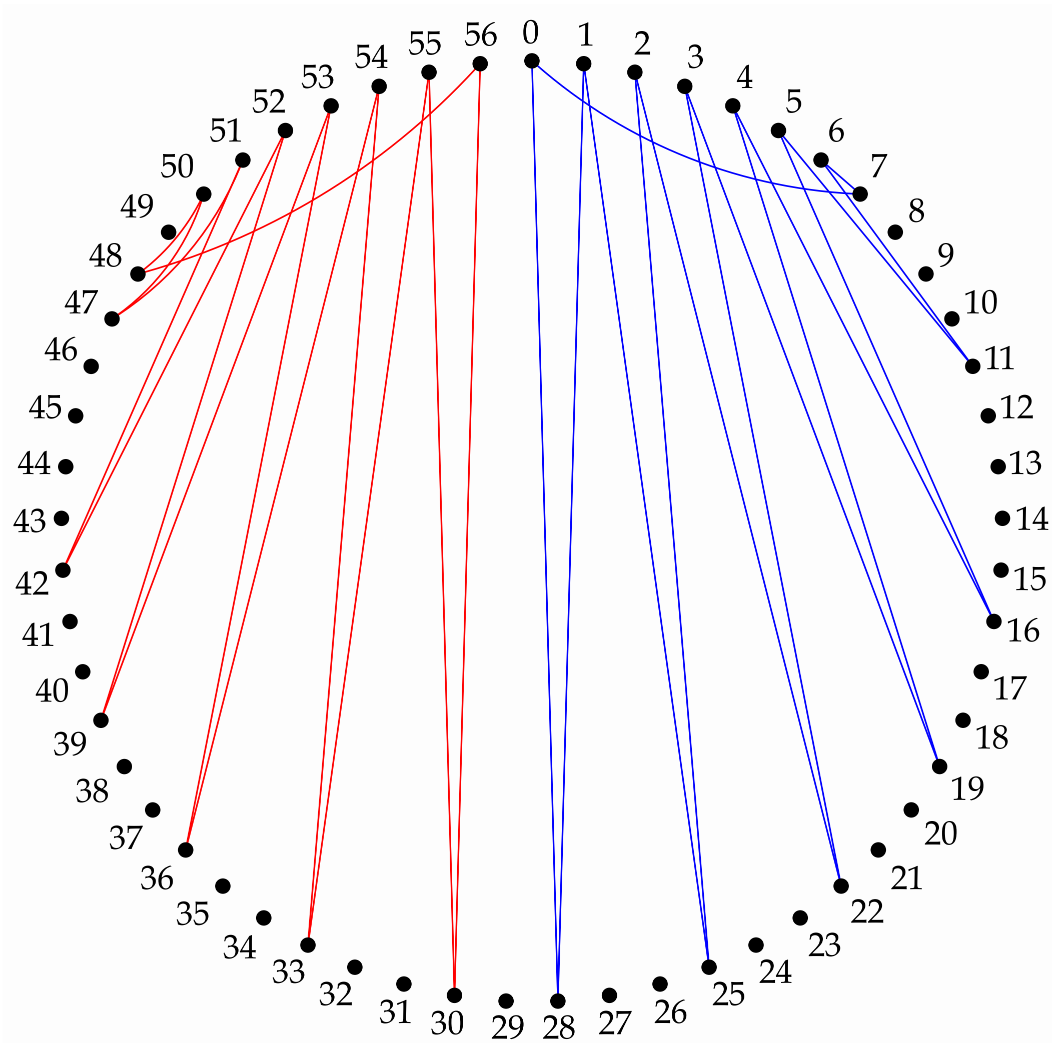



Figure 5.13: A $\mathcal{C}_{14}^{2}$ base block ($p = 2$, $k = 7$)

Table 5.10: Cycle list for the $\mathcal{C}_{18}^4$ base block in Figure 5.14

| **Cycle $\mathfrak{C}_1$** (cobalt) |
|---|
| 72 [71] 68 [67] 64 [63] 60 [59] 56 |
| [55] 52 [51] 48 [47] 44 [43] 1 [9] |
| $\mathfrak{C}_1 = \big(0, 72, 1, 69, 2, 66, 3, 63, 4,$ $60, 5, 57, 6, 54, 7, 51, 8, 9\big)$ |
| **Cycle $\mathfrak{C}_2$** (red) |
| [70] 69 [66] 65 [62] 61 [58] 57 [54] |
| 53 [50] 49 [46] 45 [42] 41 [2] 10 |
| $\mathfrak{C}_2 = \big(144, 74, 143, 77, 142, 80, 141, 83, 140,$ $86, 139, 89, 138, 92, 137, 95, 136, 134\big)$ |
| **Cycle $\mathfrak{C}_3$** (forest) |
| 40 [39] 36 [35] 32 [31] 28 [27] 24 |
| [23] 20 [19] 16 [15] 8 [7] 3 [11] |
| $\mathfrak{C}_3 = \big(10, 50, 11, 47, 12, 44, 13, 41, 14,$ $38, 15, 35, 16, 32, 17, 25, 18, 21\big)$ |
| **Cycle $\mathfrak{C}_4$** (fuchsia) |
| [38] 37 [34] 33 [30] 29 [26] 25 [22] |
| 21 [18] 17 [14] 13 [6] 5 [4] 12 |
| $\mathfrak{C}_4 = \big(132, 94, 131, 97, 130, 100, 129, 103, 128,$ $106, 127, 109, 126, 112, 125, 119, 124, 120\big)$ |

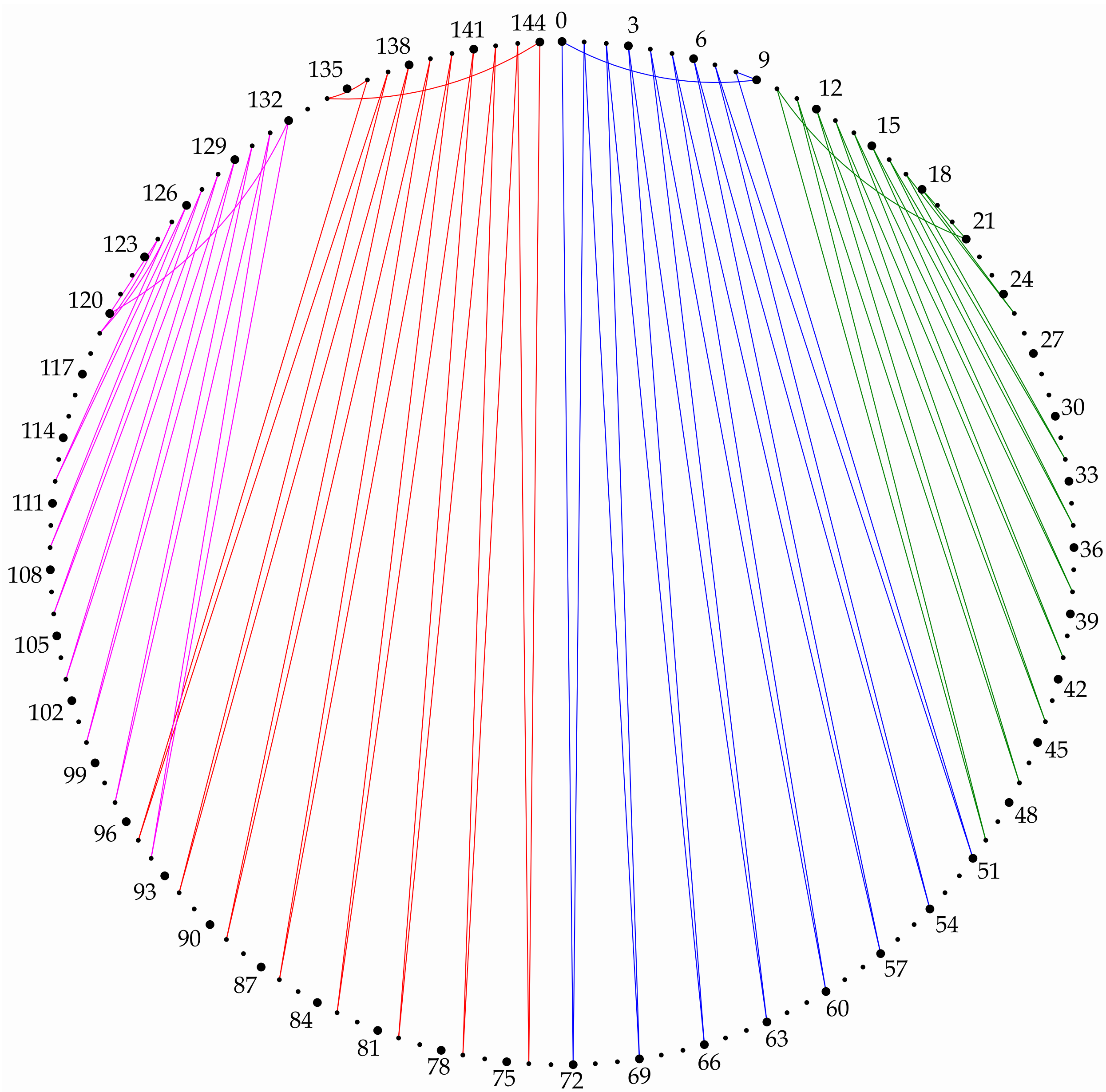



Figure 5.14: A $\mathcal{C}_{18}^{4}$ base block ($p = 4$, $k = 9$)

Table 5.11: Cycle list for the $\mathcal{C}_{22}^{6}$ base block in Figure 5.15

| **Cycle $\mathfrak{C}_1$** (cobalt) |
|---|
| 132 [131] 128 [127] 124 [123] 120 [119] 116 [115] 112 [111] 108 [107] 104 [103] 100 [99] 96 [95] 1 [11] |
| $\mathfrak{C}_1 = (0, 132, 1, 129, 2, 126, 3, 123, 4, 120, 5, 117, 6, 114, 7, 111, 8, 108, 9, 105, 10, 11)$ |
| **Cycle $\mathfrak{C}_2$** (red) |
| [130] 129 [126] 125 [122] 121 [118] 117 [114] 113 [110] 109 [106] 105 [102] 101 [98] 97 [94] 93 [2] 12 |
| $\mathfrak{C}_2 = (264, 134, 263, 137, 262, 140, 261, 143, 260, 146, 259, 149, 258, 152, 257, 155, 256, 158, 255, 161, 254, 252)$ |
| **Cycle $\mathfrak{C}_3$** (forest) |
| 92 [91] 88 [87] 84 [83] 80 [79] 76 [75] 72 [71] 68 [67] 64 [63] 60 [59] 56 [55] 3 [13] |
| $\mathfrak{C}_3 = (12, 104, 13, 101, 14, 98, 15, 95, 16, 92, 17, 89, 18, 86, 19, 83, 20, 80, 21, 77, 22, 25)$ |
| **Cycle $\mathfrak{C}_4$** (fuchsia) |
| [90] 89 [86] 85 [82] 81 [78] 77 [74] 73 [70] 69 [66] 65 [62] 61 [58] 57 [54] 53 [4] 14 |
| $\mathfrak{C}_4 = (250, 160, 249, 163, 248, 166, 247, 169, 246, 172, 245, 175, 244, 178, 243, 181, 242, 184, 241, 187, 240, 236)$ |
| **Cycle $\mathfrak{C}_5$** (orange) |
| 52 [51] 48 [47] 44 [43] 40 [39] 36 [35] 32 [31] 28 [27] 24 [23] 20 [19] 10 [9] 5 [15] |
| $\mathfrak{C}_5 = (26, 78, 27, 75, 28, 72, 29, 69, 30, 66, 31, 63, 32, 60, 33, 57, 34, 54, 35, 45, 36, 41)$ |
| **Cycle $\mathfrak{C}_6$** (plum) |
| [50] 49 [46] 45 [42] 41 [38] 37 [34] 33 [30] 29 [26] 25 [22] 21 [18] 17 [8] 7 [6] 16 |
| $\mathfrak{C}_6 = (233, 183, 232, 186, 231, 189, 230, 192, 229, 195, 228, 198, 227, 201, 226, 204, 225, 207, 224, 216, 223, 217)$ |

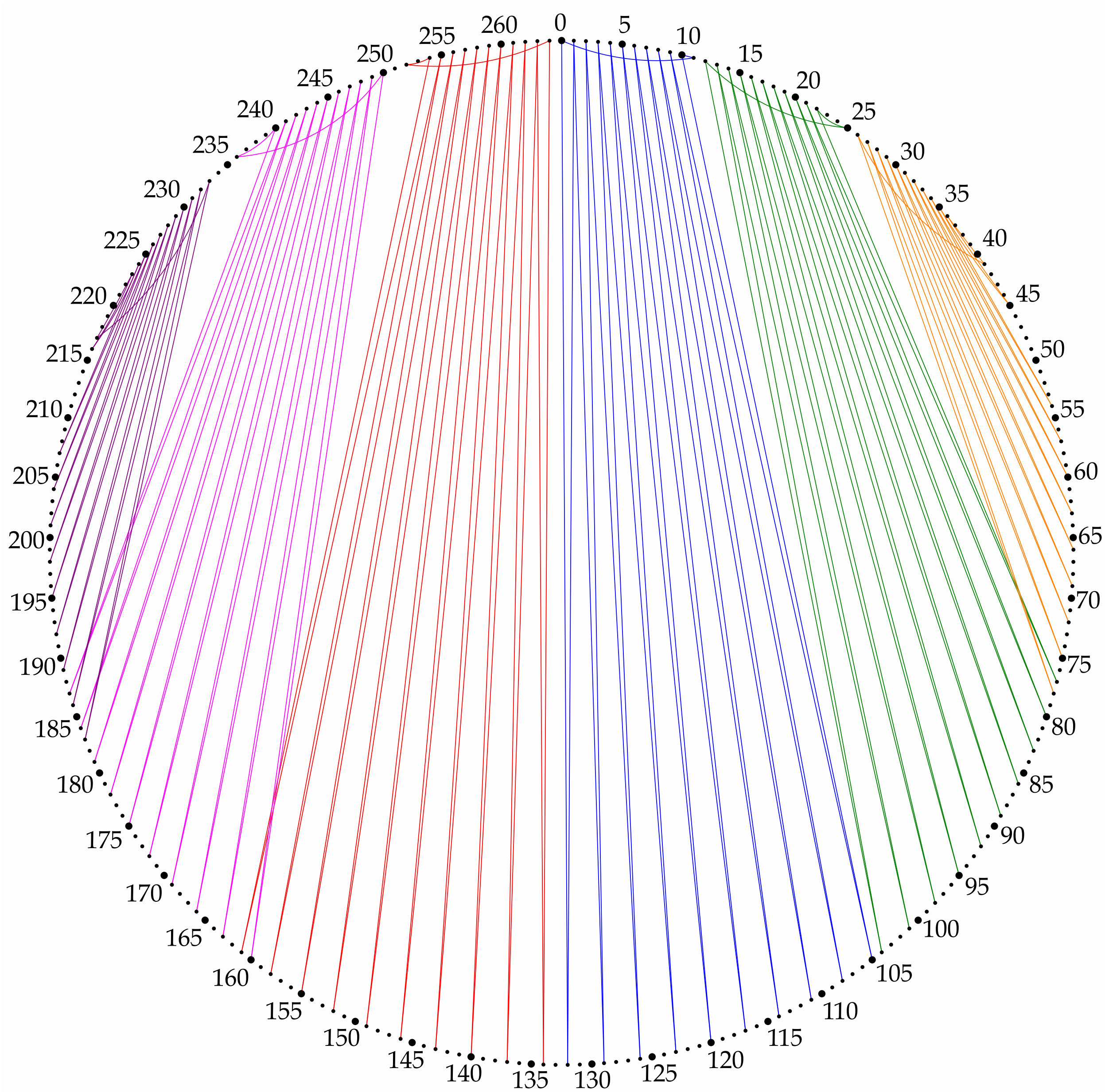



Figure 5.15: A $\mathcal{C}_{22}^{6}$ base block ($p = 6$, $k = 11$)

Table 5.12: The odd-index cycles for a $\mathcal{C}_{26}^{8}$ base block, as shown in Figure 5.16

| |
|---|
| **Cycle $\mathfrak{C}_1$** (cobalt) |
| 208 [207] 204 [203] 200 [199] 196 [195] 192 [191] 188 [187] 184 |
| [183] 180 [179] 176 [175] 172 [171] 168 [167] 164 [163] 1 [13] |
| $\mathfrak{C}_1 = \big(0, 208, 1, 205, 2, 202, 3, 199, 4, 196, 5, 193, 6,$ |
| $190, 7, 187, 8, 184, 9, 181, 10, 178, 11, 175, 12, 13\big)$ |
| **Cycle $\mathfrak{C}_3$** (forest) |
| 160 [159] 156 [155] 152 [151] 148 [147] 144 [143] 140 [139] 136 |
| [135] 132 [131] 128 [127] 124 [123] 120 [119] 116 [115] 3 [15] |
| $\mathfrak{C}_3 = \big(14, 174, 15, 171, 16, 168, 17, 165, 18, 162, 19, 159, 20,$ |
| $156, 21, 153, 22, 150, 23, 147, 24, 144, 25, 141, 26, 29\big)$ |
| **Cycle $\mathfrak{C}_5$** (orange) |
| 112 [111] 108 [107] 104 [103] 100 [99] 96 [95] 92 [91] 88 |
| [87] 84 [83] 80 [79] 76 [75] 72 [71] 68 [67] 5 [17] |
| $\mathfrak{C}_5 = \big(30, 142, 31, 139, 32, 136, 33, 133, 34, 130, 35, 127, 36,$ |
| $124, 37, 121, 38, 118, 39, 115, 40, 112, 41, 109, 42, 47\big)$ |
| **Cycle $\mathfrak{C}_7$** (sky) |
| 64 [63] 60 [59] 56 [55] 52 [51] 48 [47] 44 [43] 40 |
| [39] 36 [35] 32 [31] 28 [27] 24 [23] 12 [11] 7 [19] |
| $\mathfrak{C}_7 = \big(49, 113, 50, 110, 51, 107, 52, 104, 53, 101, 54, 98, 55,$ |
| $95, 56, 92, 57, 89, 58, 86, 59, 83, 60, 72, 61, 68\big)$ |

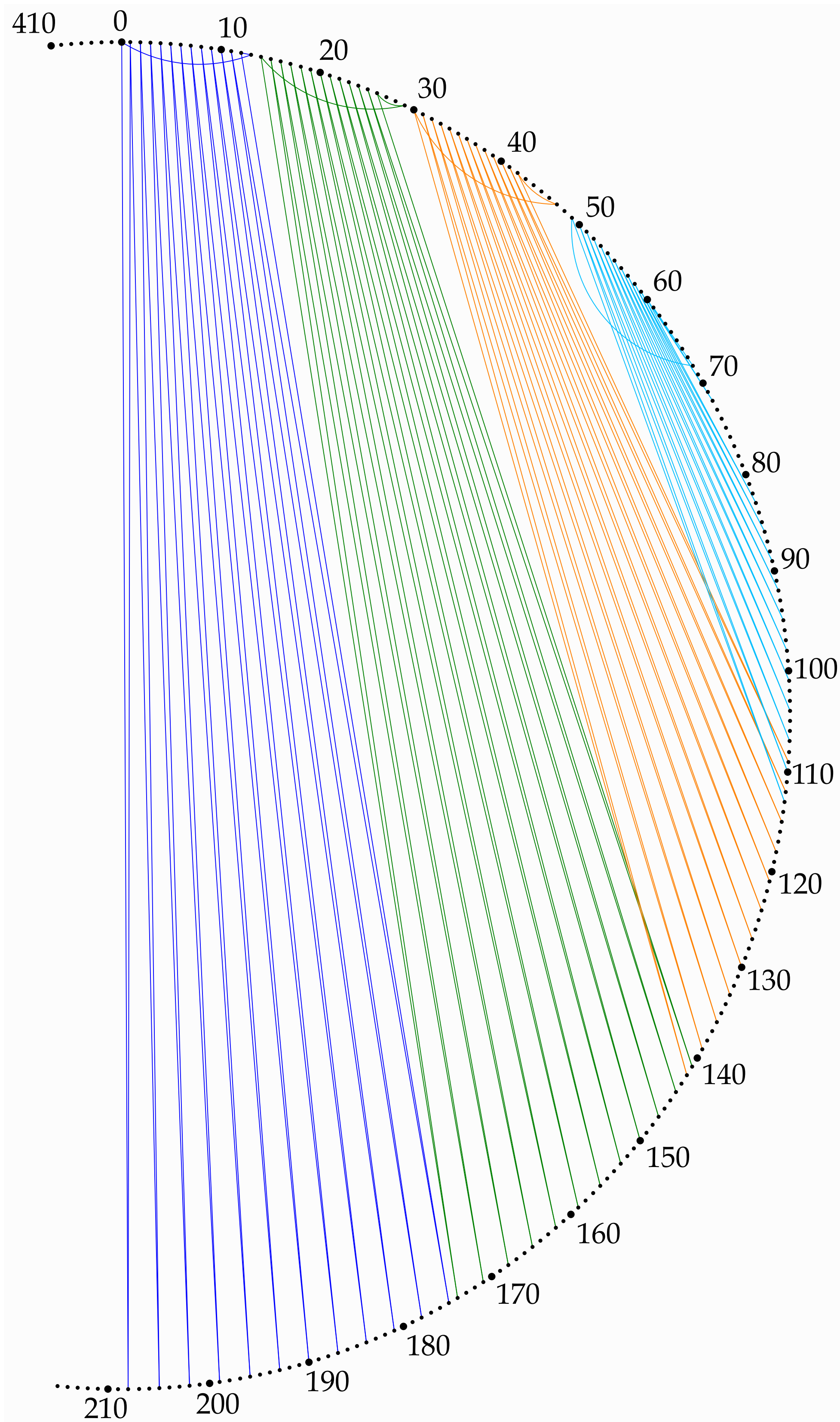



Figure 5.16: The right side of a $\mathcal{C}_{26}^{8}$ base block ($p = 8$, $k = 13$): $\mathfrak{C}_1$, $\mathfrak{C}_3$, $\mathfrak{C}_5$, and $\mathfrak{C}_7$

Table 5.13: The even-index cycles for a $\mathcal{C}_{26}^{8}$ base block, as shown in Figure 5.17

| |
|---|
| **Cycle** $\mathfrak{C}_2$ (red) |
| [206] 205 [202] 201 [198] 197 [194] 193 [190] 189 [186] 185 [182] 181 [178] 177 [174] 173 [170] 169 [166] 165 [162] 161 [2] 14 |
| $\mathfrak{C}_2 = \big(416, 210, 415, 213, 414, 216, 413, 219, 412, 222, 411, 225, 410, 228, 409, 231, 408, 234, 407, 237, 406, 240, 405, 243, 404, 402\big)$ |
| **Cycle** $\mathfrak{C}_4$ (fuchsia) |
| [158] 157 [154] 153 [150] 149 [146] 145 [142] 141 [138] 137 [134] 133 [130] 129 [126] 125 [122] 121 [118] 117 [114] 113 [4] 16 |
| $\mathfrak{C}_4 = \big(400, 242, 399, 245, 398, 248, 397, 251, 396, 254, 395, 257, 394, 260, 393, 263, 392, 266, 391, 269, 390, 272, 389, 275, 388, 384\big)$ |
| **Cycle** $\mathfrak{C}_6$ (plum) |
| [110] 109 [106] 105 [102] 101 [98] 97 [94] 93 [90] 89 [86] 85 [82] 81 [78] 77 [74] 73 [70] 69 [66] 65 [6] 18 |
| $\mathfrak{C}_6 = \big(381, 271, 380, 274, 379, 277, 378, 280, 377, 283, 376, 286, 375, 289, 374, 292, 373, 295, 372, 298, 371, 301, 370, 304, 369, 363\big)$ |
| **Cycle** $\mathfrak{C}_8$ (lime) |
| [62] 61 [58] 57 [54] 53 [50] 49 [46] 45 [42] 41 [38] 37 [34] 33 [30] 29 [26] 25 [22] 21 [10] 9 [8] 20 |
| $\mathfrak{C}_8 = \big(359, 297, 358, 300, 357, 303, 356, 306, 355, 309, 354, 312, 353, 315, 352, 318, 351, 321, 350, 324, 349, 327, 348, 338, 347, 339\big)$ |

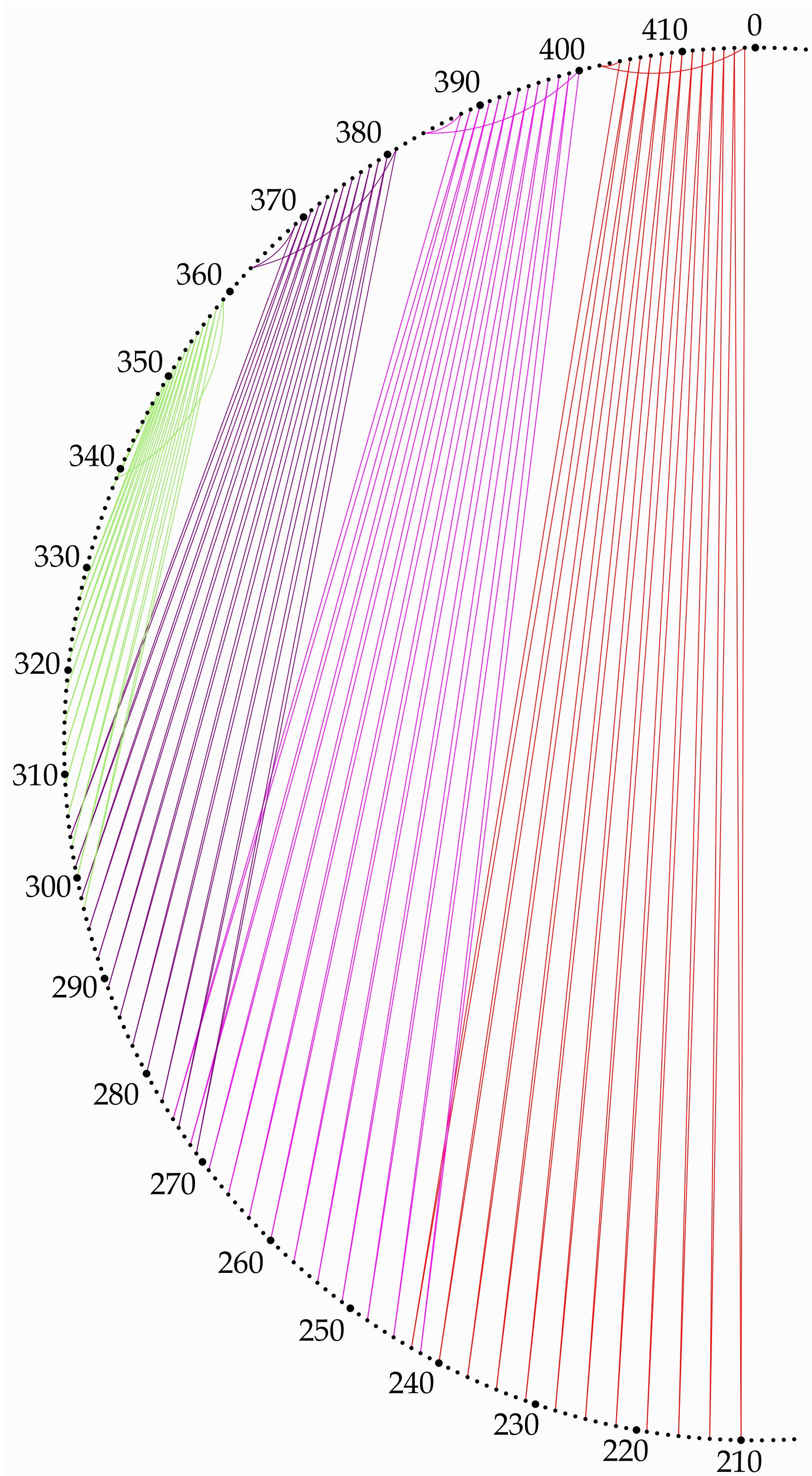



Figure 5.17: The left side of a $\mathcal{C}_{26}^{8}$ base block ($p = 8$, $k = 13$): $\mathfrak{C}_2$, $\mathfrak{C}_4$, $\mathfrak{C}_6$, and $\mathfrak{C}_8$

We now address the cases in which $k$ is too small for the general construction; these are the cases in which $k < p+4$. In these small cases, we must make slight alterations to preserve the general idea of the construction. In the smallest cases, namely those in which $k < p$, the set, $D$, of differences reserved for closing cycles cannot have the exact form described in the general construction, because the sets $[\![1,\, p]\!]$ and $[\![k,\, k+p-1]\!]$ are not disjoint. We take instead the smallest possible differences that allow us to form $p$ distinct pairs of the form $\{i,\, k+i-1\}$. In the cases in which $p < k < p+4$, we may use the set $D$ as described in the general construction, but the parts of the construction dedicated to differences $d$ such that $p < d < k$ must be modified or omitted. There are fourteen small cases in total. We exhibit the base blocks for each of these cases; see Table 5.14 for a list of the tables and figures that contain these base blocks.

Table 5.14: Directory of Tables and Figures for Small Cases ($k < p+4$)

| $k$ | $p$ | List of Cycles | Base Block |
|---|---|---|---|
| 3 | 2 | Table 5.15 | Figure 5.18 |
| | 4 | Table 5.16 | Figure 5.19 |
| | 6 | Table 5.17 | Figure 5.20 |
| | 8 | Table 5.18 | Figure 5.21 |
| 5 | 2 | Table 5.19 | Figure 5.22 |
| | 4 | Table 5.20 | Figure 5.23 |
| | 6 | Table 5.21 | Figure 5.24 |
| | 8 | Table 5.22 | Figure 5.25 |
| 7 | 4 | Table 5.23 | Figure 5.26 |
| | 6 | Table 5.24 | Figure 5.27 |
| | 8 | Table 5.25 | Figure 5.28 |
| 9 | 6 | Table 5.26 | Figure 5.29 |
| | 8 | Table 5.27 | Figure 5.30 |
| 11 | 8 | Tables 5.28 and 5.29 | Figures 5.31 and 5.32 |

Table 5.15: Cycle list for the $\mathcal{C}_6^2$ base block in Figure 5.18

| **Cycle** $\mathfrak{C}_1$ (cobalt) | **Cycle** $\mathfrak{C}_2$ (red) |
|---|---|
| 12 [11] 8 [7] 1 [3] | [10] 9 [6] 5 [2] 4 |
| $\mathfrak{C}_1 = (0, 12, 1, 9, 2, 3)$ | $\mathfrak{C}_2 = (24, 14, 23, 17, 22, 20)$ |

Table 5.16: Cycle list for the $\mathcal{C}_6^4$ base block in Figure 5.19

| **Cycle** $\mathfrak{C}_1$ (cobalt) | **Cycle** $\mathfrak{C}_3$ (forest) |
|---|---|
| 24 [23] 20 [19] 1 [3] | 16 [15] 12 [11] 5 [7] |
| $\mathfrak{C}_1 = (0, 24, 1, 21, 2, 3)$ | $\mathfrak{C}_3 = (4, 20, 5, 17, 6, 11)$ |
| **Cycle** $\mathfrak{C}_2$ (red) | **Cycle** $\mathfrak{C}_4$ (fuchsia) |
| [22] 21 [18] 17 [2] 4 | [14] 13 [10] 9 [6] 8 |
| $\mathfrak{C}_2 = (48, 26, 47, 29, 46, 44)$ | $\mathfrak{C}_4 = (42, 28, 41, 31, 40, 34)$ |

Table 5.17: Cycle list for the $\mathcal{C}_6^6$ base block in Figure 5.20

| **Cycle** $\mathfrak{C}_1$ (cobalt) | **Cycle** $\mathfrak{C}_4$ (fuchsia) |
|---|---|
| 36 [35] 32 [31] 1 [3] | [26] 25 [22] 21 [6] 8 |
| $\mathfrak{C}_1 = (0,\ 36,\ 1,\ 33,\ 2,\ 3)$ | $\mathfrak{C}_4 = (66,\ 40,\ 65,\ 43,\ 64,\ 58)$ |
| **Cycle** $\mathfrak{C}_2$ (red) | **Cycle** $\mathfrak{C}_5$ (orange) |
| [34] 33 [30] 29 [2] 4 | 20 [19] 16 [15] 9 [11] |
| $\mathfrak{C}_2 = (72,\ 38,\ 71,\ 41,\ 70,\ 68)$ | $\mathfrak{C}_5 = (8,\ 28,\ 9,\ 25,\ 10,\ 19)$ |
| **Cycle** $\mathfrak{C}_3$ (forest) | **Cycle** $\mathfrak{C}_6$ (plum) |
| 28 [27] 24 [23] 5 [7] | [18] 17 [14] 13 [10] 12 |
| $\mathfrak{C}_3 = (4,\ 32,\ 5,\ 29,\ 6,\ 11)$ | $\mathfrak{C}_6 = (62,\ 44,\ 61,\ 47,\ 60,\ 50)$ |

Table 5.18: Cycle list for the $\mathcal{C}_6^8$ base block in Figure 5.21

| **Cycle $\mathfrak{C}_1$** (cobalt) | **Cycle $\mathfrak{C}_5$** (orange) |
|---|---|
| 48 [47] 44 [43] 1 [3] | 32 [31] 28 [27] 9 [11] |
| $\mathfrak{C}_1 = \big(0, 48, 1, 45, 2, 3\big)$ | $\mathfrak{C}_5 = \big(8, 40, 9, 37, 10, 19\big)$ |
| **Cycle $\mathfrak{C}_2$** (red) | **Cycle $\mathfrak{C}_6$** (plum) |
| [46] 45 [42] 41 [2] 4 | [30] 29 [26] 25 [10] 12 |
| $\mathfrak{C}_2 = \big(96, 50, 95, 53, 94, 92\big)$ | $\mathfrak{C}_6 = \big(87, 57, 86, 60, 85, 75\big)$ |
| **Cycle $\mathfrak{C}_3$** (forest) | **Cycle $\mathfrak{C}_7$** (sky) |
| 40 [39] 36 [35] 5 [7] | 24 [23] 20 [19] 13 [15] |
| $\mathfrak{C}_3 = \big(4, 44, 5, 41, 6, 11\big)$ | $\mathfrak{C}_7 = \big(12, 36, 13, 33, 14, 27\big)$ |
| **Cycle $\mathfrak{C}_4$** (fuchsia) | **Cycle $\mathfrak{C}_8$** (lime) |
| [38] 37 [34] 33 [6] 8 | [22] 21 [18] 17 [14] 16 |
| $\mathfrak{C}_4 = \big(90, 52, 89, 55, 88, 82\big)$ | $\mathfrak{C}_8 = \big(81, 59, 80, 62, 79, 65\big)$ |

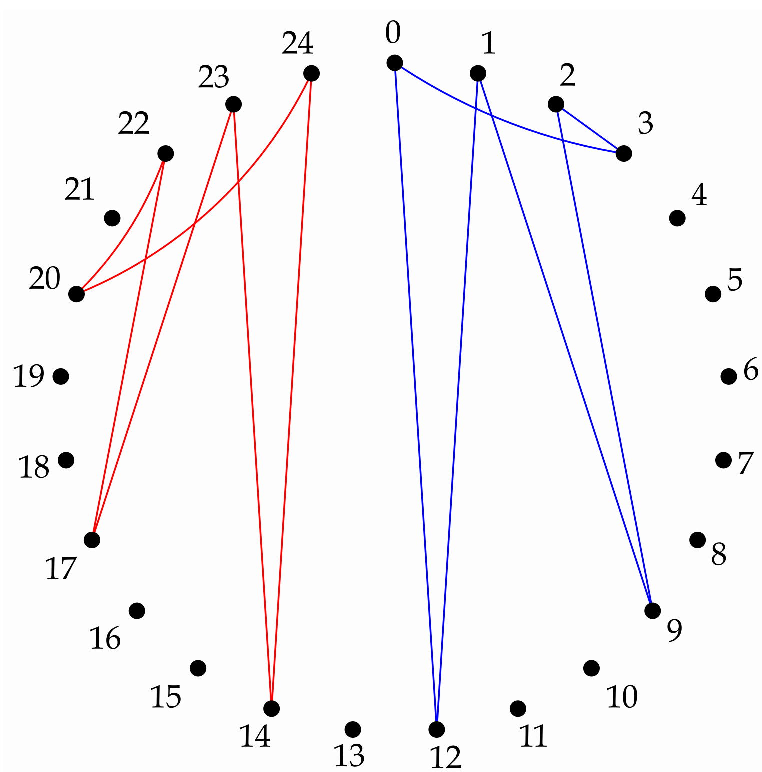



Figure 5.18: A $\mathcal{C}_6^2$ base block ($p = 2$, $k = 3$)

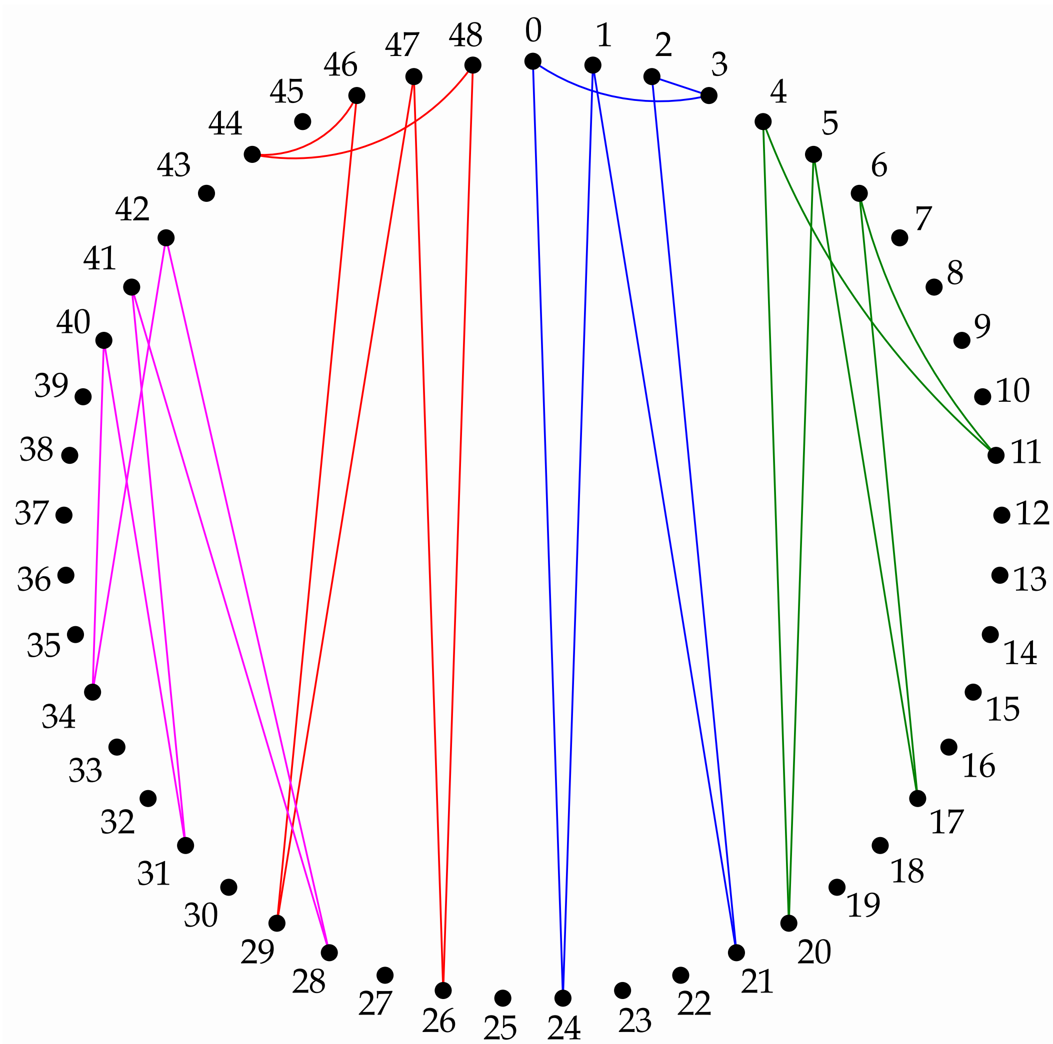



Figure 5.19: A $\mathcal{C}_6^4$ base block ($p = 4$, $k = 3$)

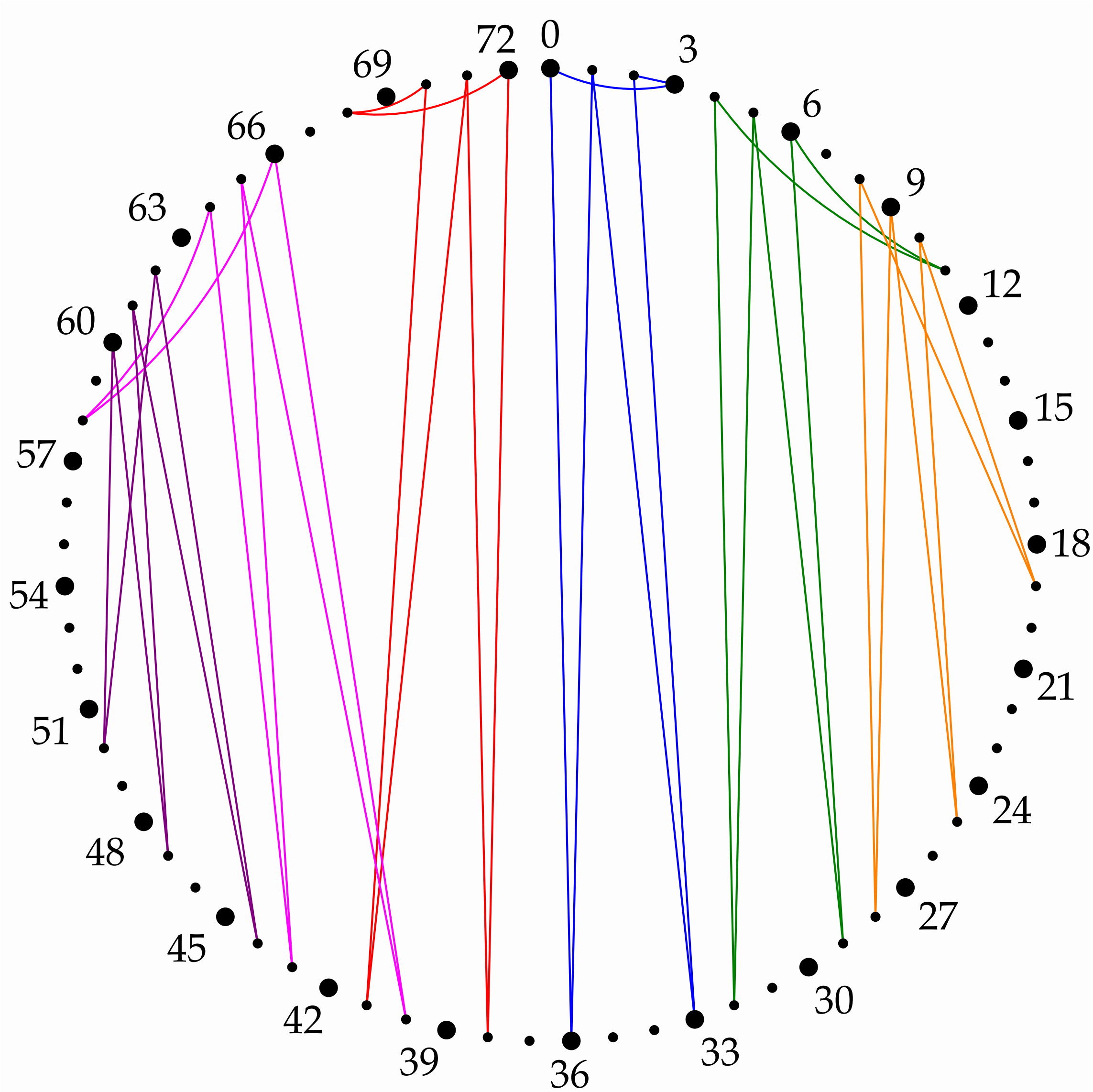



Figure 5.20: A $\mathcal{C}_6^6$ base block ($p = 6$, $k = 3$)

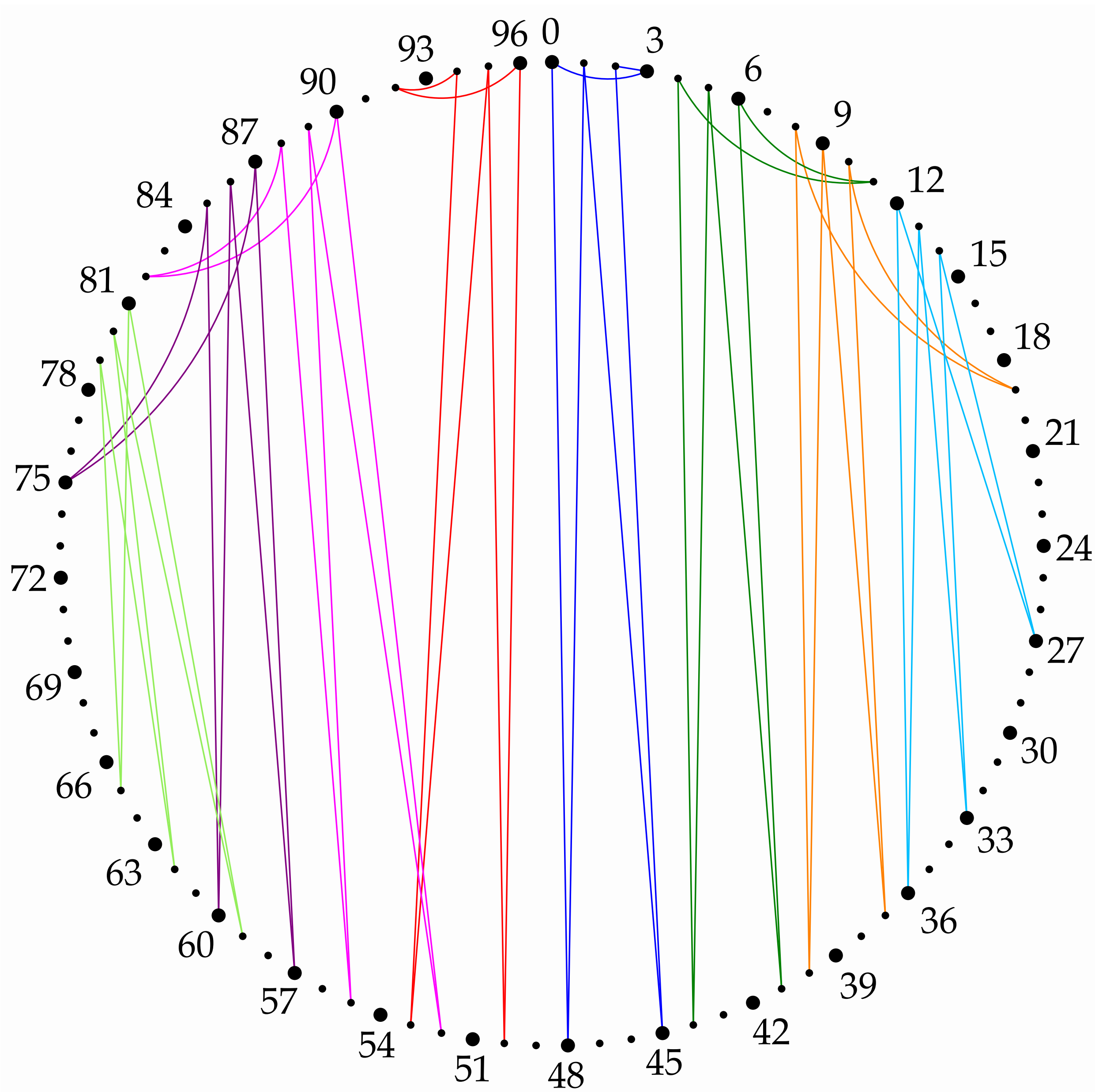



Figure 5.21: A $\mathcal{C}_6^8$ base block ($p = 8$, $k = 3$)

Table 5.19: Cycle list for the $\mathcal{C}_{10}^2$ base block in Figure 5.22

| **Cycle $\mathfrak{C}_1$** | (cobalt) | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 20 | [19] | 16 | [15] | 12 | [11] | 8 | [7] | 1 | [5] |
| $\mathfrak{C}_1 = \big(0, 20, 1, 17, 2, 14, 3, 11, 4, 5\big)$ | | | | | | | | | |
| **Cycle $\mathfrak{C}_2$** | (red) | | | | | | | | |
| [18] | 17 | [14] | 13 | [10] | 9 | [4] | 3 | [2] | 6 |
| $\mathfrak{C}_2 = \big(40, 22, 39, 25, 38, 28, 37, 33, 36, 34\big)$ | | | | | | | | | |

Table 5.20: Cycle list for the $\mathcal{C}_{10}^4$ base block in Figure 5.23

| **Cycle $\mathfrak{C}_1$** | (cobalt) | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 40 | [39] | 36 | [35] | 32 | [31] | 28 | [27] | 1 | [5] |
| $\mathfrak{C}_1 = \big(0, 40, 1, 37, 2, 34, 3, 31, 4, 5\big)$ | | | | | | | | | |
| **Cycle $\mathfrak{C}_2$** | (red) | | | | | | | | |
| [38] | 37 | [34] | 33 | [30] | 29 | [26] | 25 | [2] | 6 |
| $\mathfrak{C}_2 = \big(80, 42, 79, 45, 78, 48, 77, 51, 76, 74\big)$ | | | | | | | | | |
| **Cycle $\mathfrak{C}_3$** | (forest) | | | | | | | | |
| 24 | [23] | 20 | [19] | 16 | [15] | 12 | [11] | 3 | [7] |
| $\mathfrak{C}_3 = \big(6, 30, 7, 27, 8, 24, 9, 21, 10, 13\big)$ | | | | | | | | | |
| **Cycle $\mathfrak{C}_4$** | (fuchsia) | | | | | | | | |
| [22] | 21 | [18] | 17 | [14] | 13 | [10] | 9 | [4] | 8 |
| $\mathfrak{C}_4 = \big(72, 50, 71, 53, 70, 56, 69, 59, 68, 64\big)$ | | | | | | | | | |

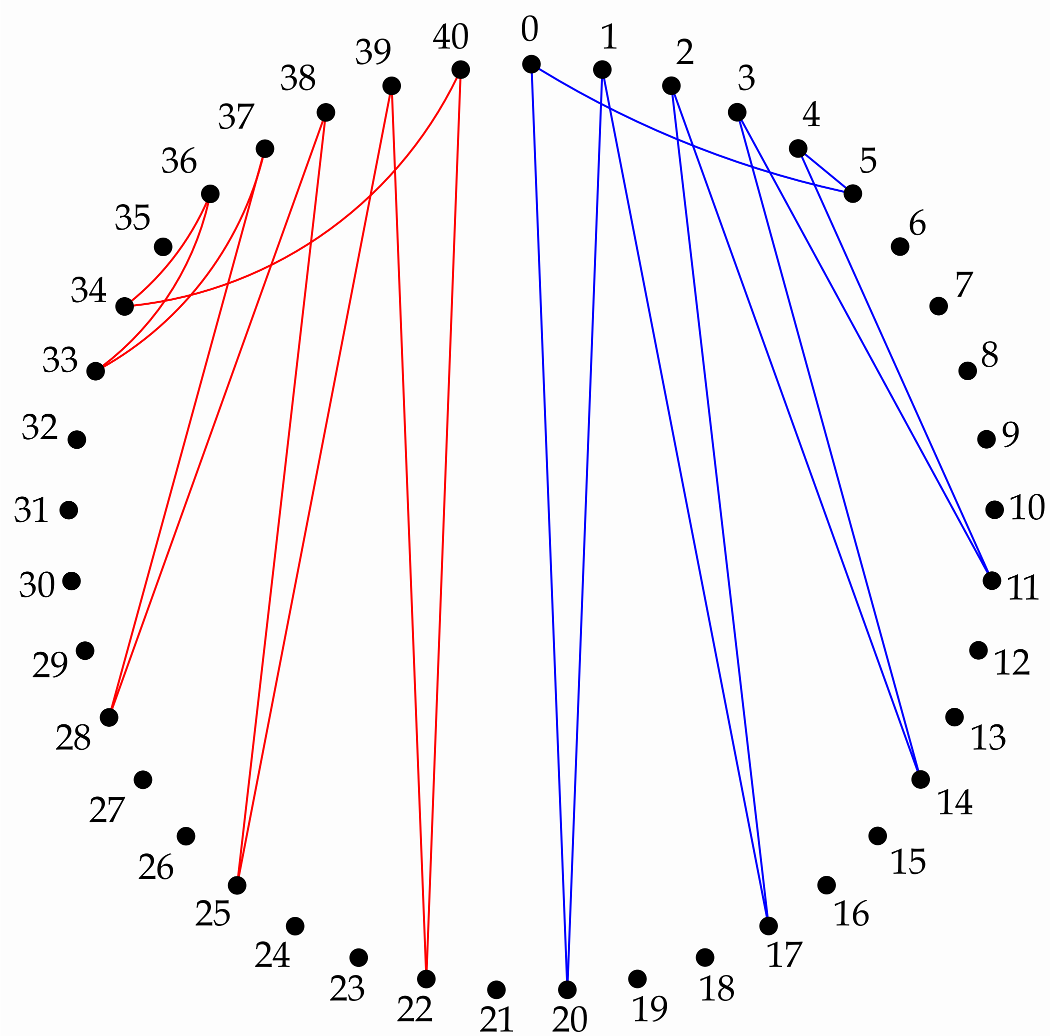



Figure 5.22: A $\mathcal{C}_{10}^{2}$ base block ($p = 2$, $k = 5$)

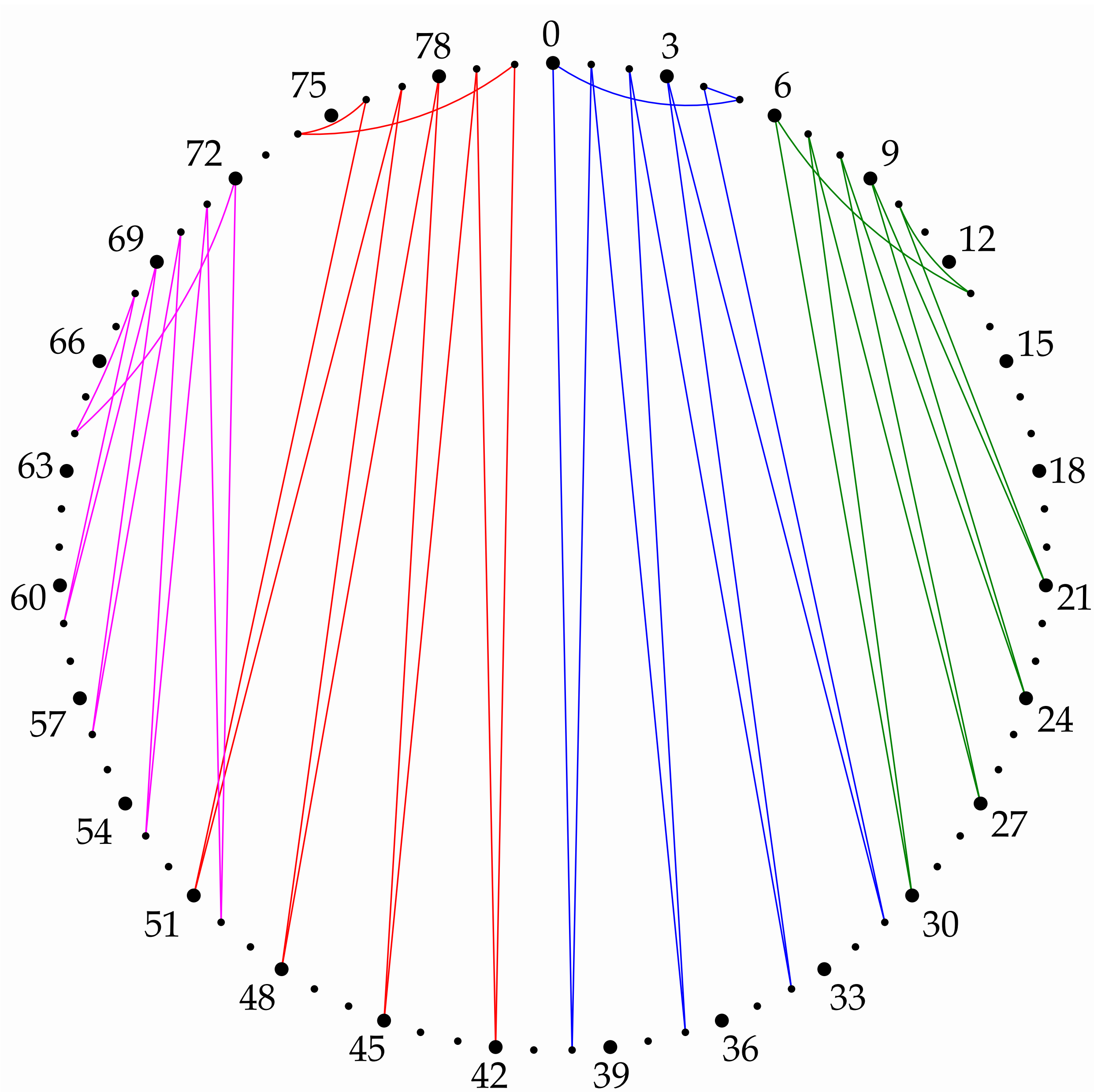



Figure 5.23: A $\mathcal{C}_{10}^{4}$ base block ($p = 4$, $k = 5$)

Table 5.21: Cycle list for the $\mathcal{C}_{10}^{6}$ base block in Figure 5.24

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **Cycle $\mathfrak{C}_1$** | (cobalt) | | | | | | | | |
| 60 | [59] | 56 | [55] | 52 | [51] | 48 | [47] | 1 | [5] |
| $\mathfrak{C}_1 = \big(0, 60, 1, 57, 2, 54, 3, 51, 4, 5\big)$ | | | | | | | | | |
| **Cycle $\mathfrak{C}_2$** | (red) | | | | | | | | |
| [58] | 57 | [54] | 53 | [50] | 49 | [46] | 45 | [2] | 6 |
| $\mathfrak{C}_2 = \big(120, 62, 119, 65, 118, 68, 117, 71, 116, 114\big)$ | | | | | | | | | |
| **Cycle $\mathfrak{C}_3$** | (forest) | | | | | | | | |
| 44 | [43] | 40 | [39] | 36 | [35] | 32 | [31] | 3 | [7] |
| $\mathfrak{C}_3 = \big(6, 50, 7, 47, 8, 44, 9, 41, 10, 13\big)$ | | | | | | | | | |
| **Cycle $\mathfrak{C}_4$** | (fuchsia) | | | | | | | | |
| [42] | 41 | [38] | 37 | [34] | 33 | [30] | 29 | [4] | 8 |
| $\mathfrak{C}_4 = \big(112, 70, 111, 73, 110, 76, 109, 79, 108, 104\big)$ | | | | | | | | | |
| **Cycle $\mathfrak{C}_5$** | (orange) | | | | | | | | |
| 28 | [27] | 24 | [23] | 20 | [19] | 16 | [15] | 9 | [13] |
| $\mathfrak{C}_5 = \big(14, 42, 15, 39, 16, 36, 17, 33, 18, 27\big)$ | | | | | | | | | |
| **Cycle $\mathfrak{C}_6$** | (plum) | | | | | | | | |
| [26] | 25 | [22] | 21 | [18] | 17 | [12] | 11 | [10] | 14 |
| $\mathfrak{C}_6 = \big(101, 75, 100, 78, 99, 81, 98, 84, 97, 87\big)$ | | | | | | | | | |

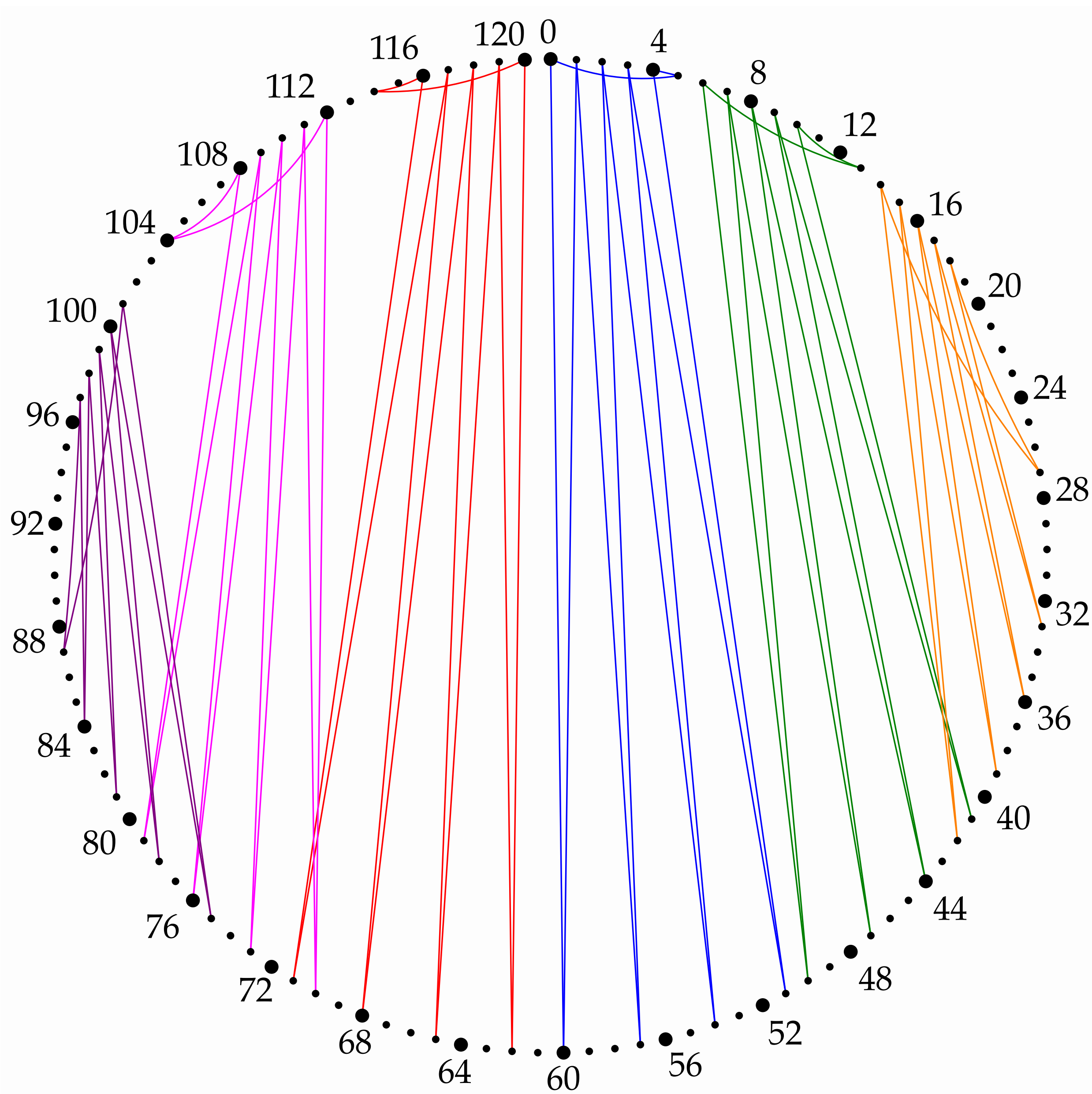



Figure 5.24: A $\mathcal{C}_{10}^{6}$ base block ($p = 6$, $k = 5$)

Table 5.22: Cycle list for the $\mathcal{C}_{10}^{8}$ base block in Figure 5.25

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **Cycle $\mathfrak{C}_1$** | (cobalt) | | | | | | | | |
| 80 | [79] | 76 | [75] | 72 | [71] | 68 | [67] | 1 | [5] |
| $\mathfrak{C}_1 = (0, 80, 1, 77, 2, 74, 3, 71, 4, 5)$ | | | | | | | | | |
| **Cycle $\mathfrak{C}_2$** | (red) | | | | | | | | |
| [78] | 77 | [74] | 73 | [70] | 69 | [66] | 65 | [2] | 6 |
| $\mathfrak{C}_2 = (160, 82, 159, 85, 158, 88, 157, 91, 156, 154)$ | | | | | | | | | |
| **Cycle $\mathfrak{C}_3$** | (forest) | | | | | | | | |
| 64 | [63] | 60 | [59] | 56 | [55] | 52 | [51] | 3 | [7] |
| $\mathfrak{C}_3 = (6, 70, 7, 67, 8, 64, 9, 61, 10, 13)$ | | | | | | | | | |
| **Cycle $\mathfrak{C}_4$** | (fuchsia) | | | | | | | | |
| [62] | 61 | [58] | 57 | [54] | 53 | [50] | 49 | [4] | 8 |
| $\mathfrak{C}_4 = (152, 90, 151, 93, 150, 96, 149, 99, 148, 144)$ | | | | | | | | | |
| **Cycle $\mathfrak{C}_5$** | (orange) | | | | | | | | |
| 48 | [47] | 44 | [43] | 40 | [39] | 36 | [35] | 9 | [13] |
| $\mathfrak{C}_5 = (14, 62, 15, 59, 16, 56, 17, 53, 18, 27)$ | | | | | | | | | |
| **Cycle $\mathfrak{C}_6$** | (plum) | | | | | | | | |
| [46] | 45 | [42] | 41 | [38] | 37 | [34] | 33 | [10] | 14 |
| $\mathfrak{C}_6 = (141, 95, 140, 98, 139, 101, 138, 104, 137, 127)$ | | | | | | | | | |
| **Cycle $\mathfrak{C}_7$** | (sky) | | | | | | | | |
| 32 | [31] | 28 | [27] | 24 | [23] | 20 | [19] | 11 | [15] |
| $\mathfrak{C}_7 = (20, 52, 21, 49, 22, 46, 23, 43, 24, 35)$ | | | | | | | | | |
| **Cycle $\mathfrak{C}_8$** | (lime) | | | | | | | | |
| [30] | 29 | [26] | 25 | [22] | 21 | [18] | 17 | [12] | 16 |
| $\mathfrak{C}_8 = (136, 106, 135, 109, 134, 112, 133, 115, 132, 120)$ | | | | | | | | | |

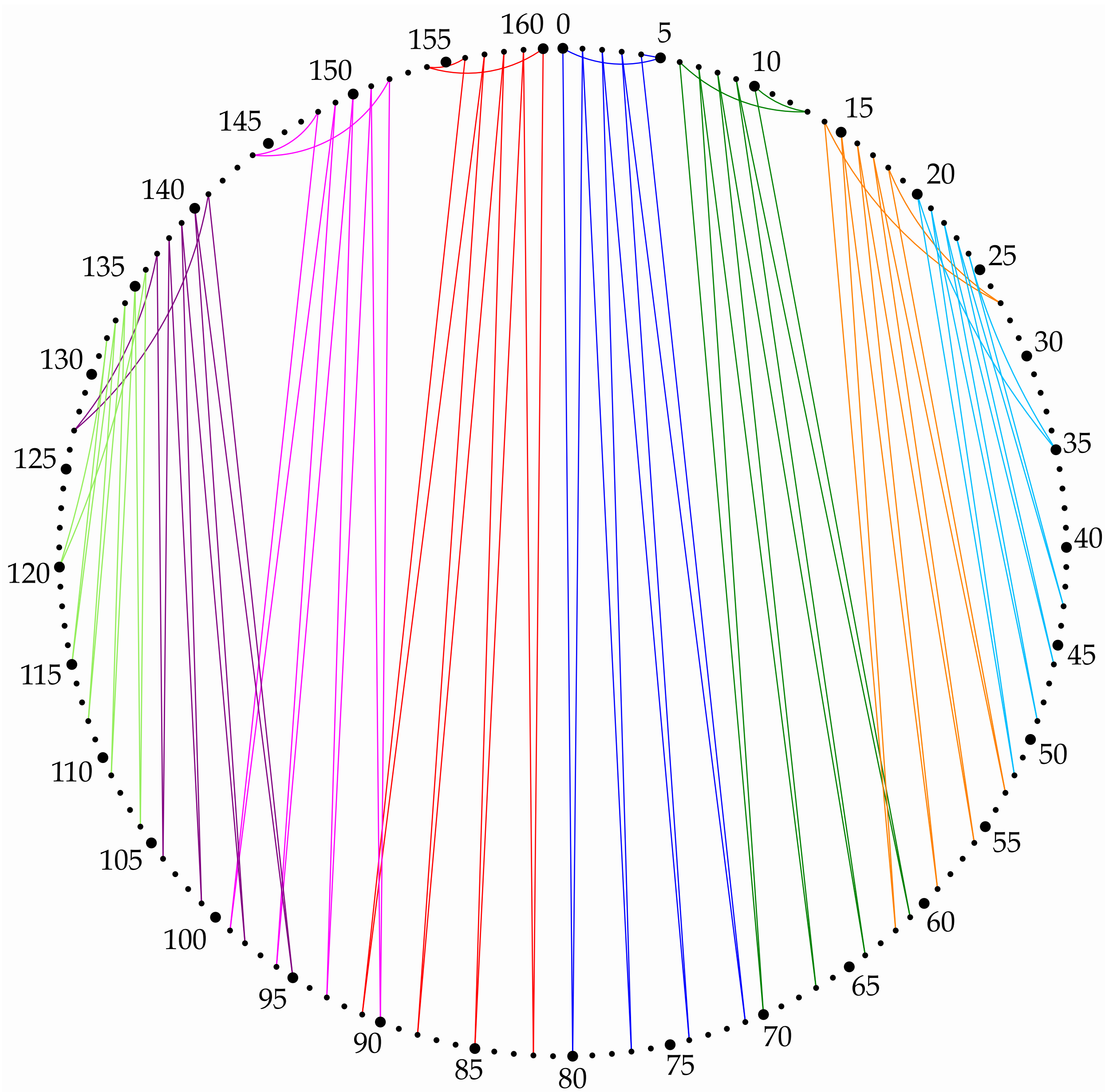



Figure 5.25: A $\mathcal{C}_{10}^{8}$ base block ($p = 8$, $k = 5$)

Table 5.23: Cycle list for the $\mathcal{C}_{14}^4$ base block in Figure 5.26

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Cycle** $\mathfrak{C}_1$ | (cobalt) | | | | | | | | | | | | |
| 56 | [55] | 52 | [51] | 48 | [47] | 44 | [43] | 40 | [39] | 36 | [35] | 1 | [7] |
| $\mathfrak{C}_1 = \big(0, 56, 1, 53, 2, 50, 3, 47, 4, 44, 5, 41, 6, 7\big)$ | | | | | | | | | | | | | |
| **Cycle** $\mathfrak{C}_2$ | (red) | | | | | | | | | | | | |
| [54] | 53 | [50] | 49 | [46] | 45 | [42] | 41 | [38] | 37 | [34] | 33 | [2] | 8 |
| $\mathfrak{C}_2 = \big(112, 58, 111, 61, 110, 64, 109, 67, 108, 70, 107, 73, 106, 104\big)$ | | | | | | | | | | | | | |
| **Cycle** $\mathfrak{C}_3$ | (forest) | | | | | | | | | | | | |
| 32 | [31] | 28 | [27] | 24 | [23] | 20 | [19] | 16 | [15] | 12 | [11] | 3 | [9] |
| $\mathfrak{C}_3 = \big(8, 40, 9, 37, 10, 34, 11, 31, 12, 28, 13, 25, 14, 17\big)$ | | | | | | | | | | | | | |
| **Cycle** $\mathfrak{C}_4$ | (fuchsia) | | | | | | | | | | | | |
| [30] | 29 | [26] | 25 | [22] | 21 | [18] | 17 | [14] | 13 | [6] | 5 | [4] | 10 |
| $\mathfrak{C}_4 = \big(102, 72, 101, 75, 100, 78, 99, 81, 98, 84, 97, 91, 96, 92\big)$ | | | | | | | | | | | | | |

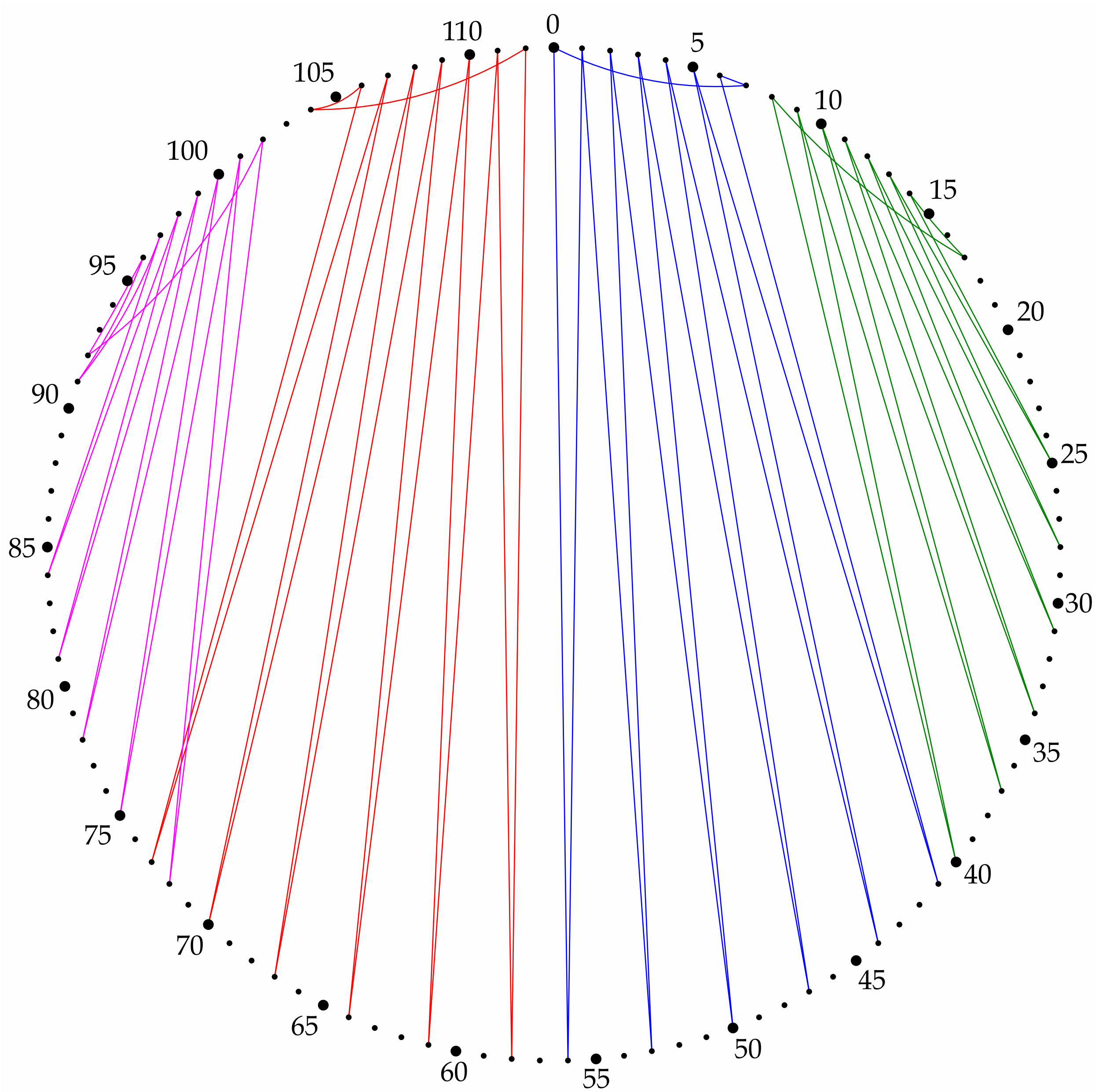



Figure 5.26: A $\mathcal{C}_{14}^4$ base block ($p = 4$, $k = 7$)

Table 5.24: Cycle list for the $\mathcal{C}_{14}^{6}$ base block in Figure 5.27

| **Cycle $\mathfrak{C}_1$** | (cobalt) |
|---|---|
| 84 [83] 80 [79] 76 [75] 72 [71] 68 [67] 64 [63] 1 [7] | |
| $\mathfrak{C}_1 = (0, 84, 1, 81, 2, 78, 3, 75, 4, 72, 5, 69, 6, 7)$ | |
| **Cycle $\mathfrak{C}_2$** | (red) |
| [82] 81 [78] 77 [74] 73 [70] 69 [66] 65 [62] 61 [2] 8 | |
| $\mathfrak{C}_2 = (168, 86, 167, 89, 166, 92, 165, 95, 164, 98, 163, 101, 162, 160)$ | |
| **Cycle $\mathfrak{C}_3$** | (forest) |
| 60 [59] 56 [55] 52 [51] 48 [47] 44 [43] 40 [39] 3 [9] | |
| $\mathfrak{C}_3 = (8, 68, 9, 65, 10, 62, 11, 59, 12, 56, 13, 53, 14, 17)$ | |
| **Cycle $\mathfrak{C}_4$** | (fuchsia) |
| [58] 57 [54] 53 [50] 49 [46] 45 [42] 41 [38] 37 [4] 10 | |
| $\mathfrak{C}_4 = (158, 100, 157, 103, 156, 106, 155, 109, 154, 112, 153, 115, 152, 148)$ | |
| **Cycle $\mathfrak{C}_5$** | (orange) |
| 36 [35] 32 [31] 28 [27] 24 [23] 20 [19] 16 [15] 5 [11] | |
| $\mathfrak{C}_5 = (18, 54, 19, 51, 20, 48, 21, 45, 22, 42, 23, 39, 24, 29)$ | |
| **Cycle $\mathfrak{C}_6$** | (plum) |
| [34] 33 [30] 29 [26] 25 [22] 21 [18] 17 [14] 13 [6] 12 | |
| $\mathfrak{C}_6 = (145, 111, 144, 114, 143, 117, 142, 120, 141, 123, 140, 126, 139, 133)$ | |

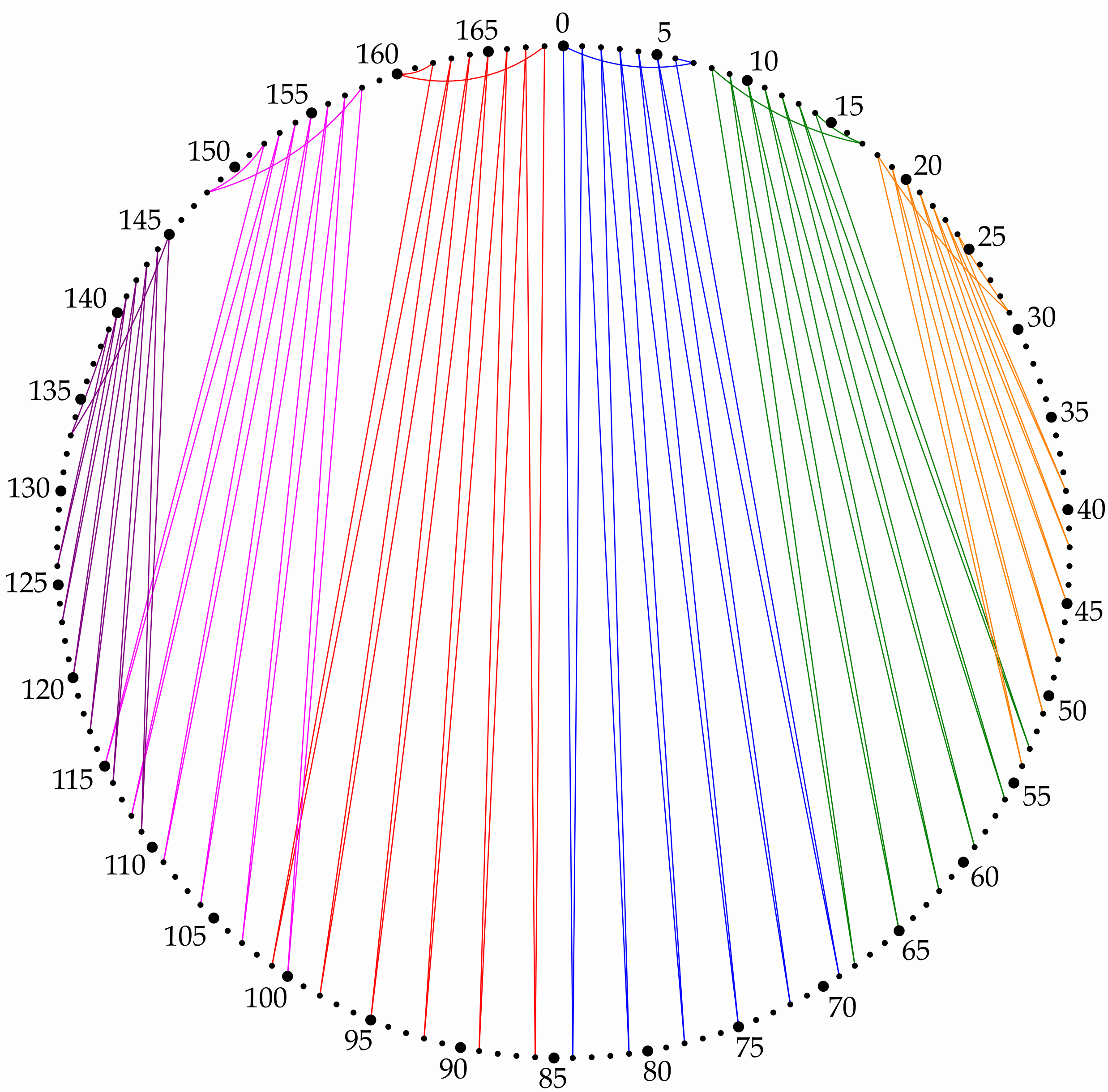



Figure 5.27: A $\mathcal{C}_{14}^{6}$ base block ($p = 6$, $k = 7$)

Table 5.25: Cycle list for the $\mathcal{C}_{14}^{8}$ base block in Figure 5.28

| |
|---|
| **Cycle $\mathfrak{C}_1$** (cobalt) |
| 112 [111] 108 [107] 104 [103] 100 [99] 96 [95] 92 [91] 1 [7] |
| $\mathfrak{C}_1 = \big(0, 112, 1, 109, 2, 106, 3, 103, 4, 100, 5, 97, 6, 7\big)$ |
| **Cycle $\mathfrak{C}_2$** (red) |
| [110] 109 [106] 105 [102] 101 [98] 97 [94] 93 [90] 89 [2] 8 |
| $\mathfrak{C}_2 = \big(224, 114, 223, 117, 222, 120, 221, 123, 220, 126, 219, 129, 218, 216\big)$ |
| **Cycle $\mathfrak{C}_3$** (forest) |
| 88 [87] 84 [83] 80 [79] 76 [75] 72 [71] 68 [67] 3 [9] |
| $\mathfrak{C}_3 = \big(8, 96, 9, 93, 10, 90, 11, 87, 12, 84, 13, 81, 14, 17\big)$ |
| **Cycle $\mathfrak{C}_4$** (fuchsia) |
| [86] 85 [82] 81 [78] 77 [74] 73 [70] 69 [66] 65 [4] 10 |
| $\mathfrak{C}_4 = \big(214, 128, 213, 131, 212, 134, 211, 137, 210, 140, 209, 143, 208, 204\big)$ |
| **Cycle $\mathfrak{C}_5$** (orange) |
| 64 [63] 60 [59] 56 [55] 52 [51] 48 [47] 44 [43] 5 [11] |
| $\mathfrak{C}_5 = \big(18, 82, 19, 79, 20, 76, 21, 73, 22, 70, 23, 67, 24, 29\big)$ |
| **Cycle $\mathfrak{C}_6$** (plum) |
| [62] 61 [58] 57 [54] 53 [50] 49 [46] 45 [42] 41 [6] 12 |
| $\mathfrak{C}_6 = \big(201, 139, 200, 142, 199, 145, 198, 148, 197, 151, 196, 154, 195, 189\big)$ |
| **Cycle $\mathfrak{C}_7$** (sky) |
| 40 [39] 36 [35] 32 [31] 28 [27] 24 [23] 18 [17] 13 [19] |
| $\mathfrak{C}_7 = \big(31, 71, 32, 68, 33, 65, 34, 62, 35, 59, 36, 54, 37, 50\big)$ |
| **Cycle $\mathfrak{C}_8$** (lime) |
| [38] 37 [34] 33 [30] 29 [26] 25 [22] 21 [16] 15 [14] 20 |
| $\mathfrak{C}_8 = \big(185, 147, 184, 150, 183, 153, 182, 156, 181, 159, 180, 164, 179, 165\big)$ |

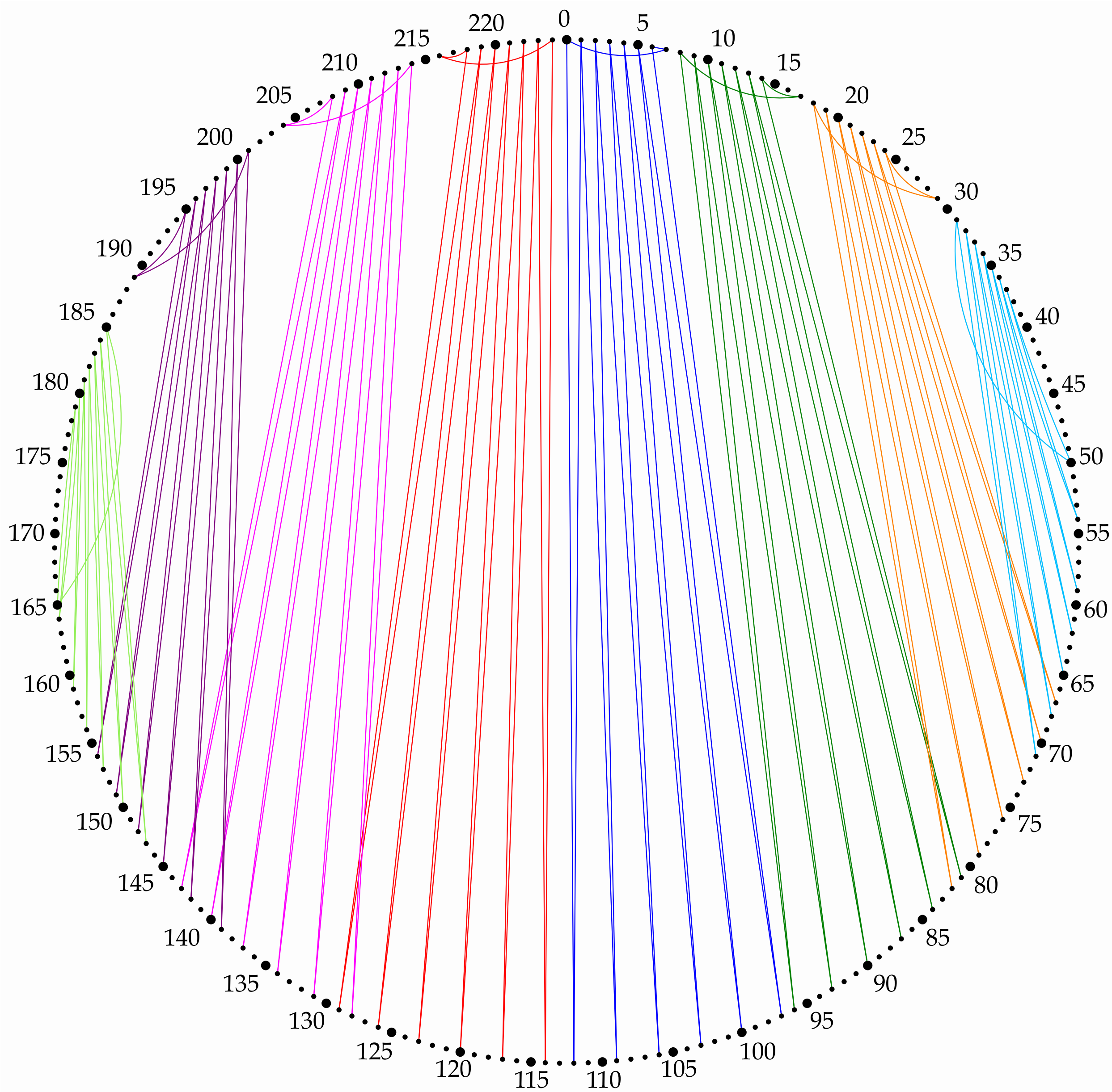



Figure 5.28: A $\mathcal{C}_{14}^{8}$ base block ($p = 8$, $k = 7$)

Table 5.26: Cycle list for the $\mathcal{C}_{18}^{6}$ base block in Figure 5.29

| | |
|---|---|
| **Cycle $\mathfrak{C}_1$** | (cobalt) |
| 108 [107] 104 [103] 100 [99] 96 [95] 92<br>[91] 88 [87] 84 [83] 80 [79] 1 [9] | |
| $\mathfrak{C}_1 = \big(0, 108, 1, 105, 2, 102, 3, 99, 4, 96, 5, 93, 6, 90, 7, 87, 8, 9\big)$ | |
| **Cycle $\mathfrak{C}_2$** | (red) |
| [106] 105 [102] 101 [98] 97 [94] 93 [90]<br>89 [86] 85 [82] 81 [78] 77 [2] 10 | |
| $\mathfrak{C}_2 = \big(216, 110, 215, 113, 214, 116, 213, 119, 212, 122, 211, 125, 210, 128, 209, 131, 208, 206\big)$ | |
| **Cycle $\mathfrak{C}_3$** | (forest) |
| 76 [75] 72 [71] 68 [67] 64 [63] 60<br>[59] 56 [55] 52 [51] 48 [47] 3 [11] | |
| $\mathfrak{C}_3 = \big(10, 86, 11, 83, 12, 80, 13, 77, 14, 74, 15, 71, 16, 68, 17, 65, 18, 21\big)$ | |
| **Cycle $\mathfrak{C}_4$** | (fuchsia) |
| [74] 73 [70] 69 [66] 65 [62] 61 [58]<br>57 [54] 53 [50] 49 [46] 45 [4] 12 | |
| $\mathfrak{C}_4 = \big(204, 130, 203, 133, 202, 136, 201, 139, 200, 142, 199, 145, 198, 148, 197, 151, 196, 192\big)$ | |
| **Cycle $\mathfrak{C}_5$** | (orange) |
| 44 [43] 40 [39] 36 [35] 32 [31] 28<br>[27] 24 [23] 20 [19] 16 [15] 5 [13] | |
| $\mathfrak{C}_5 = \big(22, 66, 23, 63, 24, 60, 25, 57, 26, 54, 27, 51, 28, 48, 29, 45, 30, 35\big)$ | |
| **Cycle $\mathfrak{C}_6$** | (plum) |
| [42] 41 [38] 37 [34] 33 [30] 29 [26]<br>25 [22] 21 [18] 17 [8] 7 [6] 14 | |
| $\mathfrak{C}_6 = \big(189, 147, 188, 150, 187, 153, 186, 156, 185, 159, 184, 162, 183, 165, 182, 174, 181, 175\big)$ | |

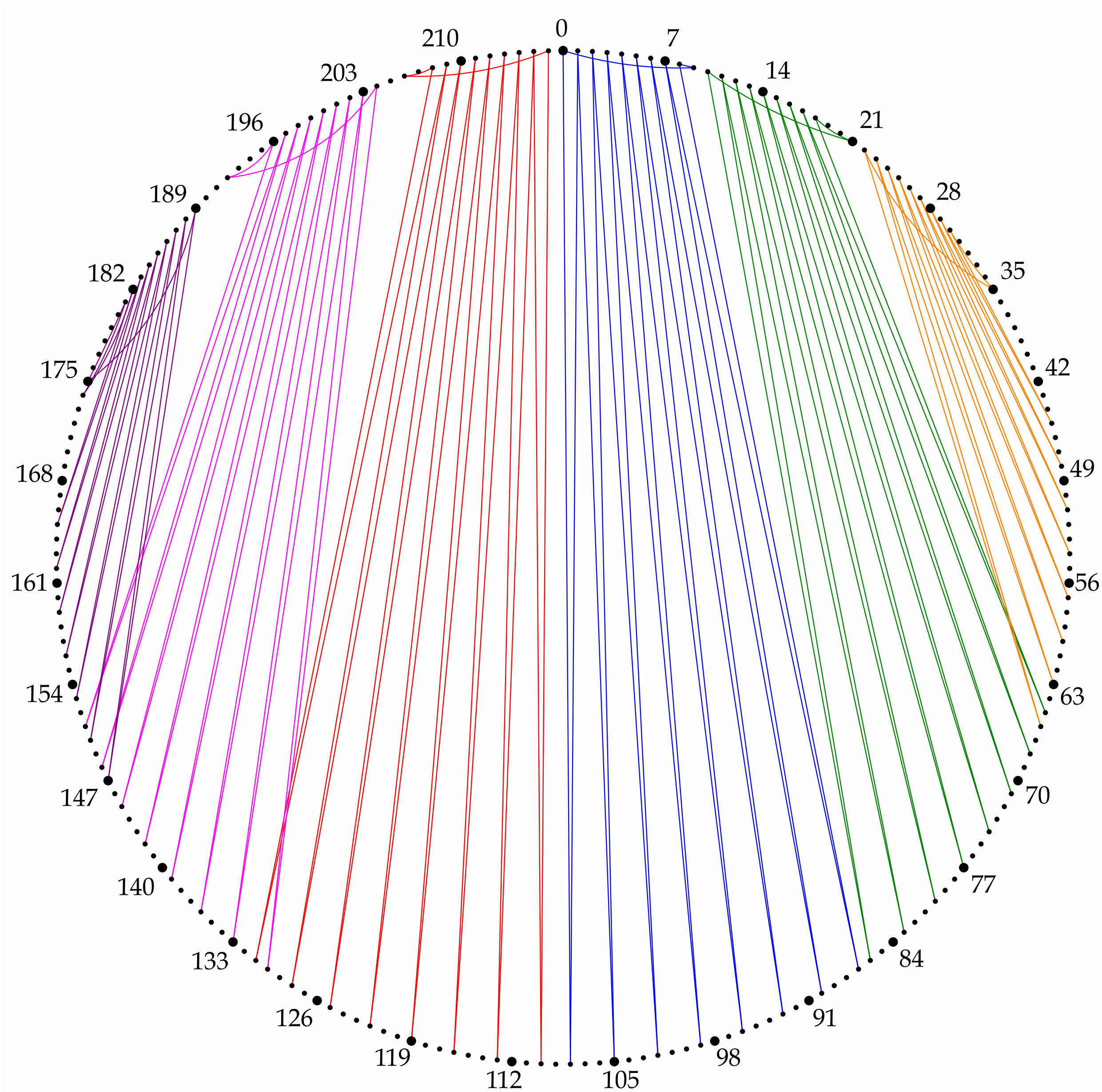



Figure 5.29: A $\mathcal{C}_{18}^{6}$ base block ($p = 6$, $k = 9$)

Table 5.27: Cycle list for the $\mathcal{C}_{18}^{8}$ base block in Figure 5.30

| **Cycle $\mathfrak{C}_1$** (cobalt) | **Cycle $\mathfrak{C}_5$** (orange) |
|---|---|
| 144 [143] 140 [139] 136 [135]<br>132 [131] 128 [127] 124 [123]<br>120 [119] 116 [115] 1 [9]<br>$\mathfrak{C}_1 = \big(0, 144, 1, 141, 2, 138, 3, 135, 4, 132, 5, 129, 6, 126, 7, 123, 8, 9\big)$ | 80 [79] 76 [75] 72 [71]<br>68 [67] 64 [63] 60 [59]<br>56 [55] 52 [51] 5 [13]<br>$\mathfrak{C}_5 = \big(22, 102, 23, 99, 24, 96, 25, 93, 26, 90, 27, 87, 28, 84, 29, 81, 30, 35\big)$ |
| **Cycle $\mathfrak{C}_2$** (red) | **Cycle $\mathfrak{C}_6$** (plum) |
| [142] 141 [138] 137 [134] 133<br>[130] 129 [126] 125 [122] 121<br>[118] 117 [114] 113 [2] 10<br>$\mathfrak{C}_2 = \big(288, 146, 287, 149, 286, 152, 285, 155, 284, 158, 283, 161, 282, 164, 281, 167, 280, 278\big)$ | [78] 77 [74] 73 [70] 69<br>[66] 65 [62] 61 [58] 57<br>[54] 53 [50] 49 [6] 14<br>$\mathfrak{C}_6 = \big(261, 183, 260, 186, 259, 189, 258, 192, 257, 195, 256, 198, 255, 201, 254, 204, 253, 247\big)$ |
| **Cycle $\mathfrak{C}_3$** (forest) | **Cycle $\mathfrak{C}_7$** (sky) |
| 112 [111] 108 [107] 104 [103]<br>100 [99] 96 [95] 92 [91]<br>88 [87] 84 [83] 3 [11]<br>$\mathfrak{C}_3 = \big(10, 122, 11, 119, 12, 116, 13, 113, 14, 110, 15, 107, 16, 104, 17, 101, 18, 21\big)$ | 48 [47] 44 [43] 40 [39]<br>36 [35] 32 [31] 28 [27]<br>24 [23] 20 [19] 7 [15]<br>$\mathfrak{C}_7 = \big(37, 85, 38, 82, 39, 79, 40, 76, 41, 73, 42, 70, 43, 67, 44, 64, 45, 52\big)$ |
| **Cycle $\mathfrak{C}_4$** (fuchsia) | **Cycle $\mathfrak{C}_8$** (lime) |
| [110] 109 [106] 105 [102] 101<br>[98] 97 [94] 93 [90] 89<br>[86] 85 [82] 81 [4] 12<br>$\mathfrak{C}_4 = \big(276, 166, 275, 169, 274, 172, 273, 175, 272, 178, 271, 181, 270, 184, 269, 187, 268, 264\big)$ | [46] 45 [42] 41 [38] 37<br>[34] 33 [30] 29 [26] 25<br>[22] 21 [18] 17 [8] 16<br>$\mathfrak{C}_8 = \big(243, 197, 242, 200, 241, 203, 240, 206, 239, 209, 238, 212, 237, 215, 236, 218, 235, 227\big)$ |

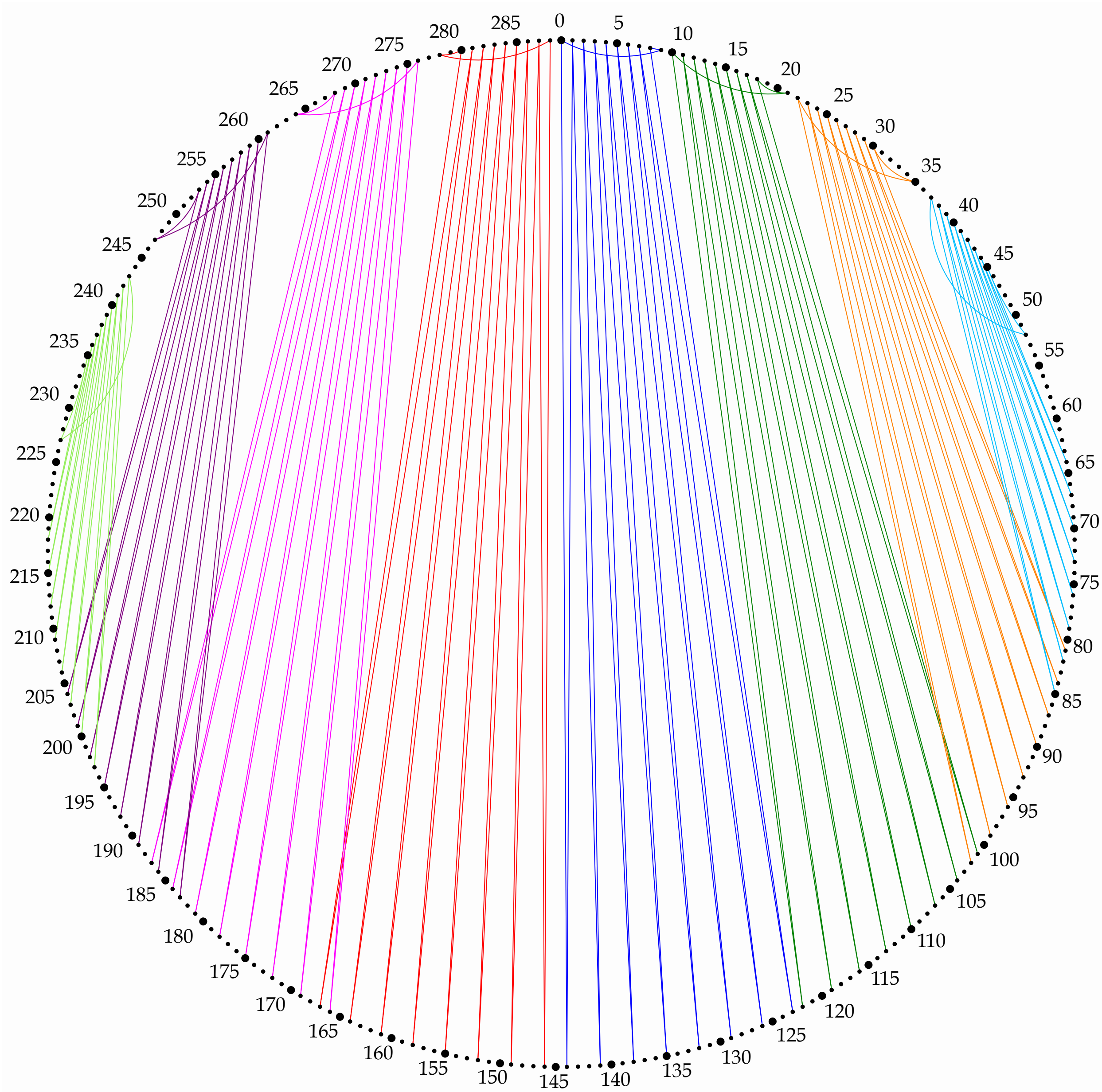



Figure 5.30: A $\mathcal{C}_{18}^{8}$ base block ($p = 8$, $k = 9$)

Table 5.28: The odd-index cycles for a $\mathcal{C}_{22}^{8}$ base block, as shown in Figure 5.31

| **Cycle** $\mathfrak{C}_1$ (cobalt) |
|---|
| 176 [175] 172 [171] 168 [167] 164 [163] 160 [159] 156<br>[155] 152 [151] 148 [147] 144 [143] 140 [139] 1 [11]<br>$\mathfrak{C}_1 = \big(0, 176, 1, 173, 2, 170, 3, 167, 4, 164, 5, 161, 6, 158, 7, 155, 8, 152, 9, 149, 10, 11\big)$ |
| **Cycle** $\mathfrak{C}_3$ (forest) |
| 136 [135] 132 [131] 128 [127] 124 [123] 120 [119] 116<br>[115] 112 [111] 108 [107] 104 [103] 100 [99] 3 [13]<br>$\mathfrak{C}_3 = \big(12, 148, 13, 145, 14, 142, 15, 139, 16, 136, 17, 133, 18, 130, 19, 127, 20, 124, 21, 121, 22, 25\big)$ |
| **Cycle** $\mathfrak{C}_5$ (orange) |
| 96 [95] 92 [91] 88 [87] 84 [83] 80 [79] 76<br>[75] 72 [71] 68 [67] 64 [63] 60 [59] 5 [15]<br>$\mathfrak{C}_5 = \big(26, 122, 27, 119, 28, 116, 29, 113, 30, 110, 31, 107, 32, 104, 33, 101, 34, 98, 35, 95, 36, 41\big)$ |
| **Cycle** $\mathfrak{C}_7$ (sky) |
| 56 [55] 52 [51] 48 [47] 44 [43] 40 [39] 36<br>[35] 32 [31] 28 [27] 24 [23] 20 [19] 7 [17]<br>$\mathfrak{C}_7 = \big(43, 99, 44, 96, 45, 93, 46, 90, 47, 87, 48, 84, 49, 81, 50, 78, 51, 75, 52, 72, 53, 60\big)$ |

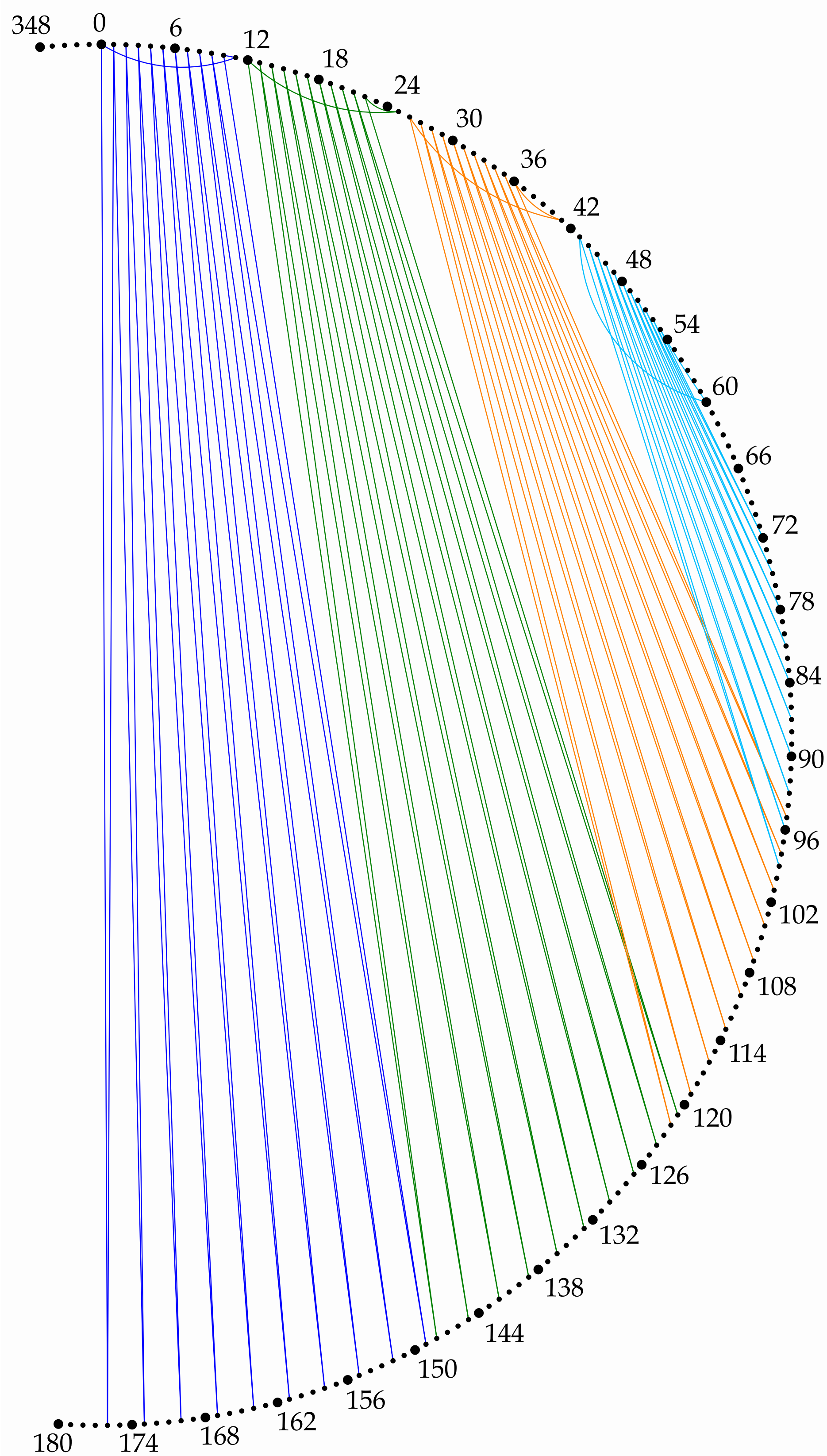



Figure 5.31: The right half of a $\mathcal{C}_{22}^{8}$ base block ($p = 8$, $k = 11$): $\mathfrak{C}_1$, $\mathfrak{C}_3$, $\mathfrak{C}_5$, and $\mathfrak{C}_7$

Table 5.29: The even-index cycles for a $\mathcal{C}_{22}^{8}$ base block, as shown in Figure 5.32

| |
|---|
| **Cycle** $\mathfrak{C}_2$ (red) |
| [174] 173 [170] 169 [166] 165 [162] 161 [158] 157 [154] 153 [150] 149 [146] 145 [142] 141 [138] 137 [2] 12 |
| $\mathfrak{C}_2 = (352, 178, 351, 181, 350, 184, 349, 187, 348, 190, 347, 193, 346, 196, 345, 199, 344, 202, 343, 205, 342, 340)$ |
| **Cycle** $\mathfrak{C}_4$ (fuchsia) |
| [134] 133 [130] 129 [126] 125 [122] 121 [118] 117 [114] 113 [110] 109 [106] 105 [102] 101 [98] 97 [4] 14 |
| $\mathfrak{C}_4 = (338, 204, 337, 207, 336, 210, 335, 213, 334, 216, 333, 219, 332, 222, 331, 225, 330, 228, 329, 231, 328, 324)$ |
| **Cycle** $\mathfrak{C}_6$ (plum) |
| [94] 93 [90] 89 [86] 85 [82] 81 [78] 77 [74] 73 [70] 69 [66] 65 [62] 61 [58] 57 [6] 16 |
| $\mathfrak{C}_6 = (321, 227, 320, 230, 319, 233, 318, 236, 317, 239, 316, 242, 315, 245, 314, 248, 313, 251, 312, 254, 311, 305)$ |
| **Cycle** $\mathfrak{C}_8$ (lime) |
| [54] 53 [50] 49 [46] 45 [42] 41 [38] 37 [34] 33 [30] 29 [26] 25 [22] 21 [10] 9 [8] 18 |
| $\mathfrak{C}_8 = (301, 247, 300, 250, 299, 253, 298, 256, 297, 259, 296, 262, 295, 265, 294, 268, 293, 271, 292, 282, 291, 283)$ |

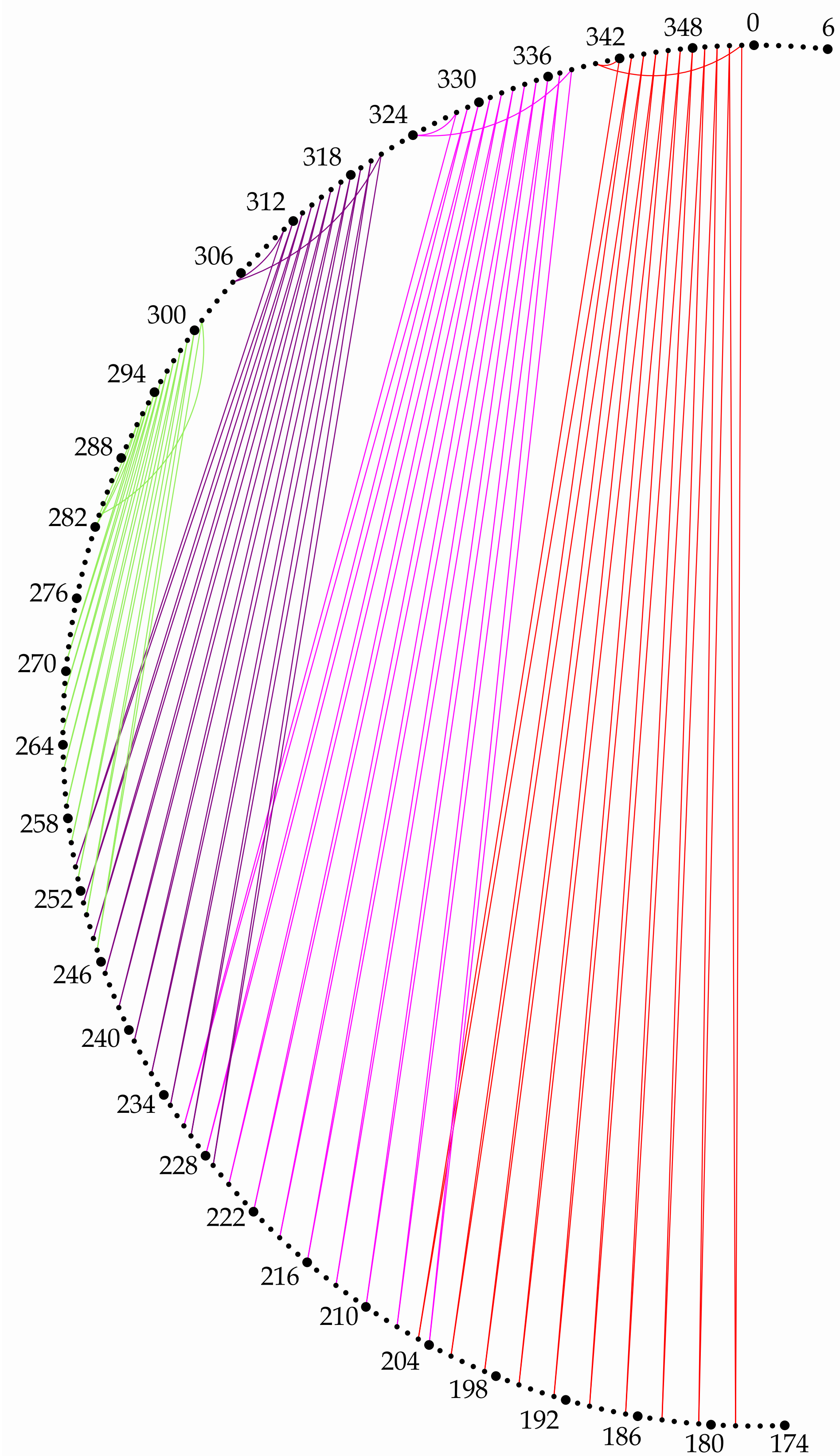



Figure 5.32: The left half of a $\mathcal{C}_{22}^{8}$ base block ($p = 8$, $k = 11$): $\mathfrak{C}_2$, $\mathfrak{C}_4$, $\mathfrak{C}_6$, and $\mathfrak{C}_8$

### 5.3.2 Our Construction for Odd $k$ and Even $p$, Variation II

The second variation of our construction is similar to the first in some respects. We continue to use the parameters $q$, $a$, and $b$ from the previous construction, which are chosen so that: $p = 2q$; if $k \equiv 3 \pmod 4$, then $k = 4a+3$; and if $k \equiv 1 \pmod 4$, then $k = 4b+1$. We again partition the set of differences into $p$ subsets $\mathcal{S}_1, \ldots, \mathcal{S}_p$ of size $2k$ and use each subset $\mathcal{S}_r$ to form a cycle $\mathfrak{C}_r$. We describe the construction in two cases, according to whether $k$ is congruent to one or congruent to three modulo four. We emphasize that this construction applies to all even values of $p$.

If $k \equiv 3 \pmod 4$, we define, for all $z \in [\![1, q]\!]$,

$$\begin{aligned}
\mathcal{S}_{2z-1} = &\Big\{(q-z)4k+2,\ (q-z)4k+k+1\Big\} \\
&\cup \Big\{(q+1-z)4k+4-4i \;\Big|\; i \in [\![1,\ 3a+2]\!]\Big\} \\
&\cup \Big\{(q+1-z)4k+3-4i \;\Big|\; i \in [\![1,\ 3a+2]\!]\Big\} \\
&\quad \cup \Big\{(q+1-z)4k+2-4i \;\Big|\; i \in [\![3a+3,\ k-1]\!]\Big\} \\
&\quad \cup \Big\{(q+1-z)4k+1-4i \;\Big|\; i \in [\![3a+3,\ k-1]\!]\Big\}
\end{aligned} \tag{5.100}$$

and

$$\begin{aligned}
\mathcal{S}_{2z} = &\Big\{(q-z)4k+1,\ (q-z)4k+k\Big\} \\
&\cup \Big\{(q+1-z)4k+2-4i \;\Big|\; i \in [\![1,\ 3a+2]\!]\Big\} \\
&\cup \Big\{(q+1-z)4k+1-4i \;\Big|\; i \in [\![1,\ 3a+2]\!]\Big\} \\
&\quad \cup \Big\{(q+1-z)4k-4i \;\Big|\; i \in [\![3a+3,\ k-1]\!]\Big\} \\
&\quad \cup \Big\{(q+1-z)4k-1-4i \;\Big|\; i \in [\![3a+3,\ k-1]\!]\Big\}\ .
\end{aligned} \tag{5.101}$$

We use the differences in set $\mathcal{S}_{2z-1}$ in the pattern $\left\{\, d_{2i-1} \;\; [d_{2i}] \,\right\}_{i=1}^{k}$, where

$$d_{2i-1} = \begin{cases} (q+1-z)4k+4-4i\,, & \text{if } 1 \le i \le 3a+2\,,\\ (q+1-z)4k+2-4i\,, & \text{if } 3a+3 \le i \le k-1\,,\\ (q-z)4k+2\,, & \text{if } i = k\,, \end{cases} \tag{5.102}$$

and

$$d_{2i} = \begin{cases} (q+1-z)4k+3-4i\,, & \text{if } 1 \le i \le 3a+2\,,\\ (q+1-z)4k+1-4i\,, & \text{if } 3a+3 \le i \le k-1\,,\\ (q-z)4k+k+1\,, & \text{if } i = k\,. \end{cases} \tag{5.103}$$

We form cycle $\mathfrak{C}_{2z-1}$ from set $\mathcal{S}_{2z-1}$ as follows. For each integer $j \in [\![1, 2k]\!]$, we denote the $j$th term of cycle $\mathfrak{C}_{2z-1}$ by $x_j$.

If $j = 2i-1$ is odd, then

$$x_j = x_{2i-1} = (z-1)k+i-1\,. \tag{5.104}$$

If $j = 2i$ is even, then

$$x_j = x_{2i} = \begin{cases} \big(2p-3z+3\big)k+3-3i\,, & \text{if } 1 \le i \le 3a+2\,,\\ \big(2p-3z+3\big)k+1-3i\,, & \text{if } 3a+3 \le i \le k\,. \end{cases} \tag{5.105}$$

We use the differences in set $\mathcal{S}_{2z}$ in the pattern $\left\{\, [d_{2i-1}] \;\; d_{2i} \,\right\}_{i=1}^{k}$, where

$$d_{2i-1} = \begin{cases} (q+1-z)4k+2-4i\,, & \text{if } 1 \le i \le 3a+2\,,\\ (q+1-z)4k-4i\,, & \text{if } 3a+3 \le i \le k-1\,,\\ (q-z)4k+1\,, & \text{if } i = k\,, \end{cases} \tag{5.106}$$

and

$$d_{2i} = \begin{cases} (q+1-z)4k+1-4i\,, & \text{if } 1 \le i \le 3a+2\,, \\ (q+1-z)4k-1-4i\,, & \text{if } 3a+3 \le i \le k-1\,, \\ (q-z)4k+k\,, & \text{if } i = k\,. \end{cases} \tag{5.107}$$

We form cycle $\mathfrak{C}_{2z}$ from set $\mathcal{S}_{2z}$ as follows. For each integer $j \in [\![1, 2k]\!]$, we denote the $j$th term of cycle $\mathfrak{C}_{2z}$ by $x_j$.

If $j = 2i-1$ is odd, then

$$x_j = x_{2i-1} = \big(4p - z + 1\big)k - i + 1\,. \tag{5.108}$$

If $j = 2i$ is even, then

$$x_j = x_{2i} = \begin{cases} \big(2p+3z-3\big)k - 1 + 3i\,, & \text{if } 1 \le i \le 3a+2\,, \\ \big(2p+3z-3\big)k + 1 + 3i\,, & \text{if } 3a+3 \le i \le k-1\,, \\ \big(2p+3z\big)k\,, & \text{if } i = k\,. \end{cases} \tag{5.109}$$

We form the base block $B$ by defining $B = \biguplus_{r=1}^{p} \mathfrak{C}_r$. This completes the construction for $k \equiv 3 \pmod 4$.

If $k \equiv 1 \pmod 4$, we define, for all $z \in [\![1, q]\!]$,

$$\begin{aligned} \mathcal{S}_{2z-1} = & \Big\{(q-z)4k+2,\ (q-z)4k+k+1\Big\} \\ & \cup \Big\{(q+1-z)4k+4-4i \;\Big|\; i \in [\![1,\ 3b+1]\!]\Big\} \\ & \cup \Big\{(q+1-z)4k+3-4i \;\Big|\; i \in [\![1,\ 3b+1]\!]\Big\} \\ & \quad \cup \Big\{(q+1-z)4k+2-4i \;\Big|\; i \in [\![3b+2,\ k-1]\!]\Big\} \\ & \quad \cup \Big\{(q+1-z)4k+1-4i \;\Big|\; i \in [\![3b+2,\ k-1]\!]\Big\} \end{aligned} \tag{5.110}$$

and

$$\begin{aligned}\mathcal{S}_{2z} = &\Big\{(q-z)4k+1,\ (q-z)4k+k\Big\}\\ &\cup\Big\{(q+1-z)4k+2-4i\ \Big|\ i\in[\![1,\ 3b]\!]\Big\}\\ &\cup\Big\{(q+1-z)4k+1-4i\ \Big|\ i\in[\![1,\ 3b]\!]\Big\}\\ &\cup\Big\{(q+1-z)4k-4i\ \Big|\ i\in[\![3b+1,\ k-1]\!]\Big\}\\ &\cup\Big\{(q+1-z)4k-1-4i\ \Big|\ i\in[\![3b+1,\ k-1]\!]\Big\}\ . \end{aligned} \tag{5.111}$$

We use the differences in set $\mathcal{S}_{2z-1}$ in the pattern $\Big\{\ d_{2i-1}\ \ [d_{2i}]\ \Big\}_{i=1}^{k}$, where

$$d_{2i-1} = \begin{cases} (q+1-z)4k+4-4i\,, & \text{if } 1\le i\le 3b+1\,,\\ (q+1-z)4k+2-4i\,, & \text{if } 3b+2\le i\le k-1\,,\\ (q-z)4k+2\,, & \text{if } i=k\,, \end{cases} \tag{5.112}$$

and

$$d_{2i} = \begin{cases} (q+1-z)4k+3-4i\,, & \text{if } 1\le i\le 3b+1\,,\\ (q+1-z)4k+1-4i\,, & \text{if } 3b+2\le i\le k-1\,,\\ (q-z)4k+k+1\,, & \text{if } i=k\,. \end{cases} \tag{5.113}$$

We form cycle $\mathfrak{C}_{2z-1}$ from set $\mathcal{S}_{2z-1}$ as follows. For each integer $j\in[\![1,2k]\!]$, we denote the $j$th term of cycle $\mathfrak{C}_{2z-1}$ by $x_j$.

If $j=2i-1$ is odd, then

$$x_j = x_{2i-1} = (z-1)k+i-1\,. \tag{5.114}$$

If $j = 2i$ is even, then

$$x_j = x_{2i} = \begin{cases} \big(2p - 3z + 3\big)k + 3 - 3i\,, & \text{if } 1 \le i \le 3b + 1\,, \\ \big(2p - 3z + 3\big)k + 1 - 3i\,, & \text{if } 3b + 2 \le i \le k\,. \end{cases} \tag{5.115}$$

We use the differences in set $\mathcal{S}_{2z}$ in the pattern $\Big\{ \,[d_{2i-1}]\;\; d_{2i}\, \Big\}_{i=1}^{k}$, where

$$d_{2i-1} = \begin{cases} (q + 1 - z)4k + 2 - 4i\,, & \text{if } 1 \le i \le 3b\,, \\ (q + 1 - z)4k - 4i\,, & \text{if } 3b + 1 \le i \le k - 1\,, \\ (q - z)4k + 1\,, & \text{if } i = k\,, \end{cases} \tag{5.116}$$

and

$$d_{2i} = \begin{cases} (q + 1 - z)4k + 1 - 4i\,, & \text{if } 1 \le i \le 3b\,, \\ (q + 1 - z)4k - 1 - 4i\,, & \text{if } 3b + 1 \le i \le k - 1\,, \\ (q - z)4k + k\,, & \text{if } i = k\,. \end{cases} \tag{5.117}$$

We form cycle $\mathfrak{C}_{2z}$ from set $\mathcal{S}_{2z}$ as follows. For each integer $j \in [\![1, 2k]\!]$, we denote the $j$th term of cycle $\mathfrak{C}_{2z}$ by $x_j$.

If $j = 2i - 1$ is odd, then

$$x_j = x_{2i-1} = \big(4p - z + 1\big)k - i + 1\,. \tag{5.118}$$

If $j = 2i$ is even, then

$$x_j = x_{2i} = \begin{cases} \big(2p + 3z - 3\big)k - 1 + 3i\,, & \text{if } 1 \le i \le 3b\,, \\ \big(2p + 3z - 3\big)k + 1 + 3i\,, & \text{if } 3b + 1 \le i \le k - 1\,, \\ \big(2p + 3z\big)k\,, & \text{if } i = k\,. \end{cases} \tag{5.119}$$

We form the base block $B$ by defining $B = \biguplus_{r=1}^{p} \mathfrak{C}_r$ ; this completes the construction for $k \equiv 1 \pmod 4$.

**Theorem 5.24.** *The subgraph $B$ of $K_{4kp+1}$ generated by the above construction is a base block for a purely cyclic $\mathcal{C}_{2k}^p$-design on $K_{4kp+1}$, and hence exhibits a $\rho$-labeling of $\mathcal{C}_{2k}^p$. There is therefore a $\mathcal{C}_{2k}^p$-design on $K_{4kp+1}$ for each pair of integers $p$ and $k$ such that $p$ is even and at least two and $k$ is odd and at least three.*

*Proof.* It is clear from the construction that each difference in $\mathcal{D}_{4kp+1}$ occurs on exactly one edge in the subgraph $B$, and that each cycle in $B$ has length $2k$. It remains to verify that the cycles $\mathfrak{C}_1, \ldots, \mathfrak{C}_p$ in $B$ are pairwise vertex-disjoint.

For each $z \in [\![1, q]\!]$, we define four sets:

$$\begin{aligned} U_{2z-1} &= [\![(z-1)k,\ zk-1]\!]\,, \\ V_{2z-1} &= [\![(2p-3z)k+1,\ (2p-3z+3)k]\!]\,, \\ U_{2z} &= [\![(4p-z)k+1,\ (4p-z+1)k]\!]\,, \text{ and} \\ V_{2z} &= [\![(2p+3z-3)k+1,\ (2p+3z)k]\!]\,. \end{aligned}$$

We observe that the sets $U_{2z-1}$ partition the set $\mathcal{U}_1 = [\![0,\ qk-1]\!]$; the sets $V_{2z-1}$ partition the set $\mathcal{V}_1 = [\![qk+1,\ 4qk]\!]$; the sets $V_{2z}$ partition the set $\mathcal{V}_2 = [\![4qk+1,\ 7qk]\!]$; and the sets $U_{2z}$ partition the set $\mathcal{U}_2 = [\![7qk+1,\ 8qk]\!]$. The sets $\mathcal{U}_1$, $\mathcal{V}_1$, $\mathcal{V}_2$, and $\mathcal{U}_2$ are clearly disjoint; they fail to partition $[\![0, 8qk]\!]$ only because none of them contains the element $qk$. Thus, for each $r \in [\![1, p]\!]$, the sets $U_r$ and $V_r$ are pairwise disjoint, so we define $W_r = U_r \cup V_r$; note that the sets $W_r$ are pairwise disjoint by construction. Furthermore, $V(\mathfrak{C}_r) \subseteq W_r$ for each $r \in [\![1, p]\!]$. Hence the cycles $\mathfrak{C}_r$ are pairwise vertex-disjoint, as desired. □

We now offer several examples of this construction: three examples each for $k = 3$, $k = 5$, and $k = 7$, and one example each for $k = 9$, $k = 11$, $k = 13$, and $k = 19$.

**Example 5.25.** We consider $\mathcal{C}_6^4$; for this graph, we have $p = 4$ and $k = 3$, so $4kp + 1 = 49$. The cycles in the base block and the difference patterns that generate them are listed in Table 5.30; the base block itself is shown in Figure 5.33. ■

**Example 5.26.** We consider $\mathcal{C}_6^8$; for this graph, we have $p = 8$ and $k = 3$, so $4kp + 1 = 97$. The cycles in the base block and the difference patterns that generate them are listed in Table 5.31; the base block itself is shown in Figure 5.34. ■

**Example 5.27.** We consider $\mathcal{C}_6^{10}$; for this graph, we have $p = 10$ and $k = 3$, so $4kp+1 = 121$. The cycles in the base block and the difference patterns that generate them are listed in Table 5.32; the base block itself is shown in Figure 5.35. ■

**Example 5.28.** We consider $\mathcal{C}_{10}^2$; for this graph, we have $p = 2$ and $k = 5$, so $4kp + 1 = 41$. The cycles in the base block and the difference patterns that generate them are listed in Table 5.33; the base block itself is shown in Figure 5.36. ■

**Example 5.29.** We consider $\mathcal{C}_{10}^6$; for this graph, we have $p = 6$ and $k = 5$, so $4kp+1 = 121$. The cycles in the base block and the difference patterns that generate them are listed in Table 5.34; the base block itself is shown in Figure 5.37. ■

**Example 5.30.** We consider $\mathcal{C}_{10}^8$; for this graph, we have $p = 8$ and $k = 5$, so $4kp+1 = 161$. The cycles in the base block and the difference patterns that generate them are listed in Table 5.35; the base block itself is shown in Figure 5.38. ■

**Example 5.31.** We consider $\mathcal{C}_{14}^4$; for this graph, we have $p = 4$ and $k = 7$, so $4kp+1 = 113$. The cycles in the base block and the difference patterns that generate them are listed in Table 5.36; the base block itself is shown in Figure 5.39. ■

**Example 5.32.** We consider $\mathcal{C}_{14}^6$; for this graph, we have $p = 6$ and $k = 7$, so $4kp+1 = 169$. The cycles in the base block and the difference patterns that generate them are listed in Table 5.37; the base block itself is shown in Figure 5.40. ■

**Example 5.33.** We consider $\mathcal{C}_{14}^{8}$; for this graph, we have $p = 8$ and $k = 7$, so $4kp+1 = 225$. The cycles in the base block and the difference patterns that generate them are listed in Table 5.38; the base block itself is shown in Figure 5.41. ■

**Example 5.34.** We consider $\mathcal{C}_{18}^{6}$; for this graph, we have $p = 6$ and $k = 9$, so $4kp+1 = 217$. The cycles in the base block and the difference patterns that generate them are listed in Table 5.39; the base block itself is shown in Figure 5.42. ■

**Example 5.35.** We consider $\mathcal{C}_{22}^{6}$; for this graph, we have $p = 6$ and $k = 11$, so $4kp+1 = 265$. The cycles in the base block and the difference patterns that generate them are listed in Table 5.40; the base block itself is shown in Figure 5.43. ■

**Example 5.36.** We consider $\mathcal{C}_{26}^{4}$; for this graph, we have $p = 4$ and $k = 13$, so $4kp+1 = 209$. The cycles in the base block and the difference patterns that generate them are listed in Table 5.41; the base block itself is shown in Figure 5.44. ■

**Example 5.37.** We consider $\mathcal{C}_{38}^{12}$; for this graph, we have $p = 12$ and $k = 19$, so $4kp+1 = 913$. The cycles in the base block and the difference patterns that generate them are listed in Tables 5.42, 5.43, and 5.44. Due to the large size of this example, we do not attempt to show the base block as a subgraph of $K_{913}$. ■

Table 5.30: Cycle list for the $\mathcal{C}_6^4$ base block in Figure 5.33

| **Cycle $\mathfrak{C}_1$** (cobalt) | **Cycle $\mathfrak{C}_3$** (lilac) |
|---|---|
| 24 [23] 20 [19] 14 [16] | 12 [11] 8 [7] 2 [4] |
| $\mathfrak{C}_1 = (0, 24, 1, 21, 2, 16)$ | $\mathfrak{C}_3 = (3, 15, 4, 12, 5, 7)$ |
| **Cycle $\mathfrak{C}_2$** (sky) | **Cycle $\mathfrak{C}_4$** (plum) |
| [22] 21 [18] 17 [13] 15 | [10] 9 [6] 5 [1] 3 |
| $\mathfrak{C}_2 = (48, 26, 47, 29, 46, 33)$ | $\mathfrak{C}_4 = (45, 35, 44, 38, 43, 42)$ |

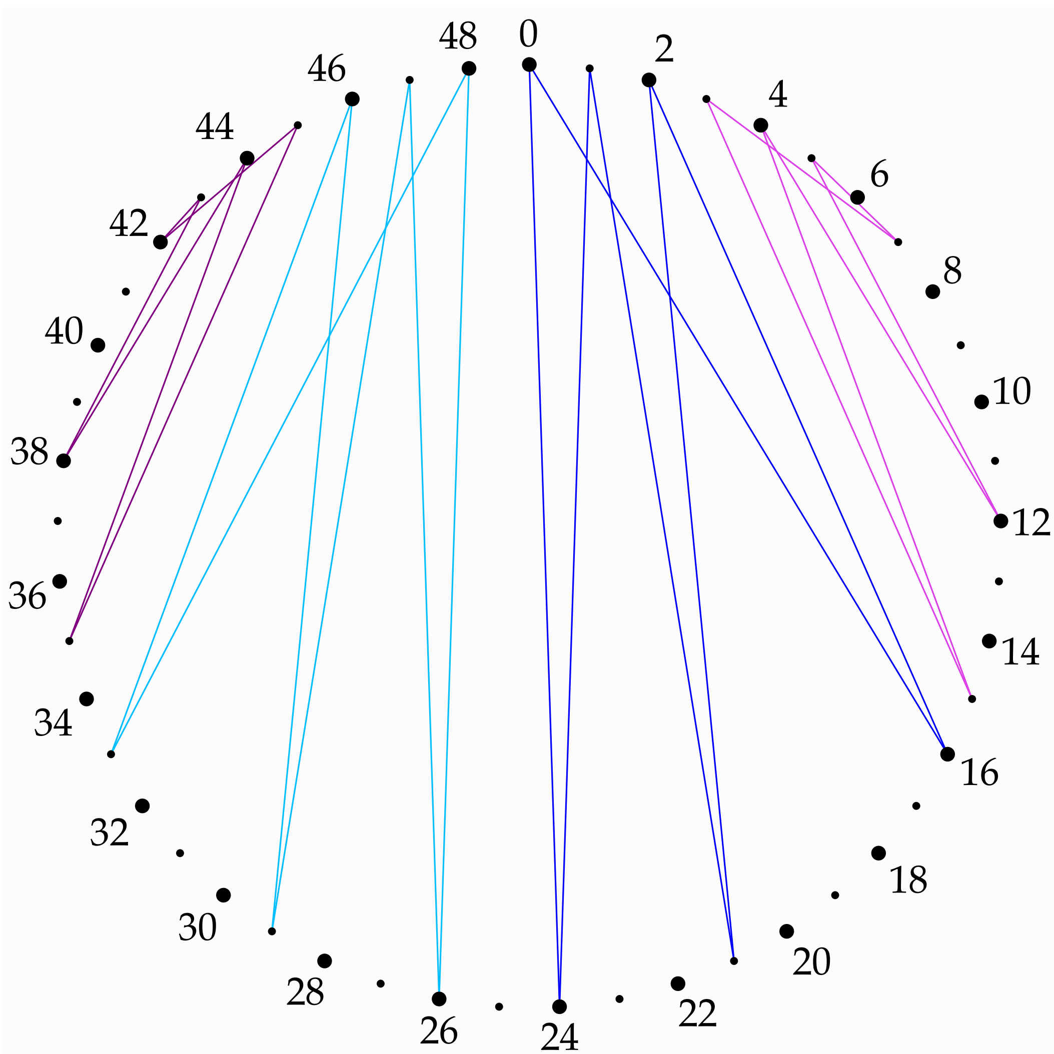



Figure 5.33: A $\mathcal{C}_6^4$ base block ($p = 4$, $k = 3$)

Table 5.31: Cycle list for the $\mathcal{C}_6^8$ base block in Figure 5.34

| **Cycle** $\mathfrak{C}_1$ (lime) | **Cycle** $\mathfrak{C}_5$ (cobalt) |
|---|---|
| 48 [47] 44 [43] 38 [40] | 24 [23] 20 [19] 14 [16] |
| $\mathfrak{C}_1 = \big(0, 48, 1, 45, 2, 40\big)$ | $\mathfrak{C}_5 = \big(6, 30, 7, 27, 8, 22\big)$ |
| **Cycle** $\mathfrak{C}_2$ (forest) | **Cycle** $\mathfrak{C}_6$ (sky) |
| [46] 45 [42] 41 [37] 39 | [22] 21 [18] 17 [13] 15 |
| $\mathfrak{C}_2 = \big(96, 50, 95, 53, 94, 57\big)$ | $\mathfrak{C}_6 = \big(90, 68, 89, 71, 88, 75\big)$ |
| **Cycle** $\mathfrak{C}_3$ (pink) | **Cycle** $\mathfrak{C}_7$ (lilac) |
| 36 [35] 32 [31] 26 [28] | 12 [11] 8 [7] 2 [4] |
| $\mathfrak{C}_3 = \big(3, 39, 4, 36, 5, 31\big)$ | $\mathfrak{C}_7 = \big(9, 21, 10, 18, 11, 13\big)$ |
| **Cycle** $\mathfrak{C}_4$ (orange) | **Cycle** $\mathfrak{C}_8$ (plum) |
| [34] 33 [30] 29 [25] 27 | [10] 9 [6] 5 [1] 3 |
| $\mathfrak{C}_4 = \big(93, 59, 92, 62, 91, 66\big)$ | $\mathfrak{C}_8 = \big(87, 77, 86, 80, 85, 84\big)$ |

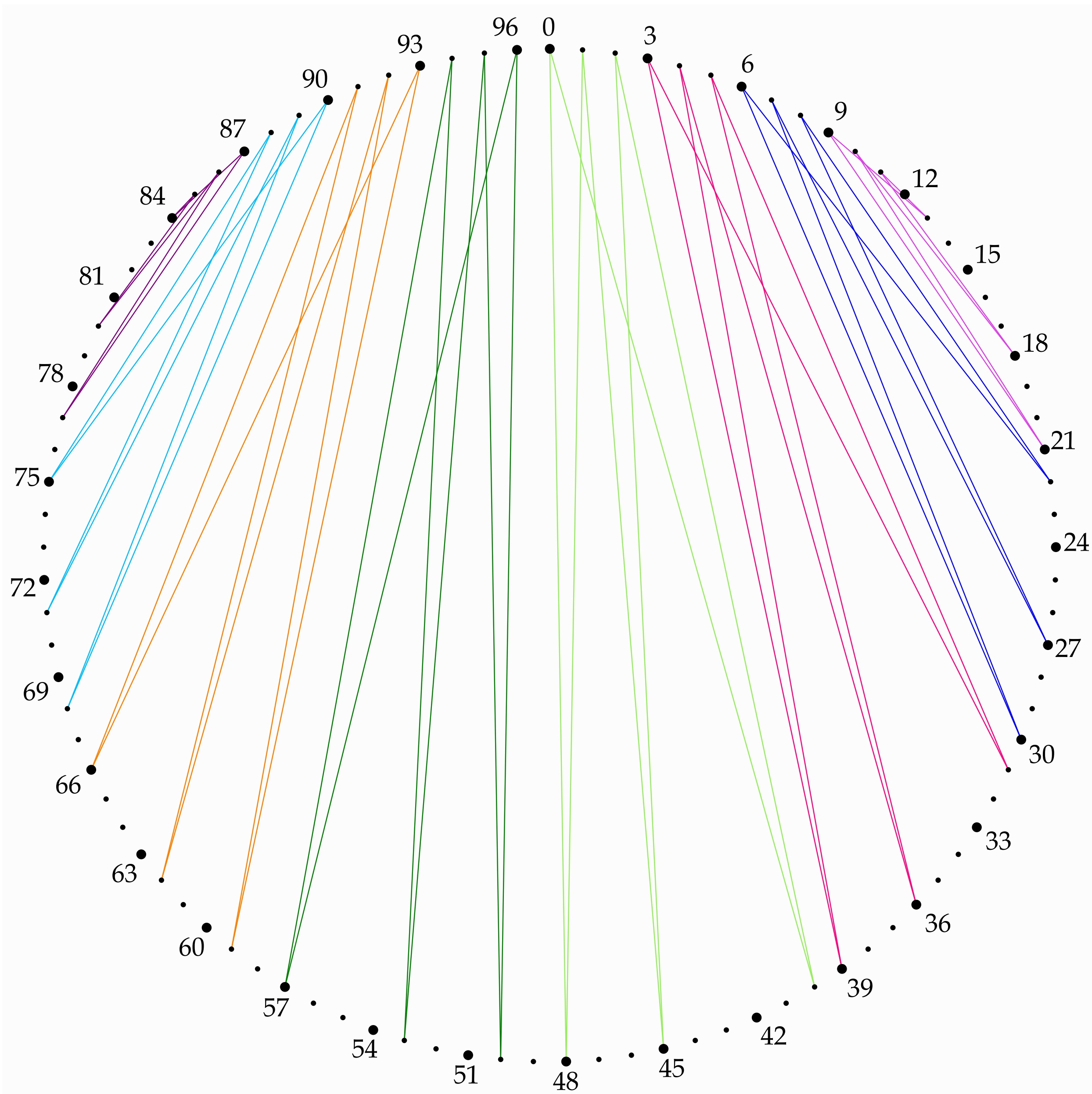



Figure 5.34: A $\mathcal{C}_6^8$ base block ($p = 8$, $k = 3$)

Table 5.32: Cycle list for the $\mathcal{C}_6^{10}$ base block in Figure 5.35

| | |
|---|---|
| **Cycle** $\mathfrak{C}_1$ (violet) | **Cycle** $\mathfrak{C}_6$ (orange) |
| 60 [59] 56 [55] 50 [52] | [34] 33 [30] 29 [25] 27 |
| $\mathfrak{C}_1 = (0, 60, 1, 57, 2, 52)$ | $\mathfrak{C}_6 = (114, 80, 113, 83, 112, 87)$ |
| **Cycle** $\mathfrak{C}_2$ (fuchsia) | **Cycle** $\mathfrak{C}_7$ (cobalt) |
| [58] 57 [54] 53 [49] 51 | 24 [23] 20 [19] 14 [16] |
| $\mathfrak{C}_2 = (120, 62, 119, 65, 118, 69)$ | $\mathfrak{C}_7 = (9, 33, 10, 30, 11, 25)$ |
| **Cycle** $\mathfrak{C}_3$ (lime) | **Cycle** $\mathfrak{C}_8$ (sky) |
| 48 [47] 44 [43] 38 [40] | [22] 21 [18] 17 [13] 15 |
| $\mathfrak{C}_3 = (3, 51, 4, 48, 5, 43)$ | $\mathfrak{C}_8 = (111, 89, 110, 92, 109, 96)$ |
| **Cycle** $\mathfrak{C}_4$ (forest) | **Cycle** $\mathfrak{C}_9$ (lilac) |
| [46] 45 [42] 41 [37] 39 | 12 [11] 8 [7] 2 [4] |
| $\mathfrak{C}_4 = (117, 71, 116, 74, 115, 78)$ | $\mathfrak{C}_9 = (12, 24, 13, 21, 14, 16)$ |
| **Cycle** $\mathfrak{C}_5$ (pink) | **Cycle** $\mathfrak{C}_{10}$ (plum) |
| 36 [35] 32 [31] 26 [28] | [10] 9 [6] 5 [1] 3 |
| $\mathfrak{C}_5 = (6, 42, 7, 39, 8, 34)$ | $\mathfrak{C}_{10} = (108, 98, 107, 101, 106, 105)$ |

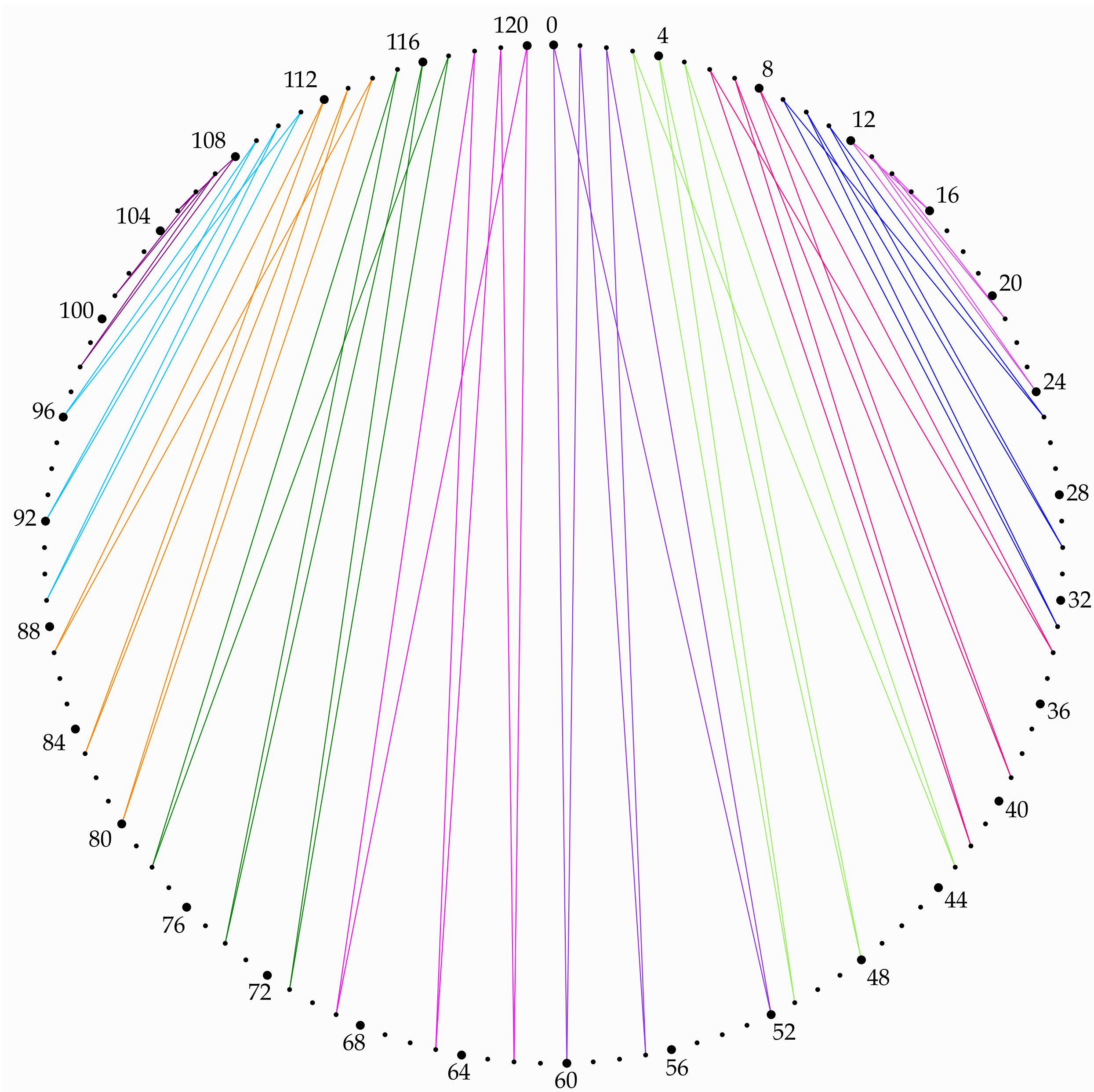



Figure 5.35: A $\mathcal{C}_6^{10}$ base block ($p = 10$, $k = 3$)

Table 5.33: Cycle list for the $\mathcal{C}_{10}^2$ base block in Figure 5.36

| **Cycle $\mathfrak{C}_1$** | | (lilac) | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 20 | [19] | 16 | [15] | 12 | [11] | 8 | [7] | 2 | [6] |
| $\mathfrak{C}_1 = \big(0, 20, 1, 17, 2, 14, 3, 11, 4, 6\big)$ | | | | | | | | | |
| **Cycle $\mathfrak{C}_2$** | | (plum) | | | | | | | |
| [18] | 17 | [14] | 13 | [10] | 9 | [4] | 3 | [1] | 5 |
| $\mathfrak{C}_2 = \big(40, 22, 39, 25, 38, 28, 37, 33, 36, 35\big)$ | | | | | | | | | |

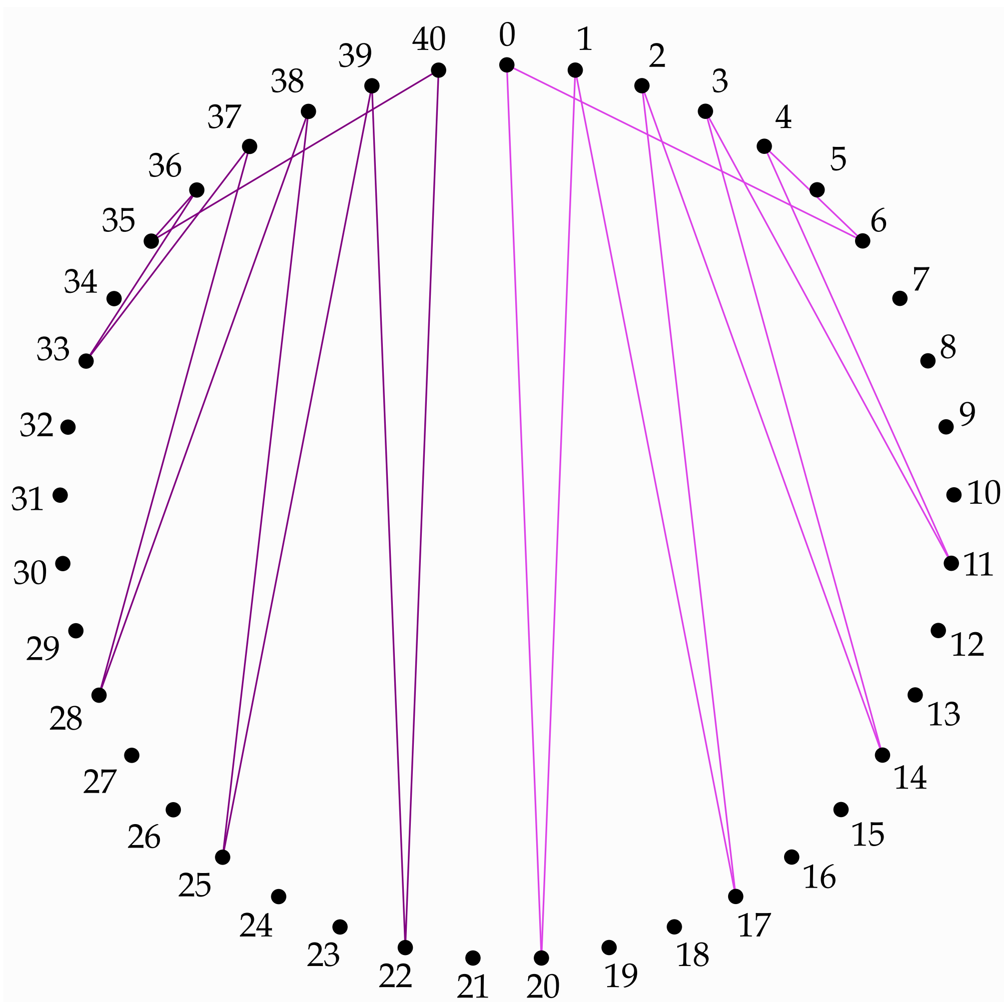



Figure 5.36: A $\mathcal{C}_{10}^2$ base block ($p = 2$, $k = 5$)

Table 5.34: Cycle list for the $\mathcal{C}_{10}^{6}$ base block in Figure 5.37

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **Cycle $\mathfrak{C}_1$** | (pink) | | | | | | | | |
| 60 | [59] | 56 | [55] | 52 | [51] | 48 | [47] | 42 | [46] |
| $\mathfrak{C}_1 = (0, 60, 1, 57, 2, 54, 3, 51, 4, 46)$ | | | | | | | | | |
| **Cycle $\mathfrak{C}_2$** | (orange) | | | | | | | | |
| [58] | 57 | [54] | 53 | [50] | 49 | [44] | 43 | [41] | 45 |
| $\mathfrak{C}_2 = (120, 62, 119, 65, 118, 68, 117, 73, 116, 75)$ | | | | | | | | | |
| **Cycle $\mathfrak{C}_3$** | (cobalt) | | | | | | | | |
| 40 | [39] | 36 | [35] | 32 | [31] | 28 | [27] | 22 | [26] |
| $\mathfrak{C}_3 = (5, 45, 6, 42, 7, 39, 8, 36, 9, 31)$ | | | | | | | | | |
| **Cycle $\mathfrak{C}_4$** | (sky) | | | | | | | | |
| [38] | 37 | [34] | 33 | [30] | 29 | [24] | 23 | [21] | 25 |
| $\mathfrak{C}_4 = (115, 77, 114, 80, 113, 83, 112, 88, 111, 90)$ | | | | | | | | | |
| **Cycle $\mathfrak{C}_5$** | (lilac) | | | | | | | | |
| 20 | [19] | 16 | [15] | 12 | [11] | 8 | [7] | 2 | [6] |
| $\mathfrak{C}_5 = (10, 30, 11, 27, 12, 24, 13, 21, 14, 16)$ | | | | | | | | | |
| **Cycle $\mathfrak{C}_6$** | (plum) | | | | | | | | |
| [18] | 17 | [14] | 13 | [10] | 9 | [4] | 3 | [1] | 5 |
| $\mathfrak{C}_6 = (110, 92, 109, 95, 108, 98, 107, 103, 106, 105)$ | | | | | | | | | |

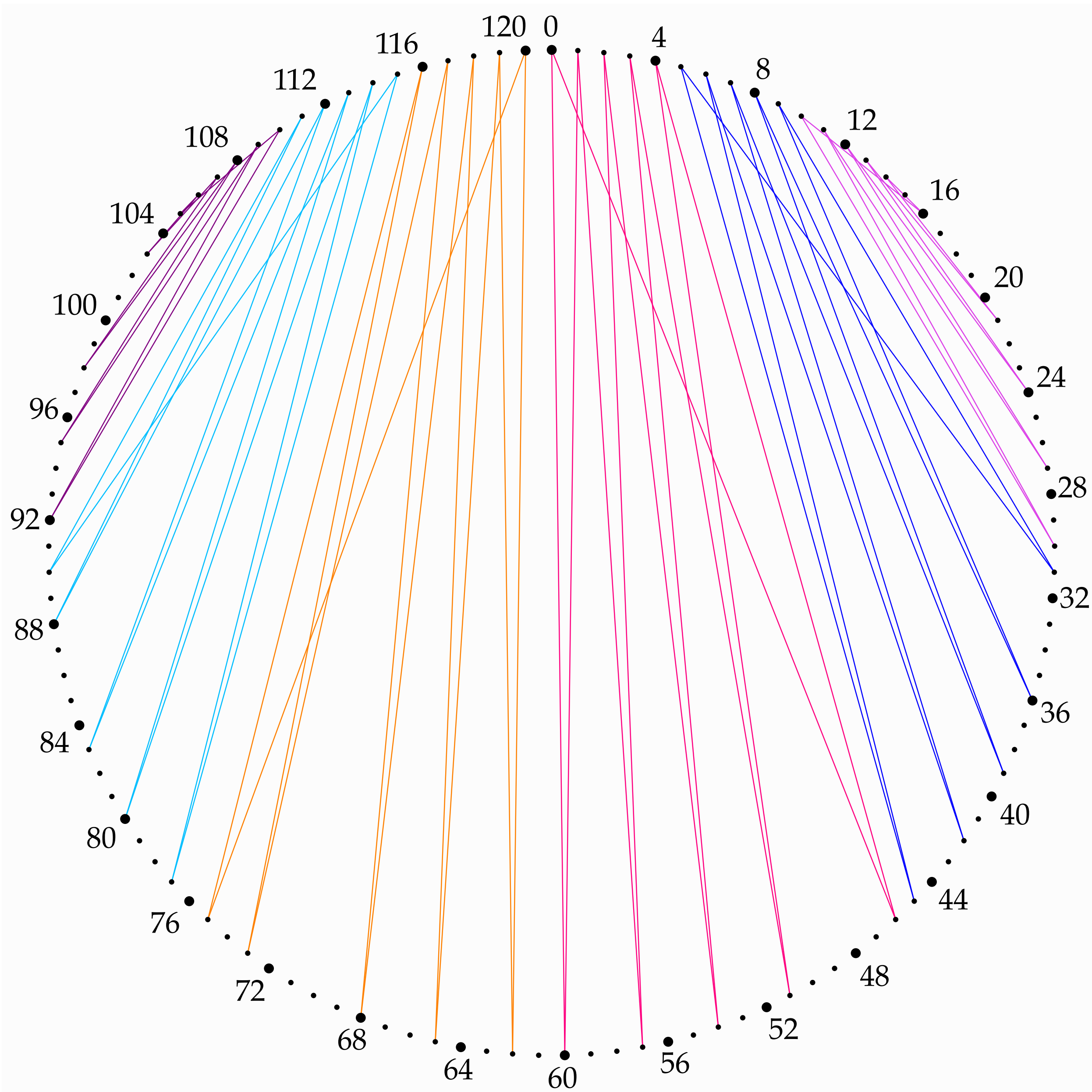



Figure 5.37: A $\mathcal{C}_{10}^{6}$ base block ($p = 6$, $k = 5$)

Table 5.35: Cycle list for the $\mathcal{C}^{8}_{10}$ base block in Figure 5.38

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **Cycle $\mathfrak{C}_1$** | (lime) | | | | | | | | |
| 80 | [79] | 76 | [75] | 72 | [71] | 68 | [67] | 62 | [66] |
| $\mathfrak{C}_1 = \big(0, 80, 1, 77, 2, 74, 3, 71, 4, 66\big)$ | | | | | | | | | |
| **Cycle $\mathfrak{C}_2$** | (forest) | | | | | | | | |
| [78] | 77 | [74] | 73 | [70] | 69 | [64] | 63 | [61] | 65 |
| $\mathfrak{C}_2 = \big(160, 82, 159, 85, 158, 88, 157, 93, 156, 95\big)$ | | | | | | | | | |
| **Cycle $\mathfrak{C}_3$** | (pink) | | | | | | | | |
| 60 | [59] | 56 | [55] | 52 | [51] | 48 | [47] | 42 | [46] |
| $\mathfrak{C}_3 = \big(5, 65, 6, 62, 7, 59, 8, 56, 9, 51\big)$ | | | | | | | | | |
| **Cycle $\mathfrak{C}_4$** | (orange) | | | | | | | | |
| [58] | 57 | [54] | 53 | [50] | 49 | [44] | 43 | [41] | 45 |
| $\mathfrak{C}_4 = \big(155, 97, 154, 100, 153, 103, 152, 108, 151, 110\big)$ | | | | | | | | | |
| **Cycle $\mathfrak{C}_5$** | (cobalt) | | | | | | | | |
| 40 | [39] | 36 | [35] | 32 | [31] | 28 | [27] | 22 | [26] |
| $\mathfrak{C}_5 = \big(10, 50, 11, 47, 12, 44, 13, 41, 14, 36\big)$ | | | | | | | | | |
| **Cycle $\mathfrak{C}_6$** | (sky) | | | | | | | | |
| [38] | 37 | [34] | 33 | [30] | 29 | [24] | 23 | [21] | 25 |
| $\mathfrak{C}_6 = \big(150, 112, 149, 115, 148, 118, 147, 123, 146, 125\big)$ | | | | | | | | | |
| **Cycle $\mathfrak{C}_7$** | (lilac) | | | | | | | | |
| 20 | [19] | 16 | [15] | 12 | [11] | 8 | [7] | 2 | [6] |
| $\mathfrak{C}_7 = \big(15, 35, 16, 32, 17, 29, 18, 26, 19, 21\big)$ | | | | | | | | | |
| **Cycle $\mathfrak{C}_8$** | (plum) | | | | | | | | |
| [18] | 17 | [14] | 13 | [10] | 9 | [4] | 3 | [1] | 5 |
| $\mathfrak{C}_8 = \big(145, 127, 144, 130, 143, 133, 142, 138, 141, 140\big)$ | | | | | | | | | |

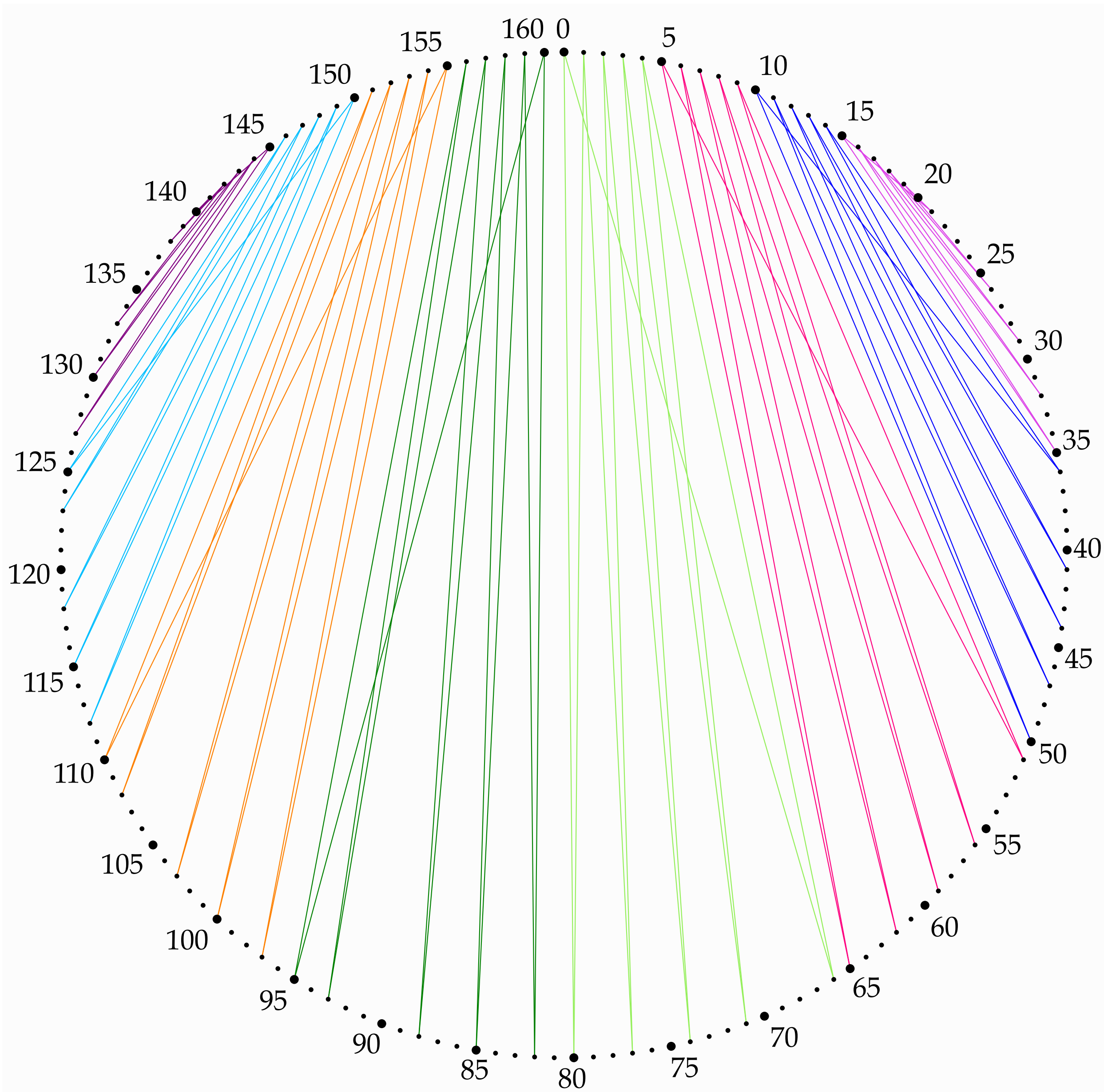



Figure 5.38: A $\mathcal{C}_{10}^{8}$ base block ($p = 8$, $k = 5$)

Table 5.36: Cycle list for the $\mathcal{C}_{14}^{4}$ base block in Figure 5.39

| **Cycle** $\mathfrak{C}_1$ | (cobalt) | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 56 | [55] | 52 | [51] | 48 | [47] | 44 | [43] | 40 | [39] | 34 | [33] | 30 | [36] |
| $\mathfrak{C}_1 = (0, 56, 1, 53, 2, 50, 3, 47, 4, 44, 5, 39, 6, 36)$ | | | | | | | | | | | | | |
| **Cycle** $\mathfrak{C}_2$ | (sky) | | | | | | | | | | | | |
| [54] | 53 | [50] | 49 | [46] | 45 | [42] | 41 | [38] | 37 | [32] | 31 | [29] | 35 |
| $\mathfrak{C}_2 = (112, 58, 111, 61, 110, 64, 109, 67, 108, 70, 107, 75, 106, 77)$ | | | | | | | | | | | | | |
| **Cycle** $\mathfrak{C}_3$ | (lilac) | | | | | | | | | | | | |
| 28 | [27] | 24 | [23] | 20 | [19] | 16 | [15] | 12 | [11] | 6 | [5] | 2 | [8] |
| $\mathfrak{C}_3 = (7, 35, 8, 32, 9, 29, 10, 26, 11, 23, 12, 18, 13, 15)$ | | | | | | | | | | | | | |
| **Cycle** $\mathfrak{C}_4$ | (plum) | | | | | | | | | | | | |
| [26] | 25 | [22] | 21 | [18] | 17 | [14] | 13 | [10] | 9 | [4] | 3 | [1] | 7 |
| $\mathfrak{C}_4 = (105, 79, 104, 82, 103, 85, 102, 88, 101, 91, 100, 96, 99, 98)$ | | | | | | | | | | | | | |

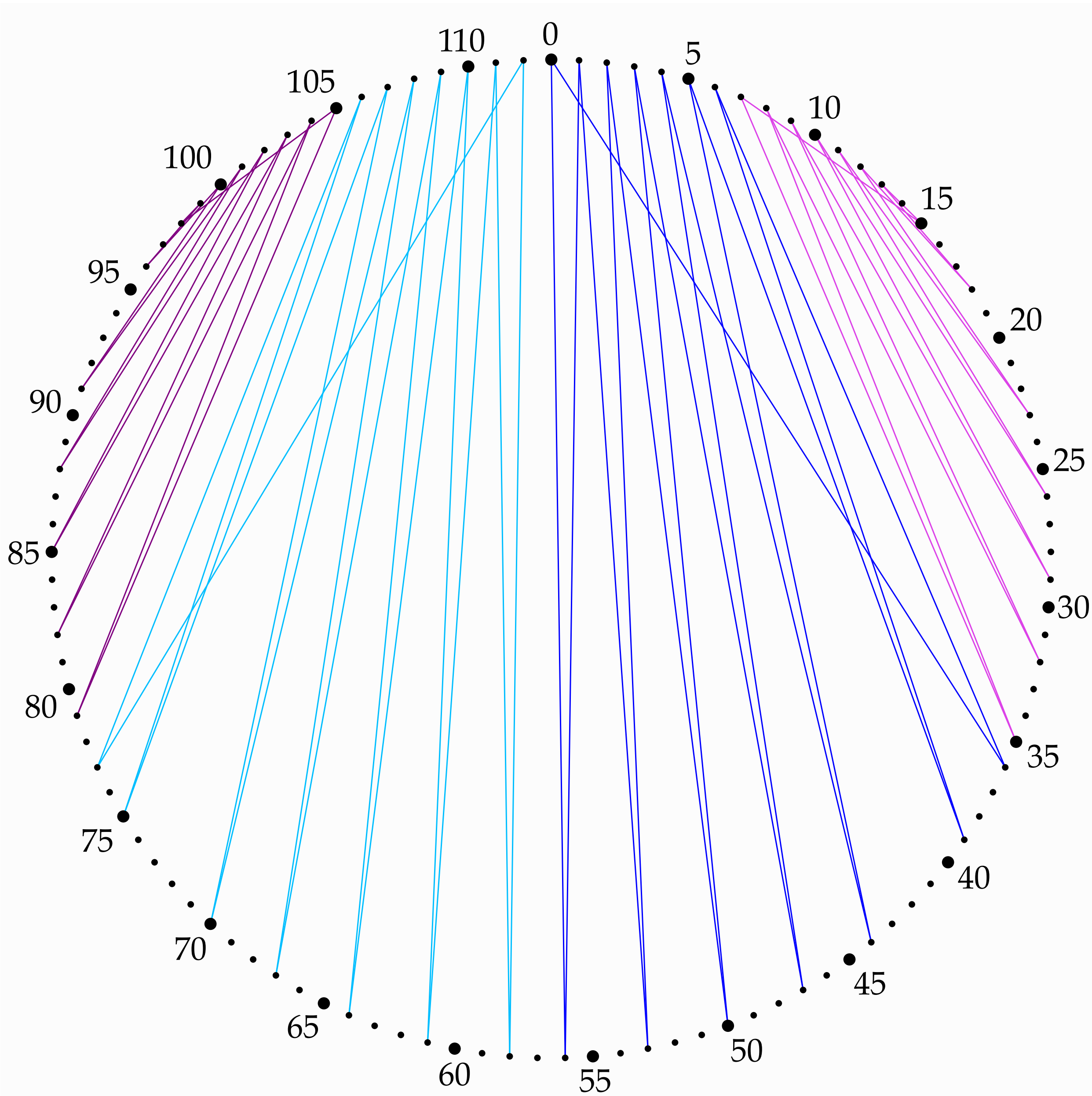


Figure 5.39: A $\mathcal{C}_{14}^4$ base block ($p = 4$, $k = 7$)

Table 5.37: Cycle list for the $\mathcal{C}_{14}^{6}$ base block in Figure 5.40

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Cycle** $\mathfrak{C}_1$ | (pink) | | | | | | | | | | | | |
| 84 | [83] | 80 | [79] | 76 | [75] | 72 | [71] | 68 | [67] | 62 | [61] | 58 | [64] |
| $\mathfrak{C}_1 = (0, 84, 1, 81, 2, 78, 3, 75, 4, 72, 5, 67, 6, 64)$ | | | | | | | | | | | | | |
| **Cycle** $\mathfrak{C}_2$ | (orange) | | | | | | | | | | | | |
| [82] | 81 | [78] | 77 | [74] | 73 | [70] | 69 | [66] | 65 | [60] | 59 | [57] | 63 |
| $\mathfrak{C}_2 = (168, 86, 167, 89, 166, 92, 165, 95, 164, 98, 163, 103, 162, 105)$ | | | | | | | | | | | | | |
| **Cycle** $\mathfrak{C}_3$ | (cobalt) | | | | | | | | | | | | |
| 56 | [55] | 52 | [51] | 48 | [47] | 44 | [43] | 40 | [39] | 34 | [33] | 30 | [36] |
| $\mathfrak{C}_3 = (7, 63, 8, 60, 9, 57, 10, 54, 11, 51, 12, 46, 13, 43)$ | | | | | | | | | | | | | |
| **Cycle** $\mathfrak{C}_4$ | (sky) | | | | | | | | | | | | |
| [54] | 53 | [50] | 49 | [46] | 45 | [42] | 41 | [38] | 37 | [32] | 31 | [29] | 35 |
| $\mathfrak{C}_4 = (161, 107, 160, 110, 159, 113, 158, 116, 157, 119, 156, 124, 155, 126)$ | | | | | | | | | | | | | |
| **Cycle** $\mathfrak{C}_5$ | (lilac) | | | | | | | | | | | | |
| 28 | [27] | 24 | [23] | 20 | [19] | 16 | [15] | 12 | [11] | 6 | [5] | 2 | [8] |
| $\mathfrak{C}_5 = (14, 42, 15, 39, 16, 36, 17, 33, 18, 30, 19, 25, 20, 22)$ | | | | | | | | | | | | | |
| **Cycle** $\mathfrak{C}_6$ | (plum) | | | | | | | | | | | | |
| [26] | 25 | [22] | 21 | [18] | 17 | [14] | 13 | [10] | 9 | [4] | 3 | [1] | 7 |
| $\mathfrak{C}_6 = (154, 128, 153, 131, 152, 134, 151, 137, 150, 140, 149, 145, 148, 147)$ | | | | | | | | | | | | | |

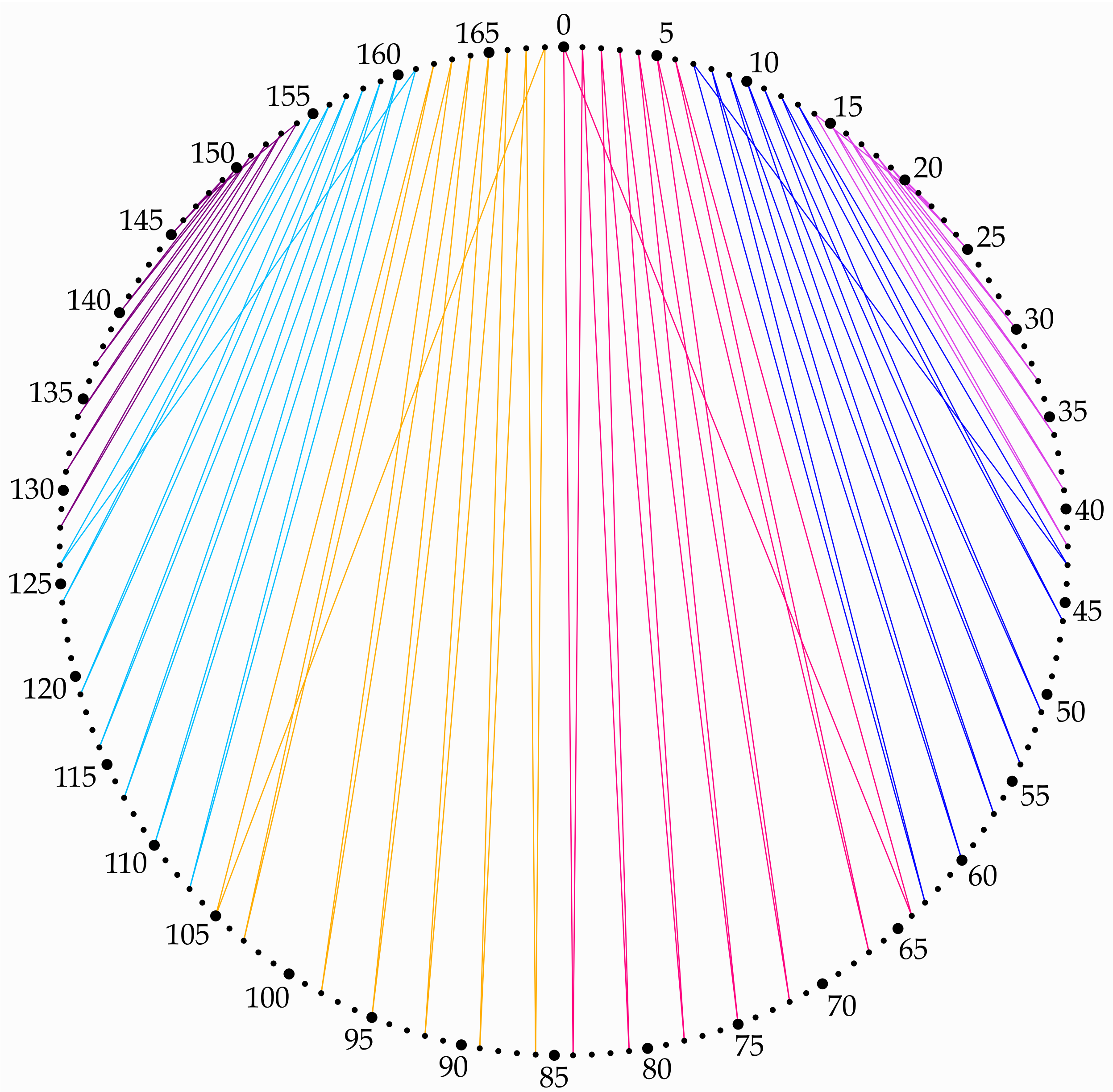



Figure 5.40: A $\mathcal{C}_{14}^{6}$ base block ($p = 6$, $k = 7$)

Table 5.38: Cycle list for the $\mathcal{C}_{14}^{8}$ base block in Figure 5.41

| | |
|---|---|
| **Cycle $\mathfrak{C}_1$** (lime) | |
| 112 [111] 108 [107] 104 [103] 100 [99] 96 [95] 90 [89] 86 [92] | $\mathfrak{C}_1 = (0, 112, 1, 109, 2, 106, 3, 103, 4, 100, 5, 95, 6, 92)$ |
| **Cycle $\mathfrak{C}_2$** (forest) | |
| [110] 109 [106] 105 [102] 101 [98] 97 [94] 93 [88] 87 [85] 91 | $\mathfrak{C}_2 = (224, 114, 223, 117, 222, 120, 221, 123, 220, 126, 219, 131, 218, 133)$ |
| **Cycle $\mathfrak{C}_3$** (pink) | |
| 84 [83] 80 [79] 76 [75] 72 [71] 68 [67] 62 [61] 58 [64] | $\mathfrak{C}_3 = (7, 91, 8, 88, 9, 85, 10, 82, 11, 79, 12, 74, 13, 71)$ |
| **Cycle $\mathfrak{C}_4$** (orange) | |
| [82] 81 [78] 77 [74] 73 [70] 69 [66] 65 [60] 59 [57] 63 | $\mathfrak{C}_4 = (217, 135, 216, 138, 215, 141, 214, 144, 213, 147, 212, 152, 211, 154)$ |
| **Cycle $\mathfrak{C}_5$** (cobalt) | |
| 56 [55] 52 [51] 48 [47] 44 [43] 40 [39] 34 [33] 30 [36] | $\mathfrak{C}_5 = (14, 70, 15, 67, 16, 64, 17, 61, 18, 58, 19, 53, 20, 50)$ |
| **Cycle $\mathfrak{C}_6$** (sky) | |
| [54] 53 [50] 49 [46] 45 [42] 41 [38] 37 [32] 31 [29] 35 | $\mathfrak{C}_6 = (210, 156, 209, 159, 208, 162, 207, 165, 206, 168, 205, 173, 204, 175)$ |
| **Cycle $\mathfrak{C}_7$** (lilac) | |
| 28 [27] 24 [23] 20 [19] 16 [15] 12 [11] 6 [5] 2 [8] | $\mathfrak{C}_7 = (21, 49, 22, 46, 23, 43, 24, 40, 25, 37, 26, 32, 27, 29)$ |
| **Cycle $\mathfrak{C}_8$** (plum) | |
| [26] 25 [22] 21 [18] 17 [14] 13 [10] 9 [4] 3 [1] 7 | $\mathfrak{C}_8 = (203, 177, 202, 180, 201, 183, 200, 186, 199, 189, 198, 194, 197, 196)$ |

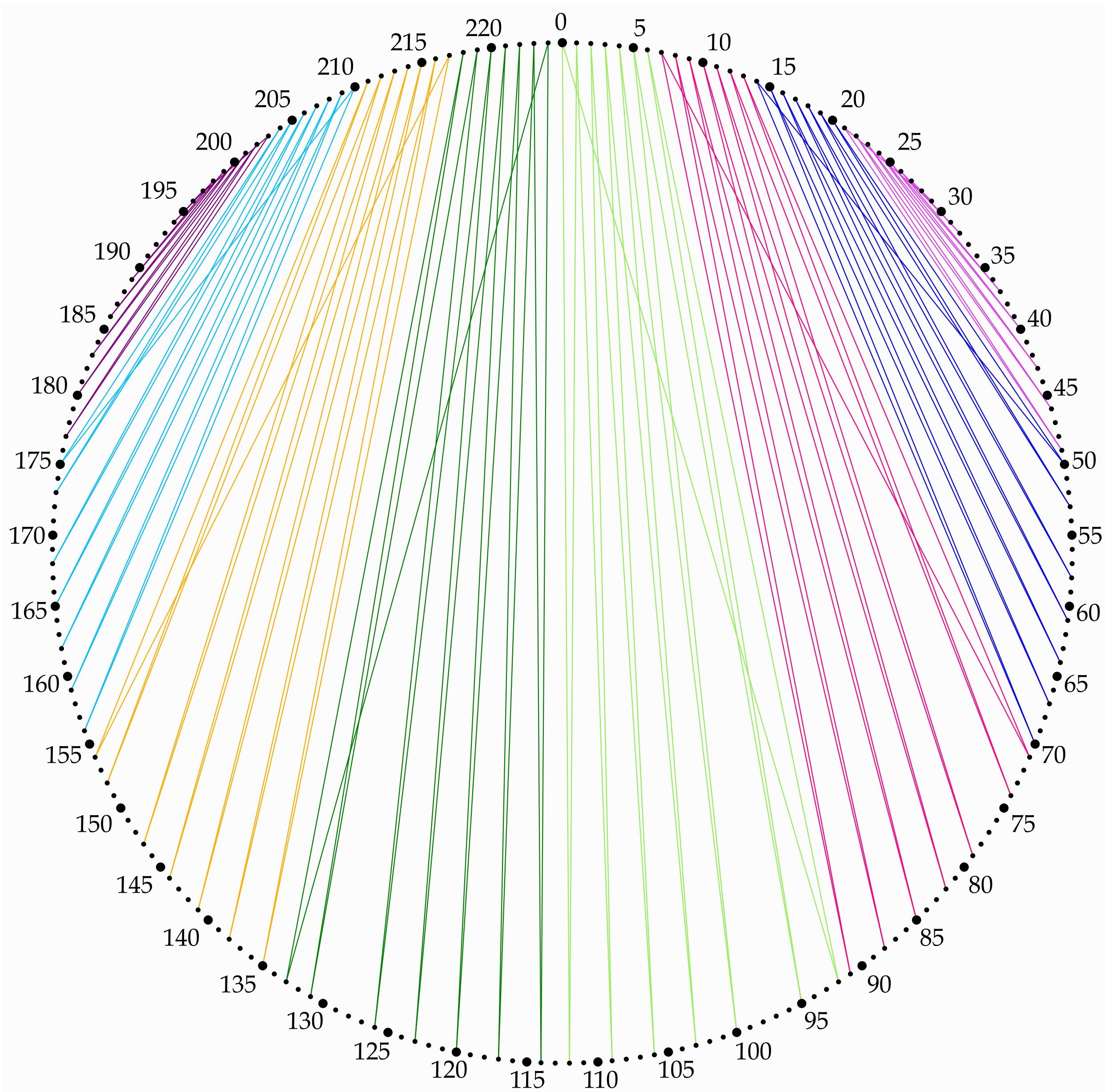



Figure 5.41: A $\mathcal{C}_{14}^{8}$ base block ($p = 8$, $k = 7$)

Table 5.39: Cycle list for the $\mathcal{C}_{18}^{6}$ base block in Figure 5.42

**Cycle $\mathfrak{C}_1$** (pink)

108 [107] 104 [103] 100 [99] 96 [95] 92
[91] 88 [87] 84 [83] 78 [77] 74 [82]

$\mathfrak{C}_1 = \big(0, 108, 1, 105, 2, 102, 3, 99, 4, 96, 5, 93, 6, 90, 7, 85, 8, 82\big)$

**Cycle $\mathfrak{C}_2$** (orange)

[106] 105 [102] 101 [98] 97 [94] 93 [90]
89 [86] 85 [80] 79 [76] 75 [73] 81

$\mathfrak{C}_2 = \big(216, 110, 215, 113, 214, 116, 213, 119, 212, 122, 211, 125, 210, 130, 209, 133, 208, 135\big)$

**Cycle $\mathfrak{C}_3$** (cobalt)

72 [71] 68 [67] 64 [63] 60 [59] 56
[55] 52 [51] 48 [47] 42 [41] 38 [46]

$\mathfrak{C}_3 = \big(9, 81, 10, 78, 11, 75, 12, 72, 13, 69, 14, 66, 15, 63, 16, 58, 17, 55\big)$

**Cycle $\mathfrak{C}_4$** (sky)

[70] 69 [66] 65 [62] 61 [58] 57 [54]
53 [50] 49 [44] 43 [40] 39 [37] 45

$\mathfrak{C}_4 = \big(207, 137, 206, 140, 205, 143, 204, 146, 203, 149, 202, 152, 201, 157, 200, 160, 199, 162\big)$

**Cycle $\mathfrak{C}_5$** (lilac)

36 [35] 32 [31] 28 [27] 24 [23] 20
[19] 16 [15] 12 [11] 6 [5] 2 [10]

$\mathfrak{C}_5 = \big(18, 54, 19, 51, 20, 48, 21, 45, 22, 42, 23, 39, 24, 36, 25, 31, 26, 28\big)$

**Cycle $\mathfrak{C}_6$** (plum)

[34] 33 [30] 29 [26] 25 [22] 21 [18]
17 [14] 13 [8] 7 [4] 3 [1] 9

$\mathfrak{C}_6 = \big(198, 164, 197, 167, 196, 170, 195, 173, 194, 176, 193, 179, 192, 184, 191, 187, 190, 189\big)$

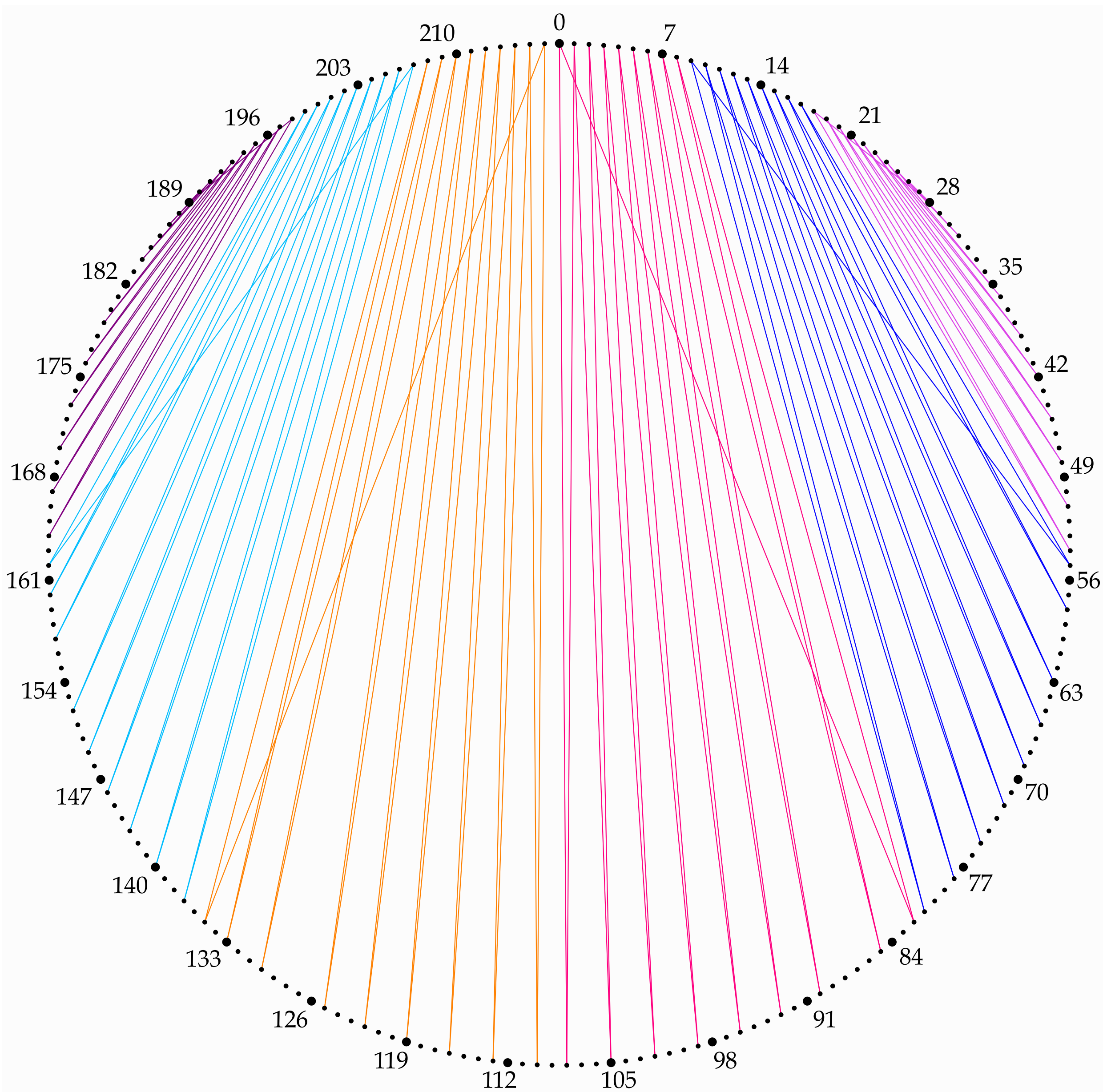



Figure 5.42: A $\mathcal{C}_{18}^{6}$ base block ($p = 6$, $k = 9$)

Table 5.40: Cycle list for the $\mathcal{C}_{22}^{6}$ base block in Figure 5.43

| **Cycle** $\mathfrak{C}_1$ (pink) |
|---|
| 132 [131] 128 [127] 124 [123] 120 [119] 116 [115] |
| 112 [111] 108 [107] 104 [103] 98 [97] 94 [93] 90 [100] |
| $\mathfrak{C}_1 = (0, 132, 1, 129, 2, 126, 3, 123, 4, 120, 5,$ $117, 6, 114, 7, 111, 8, 106, 9, 103, 10, 100)$ |
| **Cycle** $\mathfrak{C}_2$ (orange) |
| [130] 129 [126] 125 [122] 121 [118] 117 [114] 113 |
| [110] 109 [106] 105 [102] 101 [96] 95 [92] 91 [89] 99 |
| $\mathfrak{C}_2 = (264, 134, 263, 137, 262, 140, 261, 143, 260, 146, 259,$ $149, 258, 152, 257, 155, 256, 160, 255, 163, 254, 165)$ |
| **Cycle** $\mathfrak{C}_3$ (cobalt) |
| 88 [87] 84 [83] 80 [79] 76 [75] 72 [71] |
| 68 [67] 64 [63] 60 [59] 54 [53] 50 [49] 46 [56] |
| $\mathfrak{C}_3 = (11, 99, 12, 96, 13, 93, 14, 90, 15, 87, 16,$ $84, 17, 81, 18, 78, 19, 73, 20, 70, 21, 67)$ |
| **Cycle** $\mathfrak{C}_4$ (sky) |
| [86] 85 [82] 81 [78] 77 [74] 73 [70] 69 |
| [66] 65 [62] 61 [58] 57 [52] 51 [48] 47 [45] 55 |
| $\mathfrak{C}_4 = (253, 167, 252, 170, 251, 173, 250, 176, 249, 179, 248,$ $182, 247, 185, 246, 188, 245, 193, 244, 196, 243, 198)$ |
| **Cycle** $\mathfrak{C}_5$ (lilac) |
| 44 [43] 40 [39] 36 [35] 32 [31] 28 [27] |
| 24 [23] 20 [19] 16 [15] 10 [9] 6 [5] 2 [12] |
| $\mathfrak{C}_5 = (22, 66, 23, 63, 24, 60, 25, 57, 26, 54, 27,$ $51, 28, 48, 29, 45, 30, 40, 31, 37, 32, 34)$ |
| **Cycle** $\mathfrak{C}_6$ (plum) |
| [42] 41 [38] 37 [34] 33 [30] 29 [26] 25 |
| [22] 21 [18] 17 [14] 13 [8] 7 [4] 3 [1] 11 |
| $\mathfrak{C}_6 = (242, 200, 241, 203, 240, 206, 239, 209, 238, 212, 237,$ $215, 236, 218, 235, 221, 234, 226, 233, 229, 232, 231)$ |

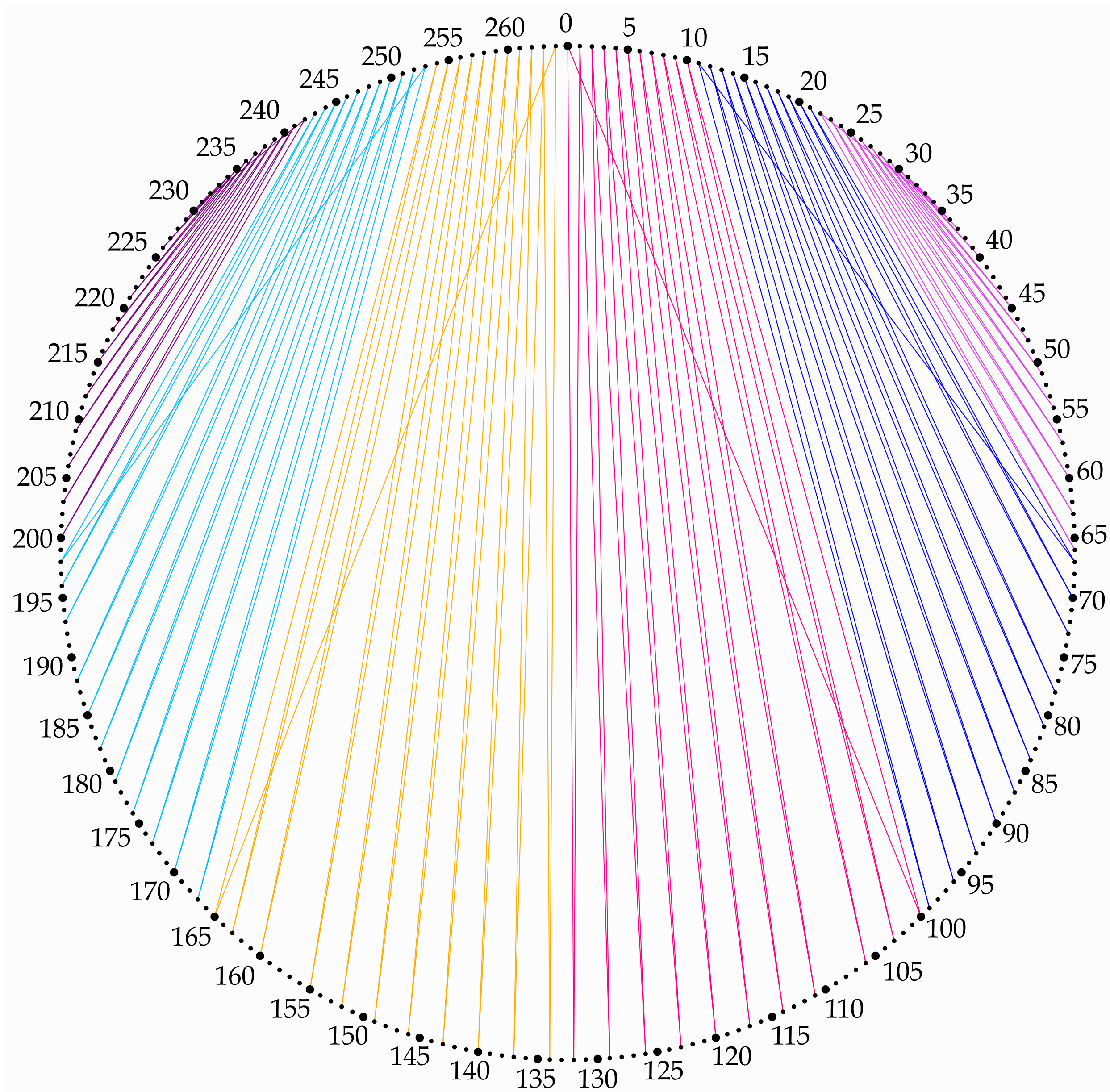



Figure 5.43: A $\mathcal{C}_{22}^{6}$ base block ($p = 6$, $k = 11$)

Table 5.41: Cycle list for the $\mathcal{C}_{26}^4$ base block in Figure 5.44

| |
|---|
| **Cycle** $\mathfrak{C}_1$ (cobalt) |
| 104 [103] 100 [99] 96 [95] 92 [91] 88 [87] 84 [83]<br>80 [79] 76 [75] 72 [71] 68 [67] 62 [61] 58 [57] 54 [66] |
| $\mathfrak{C}_1 = (0, 104, 1, 101, 2, 98, 3, 95, 4, 92, 5, 89, 6, 86, 7, 83, 8, 80, 9, 77, 10, 72, 11, 69, 12, 66)$ |
| **Cycle** $\mathfrak{C}_2$ (sky) |
| [102] 101 [98] 97 [94] 93 [90] 89 [86] 85 [82] 81<br>[78] 77 [74] 73 [70] 69 [64] 63 [60] 59 [56] 55 [53] 65 |
| $\mathfrak{C}_2 = (208, 106, 207, 109, 206, 112, 205, 115, 204, 118, 203, 121, 202, 124, 201, 127, 200, 130, 199, 135, 198, 138, 197, 141, 196, 143)$ |
| **Cycle** $\mathfrak{C}_3$ (lilac) |
| 52 [51] 48 [47] 44 [43] 40 [39] 36 [35] 32 [31]<br>28 [27] 24 [23] 20 [19] 16 [15] 10 [9] 6 [5] 2 [14] |
| $\mathfrak{C}_3 = (13, 65, 14, 62, 15, 59, 16, 56, 17, 53, 18, 50, 19, 47, 20, 44, 21, 41, 22, 38, 23, 33, 24, 30, 25, 27)$ |
| **Cycle** $\mathfrak{C}_4$ (plum) |
| [50] 49 [46] 45 [42] 41 [38] 37 [34] 33 [30] 29<br>[26] 25 [22] 21 [18] 17 [12] 11 [8] 7 [4] 3 [1] 13 |
| $\mathfrak{C}_4 = (195, 145, 194, 148, 193, 151, 192, 154, 191, 157, 190, 160, 189, 163, 188, 166, 187, 169, 186, 174, 185, 177, 184, 180, 183, 182)$ |

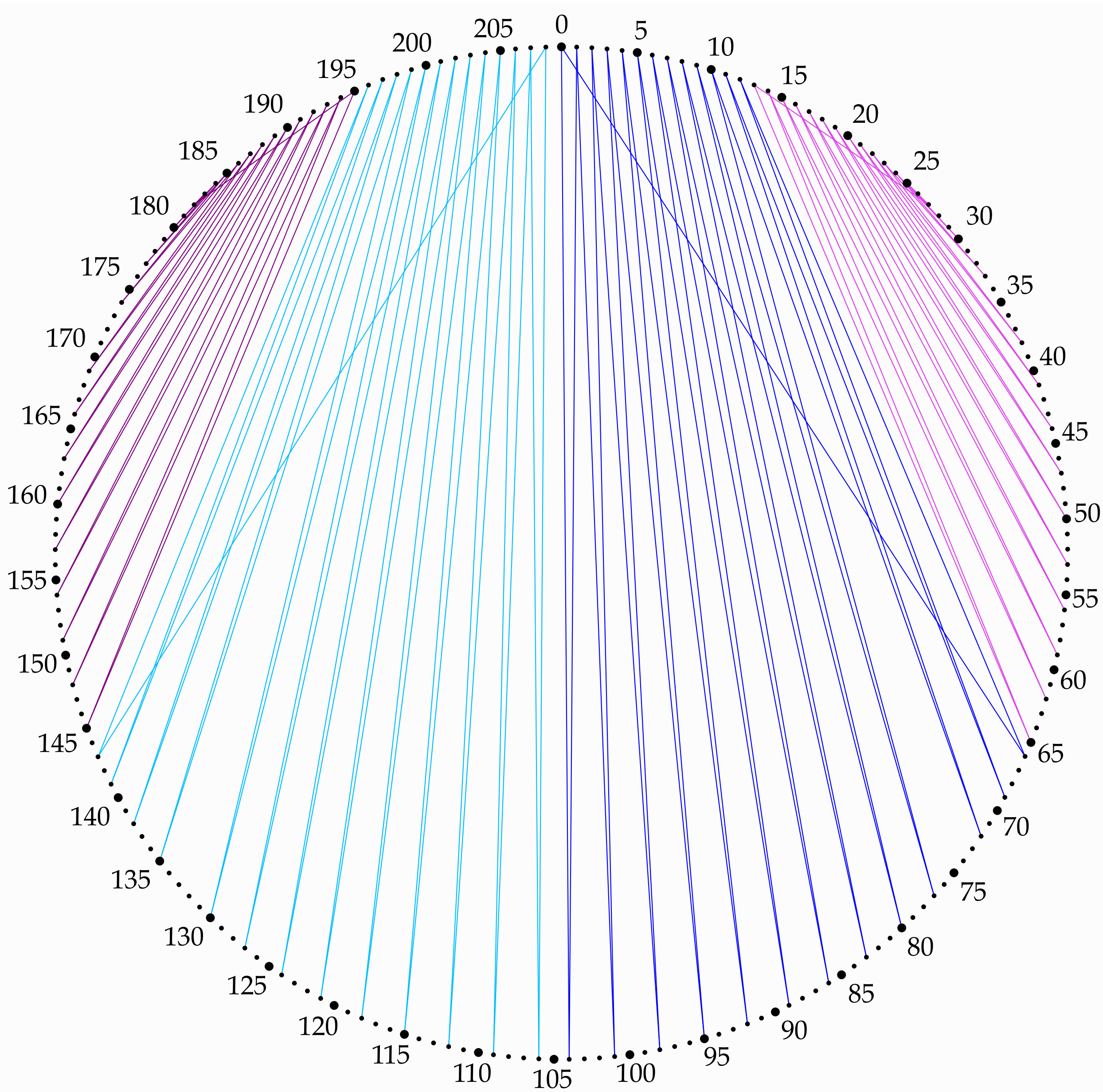



Figure 5.44: A $\mathcal{C}_{26}^{4}$ base block ($p = 4$, $k = 13$)

Table 5.42: Cycles $\mathfrak{C}_1$, $\mathfrak{C}_2$, $\mathfrak{C}_3$, and $\mathfrak{C}_4$ for Example 5.37 ($p = 12$, $k = 19$)

**Cycle $\mathfrak{C}_1$**

456 [455] 452 [451] 448 [447] 444 [443] 440 [439] 436 [435] 432 [431]
428 [427] 424 [423] 420 [419] 416 [415] 412 [411] 408 [407]
404 [403] 398 [397] 394 [393] 390 [389] 386 [385] 382 [400]

$\mathfrak{C}_1 = \big($0, 456, 1, 453, 2, 450, 3, 447, 4, 444, 5, 441, 6, 438,
7, 435, 8, 432, 9, 429, 10, 426, 11, 423, 12, 420,
13, 417, 14, 412, 15, 409, 16, 406, 17, 403, 18, 400$\big)$

**Cycle $\mathfrak{C}_2$**

[454] 453 [450] 449 [446] 445 [442] 441 [438] 437 [434] 433 [430] 429
[426] 425 [422] 421 [418] 417 [414] 413 [410] 409 [406] 405
[402] 401 [396] 395 [392] 391 [388] 387 [384] 383 [381] 399

$\mathfrak{C}_2 = \big($912, 458, 911, 461, 910, 464, 909, 467, 908, 470, 907, 473, 906, 476,
905, 479, 904, 482, 903, 485, 902, 488, 901, 491, 900, 494,
899, 497, 898, 502, 897, 505, 896, 508, 895, 511, 894, 513$\big)$

**Cycle $\mathfrak{C}_3$**

380 [379] 376 [375] 372 [371] 368 [367] 364 [363] 360 [359] 356 [355]
352 [351] 348 [347] 344 [343] 340 [339] 336 [335] 332 [331]
328 [327] 322 [321] 318 [317] 314 [313] 310 [309] 306 [324]

$\mathfrak{C}_3 = \big($19, 399, 20, 396, 21, 393, 22, 390, 23, 387, 24, 384, 25, 381,
26, 378, 27, 375, 28, 372, 29, 369, 30, 366, 31, 363,
32, 360, 33, 355, 34, 352, 35, 349, 36, 346, 37, 343 $\big)$

**Cycle $\mathfrak{C}_4$**

[378] 377 [374] 373 [370] 369 [366] 365 [362] 361 [358] 357 [354] 353
[350] 349 [346] 345 [342] 341 [338] 337 [334] 333 [330] 329
[326] 325 [320] 319 [316] 315 [312] 311 [308] 307 [305] 323

$\mathfrak{C}_4 = \big($893, 515, 892, 518, 891, 521, 890, 524, 889, 527, 888, 530, 887, 533,
886, 536, 885, 539, 884, 542, 883, 545, 882, 548, 881, 551,
880, 554, 879, 559, 878, 562, 877, 565, 876, 568, 875, 570$\big)$

Table 5.43: Cycles $\mathfrak{C}_5$, $\mathfrak{C}_6$, $\mathfrak{C}_7$, and $\mathfrak{C}_8$ for Example 5.37 ($p = 12$, $k = 19$)

**Cycle $\mathfrak{C}_5$**

304 [303] 300 [299] 296 [295] 292 [291] 288 [287] 284 [283] 280 [279]
276 [275] 272 [271] 268 [267] 264 [263] 260 [259] 256 [255]
252 [251] 246 [245] 242 [241] 238 [237] 234 [233] 230 [248]

$$\mathfrak{C}_5 = \big(38, 342, 39, 339, 40, 336, 41, 333, 42, 330, 43, 327, 44, 324, 45, 321, 46, 318, 47, 315, 48, 312, 49, 309, 50, 306, 51, 303, 52, 298, 53, 295, 54, 292, 55, 289, 56, 286\big)$$

**Cycle $\mathfrak{C}_6$**

[302] 301 [298] 297 [294] 293 [290] 289 [286] 285 [282] 281 [278] 277
[274] 273 [270] 269 [266] 265 [262] 261 [258] 257 [254] 253
[250] 249 [244] 243 [240] 239 [236] 235 [232] 231 [229] 247

$$\mathfrak{C}_6 = \big(874, 572, 873, 575, 872, 578, 871, 581, 870, 584, 869, 587, 868, 590, 867, 593, 866, 596, 865, 599, 864, 602, 863, 605, 862, 608, 861, 611, 860, 616, 859, 619, 858, 622, 857, 625, 856, 627\big)$$

**Cycle $\mathfrak{C}_7$**

228 [227] 224 [223] 220 [219] 216 [215] 212 [211] 208 [207] 204 [203]
200 [199] 196 [195] 192 [191] 188 [187] 184 [183] 180 [179]
176 [175] 170 [169] 166 [165] 162 [161] 158 [157] 154 [172]

$$\mathfrak{C}_7 = \big(57, 285, 58, 282, 59, 279, 60, 276, 61, 273, 62, 270, 63, 267, 64, 264, 65, 261, 66, 258, 67, 255, 68, 252, 69, 249, 70, 246, 71, 241, 72, 238, 73, 235, 74, 232, 75, 229\big)$$

**Cycle $\mathfrak{C}_8$**

[226] 225 [222] 221 [218] 217 [214] 213 [210] 209 [206] 205 [202] 201
[198] 197 [194] 193 [190] 189 [186] 185 [182] 181 [178] 177
[174] 173 [168] 167 [164] 163 [160] 159 [156] 155 [153] 171

$$\mathfrak{C}_8 = \big(855, 629, 854, 632, 853, 635, 852, 638, 851, 641, 850, 644, 849, 647, 848, 650, 847, 653, 846, 656, 845, 659, 844, 662, 843, 665, 842, 668, 841, 673, 840, 676, 839, 679, 838, 682, 837, 684\big)$$

Table 5.44: Cycles $\mathfrak{C}_9$, $\mathfrak{C}_{10}$, $\mathfrak{C}_{11}$, and $\mathfrak{C}_{12}$ for Example 5.37 ($p = 12$, $k = 19$)

**Cycle $\mathfrak{C}_9$**

152 [151] 148 [147] 144 [143] 140 [139] 136 [135] 132 [131] 128 [127]
124 [123] 120 [119] 116 [115] 112 [111] 108 [107] 104 [103]
100 [99] 94 [93] 90 [89] 86 [85] 82 [81] 78 [96]

$$\mathfrak{C}_9 = \big(76, 228, 77, 225, 78, 222, 79, 219, 80, 216, 81, 213, 82, 210, 83, 207, 84, 204, 85, 201, 86, 198, 87, 195, 88, 192, 89, 189, 90, 184, 91, 181, 92, 178, 93, 175, 94, 172\big)$$

**Cycle $\mathfrak{C}_{10}$**

[150] 149 [146] 145 [142] 141 [138] 137 [134] 133 [130] 129 [126] 125
[122] 121 [118] 117 [114] 113 [110] 109 [106] 105 [102] 101
[98] 97 [92] 91 [88] 87 [84] 83 [80] 79 [77] 95

$$\mathfrak{C}_{10} = \big(836, 686, 835, 689, 834, 692, 833, 695, 832, 698, 831, 701, 830, 704, 829, 707, 828, 710, 827, 713, 826, 716, 825, 719, 824, 722, 823, 725, 822, 730, 821, 733, 820, 736, 819, 739, 818, 741\big)$$

**Cycle $\mathfrak{C}_{11}$**

76 [75] 72 [71] 68 [67] 64 [63] 60 [59] 56 [55] 52 [51]
48 [47] 44 [43] 40 [39] 36 [35] 32 [31] 28 [27]
24 [23] 18 [17] 14 [13] 10 [9] 6 [5] 2 [20]

$$\mathfrak{C}_{11} = \big(95, 171, 96, 168, 97, 165, 98, 162, 99, 159, 100, 156, 101, 153, 102, 150, 103, 147, 104, 144, 105, 141, 106, 138, 107, 135, 108, 132, 109, 127, 110, 124, 111, 121, 112, 118, 113, 115\big)$$

**Cycle $\mathfrak{C}_{12}$**

[74] 73 [70] 69 [66] 65 [62] 61 [58] 57 [54] 53 [50] 49
[46] 45 [42] 41 [38] 37 [34] 33 [30] 29 [26] 25
[22] 21 [16] 15 [12] 11 [8] 7 [4] 3 [1] 19

$$\mathfrak{C}_{12} = \big(817, 743, 816, 746, 815, 749, 814, 752, 813, 755, 812, 758, 811, 761, 810, 764, 809, 767, 808, 770, 807, 773, 806, 776, 805, 779, 804, 782, 803, 787, 802, 790, 801, 793, 800, 796, 799, 798\big)$$

### 5.3.3 Comparative Analysis for Odd $k$ and Even $p$

We observe that our construction of a $\mathcal{C}_{2k}^{p}$-base block in the case that $k$ is odd and $p$ is even has significant differences from the construction by Blinco and El-Zanati. We have reproduced (at a reduced size) in Figure 5.45 the images of the base blocks from Examples 5.9, 5.29, 5.10, and 5.30, to facilitate direct visual comparison.

**Remark 5.38.** We can obtain a $\sigma^{++}$-labeling both from the $\sigma^{+}$-labeling constructed by Blinco and El-Zanati and from our base block, which induces either a $\sigma$-labeling or a $\sigma^{+}$-labeling, depending on the bipartition used. We obtain the $\sigma^{++}$-labeling in a simple way: by adding a constant to all vertex labels, which has the effect of rotating the base blocks.

Recall that, in the construction by Blinco and El-Zanati, we take $G_i = C_{2k} \uplus C_{2k}$ for each $i \in [\![1, q]\!]$, and we take $G_i$ to have vertex set $[\![1, 4k]\!] \times \{i\}$ with bipartition $[A_i, B_i]$, where $A_i = \{(a, i) \in V(G_i) \mid a \text{ is odd}\}$ and $B_i = \{(b, i) \in V(G_i) \mid b \text{ is even}\}$. $G$ is then assumed to have bipartition $[A, B]$, where $A$ and $B$ are obtained from the bipartitions $[A_i, B_i]$ of the graphs $G_i$ by

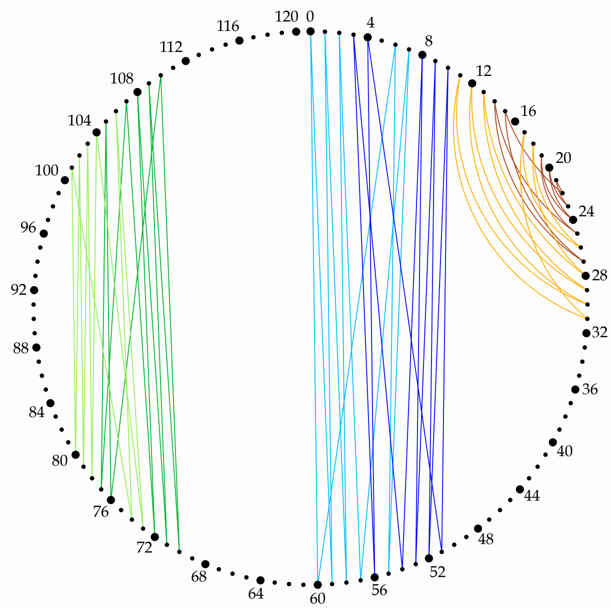
$$A = \bigcup_{i=1}^{q} A_i \text{ and } B = \bigcup_{i=1}^{q} B_i .$$

We also discuss the bipartition $[A^*, B^*]$ of $G$, where

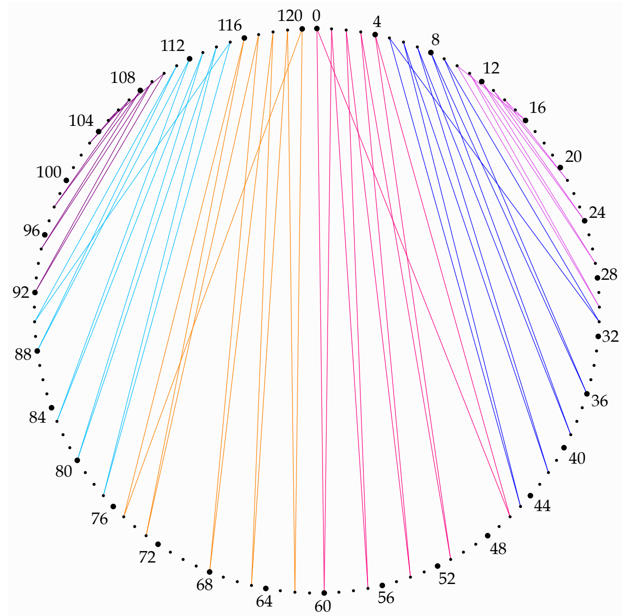
$$A^* = \bigcup_{\substack{i \text{ odd}, \\ 1 \le i \le q}} A_i \cup \bigcup_{\substack{i \text{ even}, \\ 1 \le i \le q}} B_i \quad \text{and} \quad B^* = \bigcup_{\substack{i \text{ odd}, \\ 1 \le i \le q}} B_i \cup \bigcup_{\substack{i \text{ even}, \\ 1 \le i \le q}} A_i .$$

For all $i \in [\![1, p]\!]$, we define $\hat{G}_i = C_{2k}$, with vertex set $[\![1, 2k]\!] \times \{i\}$ and bipartition $[\hat{A}_i, \hat{B}_i]$, where $\hat{A}_i = \{(a, i) \in V(\hat{G}_i) \mid a \text{ is odd}\}$ and $\hat{B}_i = \{(b, i) \in V(\hat{G}_i) \mid b \text{ is even}\}$. We define bipartition $[\hat{A}, \hat{B}]$ of $G$ by

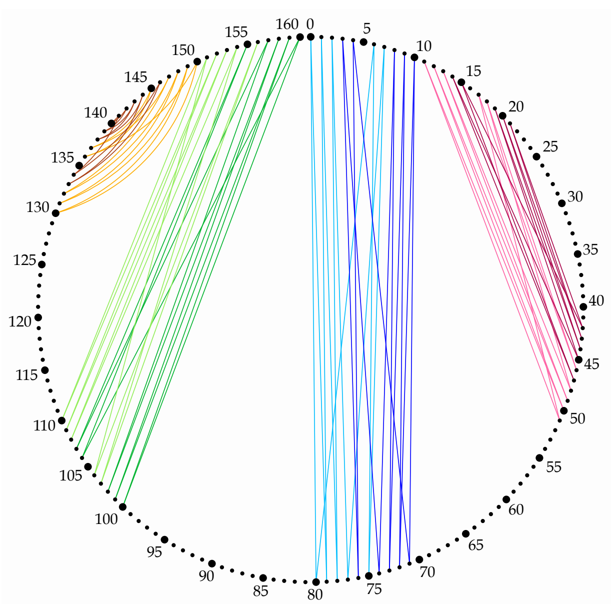
$$\hat{A} = \bigcup_{i=1}^{2q} \hat{A}_i \text{ and } \hat{B} = \bigcup_{i=1}^{2q} \hat{B}_i ,$$



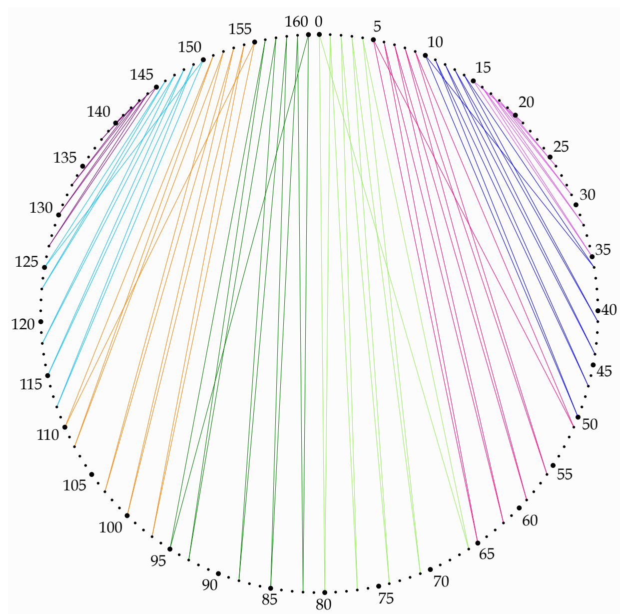
(a) Example 5.9



(b) Example 5.29



(c) Example 5.10



(d) Example 5.30

Figure 5.45: Small reproductions of $\mathcal{C}_{10}^{6}$ and $\mathcal{C}_{10}^{8}$ base blocks from Examples 5.9, 5.29, 5.10, and 5.30

and we define bipartition $[\hat{A}^*, \hat{B}^*]$ of $G$ by

$$\hat{A}^* = \bigcup_{j=1}^{q} \hat{A}_{2j-1} \ \cup \ \bigcup_{j=1}^{q} \hat{B}_{2j} \quad \text{and} \quad \hat{B}^* = \bigcup_{j=1}^{q} \hat{B}_{2j-1} \ \cup \ \bigcup_{j=1}^{q} \hat{A}_{2j}\,.$$

The construction by Blinco and El-Zanati produces a $\sigma^+$-labeling of $G = \mathcal{C}_{2k}^p$ on bipartition $[A, B]$. If we use instead bipartition $[A^*, B^*]$ and add $2k \cdot \lceil q/2 \rceil$ to each vertex label, with computations done modulo $(4kp+1)$, we obtain a $\sigma^{++}$-labeling of $\mathcal{C}_{2k}^p$ with critical value $\lambda = 4k \cdot \lceil q/2 \rceil - 1 + \lceil q/2 \rceil$.

Our base block induces a $\sigma$-labeling of $G = \mathcal{C}_{2k}^p$; this labeling is not a $\sigma^+$-labeling of $G$ on bipartition $[\hat{A}, \hat{B}]$, but it is a $\sigma^+$-labeling of $G$ on bipartition $[\hat{A}^*, \hat{B}^*]$. If we use the bipartition $[\hat{A}, \hat{B}]$ and add $kq$ to each vertex label, again computing modulo $(4kp + 1)$, we obtain a $\sigma^{++}$-labeling of $\mathcal{C}_{2k}^p$ having critical value $\lambda = kp - 1$. ■

We observe further that there is a fundamental difference between the construction by Blinco and El-Zanati and ours: separation of cycles. Since the construction by Blinco and El-Zanati requires $\alpha$-labelings, and since $C_{2k}$ cannot admit an $\alpha$-labeling when $k$ is odd, it is necessary to break $G = \mathcal{C}_{2k}^p$ into $C_{2k} \uplus C_{2k}$-subgraphs, which admit $\alpha$-labelings. Doing so causes the cycles in the final base block to be intertwined in pairs, because the $\alpha$-labeling of $C_{2k} \uplus C_{2k}$ intertwines the two cycles. In contrast, the cycles in our base block are not intertwined. In fact, there is a linear ordering of the sets $\hat{A}_i$ and $\hat{B}_i$ such that, if set $X$ occurs before set $Y$ in the ordering, then $x < y$ for all $x \in X$ and all $y \in Y$. The ordering is obtained by taking the sets $\hat{A}_i$ of odd index in increasing order, followed by the sets $\hat{B}_i$ of odd index in decreasing order, then the sets $\hat{B}_i$ of even index in increasing order, and last the sets $\hat{A}_i$ of even index in decreasing order. This separation property may be useful in applications in which the intended method is to remove some of the cycles in the base block and replace them with other graphs to produce a base block for a different type of design.

## 5.4 Complete Designs of Order $4kp+1$ for Odd $k$ and Odd $p$

In this section, we present base block constructions for $\mathcal{C}_{2k}^{p}$ in the case that $k$ and $p$ are both odd. The designs in this section are built using two methods; one method relies on group actions, while the other relies on the formation of a partition of the difference set so that the differences in each subset of the partition can be used to achieve a prescribed sum. Both methods currently require separate constructions for each pair of values of $k$ and $p$.

### 5.4.1 Constructions by Group Actions

The approach in this method is to select one cycle of length $2k$ in $K_{4kp+1}$ so that it has certain properties, and then allow a particular cyclic group of order $p$ to act on this cycle in order to create the remaining cycles in the base block. Before we describe this method in greater detail, we pause to state essential definitions and results in the theory of group actions. For all terminology of algebraic structures not defined in the discussion that follows, we refer the reader to Hungerford's text [19]. In our discussion, we need only consider actions of finite groups on finite sets, so we restrict our definitions to these groups and sets.

**Definition 5.39.** Let $G$ be a finite group, and let $X$ be a finite set. Let $e$ denote the identity element of the group $G$. An ***action*** of $G$ on $X$ is a function that associates with every $(g, x) \in G \times X$ an element $gx$ of $X$, such that the following two properties are satisfied.

(i) For all $x \in X$, $ex = x$.

(ii) For all $g, h \in G$ and all $x \in X$, $g(hx) = (gh)x$.

When an action of $G$ on $X$ is given, we say that $G$ ***acts*** on $X$. ■

An action of $G$ on $X$ induces a relation on $X$: if $x, y \in X$, we say $x \sim y$ if and only if $y = gx$ for some $g \in G$. It is easily verified that this is an equivalence relation on $X$; the equivalence classes are called *orbits*. We note that these orbits, being equivalence classes, form a partition of the set $X$; this fact is important to our construction.

**Definition 5.40.** Let $G$ be a finite group and $X$ a finite set; suppose $G$ acts on $X$. For each $x \in X$, the ***orbit*** of $x$, denoted $\operatorname{Orb}(x)$, is the subset $\operatorname{Orb}(x) = \{gx \mid g \in G\}$ of $X$. ■

Now we return to the task of forming a base block for a $\mathcal{C}_{2k}^p$-design on $K_{4kp+1}$. We recall that we are considering only odd values of $k$ and $p$ for these constructions, and we note that the integers $p$ and $4kp+1$ are relatively prime for any choice of $k$ and $p$. In our description of this construction, we use the integers from $-2kp$ to $2kp$ as the representatives of the congruence classes in the ring $\mathbb{Z}_{4kp+1}$. We denote the set of nonzero elements of this ring by $\mathbb{Z}^*_{4kp+1}$.

In order to form a base block for $\mathcal{C}_{2k}^p$, we first identify a number $x \in \mathbb{Z}^*_{4kp+1}$ such that the set

$$H = \left\{x^i \;\middle|\; i \in [\![1,p]\!]\right\} \tag{5.120}$$

is a group of order $p$ under multiplication modulo $4kp+1$. In order to proceed with the construction, we must have, for all $z \in \mathbb{Z}^*_{4kp+1}$, that the $p$ numbers in the set $\{x^i z \mid i \in [\![1,p]\!]\}$ are distinct modulo $(4kp+1)$. We assume that this is the case in what follows; we defer to Chapter 6 the discussion of when these conditions are satisfied.

We now define two actions of $H$. We define the action $\alpha : H \times \mathbb{Z}^*_{4kp+1} \to \mathbb{Z}^*_{4kp+1}$ by $\alpha(h,z) = \hat{z}$, where $\hat{z}$ is the unique element of $[\![-2kp,-1]\!] \cup [\![1,2kp]\!]$ satisfying the congruence $h \cdot z \equiv \hat{z} \,(\operatorname{mod} 4kp+1)$. We define the action $\beta : H \times \mathcal{D}_{4kp+1} \to \mathcal{D}_{4kp+1}$ by $\beta(h,d) = \left|\alpha(h,d)\right|$. We avoid use of the standard notations $hz$ for $\alpha(h,z)$ and $hd$ for $\beta(h,d)$ due to their ambiguity in this context.

The fact that $\left|\operatorname{Orb}(z)\right| = p$ for each $z \in \mathbb{Z}^*_{4kp+1}$ and $\left|\operatorname{Orb}(d)\right| = p$ for each $d \in \mathcal{D}_{4kp+1}$ follows from the (assumed) fact that the $p$ numbers in the set $\{x^i z \mid i \in [\![1,p]\!]\}$ are distinct modulo $(4kp+1)$. Then, since the collection of orbits generated by a group action on a set is a partition of the set, there are exactly $2k$ distinct orbits in $\mathcal{D}_{4kp+1}$ and exactly $4k$ distinct orbits in $\mathbb{Z}^*_{4kp+1}$.

We next choose a set $T$ of differences so that each orbit in $\mathcal{D}_{4kp+1}$ contributes exactly one element to $T$, and so that, for some function $f : T \to \{1, -1\}$, we have

$$\sum_{d \in T} f(d) \cdot d \equiv 0 \pmod{4kp+1} \; . \tag{5.121}$$

The penultimate stage of the construction is to use the differences in the set $T$ to form a cycle $C$ (as a subgraph of $K_{4kp+1}$) that will generate the base block under the action of $H$. In the formation of the cycle $C$, we require that each difference $d$ in $T$ occurs on exactly one edge in the cycle $C$, achieved in the direction dictated by $f(d)$ (clockwise if $f(d) = 1$ and counterclockwise otherwise), and that no two vertices of $C$ lie in the same orbit in $\mathbb{Z}^*_{4kp+1}$.

We complete the construction by letting $H$ act on the cycle $C$ to generate the base block. Specifically, if $C = \big(c_1, c_2, \ldots, c_{2k}\big)$, we define, for any $h \in H$,

$$hC = \Big(\alpha(h, c_1)\,,\ \alpha(h, c_2)\,,\ \ldots\,,\ \alpha(h, c_{2k})\Big)\,. \tag{5.122}$$

Since no two vertices of $C$ belong to the same orbit in $\mathbb{Z}^*_{4kp+1}$, no two vertices of $hC$ belong to the same orbit in $\mathbb{Z}^*_{4kp+1}$, so $hC$ is a cycle of length $2k$ for all $h \in H$. For distinct $h_1, h_2 \in H$ and any vertex $c_i$ of $C$, the vertices $\alpha(h_1, c_i)$ and $\alpha(h_2, c_i)$ are distinct members of the same orbit. Thus, for distinct $h_1, h_2 \in H$, the cycles $h_1C$ and $h_2C$ are vertex-disjoint. So the graph $B$ whose components are the cycles $hC$, where $h$ ranges over all elements of $H$, is indeed isomorphic to $\mathcal{C}^p_{2k}$. We observe that, if the edge $\{c_i, c_{i+1}\}$ of $C$ has difference $d$, then the edge $\big\{\alpha(h, c_i)\,,\ \alpha(h, c_{i+1})\big\}$ of $hC$ has difference $\beta(h, d)$. So the set of differences used on edges of the cycle $hC$ is $hT = \big\{\beta(h, d) \;\big|\; d \in T\big\}$. Since $T$ has exactly one element from each orbit in $\mathcal{D}_{4kp+1}$, so does $hT$ for each $h \in H$. Thus, by our careful choice of $T$ and $C$, every difference in $\mathcal{D}_{4kp+1}$ occurs on exactly one edge of the graph $B$; hence $B$ is the desired base block.

We give full details for the construction of the design for $p = 3$ and $k = 5$ as an illustrative example; we state the other designs we have achieved without these details.

#### 5.4.1.1 The Design for $p = 3$ and $k = 5$

If $p = 3$ and $k = 5$, then $n = 4pk + 1 = 61$. Note that $13 \not\equiv 1 \pmod{61}$, $13^2 \not\equiv 1 \pmod{61}$, and $13^3 \equiv 1 \pmod{61}$, so $H = \{13^i \mid 1 \le i \le 3\}$ is a group of order 3 under multiplication modulo 61. Since $13^2 \equiv -14 \pmod{61}$, we write $H = \{1, 13, -14\}$. The orbits generated by the actions of $H$ on $\mathbb{Z}_{61}^*$ and on $\mathcal{D}_{61}$ are listed in Table 5.45.

Table 5.45: Orbits generated by the actions of $H$

| **Orbits in $\mathbb{Z}_{61}^*$** | | **Orbits in $\mathcal{D}_{61}$** |
|---|---|---|
| $\{1, 13, -14\}$ | $\{-1, -13, 14\}$ | $\{1, 13, 14\}$ |
| $\{2, 26, -28\}$ | $\{-2, -26, 28\}$ | $\{2, 26, 28\}$ |
| $\{3, -22, 19\}$ | $\{-3, 22, -19\}$ | $\{3, 22, 19\}$ |
| $\{4, -9, 5\}$ | $\{-4, 9, -5\}$ | $\{4, 9, 5\}$ |
| $\{6, 17, -23\}$ | $\{-6, -17, 23\}$ | $\{6, 17, 23\}$ |
| $\{7, 30, 24\}$ | $\{-7, -30, -24\}$ | $\{7, 30, 24\}$ |
| $\{8, -18, 10\}$ | $\{-8, 18, -10\}$ | $\{8, 18, 10\}$ |
| $\{11, 21, 29\}$ | $\{-11, -21, -29\}$ | $\{11, 21, 29\}$ |
| $\{12, -27, 15\}$ | $\{-12, 27, -15\}$ | $\{12, 27, 15\}$ |
| $\{16, 25, 20\}$ | $\{-16, -25, -20\}$ | $\{16, 25, 20\}$ |

We choose the set of differences $T = \{1, 2, 19, 5, 6, 30, 10, 21, 12, 16\}$; define the function $f$ by $f(d) = 1$ for all $d \in T$, and observe that

$$\sum_{d \in T} f(d) \cdot d = 1 + 2 + 19 + 5 + 6 + 30 + 10 + 21 + 12 + 16 = 122 \equiv 0 \pmod{61} . \quad (5.123)$$

Since $f(d) = 1$ for every $d \in T$, every difference in $T$ is to be achieved by moving in the clockwise direction. We use these differences to form the cycle

$$C = (1, 17, 18, 30, -29, -19, 2, 8, 27, -4) .$$

Note that, as required, no two vertices of $C$ lie in the same orbit in $\mathbb{Z}_{61}^*$. Allowing $H$ to act on $C$, we obtain the additional cycles $13C = \big(13, -23, -10, 24, -11, -3, 26, -18, -15, 9\big)$ and $-14C = \big(-14, 6, -8, 7, -21, 22, -28, 10, -12, -5\big)$. Since the standard labels for the vertices of $K_{61}$ are the canonical representatives of the congruence classes, we revert to these representatives to state the base block. The cycles in the base block and their corresponding difference sets are given in Table 5.46; the base block is shown in Figure 5.46.

Table 5.46: Cycle list for the $\mathcal{C}_{10}^3$ base block in Figure 5.46

| | |
|---|---|
| **Cycle** $\mathfrak{C}_1$ | (cobalt) |
| $C = \mathfrak{C}_1 = \big(1,\ 17,\ 18,\ 30,\ 32,\ 42,\ 2,\ 8,\ 27,\ 57\big)$ | |
| $T = \big\{1, 2, 19, 5, 6, 30, 10, 21, 12, 16\big\}$ | |
| **Cycle** $\mathfrak{C}_2$ | (sky) |
| $13C = \mathfrak{C}_2 = \big(13,\ 38,\ 51,\ 24,\ 50,\ 58,\ 26,\ 43,\ 46,\ 9\big)$ | |
| $13T = \big\{13, 26, 3, 4, 17, 24, 8, 29, 27, 25\big\}$ | |
| **Cycle** $\mathfrak{C}_3$ | (pink) |
| $-14C = \mathfrak{C}_3 = \big(47,\ 6,\ 53,\ 7,\ 40,\ 22,\ 33,\ 10,\ 49,\ 56\big)$ | |
| $-14T = \big\{14, 28, 22, 9, 23, 7, 18, 11, 15, 20\big\}$ | |

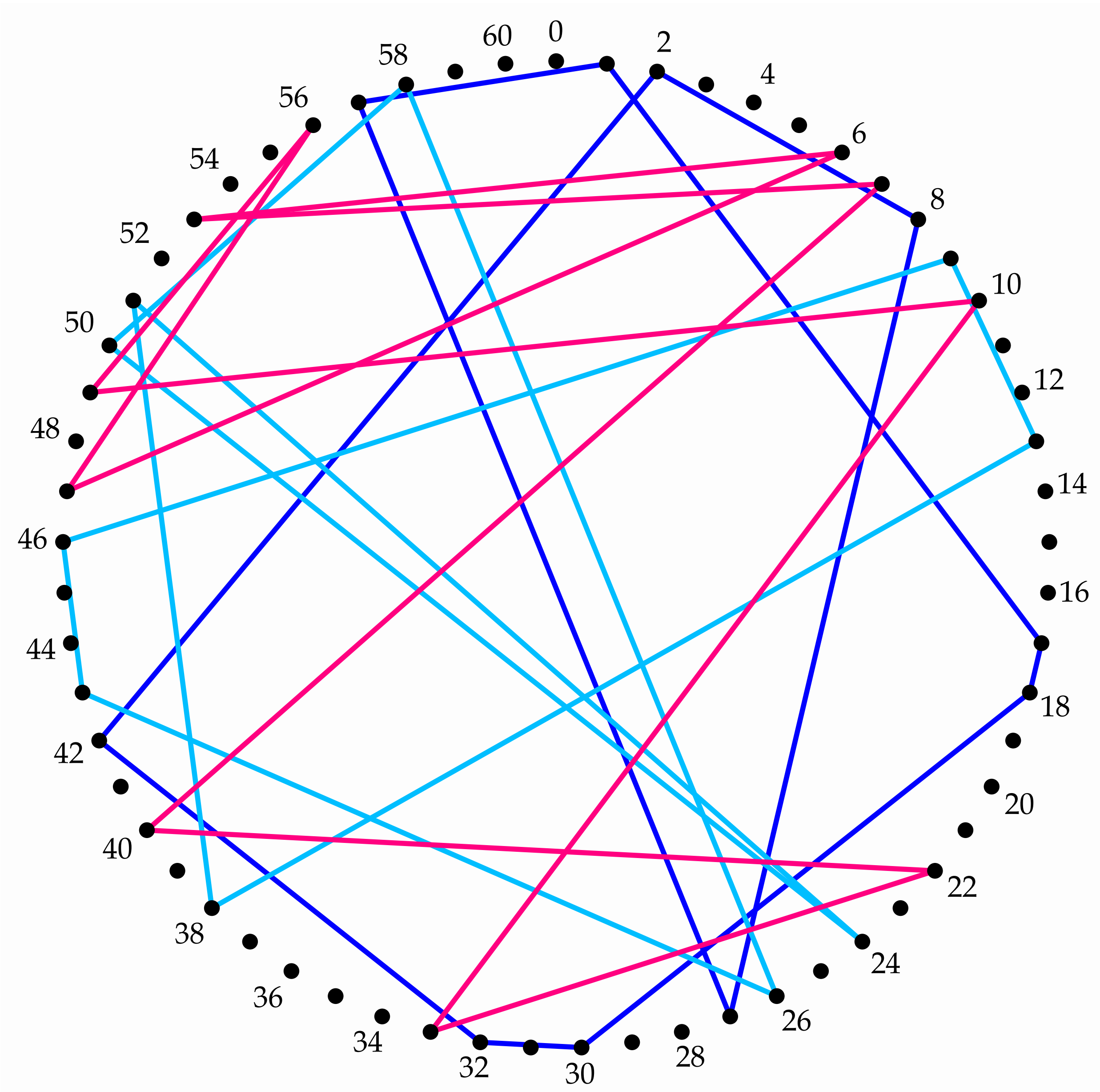



Figure 5.46: A $\mathcal{C}_{10}^{3}$ base block ($p = 3$, $k = 5$)

#### 5.4.1.2 The Design for $p = 3$ and $k = 3$

If $p = 3$ and $k = 3$, then $n = 4pk + 1 = 37$. Construction of the base block is facilitated by the action of the group $H = \{1, 10, -11\}$ on $\mathcal{D}_{37}$ and on $\mathbb{Z}_{37}^{*}$. The cycles in the base block and their corresponding difference sets are given in Table 5.47; the base block is shown in Figure 5.47.

Table 5.47: Cycle list for the $\mathcal{C}_6^3$ base block in Figure 5.47

| **Cycle $\mathfrak{C}_1$** (cobalt) |
| --- |
| $C = \mathfrak{C}_1 = (1,\ 6,\ 7,\ 14,\ 16,\ 32)$ |
| $T = \{1, 2, 7, 5, 6, 16\}$ |
| **Cycle $\mathfrak{C}_2$** (sky) |
| $10C = \mathfrak{C}_2 = (10,\ 23,\ 33,\ 29,\ 12,\ 24)$ |
| $10T = \{10, 17, 4, 13, 14, 12\}$ |
| **Cycle $\mathfrak{C}_3$** (pink) |
| $-11C = \mathfrak{C}_3 = (26,\ 8,\ 34,\ 31,\ 9,\ 18)$ |
| $-11T = \{11, 15, 3, 18, 8, 9\}$ |

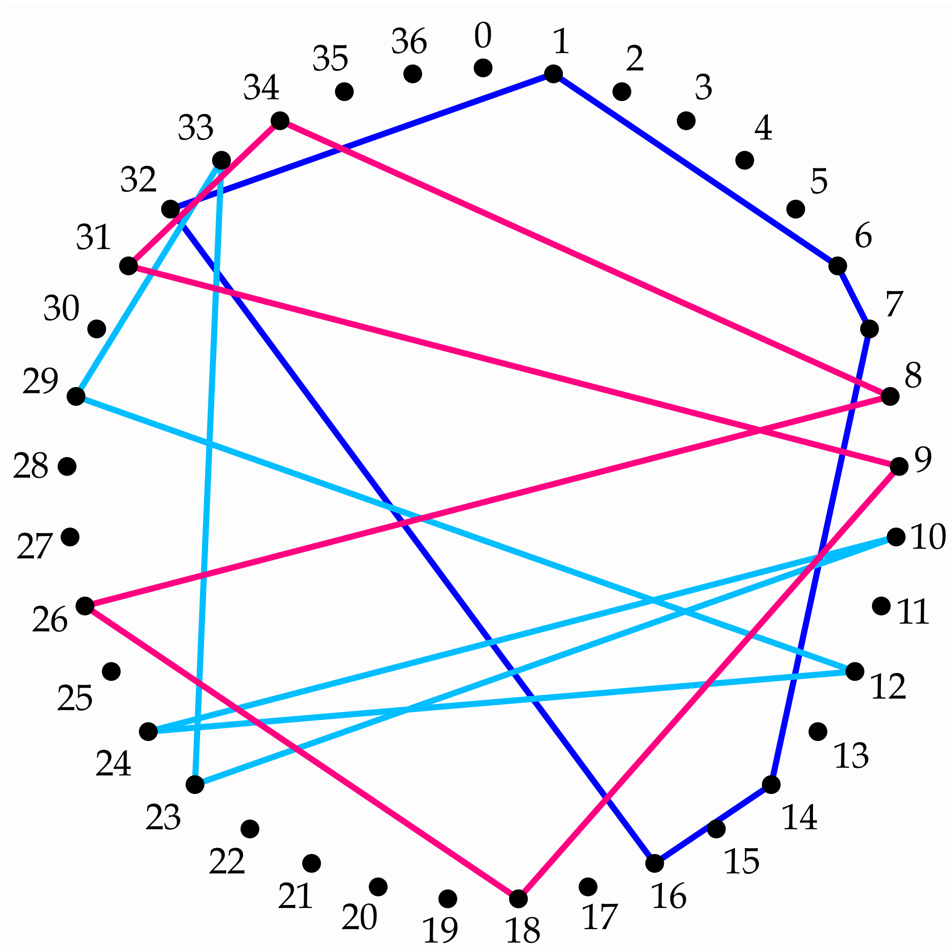



Figure 5.47: A $\mathcal{C}_6^3$ base block ($p = 3$, $k = 3$)

#### 5.4.1.3 The Design for $p = 3$ and $k = 9$

If $p = 3$ and $k = 9$, then $n = 4pk + 1 = 109$. Construction of the base block is facilitated by the action of the group $H = \{1, 45, -46\}$ on $\mathcal{D}_{109}$ and on $\mathbb{Z}^*_{109}$. The cycles in the base block and their corresponding difference sets are given in Table 5.48; the base block is shown in Figure 5.48.

Table 5.48: Cycle list for the $\mathcal{C}^3_{18}$ base block in Figure 5.48

| **Cycle $\mathfrak{C}_1$** (cobalt) |
|---|
| $C = \mathfrak{C}_1 = (1,\ 2,\ 19,\ 22,\ 26,\ 38,\ 44,\ 52,\ 74,\ 88,\ 99,\ 8,\ 21,\ 57,\ 106,\ 34,\ 50,\ 82)$ |
| $T = \{1, 17, 3, 4, 12, 6, 8, 22, 14, 11, 13, 36, 16, 18, 28, 32, 49, 37\}$ |
| **Cycle $\mathfrak{C}_2$** (sky) |
| $45C = \mathfrak{C}_2 = (45,\ 90,\ 92,\ 9,\ 80,\ 75,\ 18,\ 51,\ 60,\ 36,\ 95,\ 33,\ 73,\ 58,\ 83,\ 4,\ 70,\ 93)$ |
| $45T = \{45, 2, 26, 38, 5, 52, 33, 9, 24, 50, 40, 15, 43, 47, 48, 23, 25, 30\}$ |
| **Cycle $\mathfrak{C}_3$** (pink) |
| $-46C = \mathfrak{C}_3 = (63,\ 17,\ 107,\ 78,\ 3,\ 105,\ 47,\ 6,\ 84,\ 94,\ 24,\ 68,\ 15,\ 103,\ 29,\ 71,\ 98,\ 43)$ |
| $-46T = \{46, 19, 29, 34, 7, 51, 41, 31, 10, 39, 53, 21, 27, 44, 20, 54, 35, 42\}$ |

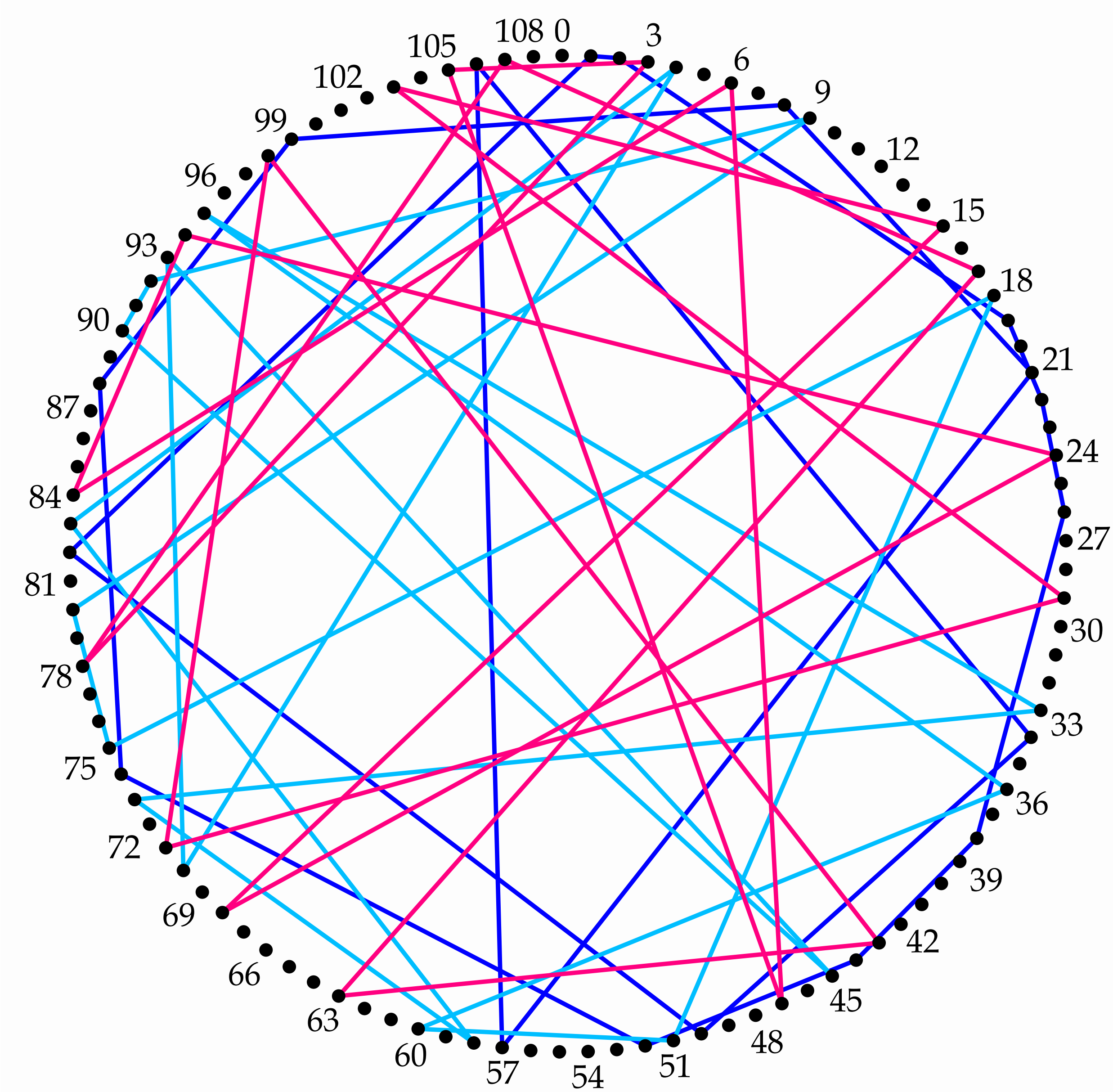



Figure 5.48: A $\mathcal{C}_{18}^{3}$ base block ($p = 3$, $k = 9$)

#### 5.4.1.4 The Design for $p = 3$ and $k = 11$

If $p = 3$ and $k = 11$, then $n = 4pk + 1 = 133$. Construction of the base block is facilitated by the action of the group $H = \{1, 11, -12\}$ on $\mathcal{D}_{133}$ and on $\mathbb{Z}^*_{133}$. The cycles in the base block and their corresponding difference sets are given in Table 5.49; the base block is shown in Figure 5.49.

Table 5.49: Cycle list for the $\mathcal{C}^3_{22}$ base block in Figure 5.49

| **Cycle $\mathfrak{C}_1$** (cobalt) |
| --- |
| $C = \mathfrak{C}_1 = (1, 3, 6, 16, 36, 41, 50, 64, 82, 112, 23, 24, 32, 49, 68, 96, 21, 27, 42, 58, 87, 127)$ |
| $T = \{ 1, 2, 3, 44, 5, 6, 7, 8, 9, 10, 14, 15, 16, 17, 18, 19, 20, 58, 28, 29, 30, 40 \}$ |
| **Cycle $\mathfrak{C}_2$** (sky) |
| $11C = \mathfrak{C}_2 = (11, 33, 66, 43, 130, 52, 18, 39, 104, 35, 120, 131, 86, 7, 83, 125, 98, 31, 63, 106, 26, 67)$ |
| $11T = \{ 11, 22, 33, 48, 55, 66, 56, 45, 34, 23, 21, 32, 43, 54, 65, 57, 46, 27, 42, 53, 64, 41 \}$ |
| **Cycle $\mathfrak{C}_3$** (pink) |
| $-12C = \mathfrak{C}_3 = (121, 97, 61, 74, 100, 40, 65, 30, 80, 119, 123, 111, 15, 77, 115, 45, 14, 75, 28, 102, 20, 72)$ |
| $-12T = \{ 12, 24, 36, 4, 60, 61, 49, 37, 25, 13, 35, 47, 59, 62, 50, 38, 26, 31, 63, 51, 39, 52 \}$ |

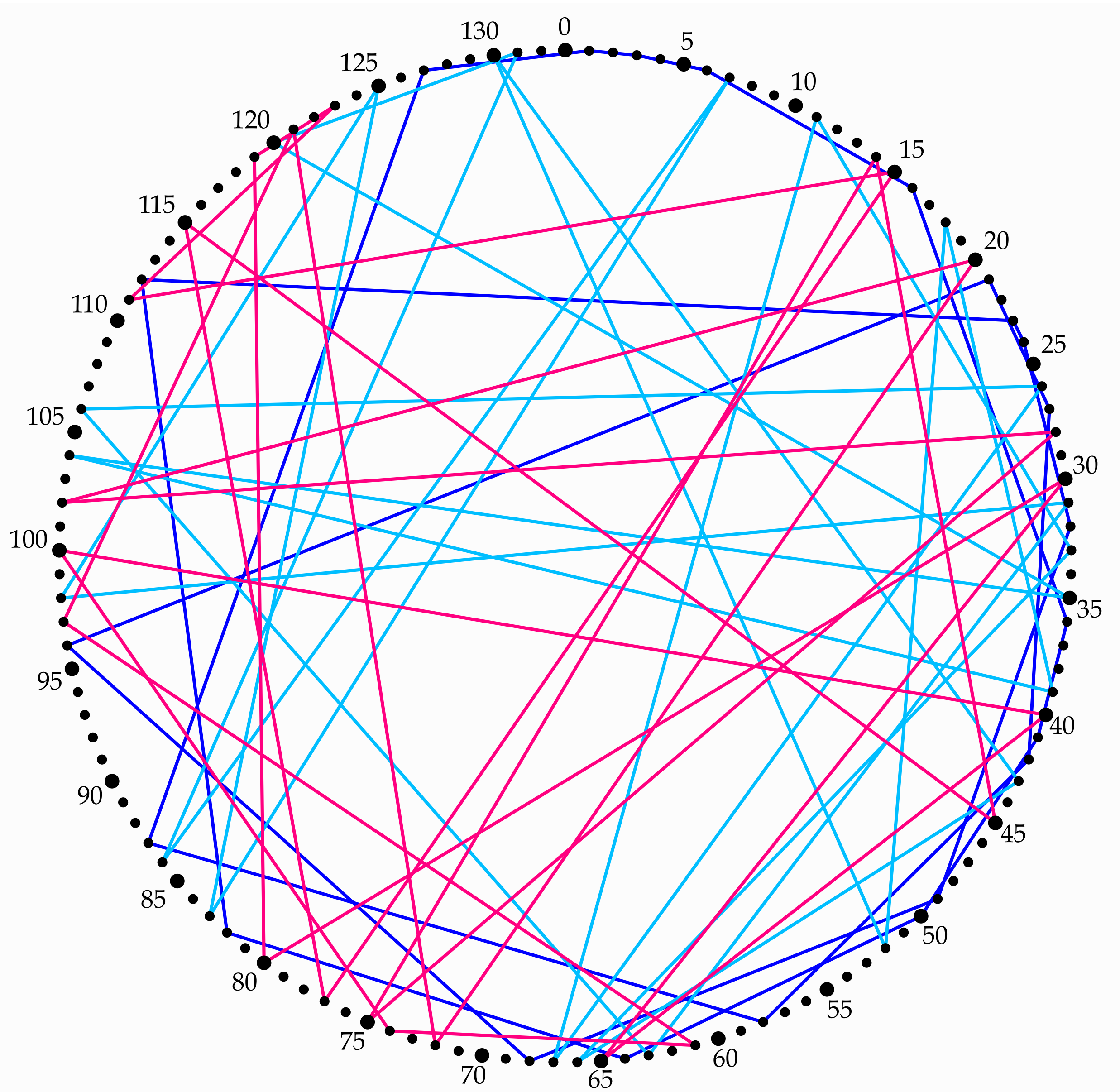



Figure 5.49: A $\mathcal{C}_{22}^{3}$ base block ($p = 3$, $k = 11$)

#### 5.4.1.5 The Design for $p = 3$ and $k = 13$

If $p = 3$ and $k = 13$, then $n = 4pk + 1 = 157$. Construction of the base block is facilitated by the action of the group $H = \{1, 12, -13\}$ on $\mathcal{D}_{157}$ and on $\mathbb{Z}^*_{157}$. The cycles in the base block and their corresponding difference sets are given in Table 5.50; the base block is shown in Figure 5.50.

Table 5.50: Cycle list for the $\mathcal{C}^3_{26}$ base block in Figure 5.50

| **Cycle $\mathfrak{C}_1$** (cobalt) |
|---|
| $C = \mathfrak{C}_1 = \big(1, 11, 34, 67, 75, 82, 88, 110, 131, 151, 10, 54, 99, 129, 3, 35, 52, 70, 73, 84, 103, 108, 117, 121, 122, 156\big)$ |
| $T = \big\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 23, 16, 17, 18, 19, 20, 21, 22, 34, 30, 31, 32, 33, 45, 44\big\}$ |
| **Cycle $\mathfrak{C}_2$** (sky) |
| $12C = \mathfrak{C}_2 = \big(12, 132, 94, 19, 115, 42, 114, 64, 2, 85, 120, 20, 89, 135, 36, 106, 153, 55, 91, 66, 137, 40, 148, 39, 51, 145\big)$ |
| $12T = \big\{12, 24, 36, 48, 60, 72, 73, 61, 49, 37, 25, 38, 35, 47, 59, 71, 74, 62, 50, 63, 46, 58, 70, 75, 69, 57\big\}$ |
| **Cycle $\mathfrak{C}_3$** (pink) |
| $-13C = \mathfrak{C}_3 = \big(144, 14, 29, 71, 124, 33, 112, 140, 24, 78, 27, 83, 126, 50, 118, 16, 109, 32, 150, 7, 74, 9, 49, 154, 141, 13\big)$ |
| $-13T = \big\{13, 26, 39, 52, 65, 78, 66, 53, 40, 27, 14, 15, 51, 64, 77, 67, 54, 41, 28, 29, 76, 68, 55, 42, 43, 56\big\}$ |

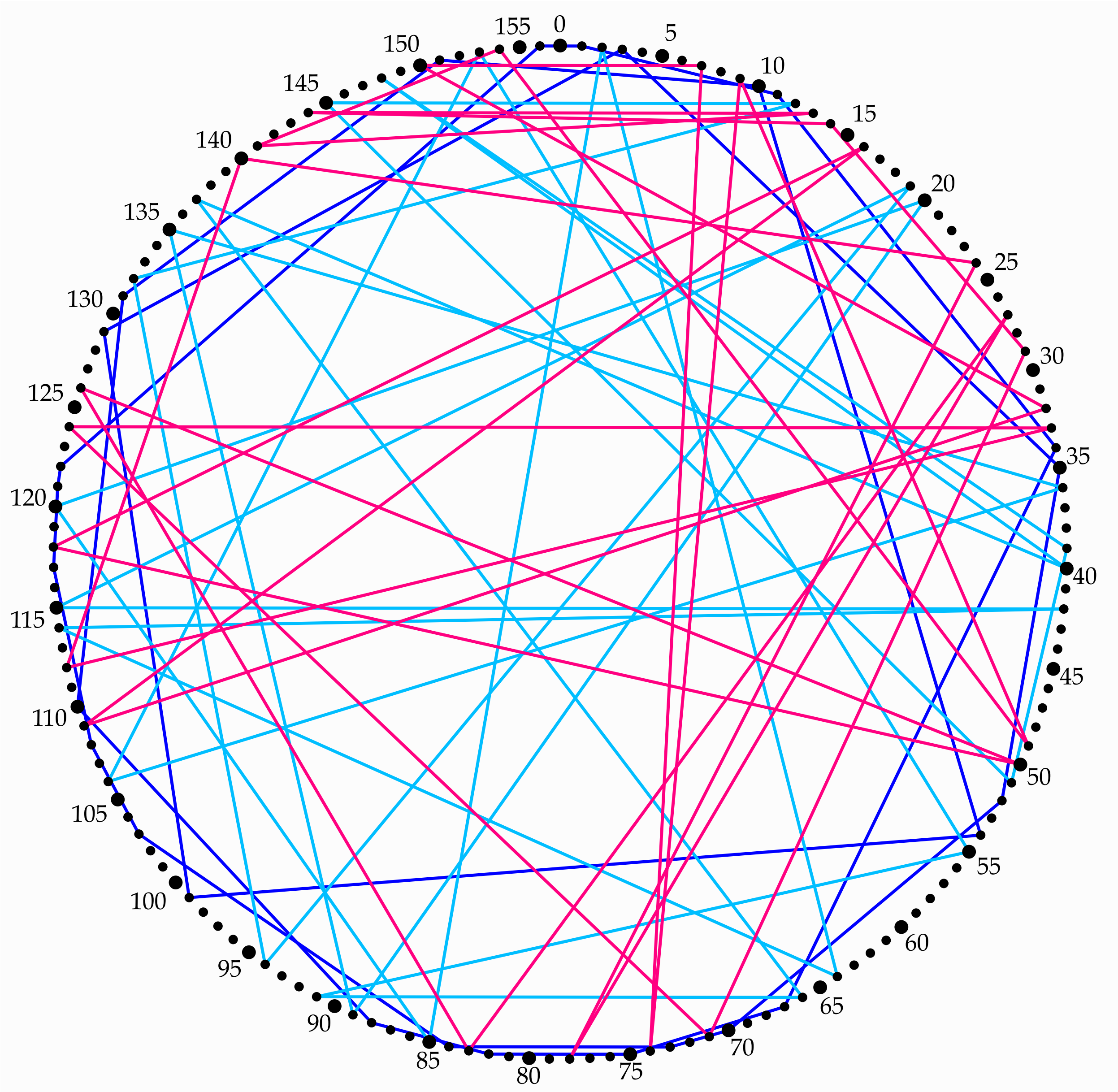



Figure 5.50: A $\mathcal{C}_{26}^3$ base block ($p = 3$, $k = 13$)

#### 5.4.1.6 The Design for $p = 5$ and $k = 3$

If $p = 5$ and $k = 3$, then $n = 4pk + 1 = 61$. Construction of the base block is facilitated by the action of the group $H = \{1, -3, 9, -27, 20\}$ on $\mathcal{D}_{61}$ and on $\mathbb{Z}_{61}^{*}$. The cycles in the base block and their corresponding difference sets are given in Table 5.51; the base block is shown in Figure 5.51.

Table 5.51: Cycle list for the $\mathcal{C}_6^5$ base block in Figure 5.51

| | |
|---|---|
| **Cycle** $\mathfrak{C}_1$ | (cobalt) |
| $C = \mathfrak{C}_1 = (1,\ 13,\ 19,\ 45,\ 50,\ 51)$ | |
| $T = \{1, 6, 12, 5, 11, 26\}$ | |
| **Cycle** $\mathfrak{C}_2$ | (sky) |
| $-3C = \mathfrak{C}_2 = (58,\ 22,\ 4,\ 48,\ 33,\ 30)$ | |
| $-3T = \{3, 18, 25, 15, 28, 17\}$ | |
| **Cycle** $\mathfrak{C}_3$ | (pink) |
| $9C = \mathfrak{C}_3 = (9,\ 56,\ 49,\ 39,\ 23,\ 32)$ | |
| $9T = \{9, 7, 14, 16, 23, 10\}$ | |
| **Cycle** $\mathfrak{C}_4$ | (lilac) |
| $-27C = \mathfrak{C}_4 = (34,\ 15,\ 36,\ 5,\ 53,\ 26)$ | |
| $-27T = \{27, 21, 19, 13, 8, 30\}$ | |
| **Cycle** $\mathfrak{C}_5$ | (apricot) |
| $20C = \mathfrak{C}_5 = (20,\ 16,\ 14,\ 46,\ 24,\ 44)$ | |
| $20T = \{20, 2, 4, 22, 24, 29\}$ | |

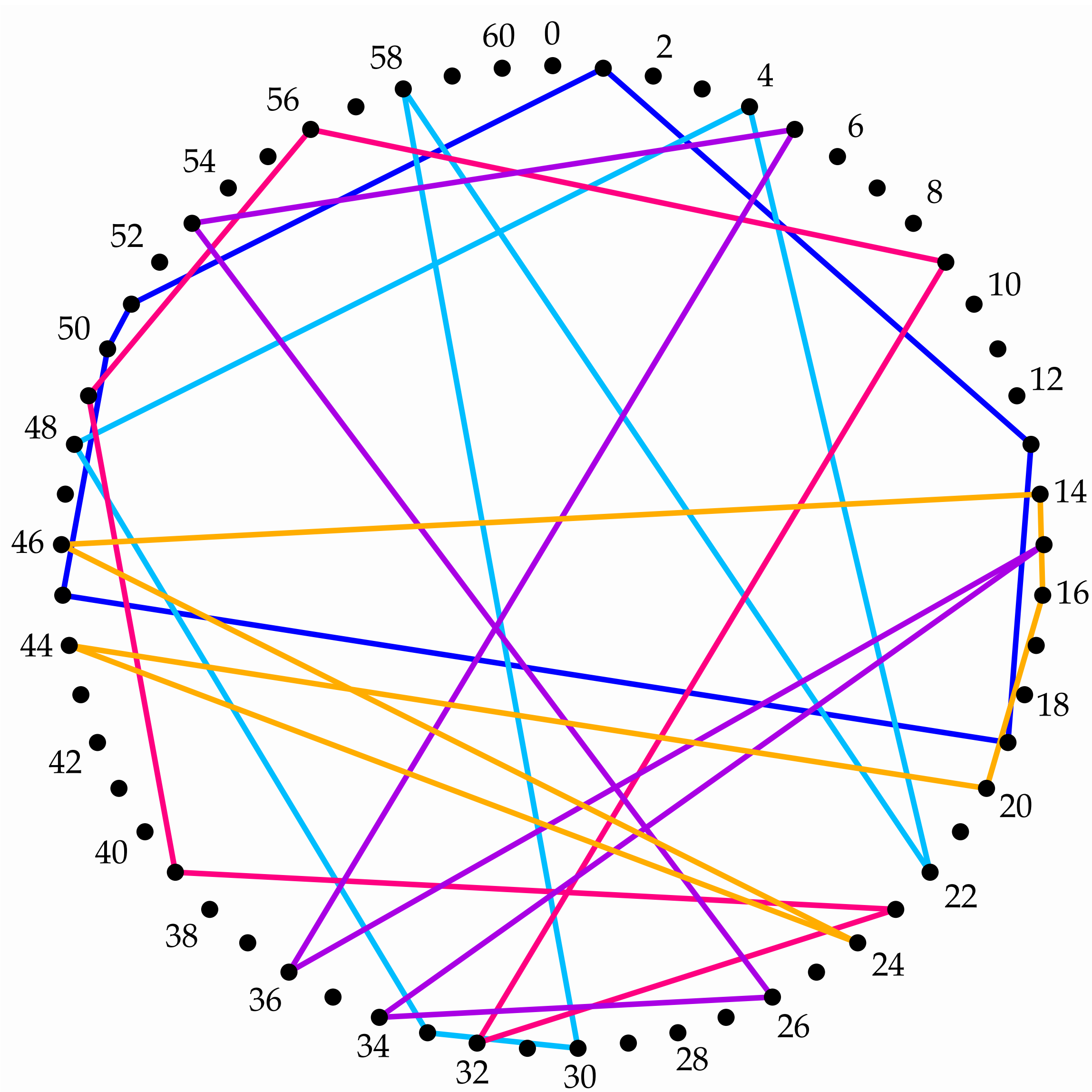



Figure 5.51: A $\mathcal{C}_6^5$ base block ($p = 5$, $k = 3$)

### 5.4.2 Constructions by the Prescribed Sum Method

The general approach in this method stems from a straightforward observation about differences occurring in cycles. Suppose that each of the differences in the set $\{d_1, \ldots, d_s\}$ occurs on exactly one edge of a cycle of length $s$ in $K_n$; then there exists a function $f : [\![1, s]\!] \to \{1, -1\}$ such that

$$\sum_{i=1}^{s} f(i) \cdot d_i \equiv 0 \pmod{n} \ . \tag{5.124}$$

This observation is a direct consequence of the fact that a cycle must begin and end at the same vertex. To see this, fix a vertex $v$ in the cycle and traverse the cycle beginning at $v$, recording $f(i) = 1$ if the edge of difference $d_i$ is traversed clockwise and $f(i) = -1$ otherwise; then congruence (5.124) must hold, as the left side records the net change in the vertex label from beginning to end of the cycle, which must be 0 modulo $n$.

In the first stage of construction by this method, we create a partition of the difference set $\mathcal{D}_{4kp+1} = [\![1, 2kp]\!]$ with $p$ subsets of size $2k$ and arrange the differences in each subset into a sum of the form in congruence (5.124) with value a multiple of $(4kp + 1)$. In the second stage, we determine an ordering of the signed differences $f(i) \cdot d_i$ for each subset and produce $p$ vertex-disjoint cycles to form the base block of a cyclic design. A pattern has emerged in the partitions of the difference sets, but no pattern is apparent in the subsequent parts of the construction.

Two of the three pairs $(p, k)$ for which designs were exhibited in the previous section are repeated here with a different design. These designs were created in the hope of discovering a general pattern in this method, and are included here for completeness.

#### 5.4.2.1 The Design for $p = 3$ and $k = 3$

If $p = 3$ and $k = 3$, then $n = 4pk + 1 = 37$. The difference patterns and cycles in the base block are given in Table 5.52; the base block is shown in Figure 5.52.

Table 5.52: Cycle list for the $\mathcal{C}_6^3$ base block in Figure 5.52

| **Cycle $\mathfrak{C}_1$** (orange) | **Cycle $\mathfrak{C}_2$** (mocha) | **Cycle $\mathfrak{C}_3$** (jade) |
|---|---|---|
| 18 15 [12] 9 4 3 | [13] [17] 7 [11] 2 [5] | 10 [1] [8] 14 6 16 |
| $\mathfrak{C}_1 = (0, 18, 33, 21, 30, 34)$ | $\mathfrak{C}_2 = (1, 25, 8, 15, 4, 6)$ | $\mathfrak{C}_3 = (2, 12, 11, 3, 17, 23)$ |

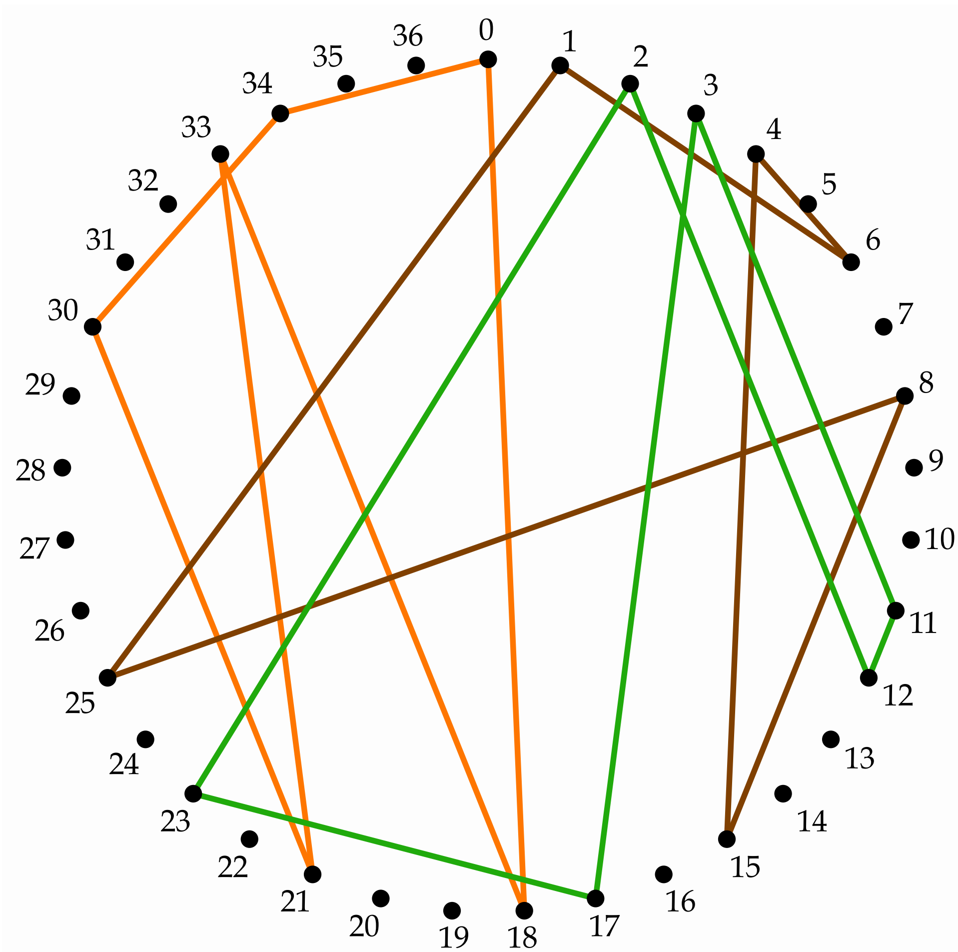



Figure 5.52: A $\mathcal{C}_6^3$ base block ($p = 3$, $k = 3$)

#### 5.4.2.2 The Design for $p = 3$ and $k = 5$

If $p = 3$ and $k = 5$, then $n = 4pk + 1 = 61$. The difference patterns and cycles in the base block are given in Table 5.53; the base block is shown in Figure 5.53.

Table 5.53: Cycle list for the $\mathcal{C}_{10}^{3}$ base block in Figure 5.53

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **Cycle $\mathfrak{C}_1$** | (orange) | | | | | | | | |
| 30 | [24] | [18] | [27] | 21 | [15] | [4] | [9] | [12] | [3] |
| $\mathfrak{C}_1 = (0, 30, 6, 49, 22, 43, 28, 24, 15, 3)$ | | | | | | | | | |
| **Cycle $\mathfrak{C}_2$** | (mocha) | | | | | | | | |
| [29] | [25] | 23 | [19] | [17] | [11] | 13 | [5] | 7 | 2 |
| $\mathfrak{C}_2 = (1, 33, 8, 31, 12, 56, 45, 58, 53, 60)$ | | | | | | | | | |
| **Cycle $\mathfrak{C}_3$** | (jade) | | | | | | | | |
| [28] | 20 | 16 | [6] | [26] | [22] | 8 | 1 | [10] | [14] |
| $\mathfrak{C}_3 = (2, 35, 55, 10, 4, 39, 17, 25, 26, 16)$ | | | | | | | | | |

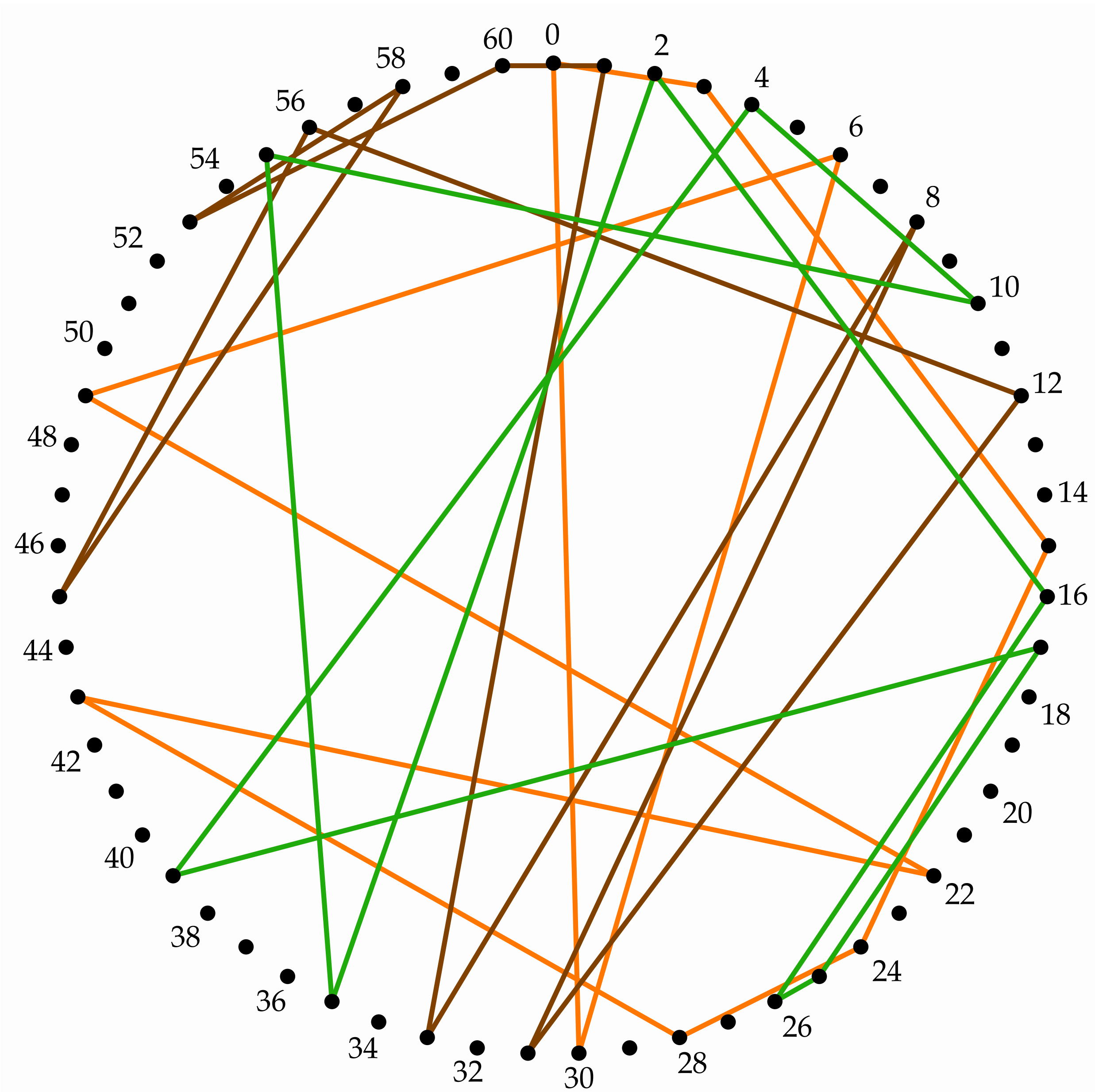



Figure 5.53: A $\mathcal{C}_{10}^{3}$ base block ($p = 3$, $k = 5$)

#### 5.4.2.3 The Design for $p = 3$ and $k = 7$

If $p = 3$ and $k = 7$, then $n = 4pk + 1 = 85$. This is the most significant design we have created using this method, because we can show that the first stage of the group action method must fail for $p = 3$ and $k = 7$. The difference patterns and cycles in the base block are given in Table 5.54; the base block is shown in Figure 5.54.

Table 5.54: Cycle list for the $\mathcal{C}_{14}^3$ base block in Figure 5.54

| |
|---|
| **Cycle** $\mathfrak{C}_1$ (orange) |
| 42 [30] [39] 21 36 27 18 3 [33] 15 [12] 24 9 4 |
| $\mathfrak{C}_1 = (0, 42, 12, 58, 79, 30, 57, 75, 78, 45, 60, 48, 72, 81)$ |
| **Cycle** $\mathfrak{C}_2$ (mocha) |
| 37 29 13 41 [17] 35 [25] [7] 19 [23] 5 [31] 11 [2] |
| $\mathfrak{C}_2 = (1, 38, 67, 80, 36, 19, 54, 29, 22, 41, 18, 23, 77, 3)$ |
| **Cycle** $\mathfrak{C}_3$ (jade) |
| [40] [38] 28 [32] 8 1 [10] [26] 20 34 [22] 14 [16] [6] |
| $\mathfrak{C}_3 = (2, 47, 9, 37, 5, 13, 14, 4, 63, 83, 32, 10, 24, 8)$ |

### 5.4.3 Comparative Analysis for Odd $k$ and Odd $p$

The comparison between our constructions and the construction by Blinco and El-Zanati is trivial in this case: our constructions have thus far yielded only nine base blocks, while the construction by Blinco and El-Zanati addresses all odd values of $p$ and $k$. We comment that our group action method, while not particularly fruitful in terms of base blocks, has spawned some interesting questions of a number-theoretic nature. We defer discussion of these questions to Chapter 6.

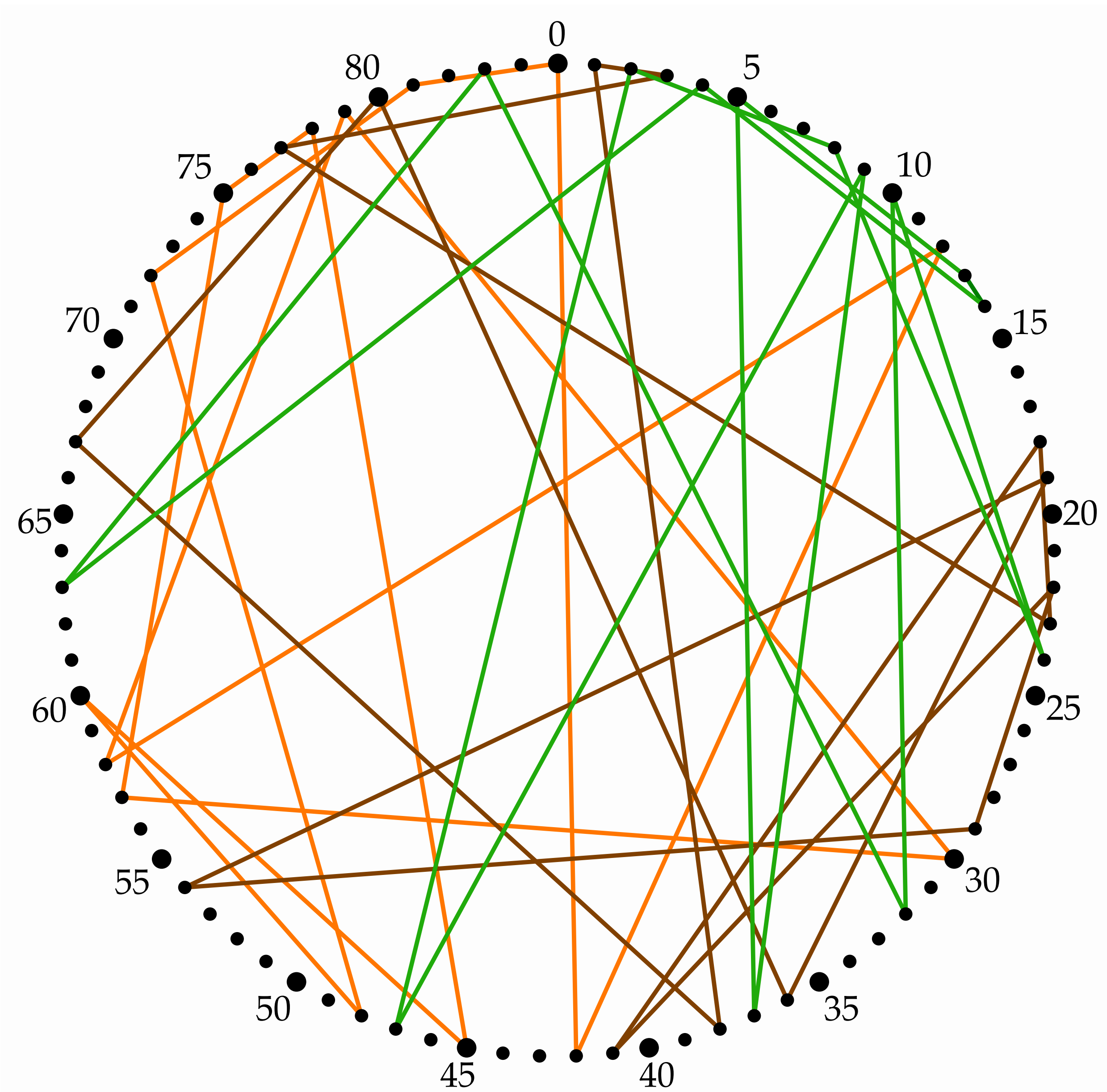



Figure 5.54: A $\mathcal{C}_{14}^{3}$ base block ($p = 3$, $k = 7$)

# Chapter 6

# Conjectures and Questions

In this chapter, we draw our discussion of bounded complete embedding graphs to a close with comments and ideas about future work on these graphs. With a view toward further progress, we summarize possible extensions and generalizations of the major ideas and approaches in our current efforts. In addition, we revisit our efforts to build $\mathcal{C}_{2k}^{p}$ base blocks for $p$ and $k$ both odd, posing some questions that arose during that work. We begin with these questions, what we know about their answers, and additional questions spurred by our investigations of these and related ideas.

## 6.1 Questions Generated by the Group Actions Method

In Section 5.4, we exhibited constructions of base blocks for purely cyclic $\mathcal{C}_{2k}^{p}$-designs on $K_{4kp+1}$ for odd values of $p$ and $k$ using two methods, one depending on group actions and the other on a fixed prescribed sum of differences. In the constructions by group actions, we used the action of a cyclic group of order $p$ to generate the $\mathcal{C}_{2k}^{p}$ base block from a single $2k$-cycle. We chose the cyclic group to be a subgroup of a particular group, called the *group of units*, that resides in the ring $\mathbb{Z}_{4kp+1}$. We pause to give two definitions and a fundamental result regarding this group.

**Definition 6.1.** Let $m \in \mathbb{P}$ such that $m \geq 2$. The set of invertible elements of $\mathbb{Z}_m$ is a group under multiplication modulo $m$, called the ***group of units of*** $\mathbb{Z}_m$; it is denoted $\mathscr{U}_m$. ■

**Definition 6.2.** For any $n \in \mathbb{P}$, let $\varphi(n)$ denote the number of positive integers less than $n$ that are relatively prime to $n$. The function $\varphi$ is known as the ***Euler*** $\varphi$***-function***. ■

The size of $\mathscr{U}_m$ is known to be $\varphi(m)$ (see [10], pp. 7–11). Furthermore, there is an explicit formula for $\varphi(m)$ in terms of the prime factors of $m$.

**Lemma 6.3.** *Let $m \in \mathbb{P}$ such that $m \geq 2$. Let $r \in \mathbb{P}$, let $q_1, \dots, q_r$ be distinct primes, and let $a_1, \dots, a_r$ be positive integers such that $m = q_1^{a_1} \cdots q_r^{a_r}$. Then*

$$\left|\mathscr{U}_m\right| = \varphi(m) = \prod_{i=1}^{r} q_i^{a_i - 1}(q_i - 1)\,.$$

The constructions of $\mathcal{C}_{2k}^{p}$-designs on $K_{4kp+1}$ by group actions were developed by considering the special case when $p$ and $k$ are odd and $(4kp+1)$ is prime; in this case, the necessary conditions for the construction are easily satisfied, as we see in the following remarks.

**Remark 6.4.** For any $p, k \in \mathbb{P}$, when $(4kp + 1)$ is prime, the ring $\mathbb{Z}_{4kp+1}$ is a field, so all of its nonzero elements are invertible, and $\mathscr{U}_{4kp+1} = \mathbb{Z}^*_{4kp+1}$ is a cyclic group of order $4kp$ (see [19], p. 116). Then, since $p$ divides $4kp$, the group $\mathscr{U}_{4kp+1}$ has a subgroup $H$ of order $p$ (see [19], p. 77). Since $\mathscr{U}_{4kp+1}$ itself is cyclic, $H$ must also be cyclic. ■

Recall that we define two actions of $H$ in the construction by group actions. For $h \in H$ and $x \in \mathbb{Z}^*_{4kp+1}$, the action $\alpha$ on $\mathbb{Z}^*_{4kp+1}$ is given by $\alpha(h, z) = \hat{z}$, where $\hat{z}$ is the unique element of $[\![-2kp, -1]\!] \cup [\![1, 2kp]\!]$ satisfying the congruence $h \cdot z \equiv \hat{z} \pmod{4kp + 1}$. The action $\beta$ on the set of differences $\mathcal{D}_{4kp+1}$ is given by $\beta(h, d) = \left|\alpha(h, d)\right|$, where $h \in H$ and $d \in \mathcal{D}_{4kp+1}$.

**Remark 6.5.** When $(4kp+1)$ is prime, since the group $H$ (from Remark 6.4) is a subgroup of $\mathscr{U}_{4kp+1} = \mathbb{Z}^*_{4kp+1}$, we have, for any $z \in \mathbb{Z}^*_{4kp+1}$, that the orbit of $z$ under the action $\alpha$ is simply the *coset* $Hz$ of $H$ in $\mathscr{U}_{4kp+1}$, and is guaranteed to have the same number of distinct elements as $H$ (see [19], p. 38). Since $H$ has order $p$, $\left|\operatorname{Orb}(z)\right| = p$ for each $z \in \mathbb{Z}^*_{4kp+1}$. Since the action $\beta$ is defined in terms of the action $\alpha$, it is easily verified that $\left|\operatorname{Orb}(d)\right| = p$ for each $d \in \mathcal{D}_{4kp+1}$. ■

In our investigations, we soon discovered that the construction by group actions can be applied successfully in some cases when $p$ and $k$ are odd but $(4kp + 1)$ is not prime. We identified two necessary conditions, which we stated in the discussion of the construction; we repeat these conditions here for reference.

(1) We must have a number $x \in \mathbb{Z}^*_{4kp+1}$ such that the set $H = \{x^i \mid i \in [\![1, p]\!]\}$ is a group of order $p$ under multiplication modulo $(4kp + 1)$; that is, the group $\mathscr{U}_{4kp+1}$ must have a cyclic subgroup of order $p$.

(2) We must have, for all $z \in \mathbb{Z}^*_{4kp+1}$, that the $p$ numbers in the set $\{x^i z \mid i \in [\![1, p]\!]\}$ are distinct modulo $(4kp + 1)$.

During our discussion of the construction by group actions, we assumed that both of these conditions held and described how to proceed if they do. Since we have found some values of $p$ and $k$ for which these conditions do not hold, we must address the issue of when these conditions hold and when they do not. We observe that it is absurd to discuss condition (2) if condition (1) does not hold, so we will assume that condition (1) holds in all subsequent discussions of condition (2). We are thus interested in the following questions.

**Question 6.6.** Given odd integers $p$ and $k$, both at least three, does $\mathscr{U}_{4kp+1}$ have a cyclic subgroup of order $p$?

**Question 6.7.** Given odd integers $p$ and $k$, both at least three, and given that $\mathscr{U}_{4kp+1}$ has a cyclic subgroup $H$ of order $p$ generated by the element $x$, do the orbits of elements of $\mathbb{Z}^*_{4kp+1}$ under the action $\alpha$ of $H$ all have size $p$?

In the remainder of this section, we summarize what we know about the answers to these questions, and pose new questions that arose during our pursuit of a particular related idea. We begin with a formal statement of our conclusions from Remarks 6.4 and 6.5 regarding conditions (1) and (2) when $(4kp + 1)$ is prime.

**Remark 6.8.** Let $p, k \in \mathbb{P}$ such that $p \geq 2$ and $k \geq 2$, and suppose that $(4kp + 1)$ is prime. Then conditions (1) and (2) both hold. ■

**Example 6.9.** If $p = 3$ and $k = 7$, then $4kp + 1 = 85 = 5 \cdot 17$ is not prime. By Lemma 6.3, the group $\mathscr{U}_{85}$ has order $\varphi(85) = (5-1)(17-1) = 64$, so it has no elements of order three; the desired subgroup $H$ therefore does not exist. ■

**Remark 6.10.** If $(4kp + 1)$ is not prime but $p$ is prime, then, by a theorem of Cauchy on group elements of prime order (see [19], p. 93), $\mathscr{U}_{4kp+1}$ has a cyclic subgroup of order $p$ if and only if $p \,\Big|\, \varphi(4kp + 1)$. Since $p$ clearly cannot divide $(4kp + 1)$, we have, by Lemma 6.3, that $p \,\Big|\, \varphi(4kp + 1)$ if and only if there is some prime factor $q$ of $(4kp + 1)$ such that $p \,\Big|\, (q - 1)$; that is, such that $q \equiv 1 \pmod{p}$.

Hence, if $p$ is prime and $(4kp + 1)$ is not prime, $\mathscr{U}_{4kp+1}$ has a cyclic subgroup of order $p$ if and only if there is at least one prime factor $q$ of $4kp + 1$ such that $q \equiv 1 \pmod{p}$. ■

In the case that $p$ and $(4kp + 1)$ are both composite, if $p \,\Big|\, \varphi(4kp + 1)$, then there is a subgroup of $\mathscr{U}_{4kp+1}$ of order $p$ (see [19], p. 77), but this subgroup is not guaranteed to be cyclic, so it may not have an element of order $p$, as is sought in condition (1). We have verified whether condition (1) holds for all pairs of odd values of $p$ and $k$ such that $p \in \{3, 5\}$ and $k \leq 75$. We exhibit these results in Tables 6.1, 6.2, and 6.3. For the pairs of values of $p$ and $k$ for which condition (1) holds, which are given in Tables 6.1 and 6.2, we include the minimum element, $x$, of order $p$ in $\mathscr{U}_{4kp+1}$ in the table entry for $p$ and $k$.

Table 6.1: Known odd values of $p$ and $k$ for which $4kp + 1$ is composite and condition (1) holds

| $p$ | $k$ | $4kp + 1$ | MIN $x$ | $p$ | $k$ | $4kp + 1$ | MIN $x$ |
|---|---|---|---|---|---|---|---|
| 3 | 11 | $133 = 7 \cdot 19$ | 11 | 5 | 17 | $341 = 11 \cdot 31$ | 4 |
| 3 | 25 | $301 = 7 \cdot 43$ | 36 | 5 | 39 | $781 = 11 \cdot 71$ | 5 |
| 3 | 27 | $325 = 5^2 \cdot 13$ | 126 | 5 | 43 | $861 = 3 \cdot 7 \cdot 41$ | 379 |
| 3 | 39 | $469 = 7 \cdot 67$ | 29 | 5 | 61 | $1221 = 3 \cdot 11 \cdot 37$ | 223 |
| 3 | 49 | $589 = 19 \cdot 31$ | 87 | | | | |
| 3 | 53 | $637 = 7^2 \cdot 13$ | 79 | | | | |
| 3 | 67 | $805 = 5 \cdot 7 \cdot 23$ | 116 | | | | |

Table 6.2: Known odd values of $p$ and $k$ for which $4kp+1$ is prime

| $p$ | $k$ | $4kp+1$ | MIN $x$ | $p$ | $k$ | $4kp+1$ | MIN $x$ | $p$ | $k$ | $4kp+1$ | MIN $x$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 3 | 37 | 10 | 3 | 51 | 613 | 65 | 5 | 23 | 461 | 88 |
| 3 | 5 | 61 | 13 | 3 | 55 | 661 | 296 | 5 | 27 | 541 | 48 |
| 3 | 9 | 109 | 45 | 3 | 59 | 709 | 227 | 5 | 33 | 661 | 197 |
| 3 | 13 | 157 | 12 | 3 | 61 | 733 | 307 | 5 | 35 | 701 | 89 |
| 3 | 15 | 181 | 48 | 3 | 63 | 757 | 27 | 5 | 41 | 821 | 51 |
| 3 | 19 | 229 | 94 | 3 | 69 | 829 | 125 | 5 | 47 | 941 | 349 |
| 3 | 23 | 277 | 116 | 3 | 71 | 853 | 220 | 5 | 51 | 1021 | 589 |
| 3 | 29 | 349 | 122 | 3 | 73 | 877 | 282 | 5 | 53 | 1061 | 220 |
| 3 | 31 | 373 | 88 | 5 | 3 | 61 | 9 | 5 | 59 | 1181 | 81 |
| 3 | 33 | 397 | 34 | 5 | 5 | 101 | 36 | 5 | 65 | 1301 | 163 |
| 3 | 35 | 421 | 20 | 5 | 9 | 181 | 42 | 5 | 69 | 1381 | 75 |
| 3 | 45 | 541 | 129 | 5 | 21 | 421 | 252 | | | | |

Table 6.3: Known odd values of $p$ and $k$ for which $4kp+1$ is composite and condition (1) fails

| $p$ | $k$ | $4kp+1$ | $p$ | $k$ | $4kp+1$ | $p$ | $k$ | $4kp+1$ |
|---|---|---|---|---|---|---|---|---|
| 3 | 7 | $85 = 5 \cdot 17$ | 5 | 7 | $141 = 3 \cdot 47$ | 5 | 49 | $981 = 9 \cdot 109$ |
| 3 | 17 | $205 = 5 \cdot 41$ | 5 | 11 | $221 = 13 \cdot 17$ | 5 | 55 | $1101 = 3 \cdot 367$ |
| 3 | 21 | $253 = 11 \cdot 23$ | 5 | 13 | $261 = 9 \cdot 29$ | 5 | 57 | $1141 = 7 \cdot 163$ |
| 3 | 37 | $445 = 5 \cdot 89$ | 5 | 15 | $301 = 7 \cdot 43$ | 5 | 63 | $1261 = 13 \cdot 97$ |
| 3 | 41 | $493 = 17 \cdot 29$ | 5 | 19 | $381 = 3 \cdot 127$ | 5 | 67 | $1341 = 9 \cdot 149$ |
| 3 | 43 | $517 = 11 \cdot 47$ | 5 | 25 | $501 = 3 \cdot 167$ | 5 | 71 | $1421 = 7^2 \cdot 29$ |
| 3 | 47 | $565 = 5 \cdot 113$ | 5 | 29 | $581 = 7 \cdot 83$ | 5 | 73 | $1461 = 3 \cdot 487$ |
| 3 | 57 | $685 = 5 \cdot 137$ | 5 | 31 | $621 = 3^3 \cdot 23$ | 5 | 75 | $1501 = 19 \cdot 79$ |
| 3 | 65 | $781 = 11 \cdot 71$ | 5 | 37 | $741 = 3 \cdot 13 \cdot 19$ | | | |
| 3 | 75 | $901 = 17 \cdot 53$ | 5 | 45 | $901 = 17 \cdot 53$ | | | |

**Example 6.11.** If $p = 3$ and $k = 11$, we have that $4kp + 1 = 133 = 7 \cdot 19$. The group $\mathscr{U}_{133}$ has order $\varphi(133) = (7-1)(19-1) = 108$, so it has a cyclic subgroup of order $p = 3$, as is required by condition (1). One such subgroup is $H = \{1, 11, -12\}$. Furthermore, for all $z \in \mathbb{Z}^*_{133}$, the $p$ elements of the set $\left\{11^i z \mid i \in [\![1,3]\!]\right\}$ are indeed distinct modulo 133, as is required by condition (2). ■

Example 6.11 is the smallest case in which $p$ and $k$ are odd, $(4kp+1)$ is composite, and conditions (1) and (2) are both satisfied. We were able to complete the construction by group actions for this case; the base block is shown in Figure 5.49. We observe that, in this case, the number $(4kp+1)$ has a special form:

$$4kp+1 = t(t+1)+1 = t^2+t+1\,, \tag{6.1}$$

in particular for $t = 11$. Having observed this, we began investigating numbers of this form for $p = 3$. We obtained the following result.

**Theorem 6.12.** *Let $t \in \mathbb{P}$, and let $N = t(t+1)+1$. If $t \geq 2$, then $\mathscr{U}_N$ has a cyclic subgroup of order six generated by $(t+1)$, and this subgroup contains the element $t$, which generates a cyclic subgroup of order three.*

*Proof.* Let $t \in \mathbb{P}$, and suppose $t \geq 2$. Let $N = t(t+1)+1$; then $N \geq 7$. We observe that 1 and $-1$ are always elements of $\mathscr{U}_m$ for any $m \geq 3$; hence 1 and $-1$ are elements of $\mathscr{U}_N$. Note that $(-t)(t+1) \equiv 1 \pmod N$ and $-(t+1)\cdot t \equiv 1 \pmod N$, so $t$, $(-t)$, $(t+1)$, and $-(t+1)$ are all invertible in $\mathbb{Z}_N$ and are thus elements of $\mathscr{U}_N$. Since $N \geq 7$, it is easily verified that $1$, $-1$, $t$, $(-t)$, $(t+1)$, and $-(t+1)$ are all distinct elements of $\mathscr{U}_N$.

Furthermore, we have

$$\begin{aligned}
(t+1)^2 &\equiv t \pmod N\\
(t+1)^3 &\equiv -1 \equiv t^2+t \pmod N\\
(t+1)^4 &\equiv -(t+1) \equiv t^2 \pmod N\\
(t+1)^5 &\equiv -t \equiv t^2+1 \pmod N\\
(t+1)^6 &\equiv 1 \pmod N
\end{aligned}$$

So $(t+1)$ generates a cyclic group of order six in $\mathscr{U}_N$, namely $\{1,\ (t+1),\ t,\ t^2+t,\ t^2,\ t^2+1\}$. Since $t \equiv (t+1)^2 \pmod N$, $t$ has order three in $\mathscr{U}_N$, as desired. □

**Remark 6.13.** If $p = 3$ and $4kp+1 = 12k+1 = t(t+1)+1$ for some $t \in \mathbb{P}$, then condition (1) is satisfied. ■

Our progress on condition (2) is more limited than our progress on condition (1). Thus far, we have verified that condition (2) holds for two pairs of odd values of $p$ and $k$ for which $(4kp+1)$ is composite: the pair in Example 6.11 and the pair in Example 6.14.

**Example 6.14.** If $p = 3$ and $k = 105$, we have that $4kp+1 = 1261 = t(t+1)+1$ for $t = 35$. The group $\mathscr{U}_{1261}$ thus has a cyclic subgroup of order $p = 3$, as is required by condition (1). One such subgroup is $H = \{1, 35, -36\}$. Furthermore, for all $z \in \mathbb{Z}^{*}_{1261}$, the $p$ elements of the set $\left\{35^i z \mid i \in [\![1, 3]\!]\right\}$ are distinct modulo 1261, as is required by condition (2). ■

Our initial investigation into the integers of the form $12k+1 = t(t+1)+1$ has expanded into a general exploration of numbers of the form $t(t+1)+1$. We note that, if $t \equiv 2 \pmod 3$ or $t \equiv 0 \pmod 3$, then $t(t+1)+1 \equiv 1 \pmod 3$, and if $t \equiv 1 \pmod 3$, then $t(t+1)+1 \equiv 0 \pmod 3$. We have explored the prime factorizations of these integers up to $t = 60$; none of them has a prime factor $q$ satisfying $q \equiv 2 \pmod 3$. We are curious whether this is true of all integers of this form. We are also interested in the prime factors of integers of a more general form: $t(rt+1)+1$, for positive integers $r$ and $t$.

**Question 6.15.** Is there a positive integer $t$ such that $t(t+1)+1$ has a prime factor $q$ satisfying $q \equiv 2 \pmod 3$?

**Question 6.16.** Let $r, t \in \mathbb{P}$, and let $Q = t(rt+1)+1$. Can we characterize the prime factors of $Q$ in terms of their congruence classes modulo some positive integer $m$?

These questions provide avenues for future research that extend far beyond the study of bounded complete embedding graphs. We now close our discussion of number-theoretic questions and return to discussion of our central topic.

## 6.2 Future Work on Bounded Complete Embedding Graphs

In Chapters 1, 3, and 4, we identified several families of graphs as bounded complete embedding graphs. These families are summarized below.

- Even cycles are bounded complete embedding graphs.
  (Theorem 1.40, Corollary 1.41, and Corollary 1.43, Rodger, 1990)

- Paths are bounded complete embedding graphs. (Theorem 3.10)

- Complete bipartite graphs are bounded complete embedding graphs.
  (Theorem 3.14)

- Any graph $G$ having all vertices of even degree and $e(G) = 2^k$ for some $k \in \mathbb{P}$ that admits a $\beta^+$-labeling is a bounded complete embedding graph. (Theorem 3.17)

  In particular,

  * the cubes $Q_n$, where $n = 2^t$ for some $t \in \mathbb{P}$, are bounded complete embedding graphs; (Corollary 3.18)

  * the graphs $C_{2a} \uplus C_{2b}$, where the positive integers $a$ and $b$ are not equal and are each at least two, and where $2a + 2b = 2^t$ for some $t \in \mathbb{P}$, are bounded complete embedding graphs; (Corollary 3.19)

  and

  * the graphs $C_{2a} \uplus C_{2b} \uplus C_{2c}$, where the positive integers $a$, $b$, and $c$ are not all equal and are each at least two, and where $2a + 2b + 2c = 2^t$ for some $t \in \mathbb{P}$, are bounded complete embedding graphs. (Corollary 3.20)

- The graphs $\mathcal{C}_{2k}^p$ for which $p, k \in [\![2, 128]\!]$ are bounded complete embedding graphs.
  (Theorem 4.16)

- The graphs $\mathcal{C}_{2k}^p$ for which $p \in \left\{2^t \mid t \in \mathbb{P}\right\}$ are bounded complete embedding graphs.
  (Theorem 4.18)

Numerous infinite families of bipartite graphs remain open to investigation as potential bounded complete embedding graphs. We wish to pursue many of these families.

Of particular interest are the remaining graphs $\mathcal{C}_{2k}^{p}$. The choice of the bound 128 for terminating computation of divisors for the Dovetail Construction is somewhat arbitrary; we are confident that continued pursuit of these computations will yield further results. It is clearly impossible, however, to verify these divisors for infinitely many values of $p$ and $k$ by direct computation. We hope to achieve additional results, possibly including new constructions, to complete the proof of the following conjecture.

**Conjecture 6.17.** *The graphs $\mathcal{C}_{2k}^{p}$ are bounded complete embedding graphs for all $p, k \in \mathbb{P}$ such that $p \geq 2$ and $k \geq 2$.*

We also intend to study the broader family of 2-regular bipartite graphs, since existing labeling results for these graphs reduce the task of showing that such a graph, $G$, is a bounded complete embedding graph to exhibiting $G$-designs on $K_{2e(G), n-1}$ for all $n \in \mathrm{SSpec}(G)$ such that $n \not\equiv 1 \pmod{2e(G)}$. Further success with the cohorts of even cycles may provide constructions that generalize nicely to accommodate these graphs.

The literature on graph designs is rich with results about path designs and complete bipartite graph designs, and the comets $S_{k,2}$ admit $\beta^{+}$-labelings, so we are also interested in the following graphs.

- cohorts of paths and, more generally, graphs whose components are paths
- cohorts of stars and graphs whose components are stars
- cohorts of complete bipartite graphs and graphs whose components are complete bipartite graphs
- the comets $S_{k,2}$ and the cohorts of $S_{k,2}$

The graphs listed above are natural choices for future pursuit, since much is known about their component graphs. In order to reach beyond these families of graphs, we identify a few characteristics of a graph, $G$, that have the potential to simplify the process of building $G$-designs. Clearly, any graph we consider must be bipartite, since all bounded complete embedding graphs are bipartite. Furthermore, regular graphs provide the strongest information from the Superspectral Conditions, especially SSC–3; graphs with most or all vertices having small degree offer more options in the building of designs. We also note that it is common, once a particular design question has been settled for cycles, to consider chorded cycles and other graphs formed by adding edges to a cycle. These considerations motivate our interest in the two families of graphs we define below.

**Definition 6.18.** Let $n$ be a positive integer, and let $D \subseteq \mathcal{D}_n$. The subgraph of $K_n$ having precisely those edges whose differences are in $D$ is called the ***difference graph corresponding to the difference set D***; we denote this graph by $K_n[D]$. If we describe $D$ by listing its elements, we omit the set brackets around this list in the notation for the difference graph; *e.g.*, we write $K_n[d_1, d_2]$, not $K_n[\{d_1, d_2\}]$. ■

We note that, for even integers $n$, some of the difference graphs $K_n[D]$ are bipartite; in particular, in order for $K_n[D]$ to be bipartite, all the differences in $D$ must have the same parity, and all must be divisors of $n$. We are, of course, interested in whether such graphs are bounded complete embedding graphs. One subfamily of these graphs is of particular interest, as they have all of the desirable characteristics we described above.

**Definition 6.19.** Let $k$ be a positive, odd integer. The ***marigold graph of order 2k***, denoted $M_{2k}$, is the difference graph $K_{2k}[1, k]$. ■

**Example 6.20.** The marigold graph $M_{14}$ is shown in Figure 6.1. ■

We observe that marigold graphs could be defined for all positive integers $k$, but the graphs obtained from even values of $k$ would not be bipartite. Since we desire only bipartite graphs, we exclude the even values of $k$ from our definition.

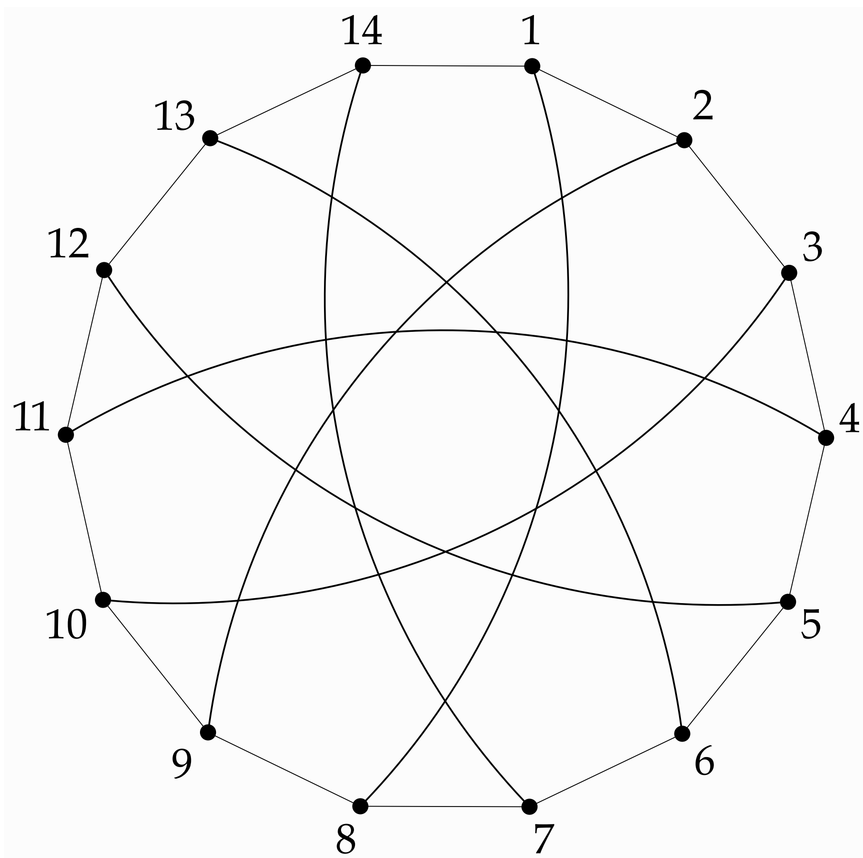


Figure 6.1: The marigold graph $M_{14}$

## 6.3 Extensions and Generalizations

We close our discussion with a few ideas for extending and generalizing our work. The first of these is, of course, the question of identifying bounded embedding graphs, since the definition of a bounded embedding graph is the natural generalization of the definition of a bounded complete embedding graph to include partial designs. Since every bounded embedding graph is also a bounded complete embedding graph, we will likely only investigate this question for those graphs that have already been identified as bounded complete embedding graphs. Even restricting ourselves in this way, we have several families of graphs to investigate: paths of length at least four, even cycles of length at least six, complete bipartite graphs with both parts of size at least two, and some of the cohorts of even cycles.

We may also generalize our current work by considering designs with holes. In order to embed a complete $G$-design of order $n$ in a complete $G$-design of order $r$, we must build a $G$-design on the graph $K_r - K_n$, which is the graph obtained by removing the edges of a $K_n$ from a $K_r$. In the context of building embeddings, we only attempt to build such designs for $n, r \in \mathrm{Spec}(G)$. We may generalize our work by considering $G$-designs on $K_r - K_n$ for all positive integers $n$ and $r$ such that $n < r$. There are also potentially interesting questions about bounds on the value of $r - n$.

Since our current work requires the building of $G$-designs on infinitely many complete bipartite graphs, another possible extension is to ask for necessary and sufficient conditions for a $G$-design on $K_{r,s}$ for particular graphs $G$ of interest. Clearly, it is necessary that $G$ be bipartite, that $v(G) \le r + s$, that $e(G) \big| rs$, and that $\gcd\big\{ \deg_G(v) \,\big|\, v \in V(G) \big\} \,\Big|\, \gcd\{r, s\}$. Results of this kind have already been achieved for $G = C_{2k}$ by Sotteau [32], for $G = P_k$ by Parker [26], and for $G = K_{a,b}$ by Hoffman and Liatti [16]; we would like to pursue similar results for cohorts of even cycles, cohorts of paths, and cohorts of complete bipartite graphs.

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# Appendices

# Appendix A

## Superspectra of the Cohorts of Even Cycles

We have computed the exact superspectrum of $\mathcal{C}_{2k}^{p}$ for all pairs of values of $p$ and $k$ such that $p, k \in [\![2, 128]\!]$. The Python code in Figure A.1 generates a text file (having roughly 96880 lines) that lists, for each pair $(p, k)$, the canonical representatives of all congruence classes modulo $4kp$ in the superspectrum of $\mathcal{C}_{2k}^{p}$ and the elements, $n$, of those congruence classes that are excluded from the superspectrum because they satisfy the double inequality $1 < n < 2kp$. The superspectrum of $\mathcal{C}_{2k}^{p}$ is given in Table A.1 for twenty-one selected pairs of values of $p$ and $k$.

Complete tables listing the exact superspectrum of $\mathcal{C}_{2k}^{p}$ for all pairs of values of $p$ and $k$ such that $p, k \in [\![2, 128]\!]$ are provided in the companion document titled *Extension A: The Superspectra of the Cohorts of Even Cycles, Revisited.* This document is available (in PDF format) through the Auburn University Repository of Research Activities (AUrora). The following is a permanent URL for the document.

<http://hdl.handle.net/11200/48522>

```
import math
import sys
import time
import itertools

f = open('SSpecP2to128K2to128','w')
PValuesToCheck = [x for x in xrange(2,129)]
KValuesToCheck = [x for x in xrange(2,129)]

f.write('Superspectra of the Cohorts of Even Cycles, ')
f.write('for p in [2, 128], k in [2, 128] \n \n')
f.write('-------------------------------------------------- \n \n')
for p in PValuesToCheck:
    for k in KValuesToCheck:
        f.write('p = ' + str(p) + '    k = ' + str(k) + '\n')
        f.write('congruence classes modulo 4kp = ' + str(4*k*p) + ': ')
        Classes = []
        for n in xrange(1,4*k*p):
            if (((n*(n-1))%(4*k*p)) == 0) and ((n-1)%2 == 0):
                Classes.append(n)
        f.write(str(Classes) + ' \n')
        MyExcludes = []
        for y in xrange(2,2*k*p):
            if (((y*(y-1))%(4*k*p)) == 0) and ((y-1)%2 == 0):
                MyExcludes.append(y)
        f.write('Exceptions (1 < n < 2kp): ' + str(MyExcludes) + ' \n \n')
        f.write('-------------------------------------------------- \n \n')
```

Figure A.1: Python code to output $\mathrm{SSpec}(\mathcal{C}_{2k}^{p})$ for all $p, k \in [\![2, 128]\!]$

Table A.1: The superspectrum of $\mathcal{C}^p_{2k}$, for selected values of $p$ and $k$

| $p$ | $k$ | $\mathrm{SSpec}(\mathcal{C}^p_{2k})$ consists of those $n \in \mathbb{P}$ such that: |
|---|---|---|
| 2 | 4 | $n \equiv 1 \pmod{32}$ |
| 2 | 11 | $n \equiv 1$ or $33 \pmod{88}$ except $n = 33$ |
| 2 | 15 | $n \equiv 1, 25, 81,$ or $105 \pmod{120}$ except $n = 25$ |
| 3 | 4 | $n \equiv 1$ or $33 \pmod{48}$ |
| 3 | 7 | $n \equiv 1, 21, 49,$ or $57 \pmod{84}$ except $n = 21$ |
| 3 | 11 | $n \equiv 1, 33, 45,$ or $121 \pmod{132}$ except $n = 33, 45$ |
| 3 | 35 | $n \equiv 1, 21, 85, 105, 141, 225, 301,$ or $385 \pmod{420}$ except $n = 21, 85, 105, 141$ |
| 5 | 17 | $n \equiv 1, 85, 205,$ or $221 \pmod{340}$ except $n = 85$ |
| 12 | 2 | $n \equiv 1$ or $33 \pmod{96}$ except $n = 33$ |
| 12 | 5 | $n \equiv 1, 81, 145,$ or $225 \pmod{240}$ except $n = 81$ |
| 12 | 12 | $n \equiv 1$ or $513 \pmod{576}$ |
| 12 | 60 | $n \equiv 1, 1665, 2241,$ or $2305 \pmod{2880}$ |
| 12 | 120 | $n \equiv 1, 1665, 2305,$ or $5121 \pmod{5760}$ except $n = 1665, 2305$ |
| 35 | 4 | $n \equiv 1, 161, 225,$ or $385 \pmod{560}$ except $n = 161, 225$ |
| 35 | 6 | $n \equiv 1, 105, 225, 385, 441, 505, 561,$ or $721 \pmod{840}$ except $n = 105, 225, 385$ |
| 42 | 13 | $n \equiv 1, 105, 169, 273, 729, 897, 1561,$ or $1729 \pmod{2184}$ except $n = 105, 169, 273, 729, 897$ |
| 42 | 65 | $n \equiv 1, 105, 1561, 2185, 3081, 3745, 4641, 5265, 6721,$ $6825, 7281, 8281, 8841, 8905, 9465,$ or $10465 \pmod{10920}$ except $n = 105, 1561, 2185, 3081, 3745, 4641, 5265$ |
| 97 | 2 | $n \equiv 1$ or $97 \pmod{776}$ except $n = 97$ |
| 97 | 11 | $n \equiv 1, 485, 2717,$ or $3201 \pmod{4268}$ except $n = 485$ |
| 97 | 35 | $n \equiv 1, 1261, 8925, 10185, 10865, 11641, 12125,$ or $12901 \pmod{13580}$ except $n = 1261$ |
| 97 | 128 | $n \equiv 1$ or $40449 \pmod{49664}$ |

## Appendix B

## Existence of Satisfactory Divisors for the Dovetail Construction

In this Appendix, we are concerned with the existence of satisfactory divisors for the Dovetail Construction (Theorem 4.8), as required in the proof of Theorem 4.15. We repeat the statements of these theorems below for reference. We also repeat a small portion of the proof of Theorem 4.15, as we will need the notation that is defined therein.

**Theorem** (The Dovetail Construction)**.** *Let $k, p \in \mathbb{P}$ such that $k \geq 2$ and $p \geq 2$, and let $n \in \mathrm{SSpec}\big(\mathcal{C}_{2k}^{p}\big)$. If there exists a divisor $z$ of $n-1$ such that $p \leq z \leq \dfrac{n-1}{k}$ and $\dfrac{n-1}{z}$ is even, then there exists a $\mathcal{C}_{2k}^{p}$-design on $K_{4kp,\,n-1}$.*

**Theorem** (4.15)**.** *Let $k, p \in \mathbb{P}$ such that $2 \leq k \leq 128$ and $2 \leq p \leq 128$. Then there is some positive integer $N(k,p)$ such that a $\mathcal{C}_{2k}^{p}$-design on $K_{4kp,\,n-1}$ exists for every $n \in \mathrm{SSpec}\big(\mathcal{C}_{2k}^{p}\big)$ such that $n \geq N(k,p)$.*

Excerpt from the Proof of Theorem 4.15:

> Let $k, p \in \mathbb{P}$ such that $k \geq 2$ and $p \geq 2$. Let $M$ denote the number of distinct modular congruence classes in $\mathrm{SSpec}\big(\mathcal{C}_{2k}^{p}\big)$, and let $\big\{n_i \;\big|\; i \in [\![1, M]\!]\big\}$ be the set of canonical representatives of those congruence classes. It suffices to show that, for each $i \in [\![1, M]\!]$, there is some $N_i \in \mathrm{SSpec}\big(\mathcal{C}_{2k}^{p}\big)$ such that $N_i \equiv n_i \pmod{4kp}$ and there is a divisor $z_i$ of $N_i - 1$ satisfying the conditions of the Dovetail Construction.

In order to verify the existence of the required values $N_i$ and their divisors, we have implemented a short Python program to complete the necessary computations. The Python code in Figure B.1 generates a text file that lists, for each pair $(p, k)$, the representatives $n_i$, the values $N_i$, and all possible values of $z_i$ for each $N_i$. As shown, the code generates output for all pairs $(p, k)$ such that $p, k \in [\![2, 32]\!]$, omitting all values of $p$ that are powers of two (since success is guaranteed for those values by Corollary 4.9). Changing the ranges of the variables *PValues* and *KValues* appropriately will produce output for the remaining pairs. The code includes an easy way to check that values $N_i$ and their corresponding divisors have been successfully identified for all pairs $(p, k)$ in the output: success has been achieved if the output file does not contain the alert message "Value Too Large: Search Aborted."

Table B.1 lists divisor information for thirty-nine selected pairs of values of $p$ and $k$; for each selected pair, the following information is given: the congruence class representatives $n_i$, the required value $N_i$ in each congruence class, the smallest possible divisor $z_i$ corresponding to each $N_i$, and the value of $N(k, p)$.

Complete tables listing divisor information for all pairs of values of $p$ and $k$ such that $p, k \in [\![2, 128]\!]$ are provided in the companion document titled *Extension B: Satisfactory Divisors for the Dovetail Construction, Revisited*. This document is available (in PDF format) through the Auburn University Repository of Research Activities (AUrora). The following is a permanent URL for the document.

<http://hdl.handle.net/11200/48522>

```
import math
import sys
import time
import itertools

f = open(’DivisorsP2to32K2to32’,’w’)
PValues = [3,5,6,7]+[x for x in xrange(9,16)]+[x for x in xrange(17,32)]
KValues = [x for x in xrange(2,33)]

def SuperSpectrum(k,p):
    numbers = []
    for n in xrange(1,4*k*p):
        if (((n*(n-1))%(4*k*p)) == 0) and ((n-1)%2 == 0):
            numbers.append(n)
    return numbers

def Divisors(c):
    CZvalues = []
    mybound = int(math.floor((c-1)/k))
    for z in xrange(p,mybound+1):
        if ((c-1)%z == 0) and ((c-1)/z)%2 == 0:
            CZvalues.append(z)
    f.write(’\n \n        n = ’ + str(c) + ’    n-1 = ’ + str(c-1))
    if len(CZvalues) != 0:
        f.write(’   Divisors:  ’ + str(CZvalues) + ’\n \n’)
    else:
        f.write(’   Divisors:   NONE’)
        if c < (400*k*p):
            Divisors(c+(4*k*p))
        else:
            f.write(’\n    Value Too Large: Search Aborted’)

f.write(’Divisors for the Dovetail Construction \n \n’)
f.write(’-------------------------------------------- \n \n’)
for p in PValues:
    for k in KValues:
        f.write(’Divisors found for p = ’ + str(p) + ’, ’)
        f.write(’k = ’ + str(k) + ’\n\n’)
        Classes = []
        Classes = SuperSpectrum(k,p)
        for c in Classes:
            f.write(’For the congruence class n == ’)
            f.write(str(c) + ’ mod ’ + str(4*k*p) + ’:’)
            if (c >= 2*k*p):
                Divisors(c)
            elif (c < (2*k*p)):
                Divisors(c+(4*k*p))
        f.write(’\n-------------------------------------------- \n \n’)
```

Figure B.1: Python code to output divisor lists for all $p, k \in [\![2, 32]\!]$

Table B.1: Divisors for selected values of $p$ and $k$

| $p$ | $k$ | $4kp$ | $i$ | $n_i$ | $N_i$ | $N_i - 1$ | $z_i$ | $N(k,p)$ |
|---|---|---|---|---|---|---|---|---|
| 3 | 2 | 24 | 1 | 1 | 25 | 24 | 3 | |
| | | | 2 | 9 | 33 | 32 | 4 | 33 |
| 3 | 3 | 36 | 1 | 1 | 37 | 36 | 3 | |
| | | | 2 | 9 | 45 | 44 | 11 | 45 |
| 3 | 4 | 48 | 1 | 1 | 49 | 48 | 3 | |
| | | | 2 | 33 | 33 | 32 | 4 | 49 |
| 3 | 5 | 60 | 1 | 1 | 61 | 60 | 3 | |
| | | | 2 | 21 | 81 | 80 | 4 | |
| | | | 3 | 25 | 85 | 84 | 3 | |
| | | | 4 | 45 | 105 | 104 | 4 | 105 |
| 3 | 6 | 72 | 1 | 1 | 73 | 72 | 3 | |
| | | | 2 | 9 | 81 | 80 | 4 | 81 |
| 3 | 7 | 84 | 1 | 1 | 85 | 84 | 3 | |
| | | | 2 | 21 | 105 | 104 | 4 | |
| | | | 3 | 49 | 49 | 48 | 3 | |
| | | | 4 | 57 | 57 | 56 | 4 | 105 |
| 3 | 34 | 408 | 1 | 1 | 409 | 408 | 3 | |
| | | | 2 | 153 | 561 | 560 | 4 | |
| | | | 3 | 273 | 273 | 272 | 4 | |
| | | | 4 | 289 | 289 | 288 | 3 | 561 |

*continued on next page*

Table B.1: Divisors for selected $p$ and $k$, *continued*

| $p$ | $k$ | $4kp$ | $i$ | $n_i$ | $N_i$ | $N_i - 1$ | $z_i$ | $N(k,p)$ |
|---|---|---|---|---|---|---|---|---|
| 3 | 35 | 420 | 1 | 1 | 421 | 420 | 3 | |
| | | | 2 | 21 | 441 | 440 | 4 | |
| | | | 3 | 85 | 505 | 504 | 3 | |
| | | | 4 | 105 | 945 | 944 | 4 | |
| | | | 5 | 141 | 561 | 560 | 4 | |
| | | | 6 | 225 | 225 | 224 | 4 | |
| | | | 7 | 301 | 301 | 300 | 3 | |
| | | | 8 | 385 | 385 | 384 | 3 | 945 |
| 3 | 36 | 432 | 1 | 1 | 433 | 432 | 3 | |
| | | | 2 | 81 | 513 | 512 | 4 | 513 |
| 5 | 29 | 580 | 1 | 1 | 581 | 580 | 5 | |
| | | | 2 | 145 | 1885 | 1884 | 6 | |
| | | | 3 | 261 | 841 | 840 | 5 | |
| | | | 4 | 465 | 465 | 464 | 8 | 1885 |
| 5 | 30 | 600 | 1 | 1 | 601 | 600 | 5 | |
| | | | 2 | 25 | 625 | 624 | 6 | |
| | | | 3 | 201 | 801 | 800 | 5 | |
| | | | 4 | 225 | 1425 | 1424 | 8 | 1425 |
| 5 | 31 | 620 | 1 | 1 | 621 | 620 | 5 | |
| | | | 2 | 125 | 745 | 744 | 6 | |
| | | | 3 | 341 | 341 | 340 | 5 | |
| | | | 4 | 465 | 465 | 464 | 8 | 745 |

*continued on next page*

Table B.1: Divisors for selected $p$ and $k$, *continued*

| $p$ | $k$ | $4kp$ | $i$ | $n_i$ | $N_i$ | $N_i - 1$ | $z_i$ | $N(k,p)$ |
|---|---|---|---|---|---|---|---|---|
| 5 | 32 | 640 | 1 | 1 | 641 | 640 | 5 | |
| | | | 2 | 385 | 385 | 384 | 6 | 641 |
| 5 | 33 | 660 | 1 | 1 | 661 | 660 | 5 | |
| | | | 2 | 45 | 705 | 704 | 8 | |
| | | | 3 | 121 | 781 | 780 | 5 | |
| | | | 4 | 165 | 1485 | 1484 | 7 | |
| | | | 5 | 265 | 925 | 924 | 6 | |
| | | | 6 | 385 | 385 | 384 | 6 | |
| | | | 7 | 441 | 441 | 440 | 5 | |
| | | | 8 | 561 | 561 | 560 | 5 | 1485 |
| 33 | 12 | 1584 | 1 | 1 | 1585 | 1584 | 36 | |
| | | | 2 | 1089 | 1089 | 1088 | 34 | |
| | | | 3 | 1233 | 1233 | 1232 | 44 | |
| | | | 4 | 1441 | 1441 | 1440 | 36 | 1585 |
| 33 | 13 | 1716 | 1 | 1 | 1717 | 1716 | 39 | |
| | | | 2 | 429 | 2145 | 2144 | 67 | |
| | | | 3 | 573 | 2289 | 2288 | 44 | |
| | | | 4 | 781 | 2497 | 2496 | 39 | |
| | | | 5 | 793 | 2509 | 2508 | 38 | |
| | | | 6 | 1353 | 1353 | 1352 | 52 | |
| | | | 7 | 1365 | 1365 | 1364 | 62 | |
| | | | 8 | 1573 | 3289 | 3288 | 137 | 3289 |

*continued on next page*

Table B.1: Divisors for selected $p$ and $k$, *continued*

| $p$ | $k$ | $4kp$ | $i$ | $n_i$ | $N_i$ | $N_i - 1$ | $z_i$ | $N(k,p)$ |
|---|---|---|---|---|---|---|---|---|
| 33 | 14 | 1848 | 1 | 1 | 1849 | 1848 | 42 | |
| | | | 2 | 385 | 2233 | 2232 | 36 | |
| | | | 3 | 441 | 2289 | 2288 | 44 | |
| | | | 4 | 561 | 2409 | 2408 | 43 | |
| | | | 5 | 1057 | 1057 | 1056 | 44 | |
| | | | 6 | 1177 | 1177 | 1176 | 42 | |
| | | | 7 | 1233 | 1233 | 1232 | 44 | |
| | | | 8 | 1617 | 1617 | 1616 | 101 | 2409 |
| 33 | 15 | 1980 | 1 | 1 | 1981 | 1980 | 33 | |
| | | | 2 | 45 | 2025 | 2024 | 44 | |
| | | | 3 | 441 | 2421 | 2420 | 55 | |
| | | | 4 | 1045 | 1045 | 1044 | 58 | |
| | | | 5 | 1441 | 1441 | 1440 | 36 | |
| | | | 6 | 1485 | 1485 | 1484 | 53 | |
| | | | 7 | 1585 | 1585 | 1584 | 33 | |
| | | | 8 | 1881 | 1881 | 1880 | 47 | 2421 |
| 33 | 16 | 2112 | 1 | 1 | 2113 | 2112 | 33 | |
| | | | 2 | 385 | 2497 | 2496 | 39 | |
| | | | 3 | 705 | 2817 | 2816 | 44 | |
| | | | 4 | 1089 | 1089 | 1088 | 34 | 2817 |

*continued on next page*

Table B.1: Divisors for selected $p$ and $k$, *continued*

| $p$ | $k$ | $4kp$ | $i$ | $n_i$ | $N_i$ | $N_i - 1$ | $z_i$ | $N(k,p)$ |
|---|---|---|---|---|---|---|---|---|
| 33 | 17 | 2244 | 1 | 1 | 2245 | 2244 | 33 | |
| | | | 2 | 561 | 9537 | 9536 | 149 | |
| | | | 3 | 969 | 3213 | 3212 | 73 | |
| | | | 4 | 1089 | 3333 | 3332 | 34 | |
| | | | 5 | 1309 | 3553 | 3552 | 37 | |
| | | | 6 | 1497 | 1497 | 1496 | 34 | |
| | | | 7 | 1717 | 1717 | 1716 | 33 | |
| | | | 8 | 1837 | 1837 | 1836 | 34 | 9537 |
| 33 | 18 | 2376 | 1 | 1 | 2377 | 2376 | 33 | |
| | | | 2 | 297 | 7425 | 7424 | 58 | |
| | | | 3 | 649 | 3025 | 3024 | 36 | |
| | | | 4 | 2025 | 2025 | 2024 | 44 | 7425 |
| 84 | 53 | 17808 | 1 | 1 | 17809 | 17808 | 84 | |
| | | | 2 | 4081 | 21889 | 21888 | 96 | |
| | | | 3 | 4929 | 22737 | 22736 | 98 | |
| | | | 4 | 5089 | 22897 | 22896 | 106 | |
| | | | 5 | 5937 | 23745 | 23744 | 106 | |
| | | | 6 | 10017 | 27825 | 27824 | 94 | |
| | | | 7 | 11025 | 11025 | 11024 | 104 | |
| | | | 8 | 16801 | 16801 | 16800 | 84 | 27825 |
| 84 | 54 | 18144 | 1 | 1 | 18145 | 18144 | 84 | |
| | | | 2 | 3969 | 40257 | 40256 | 136 | |
| | | | 3 | 7777 | 25921 | 25920 | 90 | |
| | | | 4 | 14337 | 14337 | 14336 | 112 | 40257 |

*continued on next page*

Table B.1: Divisors for selected $p$ and $k$, *continued*

| $p$ | $k$ | $4kp$ | $i$ | $n_i$ | $N_i$ | $N_i - 1$ | $z_i$ | $N(k,p)$ |
|---|---|---|---|---|---|---|---|---|
| 84 | 55 | 18480 | 1 | 1 | 18481 | 18480 | 84 | |
| | | | 2 | 385 | 18865 | 18864 | 131 | |
| | | | 3 | 561 | 19041 | 19040 | 85 | |
| | | | 4 | 3025 | 21505 | 21504 | 84 | |
| | | | 5 | 4081 | 22561 | 22560 | 94 | |
| | | | 6 | 5985 | 24465 | 24464 | 88 | |
| | | | 7 | 6721 | 25201 | 25200 | 84 | |
| | | | 8 | 8625 | 27105 | 27104 | 88 | |
| | | | 9 | 9681 | 9681 | 9680 | 88 | |
| | | | 10 | 12145 | 12145 | 12144 | 88 | |
| | | | 11 | 12321 | 12321 | 12320 | 88 | |
| | | | 12 | 12705 | 49665 | 49664 | 97 | |
| | | | 13 | 14785 | 14785 | 14784 | 84 | |
| | | | 14 | 15345 | 15345 | 15344 | 137 | |
| | | | 15 | 15841 | 15841 | 15840 | 88 | |
| | | | 16 | 16401 | 16401 | 16400 | 100 | 49665 |
| 84 | 56 | 18816 | 1 | 1 | 18817 | 18816 | 84 | |
| | | | 2 | 3969 | 22785 | 22784 | 89 | |
| | | | 3 | 6273 | 25089 | 25088 | 98 | |
| | | | 4 | 16513 | 16513 | 16512 | 86 | 25089 |
| 101 | 14 | 5656 | 1 | 1 | 5657 | 5656 | 202 | |
| | | | 2 | 505 | 6161 | 6160 | 110 | |
| | | | 3 | 1617 | 7273 | 7272 | 202 | |
| | | | 4 | 2121 | 7777 | 7776 | 108 | 7777 |

*continued on next page*

Table B.1: Divisors for selected $p$ and $k$, *continued*

| $p$ | $k$ | $4kp$ | $i$ | $n_i$ | $N_i$ | $N_i - 1$ | $z_i$ | $N(k,p)$ |
|---|---|---|---|---|---|---|---|---|
| 101 | 15 | 6060 | 1 | 1 | 6061 | 6060 | 202 | |
| | | | 2 | 405 | 6465 | 6464 | 202 | |
| | | | 3 | 505 | 12625 | 12624 | 263 | |
| | | | 4 | 2121 | 8181 | 8180 | 409 | |
| | | | 5 | 2425 | 8485 | 8484 | 202 | |
| | | | 6 | 4041 | 4041 | 4040 | 202 | |
| | | | 7 | 4141 | 4141 | 4140 | 115 | |
| | | | 8 | 4545 | 4545 | 4544 | 142 | 12625 |
| 101 | 16 | 6464 | 1 | 1 | 6465 | 6464 | 202 | |
| | | | 2 | 4545 | 4545 | 4544 | 142 | 6465 |
| 101 | 17 | 6868 | 1 | 1 | 6869 | 6868 | 202 | |
| | | | 2 | 1717 | 8585 | 8584 | 116 | |
| | | | 3 | 3333 | 10201 | 10200 | 102 | |
| | | | 4 | 5253 | 5253 | 5252 | 202 | 10201 |
| 101 | 18 | 7272 | 1 | 1 | 7273 | 7272 | 202 | |
| | | | 2 | 505 | 7777 | 7776 | 108 | |
| | | | 3 | 4041 | 4041 | 4040 | 202 | |
| | | | 4 | 4545 | 4545 | 4544 | 142 | 7777 |
| 101 | 19 | 7676 | 1 | 1 | 7677 | 7676 | 202 | |
| | | | 2 | 5757 | 13433 | 13432 | 146 | |
| | | | 3 | 6061 | 6061 | 6060 | 202 | |
| | | | 4 | 7373 | 7373 | 7372 | 194 | 13433 |

*continued on next page*

Table B.1: Divisors for selected $p$ and $k$, *continued*

| $p$ | $k$ | $4kp$ | $i$ | $n_i$ | $N_i$ | $N_i - 1$ | $z_i$ | $N(k,p)$ |
|---|---|---|---|---|---|---|---|---|
| 101 | 20 | 8080 | 1 | 1 | 8081 | 8080 | 202 | |
| | | | 2 | 4545 | 4545 | 4544 | 142 | |
| | | | 3 | 6161 | 6161 | 6160 | 110 | |
| | | | 4 | 6465 | 6465 | 6464 | 202 | 8081 |
| 117 | 64 | 29952 | 1 | 1 | 29953 | 29952 | 117 | |
| | | | 2 | 9217 | 39169 | 39168 | 128 | |
| | | | 3 | 16641 | 16641 | 16640 | 128 | |
| | | | 4 | 25857 | 25857 | 25856 | 128 | 39169 |
| 117 | 65 | 30420 | 1 | 1 | 30421 | 30420 | 117 | |
| | | | 2 | 1521 | 123201 | 123200 | 140 | |
| | | | 3 | 6085 | 36505 | 36504 | 117 | |
| | | | 4 | 7605 | 38025 | 38024 | 194 | |
| | | | 5 | 8281 | 38701 | 38700 | 129 | |
| | | | 6 | 14365 | 44785 | 44784 | 311 | |
| | | | 7 | 23661 | 23661 | 23660 | 130 | |
| | | | 8 | 29745 | 29745 | 29744 | 143 | 123201 |
| 117 | 66 | 30888 | 1 | 1 | 30889 | 30888 | 117 | |
| | | | 2 | 9153 | 40041 | 40040 | 130 | |
| | | | 3 | 10153 | 41041 | 41040 | 120 | |
| | | | 4 | 19305 | 19305 | 19304 | 127 | |
| | | | 5 | 19657 | 19657 | 19656 | 117 | |
| | | | 6 | 21385 | 21385 | 21384 | 132 | |
| | | | 7 | 28809 | 28809 | 28808 | 277 | |
| | | | 8 | 30537 | 30537 | 30536 | 347 | 41041 |

*continued on next page*

Table B.1: Divisors for selected $p$ and $k$, *continued*

| $p$ | $k$ | $4kp$ | $i$ | $n_i$ | $N_i$ | $N_i - 1$ | $z_i$ | $N(k,p)$ |
|---|---|---|---|---|---|---|---|---|
| 117 | 67 | 31356 | 1 | 1 | 31357 | 31356 | 117 | |
| | | | 2 | 469 | 31825 | 31824 | 117 | |
| | | | 3 | 23049 | 23049 | 23048 | 134 | |
| | | | 4 | 23517 | 54873 | 54872 | 361 | |
| | | | 5 | 26533 | 26533 | 26532 | 134 | |
| | | | 6 | 27001 | 27001 | 27000 | 125 | |
| | | | 7 | 27873 | 27873 | 27872 | 134 | |
| | | | 8 | 28341 | 28341 | 28340 | 130 | 54873 |
| 117 | 68 | 31824 | 1 | 1 | 31825 | 31824 | 117 | |
| | | | 2 | 3537 | 35361 | 35360 | 130 | |
| | | | 3 | 7345 | 39169 | 39168 | 128 | |
| | | | 4 | 10881 | 42705 | 42704 | 136 | |
| | | | 5 | 14977 | 46801 | 46800 | 117 | |
| | | | 6 | 18513 | 18513 | 18512 | 178 | |
| | | | 7 | 22321 | 22321 | 22320 | 120 | |
| | | | 8 | 25857 | 25857 | 25856 | 128 | 46801 |
| 117 | 69 | 32292 | 1 | 1 | 32293 | 32292 | 117 | |
| | | | 2 | 8073 | 72657 | 72656 | 152 | |
| | | | 3 | 9477 | 41769 | 41768 | 227 | |
| | | | 4 | 10557 | 42849 | 42848 | 206 | |
| | | | 5 | 11961 | 44253 | 44252 | 299 | |
| | | | 6 | 28405 | 28405 | 28404 | 263 | |
| | | | 7 | 29809 | 29809 | 29808 | 138 | |
| | | | 8 | 30889 | 30889 | 30888 | 117 | 72657 |

*continued on next page*

Table B.1: Divisors for selected $p$ and $k$, *continued*

| $p$ | $k$ | $4kp$ | $i$ | $n_i$ | $N_i$ | $N_i - 1$ | $z_i$ | $N(k,p)$ |
|---|---|---|---|---|---|---|---|---|
| 117 | 70 | 32760 | 1 | 1 | 32761 | 32760 | 117 | |
| | | | 2 | 3745 | 36505 | 36504 | 117 | |
| | | | 3 | 5265 | 38025 | 38024 | 194 | |
| | | | 4 | 7281 | 40041 | 40040 | 130 | |
| | | | 5 | 8281 | 41041 | 41040 | 120 | |
| | | | 6 | 11025 | 43785 | 43784 | 421 | |
| | | | 7 | 13105 | 45865 | 45864 | 117 | |
| | | | 8 | 15561 | 48321 | 48320 | 151 | |
| | | | 9 | 17641 | 17641 | 17640 | 126 | |
| | | | 10 | 20385 | 20385 | 20384 | 182 | |
| | | | 11 | 21385 | 21385 | 21384 | 132 | |
| | | | 12 | 23401 | 23401 | 23400 | 117 | |
| | | | 13 | 24921 | 24921 | 24920 | 140 | |
| | | | 14 | 28665 | 61425 | 61424 | 349 | |
| | | | 15 | 30681 | 30681 | 30680 | 118 | |
| | | | 16 | 30745 | 30745 | 30744 | 122 | 61425 |