Coefficient Space Properties and a Schur Algebra Generalization

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Vita

David Presnell Turner, son of Thomas E. Turner and Mary Elizabeth Shope Turner, was born June 24, 1960, in Lancaster, Pennsylvania. He graduated from North Gallia High School in Vinton, Ohio in 1978. He entered Rio Grande College in September 1978. He graduated with the degree of Bachelor of Science in mathematics in June 1986. He entered Indiana University in Bloomington, Indiana in August 1986. He graduated with the degree of Master of Arts in mathematics in May 1988. He entered Purdue University in West Lafayette, Indiana in August 1989. He graduated with the degree of Master of Science in mathematics in May 1994. He enrolled in the Ph.D. program in the Department of Mathematics, Auburn University in September, 1993. He worked as Adjunct Instructor of Mathematics for Lexington Community College in Lexington, Kentucky from August 1991 to June 1993. He has taught mathematics and physics at Faulkner University in Montgomery, Alabama from August 1993 to the present. He married Brenda White, daughter of Fred and Ruth (Dillon) White, on August 31, 1984.

Dissertation Abstract

Coefficient Space Properties and a Schur Algebra Generalization

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Let K be an infinite field and $\Gamma = \mathrm{GL}_n(K)$. If we linearly extend the natural action of Γ on the set E of n-dimensional column vectors over K to the group algebra $K\Gamma$, then E becomes a $K\Gamma$-module. We then construct the $K\Gamma$-module $E^{\otimes r}$, the r-fold tensor product of E. The image $S_r(\Gamma)$ of the corresponding representation of $K\Gamma$ is called the *Schur algebra*. If E is replaced by a different $K\Gamma$-module L, the same construction results in an algebra $S_{r,L}$. The subalgebra $A(n)$ of K^Γ generated by the *coordinate functions* $c_{\alpha\beta} : \Gamma \to K$ with $1 \leq \alpha, \beta \leq n$ is a bialgebra. $A(n)$ has a subcoalgebra A_r which consists of homogeneous polynomials of total degree r in the indeterminants $c_{\alpha\beta}$. Classically, the dual A_r^* of A_r is an algebra isomorphic to $S_r(\Gamma)$ and A_r is the coefficient space of $E^{\otimes r}$. We identify $S_{r,L}$ with the dual $A_{r,L}^*$ of the coefficient space $A_{r,L}$ of $L^{\otimes r}$ and give a description of $A_{r,L}$.

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The style manual used is *Auburn University Graduate School Guide to Preparation and Submission of Theses and Dissertations.* The bibliography uses the style in *LaTeX: A Document Preparation System* by Leslie Lamport.

The computer software packages used are LaTeX 2ε and AMS-LaTeX with diagrams generated by XY-pic. All packages are under a MiKTeX implementation with WinEdt used as the text editor. The final output was printed from Adobe Acrobat.

Table of Contents

List of Figures

Chapter 1

Preliminaries

Definitions and statements of standard results in the theory of modules, algebras, group rings, tensor products, representations, characters, and linear functionals have been drawn from [1-8, 10].

1.1 Modules, Algebras, and Group Rings

1 DEFINITION. Let R be a ring. A *left R-module* is an additive abelian group M together with a function $R \times M \to M$ $((r, m) \mapsto rm)$ which satisfies the module axioms (i) $r(m+n) = rm + rn$, (ii) $(r+s)m = rm + sm$, and (iii) $r(sm) = (rs)m$ for all $r, s \in R$ and $m, n \in M$. A *right R-module* has a similar definition with r on the right. Let M be a (left) R-module. M is called *unitary* if R has an identity 1_R and $1_R \cdot m = m$ for all $m \in M$. N is called an *R-submodule* of M if N is a subgroup of M and $rn \in N$ for all $r \in R$ and $n \in N$. If N is an R-module, a function $f : M \to N$ such that $f(m+n) = f(m) + f(n)$ and $f(rm) = rf(m)$ for all $m, n \in M$ and $r \in R$ is called an *R-module homomorphism.* The set of all R-modules homomorphisms from M to N is denoted $\mathrm{Hom}_R(M, N)$. Let K be a field. A unitary K-module V, a K-submodule of V, and a K-module homomorphism are called a *K-space*, a K-subspace, and a K-linear map, respectively.

"Module" means "left module" unless otherwise noted. K always represents a field. Since K is commutative, a K-space V can be viewed as a right K-space by defining $kv = vk$ for all $k \in K$ and $v \in V$. An injective, surjective, or bijective homomorphism is called a monomorphism, epimorphism, or isomorphism, respectively.

2 EXAMPLES. Let R be a ring and $f : M \to N$ an R-module homomorphism. Then $\ker f = f^{-1}(\{0\})$ is an R-submodule of M, $\operatorname{im} f$ is an R-submodule of N, and the quotient group $M/N = \{m + N \mid m \in M\}$ is an R-module called a *quotient module*.

3 THEOREM (First Isomorphism Theorem)**.** If $f : M \to N$ is an R-module homomorphism then $M/\ker f \cong \operatorname{im} f$.

Proof. See [5, p. 172]. □

4 DEFINITION. A K-*algebra* is a ring A with identity such that A is a K-space (with addition via the ring structure) satisfying the *algebra condition* $k(ab) = (ka)b = a(kb)$ for all $k \in K$ and $a, b \in A$. A K-*subalgebra* of a K-algebra is a subring that is also a K-subspace. If A and B are K-algebras, then a K-*algebra homomorphism* is a ring homomorphism $\varphi : A \to B$ mapping 1_A to 1_B such that $\varphi(ka) = k\varphi(a)$ for all $k \in K$ and $a \in A$.

5 LEMMA. Let A be a ring with identity. Then A is a K-algebra if and only if there is a ring homomorphism $f : K \to A$ such that $f(K) \subseteq \operatorname{cent}(A)$ and $f(1_K) = 1_A$.

Proof. ($\Longrightarrow$) Define $f : K \to A$ by $f(k) = k1_A$. We have that f is a ring homomorphism since $f(jk) = (jk)1_A = j(k1_A) = j(k(1_A 1_A)) = j(1_A(k1_A)) = (j1_A)(k1_A) = f(j)f(k)$ and $f(j + k) = (j + k)1_A = j1_A + k1_A = f(j) + f(k)$ $(j, k \in K)$ by the algebra condition and module axiom (ii). Also $f(k)a = (k1_A)a = k(1_A a) = ka = k(a1_A) = a(k1_A) = af(k)$ $(k \in K, a \in A)$ implies $f(K) \subseteq \operatorname{cent}(A)$, and $f(1_K) = 1_K 1_A = 1_A$ since A is unitary.

($\Longleftarrow$) Define $k \cdot a = f(k)a$ $(k \in K, a \in A)$ where $f(k)a$ is the multiplication in the ring A. Note $f(k)a = af(k)$ since $f(K) \subseteq \operatorname{cent}(A)$. Let $j, k \in K$ and $a, b \in A$. Since A satisfies ring distributive and associative laws, and f is a ring homomorphism,

(i) $k \cdot (a + b) = f(k)(a + b) = f(k)a + f(k)b = k \cdot a + k \cdot b$,

(ii) $(j + k) \cdot a = f(j + k)a = (f(j) + f(k))a = f(j)a + f(k)a = j \cdot a + k \cdot a$,

(iii) $j \cdot (k \cdot a) = f(j)(f(k)a) = (f(j)f(k))a = f(jk)a = (jk) \cdot a$,

(iv) $1_K \cdot a = f(1_K)a = 1_A a = a$, (v) $k \cdot (ab) = f(k)(ab) = (f(k)a)b = (k \cdot a)b$, and

(vi) $(k \cdot a)b = (f(k)a)b = (af(k))b = a(f(k)b) = a(k \cdot b)$.

Thus A is a K-space by (i) - (iv), and satisfies the algebra condition by (v) and (vi). □

6 NOTATION. Let $\Gamma = \Gamma_n$ ($n \in \mathbb{Z}^+$) denote the general linear group $\mathrm{GL}_n(K)$ and put $K^\Gamma := \{f \mid f : \Gamma \to K\}$.

7 EXAMPLES. The following are K-algebras: (a) K, (b) the set $\mathrm{Mat}_n K$ of all $n \times n$ matrices over K, (c) the set $\mathrm{End}_K(V)$ of all K-linear maps from a K-space V to itself, and (d) K^Γ with pointwise addition and multiplication, and identity $1_{K^\Gamma}(g) = 1_K$ for all $g \in \Gamma$.

8 DEFINITION. Let G be a group and R a commutative ring with identity $1_R \neq 0_R$. The *group ring* RG of G over R is the set of all (formal) sums $\sum_{g \in G} r_g g$ where only finitely many $r_g \in R$ satisfy $r_g \neq 0_R$. The equation $\sum_{g \in G} r_g g + \sum_{g \in G} s_g g = \sum_{g \in G}(r_g + s_g)g$ defines addition while $\big(\sum_{g \in G} r_g g\big)\big(\sum_{h \in G} s_h h\big) = \sum_{g,\, h \in G}(r_g s_h)(gh) = \sum_{g \in G}\big(\sum_{h \in G} r_{gh^{-1}} s_h\big)g$ defines multiplication where $r_g s_h$ is the product in R and gh is the product in G.

RG is a ring. By the definition of multiplication, RG is commutative if and only if G is abelian. We may consider G as a subset of RG by identifying $g \in G$ with $1_R g$. Similarly, $R \subseteq RG$ by identifying $r \in R$ with $r1_G$. Thus, by restriction, any KG-module may be viewed as a K-space. Further, KG is a K-space with scalar multiplication given by the ring multiplication (viewing $K \subseteq KG$).

9 LEMMA. Let H be a group. Then KH is a K-algebra.

Proof. KH is a ring by the preceding remark. It has identity $1_K 1_H$. Define $f : K \to KH$ by $f(k) = k1_H$ ($k \in K$). So $f(j + k) = (j + k)1_H = j1_H + k1_H = f(j) + f(k)$ ($j, k \in K$) and, by the definition of multiplication in KH,

$$f(jk) = (jk)1_H = (jk)(1_H 1_H) = (j1_H)(k1_H) = f(j)f(k).$$

Consequently f is a ring homomorphism. For $k \in K$ and $s \in KH$, we have

$$f(k)s = (k1_H)s = ks = sk = s(k1_H) = sf(k),$$

so $f(K) \subseteq \text{cent}\,(KH)$. Also $f(1_K) = 1_K 1_H$. Lemma (5) implies KH is a K-algebra. □

10 THEOREM. Let H be a group, A a K-algebra, and $A^\times$ the multiplicative group of invertible elements of A. Then every group homomorphism $\varphi : H \to A^\times$ has a unique extension to a K-algebra homomorphism $\overline{\varphi} : KH \to A$.

Proof. Suppose $\varphi : H \to A^\times$ is a group homomorphism. We define $\overline{\varphi} : KH \to A$ by $\overline{\varphi}\big(\sum_{h\in H} a_h h\big) = \sum_{h\in H} a_h \varphi(h)$. Then

$$\begin{aligned}\overline{\varphi}\Big(\sum_{h\in H} a_h h + \sum_{h\in H} b_h h\Big) &= \overline{\varphi}\Big(\sum_{h\in H}(a_h h + b_h h)\Big) = \sum_{h\in H}(a_h + b_h)\varphi(h)\\ &= \sum_{h\in H} a_h\varphi(h) + \sum_{h\in H} b_h\varphi(h) = \overline{\varphi}\Big(\sum_{h\in H} a_h h\Big) + \overline{\varphi}\Big(\sum_{h\in H} b_h h\Big)\end{aligned}$$

and

$$\begin{aligned}\overline{\varphi}\Big(\Big[\sum_{h\in H} a_h h\Big]\Big[\sum_{h\in H} b_h h\Big]\Big) &= \overline{\varphi}\Big(\sum_{g\in H}\Big[\sum_{h\in H} a_{gh^{-1}} b_h\Big] g\Big) = \sum_{g\in H}\Big(\sum_{h\in H} a_{gh^{-1}} b_h\Big)\varphi(g)\\ &= \sum_{h\in H}\Big(\sum_{g\in H} a_{gh^{-1}} b_h\Big)\varphi(g) = \sum_{h\in H}\Big(\sum_{g\in H} a_g b_h\Big)\varphi(gh)\\ &= \Big(\sum_{h\in H} a_h\varphi(h)\Big)\Big(\sum_{h\in H} b_h\varphi(h)\Big) = \overline{\varphi}\Big(\sum_{h\in H} a_h h\Big)\overline{\varphi}\Big(\sum_{h\in H} b_h h\Big)\end{aligned}$$

show $\overline{\varphi}$ is a ring homomorphism. Also $\overline{\varphi}(1_K 1_H) = 1_K\varphi(1_H) = 1_K 1_A = 1_A$. Now let $k \in K$ and $\sum_{h\in H} a_h h \in KH$. Then

$$\overline{\varphi}\Big(k\sum_{h\in H} a_h h\Big) = \overline{\varphi}\Big(\sum_{h\in H}(ka_h)h\Big) = \sum_{h\in H}(ka_h)\varphi(h) = k\sum_{h\in H} a_h\varphi(h) = k\overline{\varphi}\Big(\sum_{h\in H} a_h h\Big).$$

Consequently, $\overline{\varphi}$ is a K-algebra homomorphism. Finally, we establish uniqueness. Suppose that $\overline{\psi} : KH \to A$ is a K-algebra homomorphism such that $\overline{\psi}|_H = \varphi$. Then $\overline{\psi} = \overline{\varphi}$ since

$$\overline{\psi}\Big(\sum_{h\in H} a_h h\Big) = \sum_{h\in H} a_h\overline{\psi}(h) = \sum_{h\in H} a_h\varphi(h) = \overline{\varphi}\Big(\sum_{h\in H} a_h h\Big).$$

□

1.2 Tensor Products

In this section, K-spaces are assumed to be finite-dimensional.

11 DEFINITION. Let $\{v_1, v_2, \cdots, v_n\}$ and $\{w_1, w_2, \cdots, w_m\}$ be bases for K-spaces V and W, respectively. Then the *tensor product* of V and W, denoted $V \otimes W$, is the K-space with basis $\{v_i \otimes w_j \mid 1 \leq i \leq n, 1 \leq j \leq m\}$. For arbitrary $v \in V$ and $w \in W$, we may write $v = \sum_i \alpha_i v_i$ and $w = \sum_j \beta_j w_j$. We define $v \otimes w := \sum_{i,j} \alpha_i \beta_j v_i \otimes w_j \in V \otimes W$.

12 REMARKS. Let V and W be K-spaces. (a) $\dim(V \otimes W) = (\dim V)(\dim W)$ follows from the definition. (b) Let $v \in V$. Then $v \otimes 0 = v \otimes (0+0) = v \otimes 0 + v \otimes 0$. Since 0 is the only element of a group that satisfies $x + x = x$, we have $v \otimes 0 = 0$. Similarly, $0 \otimes v = 0$. (c) The tensor product of $V_1 \otimes \cdots \otimes V_n$ of n K-spaces $V_1, \ldots, V_n$ is defined similarly. We have $v_1 \otimes \cdots \otimes v_n = 0$ if any $v_i = 0$.

13 LEMMA. Let V and W be K-spaces. Suppose $u \in V \otimes W$. Then there is a positive integer n, a linearly independent subset $\{v_1, \ldots, v_n\}$ of V and a subset $\{w_1, \ldots, w_n\}$ of W such that $u = \sum_{i=1}^n v_i \otimes w_i$.

Proof. Let $\{v_\alpha\}_{\alpha \in I}$ be a basis of V. Write $u = \sum_{i=1}^n x_i \otimes y_i$ $(x_i \in V, y_i \in W)$. Thus $x_i = k_{i1} v_{\alpha_1} + \cdots + k_{in} v_{\alpha_n}$ $(k_{ij} \in K, v_{\alpha_j} \in V, 1 \leq i, j \leq n)$. Then

$$\begin{aligned}
u &= \sum_{i=1}^n (k_{i1} v_{\alpha_1} + \cdots + k_{in} v_{\alpha_n}) \otimes y_i = \sum_{i=1}^n [(k_{i1} v_{\alpha_1} \otimes y_i) + \cdots + (k_{in} v_{\alpha_n} \otimes y_i)] \\
&= \sum_{i=1}^n [(v_{\alpha_1} \otimes k_{i1} y_i) + \cdots + (v_{\alpha_n} \otimes k_{in} y_i)] \\
&= (v_{\alpha_1} \otimes k_{11} y_1 + \cdots + v_{\alpha_n} \otimes k_{1n} y_1) + \cdots + (v_{\alpha_1} \otimes k_{n1} y_n + \cdots + v_{\alpha_n} \otimes k_{nn} y_n) \\
&= [v_{\alpha_1} \otimes (k_{11} y_1 + \cdots + k_{n1} y_n)] + \cdots + [v_{\alpha_n} \otimes (k_{1n} y_1 + \cdots + k_{nn} y_n)] \\
&= \sum_{i=1}^n v_{\alpha_i} \otimes (k_{1i} y_1 + \cdots + k_{ni} y_n).
\end{aligned}$$

The result follows since each $k_{1i} y_1 + \cdots + k_{ni} y_n \in W$. □

14 DEFINITION. If R is a commutative ring with 1_R, $M_1, \ldots, M_n$, and L are R-modules, and, for all $r, r' \in R$ and $m_1, \ldots, m_n, m_i' \in M$, $f : M_1 \times \cdots \times M_n \to L$ satisfies

$$f(m_1, \ldots, m_{i-1}, rm_i + r'm_i', m_{i+1}, \ldots, m_n) = rf(m_1, \ldots, m_n) + r'f(m_1, \ldots, m_i', \ldots, m_n)$$

then f is called *n-multilinear* (or *bilinear* when $n = 2$).

15 EXAMPLES. (a) Let V and W be K-spaces. Define $\beta : V \times W \to V \otimes W$ by $\beta(v, w) = v \otimes w$ ($v \in V$, $w \in W$. Then for all $v, v_1, v_2 \in V$, $w, w_1, w_2 \in W$, and $k_1, k_2 \in K$, we have

$$\begin{aligned} \beta(k_1v_1 + k_2v_2, w) &= (k_1v_1 + k_2v_2) \otimes w = k_1v_1 \otimes w + k_2v_2 \otimes w \\ &= k_1(v_1 \otimes w) + k_2(v_2 \otimes w) = k_1\beta(v_1, w) + k_2\beta(v_2, w) \end{aligned}$$

and, similarly, $\beta(v, k_1w_1 + k_2w_2) = k_1\beta(v, w_1) + k_2\beta(v, w_2)$. Thus β is bilinear. β is called the *canonical bilinear map.* (b) We generalize (a). Let $V_1, \ldots, V_n$ be K-spaces. Define $\beta : V_1 \times \cdots \times V_n \to V_1 \otimes \cdots \otimes V_n$ by $\beta(v_1, \ldots, v_n) = v_1 \otimes \cdots \otimes v_n$ ($v_i \in V_i$, $1 \le i \le n$). Similar to (a), β is bilinear. β is called the *canonical n-multilinear map.* (c) Similar to (a), $t : V \times W \to W \otimes V$, $p_1 : V \times K \to V$ and $p_2 : K \times V \to V$ given by $t(v, w) = w \otimes v$, $p_1(v, k) = vk$ and $p_2(k, v) = kv$ ($v \in V$, $w \in W$, $k \in K$) are bilinear.

16 THEOREM. Suppose U, V, and W are K-spaces and let $f : U \times V \to W$ be bilinear. Then there exists a unique K-linear map $\bar{f} : U \otimes V \to W$ such that $\bar{f} \circ \beta = f$, where β is the canonical bilinear map.

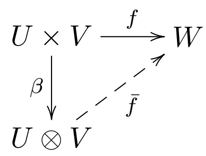

Figure 1: Tensor Product Universal Property

Proof. See [5, p. 211]. □

17 LEMMA. Let M, N, P, and Q be K-spaces and let $f : M \to P$ and $g : N \to Q$ be K-linear maps. Then there exists a unique K-linear map $f \otimes g : M \otimes N \to P \otimes Q$ such that $(f \otimes g)(m \otimes n) = f(m) \otimes g(n)$ for all $m \in M$ and $n \in N$.

Proof. Define $h : M \times N \to P \otimes Q$ by $h(m, n) = f(m) \otimes g(n)$. Then h is bilinear. By Theorem (16) there exists a unique K-linear map $f \otimes g : M \otimes N \to P \otimes Q$ such that $(f \otimes g) \circ \beta = h$ where β is the canonical bilinear map. Then for all $m \in M$ and $n \in N$,

$$(f \otimes g)(m \otimes n) = (f \otimes g)(\beta(m, n)) = [(f \otimes g) \circ \beta](m, n) = h(m, n) = f(m) \otimes g(n). \quad \square$$

18 DEFINITION. Let V and W be K-spaces with bases $\mathcal{V}$ and $\mathcal{W}$, respectively. By Theorem (16), the map t of Example (15c) induces the K-linear map $\tau : V \otimes W \to W \otimes V$ given by $\tau(v \otimes w) = w \otimes v$ for all $v \in \mathcal{V}$ and $w \in \mathcal{W}$. τ is called the *twist* map. Similarly, for all $v \in V$ and $k \in K$, the maps p_1 and p_2 of Example (15c) induce the K-linear maps $\pi_1 : V \otimes K \to V$ and $\pi_2 : K \otimes V \to V$ given by $\pi_1(v \otimes k) = vk$ and $\pi_2(k \otimes v) = kv$. π_1 and π_2 are called the *canonical projections.* $\rho_1 : V \to V \otimes K$ and $\rho_2 : V \to K \otimes V$ given by $\rho_1(v) = v \otimes 1_K$ and $\rho_2(v) = 1_K \otimes v$ are called the *canonical injections.*

19 LEMMA. Let V and W be K-spaces, $\tau : V \otimes W \to W \otimes V$ and $\tau' : W \otimes V \to V \otimes W$ twist maps, π_1 and π_2 the canonical projections, and ρ_1 and ρ_2 the canonical injections. (a) $\tau' \circ \tau = 1_{V \otimes W}$, $\tau \circ \tau' = 1_{W \otimes V}$, $\pi_1 \circ \rho_1 = 1_V$, $\rho_1 \circ \pi_1 = 1_{V \otimes K}$, $\pi_2 \circ \rho_2 = 1_V$, and $\rho_2 \circ \pi_2 = 1_{K \otimes V}$. (b) τ, π_1, π_2, ρ_1, and ρ_2 are K-space isomorphisms. (c) Let $v_1, v_2, v_3 \in V$ and $w_1, w_2, w_3 \in W$. Define $\varphi : V \otimes W \otimes V \otimes W \otimes V \otimes W \to V \otimes V \otimes V \otimes W \otimes W \otimes W$ by

$$\varphi(v_1 \otimes w_1 \otimes v_2 \otimes w_2 \otimes v_3 \otimes w_3) = v_1 \otimes v_2 \otimes v_3 \otimes w_1 \otimes w_2 \otimes w_3.$$

Then φ is a K-space isomorphism.

Proof. a. $(\tau' \circ \tau)(v \otimes w) = \tau'(w \otimes v) = v \otimes w$ for all $v \in V$, $w \in W$. So $\tau' \circ \tau = 1_{V \otimes W}$. Similarly, $\tau \circ \tau' = 1_{W \otimes V}$. $(\pi_1 \circ \rho_1)(v) = \pi_1(v \otimes 1_K) = v1_K = v = 1_V(v)$ for all $v \in V$. Thus $\pi_1 \circ \rho_1 = 1_V$. Similarly $\pi_2 \circ \rho_2 = 1_V$. For all $v \in V$ and $k \in K$, we have

$$(\rho_1 \circ \pi_1)(v \otimes k) = \rho_1(vk) = vk \otimes 1_K = v \otimes k1_K = v \otimes k = 1_{V \otimes K}(v \otimes k).$$

Thus $\rho_1 \circ \pi_1 = 1_{V \otimes K}$. Similarly $\rho_2 \circ \pi_2 = 1_{K \otimes V}$.

b. The indicated maps are all K-linear by the preceding remarks. They are K-space isomorphisms by (a).

c. Similar to the proof that τ is a K-space isomorphism. $\square$

Let U, V, and W be K-spaces. The technique proving τ is a K-space isomorphism may be applied to show that the natural identification of $(U \otimes V) \otimes W$ with $U \otimes (V \otimes W)$ is a K-space isomorphism. Thus the tensor product is associative.

1.3 Representations and Characters

In this section, K-spaces are assumed to be finite-dimensional. Also, KG-modules are assumed to be finite-dimensional as K-spaces.

20 DEFINITION. Suppose V and W are K-spaces. Denote by $\mathrm{GL}(V)$ the group of invertible K-linear maps from V to itself. If G is a finite group and $\rho : G \to \mathrm{GL}(V)$ is a group homomorphism, then ρ is called a *representation* of G. Let $\mathcal{B} = \{v_1, \cdots, v_n\}$ and $\mathcal{C} = \{w_1, \cdots, w_m\}$ be ordered bases of V and W, respectively, and $f : V \to W$ a K-linear map. For $1 \leq j \leq n$, we may write $f(v_j) = \sum_{i=1}^{m} \alpha_{ij} w_i$ for unique $\alpha_{ij} \in K$. The $m \times n$ matrix $[\alpha_{ij}]$ is called the *matrix of f relative to the bases $\mathcal{B}$ and $\mathcal{C}$*. Let $\rho : G \to \mathrm{GL}(V)$ be a representation and $[\alpha_{ij}(g)]$ the matrix of $\rho(g)$ (relative to $\mathcal{B}$) for each $g \in G$. Then $T : G \to \Gamma$ given by $T(g) = [\alpha_{ij}(g)]$ is a group homomorphism called the *matrix representation* of G afforded by V relative to $\mathcal{B}$.

Suppose V is a K-space. We establish a correspondence between representations of G and KG-modules. Let $\rho : G \to \mathrm{GL}(V)$ be a representation. Then V becomes a KG-module when we define $gv = \rho(g)(v)$ for $g \in G$ and $v \in V$ and extend linearly to all of KG via $(\sum_{g \in G} k_g g)v = \sum_{g \in G} k_g (gv) = \sum_{g \in G} k_g \rho(g)(v)$ (cf. Theorem (10)). Conversely, suppose V

is a KG-module. We then define $\rho : G \to \mathrm{GL}(V)$ by $\rho(g)(v) = gv$. For $g \in G$, $\rho(g)$ is a linear map by the module axioms. Further

$$(\rho(g)\rho(g^{-1}))(v) = \rho(g)[\rho(g^{-1})(v)] = g(g^{-1}v) = (gg^{-1})v = v = 1_V(v) \ (v \in V).$$

Hence $\rho(g)\rho(g^{-1}) = 1_V$ and $\rho(g) \in \mathrm{GL}(V)$. Consequently ρ is well-defined. Finally for $g, h \in G$, $v \in V$, $\rho(gh)(v) = (gh)v = g(hv) = \rho(g)(hv) = \rho(g)\rho(h)(v)$ since V is a KG-module. Thus ρ is a group homomorphism. It follows that ρ is a representation of G by definition. We call ρ the *representation* afforded by V.

21 DEFINITION. Let $A = [a_{ij}] \in \mathrm{Mat}_n K$, and $B \in \mathrm{Mat}_p K$. The *trace* of A is the scalar $\mathrm{tr}\, A = a_{11} + a_{22} + \cdots + a_{nn}$. The *Kronecker product* of A and B, denoted by $A \otimes B$, is a block matrix in $\mathrm{Mat}_{np} K$ whose (i, j)-block is $a_{ij}B$.

22 THEOREM. (a) If $A, B, C \in \mathrm{Mat}_n K$ with C nonsingular, then $\mathrm{tr}\,(AB) = \mathrm{tr}\,(BA)$ and $\mathrm{tr}\,(C^{-1}AC) = \mathrm{tr}\, A$. (b) If $A \in \mathrm{Mat}_n K$ and $B \in \mathrm{Mat}_p K$, then $\mathrm{tr}\,(A \otimes B) = (\mathrm{tr}\, A)(\mathrm{tr}\, B)$.

Proof. a. Let $A = [a_{ij}]$ and $B = [b_{ij}]$. Then

$$\begin{aligned}\mathrm{tr}\,(AB) &= \mathrm{tr}\left(\sum_{k=1}^{n} a_{ik}b_{kj}\right) = \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}b_{ki} = \sum_{k=1}^{n}\sum_{i=1}^{n} b_{ki}a_{ik} \\ &= \mathrm{tr}\left(\sum_{i=1}^{n} b_{ki}a_{il}\right) = \mathrm{tr}\,(BA).\end{aligned}$$

So $\mathrm{tr}\,(C^{-1}AC) = \mathrm{tr}\,([C^{-1}A]C) = \mathrm{tr}\,(C[C^{-1}A]) = \mathrm{tr}\,([CC^{-1}]A) = \mathrm{tr}\,(IA) = \mathrm{tr}\, A$.

b. Let $A = [a_{ij}]$ and $B = [b_{k\ell}]$. Consequently $A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{bmatrix}$ and $\mathrm{tr}\,(a_{ii}B) = a_{ii}(b_{11} + \cdots + b_{pp})$ for $1 \le i \le n$ imply

$$\begin{aligned}\mathrm{tr}\,(A \otimes B) &= a_{11}(b_{11} + \cdots + b_{pp}) + \cdots + a_{nn}(b_{11} + \cdots + b_{pp}) \\ &= (a_{11} + \cdots + a_{nn})(b_{11} + \cdots + b_{pp}) = (\mathrm{tr}\, A)(\mathrm{tr}\, B).\end{aligned}$$

□

Let V be a K-space, $f : V \to V$ a K-linear map, and A the matrix of f relative to some basis $\mathcal{B}$ of V. Define $\operatorname{tr} f = \operatorname{tr} A$. If a different basis $\mathcal{B}'$ is chosen, the matrix of f relative to $\mathcal{B}'$ is $C^{-1}AC$, where C is the change-of-basis matrix that changes $\mathcal{B}'$ coordinates to $\mathcal{B}$ coordinates. So $\operatorname{tr} f$ is well-defined by Theorem (22a).

23 DEFINITION. Let G be a finite group, V a KG-module, and ρ the representation afforded by V. Then $\chi : G \to K$ given by $\chi(g) = \operatorname{tr} \rho(g)$ $(g \in G)$ is called the *character of G afforded by V* (or by ρ). If V is *simple* (meaning $V \neq 0$ and 0 and V are the only submodules of V), then χ is an called an *irreducible* character.

24 REMARK. We may extend the definition of the tensor product. Let V and W be KG-modules with respective K-bases $\{v_1, \cdots, v_n\}$ and $\{w_1, \cdots, w_m\}$. Recall from Definition (11) that the *tensor product* $V \otimes W$ of V and W is the K-space with basis $\{v_i \otimes w_j \mid 1 \leq i \leq n, 1 \leq j \leq m\}$ and for arbitrary $v = \sum_i \alpha_i v_i \in V$ and $w = \sum_j \beta_j w_j \in W$ we define $v \otimes w := \sum_{i,j} \alpha_i \beta_j v_i \otimes w_j \in V \otimes W$. $V \otimes W$ becomes a KG-module by defining $g(v \otimes w) = gv \otimes gw$ for all $g \in G$, $v \in V$, and $w \in W$, and then extending linearly to KG via $(\sum_g k_g g)(v \otimes w) = \sum_g k_g (gv \otimes gw)$.

25 LEMMA. Let U, V, X, and Y be (finite-dimensional) K-spaces and let $f : U \to X$ and $g : V \to Y$ be K-linear maps. Then the Kronecker product of matrices representing f and g is a matrix representing $f \otimes g$.

Proof. Let $\mathcal{B}_1 = \{u_1, \cdots, u_m\}$ and $\mathcal{B}_2 = \{v_1, \cdots, v_n\}$ be ordered bases of U and V, respectively. Also, let $\mathcal{C}_1 = \{x_1, \cdots, x_p\}$ and $\mathcal{C}_2 = \{y_1, \cdots, y_q\}$ be ordered bases of X and Y, respectively. Then $\mathcal{B} = \{u_i \otimes v_j \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ is a basis of $U \otimes V$ and $\mathcal{C} = \{x_i \otimes y_j \mid 1 \leq i \leq p, 1 \leq j \leq q\}$ is a basis of $X \otimes Y$ by Remark (24). Now let $f(u_i) = \sum_{k=1}^{p} \alpha_{ki} x_k$ and $g(v_j) = \sum_{\ell=1}^{q} \beta_{\ell j} y_\ell$ where each $\alpha_{ki}, \beta_{\ell j} \in K$. Then

$$(f \otimes g)(u_i \otimes v_j) = f(u_i) \otimes g(v_j) = \left(\sum_{k=1}^{p} \alpha_{ki} x_k\right) \otimes \left(\sum_{\ell=1}^{q} \beta_{\ell j} y_\ell\right) = \sum_{k=1}^{p} \sum_{\ell=1}^{q} \alpha_{ki} \beta_{\ell j} (x_k \otimes y_\ell) \quad (1)$$

Note that $A = [\alpha_{ki}]$ is the matrix of f and $B = [\beta_{\ell j}]$ is the matrix of g relative to the given bases. We now order $\mathcal{B}$ into m ordered lists with the i^{th} list being $u_i \otimes v_1, \cdots, u_i \otimes v_n$ and similarly order $\mathcal{C}$ into p ordered lists with the k^{th} list being $x_k \otimes y_1, \cdots, x_k \otimes y_q$. So (1) determines the column entries for the corresponding matrix C of $f \otimes g$. Since C is a block matrix whose (k, ℓ)-block is $\alpha_{k\ell} B$, we have $C = A \otimes B$. □

26 THEOREM. Let V and W be KG-modules. Suppose V and W afford the characters χ and ψ, respectively. Then $V \otimes W$ affords the character $\chi\psi$.

Proof. Let R be the matrix representation of G afforded by V relative to the basis $\mathcal{A}$, and let S be the matrix representation of G afforded by W relative to the basis $\mathcal{B}$. Then $\mathcal{C} = \{v \otimes w \mid v \in \mathcal{A},\, w \in \mathcal{B}\}$ is a basis for $V \otimes W$ as in Remark (24). Then $T = R \otimes S$ defined by $T(g) = R(g) \otimes S(g)$ is the matrix representation of G afforded by $V \otimes W$ relative to the basis $\mathcal{C}$ by Lemma (25). Let ω be the character afforded by $V \otimes W$. Then for each $g \in G$, $\omega(g) = \operatorname{tr}(T(g)) = \operatorname{tr}(R(g) \otimes S(g)) = [\operatorname{tr}(R(g))][\operatorname{tr}(S(g))] = \chi(g)\psi(g)$. Consequently, $V \otimes W$ affords the character $\chi\psi$. □

1.4 Linear Functionals

27 DEFINITION. If A is an R-module, then the set A^* of all R-module homomorphisms from A to R is called the *dual module* of A and the elements of A^* are called *linear functionals.*

28 EXAMPLES.

a. The trace is a linear functional on $\mathrm{Mat}_n K$ since

$$\operatorname{tr}(cA + B) = \sum_{i=1}^{n} (cA_{ii} + B_{ii}) = c \sum_{i=1}^{n} A_{ii} + \sum_{i=1}^{n} B_{ii} = c \operatorname{tr} A + \operatorname{tr} B.$$

b. The function $\eta : K^* \to K$ given by $\eta(\varphi) = \varphi(1_K)$ $(\varphi \in K^*)$ is a K-linear map.

c. Recall $\Gamma := \mathrm{GL}_n(K)$. Define $\varphi : K^\Gamma \to (K\Gamma)^*$ by $\varphi(f)\big(\sum_{g\in\Gamma}\alpha_g g\big) = \sum_{g\in\Gamma}\alpha_g f(g)$. Clearly, φ is K-linear. Suppose $f \in \ker\varphi$. Then $f(g) = \varphi(f)(g) = 0$ for each $g \in \Gamma$. Consequently, $f = 0$. Hence $\ker\varphi = 0$ and φ is injective. Next let $f \in (K\Gamma)^*$. Then define $\overline{f} = f|_\Gamma$. Thus φ is surjective since $\varphi(\overline{f}) = \varphi(f|_\Gamma) = f$. Therefore φ is a K-isomorphism.

29 LEMMA. Let V be a (possibly infinite-dimensional) K-space. (a) If V is finite-dimensional then $V \cong V^*$. (b) $\rho : V^* \otimes V^* \to (V \otimes V)^*$ given by $\rho(f \otimes g)(x \otimes y) = f(x)g(y)$ where $f,\, g \in V^*$ and $x,\, y \in V$ is a K-monomorphism. (c) If V is finite-dimensional then ρ is bijective. (d) If $f_1, \cdots, f_n \in V^*$ and $x_1, \cdots, x_n \in V$ then $\theta : V^* \otimes \cdots \otimes V^* \to (V \otimes \cdots \otimes V)^*$ given by $\theta(f_1 \otimes \cdots \otimes f_n)(x_1 \otimes \cdots \otimes x_n) = f_1(x_1)\cdots f_n(x_n)$ is a K-linear map, which is a K-space isomorphism if V is finite-dimensional.

Proof. a. Let $\{v_1, v_2, \cdots, v_n\}$ be a basis of V. For each i, define $v_i^* : V \to K$ by $v_i^*(v_j) = \delta_{ij}$ (Kronecker delta). Then v_i^* is a linear functional for $1 \le i \le n$. Suppose $\sum_{i=1}^n \alpha_i v_i^* = 0$. In particular, $\alpha_j = \sum_{i=1}^n \alpha_i \delta_{ij} = \sum_{i=1}^n \alpha_i v_i^*(v_j) = 0$ for $1 \le j \le n$. Linear independence of $\{v_1^*, v_2^*, \cdots, v_n^*\}$ now follows. Next let $v^* \in V^*$ be arbitrary. Then for arbitrary $v = \sum_{i=1}^n \alpha_i v_i$ we have

$$v^*(v) = v^*\left(\sum_{i=1}^n \alpha_i v_i\right) = \sum_{i=1}^n \alpha_i v^*(v_i) = \sum_{i=1}^n v_i^*(v) v^*(v_i) = \left(\sum_{i=1}^n v^*(v_i) v_i^*\right)(v).$$

Thus $\{v_1^*, v_2^*, \cdots, v_n^*\}$ spans V^* and is a basis for V^*. Hence $\dim V = \dim V^*$. Recall that, for K-spaces V and W, $V \cong W$ if and only if $\dim V = \dim W$. So $V \cong V^*$.

b. Suppose $f,\, f_1,\, f_2,\, g,\, g_1,\, g_2 \in V^*$, $x,\, y \in V$, and $k,\, k_1,\, k_2 \in K$ are arbitrary. Define $r(f,\, g) : V \times V \to K$ by $[r(f,\, g)](x,\, y) = f(x)g(y)$. Clearly, $r(f,\, g)$ is bilinear. By Theorem (16) we obtain an induced map $V \otimes V \to K$ and hence an element of $(V \otimes V)^*$, which we also denote by $r(f,\, g)$. We have $r(f,\, g)(x \otimes y) = f(x)g(y)$. Then

$$r(k_1f_1 + k_2f_2,\, g)(x \otimes y) = (k_1f_1 + k_2f_2)(x)g(y) = (k_1f_1(x) + k_2f_2(x))g(y)$$

$$= k_1f_1(x)g(y) + k_2f_2(x)g(y) = (k_1r(f_1,\, g) + k_2r(f_2,\, g))(x \otimes y)$$

and similarly $r(f,\, k_1g_1 + k_2g_2) = k_1r(f,\, g_1) + k_2r(f,\, g_2)$. So r is bilinear. So by Theorem (16), r induces a K-linear map $\rho : V^* \otimes V^* \to (V \otimes V)^*$ such that $\rho \circ \beta = r$ where β is the canonical bilinear map. Thus ρ is given by

$$\rho(f \otimes g)(x \otimes y) = [\rho(\beta)(f,\, g)](x \otimes y) = [(\rho \circ \beta)(f,\, g)](x \otimes y) = [r(f,\, g)](x \otimes y)$$

$$= f(x)g(y).$$

Let $h \in \operatorname{Ker} \rho$. Then by Lemma (13), we may write $h = \sum_{i=1}^{n} f_i \otimes g_i$ where $\{f_1,\, \ldots,\, f_n\}$ is a linearly independent subset of V^* and $\{g_1,\, \ldots,\, g_n\} \subseteq V^*$. Then for all $u,\, v \in V$,

$$0 = \rho(h)(u,\, v) = \rho\left(\sum_{i=1}^{n} f_i \otimes g_i\right)(u,\, v) = \sum_{i=1}^{n} f_i(u)g_i(v) = \left(\sum_{i=1}^{n} g_i(v)f_i\right)(u).$$

Thus $\sum_{i=1}^{n} g_i(v)f_i = 0$ for all $v \in V$. Consequently, $g_i(v) = 0$ ($v \in V$, $1 \le i \le n$) since $\{f_1,\, \ldots,\, f_n\}$ is a linearly independent subset of V^*. So $h = \sum_{i=1}^{n} f_i \otimes g_i = 0$ and ρ is injective.

c. Let $\{v_1,\, \ldots,\, v_n\}$ be a basis of V. Then $\{v_{ij} \mid 1 \le i,\, j \le n\}$ is a basis for $V \otimes V$, where $v_{ij} := v_i \otimes v_j$. We have $\rho(v_i^* \otimes v_j^*)(v_{k\ell}) = v_i^*(v_k)v_j^*v_\ell = \delta_{ik}\delta_{j\ell} = \delta_{(i,\, j),\, (k,\, \ell)} = v_{ij}^*(v_{k\ell})$. So $\rho(v_i^* \otimes v_j^*) = v_{ij}^*$ and ρ is a K-isomorphism.

d. Apply induction to Lemma (17), (b), and (c). □

30 DEFINITION. Let V and W be K-spaces and $\varphi : V \to W$ a K-linear map. If $\varphi(v) = 0$ implies $v = 0$, then φ is called *non-singular*. The *annihilator* of $S \subseteq V$ is the set S^0 of all linear functionals f on V such that $f(\alpha) = 0$ for all $\alpha \in S$. The *dual* of φ is the map $\varphi^* : W^* \to V^*$ defined by $[\varphi^*(f)](v) = f(\varphi(v)) \in K$.

31 LEMMA. Let V be a K-space. (a) If $W \subseteq V$, then W^0 is a subspace of V^*. (b) If $W \leq V$, then $W^* \cong V^*/W^0$ and $W^0 \cong (V/W)^*$. (c) If V and W are subspaces of a K-space and $W \leq V$, then $W^0 \geq V^0$.

Proof. a. Let $w \in W$. Then $\{w\}^0 = \{f \in V^* \mid w \in \ker f\}$ by definition. So $\{w\}^0$ is a subspace of V^*. Since $W^0 = \bigcap_{w \in W} \{w\}^0$, it follows that W^0 is a subspace of V^*.

b. First, define $\varphi : V^* \to W^*$ by $\varphi(f) = f|_W$. Then φ is a K-space epimorphism with $\ker \varphi = W^0$. So $W^* \cong V^*/W^0$ by the First Isomorphism Theorem. Now define $\psi : W^0 \to (V/W)^*$ by $\psi(f)(v+W) = f(v)$. Then ψ is both well-defined and injective since, for $f \in W^0$,

$$u + W = v + W \Leftrightarrow u - v \in W \Leftrightarrow f(u) - f(v) = f(u - v) = 0 \Leftrightarrow f(u) = f(v)$$
$$\Leftrightarrow \psi(f)(u + W) = \psi(f)(v + W).$$

Let $f \in (V/W)^*$ and $v + W \in V/W$. Recall $\pi : V \to V/W$ given by $\pi(v) = v + W$ is a K-space epimorphism. Put $\overline{f} = f \circ \pi$. Then $\overline{f} \in W^0$ and

$$\psi(\overline{f})(v + W) = \overline{f}(v) = f(\pi(v)) = f(v + W).$$

Thus $f = \psi(\overline{f})$ and ψ is surjective. Finally, ψ is a K-space isomorphism since for all u, $v \in V$ and $k \in K$:

$$\psi(f)((u + W) + (v + W)) = \psi(f)((u + v) + W) = f(u + v) = f(u) + f(v)$$
$$= \psi(f)(u + W) + \psi(f)(v + W),$$
$$\psi(f)(k(v + W)) = \psi(f)(kv + W) = f(kv) = kf(v) = k\psi(f)(v + W).$$

c. Let $f \in V^0$. Then $f(w) = 0$ for all $w \in W$. Hence $f \in W^0$. □

32 LEMMA. If V and W are K-spaces and $\langle\, ,\, \rangle : V \times W \to K$ is non-singular and bilinear, then V^* and W are isomorphic.

Proof. Define $\varphi : W \to V^*$ by $[\varphi(w)](v) = \langle v, w\rangle$. Note that φ is well-defined since $\langle \alpha v, w\rangle = \alpha\langle v, w\rangle$ and $\langle v_1 + v_2, w\rangle = \langle v_1, w\rangle + \langle v_2, w\rangle$ imply that $\varphi(w) \in V^*$. Also, since

$[\varphi(w_1 + w_2)](v) = \langle v, w_1 + w_2\rangle = \langle v, w_1\rangle + \langle v, w_2\rangle = [\varphi(w_1)](v) + [\varphi(w_2)](v)$ and similarly for scalar multiplication, φ is a K-linear map. Let $x \in \ker\varphi$. Then $\langle v, x\rangle = 0$ for all $v \in V$. Hence $x = 0$ since $\langle\, ,\, \rangle$ is non-singular. So $\ker\varphi = 0$. Thus φ is injective. Finally suppose $\{v_1, \cdots, v_n\} \subseteq V$ is linearly independent and $\{w_1, \cdots, w_m\}$ is a basis of W. By the injectivity of φ, $n \geq m$. Assume $n > m$. Put $c_{ij} = \langle v_j, w_i\rangle$. Recall (linear algebra) there exist $a_1, a_2, \cdots, a_n \in K$ not all of which are zero such that $\sum_j a_j c_{ij} = 0$ for all i since $n > m$. So $v := \sum_j a_j v_j \neq 0$. We show $\langle v, w\rangle = 0$ for all $w \in W$. Thus we must show $\langle v, w_i\rangle = 0$ for each i. Then $\langle v, w_i\rangle = \left\langle \sum_j a_j v_j, w_i \right\rangle = \sum_j a_j \langle v_j, w_i\rangle = \sum_j a_j c_{ij} = 0$ for all i since $\langle\, ,\, \rangle$ is bilinear, contrary to $\langle\, ,\, \rangle$ being non-singular. Therefore $n = m$. □

Chapter 2

Algebras and Coalgebras

Definitions and statements of standard results in the theory of algebras and coalgebras have been drawn from [9-11].

2.1 Algebras and Commutative Diagrams

33 THEOREM. A is a K-algebra if and only if A is a K-space and there exist K-linear maps $\mu : A \otimes A \to A$ and $\iota : K \to A$ such that the diagrams (Figure 2) commute.

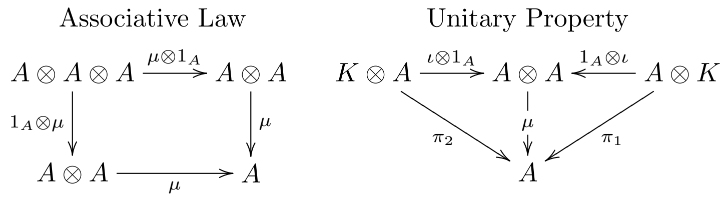

Figure 2: Associative Law and Unitary Property

Proof. ($\Longrightarrow$) Define $m : A \times A \to A$ by $m(a, b) = ab$ for all $a, b \in A$. Then m is bilinear. So by Theorem (16), m induces a K-linear map $\mu : A \otimes A \to A$ such that $\mu \circ \beta = m$ where β is the canonical bilinear map. Then $\mu(a \otimes b) = (\mu \circ \beta)(a, b) = m(a, b) = ab$ for all $a, b \in A$. Define $\iota : K \to A$ by $\iota(k) = k1_A$. Then for all $\alpha, \beta, k \in K$, we have $\iota(\alpha + \beta) = (\alpha + \beta)1_A = \alpha 1_A + \beta 1_A = \iota(\alpha) + \iota(\beta)$ and $\iota(k\alpha) = (k\alpha)1_A = k(\alpha 1_A) = k\iota(\alpha)$. Consequently ι is also a K-linear map. Let $a, b, c \in A$ and $k \in K$. The algebra condition $k(ab) = (ka)b = a(kb)$ implies $a(k1_A) = k(a1_A) = ka = k(1_A a) = (k1_A)a$. Then

$$\begin{aligned}(\mu \circ (\mu \otimes 1_A))(a \otimes b \otimes c) &= \mu(\mu(a \otimes b) \otimes 1_A(c)) = \mu(ab \otimes c) = (ab)c = a(bc) \\ &= \mu(1_A(a) \otimes \mu(b \otimes c)) = (\mu \circ (1_A \otimes \mu))(a \otimes b \otimes c),\end{aligned}$$

$$(\mu \circ (\iota \otimes 1_A))(k \otimes a) = \mu(\iota(k) \otimes 1_A(a)) = \iota(k)1_A(a) = (k1_A)a = ka = \pi_2(k \otimes a),$$

and similarly $(\mu \circ (1_A \otimes \iota))(a \otimes k) = \pi_1(a \otimes k)$. Thus the diagrams commute.

($\Longleftarrow$) Let a, b, c, $\in A$ and $k \in K$. Define a product in A by $ab := \mu(a \otimes b)$. The product is associative. Indeed, by the Associative Law diagram commutativity we have

$$a(bc) = \mu(a \otimes bc) = \mu(1_A(a) \otimes \mu(b \otimes c)) = (\mu \circ (1_A \otimes \mu))(a \otimes b \otimes c)$$

$$= (\mu \circ (\mu \otimes 1_A))(a \otimes b \otimes c) = \mu(\mu(a \otimes b) \otimes 1_A(c)) = \mu(ab \otimes c) = (ab)c.$$

Next, $(a+b)c = \mu((a+b) \otimes c) = \mu(a \otimes c + b \otimes c) = \mu(a \otimes c) + \mu(b \otimes c) = ac + bc$. Similarly, $c(a+b) = ca + cb$, so the product distributes over addition. Define $1_A := \iota(1_K)$. The (left) Unitary Property diagram yields

$$ka = \pi_2(k \otimes a) = (\mu \circ (\iota \otimes 1_A))(k \otimes a) = \mu(k1_A \otimes a) = (k1_A)a \tag{1}$$

Similarly, the (right) Unitary Property diagram yields $ak = a(k1_A)$. Thus

$$k(ab) = (k1_A)(ab) = ((k1_A)a)b = (ka)b = (ak)b = (a(k1_A))b = a((k1_A)b) = a(kb).$$

This establishes the algebra condition. Finally, by (1), $1_A a = (1_K 1_A)a = 1_K a = a$ and similarly $a1_A = a$. So 1_A is an identity. Therefore A is a K-algebra by definition. □

Theorem (33) permits (A, μ, ι) to denote a K-algebra A and its *structure maps* μ and ι, which are respectively called the *multiplication map* and *unit map*.

34 THEOREM. The tensor product of K-algebras is a K-algebra.

Proof. Suppose (A, μ_A, ι_A) and (B, μ_B, ι_B) are K-algebras, $\tau : A \otimes B \to B \otimes A$ the twist map, and $\rho_1 : K \to K \otimes K$ the canonical injection. Put $\mu_{A \otimes B} = \mu_A \otimes \mu_B \circ (1_A \otimes \tau \otimes 1_B)$ and $\iota_{A \otimes B} = (\iota_A \otimes \iota_B) \circ \rho_1$. We verify the Associative Law and Unitary Property (Figure 3).

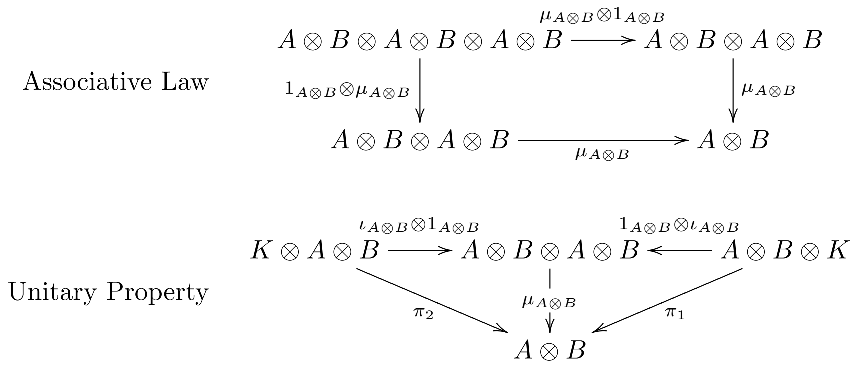

Figure 3: Tensor Product of K-algebras

Let $\mathcal{A}$ and $\mathcal{B}$ be bases for A and B, respectively. Then we have for all a_1, a_2, a_3, $a \in \mathcal{A}$, b_1, b_2, b_3, $b \in \mathcal{B}$ and $k \in K$ that

$$
\begin{aligned}
&(\mu_{A\otimes B} \circ (\mu_{A\otimes B} \otimes 1_{A\otimes B}))((a_1 \otimes b_1) \otimes (a_2 \otimes b_2) \otimes (a_3 \otimes b_3)) \\
&\quad = \mu_{A\otimes B}((\mu_A \otimes \mu_B) \circ (1_A \otimes \tau \otimes 1_B)(a_1 \otimes (b_1 \otimes a_2) \otimes b_2) \otimes (a_3 \otimes b_3)) \\
&\quad = \mu_{A\otimes B}((\mu_A \otimes \mu_B)(((a_1 \otimes a_2) \otimes (b_1 \otimes b_2)) \otimes (a_3 \otimes b_3))) \\
&\quad = \mu_{A\otimes B}((a_1 a_2 \otimes b_1 b_2) \otimes (a_3 \otimes b_3)) \\
&\quad = ((\mu_A \otimes \mu_B) \circ (1_A \otimes \tau \otimes 1_B))(a_1 a_2 \otimes (b_1 b_2 \otimes a_3) \otimes b_3) \\
&\quad = (\mu_A \otimes \mu_B)(a_1 a_2 \otimes (a_3 \otimes b_1 b_2) \otimes b_3) = (a_1 a_2) a_3 \otimes (b_1 b_2) b_3 = a_1 (a_2 a_3) \otimes b_1 (b_2 b_3),
\end{aligned}
$$

similarly $(\mu_{A\otimes B} \circ (1_{A\otimes B} \otimes \mu_{A\otimes B}))((a_1 \otimes b_1) \otimes (a_2 \otimes b_2) \otimes (a_3 \otimes b_3)) = a_1(a_2 a_3) \otimes b_1(b_2 b_3)$,

$$
\begin{aligned}
&(\mu_{A\otimes B} \circ (\iota_{A\otimes B} \otimes 1_{A\otimes B}))(k \otimes a \otimes b) = \mu_{A\otimes B}(\iota_{A\otimes B}(k) \otimes 1_{A\otimes B}(a \otimes b)) \\
&\quad = \mu_{A\otimes B}((\iota_A \otimes \iota_B)(k \otimes 1_K) \otimes a \otimes b) = \mu_{A\otimes B}(\iota_A(k) \otimes \iota_B(1_K) \otimes a \otimes b) \\
&\quad = (\mu_A \otimes \mu_B \circ (1_A \otimes \tau \otimes 1_B))(\iota_A(k) \otimes (1_B \otimes a) \otimes b) \\
&\quad = (\mu_A \otimes \mu_B)(\iota_A(k) \otimes a \otimes 1_B \otimes b) = \mu_A(\iota_A(k) \otimes a) \otimes \mu_B(1_B \otimes b) \\
&\quad = \iota_A(k) a \otimes 1_B b = ka \otimes b = \pi_2(k \otimes a \otimes b),
\end{aligned}
$$

and similarly $(\mu_{A\otimes B} \circ (1_{A\otimes B} \otimes \iota_{A\otimes B}))(a \otimes b \otimes k) = \pi_1(a \otimes b \otimes k)$. Extend linearly. Apply Theorem (33). $\square$

2.2 Coalgebras and Bialgebras

35 DEFINITION. If C is a K-space, $\Delta_C : C \to C \otimes C$ and $\varepsilon_C : C \to K$ are K-linear maps, and ρ_1 and ρ_2 the canonical injections, then $(C, \Delta_C, \varepsilon_C)$ is called a *K-coalgebra* whenever the diagrams (Figure 4) commute. Δ_C and ε_C are respectively called the *comultiplication* and *counit* maps and together are called the *structure maps* of C.

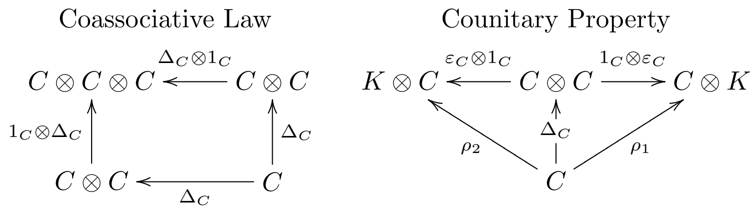

Figure 4: Coassociative Law and Counitary Property

A K-subspace D of a K-coalgebra $(C, \Delta_C, \varepsilon_C)$ that satisfies $\Delta_C(D) \subseteq D \otimes D$ is called a *K-subcoalgebra* of C whose structure maps are the restrictions of Δ_C and ε_C to D.

36 EXAMPLE. Let H be a group. $A := KH \otimes KH$ is a K-algebra by Theorem (34). Define $\varphi : H \to A^\times$ by $\varphi(g) = g \otimes g$. Then $\varphi(gh) = gh \otimes gh = (g \otimes g)(h \otimes h) = \varphi(g)\varphi(h)$ for all $g, h \in H$. Thus the group homomorphisms φ and $\psi : H \to K^\times$ given by $\psi(g) = 1_K$ respectively extend uniquely to K-algebra homomorphisms $\Delta : KH \to A$ and $\varepsilon : KH \to K$ by Theorem (10). Then $(KH, \Delta, \varepsilon)$ is a K-coalgebra since

$$\begin{aligned}
((1_{KH} \otimes \Delta) \circ \Delta)\Big(\sum_{g \in H} a_g g\Big) &= (1_{KH} \otimes \Delta)\Big(\sum_{g \in H} a_g g \otimes g\Big) = \sum_{g \in H} a_g g \otimes (g \otimes g) \\
&= \sum_{g \in H} a_g (g \otimes g) \otimes g = \sum_{g \in H} a_g \Delta(g) \otimes 1_{KH}(g), \\
&= \Delta\Big(\sum_{g \in H} (\Delta \otimes 1_{KH})(a_g g)\Big) = (\Delta \circ (\Delta \otimes 1_{KH}))\Big(\sum_{g \in H} a_g g\Big), \\
((\varepsilon \otimes 1_{KH}) \circ \Delta)\Big(\sum_{g \in H} a_g g\Big) &= (\varepsilon \otimes 1_{KH})\Big(\sum_{g \in H} a_g g \otimes g\Big) = \sum_{g \in H} a_g 1_K \otimes g \\
&= 1_K \otimes \Big(\sum_{g \in H} a_g g\Big) = \rho_2\Big(\sum_{g \in H} a_g g\Big),
\end{aligned}$$

and similarly $((1_{KH} \otimes \varepsilon) \circ \Delta)(\sum_{g \in H} a_g g) = \rho_1(\sum_{g \in H} a_g g)$.

37 THEOREM. The dual of a K-coalgebra is a K-algebra.

Proof. Let (C, Δ, ε) be a K-coalgebra. By Definition (30), $\Delta^* : (C \otimes C)^* \to C^*$ is given by $[\Delta^*(f)](c) = f(\Delta(c))$ for $c \in C$. Define $\mu : C^* \otimes C^* \to C^*$ and $\iota : K \to C^*$ by $\mu(f \otimes g)(c) = [\Delta^* \circ \rho](f \otimes g)(c)$ and $\iota(k)(c) = k\varepsilon(c)$ for $f, g \in C^*$, $c \in C$, and $k \in K$ where $\rho : C^* \otimes C^* \to (C \otimes C)^*$ is the K-space isomorphism of Lemma (29c). We verify the Associative Law and Unitary Property (Figure 5).

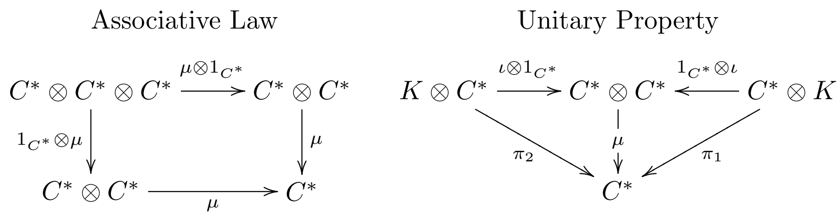

Figure 5: Dual of a K-Coalgebra

For $c \in C$, write $\Delta(c) = \sum_i c_i \otimes d_i$, $\Delta(c_i) = \sum_j a_{ij} \otimes b_{ij}$, $\Delta(d_i) = \sum_j e_{ij} \otimes f_{ij}$, and let $\theta : C^* \otimes C^* \otimes C^* \to (C \otimes C \otimes C)^*$ be the 3-fold analog of ρ (see Lemma (29d)). Then

$$\mu(f \otimes g)(c) = [\Delta^* \circ \rho](f \otimes g)(c) = \rho(f \otimes g)(\Delta(c)) = \sum_i f(c_i)g(d_i)$$

for $f, g \in C^*$ and $c \in C$. This implies that for $f, g, h \in C^*$ and $c \in C$ we have

$$\begin{aligned}
(\mu \circ (\mu \otimes 1_{C^*}))(f \otimes g \otimes h)(c) &= (\mu(\mu(f \otimes g) \otimes h))(c) = \sum_i \mu(f \otimes g)(c_i)h(d_i) \\
&= \sum_{i,\,j} f(a_{ij})g(b_{ij})h(d_i) = \theta(f \otimes g \otimes h)((\Delta \otimes 1_C) \circ \Delta)(c) \\
&= \theta(f \otimes g \otimes h)((1_C \otimes \Delta) \circ \Delta)(c) = \sum_{i,\,j} f(c_i)g(e_{ij})h(f_{ij}) \\
&= \sum_i f(c_i)\mu(g \otimes h)(d_i) = (1_{C^*} \otimes \mu)\Big(\sum_i f(c_i)(g \otimes h)(d_i)\Big) \\
&= (1_{C^*} \otimes \mu)(\mu(f \otimes (g \otimes h))(c)) = ((1_{C^*} \otimes \mu) \circ \mu)(f \otimes g \otimes h)(c)
\end{aligned}$$

This establishes the Associative law. Next, for any $c \in C$, the commutativity of the Counitary Property diagrams and Lemma (19c) yields $\sum_i \varepsilon(c_i)d_i = c = \sum_i c_i\varepsilon(d_i)$ from

$$c = 1_C(c) = (\pi_2 \circ \rho_2)(c) = (\pi_2 \circ (\varepsilon \otimes 1_C) \circ \Delta)(c) = \pi_2 \circ (\varepsilon \otimes 1_C)\left(\sum_i c_i \otimes d_i\right)$$

$$= \pi_2\left(\sum_i \varepsilon(c_i) \otimes d_i\right) = \sum_i \varepsilon(c_i) d_i$$

and similarly $c = \sum_i c_i \varepsilon(d_i)$. Then for all $k \in K$, $f \in C^*$, and $c \in C$,

$$(\mu \circ (\iota \otimes 1_{C^*}))(k \otimes f)(c) = \mu(\iota(k) \otimes f)(c) = \sum_i \iota(k)(c_i) f(d_i) = \sum_i k\varepsilon(c_i) f(d_i)$$

$$= kf\left(\sum_i \varepsilon(c_i) d_i\right) = kf(c) = \pi_2(k \otimes f)(c)$$

and similarly $(\mu \circ (1_{C^*} \otimes \iota))(f \otimes k)(c) = \pi_1(f \otimes k)(c)$. This establishes the Unitary Property and (C^*, μ, ι) is a K-algebra by Theorem (33). □

38 THEOREM. The dual of a finite-dimensional K-algebra is a K-coalgebra.

Proof. Suppose (A, μ, ι) is a finite-dimensional K-algebra. Then $\mu^* : A^* \to (A \otimes A)^*$ is given by $[\mu^*(f)](a \otimes b) = f(\mu(a \otimes b))$ and $\iota^* : A^* \to K^*$ is given by $\iota^*(f)(k) = f(\iota(k))$ for $f \in A^*$, $a \in A$, and $k \in K$ by Definition (30). Recall $\eta : K^* \to K$ given by $\eta(\varphi) = \varphi(1_K)$ for $\varphi \in K^*$ is a K-linear map . We may now define $\Delta_{A^*} : A^* \to A^* \otimes A^*$ and $\varepsilon_{A^*} : A^* \to K$ by $\Delta_{A^*}(f)(a) = [\rho^{-1} \circ \mu^*](f)(a)$ and $\varepsilon_{A^*}(f)(k) = [\eta \circ \iota^*(f)](k)$ for $f \in A^*$, $a \in A^* \otimes A^*$, where $\rho : A^* \otimes A^* \to (A \otimes A)^*$ is the K-space isomorphism of Lemma (29c) ($\dim A < \infty$ is required). We verify the Coassociative Law and Counitary Property (Figure 6).

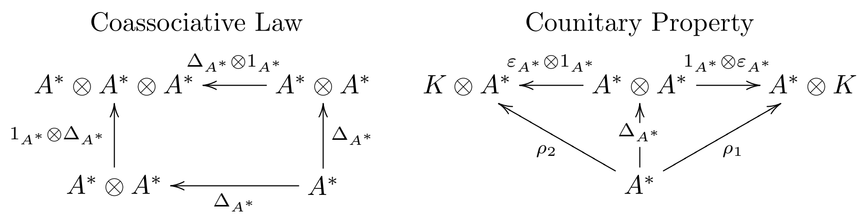

Figure 6: Dual of a Finite-Dimensional K-Algebra

Write $\Delta_{A^*}(f) = \sum_i g_i \otimes h_i$, $\Delta_{A^*}(g_i) = \sum_j m_{i,j} \otimes n_{i,j}$, and $\Delta_{A^*}(h_i) = \sum_j p_{i,j} \otimes q_{i,j}$ where g_i, h_i, $m_{i,j}$, $n_{i,j}$, $p_{i,j}$, $q_{i,j} \in A^*$. Then:

$$(\Delta_{A^*} \otimes 1_{A^*})\Delta_{A^*}(f) = (\Delta_{A^*} \otimes 1_{A^*})\left(\sum_i g_i \otimes h_i\right) = \sum_{i,\,j} m_{i,\,j} \otimes n_{i,\,j} \otimes h_i,$$

$$(1_{A^*} \otimes \Delta_{A^*})\Delta_{A^*}(f) = (1_{A^*} \otimes \Delta_{A^*})\left(\sum_i g_i \otimes h_i\right) = \sum_{i,\,j} g_i \otimes p_{i,\,j} \otimes q_{i,\,j}.$$

Note that for all $f \in A^*$ and $a,\, b \in A$, we have

$$\begin{aligned} f(ab) &= [\mu^*(f)](a \otimes b) = [\rho \circ (\rho^{-1} \circ \mu^*)(f))](a \otimes b) = [\rho(\Delta_{A^*}(f))](a \otimes b) \\ &= \left[\rho\left(\sum_i g_i \otimes h_i\right)\right](a \otimes b) = \sum_i g_i(a)h_i(b) \end{aligned} \tag{1}$$

Recall $\theta : A^* \otimes A^* \otimes A^* \to (A \otimes A \otimes A)^*$ given by $\theta(u \otimes v \otimes w)(a \otimes b \otimes c) = u(a)v(b)w(c)$ where $u,\, v,\, w \in A^*$ and $a,\, b,\, c \in A$ is a K-space isomorphism by Lemma (29d). It follows from the definition of θ and (1) that

$$\begin{aligned} &\left[\theta\left(\sum_{i,\,j} m_{i,\,j} \otimes n_{i,\,j} \otimes h_i\right)\right](a \otimes b \otimes c) = \sum_{i,\,j} m_{i,\,j}(a)n_{i,\,j}(b)h_i(c) = \sum_i g_i(ab)h_i(c) = f(abc) \\ &= \sum_i g_i(a)h_i(bc) = \sum_{i,\,j} g_i(a)p_{i,\,j}(b)q_{i,\,j}(c) = \left[\theta\left(\sum_{i,\,j} g_i \otimes p_{i,\,j} \otimes q_{i,\,j}\right)\right](a \otimes b \otimes c) \end{aligned}$$

Since θ is injective, $\sum_{i,\,j} m_{i,\,j} \otimes n_{i,\,j} \otimes h_i = \sum_{i,\,j} g_i \otimes p_{i,\,j} \otimes q_{i,\,j}$. Consequently, the Coassociative Law holds. Next, for all $f \in A^*$, we have

$$\begin{aligned} ((\varepsilon_{A^*} \otimes 1_{A^*}) \circ \Delta_{A^*})(f) &= (\varepsilon_{A^*} \otimes 1_{A^*})\left(\sum_i g_i \otimes h_i\right) = \sum_i (\varepsilon_{A^*} \otimes 1_{A^*})(g_i \otimes h_i) \\ &= \sum_i (\varepsilon_{A^*}(g_i) \otimes h_i) = \sum_i (\eta \circ \iota^*(g_i) \otimes h_i) = \sum_i (\iota^*(g_i)(1_K) \otimes h_i) = \sum_i (g_i(\iota(1_K)) \otimes h_i) \\ &= \sum_i (1_K g_i(1_A) \otimes h_i) = \sum_i (1_K \otimes g_i(1_A)h_i) = 1_K \otimes \sum_i g_i(1_A)h_i = 1_K \otimes f = \rho_2(f). \end{aligned}$$

For the penultimate inequality, we have used that $f(a) = f(1_A a) = \sum_i g_i(1_A)h_i(a)$ $(a \in A)$. Similarly, $((1_{A^*} \otimes \varepsilon_{A^*}) \circ \Delta_{A^*})(f) = \rho_1(f)$. Thus the check of the Counitary Property is complete and $(A^*,\, \Delta_{A^*},\, \varepsilon_{A^*})$ is a K-coalgebra by Definition (35). □

39 NOTATION. Let $(C,\, \Delta,\, \varepsilon)$ be a K-coalgebra. We write $\Delta(c) = \sum_i c_{i(1)} \otimes c_{i(2)}$ for each $c \in C$ or succinctly as $\Delta(c) = c_{(1)} \otimes c_{(2)}$ with summation implicit.

40 LEMMA. Let $(C,\, \Delta,\, \varepsilon)$ be a K-coalgebra with $\Delta(c) = c_{(1)} \otimes c_{(2)}$ for all $c \in C$.

a. $c_{(1)(1)} \otimes c_{(1)(2)} \otimes c_{(2)} = c_{(1)} \otimes c_{(2)(1)} \otimes c_{(2)(2)}$. b. $c = \varepsilon(c_{(1)})c_{(2)} = c_{(1)}\varepsilon(c_{(2)})$.

Proof. a. By the Coassociative Law

$$c_{(1)(1)} \otimes c_{(1)(2)} \otimes c_{(2)} = \Delta(c_{(1)}) \otimes c_{(2)} = (\Delta \otimes 1_C)(c_{(1)} \otimes c_{(2)}) = ((\Delta \otimes 1_C) \circ \Delta)(c)$$

$$= ((1_C \otimes \Delta) \circ \Delta)(c) = (1_C \otimes \Delta)(c_{(1)} \otimes c_{(2)}) = c_{(1)} \otimes \Delta(c_{(2)}) = c_{(1)} \otimes c_{(2)(1)} \otimes c_{(2)(2)}.$$

b. Since $\rho_1(c) = c \otimes 1_C$ and $\rho_2(c) = 1_C \otimes c$, by the Counitary Property we have:

$$1_C \otimes c = ((\varepsilon \otimes 1_C) \circ \Delta)(c) = \varepsilon(c_{(1)}) \otimes c_{(2)} = 1_C \otimes \varepsilon(c_{(1)})c_{(2)},$$

$$c \otimes 1_C = ((1_C \otimes \varepsilon) \circ \Delta)(c) = c_{(1)} \otimes \varepsilon(c_{(2)}) = c_{(1)}\varepsilon(c_{(2)}) \otimes 1_C.$$

Therefore $c = \varepsilon(c_{(1)})c_{(2)} = c_{(1)}\varepsilon(c_{(2)})$. □

41 THEOREM. The tensor product of K-coalgebras is a K-coalgebra.

Proof. Suppose $(C,\, \Delta_C,\, \varepsilon_C)$ and $(D,\, \Delta_D,\, \varepsilon_D)$ are K-coalgebras and τ is the twist map. Put $\Delta_{C\otimes D} = (1_C \otimes \tau \otimes 1_D) \circ \Delta_C \otimes \Delta_D$ and $\varepsilon_{C\otimes D} = \pi_2 \circ (\varepsilon_C \otimes \varepsilon_D)$. We will verify the Coassociative Law and Counitary Property (Figure 7).

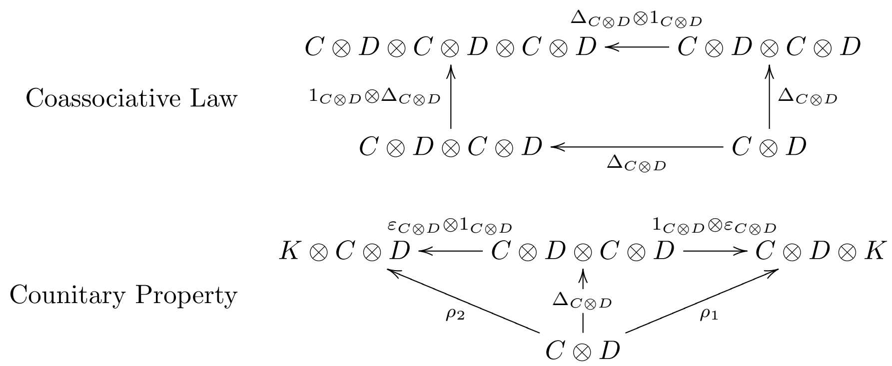

Figure 7: Tensor Product of K-coalgebras

For all $c \in C$ and $d \in D$, we have

$$((\Delta_{C\otimes D} \otimes 1_{C\otimes D}) \circ \Delta_{C\otimes D})(c \otimes d) = [(\Delta_{C\otimes D} \otimes 1_{C\otimes D}) \circ (1_C \otimes \tau \otimes 1_D) \circ (\Delta_C \otimes \Delta_D)](c \otimes d)$$

$$= [(\Delta_{C\otimes D} \otimes 1_{C\otimes D}) \circ (1_C \otimes \tau \otimes 1_D)](c_{(1)} \otimes c_{(2)} \otimes d_{(1)} \otimes d_{(2)})$$

$$= (\Delta_{C\otimes D} \otimes 1_{C\otimes D})(c_{(1)} \otimes d_{(1)} \otimes c_{(2)} \otimes d_{(2)})$$

$$= [(1_C \otimes \tau \otimes 1_D) \circ (\Delta_C \otimes \Delta_D)(c_{(1)} \otimes d_{(1)})] \otimes (c_{(2)} \otimes d_{(2)})$$

$$= [(1_C \otimes \tau \otimes 1_D)(c_{(1)(1)} \otimes c_{(1)(2)} \otimes d_{(1)(1)} \otimes d_{(1)(2)})] \otimes (c_{(2)} \otimes d_{(2)})$$

$$= c_{(1)(1)} \otimes d_{(1)(1)} \otimes c_{(1)(2)} \otimes d_{(1)(2)} \otimes c_{(2)} \otimes d_{(2)}$$

and similarly $((1_{C\otimes D} \otimes \Delta_{C\otimes D}) \circ \Delta_{C\otimes D})(c \otimes d) = c_{(1)} \otimes d_{(1)} \otimes c_{(2)(1)} \otimes d_{(2)(1)} \otimes c_{(2)(2)} \otimes d_{(2)(2)}$. Recall the K-space isomorphism φ of Lemma (19c). Then by Lemma (40a),

$$\varphi(((\Delta_{C\otimes D} \otimes 1_{C\otimes D}) \circ \Delta_{C\otimes D})(c \otimes d)) = \varphi(c_{(1)(1)} \otimes d_{(1)(1)} \otimes c_{(1)(2)} \otimes d_{(1)(2)} \otimes c_{(2)} \otimes d_{(2)})$$

$$= c_{(1)(1)} \otimes c_{(1)(2)} \otimes c_{(2)} \otimes d_{(1)(1)} \otimes d_{(1)(2)} \otimes d_{(2)}$$

$$= c_{(1)} \otimes c_{(2)(1)} \otimes c_{(2)(2)} \otimes d_{(1)} \otimes d_{(2)(1)} \otimes d_{(2)(2)}$$

$$= \varphi(c_{(1)} \otimes d_{(1)} \otimes c_{(2)(1)} \otimes d_{(2)(1)} \otimes c_{(2)(2)} \otimes d_{(2)(2)})$$

$$= \varphi(((1_{C\otimes D} \otimes \Delta_{C\otimes D}) \circ \Delta_{C\otimes D})(c \otimes d)).$$

Consequently, $((\Delta_{C\otimes D} \otimes 1_{C\otimes D}) \circ \Delta_{C\otimes D})(c \otimes d) = ((1_{C\otimes D} \otimes \Delta_{C\otimes D}) \circ \Delta_{C\otimes D})(c \otimes d)$ since φ is a K-space isomorphism. Then extending linearly establishes the Coassociative Law. Next, for all $c \in C$ and $d \in D$, applying Lemma (40b) yields

$$((\varepsilon_{C\otimes D} \otimes 1_{C\otimes D}) \circ \Delta_{C\otimes D})(c \otimes d) = [(\varepsilon_{C\otimes D} \otimes 1_{C\otimes D}) \circ (1_C \otimes \tau \otimes 1_D) \circ (\Delta_C \otimes \Delta_D)](c \otimes d)$$

$$= [(\varepsilon_{C\otimes D} \otimes 1_{C\otimes D}) \circ (1_C \otimes \tau \otimes 1_D)](c_{(1)} \otimes c_{(2)} \otimes d_{(1)} \otimes d_{(2)})$$

$$= (\varepsilon_{C\otimes D} \otimes 1_{C\otimes D})(c_{(1)} \otimes d_{(1)} \otimes c_{(2)} \otimes d_{(2)})$$

$$= [(\pi_2 \circ (\varepsilon_C \otimes \varepsilon_D))(c_{(1)} \otimes d_{(1)})] \otimes (c_{(2)} \otimes d_{(2)}) = \varepsilon_C(c_{(1)})\varepsilon_D(d_{(1)}) \otimes c_{(2)} \otimes d_{(2)}$$

$$= 1_K \otimes \varepsilon_C(c_{(1)})c_{(2)} \otimes \varepsilon_D(d_{(1)})d_{(2)} = 1_K \otimes c \otimes d = \rho_2(c \otimes d)$$

and similarly $((1_{C\otimes D} \otimes \varepsilon_{C\otimes D}) \circ \Delta_{C\otimes D})(c \otimes d) = \rho_1(c \otimes d)$. Extend linearly. Thus the Counitary Property holds and $(C \otimes D,\, \Delta_{C\otimes D},\, \varepsilon_{C\otimes D})$ is a K-coalgebra. □

42 DEFINITION. Suppose that $(C, \Delta_C, \varepsilon_C)$ and $(D, \Delta_D, \varepsilon_D)$ are K-coalgebras and there exists a K-linear map $f : C \to D$ such that $\Delta_D \circ f = (f \otimes f) \circ \Delta_C$ and $\varepsilon_D \circ f = \varepsilon_C$ (Figure 8). Then f is called a *K-coalgebra homomorphism.*

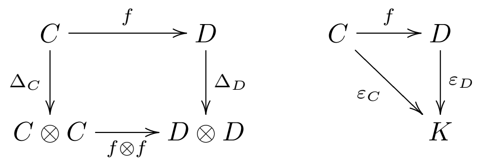

Figure 8: Coalgebra Homomorphism

43 EXAMPLE. Put $L := K \otimes K$. We have that $(K, \Delta_K, \varepsilon_K)$ is a K-coalgebra with $\Delta_K : K \to L$ and $\varepsilon_K : K \to K$ given by $\Delta_K(k) = k \otimes 1_K$ and $\varepsilon_K(k) = 1_K$ for all $k \in K$. Let $\tau : L \to L$ be the twist map. We may now define $\Delta_L : L \to L \otimes L$ and $\varepsilon_L : L \to K$ by $\Delta_L(k \otimes \ell) = (1_K \otimes \tau \otimes 1_K) \circ (\Delta_K \otimes \Delta_K)(k \otimes \ell) = k \otimes \ell \otimes 1_K \otimes 1_K$ and $\varepsilon_L(k \otimes \ell) = (\pi_1 \circ (\varepsilon_K \otimes \varepsilon_K))(k \otimes \ell) = \pi_1(1_K \otimes 1_K) = 1_K$ for all $k, \ell \in K$. Then $(L, \Delta_L, \varepsilon_L)$ is a K-coalgebra by Theorem (41). Define $\mu_K : L \to K$ and $\iota_K : K \to K$ by $\mu_K(k \otimes \ell) = k\ell$ and $\iota_K(k) = k$ for all $k, \ell \in K$. Then μ_K is a K-coalgebra homomorphism since

$$(\Delta_K \circ \mu_K)(k \otimes \ell) = \Delta_K(k\ell) = k\ell \otimes 1_K = (\mu_K \otimes \mu_K)(k \otimes \ell \otimes 1_K \otimes 1_K)$$

$$= ((\mu_K \otimes \mu_K) \circ (\Delta_L)(k \otimes \ell))$$

and

$$(\varepsilon_K \circ \mu_K)(k \otimes \ell) = \varepsilon_K(k\ell) = 1_K = \varepsilon_L(k \otimes \ell)$$

for all $k, \ell \in K$. Similarly since $\Delta_K \circ \iota_K = (\iota_K \otimes \iota_K) \circ \Delta_K$ and $\varepsilon_K \circ \iota_K = \varepsilon_K$, it follows that ι_K is a K-coalgebra homomorphism.

44 THEOREM. Let B be a K-space, (B, μ, ι) a K-algebra, and $(B, \Delta_B, \varepsilon_B)$ a K-coalgebra. The following are equivalent: (a) μ and ι are K-coalgebra homomorphisms, (b) Δ_B and ε_B are K-algebra homomorphisms, (c) $\Delta_B(bc) = \Delta_B(b)\Delta_B(c)$, $\Delta_B(1_B) = 1_B \otimes 1_B$, $\varepsilon_B(bc) = \varepsilon_B(b)\varepsilon_B(c)$, and $\varepsilon_B(1_B) = 1_K$ for all $b, c \in B$.

Proof. Consider the following four diagrams:

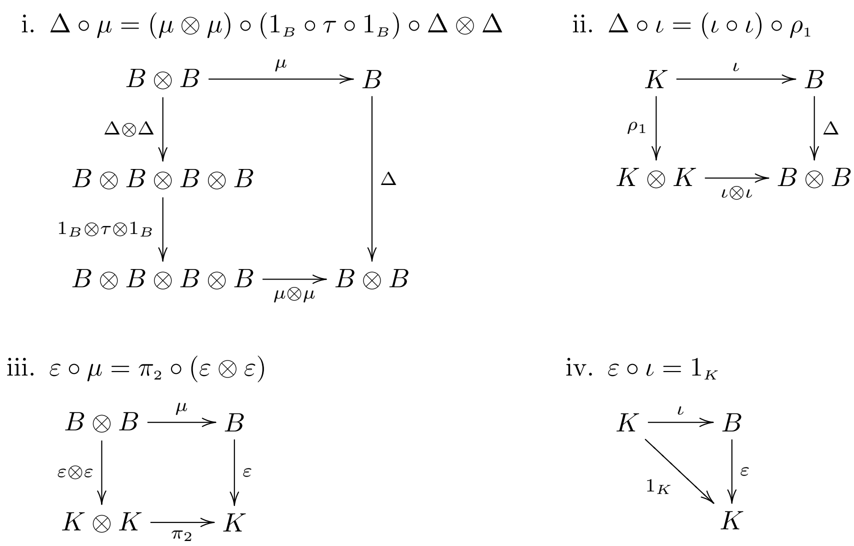

Figure 9: Bialgebra Equivalent Conditions

We have that $\Delta = \Delta_B$ is a K-algebra homomorphism when (i) and (ii) are satisfied, ε_B is a K-algebra homomorphism when (iii) and (iv) hold, μ is a K-coalgebra homomorphism when (i) and (iii) are satisfied, and ι is a K-coalgebra homomorphism when (ii) and (iv) hold. So (a) is equivalent to (b). (b) is equivalent to (c) by Definition (4). □

45 DEFINITION. Let (B, μ, ι) be a K-algebra and (B, Δ, ε) a K-coalgebra. If any condition of Theorem (44) is satisfied then $(B, \mu, \iota, \Delta, \varepsilon)$ is called a *K-bialgebra*.

46 EXAMPLES.

a. $(K, \mu_K, \iota_K, \Delta_K, \varepsilon_K)$ is a K-bialgebra. See Examples (7) and (43).

b. Let H be a group. Recall (KH, μ, ι) is a K-algebra by Lemma (9) and Theorem (33) and (KH, Δ, ϵ) is a K-coalgebra and Δ and ϵ are K-algebra homomorphisms by Example (36). Thus $(KH, \mu, \iota, \Delta, \epsilon)$ is a K-bialgebra.

Chapter 3

Results in Schur Algebras

Definitions and statements of standard results in the theory of Schur algebras have been drawn from [12, 13].

3.1 Polynomial Functions and Coefficient Space

47 DEFINITION. Let E be the set of n-dimensional column K-vectors. For $g \in \Gamma$ and $x \in E$, define gx by usual matrix multiplication. We may extend linearly to all of $K\Gamma$ via $(\sum_{g\in\Gamma} k_g g)x = \sum_{g\in\Gamma} k_g(gx)$($k_g \neq 0$ for finitely many $g \in \Gamma$ assumed throughout). Then E is called the *standard* or *natural* $K\Gamma$-module.

We write $I(n, r) := \{i = (i_1, i_2, \cdots, i_r) \mid 1 \leq i_k \leq n \text{ for } 1 \leq k \leq r\}$. Suppose $\{e_1, e_2, \cdots, e_n\}$ is the standard basis for E. Define $g(v_1 \otimes \cdots \otimes v_r) = gv_1 \otimes \cdots \otimes gv_r$ for $g \in \Gamma$. Consequently $E^{\otimes r} = E \otimes \cdots \otimes E$ (r factors) becomes a $K\Gamma$-module with K-basis $\{e_i = e_{i_1} \otimes \cdots \otimes e_{i_r} \mid i \in I(n, r)\}$.

48 PROPOSITION. Let $v, w \in E$. (a) $\tau : E^{\otimes 2} \to E^{\otimes 2}$ given by $\tau(v \otimes w) = w \otimes v$ is a KΓ-module homomorphism. (b) The sets $S^2(E) = \{x \in E^{\otimes 2} \mid \tau(x) = x\}$ and $\wedge^2(E) = \{x \in E^{\otimes 2} \mid \tau(x) = -x\}$ are KΓ-submodules of $E^{\otimes 2}$. (c) $S^2(E) = (1 + \tau)(E^{\otimes 2})$, $\wedge^2(E) = (1 - \tau)(E^{\otimes 2})$, and $E^{\otimes 2} = S^2(E) \dotplus \wedge^2(E)$ if $\operatorname{char} K \neq 2$.

Proof. a. $\tau(gx) = \tau(g(x_1 \otimes x_2)) = \tau(gx_1 \otimes gx_2) = gx_2 \otimes gx_1 = g(x_2 \otimes x_1) = g\tau(x)$ for all $g \in \Gamma$ and $x = x_1 \otimes x_2 \in E^{\otimes 2}$. Extend linearly.

b. Let $g \in \Gamma$, $x \in S^2(E)$, and $y \in \wedge^2(E)$. Note that $\tau(gx) = g\tau(x) = gx$ and that $\tau(gy) = g\tau(y) = -gy$ by (a). Extend linearly.

c. First, let $x \in S^2(E)$. Then $x = \frac{x}{2} + \frac{x}{2} = \frac{x}{2} + \tau\left(\frac{x}{2}\right) = (1+\tau)\left(\frac{x}{2}\right) \in (1+\tau)(E^{\otimes 2})$. Thus $S^2(E) \subseteq (1+\tau)(E^{\otimes 2})$. Conversely, let $x \in (1+\tau)(E^{\otimes 2})$. Then $x = y + \tau(y)$ for some $y \in E^{\otimes 2}$. We have $\tau(\tau(y)) = y$ by linear extension. It then follows that

$$\tau(x) = \tau(y + \tau(y)) = \tau(y) + \tau(\tau(y)) = \tau(y) + y = x.$$

Thus $x \in S^2(E)$. Consequently, $(1+\tau)(E^{\otimes 2}) \subseteq S^2(E)$ and the first equality is shown. The second equality is established similarly. Suppose that $x \in \wedge^2(E)$. Then $x = \frac{x}{2} + \frac{x}{2} = \frac{x}{2} - \tau\left(\frac{x}{2}\right) = (1-\tau)\left(\frac{x}{2}\right) \in (1-\tau)(E^{\otimes 2})$. Thus $\wedge^2(E) \subseteq (1-\tau)(E^{\otimes 2})$. Conversely, any $x \in (1-\tau)(E^{\otimes 2})$ may be written as $x = y - \tau(y)$ for some $y \in E^{\otimes 2}$. We also have $\tau(\tau(y)) = y$ by linear extension. Consequently,

$$\tau(x) = \tau(y - \tau(y)) = \tau(y) - \tau(\tau(y)) = \tau(y) - y = -x.$$

Thus $x \in \wedge^2(E)$. So $(1-\tau)(E^{\otimes 2}) \subseteq \wedge^2(E)$ and the second equality also holds. Finally, it is clear that $S^2(E) \cap \wedge^2(E) = \{0\}$. Let $x \in E^{\otimes 2}$. Then

$$x = \frac{1}{2}[x + \tau(x) + x - \tau(x)] = \frac{1}{2}(1+\tau)(x) + \frac{1}{2}(1-\tau)(x).$$

Since $\frac{1}{2}(x + \tau(x)) \in S^2(E)$ and $\frac{1}{2}(x - \tau(x)) \in \wedge^2(E)$, then $E^{\otimes 2} = S^2(E) + \wedge^2(E)$. □

49 DEFINITION. Let $g_{\alpha\beta}$ denote the (α, β)-entry of the matrix g. $c_{\alpha\beta} : \Gamma \to K$ where $c_{\alpha\beta}(g) = g_{\alpha\beta}$ for all $g \in \Gamma$ is called a *coordinate function.* Suppose $\mathbf{n} := \{1, 2, \cdots, n\}$ and $\mathcal{A}_n := \{c_{\alpha\beta} \mid \alpha, \beta \in \mathbf{n}\}$. We will denote by $A(n)$ the K-subalgebra of K^Γ generated by $\mathcal{A}_n$. $A(n)$ is called the *algebra of polynomial functions* and the elements of $A(n)$ are called *polynomial functions* on Γ. $\{y_1, \cdots, y_q\}$ in a K-algebra is called *algebraically independent* over K if no nonzero polynomial $p \in K[x_1, \cdots, x_q]$ exists such that $p(y_1, \cdots, y_q) = 0$.

50 LEMMA. If K is infinite then every subset of $\mathcal{A}_n$ is algebraically independent over K.

Proof. This result is well-known (see [13, page 9]). We present a proof of the case $S \subseteq \mathcal{A}_n$ with $|\,S\,| = 1$. Then $S = \{c_{ij}\}$ for some fixed i and j. Let $p(x) \in K[x]$ with $p(c_{ij}) = 0_K$. Assume $p(x)$ is not the zero polynomial. Suppose $i \neq j$. We may choose a nonzero $\alpha \in K$ with $p(\alpha) \neq 0_K$ since K is infinite. Construct matrix g where $g_{hh} = 1_K$ for $1 \leq h \leq n$, $g_{ij} = \alpha$, and $g_{\ell m} = 0_K$ for all other pairs (ℓ, m) with $\ell, m \in \mathbf{n}$. Then $g \in \Gamma$ but $0_K = p(c_{ij})(g) = p(\alpha)$. Contradiction. So $i = j$. Again choose a nonzero $\alpha \in K$ with $p(\alpha) \neq 0_K$. Construct matrix g where $g_{hh} = 1_K$ for $1 \leq h \leq n$ with $h \neq i$, $g_{ii} = \alpha$, and $g_{\ell m} = 0_K$ for all other pairs (ℓ, m) with $\ell, m \in \mathbf{n}$. Then $g \in \Gamma$ but $0_K = p(c_{ii})(g) = p(\alpha)$. Contradiction. Thus p is the zero polynomial. So S is algebraically independent over K. □

51 DEFINITION. Let V be a $K\Gamma$-module and $T : \Gamma \to \Gamma$ the matrix representation afforded by V relative to the basis $\{v_1, \cdots, v_n\}$ of V. So $T(g) = [\alpha_{ij}(g)]$ for unique $\alpha_{ij} \in K^\Gamma$ with $gv_j = \sum_i \alpha_{ij}(g)v_i$ $(g \in \Gamma)$. We extend linearly by $T(\sum_{g\in\Gamma} k_g g) = \sum_{g\in\Gamma} k_g T(g)$. The K-space $\mathrm{cf}(V)$ spanned by the α_{ij} is called the *coefficient space* of V.

52 EXAMPLES. Put $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

a. Let $\rho : \Gamma \to \mathrm{GL}_2(K)$ be the matrix representation corresponding to the natural $K\Gamma$-module E relative to the basis $\{e_1, e_2\}$ of E. Consequently ρ satisfies $\rho(g) = g$ since $ge_1 = ae_1 + ce_2$ and $ge_2 = be_1 + de_2$. Thus $\mathrm{cf}\left(E^{\otimes 1}\right) = \langle c_{11}, c_{12}, c_{21}, c_{22}\rangle$.

b. We use the convention that $c_{i_1 i_3, i_2 i_4} = c_{i_1 i_2} c_{i_3 i_4}$. $E^{\otimes 2}$ has basis $\{e_{11}, e_{12}, e_{21}, e_{22}\}$ where $e_{ij} = e_i \otimes e_j$. Then by a calculation similar to (c) below,

$$\begin{aligned}\mathrm{cf}\left(E^{\otimes 2}\right) &= \langle c_{11}^2, c_{12}^2, c_{11}c_{12}, c_{11}c_{21}, c_{11}c_{22}, c_{12}c_{21}, c_{12}c_{22}, c_{21}^2, c_{22}^2, c_{21}c_{22}\rangle \\ &= \langle c_{11,11}, c_{11,22}, c_{11,12}, c_{12,11}, c_{12,12}, c_{12,21}, c_{12,22}, c_{22,11}, c_{22,22}, c_{22,12}\rangle\end{aligned}$$

c. $S^2(E)$ has basis $\{e_{11}, e_{12} + e_{21}, e_{22}\}$. Then:

$$
\begin{aligned}
ge_{11} &= ge_1 \otimes ge_1 = (ae_1 + ce_2) \otimes (ae_1 + ce_2) = a^2 e_{11} + ac(e_{12} + e_{21}) + c^2 e_{22}, \\
g(e_{12} + e_{21}) &= ge_1 \otimes ge_2 + ge_2 \otimes ge_1 \\
&= [(ae_1 + ce_2) \otimes (be_1 + de_2)] + [(be_1 + de_2) \otimes (ae_1 + ce_2)] \\
&= 2abe_{11} + (ad + bc)(e_{12} + e_{21}) + 2cde_{22}, \\
ge_{22} &= ge_2 \otimes ge_2 = (be_1 + de_2) \otimes (be_1 + de_2) = b^2 e_{11} + bd(e_{12} + e_{21}) + d^2 e_{22}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\mathrm{cf}\,(S^2(E)) &= \langle c_{11}^2,\ 2c_{11}c_{12},\ c_{12}^2,\ c_{11}c_{21},\ c_{11}c_{22} + c_{12}c_{21},\ c_{12}c_{22},\ c_{21}^2,\ 2c_{21}c_{22},\ c_{22}^2 \rangle \\
&= \langle c_{11,11},\ 2c_{11,12},\ c_{11,22},\ c_{12,11},\ c_{12,12} + c_{12,21},\ c_{12,22},\ c_{22,11},\ 2c_{22,12},\ c_{22,22} \rangle.
\end{aligned}
$$

d. Similarly, $\wedge^2(E)$ has basis $\{e_{12} - e_{21}\}$ and $\mathrm{cf}\,(\wedge^2(E)) = \langle c_{12,12} - c_{12,21} \rangle$.

53 NOTATION. Let $\pi \in \sum_r$. Denote $\pi c_{i,\,j} := c_{i,\,j\pi}$ where $j\pi = (j_{\pi(1)}, \cdots, j_{\pi(r)})$.

54 PROPOSITION. If $\tau = (12) \in \sum_r$, then $(1 \pm \tau)\mathrm{cf}\left(E^{\otimes 2}\right) = \mathrm{cf}\,(1 \pm \tau)\left(E^{\otimes 2}\right)$.

Proof. $\mathrm{cf}\left(E^{\otimes 2}\right) = \langle c_{11,11},\ c_{11,22},\ c_{11,12},\ c_{12,11},\ c_{12,12},\ c_{12,21},\ c_{12,22},\ c_{22,11},\ c_{22,22},\ c_{22,12} \rangle$ by Example (52b). Note that:

$$
\begin{aligned}
&(1+\tau)(c_{11,11}) = c_{11,11} + c_{11,11} = 2c_{11,11}, && (1+\tau)(c_{11,22}) = c_{11,22} + c_{11,22} = 2c_{11,22}, \\
&(1+\tau)(c_{11,12}) = c_{11,12} + c_{11,21} = c_{11}c_{12} + c_{12}c_{11} = 2c_{11}c_{12} = 2c_{11,12}, \\
&(1+\tau)(c_{12,11}) = c_{12,11} + c_{12,11} = 2c_{12,11}, && (1+\tau)(c_{12,12}) = c_{12,12} + c_{12,21}, \\
&(1+\tau)(c_{12,21}) = c_{12,21} + c_{12,12}, && (1+\tau)(c_{12,22}) = c_{12,22} + c_{12,22} = 2c_{12,22}, \\
&(1+\tau)(c_{22,11}) = c_{22,11} + c_{22,11} = 2c_{22,11}, && (1+\tau)(c_{22,22}) = c_{22,22} + c_{22,22} = 2c_{22,22}, \\
&(1+\tau)(c_{22,12}) = c_{22,12} + c_{22,21} = c_{21}c_{22} + c_{22}c_{21} = 2c_{21}c_{22} = 2c_{22,12}.
\end{aligned}
$$

Thus by Example (52c) and Proposition (48c),

$$
\begin{aligned}
(1+\tau)\left(\mathrm{cf}\left(E^{\otimes 2}\right)\right) &= \langle c_{11,11},\ c_{11,22},\ c_{11,12},\ c_{12,11},\ c_{12,12} + c_{12,21},\ c_{12,22},\ c_{22,11},\ c_{22,22},\ c_{22,12} \rangle \\
&= \mathrm{cf}\,(S^2\,(E)) = \mathrm{cf}\left((1+\tau)\left(E^{\otimes 2}\right)\right).
\end{aligned}
$$

Similarly, note that:

$$(1-\tau)(c_{11,11}) = c_{11,11} - c_{11,11} = 0, \qquad (1-\tau)(c_{11,12}) = c_{11,12} - c_{11,21} = c_{11}c_{12} - c_{12}c_{11} = 0,$$

$$(1-\tau)(c_{12,12}) = c_{12,12} - c_{12,21}, \qquad (1-\tau)(c_{11,22}) = c_{11,22} - c_{11,22} = 0,$$

$$(1-\tau)(c_{12,11}) = c_{12,11} - c_{12,11} = 0, \qquad (1-\tau)(c_{12,21}) = c_{12,21} - c_{12,12},$$

$$(1-\tau)(c_{12,22}) = c_{12,22} - c_{12,22} = 0, \qquad (1-\tau)(c_{22,11}) = c_{22,11} - c_{22,11} = 0,$$

$$(1-\tau)(c_{22,12}) = c_{22,12} - c_{22,21} = c_{21}c_{22} - c_{22}c_{21} = 0, \qquad (1-\tau)(c_{22,22}) = c_{22,22} - c_{22,22} = 0.$$

Applying Example (52d) and Proposition (48c) yields

$$(1-\tau)\left(\mathrm{cf}\left(E^{\otimes 2}\right)\right) = \langle c_{12,12} - c_{12,21}\rangle = \mathrm{cf}\left(\wedge^2(E)\right) = \mathrm{cf}\left((1-\tau)\left(E^{\otimes 2}\right)\right). \qquad \square$$

55 NOTATION.

a. A polynomial is called *homogeneous* when each of its terms has the same degree. We let K be infinite hereafter. By Lemma (50), $A(n)$ may be viewed as the polynomial algebra over K in the indeterminants $c_{\alpha\beta}$. Let A_r $(r \geq 0)$ denote the K-subspace of $A(n)$ generated by the homogeneous polynomial functions of total degree r.

b. Let $I = \{f \mid f : \mathbf{r} \to \mathbf{n}\}$. $G = \sum_r$ acts on I via $i\pi = (i_{\pi(1)}, \ldots i_{\pi(r)})$ and G acts on $I \times I$ by $(i, j)\pi = (i\pi, j\pi)$ for $i, j \in I$ and $\pi \in G$. For $i, j \in I$, define $(i, j) \sim (p, q)$ for $(i, j), (p, q) \in I \times I$ when $p = i\pi$ and $q = j\pi$ for some $\pi \in G$. Let $R(n, r)$ denote a set of representatives for the equivalence classes of $I \times I$ under $\sim$.

56 REMARK. For fixed $g \in \Gamma$ and with E viewed as a K-space, define $t'_g : E^{\times r} \to E^{\otimes r}$ by $t'_g(x_1, \cdots, x_r) = g(x_1 \otimes \cdots \otimes x_r)$ for all $x_1, \cdots, x_r \in E$. Then t'_g is r-multilinear and induces a K-linear map $t_g : E^{\otimes r} \to E^{\otimes r}$ (Theorem (16) and induction) such that $t'_g = t_g \circ \beta$ where β is the canonical r-multilinear map. Then t_g gives rise to a matrix representation $T'_{n,r} : \Gamma \to \mathrm{GL}_n(E^{\otimes r})$ given by $T'_{n,r}(g) = t_g$. Extending linearly to $K\Gamma$ and using the standard basis $\{e_i \mid i \in I\}$ of $E^{\otimes r}$ yields the matrix representation $T_{n,r} : K\Gamma \to \mathrm{Mat}_I K$ given by $T_{n,r}(\kappa) = [g_{i,j}]$ for $i, j \in I$ where $\kappa e_j = \sum_{i \in I} g_{i,j} e_i$. Similarly, $c_{i,j}$ may be extended linearly to $K\Gamma$.

57 LEMMA. Let r be a nonnegative integer. Then $\sum_{i=0}^{r} \binom{n-2+i}{i} = \binom{n+r-1}{r}$.

Proof. We proceed by induction on r. The result is obvious for $r = 0$. Recall [14, p. 8] that Pascal's Rule says $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$ for $1 \leq k \leq n$. Then, using the induction hypothesis, we have

$$\begin{aligned}\sum_{i=0}^{r} \binom{n-2+i}{i} &= \left[\sum_{i=0}^{r-1} \binom{n-2+i}{i}\right] + \binom{n-2+r}{r} = \binom{n-2+r}{r-1} + \binom{n-2+r}{r} \\ &= \binom{n+r-1}{r}.\end{aligned}$$

□

58 THEOREM. (a) $\mathcal{C} = \{c_{i,\,j} = c_{i_1 j_1} \cdots c_{i_r j_r} \mid (i,\, j) \in R(n,\, r)\}$ is a K-basis for A_r. (b) $\dim A_r = \binom{n^2+r-1}{r}$. (c) $A_r = \mathrm{cf}\,(E^{\otimes r})$.

Proof.

a. A_r is spanned as a K-space by the monomials $\{c_{i,\,j} \mid i,\, j \in I\}$. Now since $c_{i,\,j} = c_{k,\,\ell}$ if and only if $(i,\, j) \sim (k,\, \ell)$, we have that this set equals $\mathcal{C}$. So $\mathcal{C}$ spans A_r, and the elements of $\mathcal{C}$ are distinct. Thus $\mathcal{C}$ is linearly independent by Lemma (50). Consequently $\mathcal{C}$ is a K-basis for A_r.

b. We show that the number of distinct monomials $x_1^{r_1} \cdots x_m^{r_m}$ in the m commuting indeterminants x_i with $\sum_i r_i = r$ is $\binom{m+r-1}{r}$. We proceed by induction on m. The result is obvious for $m = 1$. Let w_ℓ be the number of distinct monomials with $\sum_i r_i = r$ such that $r_m = \ell$. The number in question is $w = \sum_\ell w_\ell$. By Lemma (57),

$$\begin{aligned} w &= w_0 + \cdots + w_r \\ &= \binom{m-1+r-1}{r} + \binom{m-1+r-1-1}{r-1} + \cdots + \binom{m-1+0-1}{0} \\ &= \sum_{i=0}^{r} \binom{m-2+r-i}{r-i} = \sum_{i=0}^{r} \binom{m-2+i}{i} = \binom{m+r-1}{r}. \end{aligned}$$

The claim now follows from (a).

c. By Remark(56), $ge_j = \sum_{i \in I} c_{i,\,j}(g) e_i$. Thus $\mathrm{cf}\,(E^{\otimes r}) = \sum_{i,\,j \in I} K c_{i,\,j} = A_r$. □

Define $F : K^\Gamma \times K^\Gamma \to K^{\Gamma\times\Gamma}$ by $[F(f,\, g)](u,\, v) = f(u)g(v)$ $(f,\, g \in K^\Gamma,\ u,\, v \in \Gamma)$. There exists a unique K-linear map $\Phi : K^\Gamma \otimes K^\Gamma \to K^{\Gamma\times\Gamma}$ given by $[\Phi(f \otimes g)](u,\, v) = f(u)g(v)$ by Theorem (16) since F is bilinear. Φ is injective by an argument similar to that given in the proof of Lemma (29b). So, we may consider $K^\Gamma \otimes K^\Gamma$ as a K-subspace of $K^{\Gamma\times\Gamma}$.

59 LEMMA. $A(n)$ is a K-bialgebra, and A_r is a K-subcoalgebra of $A(n)$.

Proof. $A(n)$ is a K-algebra as it is a K-subalgebra of K^Γ. Then $\mu : A(n) \otimes A(n) \to A(n)$ and $\iota : K \to A(n)$ given by $\mu(c_{i,\,j} \otimes c_{k,\,\ell}) = c_{i,\,j}c_{k,\,\ell}$ and $\iota(k) = k1$ are the structure maps for $A(n)$ by the proof of Theorem (33). Define $\Delta : K^\Gamma \to K^{\Gamma\times\Gamma}$ by $[\Delta(f)](u,\, v) = f(uv)$ and $\varepsilon : K^\Gamma \to K$ by $\varepsilon(f) = f(1_\Gamma)$ for all $f \in K^\Gamma$, $u,\, v \in \Gamma$. Since for all $f,\, g \in K^\Gamma$, $u,\, v \in \Gamma$, and $k \in K$, we have

(i) $[\Delta(f + g)](u,\, v) = (f + g)(uv) = f(uv) + g(uv) = [\Delta f](u,\, v) + [\Delta g](u,\, v)$,

(ii) $[\Delta(fg)](u,\, v) = (fg)(uv) = f(uv)g(uv) = [\Delta f](u,\, v)[\Delta g](u,\, v)$,

(iii) $[\Delta(kf)](u,\, v) = (kf)(uv) = kf(uv) = k[\Delta f](u,\, v)$,

(iv) $[\Delta(1_{K^\Gamma})](u,\, v) = 1_{K^\Gamma}(uv) = 1_K = 1_{K^{\Gamma\times\Gamma}}(u,\, v)$,

(v) $\varepsilon(f + g) = (f + g)(1_\Gamma) = f(1_\Gamma) + g(1_\Gamma) = \varepsilon(f) + \varepsilon(g)$,

(vi) $\varepsilon(fg) = (fg)(1_\Gamma) = f(1_\Gamma)g(1_\Gamma) = \varepsilon(f)\varepsilon(g)$,

(vii) $\varepsilon(kf) = (kf)(1_\Gamma) = kf(1_\Gamma) = k\varepsilon(f)$, and

(viii) $\varepsilon(1_{K^\Gamma}) = 1_{K^\Gamma}(1_\Gamma) = 1_K$,

Δ and ε are K-algebra homomorphisms by (i) - (iv) and (v) - (viii), respectively. Now restrict Δ and ε to $A(n)$. Then $\Delta(c_{\alpha\beta}) = \sum_{\gamma=1}^{n} c_{\alpha\gamma} \otimes c_{\gamma\beta}$ and $\varepsilon(c_{\alpha\beta}) = \delta_{\alpha\beta}$ for $1 \le \alpha,\, \beta \le n$. We next verify the Coassociative Law and Counitary Property. Then

$$\begin{aligned}((\Delta \otimes 1) \circ \Delta)(c_{\alpha\beta}) &= (\Delta \otimes 1)\left(\sum_{\gamma} c_{\alpha\gamma} \otimes c_{\gamma\beta}\right) = \sum_{\gamma,\,\zeta} (c_{\alpha\zeta} \otimes c_{\zeta\gamma}) \otimes c_{\gamma\beta} \\ &= \sum_{\gamma,\,\zeta} c_{\alpha\zeta} \otimes (c_{\zeta\gamma} \otimes c_{\gamma\beta}) = (1 \otimes \Delta)\left(\sum_{\zeta} c_{\alpha\zeta} \otimes c_{\zeta\beta}\right) = ((1 \otimes \Delta) \circ \Delta)(c_{\alpha\beta})\end{aligned}$$

$$((\varepsilon\otimes 1)\circ\Delta)(c_{\alpha\beta}) = (\varepsilon\otimes 1)\left(\sum_\gamma c_{\alpha\gamma}\otimes c_{\gamma\beta}\right) = \sum_\gamma \varepsilon(c_{\alpha\gamma})\otimes c_{\gamma\beta} = 1_K\otimes\sum_\gamma \varepsilon(c_{\alpha\gamma})c_{\gamma\beta}$$
$$= 1_K\otimes\sum_\gamma \delta_{\alpha\gamma}c_{\gamma\beta} = 1_K\otimes c_{\alpha\beta} = \rho_2(c_{\alpha\beta}),$$

and similarly $((1\otimes\varepsilon)\circ\Delta)(c_{\alpha\beta}) = \rho_1(c_{\alpha\beta})$. So $(A(n), \Delta, \varepsilon)$ is a K-coalgebra. By Theorem (44), $(A(n), \mu, \iota, \Delta, \varepsilon)$ is a K-bialgebra. Finally, note A_r is a K-subspace of $A(n)$ by the definition of A_r. Let $c_{i,k} = c_{i_1k_1}\cdots c_{i_rk_r} \in A_r$. Then, using the fact that Δ is a K-algebra homomorphism, we find that $\Delta(c_{i,k}) = \sum_{j\in I} c_{i,j}\otimes c_{j,k} \in A_r\otimes A_r$. Thus $\Delta(A_r)\subseteq A_r\otimes A_r$. A_r is a K-subcoalgebra of $A(n)$ by Definition (35). □

3.2 Schur Algebras and Group Actions

Let $f\in K^\Gamma$ and $\kappa = \sum \kappa_g g \in K\Gamma$. Define $\overline{f}(\kappa) = \sum\kappa_g f(g)$. Then $\overline{f}$ is a unique linear extension of f. Let V be a finite-dimensional $K\Gamma$-module with basis $\{v_b \mid b\in B\}$. If Γ acts as $gv_b = \sum_B \alpha_{ab}(g)v_a$ (as in the definition of coefficient space), then $K\Gamma$ acts as $\kappa v_b = \sum_a \alpha_{ab}(\kappa)v_a$ for all $\kappa\in K\Gamma$ and all $b\in B$. Let $\rho : K\Gamma\to \mathrm{End}_K(V)$ be the representation afforded by V, and let $Y = \ker\rho$.

60 LEMMA. Let $f\in K^\Gamma$ and $\kappa\in K\Gamma$. Then (a) $\kappa\in Y$ if and only if $f(\kappa) = 0$ for all $f\in \mathrm{cf}(V)$ and (b) $f\in\mathrm{cf}(V)$ if and only if $f(\kappa)=0$ for all $\kappa\in Y$.

Proof. a. Let $\kappa\in Y$ and $f\in\mathrm{cf}(V)$. Then $f = \sum_{a,b} d_{ab}\alpha_{ab}$ for some $d_{ab}\in K$. Since $\alpha_{ab}(\kappa) = 0$ for all $a, b\in B$, we have $f(\kappa) = \sum_{a,b} d_{ab}\alpha_{ab}(\kappa) = 0$. Conversely, let $f(\kappa) = 0$ for all $f\in\mathrm{cf}(V)$. Since $\alpha_{ab}\in\mathrm{cf}(V)$ for all $a, b\in B$, we have $\alpha_{ab}(\kappa)=0$ for all $a, b\in B$. So $\rho(\kappa)(v_b) = \kappa v_b = \sum_a \alpha_{ab}(\kappa)v_a = 0$ for all $a, b\in B$. So $\kappa\in Y$.

b. Let $N := \rho(K\Gamma)$. Define $\langle\,,\,\rangle : Y^0\times N\to K$ by $\langle f, \nu\rangle = f(\kappa)$ for all $f\in Y^0$ and $\nu = \rho(\kappa)\in N$. Suppose $\rho(\kappa) = \rho(\lambda)$ for some $\kappa, \lambda\in K\Gamma$ and let $f\in Y^0$. Since ρ is a homomorphism, then $\rho(\kappa-\lambda) = 0$. Hence $\kappa-\lambda\in\ker\rho$. Thus $f(\kappa-\lambda) = 0$

since $f \in Y^0$. So $f(\kappa) - f(\lambda) = 0$ since f is linear. Hence $f(\kappa) = f(\lambda)$. Thus $\langle\,,\,\rangle$ is well-defined. Now suppose $\langle f,\, \nu\rangle = 0$ for every $\nu \in N$. So $f(\kappa) = 0$ for every $\kappa \in K\Gamma$. In particular, $0 = f(1_K g) = f(g)$ for every $g \in \Gamma$. So $f = 0$. Now let $\nu = \rho(\xi) \in N$. Suppose $\langle f,\, \nu\rangle = 0$ for every $f \in Y^0$. Then $f(\xi) = 0$ for every $f \in Y^0$ by the definition of $\langle\,,\,\rangle$. Note $Y = (Y^0)^0 = \{x \mid f(x) = 0 \text{ for every } f \in Y^0\}$. Hence $\xi \in Y$. So $\nu = 0$. So $\langle\,,\,\rangle$ is non-singular. By (a), $\mathrm{cf}(V) \subseteq Y^0$. Observe if $\nu = \rho(\kappa) \in N$ such that $f(\kappa) = \langle f,\, \nu\rangle = 0$ for all $f \in \mathrm{cf}(V)$, then $\kappa \in Y$ by (a). Hence $\nu = 0$. This implies $\langle\,,\,\rangle$ restricted to $\mathrm{cf}(V) \times N$ is non-singular. So $\mathrm{cf}(V) \cong N^* \cong Y^0$ by two applications of Lemma (32). Thus $\dim \mathrm{cf}(V) = \dim Y^0$. Therefore $\mathrm{cf}(V) = Y^0$. □

61 EXAMPLE. Let g be the 3×3 matrix with $g_{11} = g_{12} = g_{22} = g_{33} = 1$ and 0 elsewhere. We compute $T_{3,\,2}(g)$. Note $e_i = e_{i_1} \otimes e_{i_2}$ since $i = (i_1,\, i_2)$. We write $e_{jk} := e_j \otimes e_k$ and $g_{jk,\,\ell m} := \alpha(g)_{(j,\,k),\,(\ell,\,m)}$. A few calculations are included:

$ge_{11} = g(e_1 \otimes e_1) = ge_1 \otimes ge_1 = e_1 \otimes e_1 = e_{11}$

$\Rightarrow g_{11,\,11} = 1$ and $g_{i,\,11} = 0$ for $i \neq (1,\,1)$;

$ge_{12} = ge_1 \otimes ge_2 = e_1 \otimes (e_1 + e_2) = e_{11} + e_{12}$

$\Rightarrow g_{11,\,12} = g_{12,\,12} = 1$ and $g_{i,\,12} = 0$ for $i \neq (1,\,1),\,(1,\,2)$;

$ge_{13} = ge_1 \otimes ge_3 = e_1 \otimes e_3 = e_{13}$

$\Rightarrow g_{13,\,13} = 1$ and $g_{i,\,13} = 0$ for $i \neq (1,\,3)$;

$ge_{21} = ge_2 \otimes ge_1 = (e_1 + e_2) \otimes e_1 = e_{11} + e_{21}$

$\Rightarrow g_{11,\,21} = g_{21,\,21} = 1$ and $g_{i,\,21} = 0$ for $i \neq (1,\,1),\,(2,\,1)$.

We eventually obtain $T_{3,\,2}(g) = g \otimes g$ (Figure 10).

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Figure 10: $T_{3,\,2}(g)$

62 DEFINITION. The *Schur algebra*, denoted by S_r or $S_r(\Gamma)$, is the image of $K\Gamma$ under $T_{n,\,r}$ with identity $1_{S_r} = [\delta_{i,\,j}]$ where $\delta_{i,\,j} = \delta_{i_1 j_1} \cdots \delta_{i_r j_r}$.

Note that $[\delta_{i,\,j}]$ is just the identity matrix.

63 THEOREM. (a) $\langle\, ,\, \rangle : A_r \times S_r \to K$ given by $\langle c_{i,\,k},\, T_{n,\,r}(\kappa)\rangle = c_{i,\,k}(\kappa)$ is well-defined, non-singular, and bilinear where $c_{i,\,k} \in A_r$, and $\kappa \in K\Gamma$. (b) A_r^* and S_r are isomorphic as K-spaces. (c) S_r is a K-algebra with $\dim S_r = \binom{n^2+r-1}{r}$.

Proof. (a) First, if $T_{n,\,r}(\kappa) = T_{n,\,r}(\kappa')$, then $\kappa - \kappa' \in \operatorname{Ker} T_{n,\,r}$. So $c_{i,\,k}(\kappa - \kappa') = 0$ (Theorem (58c) and Lemma (60b)) and $c_{i,\,k}(\kappa) = c_{i,\,k}(\kappa')$. Consequently, the form is well-defined. Suppose $0 = \langle c_{i,\,k},\, T_{n,\,r}(\kappa)\rangle = c_{i,\,k}(\kappa)$ for all $\kappa \in K\Gamma$. Thus $c_{i,\,k} = 0$. Now suppose $c_{i,\,k}(\kappa) = \langle c_{i,\,k},\, T_{n,\,r}(\kappa)\rangle = 0$ for all $c_{i,\,k} \in A_r$. Then $\kappa \in \ker T_{n,\,r}$ by Lemma (60a). Hence $T_{n,\,r}(\kappa) = 0$. Thus $\langle, \rangle$ is non-singular. Next for all $c_{h,\,i},\, c_{j,\,k} \in A_r$, $\kappa,\, \lambda \in K\Gamma$, and $x,\, y \in K$, we have

$$\begin{aligned}\langle xc_{h,\,i} + yc_{j,\,k},\, T_{n,\,r}(\kappa)\rangle &= (xc_{h,\,i} + yc_{j,\,k})(\kappa) = xc_{h,\,i}(\kappa) + yc_{j,\,k}(\kappa)\\ &= x\langle c_{h,\,i},\, T_{n,\,r}(\kappa)\rangle + y\langle c_{j,\,k},\, T_{n,\,r}(\kappa)\rangle\end{aligned}$$

and

$$\begin{aligned}\langle c_{i,\,k},\, xT_{n,\,r}(\kappa) + yT_{n,\,r}(\lambda)\rangle &= \langle c_{i,\,k},\, T_{n,\,r}(x\kappa + y\lambda)\rangle = c_{i,\,k}(x\kappa + y\lambda) = xc_{i,\,k}(\kappa) + yc_{i,\,k}(\lambda)\\ &= x\langle c_{i,\,k},\, T_{n,\,r}(\kappa)\rangle + y\langle c_{i,\,k},\, T_{n,\,r}(\lambda)\rangle.\end{aligned}$$

Thus $\langle\, ,\, \rangle$ is bilinear. (b) $\dim A_r^*$ is finite by Theorem (58b). Then A_r^* and S_r are isomorphic as K-spaces by Lemma (32). (c) S_r is a homomorphic image of the K-algebra $K\Gamma$ so it is a K-algebra. Moreover, $\dim S_r = \dim A_r^* = \dim A_r = \binom{n^2+r-1}{r}$ by Theorem (58b). $\square$

64 LEMMA. Let $\xi,\, \eta \in S_r$ and $i,\, j \in I$.

a. $\langle c_{i,\,j},\, \xi\rangle =$ the $(i,\, j)$th entry of ξ.

b. $\langle c_{i,\,j},\, \xi\eta\rangle = \sum_{h\in I}\langle c_{i,\,h},\, \xi\rangle\langle c_{h,\,j},\, \eta\rangle$.

Proof. a. Note that $\xi : E^{\otimes r} \to E^{\otimes r}$ is a linear map. We write the matrix of ξ relative to the basis $\{e_i \mid i \in I\}$ as $[\xi_{ij}]$. We must show that $\langle c_{ij},\, \xi\rangle = \xi_{ij}$. That is, we must show that $\xi(e_j) = \sum_i \langle c_{ij},\, \xi\rangle e_i$. Suppose that $\xi = T_{n,\,r}(g)$ for some $g \in \Gamma$. By Theorem (63), we must show that $\xi(e_j) = \sum_i c_{ij}(g)e_i$. But this is clear since

$$\xi(e_j) = T_{n,r}(g)(e_j) = ge_{j_1} \otimes \cdots \otimes ge_{j_r} = \sum_{i_1=1}^{n} c_{i_1 j_1}(g) e_{i_1} \otimes \cdots \otimes \sum_{i_r=1}^{n} c_{i_r j_r}(g) e_{i_r}$$

$$= \sum_i c_{ij}(g) e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_r} = \sum_i c_{ij}(g) e_i.$$

Since $T_{n,r}$ and c_{ij} are linear, we obtain $T_{n,r}(\kappa)(e_j) = \sum_i c_{ij}(\kappa)(e_j)$ for each $\kappa \in K\Gamma$, and the claim follows.

b. By (a), $\langle c_{ij}, \xi\eta \rangle = (i, j)$th entry of $\xi\eta = \sum_{h \in I} \xi_{i,h}\eta_{h,j} = \sum_{h \in I} \langle c_{ih}, \xi \rangle \langle c_{hj}, \eta \rangle$. □

65 THEOREM. $\psi : S_r(\Gamma) \to A_r^*$ given by $\psi(\xi)(f) = \langle f, \xi \rangle$ is a K-algebra isomorphism.

Proof. First, ψ is a K-space isomorphism by the proofs of Theorem (63b) and Lemma (32). Multiplication in the algebra A_r^* is defined by $(\alpha\beta)(c) = \sum \alpha(c_i)\beta(d_i)$ ($\alpha, \beta \in A_r^*$, $c \in A_r$), where $\Delta(c) = \sum_i c_i \otimes d_i$. Indeed,

$$(\alpha\beta)(c_{i,j}) = [\Delta^*(\alpha \otimes \beta)](c_{i,j}) = (\alpha \otimes \beta)(\Delta(c_{i,j})) = (\alpha \otimes \beta)\left(\sum_{h \in I} c_{i,h} \otimes c_{h,j}\right)$$

$$= \sum_{h \in I} \alpha(c_{i,h})\beta(c_{h,j}).$$

Now let $\xi, \eta \in S_r(\Gamma)$ and $i, j \in I$. Then by the definition of ψ and Lemma (64b)

$$\psi(\xi\eta)(c_{i,j}) = \langle c_{i,j}, \xi\eta \rangle = \sum_{h \in I} \langle c_{i,h}, \xi \rangle \langle c_{h,j}, \eta \rangle = \sum_{h \in I} \psi(\xi)(c_{i,h})\psi(\eta)(c_{h,j}).$$

Since the $c_{i,j}$ span A_r, we have $\psi(\xi\eta)(c) = \sum_{i \in I} \psi(\xi)(c_i)\psi(\eta)(d_i)$ for all $c \in A_r$. Next, let $\alpha = \psi(\xi)$ and $\beta = \psi(\eta)$. Consequently,

$$\psi(\xi)\psi(\eta)(c) = (\alpha\beta)(c) = \sum_{i \in I} \alpha(c_i)\beta(d_i) = \sum_{i \in I} \psi(\xi)(c_i)\psi(\eta)(d_i) = \psi(\xi\eta)(c)$$

for all $c \in A_r$. Thus ψ is a homomorphism. Therefore ψ is an algebra map since $\psi(1_{S_r(\Gamma)})(c_{i,j}) = \langle c_{i,j}, 1_{S_r(\Gamma)} \rangle = \delta_{i,j} = \varepsilon(c_{i,j})$. □

66 DEFINITION. Let S be a set, G a group, and e the identity of G. An *action* of G on S is a function $G \times S \to S$ given by $(g, x) \mapsto gx$ such that $ex = x$ and $(gh)x = g(hx)$

for all $x \in S$ and $g, h \in G$. A *right action* of G has a similar definition with g appearing on the right. S is called a (*right*) *G-set* when an (right) action exists.

67 NOTATION. We set $N := \mathrm{End}_K(E^{\otimes r})$. Then N has basis $\{e_{i,j} \mid i, j \in I\}$ where $e_{i,j} : E^{\otimes r} \to E^{\otimes r}$ is given by $e_{i,j}(e_k) = \delta_{j,k} e_i$.

68 EXAMPLES. Let $G = \sum_r$, $\sigma \in G$, $i = (i_1, i_2, \cdots, i_r)$, and recall $I := I(n, r)$.

a. Consider $\sigma i = (i_{\sigma^{-1}(1)}, \cdots, i_{\sigma^{-1}(r)})$. This makes I a G-set since

$$\begin{aligned} (1)i &= (i_{(1)^{-1}(1)}, \cdots, i_{(1)^{-1}(r)}) = (i_1, \cdots, i_r) = i, \text{ and} \\ \sigma(\tau i) &= \sigma(i_{\tau^{-1}(1)}, \cdots, i_{\tau^{-1}(r)}) = \sigma(j_1, \cdots, j_r) = (j_{\sigma^{-1}(1)}, \cdots, j_{\sigma^{-1}(r)}) \\ &= (i_{\tau^{-1}(\sigma^{-1}(1))}, \cdots, i_{\tau^{-1}(\sigma^{-1}(r))}) = (i_{(\sigma\tau)^{-1}(1)}, \cdots, i_{(\sigma\tau)^{-1}(r)}) = (\sigma\tau)i, \end{aligned}$$

where $j_k := i_{\tau^{-1}(k)}$.

b. Next, consider $E^{\otimes r}$ and $\sigma e_i = e_{\sigma i} = e_{\sigma i_1} \otimes \cdots \otimes e_{\sigma i_r}$. Then $(1)e_i = e_{1i} = e_i$ and $\sigma(\rho e_i) = \sigma(e_{\rho i}) = e_{\sigma(\rho i)} = e_{(\sigma\rho)(i)} = (\sigma\rho)e_i$ by (a). Hence $E^{\otimes r}$ is a G-set with the above action extended linearly.

c. Define $e_{i,j}\sigma$ by $(e_{i,j}\sigma)(e) = e_{i,j}(\sigma e)$ $(e \in E^{\otimes r})$. Consequently, extending this action linearly yields that N a right G-set because $(e_{i,j}1)(e) = e_{i,j}(1e) = e_{i,j}(e)$ and $((e_{i,j}\sigma)\tau)(e) = (e_{i,j}\sigma)(\tau e) = e_{i,j}((\sigma\tau)e) = (e_{i,j}(\sigma\tau))(e)$. Moreover, we have $(e_{i,j}\sigma)e_k = e_{i,j}(\sigma e_k) = e_{i,j}(e_{\sigma k}) = \delta_{j,\sigma k} e_i = \delta_{\sigma^{-1}j,k} e_i = (e_{i,\sigma^{-1}j})e_k$. It therefore follows that $e_{i,j}\sigma = e_{i,\sigma^{-1}j}$.

d. Arguing as in (c), we find that N^* is a G-set with action $(\sigma e^*_{i,j})e_{k,\ell} = e^*_{i,j}(e_{k,\ell}\sigma)$ where $e^*_{i,j}(e_{k,\ell}) = \delta_{i,k}\delta_{j,\ell}$. Moreover,

$$(\sigma e^*_{i,j})e_{k,\ell} = e^*_{i,j}(e_{k,\ell}\sigma) = e^*_{i,j}(e_{k,\sigma^{-1}\ell}) = \delta_{i,k}\delta_{j,\sigma^{-1}\ell} = \delta_{i,k}\delta_{\sigma j,\ell} = e^*_{i,\sigma j}(e_{k,\ell})$$

by (c). So we have $\sigma e^*_{i,j} = e^*_{i,\sigma j}$.

3.3 Main Results

69 NOTATION. Suppose χ is a character of $G = \sum_r$. We set $t_\chi := \sum_{\sigma\in G}\chi(\sigma)\sigma \in KG$, $L := t_\chi E^{\otimes r}$, $N_L := \mathrm{End}_K(L)$, and $A_L := \mathrm{cf}\,(L)$.

70 DEFINITION. Let $T : K\Gamma \to N$ be the representation corresponding to the $K\Gamma$-module $E^{\otimes r}$. We define $T_L : K\Gamma \to N_L$ by $T_L(\kappa) = T(\kappa)\big|_L$.

The action of Γ on $E^{\otimes r}$ clearly commutes with the action of G. So $T(\kappa)(L) \subseteq L$ and T_L is well-defined.

71 LEMMA. $\psi : (\mathrm{im}\,T)t_\chi \to \mathrm{im}\,T_L$ given by $\psi(T(\kappa)t_\chi) = T(\kappa)\big|_L$ is a K-isomorphism.

Proof. We have that ψ is well-defined and injective since,

$$\begin{aligned} T(\kappa)t_\chi = T(\lambda)t_\chi &\Leftrightarrow T(\kappa)t_\chi(e) = T(\lambda)t_\chi(e) \text{ for all } e \in E^{\otimes r} \\ &\Leftrightarrow T(\kappa)(t_\chi e) = T(\lambda)(t_\chi e) \text{ for all } e \in E^{\otimes r} \\ &\Leftrightarrow T(\kappa)\big|_L = T(\lambda)\big|_L \Leftrightarrow \psi(T(\kappa)t_\chi) = \psi(T(\lambda)t_\chi). \end{aligned}$$

ψ is also surjective since $\psi(T(\kappa)t_\chi) = T(\kappa)\big|_L$ for any $T(\kappa)\big|_L \in \mathrm{im}\,T_L$. Finally ψ is a K-space isomorphism since for all $k \in K$ and for all $T(\kappa)\big|_L$, $T(\lambda)\big|_L \in \mathrm{im}\,T_L$:

$$\begin{aligned} \psi(T(\kappa)t_\chi + T(\lambda)t_\chi) &= \psi(T(\kappa+\lambda)t_\chi) = T(\kappa+\lambda)\big|_L = T(\kappa)\big|_L + T(\lambda)\big|_L \\ &= \psi(T(\kappa)t_\chi) + \psi(T(\lambda)t_\chi), \\ \psi(kT(\kappa)t_\chi) &= \psi(T(k\kappa)t_\chi) = T(k\kappa)\big|_L = kT(\kappa)\big|_L = k\psi(T(\kappa)t_\chi. \end{aligned}$$ □

72 REMARK. The dual of ψ in Lemma (71) is the map $\psi^* : (\mathrm{im}\,T_L)^* \to ((\mathrm{im}\,T)t_\chi)^*$ defined by $\psi^*(f)(T(\kappa)t_\chi) = f(\psi(T(\kappa)t_\chi)) = f(T(\kappa)\big|_L)$ by Definition (30). Also since N is a right G-set by Example (68c), it follows that $\mathrm{Hom}_K(K\Gamma, N)$ is a right G-set by $(f\sigma)(\kappa) = f(\kappa)\sigma$. In particular, $(Tt_\chi)(\kappa) = T(\kappa)t_\chi$.

73 LEMMA. (a) $\gamma : A_L \to (\operatorname{im} T_L)^*$ given by $(\gamma(a))(T_L(\kappa)) = a(\kappa)$ is a K-isomorphism. (b) If $\nu = \psi^* \circ \gamma$ then $\nu(a) \circ (Tt_\chi) = a$ as functions from $K\Gamma$ to K for every $a \in A_L$. (c) $A_L = \langle e^*_{i,j} \circ Tt_\chi \rangle$.

Proof. a. By Lemma (60b) we have $A_L = (\ker T_L)^\circ$. Note that there exists an isomorphism $F : (\ker T_L)^\circ \to (K\Gamma / \ker T_L)^*$ by Lemma (31b). Similarly, there exists an isomorphism $G : (K\Gamma/\ker T_L)^* \to (\operatorname{im} T_L)^*$ by the First Isomorphism Theorem. Now define $\gamma = G \circ F$. Consequently γ is an isomorphism with

$$\gamma(a)(T_L(\kappa)) = G(F(a))(T_L(\kappa)) = F(a)(\kappa + \ker T_L(\kappa)) = a(\kappa).$$

b. Let $\kappa \in K\Gamma$. Then by (a) $(\nu(a))(T(\kappa)t_\chi) = \psi^*(\gamma(a))(T(\kappa)t_\chi) = \gamma(a)(T_L(\kappa)) = a(\kappa)$ $(T(\kappa)t_\chi \in (\operatorname{im} T)t_\chi)$. Thus $a(\kappa) = (\nu(a))(T(\kappa)t_\chi) = \nu(a)\big((Tt_\chi)(\kappa)\big) = (\nu(a) \circ Tt_\chi)(\kappa)$ by the last sentence of Remark (72). Consequently, $\nu(a) \circ (Tt_\chi) = a$.

c. First, $A_L \subseteq \langle e^*_{i,j} \circ Tt_\chi \rangle$ since, using (b),we have for each $a \in A_L$

$$a = \nu(a) \circ Tt_\chi = (\psi^* \circ \gamma)(a) \circ Tt_\chi = \left(\sum_{i,j} a_{i,j} e^*_{i,j} \Big|_{(\operatorname{im} T)t_\chi} \right) \circ Tt_\chi$$

$$= \sum_{i,j} a_{i,j} (e^*_{i,j} \circ Tt_\chi) \in \langle e^*_{i,j} \circ Tt_\chi \rangle.$$

where we have used that $((\operatorname{im} T)t_\chi)^*$ is spanned by the restrictions of the $e^*_{i,j}$ to $(\operatorname{im} T)t_\chi$ to express $(\psi^* \circ \gamma)(a)$ as indicated. For the converse, let $\kappa \in \ker T_L$. Consequently, $T(\kappa)t_\chi = \psi^{-1}(T(\kappa)\big|_L) = \psi^{-1}(0) = 0$ by Lemma (71). Then since

$$(e^*_{i,j} \circ Tt_\chi)(\kappa) = e^*_{i,j}(Tt_\chi(\kappa)) = e^*_{i,j}(T(\kappa)t_\chi) = e^*_{i,j}(0) = 0,$$

we may conclude that $e^*_{i,j} \circ Tt_\chi \in ((\operatorname{im} T)t_\chi)^\circ = A_L$. Thus $\langle e^*_{i,j} \circ Tt_\chi \rangle \subseteq A_L$. $\square$

74 LEMMA. There exists a well-defined K-endomorphism t_χ of A_r with the property $t_\chi c_{i,\,j} = \sum_{\sigma\in G}\chi(\sigma)c_{i,\,\sigma j}$ $(i,\, j \in I)$.

Proof. Since A_r is spanned by the $c_{i,\,j}$, it is enough to check that the assignment is well-defined. Suppose $c_{i,\,j} = c_{k,\,\ell}$. Then $k = i\pi$ and $\ell = j\pi$ for some $\pi \in G$ (see Notation (55b)). Then $c_{k,\,\sigma\ell} = c_{i\pi,\,\sigma(j\pi)} = c_{\pi^{-1}i,\,\sigma\pi^{-1}j} = c_{i,\,\pi\sigma\pi^{-1}j}$. So

$$t_\chi c_{k,\,\ell} = \sum_{\sigma\in G}\chi(\sigma)c_{k,\,\sigma\ell} = \sum_{\sigma\in G}\chi(\sigma)c_{i,\,\pi\sigma\pi^{-1}j} = \sum_{\rho\in G}\chi(\pi^{-1}\rho\pi)c_{i,\,\rho j} = \sum_{\rho\in G}\chi(\rho)c_{i,\,\rho j} = t_\chi c_{i,\,j},$$

where we have used the fact that characters are constant on conjugacy classes. □

75 NOTATION. If E is replaced by L, the same construction (see Remark (56) and Definition (62)) which yielded S_r results in a K-algebra, which we denote by $S_{s,\,L}$. Put $A_{s,\,L} := \mathrm{cf}\,(L^{\otimes s})$.

Theorem (76), Theorem (78), and Theorem (80) below are the main results. Theorem (76) generalizes Proposition (54), Theorem (78) generalizes Theorem (58c), and Theorem (80) generalizes Theorem (63b).

76 THEOREM. $\mathrm{cf}\,(t_\chi E^{\otimes r}) = t_\chi \mathrm{cf}\,(E^{\otimes r})$.

Proof. Note

$$(e^*_{i,\,j}\circ Tt_\chi)(\kappa) = e^*_{i,\,j}((Tt_\chi)(\kappa)) = e^*_{i,\,j}((T(\kappa)t_\chi) = (t_\chi e^*_{i,\,j})(T(\kappa)) = ((t_\chi e^*_{i,\,j})\circ T)(\kappa).$$

Thus $e^*_{i,\,j}\circ Tt_\chi = t_\chi e^*_{i,\,j}\circ T$. Now $t_\chi e^*_{i,\,j} = \sum_{\sigma\in G}\chi(\sigma)\sigma e^*_{i,\,j} = \sum_{\sigma\in G}\chi(\sigma)e^*_{i,\,\sigma j}$. Then by Lemma (58c), Lemma (73c), Lemma (74), and since $c_{i,\,j} = e^*_{i,\,j}\circ T$, we have

$$\begin{aligned}\mathrm{cf}\,(t_\chi E^{\otimes r}) = A_L = \langle e^*_{i,\,j}\circ Tt_\chi\rangle = \langle (t_\chi e^*_{i,\,j})\circ T\rangle &= \left\langle \left(\sum_{\sigma\in G}\chi(\sigma)e^*_{i,\,\sigma j}\right)\circ T\right\rangle \\ &= \left\langle \sum_{\sigma\in G}\chi(\sigma)(e^*_{i,\,\sigma j}\circ T)\right\rangle = \left\langle \sum_{\sigma\in G}\chi(\sigma)c_{i,\,\sigma j}\right\rangle = \langle t_\chi c_{i,\,j}\rangle \\ &= t_\chi\langle c_{i,\,j}\rangle = t_\chi A_r = t_\chi \mathrm{cf}\,(E^{\otimes r}).\end{aligned}$$

□

77 NOTATION. Let $r_1, \cdots, r_u \in \mathbb{Z}^+$. For each i, let χ_i be a character of $\sum_{r_i}$ and put $L_{\chi_i} = t_{\chi_i} E^{\otimes r_i}$. We write $\prod_i t_{\chi_i} A_{r_i}$ to mean the set of all products $\prod_i c_i$ with $c_i \in t_{\chi_i} A_{r_i}$.

78 THEOREM. $\mathrm{cf}\left(\bigotimes_i L_{\chi_i}\right) = \prod_i t_{\chi_i} A_{r_i}$.

Proof. The matrix representation of a tensor product of modules is the Kronecker product of the matrix representations of the factors (see the proof of Theorem (26)). By Theorem (76), $\mathrm{cf}\left(\bigotimes_i L_{\chi_i}\right) = \prod_i \mathrm{cf}\left(L_{\chi_i}\right) = \prod_i \mathrm{cf}\left(t_{\chi_i} E^{\otimes r_i}\right) = \prod_i t_{\chi_i} \mathrm{cf}\left(E^{\otimes r_i}\right) = \prod_i t_{\chi_i} A_{r_i}$. □

79 COROLLARY. $\mathrm{cf}\left(L^{\otimes s}\right) = (t_\chi A_r)^s$ for any $s \in \mathbb{Z}^+$.

Proof. Immediate from Theorem (78). □

80 THEOREM. $S_{s,L} \cong A^*_{s,L}$.

Proof. Let $T_L : K\Gamma \to \mathrm{End}(L^{\otimes s})$ be the representation afforded by $L^{\otimes s}$ (extended to $K\Gamma$). Then $K\Gamma / \ker T_L \cong \mathrm{im}\, T_L = S_{s,L}$ by the First Isomorphism Theorem. Therefore $A_{s,L} = \mathrm{cf}\left(L^{\otimes s}\right) = (\ker T_L)^0 \cong (K\Gamma / \ker T_L)^* = S^*_{s,L}$ by Lemma (60b) and Lemma (31b). □

Bibliography

[1] David S. Dummit and Richard M. Foote. *Abstract Algebra.* 2nd ed. John Wiley, Hoboken, NJ, 1999.

[2] Paul Halmos. *Linear Algebra Problem Book.* Mathematical Association of America, Washington, DC, 1996.

[3] Kenneth Hoffman and Ray Kunze. *Linear Algebra.* 2nd ed., Prentice Hall, Englewood Cliffs, NJ, 1971.

[4] Randall R. Holmes. *Linear Representations of Finite Groups.* Lecture notes, Auburn University, 2000.

[5] Thomas W. Hungerford. *Algebra.* Springer-Verlag, New York, NY 1974.

[6] Nathan Jacobson. *Basic Algebra I.* 2nd ed. W H Freeman, New York, NY, 1985.

[7] Nathan Jacobson. *Basic Algebra II.* 2nd ed. W H Freeman, New York, NY, 1989.

[8] Jin Ho Kwak and Sungpyo Hong. *Linear Algebra.* Birkhäuser, Boston, MA 1997.

[9] Eiichi Abe. *Hopf Algebras.* Cambridge University Press, Cambridge, UK, 1980.

[10] Sorin Dascalesau, Constantin Nastesescu, Serban Raianu, and C. Nastasescu. *Hopf Algebras: An Introduction.* Marcel Dekker, New York, NY, 2000.

[11] D.G. Northcott. *Multilinear Algebra.* Cambridge University Press, Cambridge, UK, 1984.

[12] James A. Green. *Polynomial Representations of GL_n.* Springer-Verlag, New York, NY, 1980.

[13] Stuart Martin. *Schur Algebras and Representation Theory.* Cambridge University Press, Cambridge, UK, 1993.

[14] David M. Burton. *Elementary Number Theory.* 5th ed., McGraw-Hill, New York, NY, 2002.

Index