Geometry of Zeros and Bernstein Type Inequalities concerning Growth for Polynomials

by

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Abstract

In this dissertation, we study the problems related to geometry of zeros and Bernstein type inequalities concerning growth for polynomials. The dissertation consists of three chapters followed by list of references used in the text.

In Chapter one, besides presenting a brief introduction of the subject of geometry of zeros of polynomials, we mentioned briefly how the subject of location of zeros of polynomials can be useful in the study of problems in a dynamical system. Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with real coefficients. The well known Eneström-Kakeya theorem states that if $0 < a_0 \leq a_1 \leq a_2 \leq a_3 \ldots \leq a_n$, then all the zeros of $p(z)$ lie in the closed unit disk. Here, we generalize and extend this theorem. Also, we derive conditions on coefficients of $p(z)$ and estimate the number of zeros that the polynomial has in a prescribed region.

In Chapter two, we obtain several results which provide annuli containing all the zeros of a complex polynomial. Our result are explicit and the radii obtained are in terms of the coefficients of the polynomial. Also, we develop MATLAB code to construct examples of polynomials for which our results give sharper bound than obtainable from some well known results. The problems of this type were initiated by Gauss and Cauchy. In addition, we considered polynomial of the type $a_n z^n + a_m z^m + a_2 z^2 + a_1 z + a_0,\ 3 \leq m < n$ and obtained a disk centered at the origin that has at least one zeros of the polynomial. Problems of this type were initiated by Landau.

If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n, then it was proved by Bernstein that $\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|$, and for $R \geq 1$, $\max_{|z|=R} |p(z)| \leq R^n \max_{|z|=1} |p(z)|$. If $p(z) \neq 0$ in $|z| \leq 1$, then $\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \max_{|z|=1} |p(z)|$ and $\max_{|z|=R\geq 1} |p(z)| \leq \frac{R^n + 1}{2} \max_{|z|=1} |p(z)|$. The first result was conjectured by Erdös and proved by P. D. Lax and the second result is due to Ankeny and Rivlin. If $0 < r < 1$, then Rivlin proved that $\max_{|z|=r} |p(z)| \geq \left(\frac{1+r}{2}\right)^n \max_{|z|=1} |p(z)|$. Chapter

three deals with results in this direction where we prove a refinement of this result of Rivlin. Our result is best possible and gives a sharper result for all polynomials of this class except for polynomials with $\min\limits_{|z|=1} |p(z)| = 0$. This chapter also contains some other results in this direction.

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Chapter 1

Distribution of Zeros for Polynomials with Monotonicity Condition on Coefficients

1.1 Introduction

Let $p(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots + a_n z^n$ be a polynomial of degree n. By the Fundamental Theorem of Algebra (historically, the first important result concerning the roots of an algebraic equation), $p(z)$ has exactly n zeros in the complex plane, counting multiplicity. But this theorem does not say anything regarding the location of zeros of the polynomial, that is, the region which contains some or all of the zeros of a polynomial. Problems involving location of the zeros of a polynomial, besides being of theoretical interest, find applications in many areas of applied mathematics such as coding theory, cryptography, combinatorics, number theory, mathematical biology and engineering [5, 22, 70, 93, 99, 104]. In particular, problems dealing with location of zeros of the polynomial play an important role, for example, in solving digital audio signal processing problems [129], control engineering problems [20], and eigenvalue problems in mathematical physics [115].

Since Abel and Ruffini proved that there is no general algebraic solution to polynomial equations of degree five or higher, the problem of finding a region containing all the zeros of a polynomial became much more interesting, and over a period a large number of results have been provided in this direction. It may be remarked that there are methods, for example Ehrlich-Aberth's type (see [1, 44, 98]) for the simultaneous determination of the zeros of algebraic polynomials, and there are studies to accelerate convergence and increase computational efficiency of these methods (for example, see [92, 101]). These methods which are of course very useful, because they give approximations to the zeros of a polynomial can possibly become more efficient when combined with the results dealing with the region containing all the zeros of a polynomial, because an accurate estimate of the annulus containing all the zeros of a polynomial can considerably reduce the amount of work needed to find exact zeros, and so there is always a need for better estimates for the region containing all the zeros of a polynomial. Several monographs have been written on this subject and related subject of approximation theory (for example, see [87, 93, 94, 110]).

To see how the study of the location of zeros of a polynomial can be useful in control theory, let us consider a transfer function $H(s)$ in a dynamical system. If we have an input function, say, $X(s)$, and an output function $Y(s)$, we define $H(s) = \frac{Y(s)}{X(s)}$. In discrete time systems, the function can also be written as $H(z) = \frac{Y(z)}{X(z)}$ and is often referred to as the pulse transfer function. The zeros z_i of the system satisfy $Y(z_i) = 0$, and poles z_j of the system satisfy $X(z_j) = 0$. Poles and zeros of a transfer function are the frequencies for which the value of the transfer function becomes infinity or zero, respectively. The values of the poles and the zeros determine whether the system is stable, and how well the system performs. Control systems, in the simplest sense, can be designed by assigning simple values to the poles and zeros of the system. Physically reliable control systems must have a number of poles greater than or equal to the number of zeros. Systems that satisfy this relationship are called proper. So, the problem of finding the roots of either $Y(z_i) = 0$ or $X(z_j) = 0$, and the location of these roots are very important from a stability point of view. As a matter of fact, the closer the zeros are to the imaginary axis, the greater the stabilizing effect. This, for example, somewhat illustrates how the problem of finding the location of zeros can be of great importance.

The problems concerning the location of the zeros of a polynomial can mainly be divided in two categories, namely:

- Given an integer k, $1 \leq k \leq n$, find a region $R = R(a_0, a_1, a_2, \ldots, a_n)$ containing at least or exactly k zeros of $p(z)$. In particular, one would like to find the smallest circle $|z| = r$ which will enclose the k zeros of the polynomial. Such results are very useful for solving practical problems in numerical analysis; for example, in finding the roots of an algebraic equation by using Newton-Raphson Method, and finding eigenvalues. Note that such results can be of great help in problems dealing with eigenvalues because one is often not interested in computing all eigenvalues precisely.

- Given a region R, to find the number $k = k(a_0, a_1, a_2, \ldots, a_n)$ such that k number of zeros lie in the region R. In particular, to find the number k of zeros whose moduli do not exceed some prescribed value, say r.

This subject has been studied extensively. Due to the limited space, it would not be possible to include all the results in this subject, and therefore many important results in this area, which we would have liked to include, had to be excluded (for a more detailed study of the subject, we refer, in particular, to the monograph and books written by Dieudonné [42], Marden [87], Milovanović, Mitrinović, and Rassias [93], Rahman and Schmeisser [110], and recent articles due to Gardner and Govil [50], and Govil and Nwaeze [62]).

This chapter contains three sections. Section 1.1 lays a historical background coupled with a motivational interest to the subject matter. Section 1.2 discusses the first category of problem, stated above, with restriction on the coefficients of the given polynomial under consideration, while Section 1.3 deals with results of the second category as discussed above.

1.2 Eneström-Kakeya Theorem and its Extensions

We start by stating a classical result due to Eneström [45] and Kakeya [76] concerning the bounds for the moduli of zeros of polynomials having positive coefficients. It is often stated as:

Theorem 1.2.1 (Eneström-Kakeya). *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *with real coefficients satisfying* $0 < a_0 \leq a_1 \leq a_2 \leq a_3 \ldots \leq a_n$. *Then all the zeros of* $p(z)$ *lie in* $|z| \leq 1$.

For the sake of completeness, we present here the proof of the above theorem

Proof. Define f by the equation

$$p(z)(1-z) = a_0 + (a_1 - a_0)z + (a_2 - a_1)z^2 + (a_3 - a_2)z^3 + \cdots + (a_n - a_{n-1})z^n - a_n z^{n+1}$$
$$= f(z) - a_n z^{n+1}.$$

Then for $|z| = 1$, we have

$$|f(z)| \leq |a_0| + |a_1 - a_0| + |a_2 - a_1| + \cdots + |a_n - a_{n-1}|$$
$$= a_0 + (a_1 - a_0) + (a_2 - a_1) + \cdots + (a_n - a_{n-1})$$
$$= a_n.$$

Notice that the function $z^n f(1/z) = \sum_{j=0}^{n}(a_j - a_{j-1})z^{n-j}$, $a_{-1} = 0$ has the same bound on $|z| = 1$ as the function f. Namely, $|z^n f(1/z)| \leq a_n$ for $|z| = 1$. Since $z^n f(1/z)$ is analytic in $|z| \leq 1$, we have $|z^n f(1/z)| \leq a_n$ for $|z| \leq 1$ by the maximum modulus principle. Hence, $|f(1/z)| \leq a_n/|z|^n$ for $|z| \leq 1$. Replacing z by $1/z$, we see that $|f(z)| \leq a_n|z|^n$ for $|z| \geq 1$, and making use of this, we get

$$|(1-z)p(z)| = |f(z) - a_n z^{n+1}|$$
$$\geq a_n|z|^{n+1} - |f(z)|$$
$$\geq a_n|z|^{n+1} - a_n|z|^n$$
$$= a_n|z|^n(|z| - 1).$$

So if $|z| > 1$ then $(1-z)p(z) \neq 0$. Therefore, all the zeros of p lie in $|z| \leq 1$. □

An equivalent, but perhaps more useful, statement of the above theorem due, in fact, to Eneström [45], is the following:

Theorem 1.2.2. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *with real coefficients satisfying* $a_j > 0$ *for all* $j \in \{0, 1, 2, \cdots, n\}$*. Then all the zeros of* $p(z)$ *lie in the annulus* $\alpha \leq |z| \leq \beta$*, where* $\alpha = \min_{0 \leq j \leq n} \{a_j/a_{j+1}\}$*, and* $\beta = \max_{0 \leq j \leq n} \{a_j/a_{j+1}\}$*.*

Following this line of argument, Anderson, Saff and Varga [6] showed that

Theorem 1.2.3. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* $n \geq 1$ *with real coefficients satisfying* $\beta a_1 - a_0 > 0$ *and* $a_j > 0$ *for all* $j \in \{0, 1, 2, \cdots, n\}$*. Then all the zeros of* $p(z)$ *lie in the disk* $|z| \leq \beta$*, where* β *is as defined above in Theorem 1.2.2.*

In the literature there exist several extensions and generalizations of Theorem 1.2.1 (see [7], [75] and [81]). Joyal, Labelle and Rahman [75] extended Theorem 1.2.1 to the polynomials whose coefficients are monotonic but not necessarily nonnegative. In fact, they proved the following result:

Theorem 1.2.4. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *with real coefficients satisfying* $a_0 \leq a_1 \leq a_2 \leq a_3 \ldots \leq a_n$*. Then all the zeros of* $p(z)$ *lie in the disk* $|z| \leq \frac{1}{|a_n|}(a_n - a_0 + |a_0|)$*.*

Of course, when $a_0 > 0$ then Theorem 1.2.4 reduces to Theorem 1.2.1. It is important to note here that Theorem 1.2.4, just like the original Eneström-Kakeya theorem, is only applicable to polynomials with real coefficients. In 1968, Govil and Rahman [65] extended the Eneström-Kakeya theorem to polynomials with complex coefficients:

Theorem 1.2.5. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a complex polynomial with coefficients satisfying* $|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2}$ *for* $j \in \{0, 1, 2, \ldots, n\}$ *and for some real* α *and* β*, and* $|a_0| \leq |a_1| \leq |a_2| \leq |a_3| \ldots \leq |a_n|$*. Then all the zeros of* $p(z)$ *lie in* $|z| \leq \cos\alpha + \sin\alpha + \frac{2\sin\alpha}{|a_n|} \sum_{j=0}^{n-1} |a_j|$*.*

With $\alpha = \beta = 0$, Theorem 1.2.5 reduces to Theorem 1.2.1. In the same paper, Govil and Rahman gave a result for polynomials with complex coefficients but impose a nonnegativity and monotonicity condition on the real or imaginary parts of the coefficients of the polynomial. Specifically, they [65] proved

Theorem 1.2.6. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a complex polynomial with coefficients where* $Re a_j = \alpha_j$ *and* $Im a_j = \beta_j$ *for* $j \in \{0, 1, 2, \dots, n\}$ *satisfying* $0 \le \alpha_0 \le \alpha_1 \le \alpha_2 \le \alpha_3 \dots \le \alpha_n,\ \alpha_n \neq 0$. *Then all the zeros of* $p(z)$ *lie in* $|z| \le 1 + \frac{2}{|\alpha_n|} \sum_{j=0}^{n} |\beta_j|$.

In 1996, Aziz and Zarger [13] further extended Theorem 1.2.1 by relaxing the hypothesis in several ways and among other things proved the following result:

Theorem 1.2.7. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n such that for some* $k \ge 1$, $0 < a_0 \le a_1 \le a_2 \le a_3 \dots \le a_{n-1} \le k a_n$. *Then all the zeros of* $p(z)$ *lie in the disk* $|z + k - 1| \le k$.

In 2012, they further generalized Theorem 1.2.7 which is an interesting extension of Theorem 1.2.1. In particular, they [15] proved the following results:

Theorem 1.2.8. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n. If for some positive numbers k and* ρ *with* $k \ge 1$, $0 < \rho \le 1$, $0 \le \rho a_0 \le a_1 \le a_2 \le a_3 \dots \le a_{n-1} \le k a_n$. *Then all the zeros of* $p(z)$ *lie in the disk*

$$|z + k - 1| \le k + \frac{2a_0}{a_n}(1 - \rho).$$

Theorem 1.2.9. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n. If for some positive number* ρ, $0 < \rho \le 1$, *and for some nonnegative integer* λ, $0 \le \lambda \le n$, $\rho a_0 \le a_1 \le a_2 \le a_3 \dots \le a_{\lambda-1} \le a_\lambda \ge a_{\lambda+1} \ge \cdots \ge a_{n-1} \ge a_n$, *then all the zeros of* $p(z)$ *lie in the disk*

$$\left| z + \frac{a_{n-1}}{a_n} - 1 \right| \le \frac{1}{|a_n|}\Big[2a_\lambda - a_{n-1} + (2 - \rho)|a_0| - \rho a_0 \Big].$$

.

Looking at Theorem 1.2.8, one might want to know what happens if ρa_0 is NOT non-negative. In this section, we prove some extensions and generalization of Theorems 1.2.8 and 1.2.9 which in turns gives an answer to our enquiry.

1.2.1 Statement of New Results

Theorem 1.2.10. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n. *If for some real numbers* α *and* β, $a_0 - \beta \leq a_1 \leq a_2 \leq \ldots \leq a_{n-1} \leq a_n + \alpha$, *then all the zeros of* $p(z)$ *lie in the disk*

$$\left|z + \frac{\alpha}{a_n}\right| \leq \frac{1}{|a_n|}\Big[a_n + \alpha - a_0 + \beta + |\beta| + |a_0|\Big].$$

If $\alpha = (k-1)a_n$ with $k \geq 1$ and $\beta = (1-\rho)a_0$ with $0 < \rho \leq 1$, we get the following.

Corollary 1.2.11. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n. *If for some positive numbers* $k \geq 1$ *and* ρ, *with* $0 < \rho \leq 1$, $\rho\, a_0 \leq a_1 \leq a_2 \leq \ldots \leq a_{n-1} \leq k a_n$, *then all the zeros of* $p(z)$ *lie in the disk*

$$|z + k - 1| \leq \frac{1}{|a_n|}\Big[(k a_n - \rho a_0) + |a_0|(2-\rho)\Big].$$

If $a_0 > 0$, then Corollary 1.2.11 reduces to Theorem 1.2.8.

Theorem 1.2.12. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n. *If for some real number* s *and for some nonnegative integer* λ, $0 \leq \lambda \leq n$, $a_0 - s \leq a_1 \leq a_2 \leq a_3 \ldots \leq a_{\lambda-1} \leq a_\lambda \geq a_{\lambda+1} \geq \cdots \geq a_{n-1} \geq a_n$, *then all the zeros of* $p(z)$ *lie in the disk*

$$\left|z + \frac{a_{n-1}}{a_n} - 1\right| \leq \frac{1}{|a_n|}\Big[2a_\lambda - a_{n-1} + s - a_0 + |s| + |a_0|\Big].$$

.

If we take $s = (1-\rho)a_0$, with $0 < \rho \leq 1$, then Theorem 1.2.12 becomes Theorem 1.2.9. Instead of proving Theorem 1.2.12, we shall prove a more general case. In fact, we prove the following result:

Theorem 1.2.13. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n. *If for some real number* t, s *and for some nonnegative integer* λ, $0 \leq \lambda \leq n$, $a_0 - s \leq a_1 \leq a_2 \leq a_3 \ldots \leq a_{\lambda-1} \leq$

$a_\lambda \geq a_{\lambda+1} \geq \cdots \geq a_{n-1} \geq a_n + t$, then all the zeros of $p(z)$ lie in the disk

$$\left|z + \frac{a_{n-1}}{a_n} - \left(1 + \frac{t}{a_n}\right)\right| \leq \frac{1}{|a_n|}\Big[2a_\lambda - a_{n-1} + s - a_0 + |s| + |a_0| + |t|\Big].$$

.

We now present some examples to illustrate the importance of Theorems 1.2.10 and 1.2.13 over Theorems 1.2.1, 1.2.4, 1.2.7 and 1.2.8.

1.2.2 Demonstrating Examples

Example 1.2.14. Let us consider the polynomial

$$p(z) = 3z^5 + 4z^4 + 3z^3 + 2z^2 + z - 1.$$

The coefficients here are $a_5 = 3$, $a_4 = 4$, $a_3 = 3$, $a_2 = 2$, $a_1 = 1$ and $a_0 = -1$. We cannot apply Theorems 1.2.1, 1.2.4, 1.2.7 and 1.2.8. But we can apply Theorem 1.2.10 to determine where all the zeros of the polynomial lie. Using MATLAB, we obtain the following zeros : $-0.9154 + 0.4962i,\ -0.9154 - 0.4962i,\ 0.0530 + 0.8845i,\ 0.0530 - 0.8845i,\ 0.3916$. These zeros actually lie in the disk $|z + 2/3| \leq 1.1403$. But if we take $\alpha = 2$ and $\beta = 0$, Theorem 1.2.10 gives that all the zeros of the polynomial lie in the closed disk $|z + 2/3| \leq 7/3$.

Example 1.2.15. Next, consider

$$q(z) = -z^6 + 2z^5 + 2z^4 + 3z^3 + z^2 - 2.$$

The coefficients of $q(z)$ are $a_6 = -1$, $a_5 = 2$, $a_4 = 2$, $a_3 = 3$, $a_2 = 1$, $a_1 = 0$ and $a_0 = -2$. Using MATLAB, we obtain the following zeros: $3.0197,\ -0.7682 + 0.5814i,\ -0.7682 - 0.5814i,\ -0.0803 + 1.0233i,\ -0.0803 - 1.0233i,\ 0.6773$. Taking $\lambda = 3$, $t = 1$ and $s = 0$, Theorem 1.2.13 gives that the zeros lie in $|z - 2| \leq 9$.

1.2.3 Proofs of the Theorems

Proof of Theorem 1.2.10. Consider the polynomial

$$\begin{aligned}
g(z) &= (1-z)p(z) \\
&= -a_n z^{n+1} + (a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \cdots + (a_1 - a_0)z + a_0 \\
&= -a_n z^{n+1} - \alpha z^n + (a_n + \alpha - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \cdots + (a_1 - a_0 + \beta)z - \beta z + a_0 \\
&= -z^n(a_n z + \alpha) + (a_n + \alpha - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \cdots + (a_1 - a_0 + \beta)z - \beta z + a_0 \\
&= -z^n(a_n z + \alpha) + \phi(z)
\end{aligned}$$

where

$$\phi(z) = (a_n + \alpha - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \cdots + (a_1 - a_0 + \beta)z - \beta z + a_0.$$

Now for $|z| = 1$, we have

$$\begin{aligned}
|\phi(z)| &\leq |a_n + \alpha - a_{n-1}| + |a_{n-1} - a_{n-2}| + \cdots + |a_1 - a_0 + \beta| + |\beta| + |a_0| \\
&= a_n + \alpha - a_{n-1} + a_{n-1} - a_{n-2} + \cdots + a_1 - a_0 + \beta + |\beta| + |a_0| \\
&= a_n + \alpha - a_0 + \beta + |\beta| + |a_0|.
\end{aligned}$$

Since this is true for all complex number with a unit modulus, then it must also be true for $1/z$. With this in mind, we have

$$|z^n \phi(1/z)| \leq a_n + \alpha - a_0 + \beta + |\beta| + |a_0| \qquad \text{for all } z \text{ with } |z| = 1. \tag{1.1}$$

Also, the function $\Phi(z) = z^n\phi(1/z)$ is analytic in $|z| \leq 1$, hence, the Inequality (1.1) holds also inside the unit circle. That is,

$$|\phi(1/z)| \leq \frac{a_n + \alpha - a_0 + \beta + |\beta| + |a_0|}{|z|^n} \qquad \text{for all } z \text{ with } |z| \leq 1.$$

Replacing z by $1/z$, we get

$$|\phi(z)| \leq \Big[a_n + \alpha - a_0 + \beta + |\beta| + |a_0|\Big]|z|^n \qquad \text{for all } z \text{ with } |z| \geq 1.$$

Now for $|z| \geq 1$, we obtain

$$\begin{aligned}
|g(z)| &= | - z^n(a_n z + \alpha) + \phi(z)| \\
&\geq |z^n||a_n z + \alpha| - |\phi(z)| \\
&\geq |z^n||a_n z + \alpha| - \Big[a_n + \alpha - a_0 + \beta + |\beta| + |a_0|\Big]|z|^n \\
&= |z^n|\Big(|a_n z + \alpha| - \Big[a_n + \alpha - a_0 + \beta + |\beta| + |a_0|\Big]\Big) \\
&> 0,
\end{aligned}$$

which is true if and only if

$$|a_n z + \alpha| > \Big[a_n + \alpha - a_0 + \beta + |\beta| + |a_0|\Big];$$

that is,

$$|z + \frac{\alpha}{a_n}| > \frac{1}{|a_n|}\Big[a_n + \alpha - a_0 + \beta + |\beta| + |a_0|\Big].$$

Thus, all the zeros of $g(z)$ whose modulus is greater than or equal to 1 lie in

$$|z + \frac{\alpha}{a_n}| \leq \frac{1}{|a_n|}\Big[a_n + \alpha - a_0 + \beta + |\beta| + |a_0|\Big]. \tag{1.2}$$

But those zeros of $p(z)$ whose modulus is less than 1 satisfy (1.2). Also, all the zeros of $p(z)$ are zeros of $g(z)$. That completes the proof of Theorem 1.2.10. $\square$

Proof of Theorem 1.2.13. Consider the polynomial

$$\begin{aligned}
g(z) &= (1-z)p(z) \\
&= -a_n z^{n+1} + (a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \cdots + (a_1 - a_0)z + a_0 \\
&= -a_n z^{n+1} + (a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \cdots + (a_{\lambda+1} - a_\lambda)z^{\lambda+1} \\
&+ (a_\lambda - a_{\lambda-1})z^\lambda + \cdots + (a_1 - a_0)z + a_0 \\
&= -z^n[a_n z - a_n + a_{n-1} - t] - tz^n + (a_{n-1} - a_{n-2})z^{n-1} + \cdots + (a_{\lambda+1} - a_\lambda)z^{\lambda+1} \\
&+ (a_\lambda - a_{\lambda-1})z^\lambda + \cdots + (a_1 - a_0 + s)z - sz + a_0 \\
&= -z^n[a_n z - a_n + a_{n-1} - t] + \psi(z)
\end{aligned}$$

where

$$\psi(z) = -tz^n + (a_{n-1} - a_{n-2})z^{n-1} + \cdots + (a_{\lambda+1} - a_\lambda)z^{\lambda+1} + (a_\lambda - a_{\lambda-1})z^\lambda + \cdots + (a_1 - a_0 + s)z - sz + a_0.$$

For $|z| = 1$, we get

$$\begin{aligned}
|\psi(z)| &\le |t| + |a_{n-1} - a_{n-2}| + \cdots + |a_{\lambda+1} - a_\lambda| + |a_\lambda - a_{\lambda-1}| + \cdots + |a_1 - a_0 + s| + |s| + |a_0| \\
&= |t| + a_{n-2} - a_{n-1} + \cdots + a_\lambda - a_{\lambda+1} + a_\lambda - a_{\lambda-1} + \cdots + a_1 - a_0 + s + |s| + |a_0| \\
&= |t| - a_{n-1} + 2a_\lambda - a_0 + s + |s| + |a_0|.
\end{aligned}$$

It is clear that

$$|z^n \psi(1/z)| \le |t| - a_{n-1} + 2a_\lambda - a_0 + s + |s| + |a_0| \tag{1.3}$$

on the unit circle. Since the function $\Psi(z) = z^n\psi(1/z)$ is analytic in $|z| \leq 1$, the Inequality (1.3) holds also inside the unit circle. That is,

$$|\psi(1/z)| \leq \frac{|t| - a_{n-1} + 2a_\lambda - a_0 + s + |s| + |a_0|}{|z|^n}$$

for $|z| \leq 1$. Replacing z by $1/z$ we get

$$|\psi(z)| \leq \Big[|t| - a_{n-1} + 2a_\lambda - a_0 + s + |s| + |a_0|\Big]|z|^n$$

for $|z| \geq 1$.

Now for $|z| \geq 1$, we have

$$\begin{aligned}
|g(z)| &\geq |z^n||a_n z - a_n + a_{n-1} - t| - |\psi(z)| \\
&\geq |z^n||a_n z - a_n + a_{n-1} - t| - \Big[|t| - a_{n-1} + 2a_\lambda - a_0 + s + |s| + |a_0|\Big]|z|^n \\
&= |z^n|\Big(|a_n z - a_n + a_{n-1} - t| - \Big[|t| - a_{n-1} + 2a_\lambda - a_0 + s + |s| + |a_0|\Big]\Big) \\
&> 0,
\end{aligned}$$

which holds if and only if

$$|a_n z - a_n + a_{n-1} - t| > \Big[|t| - a_{n-1} + 2a_\lambda - a_0 + s + |s| + |a_0|\Big];$$

that is,

$$\left|z + \frac{a_{n-1}}{a_n} - \Big(1 + \frac{t}{a_n}\Big)\right| > \frac{1}{|a_n|}\Big[|t| - a_{n-1} + 2a_\lambda - a_0 + s + |s| + |a_0|\Big].$$

Hence, the zeros of $p(z)$ are in the closed disk

$$\left|z + \frac{a_{n-1}}{a_n} - \Big(1 + \frac{t}{a_n}\Big)\right| \leq \frac{1}{|a_n|}\Big[|t| - a_{n-1} + 2a_\lambda - a_0 + s + |s| + |a_0|\Big].$$

That completes the proof. $\square$

1.3 Number of Zeros in a specified Domain

By putting a restriction on the coefficients of a polynomial similar to that of the Eneström-Kakeya theorem, Mohammad [95] proved the following on the number of zeros that can be found in a specified disk.

Theorem 1.3.1. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial with real coefficients satisfying* $0 < a_0 \leq a_1 \leq a_2 \leq a_3 \ldots \leq a_n$. *Then the number of zeros of* $p(z)$ *in* $|z| \leq \frac{1}{2}$ *does not exceed*

$$1 + \frac{1}{\log 2} \log\left(\frac{a_n}{a_0}\right)$$

In her dissertation work, Dewan [34] weakens the hypotheses of Theorem 1.3.1 and proved the following two results for polynomials with complex coefficients .

Theorem 1.3.2. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial such that* $|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2}$ *for* $j \in \{0, 1, 2, \ldots, n\}$ *and for some real* α *and* β, *and* $0 < |a_0| \leq |a_1| \leq |a_2| \leq |a_3| \ldots \leq |a_n|$. *Then the number of zeros of* $p(z)$ *in* $|z| \leq 1/2$ *does not exceed*

$$\frac{1}{\log 2} \log \frac{|a_n|(\cos\alpha + \sin\alpha + 1) + 2\sin\alpha \sum_{j=0}^{n-1} |a_j|}{|a_0|}.$$

For $\alpha = 0$, Theorem 1.3.2 reduces to Theorem 1.3.1.

Theorem 1.3.3. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *where* $Re(a_j) = \alpha_j$ *and* $Im(a_j) = \beta_j$ *for all* j *and* $0 < \alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$. *Then the number of zeros of* $p(z)$ *in* $|z| \leq 1/2$ *does not exceed*

$$1 + \frac{1}{\log 2} \log \frac{\alpha_n + \sum_{j=0}^{n} |\beta_j|}{|a_0|}.$$

Pukhta [105] generalized Theorems 1.3.2 and 1.3.3 by finding the number of zeros in $|z| \leq \delta$ for $0 < \delta < 1$. The next theorem, due to Pukhta [105], deals with a monotonicity condition on the moduli of the coefficients.

Theorem 1.3.4. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial such that* $|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2}$ *for* $j \in \{0, 1, 2, \ldots, n\}$ *and for some real* α *and* β, *and* $0 < |a_0| \leq |a_1| \leq |a_2| \leq |a_3| \ldots \leq |a_n|$. *Then the number of zeros of* $p(z)$ *in* $|z| \leq \delta$, $0 < \delta < 1$, *does not exceed*

$$\frac{1}{\log 1/\delta} \log \frac{|a_n|(\cos\alpha + \sin\alpha + 1) + 2\sin\alpha \sum_{j=0}^{n-1} |a_j|}{|a_0|}.$$

Pukhta [105] also gave a result which involved a monotonicity condition on the real part of the coefficients. Though the proof presented by Pukhta is correct, there was a slight typographical error in the statement of the result as it appeared in print. The correct statement of the theorem is as follows.

Theorem 1.3.5. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *where* $Re(a_j) = \alpha_j$ *and* $Im(a_j) = \beta_j$ *for all* j *and* $0 < \alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$. *Then the number of zeros of* $p(z)$ *in* $|z| \leq \delta$, $0 < \delta < 1$, *does not exceed*

$$\frac{1}{\log 1/\delta} \log 2 \left[\frac{\alpha_n + \sum_{j=0}^{n} |\beta_j|}{|a_0|} \right].$$

In this section we generalize Theorem 1.3.5 and prove the following. It may be remarked that for values of δ very close to one, the above theorems and Theorem 1.3.6 given below do not give a satisfactory bound.

1.3.1 Statement and Proof of New Results

Theorem 1.3.6. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$, $a_0 \neq 0$, *be a polynomial of degree* n *with complex coefficients. If* $Re(a_j) = \alpha_j$ *and* $Im(a_j) = \beta_j$ *for all* j. *If for some real numbers* t, *and for some* $\lambda \in \{0, 1, 2, \cdots n\}$, $t + \alpha_n \leq \alpha_{n-1} \leq \ldots \leq \alpha_\lambda \geq \alpha_{\lambda-1} \geq \alpha_{\lambda-2} \geq \ldots \geq \alpha_1 \geq \alpha_0$, *then* *the number of zeros of* $p(z)$ *in* $|z| \leq \delta$, $0 < \delta < 1$, *does not exceed*

$$\frac{1}{\log 1/\delta} \log \frac{M_1}{|a_0|},$$

where

$$M_1 = |\alpha_0| - \alpha_0 + |\alpha_n| - \alpha_n + |t| - t + 2\alpha_\lambda + 2\sum_{j=0}^{n} |\beta_j|.$$

For $t = 0$ we get the following.

Corollary 1.3.7. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$, $a_0 \neq 0$, *be a polynomial of degree* n *with complex coefficients. If* $Re(a_j) = \alpha_j$ *and* $Im(a_j) = \beta_j$ *for all* j*. If for some* $\lambda \in \{0, 1, 2, \cdots n\}$,

$$\alpha_n \leq \alpha_{n-1} \leq \ldots \leq \alpha_\lambda \geq \alpha_{\lambda-1} \geq \alpha_{\lambda-2} \geq \ldots \geq \alpha_1 \geq \alpha_0,$$

then the number of zeros of $p(z)$ *in* $|z| \leq \delta$, $0 < \delta < 1$, *does not exceed*

$$\frac{1}{\log 1/\delta} \log \frac{M_1}{|a_0|},$$

where

$$M_1 = |\alpha_0| - \alpha_0 + |\alpha_n| - \alpha_n + 2\alpha_\lambda + 2\sum_{j=0}^{n} |\beta_j|.$$

If $\lambda = 0$, then Corollary 1.3.7 reduces to

Corollary 1.3.8. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$, $a_0 \neq 0$, *be a polynomial of degree* n *with complex coefficients. If* $Re(a_j) = \alpha_j$ *and* $Im(a_j) = \beta_j$ *for all* j*. Suppose* $\alpha_n \leq \alpha_{n-1} \leq \ldots \leq \alpha_0$, *then the number of zeros of* $p(z)$ *in* $|z| \leq \delta$, $0 < \delta < 1$, *does not exceed*

$$\frac{1}{\log 1/\delta} \log \frac{M_1}{|a_0|},$$

where

$$M_1 = |\alpha_0| + \alpha_0 + |\alpha_n| - \alpha_n + 2\sum_{j=0}^{n} |\beta_j|.$$

If, also, $\lambda = n$, then Corollary 1.3.7 reduces to

Corollary 1.3.9. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$, $a_0 \neq 0$, *be a polynomial of degree* n *with complex coefficients. If* $Re(a_j) = \alpha_j$ *and* $Im(a_j) = \beta_j$ *for all* j. *Suppose* $\alpha_n \geq \alpha_{n-1} \geq \alpha_{n-2} \geq \ldots \geq \alpha_1 \geq \alpha_0$, *then the number of zeros of* $p(z)$ *in* $|z| \leq \delta$, $0 < \delta < 1$, *does not exceed*

$$\frac{1}{\log 1/\delta} \log \frac{M_1}{|a_0|},$$

where

$$M_1 = |\alpha_0| - \alpha_0 + |\alpha_n| + \alpha_n + 2\sum_{j=0}^{n} |\beta_j|.$$

Suppose we assume $\alpha_0 > 0$ then Corollary 1.3.9 becomes Theorem 1.3.5. Instead of proving Theorem 1.3.6, we prove the following more general result.

Theorem 1.3.10. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$, $a_0 \neq 0$, *be a polynomial of degree* n *with complex coefficients. If* $Re(a_j) = \alpha_j$ *and* $Im(a_j) = \beta_j$ *for all* j *and if for some real numbers* t, s, *and for some* $\lambda \in \{0, 1, 2, \cdots n\}$, $t + \alpha_n \leq \alpha_{n-1} \leq \ldots \leq \alpha_\lambda \geq \alpha_{\lambda-1} \geq \alpha_{\lambda-2} \geq \ldots \geq \alpha_1 \geq \alpha_0 - s$, *then the number of zeros of* p *in* $|z| \leq \delta$, $0 < \delta < 1$, *does not exceed*

$$\frac{1}{\log 1/\delta} \log \frac{M_2}{|a_0|},$$

where

$$M_2 = |\alpha_0| - \alpha_0 + |\alpha_n| - \alpha_n + |t| - t + |s| + s + 2\alpha_\lambda + 2\sum_{j=0}^{n} |\beta_j|.$$

Clearly M_2 is nonnegative.

For the proof of our result we shall make use of the following lemma (see [120, p. 171]).

Lemma 1.3.11. *Let $F(z)$ be analytic in $|z| \le R$. Let $|F(z)| \le M$ in the disk $|z| \le R$ and suppose $F(0) \ne 0$. Then, for $0 < \delta < 1$, the number of zeros of $F(z)$ in the disk $|z| \le \delta R$ is less than*

$$\frac{1}{\log 1/\delta} \log \frac{M}{|F(0)|}.$$

Proof of Theorem 1.3.10 . Consider the polynomial

$$\begin{aligned} g(z) &= (1-z)p(z) \\ &= -a_n z^{n+1} + \sum_{j=1}^{n} (a_j - a_{j-1}) z^j + a_0. \end{aligned}$$

For $|z| = 1$,

$$\begin{aligned} |g(z)| &\le |a_n| + \sum_{j=1}^{n} |a_j - a_{j-1}| + |a_0| \\ &\le |\alpha_n| + |\beta_n| + \sum_{j=1}^{n} |\alpha_j - \alpha_{j-1}| + \sum_{j=1}^{n} |\beta_j - \beta_{j-1}| + |\alpha_0| + |\beta_0| \\ &\le |\alpha_n| + |\alpha_0| + \sum_{j=1}^{n} |\alpha_j - \alpha_{j-1}| + 2\sum_{j=0}^{n} |\beta_j| \\ &= |\alpha_n| + |\alpha_0| + \sum_{j=2}^{n-2} |\alpha_j - \alpha_{j-1}| + |\alpha_{n-1} - \alpha_{n-2}| + |\alpha_n - \alpha_{n-1}| + |\alpha_1 - \alpha_0| + 2\sum_{j=0}^{n} |\beta_j| \\ &\le |\alpha_0| - \alpha_0 + |\alpha_n| - \alpha_n + \alpha_{n-2} + \alpha_1 + |t| - t + |s| + s + \sum_{j=2}^{\lambda} |\alpha_j - \alpha_{j-1}| + \sum_{j=\lambda+1}^{n-2} |\alpha_j - \alpha_{j-1}| \\ &\quad + 2\sum_{j=0}^{n} |\beta_j| \\ &= |\alpha_0| - \alpha_0 + |\alpha_n| - \alpha_n + |t| - t + |s| + s + 2\alpha_\lambda + 2\sum_{j=0}^{n} |\beta_j| \\ &= M_2. \end{aligned}$$

Now $g(z)$ is analytic in $|z| \le 1$, and $|g(z)| \le M_2$ for $|z| = 1$. So by Lemma 1.3.11 and the maximum modulus principle, the number of zeros of g (and hence of p) in $|z| \le \delta$ is less than or equal to

$$\frac{1}{\log 1/\delta} \log \frac{M_2}{|a_0|}.$$

Hence, the theorem follows. $\square$

Chapter 2

Distribution of Zeros for Polynomials with no Monotonicity Condition on Coefficients

The earliest result concerning the location of the zeros of a polynomial is probably due to Gauss who incidental to his proofs of the Fundamental Theorem of Algebra showed in 1816 that a polynomial $p(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots + a_n z^n$, with all a_j real, has no zeros outside the circle $|z| = R$, where $R = \max_{1 \le j \le n} (n 2^{1/2} |a_j|)^{1/j}$.

However, in the case of arbitrary real or complex a_j, Gauss [54] in 1849 showed that R may be taken as the positive root of the equation: $z^n - 2^{1/2}(|a_1| z^{n-1} + \cdots + |a_n|) = 0$.

As a further indication of Gauss' interest in the location of the zeros of a polynomial, we have his letter (see collected works of Gauss) to Schumacher dated April 2, 1833, in which he tells of having written enough on this topic to fill several volumes, but the only results he published are those in Gauss [54]. Even, his important result, Theorem 2.0.12 stated below on the mechanical interpretation of the zeros of the derivative of a polynomial comes to us only by a brief entry he made presumably in about 1836 in a notebook otherwise devoted to astronomy.

Theorem 2.0.12. *The zeros of the function* $F(z) = \sum_{j=1}^{k} \frac{m_j}{z - z_j}$, *where all* m_j *are real, are the points of the equilibrium in the field of force due the system of* k *masses* m_j *at the fixed points* z_j *repelling a unit movable mass at* z *according to the inverse distance law.*

Around 1829, Cauchy [24] (also, see the book of Marden [87, Theorem 27.1, p. 122]) derived more exact bounds for the moduli of the zeros of a polynomial than those given by Gauss, by proving the following

Theorem 2.0.13. *Let* $p(z) = z^n + \sum_{j=0}^{n-1} a_j z^j$, *be a complex polynomial. Then all the zeros of* $p(z)$ *lie in the disc*

$$\{z : |z| \leq \eta\} \subset \{z : |z| < 1 + A\}, \tag{2.1}$$

where $A = \max_{0 \leq j \leq n-1} |a_j|$, *and* η *is the unique positive root of the real coefficient equation*

$$z^n - |a_{n-1}|z^{n-1} - |a_{n-2}|z^{n-2} - \cdots - |a_1|z - |a_0| = 0 \tag{2.2}$$

The result is best possible and the bound is attained when $p(z)$ *is the polynomial on the left hand side of* (2.2).

The proof follows easily from the inequality

$$|p(z)| \geq |z|^n - (|a_{n-1}||z|^{n-1} + |a_{n-2}||z|^{n-2} + \cdots + |a_1||z| + |a_0|) = 0, \tag{2.3}$$

which can be derived easily on applying Triangle Inequality to $p(z) = z^n + \sum_{j=0}^{n-1} a_j z^j$.

2.1 Annuli containing all the Zeros of a Polynomial

If one applies the above Theorem 2.0.13 of Cauchy to the polynomial $P(z) = z^n p(1/z)$ and combine it with Theorem 2.0.13, one easily gets

Theorem 2.1.1 (Cauchy). *All the zeros of the polynomial* $p(z) = a_0 + a_1 z + \cdot + a_n z^n$, $a_n \neq 0$, *lie in the annulus* $r_1 \leq |z| \leq r_2$, *where* r_1 *is the unique positive root of the equation*

$$|a_n|z^n + |a_{n-1}|z^{n-1} + \cdot + |a_1|z - |a_0| = 0, \tag{2.4}$$

and r_2 *is the unique positive root of the equation*

$$|a_0| + |a_1|z + \cdot + |a_{n-1}|z^{n-1} - |a_n|z^n = 0. \tag{2.5}$$

Although the above result of Cauchy gives an annulus containing all the zeros of a polynomial, it is implicit, in the sense, that in order to find the annulus containing all the zeros of a polynomial, one needs to compute the zeros of two other polynomials.

In a bid to get an explicit bound, Datt and Govil [28] (see also Dewan [35]) proved

Theorem 2.1.2. *Let* $p(z) = z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0$, *be a polynomial of degree* n *and* $A = \max_{0 \le j \le n-1} |a_j|$, *as defined in Theorem* 2.0.13. *Then p(z) has all its zeros in the ring shaped region*

$$\frac{|a_0|}{2\left(1+A\right)^{n-1}\left(An+1\right)} \le |z| \le 1 + \lambda_0 A, \tag{2.6}$$

where λ_0 *is the unique positive root of the equation* $x = 1 - 1/\left(1+Ax\right)^n$ *in the interval* $(0,1)$. *The upper bound* $1 + \lambda_0 A$ *in the above given region* (2.6) *is best possible and is attained for the polynomial* $p(z) = z^n - A\left(z^{n-1} + ... + z + 1\right)$.

In case one does not wish to solve the equation $x = 1 - 1/\left(1+Ax\right)^n$, then in order to apply the above result of Datt and Govil [28], one can apply the following result also due to Datt and Govil [28], which in every case clearly gives an improvement over Theorem 2.0.13 of Cauchy [24].

Theorem 2.1.3. *Let* $p(z) = z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0$, *be a polynomial of degree* n *and*

$$A = \max_{0 \le j \le n-1} |a_j|.$$

Then p(z) has all its zeros in the ring shaped region

$$\frac{|a_0|}{2\left(1+A\right)^{n-1}\left(An+1\right)} \le |z| \le 1 + \left(1 - \frac{1}{(1+A)^n}\right)A. \tag{2.7}$$

Since, always $\left(1 - \frac{1}{(1+A)^n}\right) < 1$, the above Theorem 2.1.3 in every situation sharpens Theorem 2.0.13 due to Cauchy.

Although, since the beginning, binomial coefficients defined by $C(n,k) = \frac{n!}{k!(n-k)!}$, $0! = 1$ have appeared in the derivation or as a part of closed expressions of bounds, the Fibonacci's numbers defined by $F_0 = 0$, $F_1 = 1$, and $F_j = F_{j-1} + F_{j-2}$ for $j \geq 2$ have not appeared either in implicit bounds or explicit bounds for the moduli of the zeros. Diaz-Barrero [39] proved the following result, which gives circular domains containing all the zeros of a polynomial where binomial coefficients and Fibonacci's numbers appear. He also gives an example of a polynomial for which the above theorem gives a better bound than the bound obtainable from Theorem 2.0.13 of Cauchy [24].

In the sequel, we will interchange between $C(n,j)$ and C_j^n as it deems convenient. We now state the result due to Diaz-Barrero [39].

Theorem 2.1.4. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ $(a_j \neq 0,\ 0 \leq j \leq n)$ *be a complex monic polynomial. Then all its zeros lie in the disk* $C_1 = \{z \in \mathbb{C} : |z| \leq r_1\}$ *or* $C_2 = \{z \in \mathbb{C} : |z| \leq r_2\}$, *where*

$$\begin{aligned} r_1 &= \max_{1\leq k\leq n} \left\{ \sqrt[k]{\frac{2^{n-1}C_2^{n+1}}{k^2 C_k^n}|a_{n-k}|} \right\}, \\ r_2 &= \max_{1\leq k\leq n} \left\{ \sqrt[k]{\frac{F_{3n}}{C_k^n 2^k F_k}|a_{n-k}|} \right\}. \end{aligned}$$

The proof of the above theorem depends on the identities

$$\sum_{k=1}^{n} k^2 C_k^n = 2^{n-2} n(n+1) \tag{2.8}$$

and

$$\sum_{k=1}^{n} C_k^n 2^k F_k = F_{3n}, \tag{2.9}$$

where F_j are the Fibonacci's numbers, and C_k^n the binomial coefficients.

The following result, which provides an annulus region containing all the zeros of a polynomial is also due to Diaz-Barrero [40].

Theorem 2.1.5. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ $(a_j \neq 0, \quad 0 \leq j \leq n)$ *be a nonconstant complex polynomial. Then all its zeros lie in the annulus* $C = \{z \in \mathbb{C} : r_1 \leq |z| \leq r_2\}$*, where*

$$\begin{aligned} r_1 &= \frac{3}{2} \min_{1 \leq j \leq n} \left\{ \frac{2^n F_j C_j^n}{F_{4n}} \left| \frac{a_0}{a_j} \right| \right\}^{1/j}, \\ r_2 &= \frac{2}{3} \max_{1 \leq j \leq n} \left\{ \frac{F_{4n}}{2^n F_j C_j^n} \left| \frac{a_{n-j}}{a_n} \right| \right\}^{1/j}. \end{aligned}$$

Here F_j *being the Fibonacci's numbers, and* C_j^n *the binomial coefficients.*

The following result of Kim [79], whose proof depends on the use of the identity

$$\sum_{k=0}^{n} C_k^n = 2^n - 1 \tag{2.10}$$

also provides an annulus containing all the zeros of a polynomial.

Theorem 2.1.6. *Let* $p(z) = \sum_{k=0}^{n} a_k z^k$ $(a_k \neq 0, \quad 0 \leq k \leq n)$ *be a nonconstant polynomial with complex coefficients. Then all the zeros of* $p(z)$ *lie in the annulus* $A = \{z : r_1 \leq |z| \leq r_2\}$*, where*

$$r_1 = \min_{1 \leq k \leq n} \left\{ \frac{C_k^n}{2^n - 1} \left| \frac{a_0}{a_k} \right| \right\}^{1/k}, \quad r_2 = \max_{1 \leq k \leq n} \left\{ \frac{2^n - 1}{C_k^n} \left| \frac{a_{n-k}}{a_n} \right| \right\}^{1/k}. \tag{2.11}$$

Here again, as usual, C_k^n *denote the binomial coefficients.*

Theorem 2.0.13 of Cauchy has also been refined by Sun and Hsieh [118], who proved

Theorem 2.1.7. *All the zeros of the polynomial* $p(z) = z^n + \sum_{j=0}^{n-1} a_j z^j$ *lie in the disks*

$$\{z : |z| < \eta\} \subset \{z : |z| < 1 + \delta_3\} \subseteq \{z : |z| < 1 + A\},$$

where δ_3 *is the unique positive root of the equation,*

$$Q_3(x) \equiv x^3 + (2 - |a_{n-1}|)x^2 + (1 - |a_{n-1}| - |a_{n-2}|)x - A = 0, \tag{2.12}$$

and

$$A = \max_{0 \le j \le n-1} |a_j|.$$

Using the method similar to that of Sun and Hsieh [118], Jain [73] refined the above result of Sun and Hsieh [118], and proved

Theorem 2.1.8. *All the zeros of the polynomial* $p(z) = z^n + \sum_{j=0}^{n-1} a_j z^j$ *lie in the disks*

$$\{z : |z| < \eta\} \subset \{z : |z| < 1 + \delta_4\} \subseteq \{z : |z| < 1 + \delta_3\} \subseteq \{z : |z| < 1 + A\},$$

where δ_4 *is the unique positive root of the equation,*

$$\begin{aligned} Q_4(x) \equiv x^4 &+ (3 - |a_{n-1}|)\, x^3 + (3 - 2|a_{n-1}| - |a_{n-2}|)\, x^2 \\ &+ (1 - |a_{n-1}| - |a_{n-2}| - |a_{n-3}|)\, x - A = 0, \end{aligned} \tag{2.13}$$

and $A = \max_{0 \le j \le n-1} |a_j|$*, is same as in Theorem* 2.1.7.

In 2009, Affane-Aji, Agarwal, and Govil [2] proved the following result which not only includes the above results of Cauchy [24], Sun and Hsieh [118], and Jain [73] as special cases but also provides a tool for obtaining sharper bounds for the location of the zeros of a polynomial.

Theorem 2.1.9. *All the zeros of the polynomial* $p(z) = z^n + \sum_{j=0}^{n-1} a_j z^j$ *lie in the disks*

$$\{z : |z| < 1 + \delta_k\} \subseteq \{z : |z| < 1 + \delta_{k-1}\} \cdots$$

$$\subseteq \{z : |z| < 1 + \delta_1\} \subseteq \{z : |z| < 1 + A\},$$

where δ_k *is the unique positive root of the* k^{th} *degree equation*

$$Q_k(x) \equiv x^k + \sum_{\nu=2}^{k} \left[C_{k-\nu}^{k-1} - \sum_{j=1}^{\nu-1} C_{k-\nu}^{k-j-1} |a_{n-j}| \right] x^{k+1-\nu} - A = 0. \tag{2.14}$$

Here

$$A = \max_{0 \le j \le n-1} |a_j|, \quad a_j = 0 \text{ if } j < 0,$$

and for k*, a positive integer,* C_k^m *are the binomial coefficients.*

As is easy to verify, for $k = 1$ the above theorem reduces to Theorem 2.0.13 due to Cauchy [24], for $k = 3$ to the result of Sun and Hsieh [118], and for $k = 4$ it reduces to the result due to Jain [73]. Further, by choosing k sufficiently large we can make δ_k in the bound to our desired accuracy.

Note that by combining the above Theorem 2.1.9 with Theorem 2.1.3 of Datt and Govil [28] one can easily obtain the following result, which is a refinement of the above Theorem 2.1.9.

Theorem 2.1.10. *All the zeros of the polynomial* $p(z) = z^n + \sum_{j=0}^{n-1} a_j z^j$ *lie in the annulus*

$$\frac{|a_0|}{2(1+A)^{n-1}(nA+1)} \le |z| \le \{z : |z| < 1 + \delta_k\} \subseteq \{z : |z| < 1 + \delta_{k-1}\} \cdots$$

$$\subseteq \{z : |z| < 1 + \delta_1\} \subseteq \{z : |z| < 1 + A\},$$

where δ_k *is as defined in Theorem* 2.1.9*, and* $A = \max_{0 \le j \le n-1} |a_j|$.

Similarly, one can obtain a refinement of Theorem 2.1.9 by combining Theorem 2.1.9 with Theorem 2.1.5 of Diaz-Barrero [40].

Later in 2010, Affane-Aji, Biaz, and Govil [3] proved the following refinement of Theorem 2.1.9, and constructed examples to show that for some polynomials their theorem, stated below, gives much better bounds than obtainable from Theorem 2.1.10. More precisely, their result is

Theorem 2.1.11. *All the zeros of the polynomial* $p(z) = z^n + \sum_{j=0}^{n-1} a_j z^j$ *lie in the annulus*

$$R_1 \le |z| \le \{z : |z| < 1 + \delta_k\} \subseteq \{z : |z| < 1 + \delta_{k-1}\} \cdots$$

$$\subseteq \{z : |z| < 1 + \delta_1\} \subseteq \{z : |z| < 1 + A\},$$

where δ_k *is as defined in Theorem* 2.1.9, *and*

$$R_1 = \frac{-R^2|a_1|(M - |a_0|) + \sqrt{4R^2M^3|a_0| + \{R^2|a_1|(M - |a_0|)\}^2}}{2M^2}. \tag{2.15}$$

Here $M = \frac{R^{n+1}+(A-1)R^n-AR}{(R-1)}$ *with* $R = 1 + \delta_k$ *and* $A = \max_{0\le j\le n-1} |a_j|$.

Note that $R = 1 + \delta_k > 1$, so for every positive integer k, we have $M > 0$ and $R > 0$. It is obvious that, in general, Theorem 2.1.11 sharpens Theorem 2.1.9.

In the same paper Affane-Aji, Biaz, and Govil [3] prove some more refinements of Theorem 2.1.9, which in some cases gives bounds that are sharper than obtainable from Theorems 2.1.3, 2.1.5, and 2.1.10. This they have shown by constructing some examples of polynomials.

The following two results by Diaz-Barrero and Egozcue [41], also provide annuli containing all the zeros of a polynomial.

Theorem 2.1.12. *Let* $p(z) = \sum_{k=0}^{n} a_k z^k$ $(a_k \ne 0, 1 \le k \le n)$ *be a non-constant complex polynomial. Then for* $j \ge 2$, *all the zeros of* $p(z)$ *lie in the annulus* $C = \{z : r_1 \le |z| \le r_2\}$, *where*

$$r_1 = \min_{1\le k\le n} \left\{ \frac{C(n,k)A_k B_j^k (bB_{j-1})^{n-k}}{A_{jn}} \left| \frac{a_0}{a_k} \right| \right\}^{1/k} \tag{2.16}$$

and

$$r_2 = \max_{1\le k\le n} \left\{ \frac{A_{jn}}{C(n,k)A_k B_j^k (bB_{j-1})^{n-k}} A_{jn} \left| \frac{a_{n-k}}{a_n} \right| \right\}^{1/k}. \tag{2.17}$$

Here, $B_n = \sum_{k=0}^{n-1} r^k s^{n-1-k}$ *and* $A_n = cr^n + ds^n$*, where* c*,* d *are real constants and* r*,*s *are the roots of the equation* $x^2 - ax - b = 0$ *in which* a*,*b *are strictly positive real numbers. For* $j \geq 2$*,* $\sum_{k=0}^{n} C(n,k)(bB_{j-1})^{n-k} B_j^k A_k = A_{jn}$*. Furthermore,* $C(n,k)$ *is the binomial coefficient.*

Theorem 2.1.13. *Let* $p(z) = \sum_{k=0}^{n} a_k z^k \, (a_k \neq 0, \, 1 \leq k \leq n)$ *be a non-constant polynomial with complex coefficients. Then, all its zeros lie in the ring shaped region* $C = \{z : r_1 \leq |z| \leq r_2\}$*, where*

$$r_1 = \min_{1\leq k\leq n} \left\{ \frac{2^k P_k C(n,k)}{P_{2n}} \left| \frac{a_0}{a_k} \right| \right\}^{1/k} \tag{2.18}$$

and

$$r_2 = \max_{1\leq k\leq n} \left\{ \frac{P_{2n}}{2^k P_k C(n,k)} \left| \frac{a_{n-k}}{a_n} \right| \right\}^{1/k}. \tag{2.19}$$

Here P_k *is the* k^{th} *Pell number, namely,* $P_0 = 0$*,* $P_1 = 1$ *and for* $k \geq 2$*,* $P_k = 2P_{k-1} + P_{k-2}$*. Furthermore,* $C(n,k) = \frac{n!}{k!(n-k)!}$ *are the binomial coefficients.*

Recently, Dalal and Govil [26] unified the above results by proving the following which as special case includes several of the above results, namely : Theorems 2.1.4, 2.1.5, 2.1.6, 2.1.12, and 2.1.13.

Theorem 2.1.14. *Let* $A_k > 0$ *for* $1 \leq k \leq n$*, and be such that* $\sum_{k=1}^{n} A_k = 1$*. If* $p(z) = \sum_{k=0}^{n} a_k z^k \, (a_k \neq 0, \, 1 \leq k \leq n)$ *is a non-constant polynomial with complex coefficients, then all the zeros of* $p(z)$ *lie in the annulus* $C = \{z : r_1 \leq |z| \leq r_2\}$*, where*

$$r_1 = \min_{1\leq k\leq n} \left\{ A_k \left| \frac{a_0}{a_k} \right| \right\}^{1/k} \tag{2.20}$$

and

$$r_2 = \max_{1\leq k\leq n} \left\{ \frac{1}{A_k} \left| \frac{a_{n-k}}{a_n} \right| \right\}^{1/k}. \tag{2.21}$$

The above theorem, by appropriate choice of the numbers $A_k > 0$ for $1 \le k \le n$, includes as special case Theorems 2.1.4, 2.1.5, 2.1.6, 2.1.12 and 2.1.13, and this has been shown in the Table 1 in the paper of Dalal and Govil [26, p. 9612].

2.1.1 Statement of some New Results

In this section, by using Theorem 2.1.14 we obtain the following results stated below that provide annuli containing all the zeros of a polynomial. Also, we show, by means of examples, that for some polynomials our results sharpen some of the known results in this direction.

Our first result connects the $n^{th}-$Bell number, B_n, which counts the partitions of a set with n elements and the Stirling number (of the second kind) with parameters n and k, denoted by $S(n,k)$, that enumerates the number of partitions of a set with n elements consisting k disjoint, nonempty sets. Here, B_n is defined recursively as: $B_0 = 1,\ \ B_{n+1} = \sum_{k=0}^{n} C(n,k)B_k$, for $n \ge 0$ and $S(n,k) = \dfrac{1}{k!}\sum_{j=0}^{k}(-1)^j C(k,j)(k-j)^n$.

Theorem 2.1.15. *Let* $p(z) = \sum_{k=0}^{n} a_k z^k$ *be a non-constant complex polynomial of degree* n, *with* $a_k \neq 0,\ \ 1 \le k \le n$. *Then all the zeros of* $p(z)$ *lie in the annulus* $C = \{z : r_1 \le |z| \le r_2\}$, *where*

$$r_1 = \min_{1\le k\le n}\left\{\frac{S(n,k)}{B_n}\left|\frac{a_0}{a_k}\right|\right\}^{1/k} \tag{2.22}$$

and

$$r_2 = \max_{1\le k\le n}\left\{\frac{B_n}{S(n,k)}\left|\frac{a_{n-k}}{a_n}\right|\right\}^{1/k}. \tag{2.23}$$

Theorem 2.1.16. *Let* $p(z) = \sum_{k=0}^{n} a_k z^k$ *be a non-constant complex polynomial of degree* n, *with* $a_k \neq 0,\ \ 1 \le k \le n$. *Then all the zeros of* $p(z)$ *lie in the annulus* $C = \{z : r_1 \le |z| \le r_2\}$, *where*

$$r_1 = \min_{1\le k\le n}\left\{\frac{C(2n-k,k)C_{n-k}}{S_n - C_n}\left|\frac{a_0}{a_k}\right|\right\}^{1/k} \tag{2.24}$$

and

$$r_2 = \max_{1\le k\le n}\left\{\frac{S_n - C_n}{C(2n-k,k)C_{n-k}}\left|\frac{a_{n-k}}{a_n}\right|\right\}^{1/k}, \tag{2.25}$$

where $C_n = \dfrac{C(2n,n)}{n+1}$ *is the* $n^{th}-$*Catalan number and* S_n *the* $n^{th}-$*Schröder number given recursively by* $S_0 = 1, \quad S_n = S_{n-1} + \sum_{j=0}^{n-1} S_j\, S_{n-1-j}$, *for* $n \ge 1$.

Theorem 2.1.17. *Let* $p(z) = \sum_{k=0}^{n} a_k z^k$ *be a non-constant complex polynomial of degree* n, *with* $a_k \neq 0, \;\; 1 \le k \le n$. *Then all the zeros of* $p(z)$ *lie in the annulus* $C = \{z : r_1 \le |z| \le r_2\}$, *where*

$$r_1 = \min_{1\le k\le n}\left\{\frac{C(n,k)M_k}{C_{n+1}-1}\left|\frac{a_0}{a_k}\right|\right\}^{1/k} \tag{2.26}$$

and

$$r_2 = \max_{1\le k\le n}\left\{\frac{C_{n+1}-1}{C(n,k)M_k}\left|\frac{a_{n-k}}{a_n}\right|\right\}^{1/k}, \tag{2.27}$$

where C_n *is the same as in Theorem* 2.1.16 *and* M_k *is the* $k^{th}-$*Motzkin number defined recursively as*

$$M_0 = M_1 = M_{-1} = 1; \quad M_{k+1} = \frac{2k+3}{k+3}M_k + \frac{3k}{k+3}M_{k-1}, \;\; k \ge 1.$$

Theorem 2.1.18. *Let* $p(z) = \sum_{k=0}^{n} a_k z^k$ *be a non-constant complex polynomial of degree* n, *with* $a_k \neq 0, \;\; 1 \le k \le n$. *Then all the zeros of* $p(z)$ *lie in the annulus* $C = \{z : r_1 \le |z| \le r_2\}$, *where*

$$r_1 = \min_{1\le k\le n}\left\{\frac{C(n,k)^2}{C(2n,n)-1}\left|\frac{a_0}{a_k}\right|\right\}^{1/k} \tag{2.28}$$

and

$$r_2 = \max_{1\le k\le n}\left\{\frac{C(2n,n)-1}{C(n,k)^2}\left|\frac{a_{n-k}}{a_n}\right|\right\}^{1/k}. \tag{2.29}$$

Theorem 2.1.19. *Let* $p(z) = \sum_{k=0}^{n} a_k z^k$ *be a non-constant complex polynomial of degree* n, *with* $a_k \neq 0, \;\; 1 \le k \le n$. *Then all the zeros of* $p(z)$ *lie in the annulus* $C = \{z : r_1 \le |z| \le r_2\}$,

where

$$r_1 = \min_{1\le k\le n}\left\{\frac{F_k}{F_{n+2}-1}\left|\frac{a_0}{a_k}\right|\right\}^{1/k} \tag{2.30}$$

and

$$r_2 = \max_{1\le k\le n}\left\{\frac{F_{n+2}-1}{F_k}\left|\frac{a_{n-k}}{a_n}\right|\right\}^{1/k}, \tag{2.31}$$

where F_n denotes the $n^{th}-$Fibonacci number.

Theorem 2.1.20. *Let $p(z) = \sum_{k=0}^{n} a_k z^k$ be a non-constant complex polynomial of degree n, with $a_k \neq 0,\ \ 1 \le k \le n$. Then all the zeros of $p(z)$ lie in the annulus $C = \{z : r_1 \le |z| \le r_2\}$, where*

$$r_1 = \min_{1\le k\le n}\left\{\frac{k\ C(n,k)}{n2^{n-1}}\left|\frac{a_0}{a_k}\right|\right\}^{1/k} \tag{2.32}$$

and

$$r_2 = \max_{1\le k\le n}\left\{\frac{n2^{n-1}}{k\ C(n,k)}\left|\frac{a_{n-k}}{a_n}\right|\right\}^{1/k}, \tag{2.33}$$

For the proofs of our results, we will need the following lemmas. For the proof of the first lemma, see [69].

Lemma 2.1.21. *In combinatorics,it is known that for any $n \in \mathbb{N}$, B_n and $S(n,k)$ are connected as follows:*

$$\sum_{k=1}^{n} S(n,k) = B_n.$$

Lemma 2.1.22. *If M_n is the $n^{th}-$ Motzkin number and C_n the $n^{th}-$Catalan number, then for $n \ge 0$,*

$$C_0 = 1;\quad \sum_{k=0}^{n} C(n,k) M_k = C_{n+1}.$$

For the proof of Lemma 2.1.22 see [16, p. 99] and [33].

Lemma 2.1.23. *If S_n is the $n^{th}-$Schröder number, then for $n \ge 0$,*

$$\sum_{k=0}^{n} C(2n-k,k) C_{n-k} = S_n.$$

See [32, p. 2782] for the proof Lemma 2.1.23.

Lemma 2.1.24. *For $n \geq 0$,*

$$\sum_{k=0}^{n} C(n-k,k) = F_{n+1},$$

where F_n is the $n^{th}-$Fibonacci number.

Proof of Lemma 2.1.24 : For $n = 0$ and $n = 1$, we have that $F_1 = 1$ and $F_2 = 1 + 0 = 1$, respectively. Now , for $n \geq 2$, assume that

$$\sum_{k=0}^{n-1} C(n-1-k,k) = F_n, \quad \text{and} \quad \sum_{k=0}^{n-2} C(n-2-k,k) = F_{n-1}.$$

So by the Pascal recursion,

$$C(n-k,k) = C(n-k-1,k-1) + C(n-k-1,k),$$

we have therefore (by the induction hypothesis, Fibonacci recursion, and $C(n,k) = 0$, when either $k > n$ or $k < 0$.)

$$\begin{aligned}
\sum_{k=0}^{n} C(n-k,k) &= \sum_{k=0}^{n} C(n-k-1,k-1) + \sum_{k=0}^{n} C(n-k-1,k) \\
&= \sum_{k=1}^{n-1} C(n-k-1,k-1) + \sum_{k=0}^{n-1} C(n-k-1,k) \\
&= \sum_{k=0}^{n-2} C(n-k-2,k) + \sum_{k=0}^{n-1} C(n-k-1,k) \\
&= F_{n-1} + F_n \\
&= F_{n+1}.
\end{aligned}$$

□

Lemma 2.1.25. *Let $n, k \in \mathbb{N}$, with $n \geq k$. Then $k\ C(n,k) = n\ C(n-1,k-1)$.*

Proof of Lemma 2.1.25 :

$$\begin{aligned} k\ C(n,k) &= k\frac{n!}{(n-k)!k!} \\ &= k\frac{n(n-1)!}{k(n-k)!(k-1)!} \\ &= n\frac{(n-1)!}{(n-k)!(k-1)!} \\ &= n\ C(n-1,k-1). \end{aligned}$$

□

Lemma 2.1.26. *For $n \geq 0$,*

$$\sum_{k=1}^{n} k\ C(n,k) = n2^{n-1}.$$

Proof of Lemma 2.1.26 : From Lemma 2.1.25 we obtain that

$$\begin{aligned} \sum_{k=1}^{n} k\ C(n,k) &= \sum_{k=1}^{n} n\ C(n-1,k-1) \\ &= n\ \sum_{k=1}^{n} C(n-1,k-1) \\ &= n\ \sum_{k=0}^{n-1} C(n-1,k) \\ &= n2^{n-1}. \end{aligned}$$

□

Lemma 2.1.27. *Let n, m and r be nonnegative integers. Then*

$$\sum_{k=0}^{r} C(m,k)\ C(n,r-k) = C(n+m,r).$$

Proof of Lemma 2.1.27 : In general, the product of two polynomials with degrees m and n, respectively, is given by

$$\left(\sum_{i=0}^{m} a_i x^i\right)\left(\sum_{j=0}^{n} b_j x^j\right) = \sum_{r=0}^{m+n}\left(\sum_{k=0}^{r} a_k b_{r-k}\right)x^r;$$

where we use the convention that $a_i = 0$ for all integers $i > m$ and $b_j = 0$ for all integers $j > n$. Note by the binomial theorem,

$$(1+x)^{m+n} = \sum_{r=0}^{m+n} C(m+n,r)x^r.$$

Using the binomial theorem also for the exponents m and n, and then the above formula for the product of polynomials, we obtain

$$\begin{aligned}\sum_{r=0}^{m+n} C(m+n,r)x^r &= (1+x)^{m+n} \\ &= (1+x)^m(1+x)^n \\ &= \left(\sum_{i=0}^{m} C(m,i)x^i\right)\left(\sum_{j=0}^{n} C(n,j)x^j\right) \\ &= \sum_{r=0}^{m+n}\left(\sum_{k=0}^{r} C(m,k)C(n,r-k)\right)x^r,\end{aligned}$$

where the above convention for the coefficients of the polynomials agrees with the definition of the binomial coefficients, because both give zero for all $i > m$ and $j > n$, respectively.

By comparing coefficients of x^r, the identity follows for all integers with $0 \le r \le m+n$. For larger integer r, both sides of the identity are zeros due to the definition of the binomial coefficients.

□

Lemma 2.1.28. *Let $n \geq 0$. Then*

$$\sum_{k=0}^{n} C(n,k)^2 = C(2n,n).$$

The proof of Lemma 2.1.28 follows easily by setting $m = r = n$ in Lemma 2.1.27.

Lemma 2.1.29. *Let $n \geq 1$. Then*

$$\sum_{k=1}^{n} F_k = F_{n+2} - 1.$$

The proof of the above lemma follows by mathematical induction.

2.1.2 Proofs of Theorems

Proof of Theorem 2.1.15. From Lemma 2.1.21, we have that

$$\sum_{k=1}^{n} \frac{S(n,k)}{B_n} = 1.$$

If we take $A_k = \dfrac{S(n,k)}{B_n}$, then $A_k > 0$ and $\sum_{k=1}^{n} A_k = 1$, and hence by applying Theorem 2.1.14 for this set of values of A_k we get our desired result.

□

Proof of Theorem 2.1.16. From Lemma 2.1.23, we have that

$$\sum_{k=1}^{n} \frac{C(2n-k,k)C_{n-k}}{S_n - C_n} = 1.$$

If we take $A_k = \frac{C(2n-k,k)C_{n-k}}{S_n - C_n}$, then $A_k > 0$ and $\sum_{k=1}^{n} A_k = 1$, and hence by applying Theorem 2.1.14 for this set of values of A_k we get the required annulus and thus the proof of Theorem 2.1.16 is complete.

□

Proof of Theorem 2.1.17. From Lemma 2.1.22, we have that

$$\sum_{k=1}^{n} \frac{C(n,k)M_k}{C_{n+1}-1} = 1.$$

If we take $A_k = \frac{C(n,k)M_k}{C_{n+1}-1}$, then $A_k > 0$ and $\sum_{k=1}^{n} A_k = 1$, and hence by applying Theorem 2.1.14 for this set of values of A_k we get the desired annulus, and thus the proof of Theorem 2.1.17 is complete.

□

Proof of Theorem 2.1.18. From Lemma 2.1.28, we have that

$$\sum_{k=1}^{n} \frac{C(n,k)^2}{C(2n,n)-1} = 1.$$

If we take $A_k = \frac{C(n,k)^2}{C(2n,n)-1}$, then $A_k > 0$ and $\sum_{k=1}^{n} A_k = 1$, and hence by applying Theorem 2.1.14 for this set of values of A_k we get the desired annulus given in Theorem 2.1.18.

□

Proof of Theorem 2.1.19. From Lemma 2.1.29, we have that

$$\sum_{k=1}^{n} \frac{F_k}{F_{n+2}-1} = 1.$$

If we take $A_k = \frac{F_k}{F_{n+2}-1}$, then $A_k > 0$ and $\sum_{k=1}^{n} A_k = 1$, and hence by applying Theorem 2.1.14 for this set of values of A_k we get the desired annulus given be the radii in Theorem 2.1.19.

□

We now give examples of polynomials for which our results can compare favorably with the already known theorems as stated above.

Example 2.1.30. Consider the polynomial $p(z) = z^3 + 0.1z^2 + 0.1z + 0.7$.

Table 2.1: Computational Analysis I

Theorems	r_1	r_2	Area of the annulus
2.1.5	0.6402	1.2312	3.4730
2.1.6	0.4641	1.6984	8.382
2.1.15	0.519249	1.51829	6.39502
2.1.17	0.59943	1.31521	4.305399
2.1.19	0.7047	1.1187	2.37155
2.1.20	0.55934	1.4095	5.25812

As one can observe from Table 2.1, our Theorem 2.1.19 is giving a significantly better bound than obtainable from the known Theorems 2.1.5 and 2.1.6. In fact, the area of the annulus containing all the zeros of the polynomial $p(z)$ obtained by Theorem 2.1.19 is about 2.37155, which is about 68.29% of the area of the annulus obtained by Theorem 2.1.5 and about 28.29% of the area of the annulus obtained by Theorem 2.1.6.

Example 2.1.31. Consider the polynomial $p(z) = z^5+0.06z^4+0.29z^3+0.29z^2+0.29z+0.001$.

Table 2.2: Computational Analysis II

Theorems	r_1	r_2	Area of the annulus
2.1.5	0.00012233	1.6912	8.986
2.1.6	0.00055617	1.158	4.2125
2.1.15	0.51925	1.51829	6.3950
2.1.17	0.000132	1.5720	7.76345
2.1.18	0.000343	1.3063	5.36063
2.1.20	0.0010776	1.07703	3.6442

It is clear from Table 2.2 that our Theorem 2.1.20 gives a better lower and upper bound for the polynomial $p(z)$, hence, a smaller area of the annulus containing all the zeros of the polynomial $p(z)$. Comparing the area obtained by Theorem 2.1.20, one observe that this area is about 40.55% of the area obtained by Theorem 2.1.5 and 86.51% of the area of the annulus obtained by Theorem 2.1.6.

2.2 Landau type results concerning Location of Zeros

2.2.1 Location of Zeros of Trinomials and Quadrinomials

Quite a few results giving bound for all the zeros of a polynomial $p(z) = \sum_{j=0}^{n} a_j z^j$ were expressed (see [87, 110]) as functions of all the coeffiicients. It seems natural to ask to find a circle of the smallest radius that contains atleast one zero of the polynomial. Landau first, raised this question in connection with his study of the Picard's Theorem. In [84] and [85], Landau proved that every trinomial

$$a_n z^n + a_1 z + a_0, \quad a_1 a_n \neq 0, \quad n \geq 2,$$

has at least one zero in

$$|z| \leq 2\left|\frac{a_0}{a_1}\right| \tag{2.34}$$

and every quadrinomial

$$a_n z^n + a_m z^m + a_1 z + a_0, \quad a_1 a_m a_n \neq 0, \quad 0 \leq m < n,$$

has at least one zero in

$$|z| \leq \frac{17}{3}\left|\frac{a_0}{a_1}\right|. \tag{2.35}$$

For every $n \geq 2$, as a refinement of (2.34) the trinomial

$$a_n z^n + a_1 z + a_0, \quad a_1 a_n \neq 0,$$

is well known [46] to have a zero in both the regions

$$\left|z+\frac{a_0}{a_1}\right| \le \left|\frac{a_0}{a_1}\right| \quad \text{and} \quad \left|z+\frac{a_0}{a_1}\right| \ge \left|\frac{a_0}{a_1}\right|. \tag{2.36}$$

Joyal, Labelle and Rahman [75] gave an alternative proof of this fact by using Gauss-Lucas theorem. In literature, there exist several results about zeros distribution of trinomial equations, for example see [4] and [47]. In 2013, Aziz and Rather [12] proved certain results for quadrinomials and gave a simpler proof of (2.36), independent of Gauss-Lucas theorem. Here are their results

Theorem 2.2.1. *At least one zero of the quadrinomial*

$$a_n z^n + a_m z^m + a_1 z + a_0, \quad a_1 a_m a_n \neq 0, \quad 2 \le m < n,$$

lie in

$$|z| \le \frac{2n}{n-1}\left|\frac{a_0}{a_1}\right| \le 3\left|\frac{a_0}{a_1}\right|. \tag{2.37}$$

Applying this result to the polynomial $z^n p(1/z)$ where $p(z) = a_0 + a_p z^p + a_{n-1} z^{n-1} + z^n$, they obtained the following:

Corollary 2.2.2. *At least one zero of the quadrinomial*

$$a_0 + a_p z^p + a_{n-1} z^{n-1} + z^n, \quad a_0 a_p a_{n-1} \neq 0, \quad 1 \le p \le n-2,$$

lie in

$$|z| \ge \frac{n-1}{2n}|a_{n-1}|. \tag{2.38}$$

Theorem 2.2.3. *For every $n \ge 3$, the quadrinomial*

$$a_n z^n + a_2 z^2 + a_1 z + a_0, \quad a_2 a_n \neq 0,$$

has at least one zero in both

$$|z| \leq \left[\frac{n}{n-2}\left|\frac{a_0}{a_2}\right|\right]^{1/2} \tag{2.39}$$

and

$$\left|z + \frac{a_1}{2a_2}\right| \geq \left|\frac{a_1}{2a_2}\right|. \tag{2.40}$$

We now present some new results in this direction.

2.2.2 Statement of some New Results

Theorem 2.2.4. *At least one zero of the polynomial*

$$a_0 + a_1 z + a_2 z^2 + a_m z^m + a_n z^n, \quad a_n a_m a_2 a_1 \neq 0, \quad 3 \leq m < n,$$

lie in the circle

$$|z| \leq \sqrt{\frac{nm}{(n-2)(m-2)}\left|\frac{a_0}{a_2}\right|}.$$

Theorem 2.2.5. *At least one zero of the polynomial*

$$a_0 + a_1 z + a_2 z^2 + a_m z^m + a_n z^n, \quad a_n a_m a_2 a_1 \neq 0, \quad 3 \leq m < n,$$

lie in the circle

$$|z| \leq \frac{2nm}{(n-1)(m-1)}\left|\frac{a_0}{a_1}\right|.$$

2.2.3 Proofs of the Theorems

Proof of Theorem 2.2.4. If $a_0 = 0$, then nothing to prove. Suppose now that $a_0 \neq 0$, then write

$$S(z) = a_0 + a_1 z + a_2 z^2 + a_m z^m + a_n z^n.$$

Suppose all the zeros of $S(z)$ lie in

$$|z| > \sqrt{\frac{nm}{(n-2)(m-2)}\left|\frac{a_0}{a_2}\right|}.$$

Then all the zeros of

$$T(z) = z^n S(1/z) = a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + a_m z^{n-m} + a_n$$

lie in

$$|z| \leq \sqrt{\frac{(n-2)(m-2)}{nm}\left|\frac{a_2}{a_0}\right|}.$$

By Gauss - Lucas theorem, all the zeros of the derived polynomial

$$T'(z) = a_0 n z^{n-1} + a_1(n-1)z^{n-2} + a_2(n-2)z^{n-3} + a_m(n-m)z^{n-m-1}$$

lie in

$$|z| \leq \sqrt{\frac{(n-2)(m-2)}{nm}\left|\frac{a_2}{a_0}\right|}.$$

This shows that all the zeros of the polynomial

$$z^{n-1}T'(1/z) = a_m(n-m)z^m + a_2(n-2)z^2 + a_1(n-1)z + a_0 n$$

lie in the region

$$|z| > \sqrt{\frac{nm}{(n-2)(m-2)}\left|\frac{a_0}{a_2}\right|}.$$

But this is a contradiction because by Theorem 2.2.3, $z^{n-1}T'(1/z)$ has at least one zero in

$$|z| \leq \sqrt{\frac{nm}{(n-2)(m-2)}\left|\frac{a_0}{a_2}\right|}.$$

Thus, the polynomial $S(z)$ has at least one zero in the prescribed circle.

□

Proof of Theorem 2.2.5. If $a_0 = 0$, then nothing to prove. Suppose now that $a_0 \neq 0$, and $S(z) = a_0 + a_1 z + a_2 z^2 + a_m z^m + a_n z^n$, has all its zeros lying in

$$|z| > \frac{2nm}{(n-1)(m-1)}\left|\frac{a_0}{a_1}\right|.$$

Then all the zeros of

$$T(z) = z^n S(1/z) = a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + a_m z^{n-m} + a_n$$

lie in

$$|z| \leq \frac{(n-1)(m-1)}{2nm}\left|\frac{a_1}{a_0}\right|.$$

By Gauss - Lucas theorem, all the zeros of the derived polynomial

$$T'(z) = a_0 n z^{n-1} + a_1(n-1)z^{n-2} + a_2(n-2)z^{n-3} + a_m(n-m)z^{n-m-1}$$

lie in

$$|z| \leq \frac{(n-1)(m-1)}{2nm}\left|\frac{a_1}{a_0}\right|.$$

This shows that all the zeos of the polynomial

$$z^{n-1}T'(1/z) = a_m(n-m)z^m + a_2(n-2)z^2 + a_1(n-1)z + a_0 n$$

lie in the region

$$|z| > \frac{2nm}{(n-1)(m-1)}\left|\frac{a_0}{a_1}\right|.$$

But this is a contradiction because by Theorem 2.2.1, $z^{n-1}T'(1/z)$ has at least one zero in

$$|z| \leq \frac{2nm}{(n-1)(m-1)}\left|\frac{a_0}{a_1}\right|.$$

Thus, the polynomial $S(z)$ has at least one zero in the prescribed circle. $\square$

We present here an example to illustrate the use of our result.

Example 2.2.6. Consider the polynomial

$$p(z) = z^5 - z^3 + 2z^2 - z + 1.$$

Here $m = 3$, and $n = 5$. Using Matlab, we obtain the zeros of $p(z)$, namely: $z_1 = -1.6663$, $z_2 = 0.7908 + 0.6846i$, $z_3 = 0.7908 - 0.6846i$, $z_4 = 0.0424 + 0.7394i$, and $z_5 = 0.0424 - 0.7394i$ which lie in $0.7407 \leq |z| \leq 1.6663$ and at least one zero lie in $|z| \leq 0.7407$. Now by Theorem 2.2.4, we see that at least one zero of the polynomial $p(z)$ lie in the disk $|z| \leq \sqrt{5/2}$. Also by Theorem 2.2.5, the disks $|z| \leq 3.75$ contains at least one zero of the polynomial.

Chapter 3

Growth of Polynomials

3.1 Introduction

Several years after chemist Mendeleev invented the periodic table of elements he made a study of the specific gravity of a solution as a function of the percentage of the dissolved substance, and for this he needed an answer to the following question.

Question: *If $p(x)$ is a quadratic polynomial with real coefficients and $|p(x)| \leq 1$ on $-1 \leq x \leq 1$, then how large can $|p'(x)|$ be on $-1 \leq x \leq 1$?*

To see how an answer to the above question of Mendeleev helped him in the solution of the problem in Chemistry he was interested in, we refer to the paper of Boas [21].

Note that, even though Mendeleev was a chemist, he was able to show that for a quadratic polynomial $p(x)$ with real coefficients and $|p(x)| \leq 1$ on $-1 \leq x \leq 1$, then $|p'(x)| \leq 4$ for $-1 \leq x \leq 1$. This estimate is best possible in the sense that there is a quadratic polynomial $p(x) = 1 - 2x^2$ for which $|p(x)| \leq 1$ on $[-1, 1]$ but $|p'(\pm 1)| = 4$. In the general case when $p(x)$ is a polynomial of degree n with real coefficients the problem was solved by A. A. Markov [89], who proved the following result which is known as Markov's Theorem (see also Pinkus and de Boor [102]).

Theorem 3.1.1. *Let $p(x) = \sum_{j=0}^{n} a_j x^j$ be an algebraic polynomial of degree n such that $|p(x)| \leq 1$ for $x \in [-1, 1]$. Then*

$$|p'(x)| \leq n^2, \qquad x \in [-1, 1] \tag{3.1}$$

The inequality is sharp. Equality holds only if $p(x) = \alpha T_n(x)$, *where* α *is a complex number such that* $|\alpha| = 1$, *and*

$$T_n(x) = \cos(n\cos^{-1} x) = 2^{n-1}\prod_{j=1}^{n}\left[x - \cos((j-\frac{1}{2})\pi/n)\right]$$

is the n^{th} *degree Tchebycheff polynomial of the first kind. It can be easily verified that* $|T_n(x)| \leq 1$ *for* $x \in [-1, 1]$ *and* $|T_n'(1)| = n^2$.

It would be natural to go on and ask for an upper bound for $|p^{(k)}(x)|$ where $1 \leq k \leq n$. Iterating Markov's Theorem yields $|p^{(k)}(x)| \leq n^{2k}L$ if $|p(x)| \leq L$. However, this inequality is not sharp; the best possible inequality was found by Markov's brother, V. A. Markov [90], who proved the following

Theorem 3.1.2. *Let* $p(x) = \sum_{j=0}^{n} a_j x^j$ *be an algebraic polynomial of degree* n *with real coefficients such that* $|p(x)| \leq 1$ *for* $x \in [-1, 1]$. *Then*

$$|p^{(k)}(x)| \leq \frac{(n^2-1^2)(n^2-2^2)\cdots(n^2-(k-1)^2)}{1\cdot 3\cdots(2k-1)}, \qquad x \in [-1, 1]. \tag{3.2}$$

The inequality is sharp, and the equality holds again only for $p(x) = T_n(x)$, *where* $T_n(x) = \cos(n\cos^{-1} x)$ *is the Chebyschev polynomial of degree* n.

Several years later, around 1926, Serge Bernstein needed the analogue of the above result Theorem 3.1.1 of A. A. Markov for polynomials in the complex domain and proved the following, which in the literature is known as Bernstein's Inequality.

Theorem 3.1.3. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a complex polynomial of degree at most* n. *Then*

$$\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|. \tag{3.3}$$

The inequality is best possible and equality holds only for polynomials of the form $p(z) = \alpha z^n$, $\alpha \neq 0$ *being a complex number.*

The above theorem is, in fact, a special case of a more general result due to M. Riesz [112] for trigonometric polynomials.

For the sake of brevity, throughout in this chapter, we shall be using the following notations.

Definition 3.1.4. *Let* $p(z)=\sum_{j=0}^{n} a_j z^j$ *be a complex polynomial of degree at most* n*. We will denote*

$$M(p,r) := \max_{|z|=r} |p(z)|\,,\ \ r>0,$$

$$||p|| := \max_{|z|=1} |p(z)|\,,$$

and

$$D(0,K) := \{z : |z| < K\},\ \ K>0.$$

In 1945, S. Bernstein initiated and observed the following result, which in fact is a simple consequence of the maximum modulus principle (see [97] or [103, p. 137]). This inequality is also known as the Bernstein's inequality.

Theorem 3.1.5. *Let* $p(z)=\sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n*. Then for* $R \geq 1$,

$$M(p,R) \leq R^n ||p||. \tag{3.4}$$

Equality holds for $p(z)=\alpha z^n$, α *being a complex number.*

If one applies the above inequality to the polynomial $P(z)=z^n p(1/z)$ and use maximum modulus principle, one easily gets

Theorem 3.1.6. *Let* $p(z)=\sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n*. Then for* $0<r\leq 1$,

$$M(p,r) \geq r^n ||p||. \tag{3.5}$$

Equality holds for $p(z)=\alpha z^n$, α *being a complex number.*

The above result is due to Varga [122] who attributes it to E. H. Zarantonello. By use of the transformation $P(z) = z^n p(1/z)$ and the maximum modulus principle it is not difficult to see that Theorem 3.1.5 and Theorem 3.1.6 can be obtained from each other. The fact that Theorem 3.1.3 can be obtained from Theorem 3.1.5 was proved by Bernstein himself. However, it was not known if Theorem 3.1.5 can also be obtained from Theorem 3.1.3, and this has been shown by Govil, Qazi and Rahman [67]. Thus all the above three Theorems 3.1.3, 3.1.5 and 3.1.6 are equivalent in the sense that anyone can be obtained from any of the others.

For the sharpening of Theorem 3.1.3, 3.1.5, and 3.1.6 we refer the reader to the paper of Frappier, Rahman and Ruscheweyh [48] (also, see Sharma and Singh [117]).

For polynomial of degree n not vanishing in the interior of the unit circle, Ankeny and Rivlin [8] proved the following result.

Theorem 3.1.7 (Ankeny and Rivlin [8]). *Let* $p(z) = \sum_{j=0}^{n} a_j z^j \neq 0$ *in* $D(0,1)$. *Then for* $R \geq 1$,

$$M(p,R) \leq \left(\frac{R^n + 1}{2}\right) ||p||. \tag{3.6}$$

Here equality holds for $p(z) = \dfrac{\alpha + \beta z^n}{2}$, *where* $|\alpha| = |\beta| = 1$.

The analogue of Inequality (3.5) for polynomials not vanishing in the interior of a unit circle was proved later in 1960 by Rivlin [113], who in fact proved

Theorem 3.1.8 (Rivlin [113]). *Let* $p(z) = \sum_{j=0}^{n} a_j z^j \neq 0$ *in* $D(0,1)$. *Then for* $0 < r \leq 1$,

$$M(p,r) \geq \left(\frac{r+1}{2}\right)^n ||p||, \tag{3.7}$$

and equality holding for $p(z) = \left(\dfrac{\alpha + \beta z}{2}\right)^n$, *where* $|\alpha| = |\beta| = 1$.

The above results, Theorems 3.1.5 and 3.1.6 which are known as Bernstein inequalities concerning growth of polynomials, and Theorems 3.1.7 and 3.1.8 have been the starting point of a considerable literature in Approximation Theory. Several books and research monographs have been written on this subject of inequalities (see for example Govil and Mohapatra [61], Milovanović, Mitrinović and Rassias [93], Pinkus and de Boor [102], Rahman and Schmeisser [110], and recent article of Govil and Nwaeze [63]).

In this chapter we study some of the developments that have, over a period, taken place around these inequalities and then present some new results in that direction. In particular, we present some generalizations, improvements and extensions of Theorems 3.1.7 and 3.1.8.

3.2 Some Improvements of Result due to Ankeny and Rivlin

We begin by presenting the brief outlines of the proof of Theorem 3.1.7 as given by Ankeny and Rivlin in [8], which makes use of Erdös-Lax theorem. As is well known, the Erdös-Lax theorem which is stated below as Lemma 3.2.1, was conjectured by Erdös and proved by Lax [86].

Lemma 3.2.1 (Lax [86]). *Let* $p(z) = \sum_{j=0}^{n} a_j z^j \neq 0$ *in* $D(0,1)$. *Then*

$$M(p',1) \leq \frac{n}{2}||p||. \tag{3.8}$$

Proof of Theorem 3.1.7. Let us assume that $p(z)$ does not have the form $\dfrac{\alpha + \beta z^n}{2}$. In view of Lemma 3.2.1

$$|p'(e^{i\theta})| \leq \frac{n}{2}||p||, \qquad 0 \leq \theta < 2\pi, \tag{3.9}$$

from which we may deduce that

$$|p'(re^{i\theta})| < \frac{n}{2}r^{n-1}||p||, \qquad 0 \leq \theta < 2\pi, \quad r > 1, \tag{3.10}$$

by applying Theorem 3.1.5 to the polynomial $p'(z)/(n/2)$ and observing that we have the strict inequality in (3.10) because $p(z)$ does not have the form $\dfrac{\alpha + \beta z^n}{2}$. But for each θ,

$0 \leq \theta < 2\pi$, we have

$$p(Re^{i\theta}) - p(e^{i\theta}) = \int_1^R e^{i\theta} p'(re^{i\theta})dr.$$

Hence

$$\left|p(Re^{i\theta}) - p(e^{i\theta})\right| \leq \int_1^R |p'(re^{i\theta})|dr < \frac{n}{2}||p|| \int_1^R r^{n-1}dr = \frac{||p||}{2}(R^n - 1),$$

and

$$\left|p(Re^{i\theta})\right| < \frac{||p||}{2}(R^n - 1) + \left|p(e^{i\theta})\right| \leq \frac{||p||}{2}(1 + R^n).$$

Finally, if $p(z) = \dfrac{\alpha + \beta z^n}{2}$, $|\alpha| = |\beta| = 1$, then clearly

$$M(p, R) = \frac{1 + R^n}{2}, \ R > 1,$$

and the proof of Theorem 3.1.7 is thus complete.

□

It may be remarked that later a simpler proof of Theorem 3.1.7 which does not make use of Erdös-Lax theorem was given by Dewan [36].

Remark 3.2.2. The converse of Theorem 3.1.7 is false as the simple example $p(z) = (z + \frac{1}{2})(z + 3)$ shows. However, the following result in the converse direction, which is also due to Ankeny and Rivlin [8], is valid.

Theorem 3.2.3. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *such that* $p(1) = 1$ *and*

$$M(p, R) \leq \left(\frac{R^n + 1}{2}\right)||p||$$

for $0 < R - 1 < \delta$, *where* δ *is any positive number. Then* $p(z)$ *does not have all its zeros within the unit circle.*

In 1989, Govil [57] observed that since the equality in (3.6) holds only for polynomials $p(z) = \alpha + \beta z^n,\ |\alpha| = |\beta|$, which satisfy

$$\left|\text{coefficient of } z^n\right| = \frac{1}{2}||p||, \tag{3.11}$$

it should be possible to improve upon the bound in (3.6) for polynomials not satisfying (3.11), and therefore in this connection he proved the following refinement of (3.6).

Theorem 3.2.4. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j \neq 0$ *in* $D(0,1)$*. Then for* $R \geq 1$*,*

$$M(p,R) \leq \Big(\frac{R^n+1}{2}\Big)||p|| - \frac{n(||p||^2 - 4|a_n|^2)}{2||p||}\left\{\frac{(R-1)||p||}{||p||+2|a_n|} - \ln\left[1 + \frac{(R-1)||p||}{||p||+2|a_n|}\right]\right\} \tag{3.12}$$

Equality holding for $p(z) = \alpha + \beta z^n$*, where* $|\alpha| = |\beta|$*.*

Since $x - \ln(1+x) > 0$ for $x > 0$, the above theorem always gives a bound sharper than obtainable from Theorem 3.1.7 of Ankeny and Rivlin unless the polynomial satisfies Equation (3.11).

In 1998, Dewan and Bhat [38] sharpened the above Theorem 3.2.4 as follows

Theorem 3.2.5. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j \neq 0$ *in* $D(0,1)$*. Then for* $R \geq 1$*,*

$$M(p,R) \leq \Big(\frac{R^n+1}{2}\Big)||p|| - \Big(\frac{R^n-1}{2}\Big)m - \frac{n}{2}\left[\frac{(||p||-m)^2 - 4|a_n|^2}{(||p||-m)}\right]\left\{\frac{(R-1)(||p||-m)}{(||p||-m)+2|a_n|} - \ln\left[1 + \frac{(R-1)(||p||-m)}{(||p||-m)+2|a_n|}\right]\right\}, \tag{3.13}$$

where $m = \min_{|z|=1} |p(z)|$*. Here again, equality holds for* $p(z) = \alpha + \beta z^n$*, where* $|\alpha| = |\beta|$*.*

In 2001, Govil and Nyuydinkong [64] generalized Theorem 3.2.5, where they considered polynomials not vanishing in $D(0,K),\ K \geq 1$. More specifically, they proved

Theorem 3.2.6. *Let* $p(z)=\sum_{j=0}^{n} a_j z^j \neq 0$ *in* $D(0,K),\ K\geq 1$. *Then for* $R\geq 1$,

$$M(p,R)\leq \Big(\frac{R^n+K}{1+K}\Big)||p||-\Big(\frac{R^n-1}{1+K}\Big)m-\frac{n}{1+K}\left[\frac{(||p||-m)^2-(1+K)^2|a_n|^2}{(||p||-m)}\right] \tag{3.14}$$
$$\times\left\{\frac{(R-1)(||p||-m)}{(||p||-m)+(1+K)|a_n|}-\ln\left[1+\frac{(R-1)(||p||-m)}{(||p||-m)+(1+K)|a_n|}\right]\right\},$$

where $m=\min_{|z|=K}|p(z)|$.

Later, Gardner, Govil and Weems [51] generalized Theorem 3.2.6 by considering polynomials of the form $a_0+\sum_{j=t}^{n} a_j z^j,\ 1\leq t\leq n$, and for this, they proved the following

Theorem 3.2.7. *Let* $p(z)=a_0+\sum_{j=t}^{n} a_j z^j,\ 1\leq t\leq n$, *be a polynomial of degree* n *and* $p(z)\neq 0$ *in* $D(0,K),\ K\geq 1$. *Then for* $R\geq 1$,

$$M(p,R)\leq \Big(\frac{R^n+K^t}{1+K^t}\Big)||p||-\Big(\frac{R^n-1}{1+K^t}\Big)m-\frac{n}{1+K^t}\left[\frac{(||p||-m)^2-(1+K^t)^2|a_n|^2}{(||p||-m)}\right] \tag{3.15}$$
$$\times\left\{\frac{(R-1)(||p||-m)}{(||p||-m)+(1+K^t)|a_n|}-\ln\left[1+\frac{(R-1)(||p||-m)}{(||p||-m)+(1+K^t)|a_n|}\right]\right\},$$

where $m=\min_{|z|=K}|p(z)|$.

Clearly, for $t=1$, Theorem 3.2.7 gives Theorem 3.2.6, which for $K=1$ reduces to Theorem 3.2.5.

In 2005, Gardner, Govil and Musukula [52] proved the following generalization and sharpening of Theorem 3.2.4.

Theorem 3.2.8. *Let* $p(z)=a_0+\sum_{j=t}^{n} a_j z^j,\ 1\leq t\leq n$, *be a polynomial of degree* n *and* $p(z)\neq 0$ *in* $D(0,K),\ K\geq 1$. *Then for* $R\geq 1$,

$$M(p,R) \le \Big(\frac{R^n+s_0}{1+s_0}\Big)||p|| - \Big(\frac{R^n-1}{1+s_0}\Big)m - \frac{n}{1+s_0}\left[\frac{(||p||-m)^2-(1+s_0)^2|a_n|^2}{(||p||-m)}\right] \tag{3.16}$$
$$\times\left\{\frac{(R-1)(||p||-m)}{(||p||-m)+(1+s_0)|a_n|} - \ln\left[1+\frac{(R-1)(||p||-m)}{(||p||-m)+(1+s_0)|a_n|}\right]\right\},$$

where $m = \min_{|z|=K} |p(z)|$, *and*

$$s_0 = K^{t+1}\frac{\frac{t}{n}\cdot\frac{|a_t|}{|a_0|-m}K^{t-1}+1}{\frac{t}{n}\cdot\frac{|a_t|}{|a_0|-m}K^{t+1}+1}.$$

Dividing both sides of (3.16) by R^n, and letting $R \to \infty$, one gets

Corollary 3.2.9. *Let* $p(z) = a_0 + \sum_{j=t}^{n} a_j z^j,\ 1 \le t \le n$, *be a polynomial of degree* n *and* $p(z) \ne 0$ *in* $D(0,K),\ K \ge 1$. *Then*

$$|a_n| \le \frac{1}{1+s_0}\big(||p||-m\big), \tag{3.17}$$

where $m = \min_{|z|=K} |p(z)|$.

In case one does not have knowledge of $m = \min_{|z|=K} |p(z)|$, one could use the following result due to Gardner, Govil and Musukula [52] which does not depend on m.

Theorem 3.2.10. *Let* $p(z) = a_0 + \sum_{j=t}^{n} a_j z^j,\ 1 \le t \le n$, *be a polynomial of degree* n *and* $p(z) \ne 0$ *in* $D(0,K),\ K \ge 1$. *Then for* $R \ge 1$,

$$M(p,R) \le \Big(\frac{R^n+s_1}{1+s_1}\Big)||p|| - \frac{n}{1+s_1}\left[\frac{||p||^2-(1+s_1)^2|a_n|^2}{||p||}\right] \tag{3.18}$$
$$\times\left\{\frac{(R-1)||p||}{||p||+(1+s_1)|a_n|} - \ln\left[1+\frac{(R-1)||p||}{||p||+(1+s_1)|a_n|}\right]\right\},$$

where

$$s_1 = K^{t+1} \frac{\frac{t}{n} \cdot \frac{|a_t|}{|a_0|} K^{t-1} + 1}{\frac{t}{n} \cdot \frac{|a_t|}{|a_0|} K^{t+1} + 1}.$$

If in the above theorem, one divides both sides of (3.18) by R^n and let $R \to \infty$, one obtains the following

Corollary 3.2.11. *Let* $p(z) = a_0 + \sum_{j=t}^{n} a_j z^j,\ 1 \le t \le n$, *be a polynomial of degree* n *and* $p(z) \neq 0$ *in* $D(0, K),\ K \ge 1$. *Then*

$$|a_n| \le \frac{1}{1 + s_1} ||p||. \tag{3.19}$$

Both Corollaries 3.2.9 and 3.2.11 generalize and sharpen the well known inequality, obtainable by an application of Visser's Inequality [123], that if $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n and $p(z) \neq 0$ in $D(0, 1)$ then $|a_n| \le \frac{1}{2}||p||$.

We present some of the examples Gardner, Govil and Musukula [52] gave to illustrate the quality of Theorems 3.2.7, 3.2.8 and 3.2.10.

Example 3.2.12. Let $p(z) = 1000 + z^2 + z^3 + z^4$. Clearly, here $t = 2$ and $n = 4$, and one can take $K = 5.4$, since numerically $p \neq 0$ for $|z| < 5.4483$. For this polynomial, the bound for $M(p, 2)$ by Theorem 3.2.7 comes out to be 1447.503, and by Theorem 3.2.8, it comes out to be 1101.84, which is a significant improvement over the bound obtained from Theorem 3.2.7. Numerically, for this polynomial $M(p, 2) \approx 1028$, which is quite close to the bound 1101.84, that is obtainable by Theorem 3.2.10. The bound for $M(p, 2)$ obtained by Theorem 3.2.10 is 1105.05, which is also quite close to the actual bound ≈ 1028. However, in this case Theorem 3.2.8 gives the best bound.

Example 3.2.13. Let $p(z) = 1000 + z^2 - z^3 - z^4$. Here also, $t = 2$ and $n = 4$. Again, numerically $p(z) \neq 0$ for $|z| < 5.43003$, and thus take $K = 5.4$. If $R = 3$, then for this polynomial the bound for $M(p, 3)$ obtained by Theorem 3.2.7 comes out to be 3479.408, while by Theorem 3.2.10 it comes out to be 1545.3, and by Theorem 3.2.8 it comes out to be 1534.5, a considerable improvement. Thus again the bounds obtained from Theorem 3.2.8 and Theorem 3.2.10 are considerably smaller than the bound obtained from Theorem 3.2.7, and the bound 1534.5 obtained by Theorem 3.2.8 is much closer to the actual bound $M(p,3) \approx 1100.6$, than the bound 3479.408, obtained from Theorem 3.2.7.

While trying to obtain inequality analogous to (3.6) for polynomials not vanishing in $D(0,K), K \leq 1$, Dewan and Ahuja [37] were able to prove this only for polynomials having all the zeros on the circle $S(0,K) := \{z : |z| = K\},\ 0 < K \leq 1$.

Theorem 3.2.14. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *having all its zeros on* $S(0,K),\ K \leq 1$. *Then for* $R \geq 1$ *and for every positive integer* s,

$$\{M(p,R)\}^s \leq \left[\frac{K^{n-1}(1+K) + (R^{ns}-1)}{K^{n-1}+K^n}\right]\{M(p,1)\}^s. \tag{3.20}$$

For $s = 1$, the Theorem 3.2.14 yields

Corollary 3.2.15. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *having all its zeros on* $S(0,K),\ K \leq 1$. *Then for* $R \geq 1$,

$$M(p,R) \leq \left[\frac{K^{n-1}(1+K) + (R^{n}-1)}{K^{n-1}+K^n}\right]M(p,1). \tag{3.21}$$

In same spirit, we prove the following results

3.2.1 Statement and Proof of some New Results

In this subsection, we present some new results that sharpen the aforementioned results.

Theorem 3.2.16. *Let* $p(z) = z^m\left[a_{n-m}z^{n-m} + \sum_{j=\mu}^{n-m} a_{n-m-j}z^{n-m-j}\right],\ 1 \leq \mu \leq n-m,\ 0 \leq m \leq n-1$, *be a polynomial of degree* n, *having* $m-fold$ *zeros at origin and remaining* $n-m$ *zeros on* $S(0,K),\ K \leq 1$. *Then for every integer* s

$$[M(p,R)]^s \leq L(\mu; K, m, n, s)[M(p,1)]^s, \quad R \geq 1 \tag{3.22}$$

where

$$L(\mu; K, m, n, s) = \frac{n(K^{n-m-2\mu+1} + K^{n-m-\mu+1}) + (R^{ns} - 1)[n + mK^{n-m-2\mu+1} + mK^{n-m-\mu+1} - m]}{n(K^{n-m-2\mu+1} + K^{n-m-\mu+1})}$$

For $m = 0$, we have

Corollary 3.2.17. *Let $p(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-j}$, $1 \leq \mu \leq n$, be a polynomial of degree n, having all zeros on $|z| = K$, $K \leq 1$. Then for every integer s*

$$[M(p, R)]^s \leq L(\mu; K, n, s)[M(p, 1)]^s, \quad R \geq 1 \tag{3.23}$$

where

$$L(\mu; K, n, s) = \frac{K^{n-\mu}(K^{1-\mu} + K) + (R^{ns} - 1)}{K^{n-2\mu+1} + K^{n-\mu+1}}$$

If we set $\mu = 1$ into Corollary 3.2.17, we get the following result of Dewan and Ahuja [37].

Corollary 3.2.18. *Let $p(z) = \sum_{j=0}^{n} a_j z^j$, be a polynomial of degree n, having all zeros on $|z| = K$, $K \leq 1$. Then for every integer s,*

$$[M(p, R)]^s \leq L(1; K, n, s)[M(p, 1)]^s, \quad R \geq 1 \tag{3.24}$$

where

$$L(1; K, n, s) = \frac{K^{n-1}(1 + K) + (R^{ns} - 1)}{K^{n-1} + K^n}.$$

For the proof Theorem 3.2.16 we need the following lemmas. The first lemma is due to Kumar and Lal [82].

Lemma 3.2.19. *Let $p(z) = z^m \left[a_{n-m} z^{n-m} + \sum_{j=\mu}^{n-m} a_{n-m-j} z^{n-m-j} \right]$, $1 \leq \mu \leq n - m$, $0 \leq m \leq n - 1$, be a polynomial of degree n, having $m - fold$ zeros at origin and remaining $n - m$*

zeros on $|z| = K$, $K \leq 1$.

$$\max_{|z|=1} |p'(z)| \leq \frac{n + m(K^{n-m-2\mu+1} + K^{n-m-\mu+1} - 1)}{K^{n-m-2\mu+1} + K^{n-m-\mu+1}} \max_{|z|=1} |p(z)| \tag{3.25}$$

The next lemma is the Bernstein inequality given in Theorem 3.1.5.

Lemma 3.2.20. *Let* $p(z)$ *be a polynomial of degree* n. *Then for* $R \geq 1$,

$$M(p, R) \leq R^n M(p, 1). \tag{3.26}$$

Proof of Theorem 3.2.16. By Lemma 3.2.19, we have

$$\max_{|z|=1} |p'(z)| \leq \frac{n + m(K^{n-m-2\mu+1} + K^{n-m-\mu+1} - 1)}{K^{n-m-2\mu+1} + K^{n-m-\mu+1}} \max_{|z|=1} |p(z)|$$

Applying Lemma 3.2.20 to the polynomial $p'(z)$ which is of degree $n-1$, it follows that for all $R \geq 1$ and $\theta \in [0, 2\pi)$,

$$\begin{aligned} |p'(Re^{i\theta})| &\leq \max_{|z|=R} |p'(z)| \\ &\leq R^{n-1} \max_{|z|=1} |p'(z)| \\ &\leq R^{n-1} \left[\frac{n + m(K^{n-m-2\mu+1} + K^{n-m-\mu+1} - 1)}{K^{n-m-2\mu+1} + K^{n-m-\mu+1}} \right] \max_{|z|=1} |p(z)|. \end{aligned}$$

So for each $\theta \in [0, 2\pi)$ and $R \geq 1$, we obtain

$$\begin{aligned} \left[p(Re^{i\theta})\right]^s - \left[p(e^{i\theta})\right]^s &= \int_1^R \frac{d\left[p(te^{i\theta})\right]^s}{dt} dt \\ &= \int_1^R s\left[p(te^{i\theta})\right]^{s-1} p'(te^{i\theta}) e^{i\theta} dt. \end{aligned}$$

This implies that

$$\left|p(Re^{i\theta})\right|^s \le \left|p(e^{i\theta})\right|^s + s\int_1^R \left|p(te^{i\theta})\right|^{s-1}\left|p'(te^{i\theta})\right|dt.$$

Thus,

$$\begin{aligned}
\left[M(p,R)\right]^s &\le \left[M(p,1)\right]^s + s\int_1^R \left[t^n M(p,1)\right]^{s-1}\left|p'(te^{i\theta})\right|dt \\
&\le \left[M(p,1)\right]^s + s\int_1^R t^{ns-n}\left[M(p,1)\right]^{s-1} t^{n-1}.\frac{n+m(K^{n-m-2\mu+1}+K^{n-m-\mu+1}-1)}{K^{n-m-2\mu+1}+K^{n-m-\mu+1}}M(p.1)dt \\
&= \left[M(p,1)\right]^s + s\left[\frac{n+m(K^{n-m-2\mu+1}+K^{n-m-\mu+1}-1)}{K^{n-m-2\mu+1}+K^{n-m-\mu+1}}\right]\left[M(p,1)\right]^s\int_1^R t^{ns-1}dt \\
&= \left[M(p,1)\right]^s + \left[M(p,1)\right]^s\left[\frac{n+m(K^{n-m-2\mu+1}+K^{n-m-\mu+1}-1)}{K^{n-m-2\mu+1}+K^{n-m-\mu+1}}\right]s.\frac{R^{ns}-1}{ns} \\
&= \left[M(p,1)\right]^s\left[1+\frac{\left[n+m(K^{n-m-2\mu+1}+K^{n-m-\mu+1}-1)\right]\left(R^{ns}-1\right)}{n\left(K^{n-m-2\mu+1}+K^{n-m-\mu+1}\right)}\right]
\end{aligned}$$

This yields

$$\left[M(p,R)\right]^s \le \left[M(p,1)\right]^s\left[\frac{n\left(K^{n-m-2\mu+1}+K^{n-m-\mu+1}\right)+\left[n+m(K^{n-m-2\mu+1}+K^{n-m-\mu+1}-1)\right]\left(R^{ns}-1\right)}{n\left(K^{n-m-2\mu+1}+K^{n-m-\mu+1}\right)}\right].$$

This completes the proof.

□

3.3 Some Improvements of a Result due to Rivlin

So far we have been dealing with improvements and generalizations of Inequality (3.6), we now turn our attention to Inequality (3.7), given in Theorem 3.1.8. In this regard, Govil [56] generalized this Theorem 3.1.8 by proving

Theorem 3.3.1. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j \neq 0$ *in* $D(0,1)$. *Then for* $0 < r \leq \rho \leq 1$,

$$M(p,r) \geq \Big(\frac{1+r}{1+\rho}\Big)^n M(p,\rho). \tag{3.27}$$

The result is best possible and equality holds for the polynomial $p(z) = \Big(\frac{1+z}{1+\rho}\Big)^n$.

If polynomial $p(z)$ has all its zeros on $|z| = 1$, the polynomial $q(z) = z^n p(\frac{1}{z})$ also has its zeros on $|z| = 1$. Further, if $1 \leq \rho \leq r$, then $\frac{1}{r} \leq \frac{1}{\rho} \leq 1$, and when (3.27) is applied to $q(z)$, it yields

$$M\Big(q, \frac{1}{r}\Big) \geq \left(\frac{1+\frac{1}{r}}{1+\frac{1}{\rho}}\right)^n M\Big(q, \frac{1}{\rho}\Big),$$

which is equivalent to (3.27).

The above explanation thus leads to the following corollary.

Corollary 3.3.2. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *having all its zeros on the unit circle. Then for* $0 < r \leq \rho \leq 1$, *and for* $1 \leq \rho \leq r$,

$$M(p,r) \geq \Big(\frac{1+r}{1+\rho}\Big)^n M(p,\rho). \tag{3.28}$$

The result is best possible and equality holds for the polynomial $p(z) = (1+z)^n$.

If in Theorem 3.3.1 one also assumes that $p'(0) = 0$, the bound in (3.27) can be considerably improved. Govil [56] in the same paper obtained the following in this direction.

Theorem 3.3.3. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j \neq 0$ *in* $D(0,1)$. *Let* $p'(0) = 0$. *Then for* $0 < r \leq \rho \leq 1$,

$$M(p,r) \geq \Big(\frac{1+r}{1+\rho}\Big)^n \left\{ \frac{1}{1 - \frac{(1-\rho)(\rho-r)n}{4}\Big(\frac{1+r}{1+\rho}\Big)^{n-1}} \right\} M(p,\rho). \tag{3.29}$$

Theorem 3.3.1 is best possible, however, if $0 < r < \rho < 1$, then for any polynomial $p(z)$ having no zeros in $D(0,1)$, and $p'(0) = 0$, the bound obtained by Theorem 3.3.3 can be considerably sharper than the bound obtained by Theorem 3.3.1. Govil [56] illustrated this by means of the following examples.

Example 3.3.4. Let $p(z) = 1 + z^3, \ \rho = 0.5, \ r = 0.1$. Theorem 3.3.1 gives $M(p, r) \geq (0.3943704)M(p, \rho)$, while by Theorem 3.3.3, $M(p, r) \geq (0.4289743)M(p, \rho)$.

Example 3.3.5. Let $p(z) = 1 + z^7, \ \rho = 0.168, \ r = 0.022$. Theorem 3.3.1 gives $M(p, r) \geq (0.3926959)M(p, \rho)$, while by Theorem 3.3.3, $M(p, r) \geq (0.4341115)M(p, \rho)$.

In this section we present some further extension and sharpening of Rivlin's result. In this regard, we have the following

3.3.1 More New Results

As generalization and sharpening of Rivlin's result, we state the following

Theorem 3.3.6. *Let* $p(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j, \ 1 \leq \mu < n$. *If* $p(z) \neq 0$ *for* $|z| < 1$, *then for* $0 < r < 1$, *we have*

$$M(p, r) \geq \frac{(1+r)^{n/\mu}}{(1+r^\mu)^{n/\mu} + \mu 2^{n/\mu} - \mu(1+r)^{n/\mu}} \Big[M(p, 1) + n \min_{|z|=1} |p(z)| \ln \Big(\frac{2}{1+r} \Big) \Big]. \quad (3.30)$$

As a consequence of the above theorem, if $p(z) \neq 0$ for $|z| < K, \ K > 0$, then the polynomial $P(z) = p(Kz) \neq 0$ for $|z| < 1$. Further, if $0 < r < K$, then $0 < r/K < 1$, and applying Theorem 3.3.6 to $P(z)$, we get

$$M(P, r/K) \geq \frac{(1+r/K)^{n/\mu}}{(1+(r/K)^\mu)^{n/\mu} + \mu 2^{n/\mu} - \mu(1+r/K)^{n/\mu}} \Big[M(P, 1) + n \min_{|z|=1} |P(z)| \ln \Big(\frac{2}{1+r/K} \Big) \Big],$$

which yields

$$M(p, r) \geq \frac{\gamma^{1/\mu}(r+K)^{n/\mu}}{\gamma(r^\mu + K^\mu)^{n/\mu} + \mu 2^{n/\mu} - \mu\gamma^{1/\mu}(r+K)^{n/\mu}} \Big[M(p, K) + nm \ln \Big(\frac{2K}{r+K} \Big) \Big], \quad (3.31)$$

where $\gamma = K^{-n}$, and $m = \min_{|z|=K} |p(z)|$.

Theorem 3.3.7. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$. *If* $p(z) \neq 0$ *for* $|z| < K$, $K \geq 1$, *then for* $0 < r < R \leq 1$, *we have*

$$M(p,r) \geq \frac{(1+r)^n}{(1+r)^n + (R+K)^n - (r+K)^n}\Big[M(p,R) + nm \ln\Big(\frac{R+K}{r+K}\Big)\Big], \qquad (3.32)$$

where $m = \min_{|z|=K} |p(z)|$.

If $R = 1$, Theorem 3.3.7 reduces to

Corollary 3.3.8. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$. *If* $p(z) \neq 0$ *for* $|z| < K$, $K \geq 1$, *then for* $0 < r < 1$, *we have*

$$M(p,r) \geq \frac{(1+r)^n}{(1+r)^n + (1+K)^n - (r+K)^n}\Big[M(p,1) + n \min_{|z|=K} |p(z)| \ln\Big(\frac{1+K}{r+K}\Big)\Big]. \qquad (3.33)$$

Setting $K = 1$ in Corollary 3.3.8 gives

Corollary 3.3.9. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$. *If* $p(z) \neq 0$ *for* $|z| < 1$, *then for* $0 < r < 1$, *we have*

$$M(p,r) \geq \Big(\frac{1+r}{2}\Big)^n \Big[M(p,1) + n \min_{|z|=1} |p(z)| \ln\Big(\frac{2}{1+r}\Big)\Big]. \qquad (3.34)$$

Unless $\min_{|z|=1} |p(z)| = 0$, Corollary 3.3.9 always gives a bound sharper than the bound obtainable from Theorem 3.1.8 of the result due to Rivlin [113] which states that if $p(z) = \sum_{j=0}^{n} a_j z^j \neq 0$ in $D(0,1)$, then for $0 < r \leq 1$, $M(p,r) \geq \Big(\frac{r+1}{2}\Big)^n ||p||$.

For the proof of Theorems 3.3.6 and 3.3.7, we need the following lemmas. The first lemma is due to Govil [58].

Lemma 3.3.10. *Let $p(z)$ be a polynomial of degree n having no zeros in $|z| < K$, $K \geq 1$, then*

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1+K}\Big[\max_{|z|=1} |p(z)| - \min_{|z|=K} |p(z)|\Big]. \tag{3.35}$$

Lemma 3.3.11 (Qazi [108])**.** *Let $p(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$, $1 \leq \mu < n$. If $p(z) \neq 0$ for $|z| < 1$, then for $0 < r < R \leq 1$, we have,*

$$M(p,r) \geq \Big(\frac{1+r^\mu}{1+R^\mu}\Big)^{n/\mu} M(p,R); \tag{3.36}$$

more precisely,

$$M(p,r) \geq \exp\left(-n\int_r^R \frac{t^\mu + (\mu/n)|a_\mu/a_0|t^{\mu-1}}{t^{\mu+1} + (\mu/n)|a_\mu/a_0|(t^\mu + t) + 1} dt\right) M(p,R). \tag{3.37}$$

Recently, Jain [71] (see also Govil and Qazi [66]) proved the following generalization of Lemma 3.3.11.

Lemma 3.3.12. *Let $p(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$, $1 \leq \mu < n$. If $p(z) \neq 0$ for $|z| < K$, $K > 0$ then for $0 < r < R \leq K$, we have,*

$$M(p,r) \geq \Big(\frac{r^\mu + K^\mu}{R^\mu + K^\mu}\Big)^{n/\mu} M(p,R); \tag{3.38}$$

more precisely,

$$M(p,r) \geq \exp\left(-n\int_r^R \frac{t^\mu + (\mu/n)|a_\mu/a_0|K^{\mu+1}t^{\mu-1}}{t^{\mu+1} + K^{\mu+1} + (\mu/n)|a_\mu/a_0|(K^{\mu+1}t^\mu + K^{2\mu}t)} dt\right) M(p,R). \tag{3.39}$$

It is important to note that if $0 \neq p(z) = a_0 + a_1 z + \cdots + a_{n-\mu}z^{n-\mu} + a_n z^n$, $1 \leq \mu < n$ for $|z| \geq K$, $K > 0$, then $0 \neq q(z) = z^n p(1/z) = a_n + a_{n-\mu}z^{n-\mu} + a_{n-\mu-1}z^{\mu+1} + \cdots + a_0 z^n$ in $|z| < K$, $K > 0$. Furthermore, if we assume that $K \leq r < R$ then we have that

$1/R < 1/r \leq 1/K$. Applying Inequality (3.38) of Lemma 3.3.12, to polynomial $q(z)$ we obtain

$$M(q, 1/R) \geq \left(\frac{R^{-\mu} + K^{-\mu}}{r^{-\mu} + K^{-\mu}} \right)^{n/\mu} M(q, 1/r);$$

this implies that

$$M(p, R) \geq \left(\frac{R^{\mu} + K^{\mu}}{r^{\mu} + K^{\mu}} \right)^{n/\mu} M(p, r).$$

Putting $\mu = 1$ in Lemma 3.3.12, one obtains the following

Lemma 3.3.13. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$. *If* $p(z) \neq 0$ *for* $|z| < K,\ K > 0$ *then for* $0 < r < R \leq K$,

$$M(p, r) \geq \left(\frac{r^2 + 2\lambda r + K^2}{R^2 + 2\lambda R + K^2} \right)^{n/2} M(p, R), \tag{3.40}$$

where $\lambda := \frac{K^2}{n} \left| \frac{a_1}{a_0} \right|$.

Applying the above lemma to the polynomial $q(z) = z^n p(1/z)$ with $K \leq r < R$, where $p(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n$ has all its zeros in $|z| < K,\ K > 0$, we get that $q(z)$ has no zeros in $|z| < K,\ K > 0$, and

$$M(q, 1/R) \geq \left(\frac{R^{-2} + 2\lambda R^{-1} + K^{-2}}{r^{-2} + 2\lambda r^{-1} + K^{-2}} \right)^{n/2} M(q, 1/r);$$

this implies

$$M(p, R) \geq \left(\frac{R^2 + 2\lambda R K^2 + K^2}{r^2 + 2\lambda r K^2 + K^2} \right)^{n/2} M(p, r).$$

Proof of Theorem 3.3.6. Let $0 < r < 1$. Let $\theta \in [0, 2\pi)$ we have:

$$\left| p(e^{i\theta}) - p(re^{i\theta}) \right| = \left| \int_r^1 e^{i\theta} p'(te^{i\theta}) dt \right|.$$

This implies

$$\left|p(e^{i\theta})\right| \leq \left|p(re^{i\theta})\right| + \left|\int_r^1 e^{i\theta}p'(te^{i\theta})dt\right|. \tag{3.41}$$

If $p(z) \neq 0$ in $|z| < 1$, then $p(tz) \neq 0$ in $|z| < 1/t$. Further, if $0 < t \leq 1$, then $1/t \geq 1$ and hence by Lemma 3.3.10 we get

$$t|p'(tz)| \leq \frac{nt}{1+t}\Big[M(p,t) - \min_{|z|=1}|p(z)|\Big]$$

which is equivalent to

$$|p'(tz)| \leq \frac{n}{1+t}\Big[M(p,t) - \min_{|z|=1}|p(z)|\Big]. \tag{3.42}$$

Combining (3.41) and (3.42) yield

$$\left|p(e^{i\theta})\right| \leq \left|p(re^{i\theta})\right| + \int_r^1 \frac{n}{1+t}M(p,t)dt - n\min_{|z|=1}|p(z)|\int_r^1 \frac{1}{1+t}dt.$$

This implies

$$M(p,1) \leq M(p,r) + \int_r^1 \frac{n}{1+t}M(p,t)dt - n\min_{|z|=1}|p(z)|\int_r^1 \frac{1}{1+t}dt.$$

Now by Lemma 3.3.11, we obtain

$$M(p,1) \leq M(p,r) + \int_r^1 \frac{n}{1+t}\Big(\frac{1+t^\mu}{1+r^\mu}\Big)^{n/\mu}M(p,r)dt - n\min_{|z|=1}|p(z)|\int_r^1 \frac{1}{1+t}dt$$

$$\Rightarrow \qquad M(p,1) \leq M(p,r) + \int_r^1 \frac{n}{1+t}\Big(\frac{1+t}{1+r^\mu}\Big)^{n/\mu}M(p,r)dt - n\min_{|z|=1}|p(z)|\int_r^1 \frac{1}{1+t}dt$$

$$\Rightarrow \quad M(p,1) \leq M(p,r) + \frac{nM(p,r)}{(1+r^{\mu})^{n/\mu}} \int_r^1 \frac{(1+t)^{n/\mu}}{1+t} dt - n \min_{|z|=1} |p(z)| \int_r^1 \frac{1}{1+t} dt$$

$$\Rightarrow \quad M(p,1) \leq M(p,r) + \frac{nM(p,r)}{(1+r^{\mu})^{n/\mu}} \left[2^{n/\mu} - (1+r)^{n/\mu}\right] \frac{\mu}{n} - n \min_{|z|=1} |p(z)| \int_r^1 \frac{1}{1+t} dt$$

$$\Rightarrow \quad M(p,1) \leq M(p,r) + \frac{\mu M(p,r)}{(1+r^{\mu})^{n/\mu}} \left[2^{n/\mu} - (1+r)^{n/\mu}\right] - n \min_{|z|=1} |p(z)| \ln\left(\frac{2}{1+r}\right)$$

$$\Rightarrow \quad M(p,r) \left[1 + \frac{\mu 2^{n/\mu}}{(1+r^{\mu})^{n/\mu}} - \frac{\mu(1+r)^{n/\mu}}{(1+r^{\mu})^{n/\mu}}\right] \geq M(p,1) + n \min_{|z|=1} |p(z)| \ln\left(\frac{2}{1+r}\right)$$

$$\Rightarrow \quad M(p,r) \left[\frac{(1+r^{\mu})^{n/\mu} + \mu 2^{n/\mu} - \mu(1+r)^{n/\mu}}{(1+r^{\mu})^{n/\mu}}\right] \geq M(p,1) + n \min_{|z|=1} |p(z)| \ln\left(\frac{2}{1+r}\right)$$

$$\Rightarrow \quad M(p,r) \geq \frac{(1+r^{\mu})^{n/\mu}}{(1+r^{\mu})^{n/\mu} + \mu 2^{n/\mu} - \mu(1+r)^{n/\mu}} \left[M(p,1) + n \min_{|z|=1} |p(z)| \ln\left(\frac{2}{1+r}\right)\right].$$

That proves the theorem. □

Proof of Theorem 3.3.7. As in the proof of Theorem 3.3.6, we obtain similarly that

$$\left|p(Re^{i\theta})\right| \leq \left|p(re^{i\theta})\right| + \left|\int_r^R e^{i\theta} p'(te^{i\theta}) dt\right|. \tag{3.43}$$

Now if $p(z) \neq 0$ in $|z| < K$, $K \geq 1$, then $p(tz) \neq 0$ in $|z| < K/t$. Further, if $0 < t \leq 1$, then $1/t \geq 1$ and $K/t \geq 1$.

By Lemma 3.3.10, we get

$$|p'(tz)| \leq \frac{n}{K+t}\Big[M(p,t) - \min_{|z|=K}|p(z)|\Big]. \tag{3.44}$$

Using (3.43) and (3.44) give

$$\left|p(Re^{i\theta})\right| \leq \left|p(re^{i\theta})\right| + \int_r^R \frac{n}{K+t}M(p,t)dt - n\min_{|z|=K}|p(z)|\int_r^R \frac{1}{K+t}dt$$

$$\Rightarrow \qquad M(p,R) \leq M(p,r) + \int_r^R \frac{n}{K+t}M(p,t)dt - n\min_{|z|=K}|p(z)|\int_r^R \frac{1}{K+t}dt.$$

By Lemma 3.3.11, we obtain

$$M(p,R) \leq M(p,r) + \int_r^R \frac{n}{K+t}\Big(\frac{1+t}{1+r}\Big)^n M(p,r)dt - n\min_{|z|=K}|p(z)|\int_r^R \frac{1}{K+t}dt$$

$$\Rightarrow \qquad M(p,R) \leq M(p,r) + \frac{nM(p,r)}{(1+r)^n}\int_r^R \frac{(1+t)^n}{K+t}dt - n\min_{|z|=K}|p(z)|\int_r^R \frac{1}{K+t}dt$$

$$\Rightarrow \qquad M(p,R) \leq M(p,r) + \frac{nM(p,r)}{(1+r)^n}\int_r^R \frac{(K+t)^n}{K+t}dt - n\min_{|z|=K}|p(z)|\int_r^R \frac{1}{K+t}dt$$

$$\Rightarrow \qquad M(p,R) \leq M(p,r) + \frac{nM(p,r)}{(1+r)^n}\Big[(K+R)^n - (K+r)^n\Big]\frac{1}{n} - n\min_{|z|=K}|p(z)|\ln\Big(\frac{K+R}{K+r}\Big)$$

$$\Rightarrow \qquad M(p,r)\left[\frac{(1+r)^n+(K+R)^n-(K+r)^n}{(1+r)^n}\right] \geq M(p,R)+n\min_{|z|=K}|p(z)|\ln\left(\frac{K+R}{K+r}\right)$$

$$\Rightarrow \qquad M(p,r) \geq \frac{(1+r)^n}{(1+r)^n+(K+R)^n-(K+r)^n}\left[M(p,R)+n\min_{|z|=K}|p(z)|\ln\left(\frac{K+R}{K+r}\right)\right],$$

as required. $\square$

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