I-weight, Special Base Properties and Related Covering Properties

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Vita

Brad Bailey was born in Savannah, Georgia in 1977. He entered Armstrong Atlantic State University in 1995, where he met his future wife, Elaine LeCreurer. Each graduating from Armstrong in May of 2000, Brad and Elaine were married that same month and began graduate studies at Auburn University that fall. Brad received his Masters degree in 2002 and his PhD in 2005, both at Auburn and in the field of topology. At the time this is being written, Brad and Elaine are happily anticipating the birth of their child, who is expected to arrive in the summer of 2006.

Dissertation Abstract

I-weight, Special Base Properties and Related Covering Properties

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Alleche, Arhangel'skiĭ and Calbrix defined the notion of a sharp base and posed the question: Is there a regular space with a sharp base whose product with [0, 1] does not have a sharp base? Chapter 2 contains an example of a space P with a sharp base whose product with [0, 1] does not have a sharp base. The example in Chapter 2 also answers the following 3 questions found in the literature: Is every pseudocompact Tychonoff space with a sharp base metrizable? Is there a pseudocompact space X with a G_δ-diagonal and a point-countable base such that X is not developable? Is every Čech-complete pseudocompact space with a point-countable base metrizable? The space we construct is pseudocompact, Čech-complete, has a G_δ-diagonal, a sharp base and a point-countable base, but is not metrizable nor developable.

In Chapter 3, we study open-in-finite (OIF) bases and introduce the notion of a δ-open-in-finite (δ-OIF) base. Each δ-OIF base is also OIF. We show that a base $\mathcal{B}$ for the space X is δ-OIF if and only if for each dense subset Y of X, $\mathcal{B} \restriction Y$ is OIF. We also define OIF-metacompact, δ-OIF-metacompact, (n, κ)-metacompact, and $(< \omega, \kappa)$-metacompact and show that for generalized order spaces and $\kappa = \omega$ these properties

are equivalent. The $(<\omega,\omega)$-metacompact property is corresponds to the $<\omega$-weakly uniform base property. We show that for Moore spaces X, the space X has an OIF base (resp. δ-OIF base, $<\omega$-weakly uniform base) if and only if the space is OIF-metacompact (resp. δ-OIF-metacompact, $(<\omega,\omega)$-metacompact).

In the final chapter, we prove that for the class of linearly ordered compact spaces, i-weight reflects all cardinals. We find necessary and sufficient conditions for i-weight to reflect cardinal κ in the class of locally compact linearly ordered spaces. In the last section we calculate the i-weight of paracompact spaces in terms of the local i-weight and extent of the space. This result determines that for compact spaces i-weight and local i-weight are the same.

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I would like to thank my family for their ongoing encouragement and my wife, Elaine for being so supportive through the years.

Style manual or journal used Transactions of the American Mathematical Society (together with the style known as “auphd”). Bibliograpy follows van Leunen’s *A Handbook for Scholars.*

Computer software used The document preparation package TEX (specifically LATEX) together with the departmental style-file `auphd.sty`.

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Chapter 1

Introduction and Background

1.1 Introduction

This dissertation focuses on three different research topics. In Chapter 2 we construct a space P that answers several questions found in the literature. The space P has what is known as a sharp base, yet it's product with the space [0, 1] does not have a sharp base.

Chapter 3 is about open-in-finite (OIF) bases and δ-open-in-finite (δ-OIF) bases. In [4] OIF bases are introduced and the question is posed: Must each dense subspace of a space with an OIF base have an OIF base? That question is still open. However, we show that a base $\mathcal{B}$ for a space X is a δ-OIF base if and only if for each dense subset Y of X, the base $\mathcal{B}\restriction Y$ is an OIF base.

We demonstrate that for OIF spaces the cardinal functions weight and π-weight coincide, and that the same is true of character and π-character. Further, we prove that if X is OIF, then for each dense subset Y of X, then the weights $X$and Y are the same. Several covering properties related to OIF, δ-OIF and $< \omega$-weakly uniform bases are discussed in the final sections of Chapter 3. We prove that for the class of Moore spaces these covering properties are equivalent to the corresponding base properties. We also show that for generalized order spaces (GO spaces), these properties are all equivalent to each other and to paracompactness.

The last chapter contains reflection theorems for i-weight and formulas for the i-weight of linearly ordered compact and locally compact spaces, and for paracompact spaces. We show that i-weight reflects for linearly ordered compact. Then we find conditions under which i-weight reflects for linearly ordered locally compact spaces. We define local i-weight, denoted $liw(X)$, and the last section provides that for a paracompact space X, the i-weight of X is $\max\{log(e(X)), liw(X)\}$.

1.2 Definitions and Background Results

This section contains definitions from general topology and set theory that are used throughout this work. All the material in this chapter may be found in one of [7], [12] or [14].

First we define several cardinal functions.

Definition. The weight of space X, denoted $w(X)$ is the minimum cardinality plus ω of a base for the topology on X.

Definition. A *π-base* for X is a collection $\mathcal{V}$ of nonempty open sets in X so that if U is any nonempty open set in X, then $V \subseteq U$ for some $V \in \mathcal{V}$. The π-weight of X, denoted $\pi w(X)$, is the minimum cardinality plus ω of a π-base for X.

Definition. A *local base at x* is a collection $\mathcal{V}$ of open sets each containing x, so that if U is an open set containing x, $V \subseteq U$ for some $V \in \mathcal{V}$. The *character of X at x*, denoted $\chi(x, X)$, is the minimum cardinality of a local base at x. Then $\chi(X) = \sup\{\chi(x, X) : x \in X\} + \omega$.

Definition. A *local π-base at x* is a collection $\mathcal{V}$ of open sets, so that if U is an open set containing x, then $V \subseteq U$ for some $V \in \mathcal{V}$. Then π-character at x , denoted $\pi\chi(x, X)$, is the minimum cardinality of a local π-base at x, and $\pi\chi(X) = \sup\{\pi\chi(x, X) : x \in X\}$.

We call a cardinal function φ *monotone* if $\varphi(Y) \leq \varphi(X)$ for each subspace Y of X.

Theorem 1.1 *Weight is monotone.*

Proof. For a space X let $\mathcal{B}$ be a base for X so that $|\mathcal{B}| = w(X)$ and let Y be any subspace of X. Then $\mathcal{B}\upharpoonright Y = \{B \cap Y : B \in \mathcal{B}\}$ is a base for Y and $|\mathcal{B}\upharpoonright Y| \leq |\mathcal{B}|$. $\square$

Definition. A subset C of a space X is called *discrete* if for each $c \in C$ there is an open set U_c so that $U_c \cap C = \{c\}$. We define the *extent of* X, denoted $e(X)$ by $e(X) = \sup\{|C| : C \text{ is closed and discrete subset of } X\} + \omega$.

Definition. Let A be a set and λ be a cardinal less than $\lambda \leq |A|$ Then define $[A]^{\lambda} = \{C \in \mathcal{P}(A) : |C| = \lambda\}$, $[A]^{\leq\lambda} = \{C \in \mathcal{P}(A) : |C| \leq \lambda\}$ and $[A]^{<\lambda} = \{C \in \mathcal{P}(A) : |C| < \lambda\}$.

Definition. A *tree* is a partial order , $\langle T, \leq \rangle$, so that for each $x \in T$, $\{y \in T : y < x\}$ is well ordered by $<$. If $x \in T$, then we call x a *node* and the *height of x in T*, denoted $ht(x,T)$, is $\mathtt{type}(\{y \in T : y < x\})$. For each ordinal α, the α^{th} level is $\{x \in T : ht(x,T) = \alpha\}$. The *height of T* is the least α so that α^{th} level is empty. A *chain* in T is a set $C \subseteq T$ which is totally ordered by $<$, and an *antichain* is a set $A \subseteq T$, such that $\forall x, y \in A(x{\neq}y \to (x \not\leq y \wedge y \not\leq x))$. A *branch* of a tree is a maximal chain. If a node t is contained in a branch b, we write $t \in b$. If a node t is of height α, then we say that t is *β-branching*, where β is a cardinal, if $|\{x \in T : ht(x,T) = \alpha + 1 \text{ and } t < x\}| = \beta$.

There are two specific types of trees that we discuss.

Definition. For any regular κ, a *κ-Aronszajn tree* is a tree such that every chain and level is of cardinality $< \kappa$. A *κ-Suslin tree* is a tree T so that $|T| = \kappa$ and every chain and antichain of T has cardinality $< \kappa$.

Let T be a tree so that the 0^{th} level has cardinality 1. If each node is either 0 or 2-branching, we can identify the nodes t of T with functions $\sigma_t : \alpha \to 2$ where $\alpha = ht(t,T)$, as we describe. Let the single node at the 0^{th} level correspond to the empty sequence and the two nodes of level 1 correspond to (0) and (1). Above those are $(0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$, et cetera. So a node t at the α^{th} level corresponds to an element σ_t of ${}^{\alpha}2$. For each node $t \in T$ that is 2-branching, let $\sigma_t^\frown(0)$ and $\sigma_t^\frown(1)$ correspond to the two nodes above t in the tree ordering. Similar to nodes, a branch b corresponds to a sequence from ${}^{ht(b)}2$. We write $b(\alpha) = 0$ (respectively 1) if the node of b on level α corresponds to a sequence from ${}^{\alpha}2$ ending with 0 (respectively 1).

Let $X(T)$ denote the set of branches of T. We are able to define two different topologies on $X(T)$. The first topology is a linear order topology. For branches $c, d \in X(T)$ so that $c \neq d$, there is a minimal level n so that $c(n) \neq d(n)$. We write $c < d$ if and only if $c(n) = 0$ and $d(n) = 1$. Then $<$ orders the points of $X(T)$.

For the second topology, let $t \in T$ and define $[t] = \{b \in X(T) : t \in b\}$. Then $\{[t] : t \in T\}$ defines a clopen base for a topology on $X(T)$.

These topologies are used on $X(T)$ in different sections of this work. We rely on the different notation for the basic open sets to make it clear to the reader which topology is under discussion.

Chapter 2

An Example of a Space with a Sharp Base

2.1 Introduction

Definition. A *sharp base* is a base $\mathcal{B}$ such that whenever $(B_i)_{i<\omega}$ is an injective sequence from $\mathcal{B}$ with $x \in \bigcap_{i<\omega} B_i$, then $\{\bigcap_{i\leq n} B_i : n < \omega\}$ is a base at x.

In [1], Alleche, Arhangel'skiĭ and Calbrix introduced and studied sharp bases and asked if there is a regular space with a sharp base whose product with [0, 1] does not have a sharp base. Good, Knight and Mohamad [8] claimed to have a Tychonoff counterexample, but it turns out that their space is not regular. It is not regular because they added a closed discrete set L to the Baire metric space ${}^{\omega}\mathfrak{c}$, in such a way to make the new space P pseudocompact. Such P cannot be regular: for if it is, one may find a neighborhood of $p \in {}^{\omega}\mathfrak{c}$ whose closure misses L. That neighborhood can be assumed to come from a clopen basis for ${}^{\omega}\mathfrak{c}$, and would then be homeomorphic to ${}^{\omega}\mathfrak{c}$ and be pseudocompact, a contradiction.

In this chapter we give a modification of the Good, Knight and Mohamad space which makes the space Tychonoff; instead of taking a union of a closed discrete set with ${}^{\omega}\mathfrak{c}$, we add a σ-discrete set to ${}^{\omega}\mathfrak{c}$ so that the added set is dense in the union. The space we construct is pseudocompact but not compact, hence not metrizable; we also show it is not developable. Our space has no isolated points and a sharp base, and for T_1 spaces a sharp base is always weakly uniform. Since Heath and Lindgren show that a T_2 space with a weakly uniform base has a G_δ-diagonal [11], our space has one also. In [3], it is shown that a pseudocompact space with a G_δ-diagonal is Čech-complete, and that if a space with not more than ω_1 isolated points has a sharp base, then it has a point countable base. Therefore, the space we construct also answers these three other questions in the negative:

Is every pseudocompact Tychonoff space with a sharp base metrizable? [3]

Is every pseudocompact space X with a G_δ-diagonal and with a point-countable base developable? [2]

Is every Čech-complete pseudocompact space with a point-countable base metrizable? [2]

2.2 The Construction of Space P

Let $B = {}^{\omega}\mathfrak{c}$ and for $\sigma \in {}^{<\omega}\mathfrak{c}$ define $[\sigma] = \{g \in B : \sigma \subseteq g\}$. We also denote $\sigma \in {}^{n+1}\mathfrak{c}$ by $(\alpha_0, \alpha_1, \cdots, \alpha_n)$, where $\sigma(i) = \alpha_i$. By $\sigma_1 \perp \sigma_2$ we mean that σ_1 and σ_2 are incompatible (i.e. the two finite partial functions disagree at a point in both domains).

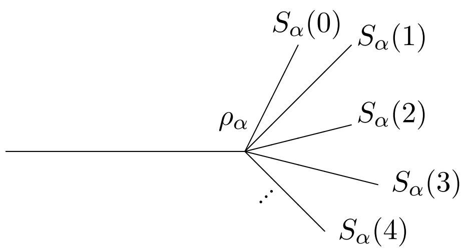

Figure 2.1: A typical S_α with root ρ_α.

Define $\mathcal{S}$ to be the collection of elements of ${}^{\omega}({}^{<\omega}\mathfrak{c})$ subject to these two conditions:

1. For all $S \in \mathcal{S}$ there exists a $k_s < \omega$ and a $\rho_s \in {}^{<\omega}\mathfrak{c}$ such that whenever $\sigma \in S$, $\sigma \restriction k_s = \rho_s$. This ρ_s will be called the *root* of S.

2. Whenever σ_1 and σ_2 are distinct elements of S, $\sigma_1(k_s) \neq \sigma_2(k_s)$.

Let $\mathcal{S} = \{S_\alpha : \alpha < \mathfrak{c}\}$, and let the root of S_α be ρ_α. Define $T_\alpha \in {}^{\omega}({}^{<\omega}\mathfrak{c})$ so that $\mathcal{T} = \{T_\alpha : \alpha < \mathfrak{c}\}$ has these three properties:

(i) for $i \neq j$, $T_\alpha(i) \perp T_\alpha(j)$

(ii) if $\beta, \alpha < \mathfrak{c}$, $\beta \neq \alpha$, with T_α and T_β defined, then $\mathtt{ran}T_\beta \cap \mathtt{ran}T_\alpha = \emptyset$, and

(iii) for $\beta, \alpha < \mathfrak{c}$, $\beta \neq \alpha$, T_α and T_β defined, if $T_\alpha(i) \supseteq T_\beta(j)$, then whenever $j' \neq j$, $T_\alpha(i') \perp T_\beta(j')$ for all $i' < \omega$.

Assume for $\alpha < \gamma$ we have either constructed a $T_\alpha \in {}^{\omega}({}^{<\omega}\mathfrak{c})$ subject to the conditions above or we have not constructed a T_α at all. Now we define T_γ. Choose a $\delta \in \mathfrak{c}$ not in $\bigcup\{\mathtt{ran}T_\alpha(j) : \alpha < \gamma, j \in \omega\}$. Then for each $i \in \omega$ let $S'_\gamma(i) = S_\gamma(i)^\frown(\delta)$. The sequence $(T_\gamma(i))_{i<\omega}$ will be a subsequence of $(S'_\gamma(i))_{i<\omega}$, so the fact that no previous T_α contains a finite partial function with δ in the range will yield property (ii) for T_γ. In addition, the fact that the elements of S'_γ are pairwise incompatible will make the elements of T_γ also incompatible, satisfying property (i). So we need only concern ourselves with property (iii).

Case 1. Suppose there exists some $\alpha < \gamma$ for which T_α was defined, such that for infinitely many j there is some $i \in \omega$ with $S_\gamma(i) \supseteq T_\alpha(j)$. If this is the case, do not define T_γ.

Case 2. If for each $\alpha < \gamma$ there are at most finitely many j for which $S_\gamma(i) \supseteq T_\alpha(j)$ for some i, we will define a T_γ.

Suppose that for $i \leq k$ we have already selected a sequence of natural numbers $0 = n_0 < n_1 < \cdots < n_k$ and defined $T_\gamma(i) = S'_\gamma(n_i)$.

There are at most finitely many different finite partial functions f such that $f \subseteq T_\gamma(i)$ for some $i \leq k$. The second induction condition implies that there are at most finitely many $\alpha < \gamma$ with such an f in the range of T_α. List these as $\alpha(0), \ldots, \alpha(m)$. We have assumed that for each $\alpha < \gamma$, there are at most finitely many j for which $S'_\gamma(i)$ extends $T_\alpha(j)$ for some i. Using this fact, we see that for each $\alpha(p)$ there is a j_p such that for all $j \geq j_p$, $S'_\gamma(i)$ does not extend any $T_{\alpha(p)}(j)$. Then define $n_{k+1} = \max\{j_p : p \leq m\}$ and $T_\gamma(k+1) = S'_\gamma(n_{k+1})$. To check property (iii), suppose that $\beta < \gamma$ and $T_\gamma(k) \supseteq T_\beta(j)$ for some $j, k < \omega$. Assume that k is the least possible for which there exists such a j. Then $\beta = \alpha(p)$ for some $p \leq m$ in the above construction. Since $n_{k+1}, n_{k+2}, \ldots$ are all greater than j_p, $T_\gamma(i)$ cannot extend $T_\beta(j')$ for any $j' \neq j$ and any i, so we have property (iii). Note that $T_\beta(j)$ could not extend $T_\gamma(i)$ because $\delta \in \mathtt{ran}T_\gamma(i) \backslash \mathtt{ran}T_\beta(j)$. Indeed, from this and from (iii) with conditions 1 and 2 of $S \in \mathcal{S}$ it is easy to see that we have the following.

(iv) If $\rho_\alpha = \rho_\beta$, then $T_\alpha(j)$ and $T_\beta(i)$ are compatible for at most one pair (i, j) in $\omega \times \omega$.

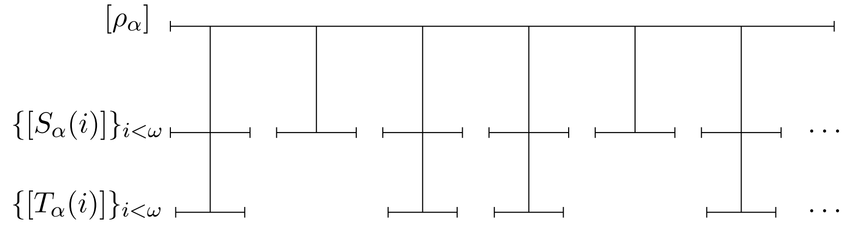

Figure 2.2: An open set $[\rho_\alpha]$ and the sequence of disjoint open sets $\{[T_\alpha(i)]\}_{i<\omega}$.

Choose L disjoint from B such that $L = \{s_\alpha : T_\alpha \text{ is defined}\}$. Let *the root of* s_α refer to ρ_α. Let $P = B \cup L$.

For $\sigma \in {}^{<\omega}\mathfrak{c}$, let $B(\sigma) = [\sigma] \cup \{s_\beta : \rho_\beta \supseteq \sigma\}$ and let $B_n(s_\alpha) = \{s_\alpha\} \cup \bigcup_{m \geq n} ([T_\alpha(m)] \cup \{s_\beta : \rho_\beta \supseteq T_\alpha(m)\})$. These will be the basic open sets for P, and call the collection of them $\mathcal{B}$.

2.3 Verifying Properties of P

First, we will observe some properties of $\mathcal{B}$.

Theorem 2.1 *The collection $\mathcal{B}$ has the following properties.*

(a) For $\sigma_1, \sigma_2 \in {}^{<\omega}\mathfrak{c}$, $\sigma_1 \perp \sigma_2$ iff $B(\sigma_1) \cap B(\sigma_2) = \emptyset$ and if $\rho_\alpha \perp \rho_\beta$ then $B_n(s_\alpha) \cap B_m(s_\beta) = \emptyset$.

(b) $\sigma_1 \subseteq \sigma_2$ iff $B(\sigma_2) \subseteq B(\sigma_1)$, and if $\rho_\alpha \supseteq \sigma$, then for each $n < \omega$, $B_n(s_\alpha) \subseteq B(\sigma)$.

(c) Suppose $B(\sigma) \cap B_n(s_\alpha) \neq \emptyset$. Then $\sigma \subseteq \rho_\alpha$ or $\rho_\alpha \subseteq \sigma$. If $\sigma \subseteq \rho_\alpha$ then $B(\sigma) \cap B_n(s_\alpha) = B_n(s_\alpha)$. If $\sigma \supsetneq \rho_\alpha$, then the intersection is either $B(\sigma)$ or $B(T_\alpha(m))$ for some $m \geq n$. Finally, if $B(\sigma) \subseteq B_n(s_\alpha)$ then for some $m \geq n$ we have $B(\sigma) \subseteq B(T_\alpha(m))$.

(d) If $B_n(s_\alpha) \cap B_{n'}(s_{\alpha'}) \neq \emptyset$ and $\rho_{\alpha'} \subseteq \rho_\alpha$, then the intersection is either $B_n(s_\alpha)$ or a set of form $B(\sigma)$, for some $\sigma \in \{T_\alpha(m), T_{\alpha'}(m') : m \geq n, m' \geq n'\}$. In particular, the latter holds if $\rho_{\alpha'} = \rho_\alpha$.

Proof. *(a)* Suppose that $\sigma_1 \perp \sigma_2$; then there is no point of B nor any finite partial function that could extend both σ_1 and σ_2. If $s_\alpha \in L$ is in $B(\sigma_1) \cap B(\sigma_2)$ then ρ_α

extends both, contradiction. Suppose for the reverse, that $B(\sigma_1) \cap B(\sigma_2) = \emptyset$; then since $[\sigma_1] \cap [\sigma_2]$ is contained in this set, it is clear that $\sigma_1 \perp \sigma_2$.

Now if the roots of s_α and s_β are incompatible then each pair of extensions of the roots will be incompatible, hence $B(T_\alpha(n')) \cap B(T_\beta(m')) = \emptyset$ for each $n' \geq n$ and $m' \geq m$. Further, $s_\alpha \in B_m(s_\beta)$ implies that ρ_α extends ρ_β, which has been assumed to be not the case. So $B_n(s_\alpha) \cap B_m(s_\beta) = \emptyset$.

(b) Clear from the definition of $B(\sigma)$ and s_α.

(c) Suppose that $B(\sigma) \cap B_n(s_\alpha) \neq \emptyset$. Since $B_n(s_\alpha) \subseteq B(\rho_\alpha)$ we have $\sigma \not\perp \rho_\alpha$, by (a). If $\sigma \subseteq \rho_\alpha$, then for each $m \geq n$, $\sigma \subseteq T_\alpha(m)$ and $s_\alpha \in B(\sigma)$, so $B_n(s_\alpha) \subseteq B(\sigma)$.

Suppose $\sigma \not\subseteq \rho_\alpha$; then for some $m \geq n$, $B(\sigma) \cap B(T_\alpha(m)) \neq \emptyset$, while property (i) of $\mathcal{T}$ implies that $B(\sigma) \cap B(T_\alpha(k)) = \emptyset$ for $k \neq m$. By (a) and (b), one of $B(\sigma)$ and $B(T_\alpha(m))$ is contained in the other, and the intersection is simply the contained set. This implies the last sentence of (c).

(d) Suppose $B_n(s_\alpha) \cap B_{n'}(s_{\alpha'}) \neq \emptyset$, where $s_\alpha \neq s_{\alpha'}$. If $\rho_{\alpha'} \subsetneq \rho_\alpha$ then $s_{\alpha'} \notin B_n(s_\alpha)$ and $[T_{\alpha'}(j)] \cap [\rho_\alpha] \neq \emptyset$ for at most one $j \in \omega$. Therefore, $B_n(s_\alpha) \cap B_{n'}(s_{\alpha'}) = B(T_{\alpha'}(j)) \cap B_n(s_\alpha)$ for some $j \geq n'$. Now the rest follows from (c).

If $\rho_\alpha = \rho_{\alpha'}$, then the conclusion follows from condition (iv). $\square$

Proposition 2.2 *The base $\mathcal{B}$ is a clopen base for P.*

Proof. Notice that the properties show immediately that $\mathcal{B}$ is a base. To see that $B_n(s_\alpha)$ is closed consider $s_\gamma \in L \setminus B_n(s_\alpha)$. Suppose that $B_j(s_\gamma)$ meets $B_n(s_\alpha)$, where j is sufficiently large that $s_\alpha \notin B_j(s_\gamma)$. Then by c) the intersection is one of $B(T_\alpha(n'))$ for some $n' \geq n$, $B(T_\gamma(j'))$ for some $j' \geq j$, $B_j(s_\gamma)$ or $B_n(s_\alpha)$.

Since $s_\gamma \notin B_n(s_\alpha)$ and $s_\alpha \notin B_j(s_\gamma)$, we know that the intersection cannot be $B_j(s_\gamma)$ or $B_n(s_\alpha)$. If the intersection is $B(T_\gamma(j'))$ then $B_{j'+1}(s_\gamma)$ misses $B_n(s_\alpha)$. So the intersection has to be some $B(T_\alpha(n'))$. We have that $\rho_\gamma \not\supseteq T_\alpha(n')$, so there exist $j'' < \omega$ so that for $i > j''$ we have $T_\gamma(i) \not\supseteq T_\alpha(n')$. So $B(T_\alpha(n')) \subseteq B(T_\gamma(j'))$ or $B_n(s_\alpha) \subseteq B_j(s_\gamma)$. So then $B_{j''+1}(s_\gamma)$ misses $B_n(s_\alpha)$.

To see that each limit point of $B_n(s_\alpha)$ in B is in $B_n(s_\alpha)$, suppose that p is such a limit point of $B_n(s_\alpha)$ not in $B_n(s_\alpha)$. Choose $k < \omega$ so that $p{\restriction}k \not\subseteq \rho_\alpha$. Then by property (d), $B(p{\restriction}k) \cap B_n(s_\alpha) = B(T_\alpha(m))$ for some $m \geq n$. Then for $k' < \omega$ so that $k' > |T_\alpha(m)|$ we have $B(p{\restriction}k') \cap B_n(s_\alpha) = \emptyset$.

Lastly, we observe that $B(\sigma)$ is clopen. Since B is dense and the subspace base is clopen, we only need to turn our attention to limit points of $B(\sigma)$ in L. Suppose then that $s_\alpha \in L$ is a limit point of $B(\sigma)$, not in $B(\sigma)$; then for all $n < \omega$, $B_n(s_\alpha)$ meets $B(\sigma)$. If $\rho_\alpha \perp \sigma$, then clearly $B(\sigma) \cap B_n(s_\alpha) = \emptyset$. If $\rho_\alpha \supseteq \sigma$, then s_α is in $B(\sigma)$ which is contrary to our assumptions. So assume that $\sigma \supsetneq \rho_a$, then there is at most one $T_\alpha(m)$ that extends σ or is extended by σ. Then $B_{m+1}(s_\alpha) \cap B(\sigma) = \emptyset$. $\square$

Proposition 2.3 *The base $\mathcal{B}$ is sharp.*

Proof. Let the injective sequence $(B(\sigma_i))_{i<\omega}$ come from $\mathcal{B}$. If $p \in B$ is contained in every $B(\sigma_i)$ then $p \supseteq \sigma_i$, so since $|\sigma_i|$ must be unbounded, it is clear that $\{\bigcap_{i \leq n} B(\sigma_i) : n < \omega\}$ is a base at p. If s_α is in every $B(\sigma_i)$, then ρ_α extends every σ_i, but since $|\rho_\alpha|$ is finite, this is not possible.

Now consider an injective sequence $(B_{n_i}(s_{\alpha_i}))_{i<\omega}$, with nonempty intersection. If there is an infinite subset J of ω such that the ρ_{α_i}, $i \in J$, are distinct, then it is easy to see that $\{B(\rho_{\alpha_i}) : i \in J\}$ is a base for a unique point $p \in B$. Hence, so is $\{\bigcap_{i \leq j} B_{n_i}(s_{\alpha_i}) : j < \omega\}$, since for each $i \in J$ we have $B_{n_i}(s_{\alpha_i}) \subseteq B(\rho_{\alpha_i})$.

Next, suppose that $s_{\alpha_i} = s_\alpha$ for all i in an infinite subset J of ω. Then $\{B_{n_i}(s_{\alpha_i}) : i < \omega\}$ is a base at s_α, therefore $\{\bigcap_{i \leq j} B_{n_i}(s_{\alpha_i}) : j < \omega\}$ is a base at s_α too.

The final case, without loss of generality, is when the s_{α_i}'s are distinct, but $\rho_{\alpha_i} = \rho$ for all $i < \omega$. Then by (d), pairwise intersections have the form $B(\sigma)$ for some σ in the range of the corresponding pair from $\mathcal{T}$. By property (ii) of $\mathcal{T}$, $\{B_{n_i}(s_{\alpha_i}) \cap B_{n_{i+1}}(s_{\alpha_{i+1}}) : i \text{ is even}, i < \omega\}$ consists of distinct $B(\sigma)$'s. Therefore, this must be a base at some $p \in B$, and $\{\bigcap_{i \leq j} B_{n_i}(s_{\alpha_i}) : j < \omega\}$ is as well. $\square$

Proposition 2.4 *The space P is not compact.*

Proof. Consider $C_0 = \{s_\alpha \in L : \rho_\alpha = \emptyset\}$. Note that $P \setminus C_0 = \bigcup_{\alpha < \mathfrak{c}} B((\alpha))$. We intend to show that the closed set C_0 is infinite and discrete. To see that this is a discrete set, notice that for $s_\alpha \in C_0$, the set $B_1(s_\alpha) \cap C_0$ can only contain s_α. Examine $\{(\alpha_i^\gamma)_{i<\omega} : \gamma < \mathfrak{c}\}$, where $\alpha_i^\gamma = \alpha_{i'}^{\gamma'}$ iff both $i = i'$ and $\gamma = \gamma'$. Call this collection $\mathcal{S}_0$; then this is a subset of $\mathcal{S}$. Note, that for each $S_\alpha \in \mathcal{S}_0$ and $i < \omega$, we have that the length of $S_\alpha(i)$ is exactly one. Each $T_\gamma(j)$ is constructed to have length at least 2. Therefore, during the induction that defined $\mathcal{T}$, for each $S_\alpha \in \mathcal{S}_0$, Case 1 does not hold. Hence, a corresponding T_α is constructed for each $S_\alpha \in \mathcal{S}_0$. This implies that $\{s_\alpha \in L : \rho_\alpha = \emptyset\}$ has size at least $\mathfrak{c}$. $\square$

A space is called perfect if every open set is F_σ. A development for a space X is a sequence $(\mathcal{G}_i)_{i<\omega}$ of open covers so that for each point $x \in X$ the set $\{G \in \mathcal{G}_i : x \in G, i < \omega\}$ is a local base for x. A space with such a development is called developable and each regular developable space is perfect.

Proposition 2.5 *The space P is not perfect, hence not developable.*

Proof. Let $U = P \setminus C_0$. We show that U is not F_σ, and hence P is not developable. Suppose that $\{F_j\}_{j<\omega}$ is a collection of closed sets so that $\bigcup_{j<\omega} F_j = U$. By the Baire property of B, each $[(\alpha)]$ is Baire. So for all $\alpha < \mathfrak{c}$ there is an n_α and an $[\tilde{\alpha}] = [(\alpha, \beta_1, \cdots, \beta_{n_\alpha})] \subseteq F_{n_\alpha}$. Choose n_0 so that $\{\alpha : [\tilde{\alpha}] \subseteq F_{n_0}\}$ is infinite. Order $\{\alpha_i\}_{i<\omega} \subseteq \{\alpha : [\tilde{\alpha}] \subseteq F_{n_0}\}$, then $S = ((\tilde{\alpha_i}))_{i<\omega} \in \mathcal{S}$, and has the empty set as its root. So an $s \in L$ was defined as a limit point of S, and σ the root of s is also the empty set. Therefore, s is a limit point of a closed set, F_{n_0}, which implies that $s \in \bigcup_{\alpha<\mathfrak{c}} B((\alpha))$. So there is a $\beta < \mathfrak{c}$ so that $\emptyset \supseteq (\beta)$, a contradiction. $\square$

Recall that a Tychonoff space is pseudocompact if every continuous real valued function is bounded.

Proposition 2.6 *The space P is pseudocompact.*

Proof. Suppose that φ is an unbounded continuous real valued function on P. Since B is dense, for each $n \in \omega$ there is an x_n such that $\varphi(x_n) > n$. Let $D = \{x_n : n \in \omega\}$ and let's note that D is closed discrete, hence not compact. If p were a cluster point of D, then every open neighborhood of p contains infinitely many elements of D. This implies that φ increases unboundedly over every neighborhood of p, contradicting the continuity of φ.

Since D is closed and not compact we can find a $k < \omega$ such that $\{x_n \restriction k : x_n \in D\}$ is infinite. Choose the minimum such k. Then there is a $\sigma \in {}^{<\omega}\mathfrak{c}$ and an infinite subset A of ω, such that $x_n \restriction (k-1) = \sigma$ for $n \in A$, and $x_n(k-1)$ is different for these infinitely many $n \in A$.

Let $D^* = \{x_n : n \in A\}$. Since $\varphi(x_n) > n$ by continuity of φ there exists $j_n > k$ so that $\varphi(B(x_n \restriction j_n)) > n$. Then for some $\alpha < \mathfrak{c}$, $\{x_n \restriction j_n : x_n \in D^*\}$ is S_α and $\rho_\alpha = \sigma$. If s_α

was not defined then for some $\beta < \alpha$, $T_\beta(j) \subseteq S_\alpha(n) = x_n \restriction k$ for infinitely many n. Then each basic open neighborhood of s_β contains infinitely many of the sets $B(x_n \restriction k)$. So φ takes on arbitrarily large values over every neighborhood of s_β contradicting continuity. If s_α was defined, then $T_\alpha(i)$ was chosen so that $T_\alpha(i) \supseteq x_{n_i} \restriction j_{n_i}$ for each $i \in \omega$, so $B(T_\alpha(i)) \subseteq B(x_{n_i} \restriction j_{n_i})$. So again, φ takes on large values over every open set containing s_α, contradicting the continuity of φ. $\square$

For a metric space, compact and pseudocompact are equivalent, so P clearly cannot be metrizable.

The following definition can be found in [5].

Definition. An *n-weakly uniform base* $\mathcal{B}$ for the space X is a base so that given any subset A of X, the set $\{B \in \mathcal{B} : A \subseteq B\}$ is finite. A $< \omega$*-weakly uniform base* $\mathcal{B}$ is a base so that given any infinite subset A of X there is a finite subset F of A so that $\{B \in \mathcal{B} : F \subseteq B\}$ is finite.

The notion of a weakly uniform base, which is due to [11], corresponds to a 2-weakly uniform base. For $n < w < \omega$ it is clear that n-weakly uniform base are m-weakly uniform, and that each n-weakly uniform base is $< \omega$-weakly uniform. Also, for any T_1 space a sharp base is a weakly uniform base.

Lemma 2.7 *Let X be a Tychonoff, pseudocompact, non-compact space with no isolated points which partitions into $B \cup L$, and has an n-weakly uniform base $\mathcal{B}$ (resp. $< \omega$-weakly uniform base $\mathcal{B}$). If*

(a) $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ where $\mathcal{B}_1$ is a σ-point finite base for B

(b) for all $x \in L$ there is a local base $\{B_n(x) : n < \omega\}$ so that $n < m$ implies $B_m(x) \subsetneq B_n(x)$ and $\mathcal{B}_2 = \{B_n(x) : n < \omega, x \in L\}$

(c) for $x \neq y \in L$, n, $m \in \omega$, $B_n(x) \neq B_m(y)$.

Then $X \times [0,1]$ does not have an n-weakly uniform base (resp. $< \omega$-weakly uniform base).

Proof. Assume, by way of contradiction, that $\mathcal{W}$ is an n-weakly uniform base for $X \times [0,1]$. Let $\mathcal{C}$ be a countable base for $[0,1]$. For each $x \in L$, choose $W_n^x \in \mathcal{W}$, $B_n^x \in \mathcal{B}$ and $C_n^x \in \mathcal{C}$ so that $(x, \frac{1}{2}) \in B_n^x \times C_n^x \subseteq W_n^x \subseteq B_n(x) \times [0,1]$. Let $\mathcal{B}_C = \{B \in \mathcal{B}$: for some $n \in \omega$ and $x \in L$, $B = B_n^x$ and $C = C_n^x\}$.

We claim that $\mathcal{B}_C$ is point-finite. Suppose not; then there exists an infinite collection $(B_j)_{j<\omega}$ from $\mathcal{B}_C$ that has nonempty intersection. Let $y \in \bigcap_{j \in \omega} B_j$; then there are $x_j \in L$ and $n_j \in \omega$ so that $B_j = B_{n_j}^{x_j}$ and $C = C_{n_j}^{x_j}$. Then $\{y\} \times C \subset \bigcap_{j \in \omega} (B_{n_j}^{x_j} \times C_{n_j}^{x_j}) \subseteq \bigcap_{j \in \omega} W_{n_j}^{x_j}$. If $x_j \neq x_k$ then $B_n(x_j) \neq B_{n'}(x_k)$.

There are two cases to consider.

Case 1. There is an infinite $J \subseteq \omega$ so that $x_j \neq x_k$ whenever $j \neq k$ with $j, k \in J$. Then $\{W_{n_j}^{x_j} : j \in J\}$ is infinite. Suppose not; then some open W is contained in infinitely many different $B_{n_j}(x_j) \times [0,1]$. That $\mathcal{B}$ is n-weakly uniform implies that $\bigcap_{j<\omega} B_{n_j}(x_j)$ is finite. Let it be F, implying $W \subseteq F \times [0,1]$, which is impossible since X has no isolated points. Hence $\{W_{n_j}^{x_j} : j \in J\}$ is infinite, and so $\{y\} \times C \subseteq \bigcap_{j \in \omega} W_{n_j}^{x_j}$ is a finite set, a contradiction.

Case 2. There is an infinite $K \subseteq \omega$ so that $x_j = x_k = x$ for $j, k \in K$. Then the set $\{n_k : k \in K\}$ is infinite, since the $B_{n_k}^{x_k}$ are distinct. Again, $\{y\} \times C \subseteq \bigcap_{k \in K} (B_{n_k}^{x_k} \times C_{n_k}^{x_k}) \subseteq$

$\bigcap_{k\in K} W_{n_k}^{x_k} = \bigcap_{k\in K} W_{n_k}^{x}$. Once again, this must be finite, so we have the same contradiction as in Case 1.

Therefore, $\mathcal{B}_C$ is point finite. Let $\mathcal{B}' = \bigcup_{C\in\mathcal{C}} \mathcal{B}_C$; then $\mathcal{B}_1 \cup \mathcal{B}'$ is a σ-point finite base for X. All pseudocompact spaces with σ-point finite bases are metrizable [17]. However, all metrizable pseudocompact spaces are also compact, contradiction. □

Theorem 2.8 *The product $P \times [0,1]$ does not have a sharp base.*

Proof. We use Lemma 2.7. Let $\mathcal{B}_1 = \bigcup_{n<\omega} \{B(\sigma) : |\sigma| = n\}$ and $\mathcal{B}_2 = \{B_n(s_\alpha) : s_\alpha \in L, n < \omega\}$. Since for T_1 spaces sharp implies weakly uniform, this means that $P \times [0,1]$ has no sharp base. □

Chapter 3

Open-in-finite bases

3.1 Introduction

Definition. A base for a topological space X is *open-in-finite* (OIF) if every nonempty open set is contained in at most finitely many elements of the base. If space X has an OIF base then we say X is an OIF space.

In [4] the base property OIF was introduced and the following question was asked: Is every dense subspace of a regular OIF space OIF? This work came out of an attempt to answer that question. In the next section, we examine the relationship between the cardinal functions of dense subsets of an OIF space and the cardinal functions of the OIF spaces. In section 4 we find properties that a space must have if it is densely contained in an OIF space.

We show that every left or right separated dense subset of an OIF space must be OIF. We also give a necessary and sufficient condition for when dense subsets of OIF spaces are themselves OIF.

In the last section of this chapter we introduce some covering properties that are related to the base properties OIF, δ-OIF and $< \omega$-weakly uniform. We show that for GO spaces these properties are equivalent to each other and to paracompact. We also show that for Moore spaces the covering properties are equivalent to the corresponding base property.

3.2 OIF Spaces and Cardinal Functions

Next, we present some results regarding cardinal functions and OIF spaces. The definitions of these cardinal functions can be found in Chapter 1.

Theorem 3.1 *If X is OIF, then $w(X) = \pi w(X)$ and $\chi(X) = \pi\chi(X)$.*

Proof. For any space X, we know $w(X) \geq \pi w(X)$, because any base is a π-base. Suppose $w(X) > \pi w(X)$. Let $\mathcal{B}$ be any base of X, and let $\mathcal{A}$ be a π-base of size $\pi w(X)$. Then if each $A \in \mathcal{A}$ is in finitely many members of $\mathcal{B}$ then $|\mathcal{B}| = |\mathcal{A}|$. So some $A \in \mathcal{A}$ is in infinitely many members of $\mathcal{B}$ and since $\mathcal{B}$ was arbitrary, X is not OIF. Hence $w(X) = \pi w(X)$ for any OIF space.

Let $p \in X$, where X is an OIF space. Suppose $\mathcal{V}$ is a local base for p where the elements of $\mathcal{V}$ are taken from an OIF base. Then let $\mathcal{U}$ be a local π-base for p of cardinality $\pi\chi(p, X)$. Any local base is a local π-base so $\pi\chi(p, X) \leq \chi(p, X)$. Suppose that $|\mathcal{U}| = \pi\chi(p, X) < \chi(X) = |\mathcal{V}|$. Then each element of $\mathcal{V}$ must contain an element of $\mathcal{U}$. For each $V \in \mathcal{V}$ assign $U_V \in \mathcal{U}$ so that $U_V \subseteq V$. Then some $U \in \mathcal{U}$ is assigned to infinitely many $V \in \mathcal{V}$, which means that $\mathcal{V}$ is not an OIF collection. So for each $p \in X$ we have $\pi\chi(p, X) = \chi(p, X)$. Therefore, $\pi\chi(X) = \chi(X)$. $\square$

Theorem 3.2 *If X is a regular OIF space with OIF dense subspace Y, then $\pi w(Y) = w(Y) = \pi w(X) = w(X)$.*

Proof. Suppose that $\mathcal{A}$ is a π-base of Y. Then let $\mathcal{A}' = \{(\overline{A})^\circ : A \in \mathcal{A}\}$. We show that $\mathcal{A}'$ is a π-base for X. Let U be an open set in X; since Y is dense, $U \cap Y \neq \emptyset$. Let $x \in U \cap Y$; then by regularity there is a V open in X so that $x \in V \subseteq \overline{V} \subseteq U$. Also, $V \cap Y$ is open in Y, so there an $A \in \mathcal{A}$ so that $A \subseteq V$. So $(\overline{A})^\circ \subseteq \overline{V} \subseteq U$. We know that

$(\overline{A})^\circ \neq \emptyset$. So $\mathcal{A}'$ is a π-base of X of the same cardinality as $\mathcal{A}$. Thus $\pi w(X) \leq \pi w(Y)$. We already know that $\pi w(Y) = w(Y) \leq w(X) = \pi w(X)$, so equality holds. $\square$

In Section 3.4 we will see that if X is an OIF space then all the dense subspaces of X have the same weight as X.

Examples. The space $\beta\omega$ is not OIF because the weight of $\beta\omega$ is $\mathfrak{c}$ and the weight of ω is ω. Also, $\beta\mathbb{R}$ is not OIF because the weight of $\mathbb{R}$ is ω while the weight of $\beta\mathbb{R}$ is $\mathfrak{c}$.

Corollary 3.3 *If X is a completely regular space and C is an infinite closed discrete subspace of X such that $2^{|C|} > w(X)$, then βX is not OIF.*

Proof. If C is an infinite closed discrete space, then βC has weight $2^{|C|}$ and $w(\beta C) \leq w(\beta X)$. Then $w(\beta X) \neq w(X)$, so $\beta(X)$ is not OIF. $\square$

This means that if βX has an OIF base and $w(X) < 2^\omega$, then X is countably compact. Recall that βX is only defined for completely regular X, so if X is also second countable, then X is metrizable. For metrizable spaces, countably compact is equivalent to compact. Therefore if X is a completely regular second countable space for which βX is OIF, then X is compact. So if βX and X are distinct OIF spaces, it must be the case that $w(X) > \omega$.

In [4] the authors noted that if X and Y are OIF then $X \times Y$ is OIF. Also, the question was raised : If $X \times X$ is OIF does that imply that X is OIF? In connection with that question we explore the relationship between the cardinal functions of X and $X \times X$.

Proposition 3.4 *If $X \times X$ is OIF, then $\pi w(X) = w(X) = w(X \times X) = \pi w(X \times X)$, and $\pi\chi(X) = \chi(X) = \chi(X \times X) = \pi\chi(X \times X)$*

Proof. Assume $X \times X$ is OIF and let $x \in X$. Then $\pi w(X) \leq w(X) = w(X \times \{x\}) \leq w(X \times X) = \pi w(X \times X)$. If $\mathcal{A}$ is a π-base of X, then $\mathcal{A} \times \mathcal{A}$ is a π-base of $X \times X$, so $\pi w(X \times X) \leq \pi w(X)$. So the claimed equality holds.

Suppose (x_1, x_2) is a point in $X \times X$. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be open neighborhood bases at x_1 and x_2 respectively, with each $\mathcal{A}_i$ having size $\pi\chi(X)$. Then $\mathcal{A}_1 \times \mathcal{A}_2$ is a local base at (x_1, x_2). It follows that $\pi\chi(X \times X) \leq \pi\chi(X)$, while it is already known that $\pi\chi(X) \leq \chi(X) \leq \chi(X \times X) = \pi\chi(X \times X)$. □

This last result is an easy observation.

Proposition 3.5 *Let X be a space with OIF base $\mathcal{B}$ and let the character of X be κ. Then $\mathcal{B}$ is a point-$\leq \kappa$ base.*

Proof. Suppose that x is a point in X contained in more than κ many elements of an OIF base $\mathcal{B}$. There is a local base at x of size cardinality less than or equal to κ. Each of the $\geq \kappa^+$ sets in $\mathcal{B}$ containing x must contain some set from the local base. Therefore some element of the local base is contained in infinitely many elements of $\mathcal{B}$, contradiction. □

Corollary 3.6 *Suppose X is a regular, first countable, countably compact space. If X is OIF, then X is a compact metrizable space.*

3.3 Stronger Base Properties

Here we primarily discuss the property called δ-OIF, but we shall also refer to a generalization of weakly uniform bases. The δ-OIF property is of interest because δ-OIF implies OIF, each example of an OIF space in [4] is also a δ-OIF space, and every dense subspace of a δ-OIF space is δ-OIF.

Definition. A base $\mathcal{B}$ for space X is called a *δ-OIF base* if every infinite intersection from $\mathcal{B}$ is nowhere dense, and X is called a *δ-OIF space* if X has a δ-OIF base.

Proposition 3.7 *If $\mathcal{B}$ is a δ-OIF base then $\mathcal{B}$ is an OIF base.*

Proof. Suppose $\mathcal{B}$ is a base as above. Let $\{B_i : i < \omega\}$ be a subset of $\mathcal{B}$. Since $\bigcap_{i<\omega} B_i$ is nowhere dense, $\bigcap_{i<\omega} B_i$ must have empty interior. So it must be that every open set is contained in at most finitely many members of $\mathcal{B}$. $\square$

Definition. A sequence of sets $\{V_i : i < \omega\}$ is called strongly decreasing if $\overline{V_{i+1}} \subseteq V_i$ for each $i < \omega$.

Lemma 3.8 *Any strongly decreasing sequence $\{V_i : i < \omega\}$ of open sets from an OIF base is a δ-OIF collection.*

Proof. Suppose that $\{V_i : i < \omega\}$ is strongly decreasing and a subset of some OIF base. Then consider $C = \overline{\bigcap_{i<\omega} V_i}$. If $x \in C$, either $x \in \bigcap_{i<\omega} V_i$ or x is a limit point of this set. In the second case, for each $i < \omega$, $x \in \overline{V_{i+1}}$. Therefore, $x \in V_i$ for each $i < \omega$. So $C \subseteq V_i$ for each i. It follows now that C must have empty interior. $\square$

Definition. A collection $\mathcal{U}$ of open sets is called *regular* if for each $U \in \mathcal{U}$ there is a $U' \in \mathcal{U}$ so that $\overline{U'} \subseteq U$.

Example. There is a regular OIF collection of open sets in $\mathbb{R}$ that is not δ-OIF. Let $U_n = \left(2 - \dfrac{1}{n}, 2 + \dfrac{1}{n}\right)$ for each $n \in \mathbb{N}$. Then $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ is a regular OIF collection of open sets in $\mathbb{R}$. Let $A_n = \left\{\frac{k}{3^n} : 1 \leq k < 3^n\right\}$. Then define $U_n^* = (U_n \cup (0,1)) \setminus A_n$.

Then $\left(\bigcap_{n\in\mathbb{N}} U_n^*\right)^\circ = \emptyset$, since it misses the set $\bigcup_{n\in\mathbb{N}} A_n$ which is dense in (0, 1). Therefore $\mathcal{U}^* = \mathcal{U} \cup \{U_n^* : n \in \mathbb{N}\}$ is OIF. Also, $\overline{U_{n+1}} \subseteq U_n \subseteq U_n^*$, so $\mathcal{U}^*$ is a regular collection. However, $\left(\overline{\bigcap_{n\in\mathbb{N}} U_n^*}\right)^\circ = (0,1)$, and so $\mathcal{U}^*$ is not δ-OIF.

Theorem 3.9 *An OIF base $\mathcal{B}$ of X is a δ-OIF base if and only if for every dense subset Y of X, the trace of $\mathcal{B}$ on Y is an OIF base for Y.*

Proof. For the forward direction, let X be a space with a δ-OIF base $\mathcal{B}$. Let Y be a dense subset of X and let $\mathcal{B}\restriction Y = \{B \cap Y : B \in \mathcal{B}\}$. Suppose that $C_i \in \mathcal{B}\restriction Y$ for each $i \in \omega$ and let $G = \bigcap_{i<\omega} C_i$. Suppose $\overline{G}^\circ \neq \emptyset$ in Y. Then $\overline{G}^\circ \neq \emptyset$ in X, and $\overline{G}^\circ \subseteq \overline{\bigcap_{i<\omega} C_i}^\circ$, which contradicts our assumption that $\mathcal{B}$ is δ-OIF. Thus, $\mathcal{B}\restriction Y$ is δ-OIF.

For the reverse direction, let $\mathcal{B}$ be an OIF base so that for each dense subset Y of X, $\mathcal{B}\restriction Y$ is OIF. Suppose that $\mathcal{B}$ is not δ-OIF, and let $\mathcal{B}_n \in \mathcal{B}$ for each $n \in \omega$ so that $W = \overline{\bigcap_{n<\omega} B_n}^\circ \neq \emptyset$.

Consider $Y' = \bigcap_{n<\omega} B_n \cup (X \setminus \overline{\bigcap_{n<\omega} B_n})$. This Y' is dense, for if U is an open set of X, then either $U \cap \bigcap_{n<\omega} B_n$ is empty or it isn't. If it is empty then $U \subseteq (X \setminus \overline{\bigcap_{n<\omega} B_n}) \subseteq Y'$. If the intersection is not empty then U meets Y'. Next we add points to Y' to form Y so that $\{B_n \cap Y : n < \omega\}$ is infinite.

Choose two disjoint open sets in W, say A_1 and A_2. Next for each $\{m, n\} \in [\omega]^2$ choose $x_{m,n} \in (B_m \setminus B_n) \cup (B_n \setminus B_m)$. Then let $P_1 = \{\{m,n\} : x_{m,n} \notin A_1\}$ and $P_2 = [\omega]^2 \setminus P_1$. Ramsey's Theorem states that if we partition $[\omega]^2$ into P_1 and P_2, then there is an infinite $P \subset \omega$ such that $[P]^2$ is contained in one partition. Let $Y = Y' \cup \{x_{m,n} : \{m,n\} \in P\}$. We show that $\{B_n \cap Y : n < \omega\}$ is infinite. Consider $B_n \cap Y$ and $B_m \cap Y$

with $\{m, n\} \in P$. The point $x_{m,n}$ is contained in Y and is contained in only one of B_n and B_m, therefore $B_n \cap Y \neq B_m \cap Y$.

If $P \subseteq P_1$ then let $U = A_1$ and let $U = A_2$ otherwise. Note that $U \subseteq W \setminus \{x_{m,n} : \{m, n\} \in P\}$ and $W \cap (X \setminus \overline{\bigcap_{i<\omega} B_i}) = \emptyset$, and therefore, $U \cap Y \subseteq \bigcap_{i<\omega} B_i \cap Y = \bigcap_{i<\omega} (B_i \cap Y)$, contradiction. □

Corollary 3.10 *If X has a δ-OIF base then every dense subset of X has a δ-OIF base.*

It is known that every metacompact Moore space has a OIF base; it is in fact true that every metacompact Moore space has a δ-OIF base.

Theorem 3.11 *Every metacompact Moore space has a δ-OIF base.*

Proof. Let $\mathcal{G} = (\mathcal{G}_i)_{n<\omega}$ be a development for metacompact Moore space X, so that $\mathcal{G}_{n+1}$ refines $\mathcal{G}_n$, each $\mathcal{G}_n$ is point-finite. Then consider $\{V_i : i < \omega\} \subseteq \bigcup_{i\in\omega} \mathcal{G}$; with $\bigcap_{i<\omega} V_i \neq \emptyset$. If some $\mathcal{G}_n$ contains infinitely many sets from $\{V_i : i < \omega\}$, then because $\mathcal{G}_n$ is point finite, $\bigcap_{i<\omega} V_i = \emptyset$, contradiction. Therefore, we have that $\{n :$ there exists $i < \omega$ so that $V_i \in \mathcal{G}_n\}$ is cofinal in ω. Therefore, $\{V_i : i < \omega\}$ either has empty intersection or contains a base for some point of $\bigcap_{i<\omega} V_i$. This means that $|\bigcap_{i<\omega} V_i| < 2$, and therefore is nowhere dense. □

In [4], the authors show that if a space has an OIF base of regular open sets, then each dense subset of that space is OIF. The OIF property is also known to be productive and the third theorem below shows the same is true for δ-OIF.

Theorem 3.12 *An OIF base of regular open sets is a δ-OIF base.*

Proof. Suppose $\mathcal{B}$ is a regular open OIF base for space X.

Consider $\overline{\bigcap_{i<\omega} B_i}^{\circ} \subseteq \left(\bigcap_{i<\omega} \overline{B_i}\right)^{\circ} \subseteq \bigcap_{i<\omega} \overline{B_i}^{\circ} = \bigcap_{i<\omega} B_i$. So each G_δ set from $\mathcal{B}$ is nowhere dense, hence $\mathcal{B}$ is δ-OIF. □

Theorem 3.13 *Suppose X is a space with an OIF base $\mathcal{B}$. Then X has a δ-OIF base if and only if for all $B \in \mathcal{B}$ there is an collection $\mathcal{C}_B = \{V_i : i \in I_B\} \subseteq \mathcal{B}$ so that*

1. $\overline{V_i}^{\circ} \subseteq B$ *for each* $i \in I_B$,
2. $B = \bigcup_{i \in I_B} V_i$, *and*
3. *for each infinite* $F \subseteq I_B$, *we have* $\overline{\bigcap_F V_i}^{\circ} = \emptyset$.

Proof. Take X, $\mathcal{B}$ and $\mathcal{C}_B$ as above and let $\mathcal{C} = \bigcup_{B \in \mathcal{B}} \mathcal{C}_B$. Then this $\mathcal{C}$ is a base for X. Suppose that $\{O_n : n < \omega\} \subseteq \mathcal{C}$ and consider $\bigcap_{n<\omega} O_n$. For each $n < \omega$, let B_n be an element of $\mathcal{B}$ so that $O_n \in \mathcal{C}_{B_n}$. If $\{B_n : n < \omega\}$ is finite then infinitely many O_n are from the same $\mathcal{C}_B$ and therefore the intersection of them is nowhere dense. If $\{B_n : n < \omega\}$ is infinite, then observe that $\overline{\bigcap_{n<\omega} O_n}^{\circ} \subseteq \left(\bigcap_{n<\omega} \overline{O_n}\right)^{\circ} \subseteq \left(\bigcap_{n<\omega} B_n\right)^{\circ} = \emptyset$. Therefore, $\mathcal{C}$ is δ-OIF.

Now suppose that X has a δ-OIF base $\mathcal{C}$ and let $\mathcal{B}$ be a base for X. Then for each $B \in \mathcal{B}$ let $\mathcal{C}_B$ be any collection from $\mathcal{C}$ of sets contained in B whose union is B. □

Corollary 3.14 *If X is a regular OIF space that is hereditarily metacompact, then X has a δ-OIF base.*

Proof. Let X be such a space and let $\mathcal{B}$ be an OIF base for X. For each $B \in \mathcal{B}$ and each $x \in B$ let $V_{x,B}$ be a set from $\mathcal{B}$ so that $x \in V_{x,B} \subseteq \overline{V_{x,B}} \subseteq B$. Then for each B the collection $\mathcal{V}_B = \{V_{x,B} : x \in B\}$ covers B. Find a point finite open refinement $\mathcal{C}_B$ of each $\mathcal{V}_B$. Then $\mathcal{C}_B$ satisfies the conditions of the previous theorem. $\square$

Theorem 3.15 *If for each $\alpha < \kappa$ the space X_α is δ-OIF, then $\prod_{\alpha<\kappa} X_\alpha$ is δ-OIF.*

Proof. Let $\mathcal{B}_\alpha$ be a δ-OIF base for X_α. For $F \in [\kappa]^{<\omega}$, a basic open set in the product would be $\bigcap_{\alpha\in F} \pi_\alpha^{-1}(A_\alpha)$ where $A_\alpha \in \mathcal{B}_\alpha$. Suppose that the open set $G = \overline{\bigcap_{i<\omega}\left(\bigcap_{\alpha\in F_i} \pi_\alpha^{-1}(A_{\alpha,i})\right)}^{\circ}$ where $\left\{\bigcap_{\alpha\in F_i} \pi_\alpha^{-1}(A_{\alpha,i})\right\}_{i\in\omega}$ is an infinite collection of basic open sets and each $A_{\alpha,i}{\neq}X_\alpha$. Let G' be a basic open set contained in G. Then for all but finitely many α, $\pi_\alpha(G') = X_\alpha$, and therefore $\pi_\alpha(G) = X_\alpha$ for all but finitely many α. Hence, there is a finite set F so that $F_i \subseteq F$ for each $i \in \omega$. To verify this claim, aiming for a contradiction, suppose that $\bigcup_{i<\omega} F_i$ is infinite. Then let $\gamma < \kappa$ so that $\gamma \in \bigcup_{i<\omega} F_i$ and $\pi_\gamma(G) = X_\gamma$. Then let $B_\gamma \in \mathcal{B}_\gamma$ be a set that misses A_γ. Then $\prod_{\alpha<\gamma} X_\alpha \times B_\gamma \times \prod_{\gamma<\alpha<\kappa} X_\alpha$ is an open set that misses $\bigcap_{i<\omega}\left(\bigcap_{\alpha\in F_i} \pi_\alpha^{-1}(A_{\alpha,i})\right)$, contradiction.

There is a $\beta \in \kappa$ so that $\beta \in F_i$ for infinitely many $i < \kappa$ and $\{A_{\beta,i} : \beta \in F_i\}$ is infinite; else $\left\{\bigcap_{\alpha_i\in F_i} \pi_a^{-1}(A_{\alpha_i})\right\}_{i\in\omega}$ is a finite collection of open sets. But notice that $\{A_{\alpha_i} : i \in I\} \subseteq \mathcal{B}_\beta$.

Then we have $\pi_\beta(G) \subseteq \pi_\beta\left(\overline{\bigcap_{i<\omega}\left(\bigcap_{\alpha\in F_i} \pi_\alpha^{-1}(A_{\alpha,i})\right)}^{\circ}\right) \subseteq \left(\overline{\bigcap_{i\in I} A_{\alpha,i}}\right)^{\circ}$. This contradicts that $\mathcal{B}_\beta$ is a δ-OIF base. $\square$

Recall from Chapter 2 that an n-weakly uniform base $\mathcal{B}$ for a space X is a base such that given any subset A of X with $|A| = n$, the collection $\{B \in \mathcal{B} : A \subseteq B\}$ is finite. Call $\mathcal{B} < \omega$-weakly uniform if, given any infinite set A, there is a finite subset $F \subseteq A$ with $\{B \in \mathcal{B} : F \subseteq B\}$ finite.

Theorem 3.16 *Suppose X has a $< \omega$-weakly uniform base that is point finite at every isolated point in X. Then X is δ-OIF.*

Proof. Let $\mathcal{B}$ be a base for X, as above. Suppose $\{B_i : i < \omega\} \subseteq \mathcal{B}$.

Consider $\bigcap_{i<\omega} B_i$. If this intersection is finite, then it is closed and does not contain an open set because any isolated point is in only finitely many elements of the base. Therefore, it is nowhere dense.

If $\bigcap_{i<\omega} B_i$ is infinite, then there exists a finite subset F of $\bigcap_{i<\omega} B_i$ so that $\{B \in \mathcal{B} : F \subseteq B\}$ is finite, contradiction. □

Theorem 3.17 *There is a space that has a δ-OIF base but does not have a $< \omega$-weakly uniform base.*

Proof. The product $P \times [0,\ 1]$, where P is the space from Chapter 2, is such a space. The space $P \times [0, 1]$ does not have a $< \omega$-weakly uniform base by Lemma 2.7, but Theorems 3.16 and 3.15 shows it does have a δ-OIF base. □

Example. If κ is an uncountable cardinal, then $[0,\ 1]^\kappa$ will be δ-OIF but not metrizable.

Definition. A space is *neighborhood OIF at a point x* if there is a local base for x that is an OIF collection. We call a space *neighborhood OIF* if it is neighborhood OIF at each point $x \in X$.

A space X is *neighborhood δ-OIF at point x* if there is a local base for x that is a δ-OIF collection. We call X a *neighborhood δ-OIF* space if it is neighborhood δ-OIF at each point $x \in X$.

Naturally, each δ-OIF space is neighborhood δ-OIF, and each neighborhood δ-OIF space is neighborhood OIF.

Proposition 3.18 *If X is a regular space that is neighborhood OIF at x, then X is neighborhood δ-OIF at x.*

Proof. Let $\mathcal{U}$ be a OIF local base at x, and order $\mathcal{U} = \{U_i : i < \kappa\}$. For each $i < \kappa$, if possible, let U'_i be a member of $\mathcal{U}$ so that $x \in U'_i$, $\overline{U'_i} \subseteq U_i$ and $U'_j \neq U'_i$ for each $j < i$. If for some $i < \kappa$ we are unable to find a U'_i that has not already been assigned to a previous member of $\mathcal{U}$, then do not define U'_i. Then $\mathcal{U}' = \{U'_i : U'_i$ defined and $i < \kappa\}$ is also a local OIF base at x. It is a δ-OIF collection, for $J \in [\kappa]^\omega$ and consider $\overline{\bigcap_{j \in J} U'_j}^{\circ} \subseteq \left(\bigcap_{j \in J} \overline{U'_j}\right)^{\circ} \subseteq \left(\bigcap_{j \in J} U_j\right)^{\circ} = \emptyset$. □

Corollary 3.19 *All regular neighborhood OIF spaces are neighborhood δ-OIF spaces.*

Corollary 3.20 *If Z is a regular neighborhood δ-OIF space, then each dense subspace of Z is a neighborhood δ-OIF space. In fact, each dense subspace of a regular OIF space is neighborhood δ-OIF.*

Proof. By the proof of Theorem 3.9, we know that each δ-OIF collection in Z will have an OIF trace in any dense subspace. By Proposition 3.18, each of these OIF local bases in the dense subset will contain a δ-OIF local base. □

Corollary 3.21 *No non-trivial regular P-space can be embedded in a regular OIF space.*

Proof. Suppose that X is a non-trivial regular P-space, and let $x \in X$ be any non-isolated point. Then let $\mathcal{B}_x$ be any local base for x in X. Let $\{B_i : i < \omega\} \subseteq \mathcal{B}_x$;

then $x \in \bigcap_{i<\omega} B_i$. So $\bigcap_{i<\omega} B_i$ is nonempty, and as a G_δ subset of a P-space it is also open. Therefore $\mathcal{B}_x$ is not an OIF collection. Thus X is not neighborhood OIF at any non-isolated point. $\square$

3.4 The ✠ Property

Recall that the question that motivates this research is: Is every dense subspace of a regular OIF space OIF? While trying to answer this question we considered the possibility of creating a counterexample by embedding a space that is known not to be OIF in a space that is regular and OIF. Naturally, this cannot be done with an arbitrary space. This section contains conditions under which a space cannot be densely embedded in a regular OIF space. In this section we prove that every left (or right) separated space that is dense in an OIF space is OIF, and every space contains left separated dense subspaces. Therefore, if X is a space that is not OIF but can be densely embedded in a regular OIF space, then each dense left separated subspace of X is OIF. Later in this section we examine some of the properties that must be enjoyed in order for space Z and some dense subspace X of Z to both be OIF. This section ends with a necessary and sufficient condition for a dense subset X of an OIF space Z to be OIF.

The following ✠ property is necessary for a space to be a dense subspace of a regular OIF space.

✠: *There exists a base $\mathcal{B}$ so that for each point-selection $p : \mathcal{B} \to X$ with $p(B) \in B$ there is a $g_p : \mathcal{B} \to \mathcal{B}$ so that $p(B) \in g_p(B) \subseteq \overline{g_p(B)} \subseteq B$ and $\mathtt{ran}(g_p)$ is OIF.*

Proposition 3.22 *If X is a dense subset of a regular OIF space then X has the ✠ property.*

Proof. Suppose that X is a dense subset of a regular OIF space, Z. Let $\mathcal{B}$ be an OIF base for Z, then $\mathcal{B}\restriction X$ is a base for X. Let $p : \mathcal{B}\restriction X \to X$ be a point-selection. For each $p(B \cap X)$ find $B' \in \mathcal{B}$ so that $p(B \cap X) \in B' \subseteq \overline{B'} \subseteq B$, where the closure in taken in Z; this is possible since Z is regular. Then define $g_p : \mathcal{B}\restriction X \to \mathcal{B}\restriction X$ by $g_p(B \cap X) = B' \cap X$.

Aiming for a contradiction suppose that $\mathtt{ran}(g_p)$ is not OIF. Then there is a $V \in \mathcal{B}$ and $\{B_i : i < \omega\}$ so that $V \cap X \subseteq g_p(B_i \cap X) = B'_i \cap X$ for each $i < \omega$. Since $\overline{B'_i} \subseteq B_i$, we see that $V \subseteq B_i$ for $i < \omega$. This contradicts the assumption that $\mathcal{B}$ is OIF. $\square$

The ✠ property arose from an examination of the proofs that the following two examples cannot be densely embedded in a regular OIF space.

Example 1. *Tangent-Disk space.* The underlying set is $X = \{(x, y) \in \mathbb{R}^2 : y \geq 0\}$. By $B_n(x, y)$ we denote the Euclidean ball of radius $\dfrac{1}{n}$ centered at (x, y). For each $(x, 0)$ we denote $D_n(x, 0) = \{(x, 0)\} \cup B_n\left(x, \dfrac{1}{n}\right)$. Then $\mathcal{B} = \{B_n(x, y) : y > 0$ and $n \in \mathbb{N}\} \cup \{D_n(x, 0) : n \in \mathbb{N}\}$ is a base for the Tangent-Disk topology.

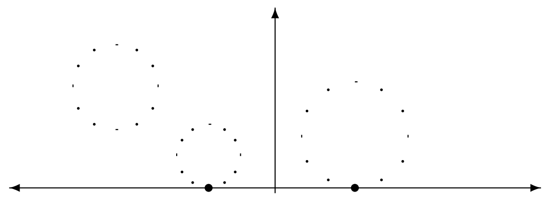

Figure 3.1: The tangent-disk space and some open sets

Let $\mathcal{B}$ be any base for X and define $p : \mathcal{B} \to X$ so that if there exists a unique $(x, 0) \in B$ then $p(B) = (x, 0)$. This is possible since for each $D_n(x, 0)$ there is an element of $\mathcal{B}$ that contains $(x, 0)$ and is contained in $D_n(x, 0)$. For other $B \in \mathcal{B}$, let $p(B)$ be arbitrary. Then for any $g_p : \mathcal{B} \to \mathcal{B}$ so that $p(B) \in g_p(B) \subseteq \overline{g_p(B)} \subseteq B$ we intend to show that $\mathtt{ran}(g_p)$ is not OIF. Since the cardinality of the real line is $\mathfrak{c}$, the range of g_p has cardinality $\mathfrak{c}$. Therefore there is at least one open set from a countable base for $\mathbb{R} \times (0, \infty)$ that is contained in infinitely many members of $\mathtt{ran}(g_p)$.

Proposition 3.23 *If X has ✠ property then X is neighborhood OIF.*

Proof. Suppose that X has the ✠ property. Let $\mathcal{B}$ be the base guaranteed by ✠. We intend to show that X is neighborhood OIF at each point. Pick $x \in X$ and let $\mathcal{B}_x = \{B \in \mathcal{B} : x \in B\}$. Then let $p : \mathcal{B} \to X$ be defined so that $p(B) = x$ for $B \in \mathcal{B}_x$ and let $p(B)$ be arbitrary for $B \notin \mathcal{B}_x$. Then there is a $g_p : \mathcal{B} \to \mathcal{B}$ so that $\mathtt{ran}(g_p)$ is OIF. Therefore, $g_p(\mathcal{B}_x)$ is OIF and a local base at x in X. $\square$

Example 2. *Sequential Fan on ω_1 many sequences.* Suppose X is the space of consisting of a point, ∞, and ω_1 many sequences converging to ∞. The points of each sequence are isolated, and each neighborhood of ∞ is made up of a tail from each sequence. Then ∞ is a point whose local base must always fail to be OIF.

Since all first countable spaces are neighborhood OIF, the Tangent-Disk space described above serves as an example of a space that is neighborhood OIF but not OIF.

We observed that if X has the ✠ property, then X is neighborhood OIF. Furthermore, this implies that each dense subset of a regular OIF space must be neighborhood OIF. For left or right separated dense subsets we can say more.

Proposition 3.24 *Suppose that a regular space X is left or right separated, and has the ✠ property. Then X is OIF.*

Proof. We will assume that X is a regular left separated space, and our proof will work analogously for a right separated space.

Since the space is left separated, there is a well-ordering, say $\prec$, under which each $\{y : y \prec x\}$ is closed. Let $\mathcal{B}$ be the base from ✠ and for each $x \in X$ choose $B_x \in \mathcal{B}$ so that $x \in B_x \subseteq [x, \infty)$. This assignment is one-to-one. For each $x \in X$ let $\mathcal{B}_x = \{B \in \mathcal{B} : x \in B \subseteq B_x\}$. Then we see that $\mathcal{B}_x$ is a local base at x and $\mathcal{B}_x \cap \mathcal{B}_y \neq \emptyset$ if and only if $x = y$. Choose an arbitrary point x_0 and define $p : \mathcal{B} \to X$ by $p(B) = x$ if $B \in \mathcal{B}_x$ and $p(B) = x_0$ otherwise. Then by ✠, there is a $g_p : \mathcal{B} \to \mathcal{B}$ so that $p(B) \in g_p(B) \subseteq \overline{g_p(B)} \subseteq B$ and $\mathtt{ran}(g_p)$ is OIF. Therefore, $\mathtt{ran}(g_p)$ is an OIF base for X. $\square$

The following lemma and its proof are both well known.

Lemma 3.25 *Every space contains a left separated dense subset.*

Proof. The subset is created recursively. At stage α, choose x_α from $X \setminus \overline{\{x_\beta : \beta < \alpha\}}$, if possible. If $X \setminus \overline{\{x_\beta : \beta < \alpha\}} = \emptyset$, then $L = \{x_\beta : \beta < \alpha\}$. This method assures that each $[x, \infty)$ in L is open in L, and that L is dense. Clearly this does terminate at some stage less than or equal to $|X|$. $\square$

Corollary 3.26 *If Z is a regular OIF space, then any dense subspace of Z will have the same weight as Z.*

Proof. Suppose X is a dense subspace of Z. We use Lemma 3.25 to find a left separated dense subspace X' of X, which will also be dense in Z. Then by Proposition 3.24 X' is OIF, and by Theorem 3.1 $w(X') = w(Z)$. Since weight is monotonic, $w(X) = w(Z)$. $\square$

Next, we present a condition that does not depend upon the points of the space, just the open sets.

✠✠: *There is $\mathcal{B}$ and a $g : \mathcal{B} \to \mathcal{B}$ satisfying $\emptyset \neq g(B) \subseteq \overline{g(B)} \subseteq B$ and $\{g(B) : B \in \mathcal{B}\}$ is OIF.*

We note that $\maltese$ implies $\maltese\maltese$. The next example shows that $\maltese\maltese$ does not imply $\maltese$.

Proposition 3.27 *There is a space with the $\maltese\maltese$ property that does not have the $\maltese$ property.*

Proof. Let L denote the set of limit ordinals less than ω_1. Define $X = (\omega_1+1)\setminus L$ with the topology inherited from the order topology. Let $\mathcal{B} = \{[\alpha, \omega_1] : \alpha \in X\} \cup \{\{\alpha\} : \alpha \in X\}$ be a base for X. Define $g : \mathcal{B} \to X$ by $g([\alpha, \omega]) = \{\alpha\}$ and $g(\{\alpha\}) = \{\alpha\}$. Then for each $B \in \mathcal{B}$ we have $\overline{g(B)} \subseteq B$ and $g(\mathcal{B})$ is OIF.

Also, X is not neighborhood OIF at the point ω_1, since any local base for ω_1 in X is would contain ω_1-many different open sets. Therefore for any local base for ω_1 some isolated point is contained in infinitely many different sets from the local base. Since X is not neighborhood OIF, X cannot have the $\maltese$ property. $\square$

Proposition 3.28 *If X is a dense subspace of a regular OIF space then X has the $\maltese\maltese$ property.*

Proof. Since $\maltese$ implies $\maltese\maltese$, this follows from Proposition 3.22 $\square$

Lemma 3.29 *If X is a space with uncountable π-weight and a π-base $\mathcal{A}$ which is an ω_1-Suslin tree under reverse inclusion, then X cannot be densely embedded in a regular OIF space.*

Proof. For contradiction, suppose that X and $\mathcal{A}$ are as above and is dense in an OIF space. Then there is a base $\mathcal{B}$ for X and $g : \mathcal{B} \to \mathcal{B}$ so that $\overline{g(B)} \subseteq B$ and $g(\mathcal{B})$ is an OIF collection. Define $g^* : \mathcal{B} \to \mathcal{A}$ by $g^*(B) \subseteq g(B)$. Because $g(\mathcal{B})$ is OIF, the tree $(g^*(B), \supseteq)$ has height ω and $|g^*(\mathcal{B})| = |g(\mathcal{B})| > \omega$. Therefore, $(g(\mathcal{B}), \supseteq)$ has an uncountable level , which is an uncountable antichain in $(\mathcal{A}, \supseteq)$, contradiction. $\square$

Theorem 3.30 *A Suslin line cannot be densely embedded in a regular OIF space.*

Proof. If X if an arbitrary Suslin line we intend to show that there is a collection of open sets in X that is an ω_1-Suslin tree ordered by reverse inclusion. If X is any Suslin line by Theorem II.4.4 in [14], there is an L which is dense in X, dense in itself and has no separable open subset. To form L, we define equivalence classes in X by letting $x \sim y$ if (x, y) or (y, x) is a separable subset of X. Then L is the set of $\sim$ equivalence classes. Since X is ccc, only countably many equivalence classes are more than just one point. For the countably many non-trivial separable intervals, there is a countable collection of open intervals that is a π-base for each interval. If the rest of X also has a π-base that is an ω_1-Suslin tree under reverse inclusion, then the union of these countably many countable trees with the ω_1-Suslin tree is still an ω_1-Suslin tree. Therefore, we work with the line L.

In [14] Theorem II.5.13 describes the construction of an ω_1-Suslin tree from a Suslin line L which is dense in itself and has no nonempty open subset which is separable. In the construction, the nodes of the tree are open intervals from the line, and the order is reverse inclusion. Kunen let $\mathcal{J}$ denote the collection of all the nonempty open intervals of L. Then for each $\beta < \omega_1$ defined $\mathcal{J}_\beta$ so that for each β,

1. the elements of $\mathcal{J}_\beta$ are pairwise disjoint,
2. $\bigcup \mathcal{J}_\beta$ is dense in L,
3. if $\alpha < \beta$, $I \in \mathcal{J}_\alpha$ and $J \in \mathcal{J}_\beta$ then either,
   a. $I \cap J = \emptyset$, or
   b. $J \subset I$ and $I \setminus \overline{J} \neq \emptyset$.

4. if $\alpha < \beta$ for each $J \in \mathcal{J}_\beta$ there exists $I \in \mathcal{J}_\alpha$ so that $J \subset I$.

These conditions ensure that $\left(\bigcup_{\beta<\omega_1} \mathcal{J}_\beta, \supset\right)$ is an ω_1-Suslin tree. Now we intend to show that $\bigcup_{\beta<\omega_1} \mathcal{J}_\beta$ is a π-base for L. Suppose that (a,b) is a nonempty open interval of L and that no element of $\bigcup_{\beta<\omega_1} \mathcal{J}_\beta$ is contained in (a,b). Then by properties 1 and 2, for each $\beta < \omega_1$ there are c_β, d_β and e_β so that $d_\beta < a < c_\beta < b < e_\beta$, and $(d_\beta, c_\beta), (c_\beta, e_\beta) \in \mathcal{J}_\beta$. Then $\{(d_\beta, c_\beta) : \beta < \omega_1\}$ is chain of cardinality ω_1 in $\left(\bigcup_{\beta<\omega_1} \mathcal{J}_\beta, \supset\right)$, which is not possible. Therefore, for some $\alpha < \omega_1$ there is a $U \in \mathcal{J}_\alpha$ so that $U \subseteq (a,b)$. $\square$

Recall that the space $X(T)$ generated by the tree T has as its points the maximal chains in the tree and in the following theorem the basic open sets are $[\sigma] = \{c : \sigma \in c\}$, where $\sigma \in T$.

Theorem 3.31 *The space generated by an ω_1-Suslin tree cannot be embedded in a regular OIF space.*

Proof. Let S be an ω_1-Suslin tree. Then the base $\mathcal{B} = \{[v] : v \in S\}$ for $X(S)$ is clearly an ω_1-Suslin tree under reverse inclusion. $\square$

Proposition 3.27 gives an example of a regular space that has ✠✠ property and not the ✠ property; hence the space is not OIF. This leaves us with the open question: Is there a regular space that has the ✠ and is not OIF?

3.5 Covering Properties

The base properties that we have studied have natural associations with covering properties, which are just as intertwined. We define OIF metacompact, and (n,κ)-metacompact, determine that they are the same for GO spaces but establish that the properties do not necessarily coincide even if the space is required to be monotonically normal. We also define δ-OIF metacompact and find some conditions for which these covering properties imply the base property.

The following definition can be found in [5].

Definition. A space X is *n-metacompact* if every open cover $\mathcal{U}$ has an open refinement $\mathcal{V}$ so that for each $A \subseteq X$ such that $|A| = n$, then $|\{V \in \mathcal{V} : A \subseteq V\}| < \omega$. A space is *$< \omega$-metacompact* if every open cover $\mathcal{U}$ has open refinement $\mathcal{V}$ so that for each $A \subseteq X$ such that $|A| = \omega$ there is a finite subset B of A so that $|\{V \in \mathcal{V} : B \subseteq V\}| < \omega$.

Thus the well-known property "metacompact" is the same as 1-metacompact. The following definitions allow us to generalize these metacompact properties to cardinals larger than ω.

Definition. Let $n < \omega$ and κ be an infinite cardinal. A space X is *(n, κ)-metacompact* if every open cover $\mathcal{U}$ has an open refinement $\mathcal{V}$ so that for each $A \subseteq X$ such that $|A| = n$, then $|\{V \in \mathcal{V} : A \subseteq V\}| < \kappa$. A space X is *$(< \omega, \kappa)$-metacompact* if every open cover $\mathcal{U}$ has an open refinement $\mathcal{V}$ so that for every infinite set $A \subseteq X$, there is a finite set B so that $|\{V \in \mathcal{V} : B \subseteq V\}| < \kappa$.

Therefore, (n, ω)-metacompact is n-metacompact and $(< \omega, \omega)$-metacompact is $< \omega$-metacompact. Also, $(1, \omega_1)$-metacompact is the same as meta-Lindelöf. Fix κ and let $n < m < \omega$. Clearly, (n, κ)-metacompact implies (m, κ)-metacompact which implies $(< \omega, \kappa)$-metacompact.

Proposition 3.32 *If a space has a $< \omega$-weakly uniform base then it is $< \omega$-metacompact.*

Proof. For any open cover, any refinement consisting of members of the $< \omega$-weakly uniform base will witness $< \omega$-metacompactness. $\square$

Definition. A space X is *OIF-metacompact* if every open cover $\mathcal{U}$ has an open refinement $\mathcal{V}$ that is an open-in-finite collection. A space X is *δ-OIF-metacompact* if $\mathcal{V}$ is a δ-open-in-finite collection.

Since each δ-OIF collection is OIF, we know that δ-OIF-metacompact implies OIF-metacompact. Also, clearly any OIF space (resp. δ-OIF space) will be OIF-metacompact (resp. δ-OIF-metacompact).

Proposition 3.33 *If X is an OIF-metacompact space with character κ, then X is $(1, \kappa^+)$-metacompact.*

Proof. Let $\mathcal{U}$ be an arbitrary open cover, and let $\mathcal{V}$ be the OIF-metacompact refinement of $\mathcal{U}$. Then since the character of X is κ, we know that each $x \in X$ is contained in not more than κ many elements of $\mathcal{V}$, because if x is contained in κ^+ many members of $\mathcal{V}$, then there is some member of the local base for x that is contained in infinitely many members of $\mathcal{V}$. $\square$

Corollary 3.34 *Every space that is OIF-metacompact and first countable is metaLindelöf.*

Definition. A *linearly ordered space* is a pair $(X, \prec)$ so that $\prec$ is a linear order on the set X and $\{(a,b) : a, b \in X, a \prec b\} \cup \{(-\infty, a) : a \in X\} \cup \{(b, \infty) : b \in X\}$ is a base for a topology on X. This topology is referred to as the *order topology.* A *generalized ordered space*, or GO space, is a subspace of a linearly ordered space.

Proposition 3.35 *For GO spaces OIF-metacompact implies paracompact.*

Proof. Let $\mathcal{U}$ be an open cover of X an OIF-metacompact space, and let $\mathcal{V}$ be an OIF-refinement of $\mathcal{U}$. We may assume that $\mathcal{V}$ consists of convex subsets of X. Then by Bennet and Lutzer [6] we may assume that $\mathcal{V}$ is a σ-point-finite collection. In fact, they show that if $\mathcal{V}_0 = \{V \in \mathcal{V} : |V| = 1\}$ and for $n \geq 0$, $\mathcal{V}_{n+1}$ is the collection of maximal subsets of $\mathcal{V} \setminus \bigcup_{0 \leq k \leq n} \mathcal{V}_k$, then each $\mathcal{V}_n$ is star-finite (e.g. each member of $\mathcal{V}_n$ meets only finitely many other members of $\mathcal{V}_n$). Then $\mathcal{V}_0 \cup \mathcal{V}_1$ covers the same set as all of $\mathcal{V}$ and is locally finite. $\square$

Proposition 3.36 *For GO spaces* $(< \omega, \kappa)$*-metacompact implies* $(1, \kappa^+)$*-metacompact.*

Proof. Let X be a $(< \omega, \kappa)$-metacompact GO space, and let $\prec$ be the order on X. Let $\mathcal{U}$ be an open cover of X. Then let $\mathcal{V}$ be a refinement of $\mathcal{U}$ witnessing $(< \omega, \kappa)$-metacompact. We may assume that the elements of $\mathcal{V}$ are convex. We define $\mathcal{V}_0$ and $\mathcal{V}_1$, $\mathcal{V}_0 = \{V \in \mathcal{V} : |V| = 1\}$ and $\mathcal{V}_1 = \mathcal{V} \setminus \mathcal{V}_0$. Obviously, $\mathcal{V}_0$ is point-finite.

We wish to show that each point of X is contained in only κ many elements of $\mathcal{V}_1$. For contradiction, assume $b \in X$ and that $\mathcal{B} = \{V \in \mathcal{V}_1 : b \in V\}$ has cardinality κ^+. Then the sets $\{\inf(V) : V \in \mathcal{B}\}$ and $\{\sup(V) : V \in \mathcal{B}\}$ cannot both have cardinality less than or equal to κ. However, all the supremums are $\prec$-greater than b and all the infimums are $\prec$-less than b.

Assume that the set of supremums is of size κ^+, call this set S. Then we will show that there is a $x \in X$ so that the cardinality of the set of supremums in S $\prec$-less than x is at least κ.

For each $x \in S$, let $L(x) = \{y \in S : y \prec x\}$ and $R(x) = \{y \in S : x \prec y\}$. These are the "left" and "right" sides of x. Notice that $L(x) \cup R(x) \cup \{x\} = S$, therefore for each x we have $\max\{|L(x)|, |R(x)|\} = \kappa^+$. Next, define $L = \{x \in S : |L(x)| \geq \kappa\}$ and $R = \{x \in S : |R(x)| \geq \kappa\}$; then clearly $L \cup R = S$. Finally, we note that $|S \setminus L| \leq \kappa$. If $|S \setminus L| = \kappa^+$, then because κ^+ is regular, no sequence of κ many elements of $S \setminus L$ is cofinal in $S \setminus L$. Therefore, for any κ sized subset of $S \setminus L$, there is an upper bound z in $S \setminus L$. But then $|L(z)| \geq \kappa$, contradiction. Similarly, $S \setminus R$ is also a set of cardinality not more than κ. Hence, $L \cap R \neq \emptyset$.

Choose $x \in L \cap R$ and let $\mathfrak{A} = \{V \in \mathcal{B} : x \prec \sup(V)\}$. Then because the elements of $\mathfrak{A}$ are convex, all contain b and have supremums $\prec$-greater than x. So for all $V \in \mathfrak{A}$ we have $[b, x] \subseteq V$, contradicting $(< \omega, \kappa)$-metacompact.

If the infimums are uncountable, the proof is analogous.

Therefore, $\mathcal{V}_1$ is point-$\leq \kappa$. So $\mathcal{V}_0 \cup \mathcal{V}_1$ is a $(1, \kappa^+)$ open refinement of $\mathcal{U}$. □

It is known that for GO spaces, meta-Lindelöf implies paracompact. Naturally, metacompact implies both $(< \omega, \omega)$-metacompact and δ-OIF-metacompact, so we have the following corollary.

Corollary 3.37 *For GO spaces, the following are equivalent:*

a) X is paracompact,

b) X is δ-OIF-metacompact,

c) X is OIF-metacompact,

d) X is $(< \omega, \omega)$-metacompact,

e) X is (n, ω)-metacompact for each $n < \omega$.

Proposition 3.38 *The (n, κ)-metacompact and $(< \omega, \kappa)$-metacompact properties are hereditary for closed sets.*

Proof. Let X be either (n, κ)-metacompact or $(< \omega, \kappa)$-metacompact, and let C be a closed subspace of X. Then let $\mathcal{U}$ be an open cover of the space C. For each $U \in \mathcal{U}$, there is an open set U' in X so that $U' \cap C = U$. Let $\mathcal{U}' = \{U' : U \in \mathcal{U}\} \cup \{X \backslash C\}$ be an open cover of X, then there is $\mathcal{V}'$ an open refinement of $\mathcal{U}'$ which is (n, κ)-metacompact or $(< \omega, \kappa)$-metacompact. So $\mathcal{V} = \{V' \cap C : V' \in \mathcal{V}'\}$ is an open refinement of $\mathcal{U}$ and is either (n, κ)-metacompact or $(< \omega, \kappa)$-metacompact, according to $\mathcal{V}'$. $\square$

Corollary 3.39 *If X is a monotonically normal $(< \omega, \omega)$-metacompact space, then X is paracompact.*

Proof. If X is not paracompact, then X contains a closed subspace C that is homeomorphic to a stationary subset of an uncountable cardinal. Then since C is closed, C is $(< \omega, \omega)$-metacompact. Also, C is a GO space therefore, C is paracompact, contradiction. $\square$

Corollary 3.40 *For GO spaces, OIF-metacompact is hereditary for closed sets.*

The following theorem can be found in [9].

Theorem 3.41 *A space is monotonically normal if and only if for each open set U and $x \in U$, one can assign an open set U_x containing x such that $U_x \cap V_y \neq \emptyset$ implies $x \in V$ or $y \in U$.*

Theorem 3.42 *Every space X is a closed subset of a δ-OIF space $O(X)$ such that if X is monotonically normal, then so is $O(X)$.*

Proof. We follow the construction found in [4]. The authors show that $O(X)$ is OIF and if X is T_1, then X is a G_δ subset of $O(X)$. They also state that $O(X)$ has the same separation axioms as X, but monotonically normal is not mentioned.

Let X be a space with base $\mathcal{A}$ and let $F[\mathcal{A}]$ be the set of all finite subsets of $\mathcal{A}$. Let $Y = \{\langle p, \mathcal{F}\rangle : p \in X \text{ and } \mathcal{F} \in F[\mathcal{A}]\}$. For each $U \in \mathcal{A}$, define $f(U) = \{\langle p, \mathcal{F}\rangle \in Y : p \in U \in \mathcal{F}\}$ and let $S(U) = U \cup f(U)$. The space is $O(X) = X \cup Y$ where the topology is generated by the subbase $\mathcal{S} = \{S(U) : U \in \mathcal{A}\} \cup \{\{x\} : x \in Y\}$.

First, we show that $\mathcal{S}$ generates a δ-OIF base. For any collection $\{B_i : i < \omega\} \subseteq \mathcal{S}$ we may consider $\bigcap\limits_{i<\omega} B_i = \bigcap\limits_{j<\omega} S(U_j) = \bigcap\limits_{j<\omega}(U_j \cup f(U_j)) = \bigcap\limits_{j<\omega} U_j \cup \bigcap\limits_{j<\omega} f(U_j)$. Note that $\bigcap\limits_{j<\omega} f(U_j) = \emptyset$. Therefore, $\overline{\bigcap\limits_{j<\omega} S(U_j)} = \overline{\bigcap\limits_{j<\omega} U_j}$, and because the points of Y are isolated $Y \cap \overline{\bigcap\limits_{j<\omega} U_j} = \emptyset$. However, Y is dense in $O(X)$, so $\overline{\bigcap\limits_{j<\omega} U_j}^{\circ} = \emptyset$.

Now we show that if X is monotonically normal then $O(X)$ is as well. Let O be open in $O(X)$, and let $x \in O$. There is a basic open set containing x and contained in O, which has the form $\bigcap\limits_{i<n_x} S(U_i^x)$. If $x \in Y$, then let $O_x = \{x\}$. If $x \in X$, then $x \in \bigcap\limits_{i<n_x} U_i^x = O^x$. Since X is monotonically normal, there is an assigned set O_x^x as in Theorem 3.41. In $O(X)$ let $O_x = S(O_x^x) \cap O$. We need to check that if V is open in $O(X)$ and $y \in V$ then $O_x \cap V_y \neq \emptyset$ implies $x \in V$ or $y \in O$.

Assume that $O_x \cap V_y \neq \emptyset$. There are essentially two cases; either both points x and y are contained in X or not.

1. If $y \in Y$, then $O_x \cap V_y \neq \emptyset$ and $V_y = \{y\}$ implies $y \in O_x \subseteq O$. Similarly, if $x \in Y$, then $x \in V_y \subseteq V$.

2. If $x, y \in X$, then let $V_y = S(V_y^y) \cap V$ be assigned to V. Let $\bigcap\limits_{i<n_y} V_i^y$ be the basic open set containing y and used to define V_y. Therefore, $O_x \cap V_y = S(O_x^x) \cap S(V_y^y) \cap O \cap V$. This means that $O_x^x \cap V_y^y \neq \emptyset$, therefore $y \in \bigcap\limits_{i<n_x} U_i^x$ or $x \in \bigcap\limits_{i<n_y} V_i^y$. Without loss of generality, let $y \in \bigcap\limits_{i<n_x} U_i^x$. Then $y \in \bigcap\limits_{i<n_x} U_i^x \subset \bigcap\limits_{i<n_x} S(U_i^x) \subseteq O$.

So we see that $O(X)$ is monotonically normal. $\square$

Corollary 3.43 *There is a monotonically normal space that is OIF, hence OIF-metacompact, but not paracompact.*

Proof. Let $X = \omega_1$ with the order topology. Then X is monotonically normal, since it is an ordered space, but not paracompact. Form $O(X)$, as in Theorem 3.42, and then $O(X)$ is OIF, hence OIF-metacompact. This cannot be a paracompact space, because all closed subspaces of paracompact spaces are paracompact, and X is such a closed subspace. $\square$

This is also an example of a space that is OIF-metacompact but not metacompact or $(< \omega, \omega)$-metacompact. For if it were $(< \omega, \omega)$-metacompact, then X would be as well, and therefore X would be paracompact.

It is already known that metacompact Moore spaces are OIF and have a n-weakly uniform base for each $n < \omega$, but we are able to be more precise in the next theorem. We are able to say, informally, that for Moore spaces "the base property is the same as the covering property".

Theorem 3.44 *Suppose X is a Moore space. Then we have the following.*

1. *X has an OIF base if and only if X is OIF-metacompact.*

2. *X has an δ-OIF base if and only if X is δ-OIF-metacompact.*

3. *For $n \geq 2$, X has an n-weakly uniform base if and only if X is (n, ω)-metacompact.*

Proof. For 1, suppose X has an OIF base $\mathcal{B}$ and let $\mathcal{U}$ be an open cover of X. Then refine $\mathcal{U}$ with elements of $\mathcal{B}$ to find an OIF-refinement of $\mathcal{U}$. This proves the forward direction, we now prove the reverse. Suppose that X has development $\mathcal{G} = (\mathcal{G}_i)_{i<\omega}$ and that X is OIF-metacompact. Then let $\mathcal{G}_0''$ be an OIF refinement of $\mathcal{G}_0$. For $\mathcal{G}_i''$ defined, let $\mathcal{G}_{i+1}' = \{V \cap U : V \in \mathcal{G}_{i+1}, U \in \mathcal{G}_i''\}$, then take $\mathcal{G}_{i+1}''$ to be an OIF refinement of $\mathcal{G}_{i+1}'$. Therefore, $\mathcal{G}'' = (\mathcal{G}_i'')_{i<\omega}$ is a development for X with each $\mathcal{G}_i''$ an OIF collection of open sets, and each $\mathcal{G}_{i+1}''$ a refinement of $\mathcal{G}_i''$. Therefore $\bigcup_{i<\omega} \mathcal{G}_i''$ is an base for X. To see that $\bigcup_{i<\omega} \mathcal{G}_i''$ is OIF, suppose that V is an open set from $\bigcup_{i<\omega} \mathcal{G}_i''$. Then for some $n < \omega$ we have that $V \in \mathcal{G}_n''$ and V is a subset of finitely many of the elements of each $\mathcal{G}_i''$ for $i \leq n$. If there are infinitely many sets from $\bigcup_{i>n} \mathcal{G}_i''$ containing V, then we may assume that every $\mathcal{G}_i''$ for $i > n$ has an element containing V. For each $i < \omega$, let U_i be an element of $\mathcal{G}_i''$ containing V. Then $(U_i)_{i<\omega}$ is a base for any point of V. This implies that V is a singleton, contradiction.

Based on the proof of 1, the proof for 2 should be clear.

For 3, the forward direction is analogous to Proposition 3.32, and the reverse direction is much the same as 1. For each $i < \omega$, choose $\mathcal{G}_i''$ to be a (n, ω) refinement. Suppose that F is a subset of X having cardinality n, and that F is contained in infinitely many sets from $\bigcup_{i<\omega} \mathcal{G}_i''$. Then we may assume that there is an $U_i \in \mathcal{G}_i''$ so that $F \subseteq U_i$ for each $i < \omega$. Yet, $(U_i)_{i<\omega}$ is a base for some point of F, implying $|F| = 1$, contradiction. $\square$

There is an example of a space that has each of these covering properties but does not have any of these base properties. Indeed, the example we present is paracompact.

Example. The sequential fan was described in Section 3.4. Suppose that X is the sequential fan, and let $\mathcal{U}$ be any covering of X. Choose $U_\infty \in \mathcal{U}$ so that $\infty \in U_\infty$. Then let $\mathcal{V} = \{U_\infty\} \cup \{\{x\} : x \in X \setminus U_\infty\}$. Then $\mathcal{V}$ refines $\mathcal{U}$ and is locally finite.

3.6 Set Theoretic and Combinatorial Conditions

In [5] the axiom CECA was introduced and proven consistent with ZFC. The authors show that CECA is equivalent to GCH plus a weakening of $\square_\lambda$ for singular λ. So in particular, CECA holds in Gödels constructible universe. In the same paper, CECA is used to prove the following result.

Theorem 3.45 *Assume CECA. Suppose that σ, τ are regular infinite cardinals, and let $\langle A_\alpha \rangle_{\alpha<\kappa}$ be a sequence of sets such that for every $I \in [\kappa]^\tau$ there is $J \in [I]^{<\tau}$ so that $|\bigcap_{\alpha \in J} A_\alpha| < \sigma$. Then there exists $\langle A'_\alpha \rangle_{\alpha<\kappa}$ such that $|A'_\alpha| \leq \sigma$ for each $\alpha < \kappa$ and the sequence $\langle A_\alpha \backslash A'_\alpha \rangle_{\alpha<\kappa}$ is point-$< \tau$.*

We have already established the next result for GO spaces without special set theoretic considerations; in [5] this result is given for $\kappa = \omega$.

Theorem 3.46 *Assume CECA. Fix a regular infinite cardinal κ. If X is $(< \omega, \kappa)$-metacompact, then X is $(1, \kappa^+)$-metacompact.*

Proof. Let $\mathcal{U}$ be an open cover of X and let $\mathcal{V}$ be an $(< \omega, \kappa)$-metacompact refinement of $\mathcal{U}$. List $X = \{x_\alpha : \alpha < \lambda\}$ and for each $\alpha < \lambda$ let $A_\alpha = \{V \in \mathcal{V} : x_\alpha \in V\}$. For each infinite collection of points, there is some finite subcollection that is contained in less than κ-many elements of $\mathcal{V}$. Hence for each $I \in [\lambda]^\omega$ there is some finite subset J of I so that $|\bigcap_{\alpha \in J} A_\alpha| < \kappa$. We apply Theorem 3.45 with $\tau = \omega$ and $\sigma = \kappa$. Therefore for each $\alpha < \lambda$ there is a $A'_\alpha \in [A_\alpha]^{\leq \kappa}$ such that $\langle A_\alpha \backslash A'_\alpha : \alpha < \kappa \rangle$ is point-finite on the set $\mathcal{V}$. For each $V \in \mathcal{V}$ let $I(V) = \{x_\alpha \in V : V \in A_\alpha \backslash A'_\alpha\}$. Each $I(V)$ is finite and if $x_\alpha \in V \backslash I(V)$, then $V \in A'_\alpha$. For each $\alpha < \lambda$ choose some $V_\alpha \in \mathcal{V}$ so that $x_\alpha \in V_\alpha$. Then for each $V \in \mathcal{V}$, define $V^* = (V \backslash I(V)) \cup \{x_\alpha \in I(V) : V = V_\alpha\}$, note that each

$V^* \subseteq V$. Therefore $\mathcal{V}^* = \{V^* : V \in \mathcal{V}\}$ is an open refinement of $\mathcal{U}$. Such a $\mathcal{V}^*$ is a $(1, \kappa^+)$-metacompact refinement, because if $x_\alpha \in V$, then $V \in A'_\alpha \cup \{V_\alpha\}$, and this is a set of size $\leq \kappa$. $\square$

Now the question is raised whether the statement "$(< \omega, \kappa)$-metacompact implies $(1, \kappa^+)$-metacompact" is independent of ZFC. If the following combinatorial principal holds (and is consistent with MA + $\omega_3 \leq 2^\omega$), then there is a space that is $(< \omega, \omega)$-metacompact and not $(1, \omega_1)$-metacompact.

(*) *If X is a set and $|X| = \omega_3$, then there exists a collection $\mathcal{H}$ of subsets of X and a partition $\{\mathcal{H}_n : n < \omega\}$ of $\mathcal{H}$ such that : (1) if $H_1, H_2 \in \mathcal{H}$ and $H_1 \neq H_2$, then $|H_1 \cap H_2| < \omega$; and (2) if $Y \subseteq X$ and $|Y| = \omega_3$, then for each $n \in \omega$, there exists $H \in \mathcal{H}_n$ such that $|Y \cap H| = \omega_2$.*

Theorem 3.47 *(MA + $\omega_3 \leq 2^\omega$ + *) There is a space with a weakly uniform base that is not $(1, \omega_2)$-metacompact.*

Proof. Assume MA + $\omega_3 \leq 2^\omega$. This construction is essentially due to [18]. Let S be a subset of the x-axis of size ω_3 and let K be a countable dense subset of the upper half plane of $\mathbb{R}^2$. Denote $K = \{p_n : n < \omega\}$. Then we let $X = S \cup K$. For $x \in S$, the neighborhoods are $B_n(x)$ define to be an open disk in the upper half plane of radius $1/n$ tangent to the axis at x intersected with K, together with $\{x\}$. The points of K are isolated. Now let $\mathcal{H} = \bigcup_{n<\omega} \mathcal{H}_n$ be the collection and partition satisfying condition (*). For each $n < \omega$ let $K'(n) = \{(p_n, H) : H \in \mathcal{H}_n\}$ and define $K' = \bigcup_{n<\omega} K'(n)$. Now let $X' = S \cup K'$.

In X' we define the open neighborhoods of $x \in S$ to be $B'_n(x) = \{x\} \cup \{(p_i, H) : (p_i, H) \in K'(i), p_i \in B_n(x)$ and $x \in H\}$. The points of K' are isolated. Let $\mathcal{B}_n =$

$\{B'_n(x) : x \in S\} \cup \{\{q\} : q \in K'\}$, then define $\mathcal{B} = \bigcup_{n<\omega} \mathcal{B}'_n$. We claim that this is a weakly uniform base for X'. Suppose that x_1 and x_2 are both in K', and let $x_1 = (p_k, H_1)$ and $x_2 = (p_j, H_2)$. We have that $|H_1 \cap H_2| < \omega$; suppose that $y_0, y_1, \cdots, y_m$ list $H_1 \cap H_2$. Then for each y_i there is an n_i so that x_1 is not in $B'_{n_i}(y_i)$. Let $n = \max\{n_i : 0 \le i \le m\}$; then we have that $\{x_1, x_2\}$ is contained in less than $n \cdot m$ many sets from $\mathcal{B}$.

Next, we claim that X' has an open cover with no point $\le \omega_1$ open refinement. Consider $\mathcal{B}'_1 = \{B'_1(x) : x \in S\} \cup \{\{q\} : q \in K'\}$, which is an open cover of X'. Suppose that $\mathcal{V}$ is an open point $\le \omega_1$ open refinement of $\mathcal{B}'_1$. Then for each $x \in S$, there is an n_x so that $B'_{n_x}(x)$ is contained in some element of $\mathcal{V}$; then $\mathcal{U}' = \{B'_{n_x}(x) : x \in S\}$ must be point $\le \omega_1$. Let $\mathcal{U} = \{B_{n_x}(x) : x \in S\}$. Since K is dense in X, there is p_i that is contained in ω_3 many sets from $\mathcal{U}$, say $\{B_{n_x}(x) : x \in Y\}$ for some subset Y of X of cardinality ω_3. So there exists $H \in \mathcal{H}_i$ so that $|H \cap Y| = \omega_2$ and the point (p_i, H) is in K'. For each $x \in Y \cap H$ we have $(p_i, H) \in G'_{n_x}(x)$. So (p_i, H) is contained in ω_2 many elements of $\mathcal{U}'$, contradiction. $\square$

Chapter 4

I-weight and separating weight

4.1 Introduction

The study of reflection was started by Tkacenko in [16], and a systematic study was made by Hodel and Vaughan in [13]. Hajnal and Juhász proved that weight reflects every infinite cardinal [10]. Ramírez-Páramo proved that under GCH for the class of compact Hausdorff spaces, i-weight reflects all infinite cardinals [15]. In the second section of this chapter we prove that for compact linearly ordered spaces i-weight reflects all infinite cardinals. We show that the point-separating weight must reflect, which implies that i-weight must reflect.

In section three, we find necessary and sufficient conditions for i-weight to reflect in the class of locally compact linearly ordered spaces. The lemmas used to determine under what conditions i-weight will reflect for these spaces provide a means of calculating the i-weight of an ordinal space.

We begin with some definitions which may be found in [15], [13].

Definition. A cardinal function ϕ is said to *reflect* cardinal κ, if when $\phi(X) \geq \kappa$ there is a subset Y of X so that $|Y| \leq \kappa$ and $\phi(Y) \geq \kappa$.

Definition. We say that X is *condensed onto* Z if there is a bijection from $f : X \to Z$ so that for each open subset U of Z, $f^{-1}(U)$ is open in X. Commonly, Z is regarded as a copy of X, and the topology on Z is considered to be contained in the topology on X.

Definition. For a Tychonoff space X the *i-weight of* X is the minimum weight of a Tychonoff space onto which X may be condensed.

So for example, the i-weight of Tychonoff space X is ω if and only if X has a weaker separable metric topology.

Definition. We say that $\mathcal{V} \subseteq \mathcal{P}(X)$ is *separating* if for each pair $(x, y) \in X^2$, $x \neq y$, there is an $U \in \mathcal{V}$ so that $x \in U$ and $y \notin U$. By *separating weight*, denoted $sw(X)$, we mean $\min\{|\mathcal{V}| : \mathcal{V}$ is a separating open cover of $X\}$.

4.2 Compact Linearly Ordered Spaces

In this section we show that for compact linearly ordered spaces, point-separating weight and i-weight reflect all cardinals. For the reader's comfort we now briefly outline how we intend to show this.

A κ^+-Aronsajn tree is a tree such that every chain and every level is of cardinality $< \kappa^+$. For a κ^+-Aronsajn tree T we construct the space $X(T)$ of chains in T, and show that there is no point-separating open cover of cardinality less than κ^+. For each compact linearly ordered space we construct the tree $T(X)$. If every subset of X that has cardinality κ^+ may be point-separated by κ or fewer open sets, then $T(X)$ will contain κ^+-Aronsajn tree A for which the points of $X(A)$ may be point-separated by κ or fewer open sets, which contradicts the earlier result.

Let A be a κ^+-Aronszajn tree where every node is 2 or 0-branching, and level 0 has only one node. For node t at level α, node t corresponds to a sequence $\sigma_t \in 2^\alpha$ and, if t is 2-branching, the two nodes above t correspond to the sequences $\sigma_t^\frown(0)$ and $\sigma_t^\frown(1)$. Then for each node t that is 2-branching let $l(t) = \sigma_t^\frown(0, 0, 0, \cdots)$ and $r(t) = \sigma_t^\frown(1, 1, 1, \cdots)$

be 2 branches passing through node t. Let c and d be two maximal branches of A. Since $c \neq d$, we know that for some $n < \kappa^+$ we have $c(n) \neq d(n)$. Then define $c < d$ if and only if for the least $n < \kappa^+$ for which $\sigma_c(n) \neq \sigma_d(n)$, $\sigma_c(n) = 0$ and $\sigma_d(n) = 1$. We give $X(A)$, the space of maximal branches of A, the topology generated by this order.

Lemma 4.1 *Let A be a κ^+-Aronszajn tree such that*

1. *A has one node at level 0.*

2. *A is 2-branching, or 0-branching at each node t*

Then less than or equal to κ many open sets in $X(A)$ cannot separate the pairs $\{\{l(t), r(t)\} : t \in A\}$.

Proof. Let A be as above and let A' be the collection of nodes that are 2-branching. Consider in $X(A)$ the branches $l(t)$ and $r(t)$ for each $t \in A'$. Suppose that $\mathcal{V}$ is a collection of $\leq \kappa$-many open sets that point separate $r(t)$ and $l(t)$ for $t \in A'$. For each $t \in A'$, let $\langle V_t, W_t \rangle$ be a pair from $\mathcal{V}$ that separates $l(t)$ and $r(t)$ with $l(t) \in V_t$, $r(t) \in W_t$, $r(t) \notin V_t$ and $l(t) \notin W_t$. Each $V \in \mathcal{V}$ can be decomposed into disjoint convex subsets, so for each $t \in A'$ let (v_t, v'_t) be the neighborhood of $l(t)$ from the decomposition of V_t. Similarly, define (w_t, w'_t) so that $r(t) \in (w_t, w'_t) \subset W_t$.

For the set (v_t, v'_t) to contain $l(t)$ and not $r(t)$, the branch v_t must contain a node a_t that is immediately to the left of a node below t and v'_t must be a branch that contains node t.

Let S be a stationary subset of κ^+, and for each $t \in A'$ so that $level(t) \in S$, let $p(t) = a_t$. Now we define a map f from S into κ^+. First, for each $s \in S$, pick $t^s \in A'$ in the s^{th} level of A, then let $f(s) = \alpha$ if and only if the level of $(p(t^s))$ is α. By the Pressing

Down Lemma, there is a level of the tree, α, so that $f^{-1}\{\alpha\}$ is a stationary subset of κ^+. Since each level has cardinality less than or equal to κ this means that there is a node a in level α so that $|p^{-1}\{a\}| = \kappa^+$. Consider $\{V_t : t \in p^{-1}\{a\}\} \subseteq \mathcal{V}$. Since $|\mathcal{V}| \leq \kappa$, there is a V so that $V = V_t$ for κ^+-many $t \in p^{-1}\{a\}$. Since a is immediately to the left of a node below each $t \in p^{-1}\{a\}$, there is one convex set (v, v') from the disjoint decomposition of V so that $v = v_t$ and $v' = v'_t$ for each $t \in p^{-1}\{a\}$. Let β be the height of branch v', and note that $|\beta| \leq \kappa$. Then the number of nodes t so that $l(t) \in (v, v')$ is less than $|\beta| \leq \kappa$, yet $p^{-1}(\{a\}) \subseteq (v, v')$, contradiction. $\square$

Lemma 4.2 *For a compact Hausdorff space X, $iw(X) = w(X) = nw(X) = sw(X)$.*

Proof. By [12], we know that $sw(X) \leq nw(X)$, and for compact Hausdorff spaces, $w(X) = nw(X) = psw(X)$. Also, $psw(X) \leq iw(X) \leq w(X)$, therefore, $iw(X) = w(X)$ if X is compact Hausdorff. We intend to show that $sw(X) \geq nw(X)$ if X is compact Hausdorff. Suppose $\mathcal{V}$ is a separating open cover of X. Then let $\mathcal{N} = \{X \setminus W : W$ is a finite union of elements of $\mathcal{V}\}$. We claim that $\mathcal{N}$ is a net for X. Suppose that U is an open set of X, then pick $p \in U$. For each $q \in X \setminus U$, find $V_q \in \mathcal{V}$ so that $q \in V_q$ and $p \notin V_q$. Then $\{U\} \cup \{V_q : q \notin U\}$ covers X, hence there is a finite subcover, $\mathcal{V}'$. Let $W = \bigcup\{V \in \mathcal{V}' : V \neq U\}$, and $N = X \setminus W$. Then $N \in \mathcal{N}$ and clearly, $p \in N \subseteq U$. Therefore, $\mathcal{N}$ is a net with the same cardinality as $\mathcal{V}$. $\square$

The following lemma and its proof are found in [13].

Lemma 4.3 *If ϕ is a monotone cardinal function that reflects successor cardinals, then ϕ reflects all infinite cardinals.*

Next, we need to observe that separating weight is monotone. We combine this with a similar lemma for i-weight.

Lemma 4.4 *I-weight is monotone, and for compact spaces point-separating weight is monotone.*

Proof. Let X and Y be topological spaces, with $Y \subset X$. If $\mathcal{B}$ is a base for X, then $\{U \cap Y : U \in \mathcal{B}\}$ is a base for Y. So the minimum cardinality of a base for X is not less than the minimum cardinality of a base for Y.

For separating weight, let X and Y have the same relationship as above. Then, if $\mathcal{V}$ is a separating open cover of X, $\mathcal{V}\restriction Y$ is a separating open cover of Y with cardinality not more than $|\mathcal{V}|$. $\square$

Theorem 4.5 *For a compact linearly ordered space, separating weight reflects κ^+. Hence, for compact linearly ordered spaces, separating weight reflects all infinite cardinals.*

Proof. Suppose X is a compact linearly ordered space and $sw(X) \geq \kappa^+$, but every subset Y of X such that $|Y| \leq \kappa^+$ has separating weight κ.

We construct a tree $T(X)$ from the space X.

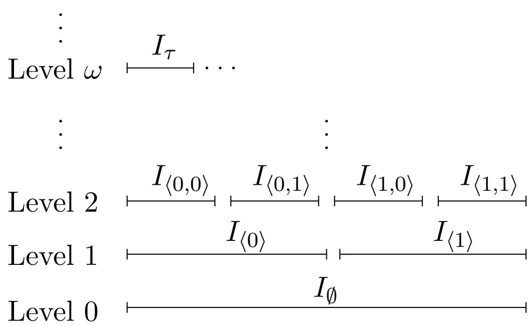

Figure 4.1: The tree $T(X)$.

Let $X = I_\emptyset$, then divide the space X into two closed intervals with at most one point in common, $I_{\langle 0\rangle}$ and $I_{\langle 1\rangle}$, so that every point of $I_{\langle 0\rangle}$ is less than or equal to every point of $I_{\langle 1\rangle}$. For a reason that will only be important near the end of this construction, we choose to split the interval at a non-isolated point, if the interval contains a non-isolated point. Next, form $I_{\langle 0,0\rangle}$, $I_{\langle 0,1\rangle}$, $I_{\langle 1,0\rangle}$ and $I_{\langle 1,1\rangle}$ by dividing each of $I_{\langle 0\rangle}$ and $I_{\langle 1\rangle}$ into two parts, again at a non-isolated point, if possible. This process is done at each successor level of the tree. At the level ω, consider $\sigma : w \to 2$. If $\bigcap\limits_{n<\omega} I_{\sigma\restriction n}$ is a non-degenerate interval, then I_σ is a node at the ω level. Likewise at each limit level, nodes appear only if the intersection of preceding intervals is a non-degenerate interval. If at some successor level an interval should have only one point in it, then that node does not branch. The ordering on the tree is that $I_\sigma \leq I_\tau$ if and only if $\sigma \subseteq \tau$.

Suppose that at some level less than κ^+, there are at least κ^+ many non-degenerate intervals (if necessary, choose κ^+ of them). Let α be the least such level.

Since $|\alpha| \leq \kappa$, the collection of nodes that precede the α level has size not more than κ. Let the κ^+ many non-degenerate intervals at the level α be $\{I_\gamma : \gamma < \kappa^+\} = \{[l_\gamma, r_\gamma] : \gamma < \kappa^+\}$. Assume that there is a collection $\mathcal{V}$ of κ-many open sets in X that T_1 separate points r_γ and l_γ for $\gamma < \kappa^+$. Then for each $\gamma < \kappa^+$ there is a pair $\langle V_\gamma, W_\gamma\rangle \in \mathcal{V} \times \mathcal{V}$ so that $l_\gamma \in V_\gamma$, $r_\gamma \in W_\gamma$, $r_\gamma \notin V_\gamma$ and $l_\gamma \notin W_\gamma$. There is a pair $\langle V, W\rangle$ and some $J \in [\kappa^+]^{\kappa^+}$ so that $\langle V, W\rangle = \langle V_\gamma, W_\gamma\rangle$ for $\gamma \in J$.

Next, decompose W into disjoint convex subsets, and for each $\gamma \in J$ let $(w_\gamma, w'_\gamma) \subset W$ be the convex neighborhood of r_γ from this disjoint decomposition. The assignment $\gamma \mapsto (w_\gamma, w'_\gamma)$ is one-to-one since if $(w_\gamma, w'_\gamma) = (w_\beta, w'_\beta)$, then either l_γ or l_β is in $(w_\gamma, w'_\gamma) \subset W$. Now each r_γ is the limit of a sequence of right end points of the intervals that precede I_γ in the ordering of the tree. Therefore, for each $\gamma \in J$, there is a right end point r'_γ of an interval preceding I_γ that is contained in (w_γ, w'_γ). However, there are only κ-many nodes below the α level of this tree, and therefore only κ-many potential r'_γ for κ^+-many disjoint (w_γ, w'_γ), contradiction. Therefore, we conclude that every level below the κ^+ level has cardinality less than κ^+.

Suppose this tree contains a branch of length κ^+. Then either the sequence of left endpoints or the sequence of right endpoints of intervals forming the branch in the tree has cardinality κ^+. Without loss of generality, the set of right end points $\{r_\alpha : \alpha < \kappa^+\}$ has cardinality κ^+. Then $\{r_\alpha\}_{\alpha<\kappa^+}$ is a non-increasing sequence, and must converge to $r \geq \sup\{l_\alpha : \alpha < \kappa^+\}$. For each $\alpha < \kappa^+$ let $U_\alpha \in \mathcal{V}$ be the open set that contains r and not r_α. Then some U is U_α for κ^+-many α. However, for some $\beta < \kappa^+$ we will have $r_\gamma \in U$ whenever $\gamma > \beta$, contradiction.

We have now shown that every level of $T(X)$ has cardinality not more than κ and that every chain has length less than or equal to κ. So either $T(X)$ is a κ^+-Aronszajn tree or the height of $T(X)$ is less than κ^+. It follows from Lemma 4.1 that $T(X)$ cannot be a κ^+-Aronszajn tree. If $T(X)$ were a κ^+-Aronszajn tree, then consider $X(T(X))$ and the points corresponding to $r(t)$ and $l(t)$ for the nodes $t \in T(X)$. By our assumptions, in X we are able to separate these $\leq \kappa^+$ points in X with κ-many open sets, contradiction.

Since the tree has height less than κ^+, without loss of generality, assume the height of the tree is κ. Then the left and right endpoints of the intervals contained as nodes in the tree form a subset of X, call this collection Y and $|Y| = \kappa$. We claim that this subset together with the isolated points is dense in X. Consider a nonempty open convex subset (a, b) of X. Either (a, b) contains a left or right endpoint of some interval contained in the tree, or (a, b) is contained in an interval from each level of the tree. Then let β be the height of the tree, and for each $\alpha < \beta$ let J_α be the interval from level α that contains (a,b). Then $(a,b) \subseteq \bigcap_{\alpha<\beta} J_\alpha$, while $|\bigcap_{\alpha<\beta} J_\alpha| = 1$. So nonempty $(a,b) = \{x\}$. So Y together with the isolated points is a dense set in X.

We now claim that the set of isolated points has cardinality at most κ. Let $\{J_m : m \in M\}$ be the collection of minimal intervals contained in the tree which we were not able to split at a non-isolated point. We claim each J_m can contain at most countably many isolated points. Let $[l_m, r_m] = J_m$, then pick $a_m \in J_m$; we have $[l_m, a_m]$ is a closed set, and is therefore compact. Pick any open neighborhood W of l_m, and then $\{W\} \cup \{\{x\} : x \in [l_m, a_m] \setminus W\}$ covers $[l_m, a_m]$. Therefore, all but finitely many of the points of $[l_m, a_m]$ must be in W. Since W is arbitrary, this implies that$[l_m, a_m]$ contains countably many isolated points. By symmetry, so does $[a_m, r_m]$. Now it remains to note that every isolated point of X is contained in J_m for some $m \in M$, and since $|M| \leq \kappa$, we have at most κ many isolated points.

We use Y to construct a base of cardinality κ for X. Let $\mathcal{B} = \{(a,b) : a, b \in Y, a < b\} \cup \{\{x\} : x \text{ is isolated }\}$. Let $Y' = Y \cup \{x : x \text{ is isolated }\}$.

Let x be a point in X and (u, u') be a convex neighborhood of x. Unless x is isolated, we have that at least one of (u, x) and (x, u') is nonempty. Without loss of generality, assume that (u, x) is nonempty, and choose $a \in Y' \cap (u, x)$. If (x, u') is nonempty, then choose b similarly, and $x \in (a, b) \subseteq (u, u')$. So assume instead that (x, u') is empty.

Consider an increasing sequence of nodes in the tree (which is a decreasing sequence of intervals) that contain the point x, say $\{K_\alpha : \alpha < \beta\}$. Then $\bigcap_{\alpha<\beta} K_\alpha = \{x\}$. Let γ be the least ordinal so that for r_γ, the right end point of K_γ, $r_\gamma \leq u'$. If $r_\gamma = u'$, then $u' \in Y$, and the set we need is (a, u'). If $r_\gamma = x$, then there is a second increasing sequence of intervals in the tree that contain x, but for this sequence x is a left end point. Then consider all the members of this second sequence that also contain u'. The intersection of them would be $[x, u']$, and would be a node of the tree. Therefore, $u' \in Y$, and again the set we want is (a, u'), for then $x \in (a, u') \subseteq (u, u')$. Therefore, X has weight κ, and hence i-weight κ, contradiction. $\square$

Corollary 4.6 *For compact linearly ordered spaces, i-weight reflects all cardinals.*

Proof. Since $iw(X) = sw(X)$ for each compact Hausdorff space, if $iw(X) \geq \kappa$ then $sw(X) \geq \kappa$. Therefore, there is $Y \subseteq X$ so that $|Y| \leq \kappa$ yet $sw(Y) \geq \kappa$. Any base for a Tychonoff topology, would also be a separating open cover, therefore, $iw(Y) \geq sw(Y) \geq \kappa$. $\square$

4.3 Locally Compact Linearly Ordered Spaces.

In this section we prove reflection theorems for locally compact linearly ordered spaces and i-weight. We begin with several lemmas that build toward the main result. We determine that the i-weight of an ordinal space is the cardinality of the ordinal. Also, we determine the i-weights of subspaces of ordinal spaces. We find necessary and sufficient conditions for i-weight to reflect cardinal κ in the class of locally compact linearly ordered space. This section ends with the presentation of several examples.

We use the following definitions throughout the rest of this section.

Definition. For a locally compact linearly ordered space X and $a, b \in X$ we write $a \sim b$ if and only if either $[a, b]$ or $[b, a]$ is compact. Then $\sim$ is an equivalence relation. Let $\tilde{a} = \{b \in X : a \sim b\}$ denote the equivalence class of a. Then for a locally compact linearly ordered space X, we define $E(X)$ to be the number of distinct equivalence classes under the $\sim$ relation, i.e., $E(X) = |\{\tilde{a} : a \in X\}|$.

Lemma 4.7 *For a locally compact linearly ordered space X, each $\tilde{a}$ is an open subset of X.*

Proof. To see that $\tilde{a}$ is open, let $p \in \tilde{a}$. We assume that $a < p$, and the proof for the case with $p < a$ is analogous. Then since X is linearly ordered and locally compact there is an open interval (c, d) of X so that $p \in (c, d)$ and $\overline{(c, d)} = [c, d]$ is compact. Then either $a < c$ or $c \leq a \leq d$. If $a < c$, then $[a, d] = [a, p] \cup [p, d]$. Since $[p, d]$ is a closed subset of the compact space $[c, d]$, $[p, d]$ is compact and $[a, d]$ is compact as a union of finitely many compact sets. Also, for each $b \in (c, d)$ the set $[a, b]$ is compact. Therefore, $(c, d) \subseteq \tilde{a}$, which implies that $\tilde{a}$ is open.

If $c \leq a \leq d$ then for each $b \in (c, d)$, either $[a, b]$ or $[b, a]$ is nonempty and compact as a closed subset of $[c, d]$, so in this case $(c, d) \subseteq \tilde{a}$. $\square$

Lemma 4.8 *For a locally compact linearly ordered space X, if $E(X)$ is infinite then $E(X) = e(X)$.*

Proof. Notice first that $e(X) \geq E(X)$. We form a closed discrete set C of cardinality $E(X)$ by taking one point from each equivalence class. We have shown that for each $a \in X$ the set $\tilde{a}$ is open, so C is discrete. Also, $\tilde{a} \setminus \{a\}$ is open, so C is closed.

Next, suppose that $e(X) > E(X)$. Since $E(X)$ is infinite, $e(X) \geq \omega_1$. Then there is at least one equivalence class, call it $\tilde{a}$, that contains at least ω_1-many members of a closed discrete set C. Choose a point $p' \in C \cap \tilde{a}$. At least one of $P = \{c \in C \cap \tilde{a} : c < p'\}$ and $S = \{c \in C \cap \tilde{a} : c > p'\}$ is uncountable. We assume the set P is uncountable, as the proof for S uncountable is analogous. We claim that we can find $p < p' \in \tilde{a}$ so that $|C \cap [p, p']| \geq \omega$. For $c \in P$, let $A_c = \{d \in P : c < d\}$. If A_c were finite for each $c \in P$, then P would be an increasing union of sets which are all finite, and so $|P| \leq \omega$, contradiction. Hence there is a p so that $|A_p| \geq \omega$, then $[p, p'] \cap C$ is infinite, $[p, p']$ is compact and cannot contain an infinite closed discrete set, contradiction. $\square$

Lemma 4.9 *For spaces X and Y, $iw(X \times Y) = \max\{iw(X),\ iw(Y)\}$.*

Proof. Suppose that X and Y are topological spaces, and consider $X \times Y$. If $\mathcal{B}_X$ and $\mathcal{B}_Y$ are bases for X and Y, then $\mathcal{B}_X \times \mathcal{B}_Y$ is a base for $X \times Y$.

So $w(X \times Y) \leq |\mathcal{B}_X \times \mathcal{B}_Y| = |\mathcal{B}_X||\mathcal{B}_Y|$. Therefore, $iw(X \times Y) \leq iw(X)iw(Y) = \max\{iw(X), iw(Y)\}$.

Next, suppose that $\mathcal{B}$ is a base for a Tychonoff topology on $X \times Y$ which is coarser than the product topology. Fix $y_0 \in Y$ and consider $\mathcal{U}_X = \{U \cap (X \times \{y_0\}) \neq \emptyset : U \in \mathcal{B}\}$. Then $\pi_1(\mathcal{U}_X)$ is a base for a Tychonoff topology on X.

The above argument is symmetric with respect to x and y, so the i-weights of X and Y are not more than $|\mathcal{B}|$. Therefore, i-weights of X and Y are not more than i-weight of $X \times Y$. $\square$

Theorem 4.10 *Let κ be a regular cardinal, and let A be a stationary subset of κ. Then A with the subspace topology inherited from the order topology on κ has i-weight κ.*

Proof. Let κ be a regular cardinal and assume that A is a stationary subset of κ. Suppose by way of a contradiction that $\mathcal{B}$ is a base for a Tychonoff topology on A so that $|\mathcal{B}| < \kappa$. For each $U \in \mathcal{B}$, there is an open subset U' of κ so that $U' \cap A = U$. Let $\mathcal{B}' = \{U' : U \in \mathcal{B}\}$. Since A is stationary, A contains stationarily many limit ordinals. Let S denote the limit ordinals contained in A.

For each $s \in S$, let p_s be any element of A so that $p_s > s$. Also, for each $s \in S$ let $(U_s, V_s) \in \mathcal{B}^2$ be such that $s \in U_s$, $p_s \in V_s$ and $U_s \cap V_s = \emptyset$. Since U'_s must be open in the order topology, we know that each U'_s contains a convex segment containing s. Let $g(s)$ be an ordinal less than s so that $(g(s), s] \subseteq U'_s$. Then since S is stationary, there is a γ so that $g^{-1}(\gamma)$ is stationary. Because $|\mathcal{B}| < \kappa$ and $|g^{-1}(\gamma)| = \kappa$, there is (U^*, V^*) so that $(U^*, V^*) = (U_s, V_s)$ for κ-many different $s \in g^{-1}(\gamma)$. Then $s \in (\gamma, s] \subseteq U^*$ and $p_s \notin U^*$ for each $s \in g^{-1}(\gamma)$. For any fixed $s \in g^{-1}(\gamma)$, let $p^* = p_s$. We claim that p^* is an upper bound on the set $g^{-1}(\gamma)$, else if there is a s' so that $p^* \leq s'$ then $p^* \in (\gamma, s'], \subseteq U^*$. $\square$

Corollary 4.11 *For any cardinal κ, the i-weight of the ordinal space κ is κ.*

Proof. If κ is not regular then κ must be a limit ordinal, since each successor ordinal is regular. Each limit cardinal is the limit of the preceding regular cardinals. Therefore, let $L = \{\alpha < \kappa : \alpha \text{ is a regular cardinal}\}$, and notice that κ is equal to $\bigcup_{\alpha \in L} \alpha$. Then $|\kappa| \geq iw(\kappa) \geq \sup\{iw(\alpha) : \alpha \in L\} = \kappa$. $\square$

Corollary 4.12 *The i-weight of any ordinal space κ is $|\kappa|$.*

Proof. Assume that κ is an ordinal but not a cardinal. Then, as ordinals, $|\kappa| < \kappa$. By monotonicity of i-weight we know that $iw(|\kappa|) \leq iw(\kappa)$. Also, $iw(\kappa) \leq w(\kappa) = |\kappa| = iw(|\kappa|)$. $\square$

Lemma 4.13 *A Tychonoff space X with $iw(X) \leq \lambda$ can be condensed into I^λ.*

Proof. Any Tychonoff space of weight m can be embedded in I^m. So if a space X has i-weight $m \leq \lambda$, then X has a Tychonoff topology τ so that (X, τ) is homeomorphic to a subset of I^m. Call the corresponding embedding f.

Then we embed each I^m into I^γ by the defining $h : I^m \to I^\gamma$ as follows. Let $x \in I^m$ be denoted as $(x_i)_{i<m}$; then $h(x) = (x_i)_{i<m} {}^\frown (0)_{m \leq j < \gamma}$. Let $G : X \to I^\gamma$ be defined by $h \circ f$. Clearly, G is one-to-one and continuous. $\square$

Recall that D_κ denotes the discrete space of cardinality κ.

Theorem 4.14 *Let κ be an ordinal, and C a club subset of κ. Suppose that $\kappa \setminus C = \bigcup_{i<\gamma}(a_i, b_i)$ so that $\gamma \leq |\kappa|$ and $(a_i, b_i) \cap (a_j, b_j) \neq \emptyset$ if and only if $i = j$. Then, $iw(\kappa \setminus C) = \max\{iw(D_\gamma),\ \sup\{iw((a_i, b_i)) : i < \gamma\}\}$.*

Proof. Let κ and C be as above, i.e. $\kappa \setminus C = \bigcup_{i<\gamma}(a_i, b_i)$. Then by monotonicity of i-weight, we have $iw(\kappa \setminus C) \geq iw((a_i, b_i))$ for each $i < \gamma$. Also, since the (a_i, b_i) are pairwise disjoint, and each is nonempty and open, we may choose $x_i \in (a_i, b_i)$ so that $X = \{x_i : i < \gamma\}$ is a discrete space homeomorphic to D_γ. Then, invoking monotonicity once again, $iw(\kappa \setminus C) \geq iw(X)$.

We prove that $iw(\kappa \setminus C) \leq \max\{iw(D_\gamma),\ \sup\{iw((a_i, b_i)) : i < \gamma\}\}$ by considering two cases.

Suppose that there is $\lambda < |\kappa|$ so that $\lambda = \sup\{iw((a_i, b_i)) : i < \gamma\}$. We condense each (a_i, b_i) into I^λ. Then, $\kappa \setminus C$ can be condensed to a subset of $D_\gamma \times I^\lambda$. So by Lemma 4.9, $iw(D_\gamma \times I^\lambda) = \max\{iw(D_\gamma), \lambda\}$. Therefore, $iw(\kappa \setminus C) \leq \{iw(D_\gamma),\ \sup\{iw((a_i, b_i) : i < \gamma\}\}$. Suppose on the other hand that $\sup\{iw((a_i, b_i)) : i < \gamma\} = |\kappa|$. Since $|\kappa| = iw(\kappa) \geq iw(\kappa \setminus C) \geq \sup\{iw((a_i, b_i)) : i < \gamma\}$, $iw(\kappa \setminus C) = |\kappa| = \max\{iw(D_\gamma),\ \sup\{iw((a_i, b_i)) : i < \gamma\}\}$. □

For the following corollary, recall that $\log(\kappa) = \min\{\lambda : 2^\lambda \geq \kappa\}$.

Corollary 4.15 *The i-weight of D_κ is $\log(\kappa)$.*

Proof. From Theorem 4.2 in [12], we know that for any Hausdorff topology on X, $|X| \leq w(X)^{\psi(X)} \leq w(X)^{w(X)} = 2^{w(X)}$. This gives us a means of bounding the i-weight of a space, in particular, $\kappa \leq 2^{iw(D_\kappa)}$. So $iw(D_\kappa) \in \{\lambda : 2^\lambda \geq \kappa\}$. Notice that for each λ so that $2^\lambda \geq \kappa$, we may consider κ to be a subset of 2^λ. Then under the product topology on 2^λ, the weight of 2^λ is λ, which means that $iw(X) \leq \lambda$. So $iw(X) = \min\{\lambda : 2^\lambda \geq \kappa\} = \log(\kappa)$. $\square$

Lemma 4.16 *If X is a locally compact linearly ordered space so that for each pair $a, b \in X$ with $a < b$ the set $[a, b]$ is compact, then $iw(X) = w(X)$. Moreover, for such a space X, i-weight reflects all cardinals.*

Proof. Let X be as above. Then pick $a \in X$. Either $(-\infty, a]$ or $[a, \infty)$ has the same weight as X. Without loss of generality, let $w([a, \infty)) = w(X)$.

We intend to show that $w([a, \infty)) = iw([a, \infty)) = \max\{\mathrm{cf}([a, \infty)),\ \sup\{w([a, b]) : b > a\}\}$.

First, suppose that $\mathcal{B}$ is a base for a Tychonoff topology on $[a, \infty)$ which is coarser than the order topology. Since weight equals i-weight for compact Hausdorff spaces, we know that $iw([a, b]) = w([a, b])$ and by monotonicity of weight, we know that $w([a, \infty)) \geq \sup\{w([a, b]) : b > a\}$. Also, suppose that $\mathrm{cf}([a, \infty)) = \kappa$. We construct a set C that is homeomorphic to κ. Let $c_0 = a$. Suppose that for $j \leq i$ each c_j has been defined, and pick $c_{i+1} > c_i$. If α is a limit ordinal so that for each $j < \alpha$, c_j has been defined, define $c_\alpha = \sup\{c_j : j < \alpha\}$. Since the cofinality of $[a, \infty)$ is κ, c_i is defined for each $i < \kappa$. Let $C = \{c_i : i < \kappa\}$. If i is a successor ordinal, $(c_{i-1}, c_{i+1}) \cap C = \{c_i\}$ and is open. If α is a limit ordinal then $(c_i, c_\alpha] \cap C = \{c_j : i < j \leq \alpha\}$ is open for $i < \alpha$. We map C homeomorphically to κ by $h(c_i) = i$. Then the i-weight of C is κ, the i-weight of κ. This implies that $iw([a, \infty)) \geq \kappa = \mathrm{cf}([a, \infty))$.

Next, we observe that $w([a, \infty)) \leq \max\{\mathrm{cf}([a, \infty)),\ \sup\{w([a, b]) : b > a\}\}$. Let K be a cofinal subset of $[a, \infty)$ of cardinality $\mathrm{cf}([a, \infty))$; so $K = \{k_i : i < \kappa\}$ and $k_i < k_j$

iff $i < j$. The set $\{[a, \kappa_\alpha) : \alpha < \kappa\}$ is an open cover of $[a, \infty)$. Also, $w([a, \kappa_\alpha)) \leq w([a, \kappa_\alpha])$. Let $\mathcal{B}_\alpha$ be a base for $[a, \kappa_\alpha)$ under the subspace topology for the order topology on $[a, \infty)$. Then $\mathcal{B} = \bigcup_{\alpha<\kappa} \mathcal{B}_\alpha$ is a base for $[a, \infty)$. The cardinality of $\mathcal{B}$ is less than $\max\{\kappa,\ \sup\{w([a, k_\alpha)) : \alpha < \kappa\}\} \leq \max\{\kappa,\ \sup\{w([a, k_\alpha]) : \alpha < \kappa\}\} \leq \max\{\kappa,\ \sup\{w([a, b]) : b > a\}\}$.

So $w([a, \infty)) = iw([a, \infty))$.

Now we will show that i-weight reflects. Suppose that $\gamma \leq iw([a, \infty))$. Then consider several quick cases.

1. If $\gamma \leq \mathrm{cf}([a, \infty))$, then let C be the cofinal subset above in this proof. Then $Y = \{c_i : i < \gamma\}$ is a subset of $[a, \infty)$ that is homeomorphic to γ, hence Y has i-weight γ.

2. If there is a $b > a$ so that $iw([a, b]) \geq \gamma$, then take Y to be a subset of $[a, b]$ that reflects γ.

3. Now assume that $\gamma > \mathrm{cf}([a, \infty))$ and that $iw([a, b]) < \gamma$ for each $b > a$. Then let $Y_i \subseteq [a, k_i]$ so that $|Y_i| \leq iw([a, k_i]) = iw(Y_i)$. Take $Y = \bigcup_{i<\kappa} Y_i$. We claim $iw([a, \infty)) \geq iw(Y) \geq \sup\{w([a, k_i]) : i < \kappa\} = \sup\{w([a, b]) : b > a\} = \gamma$. It's clear that $\sup\{w([a, k_i]) : i < \kappa\} = \sup\{w([a, b]) : b > a\}$ since K is cofinal. To verify that $\sup\{w([a, b]) : b > a\} = \gamma$ recall that $w([a, b]) = iw([a, b])$ and that $iw([a, b]) < \gamma \leq iw([a, \infty))$, so $\sup\{w([a, b]) : b > a\} = \gamma$. So $iw(Y) \geq \gamma$, because $iw(Y) = \max\{\kappa, \sup\{iw([a, \kappa_i]) : i < \kappa\}\} = \max\{\kappa, \lambda\}$ and $\kappa \leq \lambda$.

Therefore, for $[a, \infty)$, i-weight reflects all cardinals. Recall, that $w([a, \infty)) = w(X) \geq iw(X) \geq iw([a, \infty)) = w([a, \infty))$. So the i-weight of X is the i-weight of $[a, \infty)$; therefore, i-weight reflects for the space X. $\square$

Theorem 4.17 *Let X be a locally compact linearly ordered space. Then* $iw(X) = \max\{iw(D_{E(X)}),\ \sup\{iw(\tilde{a}) : a \in X\}\} = \max\{\log(e(X)),\ \sup\{iw(\tilde{a}) : a \in X\}\}$.

Proof. By monotonicity, we know that $iw(X) \geq \max\{iw(D_{E(X)}),\ \sup\{iw(\tilde{a}) : a \in X\}\}$. Now suppose that $\lambda = \sup\{iw(\tilde{a} : a \in X\}$. Then there is a condensation of X into $D_{E(X)} \times I^{\lambda}$, which has i-weight $\max\{iw(D_{E(X)}),\ \lambda\}$. So $iw(X) = \max\{iw(D_{E(X)}),$ $\sup\{iw(\tilde{a}) : a \in X\}\}$.

Also $iw(D_{E(X)}) = \log|E(X)| = \log(e(X))$, therefore, $iw(X) = \max\ \{\log(e(X)),$ $\sup\{iw(\tilde{a}) : a \in X\}\}$. □

Theorem 4.18 *Let X be a locally compact linearly ordered space. If $iw(X) = iw(D_{E(X)})$ $= \log(e(X))$, then i-weight reflects the cardinal κ if and only if either $2^{\lambda} < \kappa$ for all $\lambda < \kappa$ or $\kappa \leq \sup\{iw(\tilde{a}) : a \in X\}$. Hence, if $iw(X) = \sup\{iw(\tilde{a}) : a \in X\}$, then i-weight reflects all cardinals.*

Proof. Suppose first that X is as above, and $iw(X) = iw(E(X))$. Assume that i-weight reflects the cardinal κ and $\kappa > \sup\{iw(\tilde{a}) : a \in X\}$. There is a Y contained in X so that $|Y| \leq \kappa$ and $iw(Y) \geq \kappa$. Since $iw(Y \cap \tilde{a}) \leq iw(\tilde{a})$ we have that $iw(Y \cap \tilde{a}) < \kappa$ for each $a \in X$. Also, since $|Y| \leq \kappa$, $Y \cap \tilde{a}$ is nonempty for only κ-many different equivalence classes. So let $\{\tilde{a_i} : i < \kappa\} = \{\tilde{a} : \tilde{a} \cap Y \neq \emptyset\}$. Then for we may condense Y into $\kappa \times I^{\sup\{iw(\tilde{a_i}):i<\kappa\}}$, which has i-weight $iw(k)$ since $\kappa > \sup\{iw(\tilde{a}) : a \in X\}$. Therefore, $\kappa = iw(Y) \leq iw(\kappa) \leq \kappa$. So, for each $\lambda < \kappa$, we have $2^{\lambda} < \kappa$; else, if $2^{\lambda} \geq \kappa$ for some $\lambda < \kappa$, the i-weight of κ would be λ.

Next, assume that i-weight reflects κ and $2^{\lambda} \geq \kappa$ for some $\lambda < \kappa$. Aiming for a contradiction, further assume that $\kappa > \sup\{iw(\tilde{a}) : a \in X\}$. Since i-weight reflects

κ, there is a set Y so that $|Y| \leq \kappa$ and $iw(Y) \geq \kappa$. Then Y can be condensed into $\kappa \times I^{\sup\{iw(\tilde{a}):a\in X\}}$ which has i-weight $\max\{\lambda, \sup\{iw(\tilde{a}) : a \in X\}\} \leq \kappa$, contradiction.

Now we prove the reverse direction.

Assume that $2^\lambda < \kappa$ for all $\lambda < \kappa$ and $\kappa \leq iw(X) = iw(E(X)) \leq E(X)$. Pick κ-many different a_i so that $\{\tilde{a}_i : i < \kappa\}$ is a collection of pairwise disjoint open sets. Then for each $i < \kappa$, pick $y_i \in \tilde{a}_i$ and define $Y = \{y_i : i < \kappa\}$. So Y is a discrete space of cardinality κ, so the i-weight of Y is $\log(\kappa) = \kappa$.

If $\kappa \leq \sup\{iw(\tilde{a}) : a \in X\}$, then consider two cases. First, if $\kappa < \sup\{iw(\tilde{a}) : a \in X\}$, then let $x \in X$ be so that $iw(\tilde{x}) \geq \kappa$. Then for $\tilde{x}$, i-weight reflects κ.

Suppose that $\kappa > iw(\tilde{a})$ for each $a \in X$. Then for some $\gamma \leq \kappa$ let $\{a_i : i < \gamma\}$ be a subset of X so that $\{iw(\tilde{a}_i) : i < \gamma\}$ is cofinal in κ. Then for each $i < \gamma$ pick $Y_i \subseteq \tilde{a}_i$ so that $|Y| \leq iw(\tilde{a}_i) = iw(Y_i)$. Let $Y = \bigcup_{i<\gamma} Y_i$. Then $|Y| \leq \kappa$ and $iw(Y) = \sup\{iw(\tilde{a}_i) : i < \gamma\} = \kappa$. $\square$

Next, we present some examples of linearly ordered spaces for which i-weight does not reflect some cardinal because $E(X) > iw(X)$. So the conditions on $E(X)$ in the preceding theorem may not be omitted. There is also an example of a locally compact linearly ordered space for which $E(X) > iw(X)$ and yet i-weight will reflect all cardinals less than $iw(X)$.

Lemma 4.19 *Any infinite discrete space is homeomorphic to a linearly ordered space.*

Proof. Suppose that X is a discrete space of cardinality κ. Let $X' = \kappa \times \mathbb{Z}$ and order X' lexicographically. Then $((\alpha, n-1), (\alpha, n+1)) = \{(\alpha, n)\}$ for each point $(\alpha, n) \in \kappa \times \mathbb{Z}$, so X' is a discrete space of cardinality κ. Then X is homeomorphic to X'. $\square$

Theorem 4.20 *(GCH). There is a locally compact linearly ordered space X with $iw(X) = \omega_1$, yet for X i-weight does not reflect ω_1.*

Proof. Consider $X = D_{2^{\omega_1}}$. Then by [12], we know that $2^{\omega_1} \leq 2^{iw(X)}$. Since the order topology is coarser than the discrete topology, $iw(X) = \omega_1$.

Take any subset Y of X so that $|Y| \leq \omega_1$. Since Y is discrete, Y may be condensed onto a subset of the real line. Thus $iw(Y) = \omega$. □

We may eliminate the need for GCH if we are willing to allow the i-weight to exceed ω_1. The following is also an example of a paracompact space for which the i-weight does not reflect.

Proposition 4.21 *There is a locally compact linearly ordered space X that has $iw(X) \geq \omega_1$, yet for X i-weight does not reflect ω_1.*

Proof. Take $X = D_{2^{2^{\omega_1}}}$. Then $iw(X) \geq \omega_1$; else if $iw(X) = \omega$ then $|X| < 2^{\omega} \leq 2^{\omega_1} < 2^{2^{\omega_1}}$. Take Y to be a subset of cardinality not exceeding ω_1 and just as above, $iw(Y) = \omega$. □

Theorem 4.22 *For each κ there is an example of a linearly ordered locally compact space X so that $\kappa = iw(X) < E(X)$ and yet i-weight reflects all cardinals.*

Proof. Suppose X' is any compact linearly ordered space so that $iw(X') = \kappa$. Give $\lambda = |2^{\kappa}|$ the order topology, and let L be the set of successor ordinals in λ. Notice that L has cardinality λ and that $iw(L) \leq \kappa$. Let $X = L \times X'$ have the topology induced by the lexicographic order.

First, notice that $E(X) = e(X) = \lambda$. Next, we observe that $iw(X) \leq \kappa$. We know that the i-weight of X under the product topology is κ. We just need to show that the product topology is weaker than the order topology. Suppose that $U \times V$ is a basic open set in the product topology. Then let $(\alpha, x) \in U \times V$. So $\alpha \in \lambda$ and $x \in X'$. If α is a successor ordinal then for any (a, b) so that $x \in (a, b) \subseteq V$, the point $(\alpha, x) \in ((\alpha, a), (\alpha, b)) \subseteq U \times V$. So $iw(X) \leq \kappa$ and by monotonicity, $iw(X) \geq \kappa$; hence $iw(X) = \kappa$.

Now suppose that $iw(X) \geq \gamma$. Then $iw(\{j\} \times X') \geq \gamma$ for each $j < \lambda$, and by Corollary 4.6 i-weight reflects cardinal γ for X'. Find $Y' \subseteq X'$ so that $|Y'| \leq \gamma$ and $iw(Y') \geq \gamma$. Then $iw(\{j\} \times Y') \geq \gamma$ and $|Y'| \leq \gamma$. $\square$

4.4 Paracompact Spaces

In this section we calculate the i-weight of paracompact spaces. This gives us a formula for the i-weight of a compact space as well. We begin with a definition.

Definition. For a Tychonoff space X and $x \in X$, let the local i-weight of x in X be defined as $liw(x, X) = \min\{iw(U) : U \text{ is an open neighborhood of } x\}$. Then the local i-weight of X is $liw(X) = \sup\{liw(x, X) : x \in X\}$.

To see that the local i-weight of a space need not coincide with the i-weight, consider the ordinal space ω_1. The i-weight of ω_1 has been shown to be ω_1. However, the local i-weight of each point in ω_1 is ω, thus $liw(\omega_1) = \omega < iw(\omega_1)$. This example is not compact, indeed it is not paracompact. If we consider the space $X = \omega_1 \cup \{\omega_1\}$ we find that the $liw(\omega_1, X) = \omega_1 = iw(X)$, so in this case local i-weight and i-weight are the same. We will prove that this is true for all compact spaces.

Recall that, by monotonicity of i-weight, $iw(X) \geq \max\{\log(e(X)),$
$liw(X)\}$ for any space X.

Theorem 4.23 *If X is a paracompact, Tychonoff space then $iw(X) = \max\{log(e(X)), liw(X)\}$.*

Proof. Let X be a paracompact Tychonoff space, and cover X with open sets witnessing local i-weight. We may accomplish this by taking for each $x \in X$, a neighborhood N_x of x so that $iw(N_x) = liw(x, X)$; then $\{N_x : x \in X\}$ is the desired cover. Next, let $\mathcal{V}$ be a σ-discrete open refinement of the cover; $\mathcal{V} = \bigcup_{i<\omega} \mathcal{V}_i$ and each $\mathcal{V}_i$ is a discrete family. Then let $W_i = \bigcup\{V \in \mathcal{V}_i\}$, so that $\{W_i : i < \omega\}$ is a countable open cover of X. Notice that since for each $i < \omega$, the set W_i is the union of pairwise disjoint open sets, $iw(W_i) = \max\{log|\mathcal{V}_i|, \sup\{iw(V) : V \in \mathcal{V}_i\}\} \leq \max\{log(e(X)), liw(X)\}$.

Now for each $i < \omega$ find a open set W_i' so that $\overline{W_i'} \subseteq W_i$, and $\{W_i' : i < \omega\}$ is a cover of X; we can do this by Remark 5.1.7 in [7]. Also, let $\mathcal{R}$ be a countable base of convex sets for the interval $[0,\ 1]$.

Denote the original topology on X by T. Let $\mathcal{B}_i$ be a base for a Tychonoff topology τ_i on W_i which is weaker than the subspace topology inherited from T and $|\mathcal{B}_i| = iw(W_i)$. Since we will be discussing several different topologies and subspaces we will denote the closure of a set U as a subset of X under the topology T by $\mathrm{cl}_{X,T}(U)$. Let $\mathcal{B}_i' = \{B \cap W_i' : B \in \mathcal{B}_i\}$ and define $\mathcal{A}_i = \{(U', U) \in \mathcal{B}_i' \times \mathcal{B}_i : \mathrm{cl}_{W_i,\tau_i}(U') \subseteq U\}$.

We claim for each $x \in W_i'$ and $y \neq x$ there is a $(U', U) \in \mathcal{A}_i$ with $x \in U'$ and $y \notin U$. Take $x \in W_i' \subseteq W_i$, and suppose that $y \neq x$. If $y \in W_i'$ then there is a pair $U, V \in \mathcal{B}_i$ so that $x \in U$ and $y \in V$ and $U \cap V = \emptyset$. Next, find $U'' \in \mathcal{B}_i$ so that $x \in U'' \subseteq \mathrm{cl}_{W_i,\tau_i}(U'') \subseteq U$ and let $U' = U'' \cap W_i'$. Then $x \in U' \subseteq \mathrm{cl}_{W_i,\tau_i}(U') \subseteq U$ and $U' \in \mathcal{B}_i'$ so (U', U), so $(U', U) \in \mathcal{A}_i$.

Now assume that $y \notin W_i'$. Then find $U \in \mathcal{B}_i$ so that $x \in U$ and $y \notin U$, and find U' as above.

We claim that if $(U', U) \in \mathcal{A}_i$, then $\mathrm{cl}_{X,T}(U') \subseteq \mathrm{cl}_{W_i,\tau_i}(U')$. Aiming for a contradiction, suppose that $y \in \mathrm{cl}_{X,T}(U')$ and $y \notin \mathrm{cl}_{W_i,\tau_i}(U')$. Since $U' \subseteq W_i'$ and $\mathrm{cl}_{X,T}(W_i') \subseteq W_i$, we have $\mathrm{cl}_{X,T}(U') \subseteq \mathrm{cl}_{X,T}(W_i') \subseteq W_i$. Then there is a set V so that $y \in V \in \tau_i$ and $V \cap U' = \emptyset$. Yet, V is open in X, contradiction.

Next, since X is normal and $\mathrm{cl}_{X,T}(U') \subseteq U$ for each $(U', U) \in \mathcal{A}_i$, there is a map $f_{U',U} : X \to [0,\ 1]$ so that $f_{U',U}(U') = \{0\}$ and $f_{U',U}(U^c) = \{1\}$ for each $(U', U) \in \mathcal{A}_i$. Let $\mathfrak{F}_i = \{f_{U',U} : (U', U) \in \mathcal{A}_i\}$ and let $\mathfrak{F} = \bigcup_{i<\omega} \mathfrak{F}_i$. Suppose $x, y \in X$ and $x \neq y$. Choose $i < \omega$ so that $x \in W_i'$. We showed above that there is a $(U', U) \in \mathcal{A}_i$ with $x \in U'$ and $y \notin U$, so $f_{U',U}(x) = 0$ and $f_{U',U}(y) = 1$.

Then define $F : X \to [0,\ 1]^{\mathfrak{F}}$ by $F(x) = (f(x))_{f \in \mathfrak{F}}$. This map is continuous and one-to-one. Consider $F(X) \subseteq [0,\ 1]^{\mathfrak{F}}$. The weight of the Tychonoff topology on $F(X)$ is less than or equal to $|\mathfrak{F}| \cdot \omega$. Let T' be the topology on $F(X)$ and take $F^{-1}(T')$ as a topology on X that is Tychonoff and weaker than T.

Now note that $|\mathfrak{F}| = \max\{|\mathfrak{F}_i| : i < \omega\}$ and that $|\mathfrak{F}_i| = |\mathcal{A}_i| = |\mathcal{B}_i| = iw(W_i)$. Therefore, this will generate a Tychonoff topology whose weight is less than or equal to $\sup\{iw(W_i) : i < \omega\}$ and recall, $iw(W_i) \leq \max\{\log(e(X)), liw(X)\}$. Therefore, $iw(X) \leq \max\{log(e(X)), liw(X)\}$. $\square$

Corollary 4.24 *In the class of compact spaces, i-weight equals local i-weight. Furthermore, for a compact space X, there is a point $x \in X$ so that $liw(x, X) = liw(X) = iw(X)$.*

Proof. Suppose that X is a compact space. For each point $x \in X$, choose open sets W_x and W'_x so that $x \in W'_x \subseteq \overline{W'_x} \subseteq W_x$ and $iw(W_x) \leq liw(x, X)$. Then $\{W'_x : x \in X\}$ is an open cover of X. Choose a finite subcover and denote it $\{W'_{x_i} : i < N\}$. Then $\{W_{x_i} : i < N\}$ is also an open cover of X. By the proof of Theorem 4.23 above, $iw(X) = \sup\{iw(W_{x_i}) : i < N\} = \max\{iw(W_{x_i}) : i < N\} \leq liw(X)$. So $iw(X) = liw(X)$, and $liw(x_i, X) = liw(X)$ for some $i < N$. $\square$

Example. There is a paracompact space X so that $iw(X) = \log(e(X))$. Let X' be a paracompact space for which i-weight and local i-weight coincide. Let $iw(X') = \kappa$ and consider $D_{2^{2^\kappa}}$. Define $X = D_{2^{2^\kappa}} \times X'$. Then $\{\alpha\} \times X'$ is clopen for each $\alpha \in D_{2^{2^\kappa}}$, so X is still paracompact. Also, $liw(X) = \kappa$ while $\log(e(X)) = 2^\kappa$, so $iw(X) = \log(e(X))$.

Recall Theorems 4.17 and 4.18 for locally compact linearly ordered spaces. Theorem 4.23 somewhat parallels Theorem 4.17, so we considered the question: For the class of paracompact Tychonoff spaces, if i-weight is determined by the local i-weight, does i-weight reflect all cardinals? This next example shows that that is not necessarily the case.

Example. Let L denote the set of limit ordinals strictly less than ω_2. Then let $X = (\omega_2 \cup \{\omega_2\}) \setminus L$ with the topology inherited from the order topology. We claim that X is a paracompact GO space for which i-weight is ω_2 and i-weight does not reflect the cardinal ω_1.

Aiming for a contradiction, suppose that the i-weight of X is ω_1 and that $\mathcal{B}$ is a base for a Tychonoff topology for X of cardinality ω_1 or less. For each $x \in X$, let $U_x \in \mathcal{B}$ be so that $x \notin U_x$ and $\omega_2 \in U_x$. Then for each $x \in X$ let β_x be the least ordinal so that $(\beta_x, \omega_2] \subseteq U_x$. Since $|\mathcal{B}| \leq \omega_1$ at most ω_1 different β_x are chosen; therefore, for some

$\beta < \omega_2$ the set $\{x \in X : \beta = \beta_x\}$ has cardinality ω_2. However, $|\beta| \leq \omega_1$. Therefore, for some $x \in X$ we have $x \in (\beta_x, \omega_2] \subseteq U_x$, contradiction.

We now claim that any subset of X of cardinality ω_1 is discrete and therefore has i-weight ω. Note that if A is a subset of cardinality ω_1 that does not contain ω_2 then A is clearly discrete. Next, suppose that A is a subset of X that contains the point ω_2. Observe that since ω_2 is regular, no subset of cardinality ω_1 is cofinal in ω_2. Therefore, each subset of ω_2 with cardinality ω_1 is bounded. We find $\gamma < \omega_2$ so that $\gamma > a$ for all $a \in A \setminus \{\omega_2\}$. So $(\gamma, \omega_2] \cap A = \{\omega_2\}$; hence A is discrete.

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