The Intersection Problem for Steiner Triple Systems

by

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Abstract

A Steiner triple system of order n is a pair (S, T) where T is an edge disjoint partition of the edge set of K_n (the complete undirected graph on n vertices with vertex set S) into triangles (or triples). It is by now well-known that the spectrum for triple systems is precisely the set of all $n \equiv 1$ or $3 \pmod 6$ [2]. Let $J(n)$ denote the possible number of triples that two triple systems of order n can have in common. In this thesis, we provide an alternative proof of the following result of C.C. Lindner and A. Rosa [4] that

$$J(n) = \{0, 1, 2, \ldots, x = \frac{n(n-1)}{6}\} \backslash \{x-1, x-2, x-3, x-5\}$$

for all $n \equiv 1$ or $3 \pmod 6$ with the exceptions of $n = 9$ and $n = 13$. Our alternative proof of this result makes use of H.L. Fu's solution to the intersection problem for quasigroups [1] which was not available at the time of the original publication.

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Chapter 1

Introduction

A Steiner triple system (or more simply triple system) of order n is a pair (S,T) where T is an edge disjoint partition of the edge set of K_n (the complete undirected graph on n vertices with vertex set S) into triangles (or triples). It is by now well-known that the spectrum for triple systems is precisely the set of all $n \equiv 1$ or $3 \pmod 6$ [2]. It is trivial to see that if (S,T) is a triple system of order n that $|T| = \frac{n(n-1)}{6}$.

A problem that has been around for some time is the intersection problem for triple systems.

The Intersection Problem. *For which $k \in \{0,1,2,\ldots,\frac{n(n-1)}{6}\}$ does there exist a pair of triple systems (S,T_1) and (S,T_2) of order n such that $|T_1 \cap T_2| = k$?*

A routine computation shows that $k \in I(n) = \{0,1,2,\ldots,x = \frac{n(n-1)}{6}\} \backslash \{x-1, x-2, x-3, x-5\}$ is necessary. It turns out that this necessary condition is also sufficient except for $n = 9$ or 13. If we denote by $J(n)$ the intersection numbers for STS(n), in [4] Lindner and Rosa show that $J(n) = I(n)$ for all $n \equiv 1$ or $3 \pmod 6$ except for $n = 9$ and 13. In the case of $n = 9$ Kramer and Mesner [3] have shown that $J(9) = \{0,1,2,3,4,6,12\} = I(9)\backslash\{5,8\}$. This was done on a computer. In 1975 Lindner and Rosa [4] showed that $J(13) = I(13)\backslash\{15,17,19\}$. These are the only exceptions.

Theorem 1.1. (*Lindner and Rosa* [4])
$J(n) = I(n)$ for all $n \equiv 1$ or $3 \pmod 6$ except for $n = 9$ and 13. In these cases $J(9) = I(9)\backslash\{5,8\}$ and $J(13) = I(13)\backslash\{15,17,19\}$. □

The object of this thesis is a completely different and basic proof of this theorem using results due to H. L. Fu [1] not available at the time. We will end the introduction with an example.

Example 1.2. $J(7) = I(7) = \{0, 1, 3, 7\}$.

Define Steiner triple systems $(S, T_1), (S, T_2), (S, T_3),$ and (S, T_4) as follows:

$$T_1 = \begin{matrix} 1 & 2 & 3 \\ 1 & 4 & 5 \\ 1 & 6 & 7 \\ 2 & 4 & 6 \\ 2 & 5 & 7 \\ 3 & 4 & 7 \\ 3 & 5 & 6 \end{matrix} \quad T_2 = \begin{matrix} 1 & 2 & 3 \\ 1 & 4 & 6 \\ 1 & 5 & 7 \\ 2 & 4 & 7 \\ 2 & 5 & 6 \\ 3 & 4 & 5 \\ 3 & 6 & 7 \end{matrix} \quad T_3 = \begin{matrix} 1 & 2 & 3 \\ 1 & 4 & 5 \\ 1 & 6 & 7 \\ 2 & 4 & 7 \\ 2 & 5 & 6 \\ 3 & 4 & 6 \\ 3 & 5 & 7 \end{matrix} \quad T_4 = \begin{matrix} 1 & 3 & 4 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \\ 4 & 6 & 7 \\ 5 & 7 & 1 \\ 6 & 1 & 2 \\ 7 & 2 & 3 \end{matrix}$$

Then $|T_1 \cap T_4| = 0$, $|T_1 \cap T_2| = 1$, $|T_1 \cap T_3| = 3$, and $|T_1 \cap T_1| = 7$.

Chapter 2

Intersection of Two STS(15)

In this section we will give a construction which will allow us to construct a pair of STS(15) which intersect in n triples for any $n \in I(15)$. Before we do this, we need a theorem about quasigroups.

Theorem 2.1. (*H. L. Fu* [1])
The intersection numbers for a pair of quasigroups of order 4 are $\{0, 1, 2, 3, 4, 6, 8, 9, 12, 16\}$. *For quasigroups of order* $n \geq 5$ *the intersection numbers are* $\{0, 1, 2, \ldots, n^2\} \backslash \{n^2 - 1, n^2 - 2, n^2 - 3, n^2 - 5\}$. $\square$

With this theorem in hand we now present our constructions for STS(15).

Construction 2.2. *Let* X *be a set of size 4 and set* $S = \{\infty_1, \infty_2, \infty_3\} \cup (X \times \{1, 2, 3\})$. *Define a collection of triples* T *as follows:*

1. *Define a triple system* T_i *of order 7 on each* $\{\infty_1, \infty_2, \infty_3\} \cup (X \times \{i\})$, $i \in \{1, 2, 3\}$, *making sure that* $\{\infty_1, \infty_2, \infty_3\}$ *is a triple in each triple system, and place these triples in* T.

2. *Let* $(X, \circ)$ *be a quasigroup of order 4 and for each* $x, y \in X$, *place the triple* $\{(x, 1), (y, 2), (x \circ y, 3)\}$ *in* T.

Then (S,T) is a triple system of order 15.

Lemma 2.3. $J(15) = I(15)$.

Proof. Let (S, T_1) and (S, T_2) be two triple systems constructed using Construction 2.2. It is straight forward to see that the number of Type 1 triples that T_1 and T_2 can have in common is $a + b + c$ where $a \in \{0, 1, 3, 7\}$, $b \in \{0, 2, 6\}$, and $c \in \{0, 2, 6\}$. The number of Type 2 triples, d, in common comes from the set $\{0, 1, 2, 3, 4, 6, 8, 9, 12, 16\}$. Since the numbers a, b, c, and d can be chosen independently we can construct T_1 and T_2 so that $|T_1 \cap T_2| \in I(n)\setminus\{26\}$. In order to achieve an intersection of 26 triples we will need the following construction.

Construction 2.4. *Let X be a set of size 7 and set $S = \{\infty\} \cup (X \times \{1, 2\})$. Let $(X, \circ)$ be an idempotent commutative quasigroup of order 7 and define triples T as follows:*

1. $\{\infty, (x, 1), (x, 2)\} \in T$ *for all* $x \in X$,
2. *for all* $x \neq y \in X$, $\{(x, 1), (y, 1), (x \circ y, 2)\} \in T$, *and*
3. *define any triple system on* $X \times \{2\}$.

Let $(X, \circ_1)$ and $(X, \circ_2)$ be the idempotent commutative quasigroups of order 7 defined below:

$$(1)\quad \begin{array}{ccccccc} 1 & 3 & 2 & 5 & 4 & 7 & 6 \\ 3 & 2 & 1 & 6 & 7 & 5 & 4 \\ 2 & 1 & 3 & 7 & 6 & 4 & 5 \\ 5 & 6 & 7 & 4 & 2 & 3 & 1 \\ 4 & 7 & 6 & 2 & 5 & 1 & 3 \\ 7 & 5 & 4 & 3 & 1 & 6 & 2 \\ 6 & 4 & 5 & 1 & 3 & 2 & 7 \end{array} \qquad (2)\quad \begin{array}{ccccccc} 1 & 3 & 2 & 5 & 4 & 7 & 6 \\ 3 & 2 & 4 & 6 & 7 & 1 & 5 \\ 2 & 4 & 3 & 7 & 6 & 5 & 1 \\ 5 & 6 & 7 & 4 & 1 & 3 & 2 \\ 4 & 7 & 6 & 1 & 5 & 2 & 3 \\ 7 & 1 & 5 & 3 & 2 & 6 & 4 \\ 6 & 5 & 1 & 2 & 3 & 4 & 7 \end{array}$$

Then $(X, \circ_1)$ and $(X, \circ_2)$ have 24 entries in common (off the main diagonal). Using this pair of quasigroups gives 12 Type 2 triples in common. So taking the Type 3 triples to be the same we have $7 + 12 + 7 = 26$ triples in common. $\square$

Chapter 3

Intersection of two STS(n), $n \geq 21$ and $n \equiv 3 (\text{mod } 6)$

In this section we will give two general constructions which will allow us to construct two STS(n) intersecting in t triples for $t \in I(n)$ for $n \geq 21$ and $n \equiv 3(\text{mod } 6)$. From here on let $n = 6k + 3$. Our first construction will consider the cases when $2k + 1 \equiv 1$ or $3(\text{mod } 6)$.

Construction 3.1. *Let $(Q, \circ)$ be a guasigroup of order $2k + 1 \equiv 1$ or $3(mod\ 6)$ and define the following collection of triples T on $X = Q \times \{1, 2, 3\}$.*

1. *For each $i \in \{1, 2, 3\}$ define a STS$(2k + 1)$ T_i on $Q \times \{i\}$ and put these triples in T.*
2. *For each x,y $\in Q$ put the triple $\{(x, 1), (y, 2), (x \circ y, 3)\}$ in T.*

It is straight forward to see that (X, T) is a triple system of order $n = 6k + 3$.

Lemma 3.2. *$J(n) = I(n)$ for all $n \equiv 3(mod\ 6)$, where $n = 3(2k + 1) \geq 15$ and $2k + 1 \equiv 1$ or $3(mod\ 6)$.*

Proof. Let $(Q \times \{i\}, T_{i_1})$ and $(Q, \times\{i\}, T_{i_2})$. be a pair of STS$(2k+1)$ for $i \in \{1, 2, 3\}$ and let $(Q, \circ_1)$ and $(Q, \circ_2)$ be a pair of quasigroups. We can take $(Q\times\{i\}, T_{i_1})$ and $(Q\times\{i\}, T_{i_2})$ to be equal or disjoint [4] and $|(Q, \circ_1)\cap(Q, \circ_2)| \in \{0, 1, 2, \ldots, x = (2k+1)^2\}\backslash\{x-1, x-2, x-3, x-5\}$. Denote by T_1 the triples constructed using T_{i_1} and $(Q, \circ_1)$ and T_2 the triples constructed using T_{i_2} and $(Q, \circ_2)$. Then the triple systems $(Q \times \{1, 2, 3\}, T_1)$ and $(Q \times \{1, 2, 3\}, T_2)$ have $\{0, t\} + \{0, t\} + \{0, t\} + \{0, 1, 2, \ldots, x = (2k + 1)^2\}\backslash\{x - 1, x - 2, x - 3, x - 5\}$ in common where $t = \frac{(2k+1)2k}{6}$. A simple computation shows that $J(n) = I(n)$. □

Now we will give a construction for $n = 6k + 3$ and $2k + 1 \equiv 5(\text{mod } 6)$.

Construction 3.3. *Let $(Q, \circ)$ be a quasigroup of order $2k \equiv 4(mod\ 6)$ and define the following collection of triples T on $X = \{\infty_1, \infty_2, \infty_3\} \cup (Q \times \{1, 2, 3\})$.*

1. *For each $i \in \{1, 2, 3\}$ define a STS$(3 + 2k)$ T_i on X and place these triples in T. The triple $\{\infty_1, \infty_2, \infty_3\}$ must belong to T_2 and T_3.*

2. *For each $x, y \in Q$ place the triple $\{(x, 1), (y, 2), (x \circ y, 3)\}$ in T.*

Then (X, T) is a triple system of order $6k + 3$.

Lemma 3.4. *$J(n) = I(n)$ for all $n \equiv 3(mod\ 6)$, where $n = 3 + 6k \geq 33$ and $2k \equiv 4(mod\ 6)$.*

Proof. Let $X_i = \{\infty_1, \infty_2, \infty_3\} \cup (Q \times \{i\})$ and let (X_i, T_{i_1}) and (X_i, T_{i_2}) be a pair of STS$(3 + 2k)$ for $i \in \{1, 2, 3\}$ and let $(Q, \circ_1)$ and $(Q, \circ_2)$ be a pair of quasigroups. We can take (X_1, T_{1_1}) and (X_1, T_{1_2}) to be equal or disjoint and (X_1, T_{i_1}) and (X_i, T_{i_2}) to be equal or intersecting in $\{\infty_1, \infty_2, \infty_3\}$ for $i = 2$ and 3; and $|(Q, \circ_1) \cap (Q, \circ_2)| \in \{0, 1, 2, \ldots, x = (2k)^2\} \backslash \{x - 1, x - 2, x - 3, x - 5\}$. Denote by T_1 the triples constructed using T_{i_1} and $(Q, \circ_1)$ and T_2 the triples constructed using T_{i_2} and $(Q, \circ_2)$. Then the triple systems (X, T_1) and (X, T_2) have $\{0, t\} + \{0, t-1\} + \{0, t-1\} + \{0, 1, 2, \ldots, x = (2k)^2\} \backslash \{x - 1, x - 2, x - 3, x - 5\}$ triples in common where $t = \frac{(2k+3)(2k+2)}{2} = (2k + 1)(k + 1)$. A simple computation shows that $J(n) = I(n)$. $\square$

Combining [3], Construction 2.4, and Lemma 3.4 gives the following Theorem.

Theorem 3.5. *$J(n) = I(n)$ for all $n \equiv 3(mod\ 6)$, except for $J(9)$. In this case, $J(9) = I(9) \backslash \{5, 8\}$.* $\square$

Chapter 4

Intersection of two STS(n), $n \geq 19$ and $n \equiv 1(\text{mod } 6)$

In this section we will give two general constructions which will allow us to construct two STS(n) intersecting in t triples for $t \in I(n)$ for $n \geq 19$ and $n \equiv 1(\text{mod } 6)$. From here on let $n = 6k+1$. Our first construction will consider the cases when $2k \equiv 0$ or $2(\text{mod } 6)$.

Construction 4.1. *Let X be a set of size $2k \equiv 0$ or $2(\text{mod } 6)$ and $(X, \circ)$ a quasigroup of order $2k$. Denote by T the following collection of triples defined on $S = \{\infty\} \cup (X \times \{1,2,3\})$.*

1. *For each $i \in \{1,2,3\}$ let $\{\infty\} \cup (X \times \{i\})$ be a STS$(2k+1)$ (Remember that $1+2k \equiv 1$ or $3(\text{mod } 6)$) and place these triples in T.*

2. *For each $x, y \in X$ place the triple $\{(x,1),(y,2),(x \circ y, 3)\}$ in T.*

Then (S,T) is a triple system of order $2k+1$.

Lemma 4.2. *$J(n) = I(n)$ where $n = 6k+3$ and $2k \equiv 0$ or $2(\text{mod } 6)$.*

Proof. For each $i \in \{1,2,3\}$ let $(\{\infty\} \cup (X \times \{i\}), T_{1_i})$ and $(\{\infty\} \cup (X \times \{i\}), T_{2_i})$ be a pair of STS($2k+1$) and let $(X, \circ_1)$ and $(X, \circ_2)$ be a pair of quasigroups (any pair). We can take T_{1_i} and T_{2_i} to be equal or disjoint and $|(X, \circ_1) \cap (X, \circ_2)| \in \{0,1,2,\ldots,x=(2k)^2\} \backslash \{x-1, x-2, x-3, x-5\}$. Let T_1 be constructed from the triples T_{1_i} and T_2 constructed from the triples T_{2_i}. It is immediate that $|T_1 \cap T_2| \in \{0, \frac{(2k+1)2k}{6}\} + \{0, \frac{(2k+1)2k}{6}\} + \{0, \frac{(2k+1)2k}{6}\} + \{0,1,2,\ldots,x=(2k)^2\} \backslash \{x-1, x-2, x-3, x-5\}$ and that $J(n) = I(n)$. □

We will now give a construction for $n = 6k+1$ and $2k \equiv 4(\text{mod } 6)$.

Construction 4.3. *Let* $2k \equiv 4(mod\ 6)$*,* X *a set of size* $2k$*, and* (X, P) *a GDD*$(2k, \{4^*, 2\}, 3)$*. Define triples* T *on* $S = \{\infty\} \cup (X \times \{1, 2, 3\})$ *as follows:*

1. *Let* g^* *be the group of size 4 and let* $(\{\infty\} \cup (g^* \times \{1, 2, 3\}))$ *be a triple system of order 13.*

2. *For each group* h *of size 2 let* $\{\infty\} \cup (h \times \{1, 2, 3\})$ *be a triple system of order 7.*

3. *Finally for each triple* $\{x, y, z\}$ *let* (P, B) *be a GDD*$(9, 3, 3)$ *of order 9 with groups* $\{x\} \times \{1, 2, 3\}$*,* $\{y\} \times \{1, 2, 3\}$*,* $\{z\} \times \{1, 2, 3\}$*. This is equivalent to a quasigroup of order 3.*

Then (S, T) *is a triple system of order* $6k + 1$*, and we can take the intersection of two of the GDD*$(9, 3, 3)$*s to be* $0, 3,$ *or* 9*.*

Lemma 4.4. $J(n) = I(n)$ *for* $n = 6k + 1$ *and* $2k \equiv 4(mod\ 6)$*.*

Proof. We can now build two STS(n)s using Construction 4.3 with intersection numbers belonging to $J(13) = I(13)\setminus\{15, 17, 19\}$, $J(7) = I(7) = \{0, 1, 3, 7\}$, and $\{0, 3, 9\}$ for each triple. This gives intersection numbers $J(6k + 1) = I(6k + 1)\setminus\{x - 11, x - 9, x - 7\}$ where $x = \frac{(6k+1)6k}{6}$. To obtain the numbers $x-11$, $x-9$, $x-7$ is easy. Let (S,T) be any STS($6k+1$) constructed using Construction 4.3. To obtain $x - 7$ define a pair of disjoint triple systems on one of the groups of size 2. To obtain $x - 9$ use a pair of GDD(9,3,3)s having 0 triples in common. To obtain $x - 11$ define a pair of disjoint triple systems on one of the groups and a pair of triple systems intersecting in 3 triples on one of the groups. $\square$

Theorem 4.5. $J(n) = I(n)$ *for all* $n \equiv 1(mod\ 6)$*, except for* $J(13)$*. In this case,* $J(13) = I(13)\setminus\{15, 17, 19\}$*.* $\square$

Chapter 5

Concluding Remarks

Combining Theorems 3.5 and 4.5 gives the following theorem.

Theorem 5.1. *$J(n) = I(n)$ for all $n \equiv 1$ or $3 (mod\ 6)$ except for $n = 9$ and 13. In these cases $J(9) = I(9) \backslash \{5, 8\}$ and $J(13) = I(13) \backslash \{15, 17, 19\}$.* □

As mentioned in the introduction the object of this thesis is a new and much shorter proof of the intersection problem for Steiner triple systems. What makes the constructions easier and shorter than the original paper is the use of H. L. Fu's results [1] on the complete solution of the intersection problem for quasigroups. Fu's results were obtained 10 years after the initial solution by C. C. Lindner and A. Rosa. As the old saying goes: time marches on.

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