Techniques for Finding Homeomorphisms Between Generalized Inverse Limits

by

Alan Bertl

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Approved by

Michel Smith, Professor of Mathematics
Stewart Baldwin, Professor of Mathematics
Ziqin Feng, Professor of Mathematics
Gary Gruenhage, Professor of Mathematics
Randall R. Holmes, Professor of Mathematics

Abstract

Techniques are presented for finding homeomorphisms between generalized inverse limits, including a generalization of techniques introduced by Smith and Varagona, and a characterization in terms of category theory.

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Chapter 0

Introduction

Mahavier introduced what later became known as generalized inverse limits in [10], in which several results that hold for traditional inverse limits of compact metric spaces using continuous bonding maps are extended to the case where each factor space is the unit interval and each "bonding map" is not a map at all, but a closed subset of $[0, 1] \times [0, 1]$. Subsequent work by Mahavier with Ingram (see [4], [5]) allowed the factor spaces to be arbitrary compact metric spaces and reframed the bonding relations as upper semi-continuous set-valued maps. The work presented here shows there is value in Mahavier's original idea of viewing the "bonding maps" as relations.

In [13], Smith and Varagona showed that traditional inverse limits with factor spaces the unit interval and "N-type" bonding maps are homeomorphic to a certain class of generalized inverse limits. They showed this by modifying a technique Baldwin used in [1], showing two spaces are homeomorphic by constructing an auxiliary space with all but the initial factor space finite (and not Hausdorff in general), to which each of the two spaces can be shown to be homeomorphic. The work presented here extends the work of Smith and Varagona by giving a general set of conditions under which the technique can be applied, worked out jointly between B and Smith. Additionally, the solenoid is an example where an inverse limit is has a group structure compatible with the topology. It is shown that the techniques of S and V can be applied to topological groups under certain circumstances to establish a topological group isomorphism. This is demonstrated in the case of the solenoid.

Finally, there has been interest (see for example [2]) in characterizing generalized inverse limits in terms of category theory in such a way that generalized inverse limits are limits in the category theoretical sense. We give such a characterization in terms of well-known categories, which has the property that taking generalized inverse limits is functorial, and has a nice adjoint functor. This characterization relies on viewing the bonding relations in a generalized inverse limit as a topological space in its own right, similar to how Mahavier originally viewed them as noted above.

Chapter 1

Background

The following are well known theorems in general topology and continuum theory. Proofs can be found in [7], [8], [12], and [3].

Theorem 1.1. *If X is compact and Y is Hausdorff and $\varphi : X \to Y$ is a continuous bijection, then φ is a homeomorphism.*

Definition Let I be a set, and for each $i \in I$, let X_i be a topological space. Define $\prod_{i \in I} X_i$ to be the space having underlying set all functions $\mathbf{x}$ such that $\mathbf{x}(i) \in X_i$ for each $i \in I$, and topology generated by basis $\prod_{i \in I} U_i$, where U_i is an open subset of X_i for each $i \in I$, and $U_i = X_i$ for all but finitely many $i \in I$. $\mathbf{x}(i)$ is typically written x_i.

Definition For each $i \in \omega$, let X_i be a topological space and $f_i : X_{i+1} \to X_i$ be a continuous function. Define $\varprojlim \mathbf{f}$ to be the subspace of $\prod_{i \in \omega} X_i$ with underlying set $\{\mathbf{x} \in \prod_{i \in \omega} X_i \mid f_i(x_{i+1}) = x_i \text{ for all } i \in \omega\}$.

Definition For each $i \in \omega$, let X_i be a topological space and r_i be a subset of $X_i \times X_{i+1}$. Define $\varprojlim \mathbf{r}$ to be the subspace of $\prod_{i \in \omega} X_i$ with underlying set $\{\mathbf{x} \in \prod_{i \in \omega} X_i \mid (x_i, x_{i+1}) \in r_i \text{ for all } i \in \omega\}$. We will occasionally also use the notation $\varprojlim \{r_i\}_{i \in \omega}$ for $\varprojlim \mathbf{r}$.

Theorem 1.2. *For each $i \in \omega$ let X_i be a compact space. Then $\prod_{i \in \omega} X_i$ is compact. Moreover, if for each $i \in \omega$ $f_i : X_{i+1} \to X_i$ is a continuous function, then $\varprojlim \mathbf{f}$ is compact.*

Theorem 1.3. *For each $i \in \omega$ let X_i be a compact space and r_i be a closed subset of $X_i \times X_{i+1}$. Then $\varprojlim \mathbf{r}$ is compact.*

Theorem 1.4. *Let each of X and Y be a topological space, $f : X \to Y$ be a continuous function, U be a subset of X, and V be a subset of Y such that $f[U] \subseteq V$. Then $f|_U$ is a continuous function with respect to U and V with the subspace topologies.*

Theorem 1.5. *For each $i \in \omega$ let each of X_i and Y_i be a topological space, and $\varphi_i : X_i \to Y_i$ be a continuous function. Then $\varphi : \prod_{i\in\omega} X_i \to \prod_{i\in\omega} Y_i$ defined by $\varphi(\mathbf{x})_i = \varphi_i(x_i)$ for each $i \in \omega$, is continuous. Moreover, if for each $i \in \omega$, r_i is a subset of $X_i \times X_{i+1}$ and s_i is a subset of $Y_i \times Y_{i+1}$ such that for each $\mathbf{x} \in \varprojlim \mathbf{r}$, $\varphi(\mathbf{x}) \in \varprojlim \mathbf{s}$, then φ restricted to $\varprojlim \mathbf{r}$ is continuous into $\varprojlim \mathbf{s}$.*

Definition Suppose each of X and Y is a topological space and r is a subset of $X \times Y$. The statement that r *is upper semi-continuous* means for each $x \in X$, $\{b \mid (x, b) \in r\}$ is closed, and if V is an open subset of Y such that $\{b \mid (x, b) \in r\} \subseteq V$ then there is an open subset U of X containing x such that $\{b \mid \exists u \in U \text{ such that } (u, b) \in r\} \subseteq V$.

Theorem 1.6. *Let r be a subset of $X \times Y$ such that Y is compact and regular. Then r is upper semi-continuous if and only if r is closed under the product topology.*

Theorem 1.7. *For each $i \in \omega$ let X_i be a group. Then $\prod_{i\in\omega} X_i$ is a group under the induced operation $(\mathbf{xy})_i = x_i y_i$ for each $i \in \omega$.*

Chapter 2

Finite Domain Spaces

2.1 Definitions

Definition For each $i \in \omega$ let X_i be a topological space and r_i be a subset of $X_i \times X_{i+1}$ (r_i thus can be viewed as a relation). Define $\varprojlim \mathbf{r}$ to be the subspace of $\prod_{i\in\omega} X_i$ to which an element $\mathbf{x}$ of $\prod_{i\in\omega} X_i$ belongs only in the case that for each $i \in \omega$, $(x_i, x_{i+1}) \in r_i$. We will view an element $\mathbf{x}$ of the product as a function from ω into the corresponding factor space, but will denote $\mathbf{x}(i)$ by x_i.

Remark This is equivalent to the definition of a generalized inverse limit space. Here we have chosen to view each bonding map r_i as a subset of $X_i \times X_{i+1}$ (note the indexing by the domain factor space) rather than a set valued function, which allows us to view the bonding map as a topological space in its own right. This is in the spirit of how Mahavier defined generalized inverse limits originally in [10]. We have a choice of viewing r_i as a subset of $X_i \times X_{i+1}$ or as a subset of $X_{i+1} \times X_i$. We have chosen to go counter to the previous section since it seems more natural to proceed forwards then backwards, all else being equal. Additionally, we will be concerned with the case when a certain subset of r_i is a function, whereas if r_i were viewed as a subset of $X_{i+1} \times X_i$, then a certain subset of r_i^{-1} would be required to be a function.

Before continuing, we establish some notation for use with relations. For a relation r and a set A, define $r[A] = \{b \mid \exists a \in A \text{ such that } (a, b) \in r\}$. The square bracket notation is used to differentiate the image of sets from the operation on an element denoted by parentheses (in the case of functions). If the set A is a singleton, for example $\{x\}$, we will omit the square brackets and write $r\{x\}$.

2.2 Conditions Which Ensure Limits are Homeomorphic

We intend to examine non-Hausdorff spaces that induce a Hausdorff limit. The following condition is sufficient to ensure a limit is Hausdorff.

Definition For each $i \in \omega$ let X_i be a topological space and r_i be a subset of $X_i \times X_{i+1}$. We say $(\mathbf{X}, \mathbf{r})$ is *subsequently separable* if for each $i \in \omega$, each $p \in X_i$, and each pair $a, b \in r_i\{p\}$ such that $a \neq b$, there are disjoint open subsets of X_{i+1} separating a and b.

Lemma 2.2.1. *For each $i \in \omega$, let X_i be a topological space such that X_0 is Hausdorff, and r_i be a subset of $X_i \times X_{i+1}$ such that $(\mathbf{X}, \mathbf{r})$ is subsequently separable. Then $\varprojlim \mathbf{r}$ is Hausdorff.*

Proof. Let each of $\mathbf{u}$ and $\mathbf{v}$ be an element of $\varprojlim \mathbf{r}$, such that there do not exist disjoint open sets separating them. X_0 is a Hausdorff space, so if $u_0 \neq v_0$, basic open subsets of $\varprojlim \mathbf{r}$ can be constructed separating $\mathbf{u}$ and $\mathbf{v}$. Thus $u_0 = v_0$. Proceeding by induction, let i be in ω such that $i > 0$ and suppose that for all $j < i$, $u_j = v_j$. Note that $u_i \in r_i\{u_{i-1}\}$ and $v_i \in r_i\{v_{i-1}\} = r_i\{u_{i-1}\}$. If u_i and v_i are distinct, then since both are in $r_i\{u_{i-1}\}$ and (X, r) is subsequently separable, there are basic open subsets U and V of X_i separating u_i and v_i. So basic open subsets of $\varprojlim \mathbf{r}$ can be constructed separating $\mathbf{u}$ and $\mathbf{v}$. This contradicts our assumption, so $u_i = v_i$. So by induction, for all $i \in \omega$, $u_i = v_i$, and thus $\mathbf{u} = \mathbf{v}$. So $\varprojlim \mathbf{r}$ is Hausdorff. $\square$

Remark We will adopt the habit of defining functions between limit spaces induced by functions between the factor spaces. Henceforth if we have a limit space $\varprojlim \mathbf{r}$ and indexed functions $\{\varphi_i : X_i \to Y_i\}_{i \in \omega}$, we will define φ to be the function with domain $\varprojlim \mathbf{r}$ such that for each $\mathbf{x} \in \varprojlim \mathbf{r}$, $\varphi(\mathbf{x})$ is the element of $\prod_{i \in \omega} Y_i$ such that for each $i \in \omega$, $\varphi(\mathbf{x})_i = \varphi_i(x_i)$.

Our typical situation can be visualized as below:

$$
\begin{tikzcd}
\cdots \arrow[r, "r_{i-1}"] & X_i \arrow[r, "r_i"] \arrow[d, "\varphi_i"] & X_{i+1} \arrow[r, "r_{i+1}"] \arrow[d, "\varphi_{i+1}"] & \cdots \\
\cdots \arrow[r, "s_{i-1}"] & Y_i \arrow[r, "s_i"] & Y_{i+1} \arrow[r, "s_{i+1}"] & \cdots
\end{tikzcd}
$$

Where the arrows between the r_i's and s_i's point from the first coordinate space to the second coordinate space in each case.We will call such a system a generalized limit system.

Definition For each $i \in \omega$ let each of X_i and Y_i be a topological space, r_i be a subset of $X_i \times X_{i+1}$, s_i be a subset of $Y_i \times Y_{i+1}$, and $\varphi_i : X_i \to Y_i$ be a continuous function. We say $(\mathbf{X}, \mathbf{Y}, \mathbf{r}, \mathbf{s}, \varphi)$ *is a generalized limit system.*

It will become clear that if we desire $\varprojlim \mathbf{r}$ and $\varprojlim \mathbf{s}$ to be homeomorphic, we want the above diagram to commute. In other words, we will desire that for each $i \in \omega$, $\varphi_{i+1} \circ r_i = s_i \circ \varphi_i$ using the natural definition of composition of relations.

Definition A generalized limit system $(\mathbf{X}, \mathbf{Y}, \mathbf{r}, \mathbf{s}, \varphi)$ is said to be *commutative* if $\varphi_{i+1} \circ r_i = s_i \circ \varphi_i$ for each $i \in \omega$.

Lemma 2.2.2. *Let $(\mathbf{X}, \mathbf{Y}, \mathbf{r}, \mathbf{s}, \varphi)$ be a generalized limit system such that for each $i \in \omega$, $\varphi_{i+1} \circ r_i \subseteq s_i \circ \varphi_i$. Then for each $\mathbf{x} \in \varprojlim \mathbf{r}$, $\varphi(\mathbf{x}) \in \varprojlim \mathbf{s}$.*

Proof. Let x be in $\varprojlim \mathbf{r}$ and i be in ω. Then:

$$
\begin{aligned}
& (x_i, x_{i+1}) \in r_i \text{ and by the definition of } \varphi_{i+1},\ (x_{i+1}, \varphi_{i+1}(x_{i+1})) \in \varphi_{i+1} \\
\Longrightarrow\ & (x_i, \varphi_{i+1}(x_{i+1})) \in \varphi_{i+1} \circ r_i \subseteq s_i \circ \varphi_i \\
\Longrightarrow\ & \exists q \text{ such that } (x_i, q) \in \varphi_i \text{ and } (q, \varphi_{i+1}(x_{i+1})) \in s_i \\
\Longrightarrow\ & q = \varphi_i(x_i) \text{ and } (q, \varphi_{i+1}(x_{i+1})) \in s_i \\
\Longrightarrow\ & (\varphi_i(x_i), \varphi_{i+1}(x_{i+1})) \in s_i \\
\Longrightarrow\ & (\varphi(\mathbf{x})_i, \varphi(\mathbf{x})_{i+1}) \in s_i.
\end{aligned}
$$

So for each $\mathbf{x} \in \varprojlim \mathbf{r}$, $\varphi(\mathbf{x}) \in \varprojlim \mathbf{s}$. $\square$

Another condition that will be important is that when we consider only those pairs in r_i whose second coordinate lies in a certain range delineated by φ_{i+1}, we have a function.

Notationally, for a relation r and a set B, we denote by $r|^B$ the set $\{(a, b) \in r \mid b \in B\}$, which mirrors the notation for restriction of the domain of a function.

Definition Let r be a subset of $X \times Y$ and $\varphi : Y \to Z$ be a function. The statement that r *is function decomposable relative to* φ means for each $z \in Z$, $r|^{\varphi^{-1}\{z\}}$ is a function.

Lemma 2.2.3. *Let* $(\mathbf{X}, \mathbf{Y}, \mathbf{r}, \mathbf{s}, \varphi)$ *be a generalized limit system such that* φ_0 *is an injection and for each* $i \in \omega$, r_i *is function decomposable relative to* φ_{i+1}. *Then* φ *is an injection.*

Proof. Let each of $\mathbf{u}$ and $\mathbf{v}$ be in $\varprojlim \mathbf{r}$ such that $\varphi(\mathbf{u}) = \varphi(\mathbf{v})$.

$$\begin{aligned} & \varphi_0(u_0) = \varphi(\mathbf{u})_0 = && \varphi(\mathbf{v})_0 = \varphi_0(v_0) \\ \implies & \qquad\qquad u_0 = && v_0. \end{aligned}$$

Proceeding by induction, suppose that $i \in \omega$, $i > 0$, and for all $j < i$, $u_j = v_j$.

$$\begin{aligned} u_i = & r_{i-1}|^{\varphi_i^{-1}\{\varphi_i(u_i)\}}(u_{i-1}) = r_{i-1}|^{\varphi_i^{-1}\{\varphi(\mathbf{u})_i\}}(u_{i-1}) \\ = & r_{i-1}|^{\varphi_i^{-1}\{\varphi(\mathbf{v})_i\}}(v_{i-1}) = r_{i-1}|^{\varphi_i^{-1}\{\varphi_i(v_i)\}}(v_{i-1}) = v_i. \end{aligned}$$

So by induction, $u_i = v_i$ for all $i \in \omega$ and thus $\mathbf{u} = \mathbf{v}$. So φ is an injection. $\square$

Lemma 2.2.4. *Let* $(\mathbf{X}, \mathbf{Y}, \mathbf{r}, \mathbf{s}, \varphi)$ *be generalized limit system such that* φ_0 *is a surjection and for each* $i \in \omega$, r_i *is function decomposable relative to* φ_{i+1} *and* $s_i \circ \varphi_i \subseteq \varphi_{i+1} \circ r_i$. *Then* φ *is a surjection.*

Proof. Let $\mathbf{y}$ be in $\varprojlim \mathbf{s}$. Define $\mathbf{x}$ inductively as follows. Since φ_0 is a surjection, $\varphi_0^{-1}\{y_0\}$ is nonempty, so there is an $x_0 \in \varphi_0^{-1}\{y_0\}$. Let i be in ω such that $i > 0$ and suppose that for all $j < i$, $\varphi_j(x_j) = y_j$. Note that:

$$\begin{aligned}
& (y_{i-1}, y_i) \in s_{i-1} \text{ and } (x_{i-1}, y_{i-1}) \in \varphi_{i-1} \\
\Longrightarrow\ & (x_{i-1}, y_i) \in s_{i-1} \circ \varphi_{i-1} \subseteq \varphi_i \circ r_{i-1} \\
\Longrightarrow\ & \exists b \text{ such that } (x_{i-1}, b) \in r_{i-1} \text{ and } (b, y_i) \in \varphi_i \\
\Longleftrightarrow\ & \exists b \text{ such that } (x_{i-1}, b) \in r_{i-1} \text{ and } b \in \varphi_i^{-1}\{y_i\} \\
\Longleftrightarrow\ & \exists b \text{ such that } (x_{i-1}, b) \in r_{i-1}|^{\varphi_i^{-1}\{y_i\}} \\
\Longleftrightarrow\ & x_{i-1} \in \operatorname{dom}(r_{i-1}|^{\varphi_i^{-1}\{y_i\}}).
\end{aligned}$$

So define $x_i = r_{i-1}|^{\varphi_i^{-1}\{y_i\}}(x_{i-1})$. Then:

$$\varphi(\mathbf{x})_i = \varphi_i(x_i) = \varphi_i(r_{i-1}|^{\varphi_i^{-1}\{y_i\}}(x_{i-1})) \subseteq \varphi_i[\varphi_i^{-1}\{y_i\}] = \{y_i\}.$$

So $\varphi(\mathbf{x})_i = y_i$. Since this is true for all $i \in \omega$, $\varphi(\mathbf{x}) = \mathbf{y}$. So φ is surjection. $\square$

Theorem 2.1. *Let $(\mathbf{X}, \mathbf{Y}, \mathbf{r}, \mathbf{s}, \varphi)$ be a commutative generalized limit system such that $\varprojlim \mathbf{r}$ is compact, Y_0 is Hausdorff, $(\mathbf{Y}, \mathbf{s})$ is subsequently separable, φ_0 is a bijection, and r_i is function decomposable relative to φ_{i+1}. Then φ is a homeomorphism.*

Proof. Since Y_0 is Hausdorff and $(\mathbf{Y}, \mathbf{s})$ is subsequently separable, $\varprojlim \mathbf{s}$ is Hausdorff by Lemma 2.2.1. Since $(\mathbf{X}, \mathbf{Y}, \mathbf{r}, \mathbf{s}, \varphi)$ is commutative, we have that for each $i \in \omega$, $\varphi_{i+1} \circ r_i \subseteq s_i \circ \varphi_i$, so by Lemma 2.2.2, for each $\mathbf{x} \in \varprojlim \mathbf{r}$, $\varphi(\mathbf{x}) \in \varprojlim \mathbf{s}$. Since for each $i \in \omega$, φ_i is continuous, φ is continuous. Since φ_0 is an injection and r_i is function decomposable relative to φ_i for each $i \in \omega$, by Lemma 2.2.3, φ is an injection. Since φ_0 is a surjection and $(\mathbf{X}, \mathbf{Y}, \mathbf{r}, \mathbf{s}, \varphi)$ is commutative, we have that for each $i \in \omega$, $s_i \circ \varphi_i \subseteq \varphi_{i+1} \circ r_i$. r_i is function decomposable relative to φ_{i+1} for each $i \in \omega$, so by Lemma 2.2.4, φ is a surjection. Since $\varprojlim \mathbf{r}$ is compact, $\varprojlim \mathbf{s}$ is Hausdorff, and φ is a continuous bijection, φ is a homeomorphism. $\square$

2.3 An Induced Limit Space

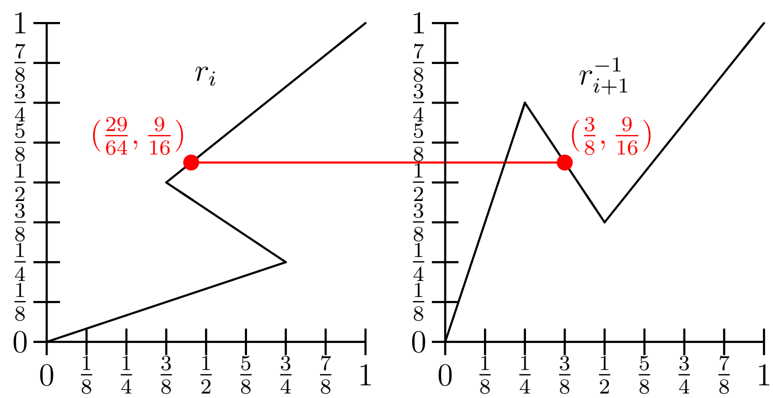
For the following, let X_i be a topological space for each $i \in \omega$ and r_i be a subset of $X_i \times X_{i+1}$, from which we form the limit space $\varprojlim \mathbf{r}$. There is a natural homeomorphism between this space and another limit space whose factor spaces are the r_i's.

For each $i \in \omega$, define t_i to be the subset of $r_i \times r_{i+1}$ to which $((a,b),(c,d))$ belongs only in case $b = c$.

When a graph is available for each r_i, we can use the following method to visualize the relation t_i.

We depict the graph of r_i horizontally next to the graph of r_{i+1}^{-1}. An ordered pair (a,b) from r_i is related to an ordered pair (c,d) from r_{i+1} if and only if a horizontal line can be drawn connecting (a,b) and (d,c) in the depiction.

This is shown below for a situation where all the r_i's are the same.



The horizontal line indicates that $(\frac{29}{64}, \frac{9}{16}) \in r_i$ and $(\frac{9}{16}, \frac{3}{8}) \in r_{i+1}$ are related by t_i.

For each $i \in \omega$, let $\phi_i : r_i \to X_i$ be the function so that for each $(a,b) \in r_i$, $\phi_i(a,b) = a$.

Define $\phi : \varprojlim \mathbf{t} \to \varprojlim \mathbf{r}$ to be the function defined so that for each $\mathbf{w} \in \varprojlim \mathbf{t}$ and each $i \in \omega$, $\phi(\mathbf{w})_i = \phi_i(w_i)$.

Theorem 2.2. *ϕ is a homeomorphism.*

Proof. For each $i \in \omega$, ϕ_i is continuous since it is a projection. So ϕ is continuous.

Claim. *Let $\mathbf{w}$ be in $\varprojlim \mathbf{t}$. Then $\phi(\mathbf{w}) \in \varprojlim \mathbf{r}$.*

Proof. For each $i \in \omega$, $\phi(\mathbf{w})_i = \phi_i(w_i) \in X_i$. For each $i \in \omega$, let $w_i = (a, b)$ and $w_{i+1} = (c, d)$. $((a, b), (c, d)) = (w_i, w_{i+1}) \in t_i$, so $b = c$. We have that

$$(\phi(\mathbf{w})_i, \phi(\mathbf{w})_{i+1}) = (\phi_i(w_i), \phi_{i+1}(w_{i+1})) = (\phi_i(a, b), \phi_i(c, d)) = (a, c) = (a, b) = w_i \in r_i.$$

So $\phi(\mathbf{w}) \in \varprojlim \mathbf{r}$.

Claim. *ϕ is a surjection.*

Proof. Let $\mathbf{y}$ be in $\varprojlim \mathbf{r}$. Define $\mathbf{w}$ so that for each $i \in \omega$, $w_i = (y_i, y_{i+1})$. Since $\mathbf{y} \in \varprojlim \mathbf{r}$, $w_i = (y_i, y_{i+1}) \in r_i$. $w_i = (y_i, y_{i+1})$ and $w_{i+1} = (y_{i+1}, y_{i+2})$, so $(w_i, w_{i+1}) \in t_i$. Thus $\mathbf{w} \in \varprojlim \mathbf{t}$. For each $i \in \omega$, we have that $\phi(\mathbf{w})_i = \phi_i(w_i) = \phi_i(y_i, y_{i+1}) = y_i$. So $\phi(\mathbf{w}) = \mathbf{y}$. So ϕ is a surjection.

Claim. *ϕ is an injection.*

Proof. Let each of $\mathbf{w}$ and $\mathbf{y}$ be in $\varprojlim \mathbf{t}$ so that $\phi(\mathbf{w}) = \phi(\mathbf{y})$. Let i be in ω and let $w_i = (a, b)$, $w_{i+1} = (b, c)$, $y_i = (t, u)$, and $y_{i+1} = (u, v)$. Then

$$a = \phi_i(a, b) = \phi_i(w_i) = \phi(\mathbf{w})_i = \phi(\mathbf{y})_i = \phi_i(y_i) = \phi_i(t, u) = t$$

and

$$b = \phi_{i+1}(b, c) = \phi_{i+1}(w_{i+1}) = \phi(\mathbf{w})_{i+1} = \phi(\mathbf{y})_{i+1} = \phi_{i+1}(y_{i+1}) = \phi_{i+1}(u, v) = u.$$

So $w_i = (a,b) = (t,u) = y_i$. Since this is true for all $i \in \omega$, $\mathbf{w} = \mathbf{y}$. So ϕ in an injection.

Claim. *ϕ is open.*

Proof. Let $U = \prod_{i\in\omega}(U_i \times V_{i+1})$ be a basic open subset of $\prod_{i\in\omega} r_i$, so $U \cap \varprojlim \mathbf{t}$ is an open subset of $\varprojlim \mathbf{t}$. Note for cofinitely many i, $U_i \times V_{i+1} = X_i \times X_{i+1}$, so $U_i = X_i$ and $V_{i+1} = X_{i+1}$.

Let $\mathbf{w}$ be in $U \cap \varprojlim \mathbf{t}$. Then for each $i \in \omega$, $w_i = (a,b)$ for some $(a,b) \in U_i \times V_{i+1}$ and $w_{i+1} = (c,d)$ for some $(c,d) \in U_{i+1} \times V_{i+2}$, and since $((a,b),(c,d)) = (w_i, w_{i+1}) \in t_i$, it must be the case that $b = c$. So U_{i+1} intersects V_{i+1} for all $i \in \omega$.

Note there is no V_0, so for convenient notation define $V_0 = U_0$.

Let $\mathbf{y}$ be in $(\prod_{i\in\omega}(U_i \cap V_i)) \bigcap \varprojlim \mathbf{r}$ (an open subset of $\varprojlim \mathbf{r}$ since $U_i \cap V_i = X_i$ for cofinitely many i). ϕ is a surjection, so there is a $\mathbf{z} \in \varprojlim \mathbf{t}$ such that $\phi(\mathbf{z}) = \mathbf{y}$. For each $i \in \omega$, $z_i = (a,b)$ for some $(a,b) \in r_i$, so $a = \phi_i(a,b) = \phi_i(z_i) = \phi(\mathbf{z})_i = y_i \in U_i \cap V_i \subseteq U_i$. $z_{i+1} = (c,d)$ for some $(c,d) \in r_{i+1}$, with $b = c$ since $((a,b),(c,d)) \in t_i$. So $b = c = \phi_{i+1}(c,d) = \phi_{i+1}(z_{i+1}) = \phi(\mathbf{z})_{i+1} = y_{i+1} \in U_{i+1} \cap V_{i+1} \subseteq V_{i+1}$.

So the open set $(\prod_{i\in\omega}(U_i \cap V_i)) \bigcap \varprojlim \mathbf{r}$ is a subset of $\phi(U \cap \varprojlim \mathbf{t})$ containing $\mathbf{y}$. Since we can find such an open set for each $\mathbf{y} \in \phi(U \cap \varprojlim \mathbf{t})$, $\phi(U \cap \varprojlim \mathbf{t})$ is open. So ϕ is an open function.

So ϕ is a continuous open bijection, and is thus a homeomorphism. □

We will also need the following lemma in the next section:

Lemma 2.3.1. *For each $i \in \omega$, suppose X_i is a compact Hausdorff space and r_i is a reversibly upper semi-continuous subset of $X_i \times X_{i+1}$. Then $\varprojlim \mathbf{t}$ is compact.*

(By reversibly upper semi-continuous, we mean that r_i^{-1} is upper semi-continuous.)

Proof. Since each X_i is compact Hausdorff, it is regular, so since r_i is reversibly upper semicontinuous, r_i is closed. Being a closed subset of the compact space $X_i \times X_{i+1}$, r_i is compact. So $\prod_{i\in\omega} r_i$ is compact.

Claim. $\varprojlim \mathbf{t}$ *is closed in* $\prod_{i\in\omega} r_i$ *under the product topology.*

Proof. Let $\mathbf{w}$ be in $\prod_{i\in\omega} r_i \setminus \varprojlim \mathbf{t}$. Since $\mathbf{w} \notin \varprojlim \mathbf{t}$, there is a $j \in \omega$ such that $(w_j, w_{j+1}) \notin t_j$. We can write $w_j = (a, b)$ and $w_{j+1} = (c, d)$ for some $a \in X_j$, $b, c \in X_{j+1}$, and $d \in X_{j+2}$. It must be the case that $b \neq c$. Now b, c are in X_{j+1}, a Hausdorff space, so there are disjoint open sets S and T separating b from c.

For each $i \in \omega$ such that $i \neq j$ and $i \neq j + 1$, define $s_i = r_i$, and define $s_j = (X_j \times S) \cap r_j$, $s_{j+1} = (T \times X_{j+2}) \cap r_{j+1}$, open subsets of r_j and r_{j+1} respectively. Each s_i is an open subset of r_i, and for cofinitely many $i \in \omega$, $s_i = r_i$, so $\prod_{i\in\omega} s_i$ is an open subset of $\prod_{i\in\omega} r_i$.

Let $\mathbf{y}$ be in $\prod_{i\in\omega} s_i$ with $y_j = (p, q)$ and $y_{j+1} = (u, v)$; $(p, q) = y_j \in s_j = (X_j \times S) \cap r_j$, so $q \in S$; $(u, v) = y_{j+1} \in s_{j+1} = (T \times X_{j+2}) \cap r_{j+1}$, so $u \in T$. Since S and T are disjoint, $q \neq u$, so $(y_i, y_{i+1}) = ((p, q), (u, v)) \notin t_i$. So $\mathbf{y} \notin \varprojlim \mathbf{t}$. So for each element $\mathbf{w}$ of $\prod_{i\in\omega} r_i \setminus \varprojlim \mathbf{t}$, there is an open subset of $\prod_{i\in\omega} r_i$ containing $\mathbf{w}$ that is a subset of $\prod_{i\in\omega} r_i \setminus \varprojlim \mathbf{t}$. So $\prod_{i\in\omega} r_i \setminus \varprojlim \mathbf{t}$ is open. So $\varprojlim \mathbf{t}$ is closed.

Thus $\varprojlim \mathbf{t}$ is a closed subset of the compact set $\prod_{i\in\omega} r_i$, and so it is compact. □

2.4 Constructing a Finite Domain Space

Given a limit space $\varprojlim \mathbf{r}$ our objective is to construct a homeomorphic space to which other "similar enough" spaces can also be shown to be homeomorphic.

For each $i \in \omega$ such that $i > 0$, let Y_i be a partition of r_i and π_i be the function that maps each element of r_i to the partition element containing it.

Since our intent is to construct bonding maps s_i that are subsets of $Y_i \times Y_{i+1}$ chosen so that $\varprojlim \mathbf{s} \cong \varprojlim \mathbf{t}$ (and hence $\varprojlim \mathbf{s}$ is homeomorphic to $\varprojlim \mathbf{r}$) it is natural to define $s_i = \{(\pi_i(a,b), \pi_{i+1}(c,d)) \mid ((a,b),(c,d)) \in t_i\}$.

$$\begin{tikzcd}
\cdots \arrow[r, "t_{i-1}"] & r_i \arrow[r, "t_i"] \arrow[d, "\pi_i"] & r_{i+1} \arrow[r, "t_{i+1}"] \arrow[d, "\pi_{i+1}"] & \cdots \\
\cdots \arrow[r, "s_{i-1}"] & Y_i \arrow[r, "s_i"] & Y_{i+1} \arrow[r, "s_{i+1}"] & \cdots
\end{tikzcd}$$

In fact, this guarantees that one inclusion of the commutivity condition for the generalized limit system is satisfied: Let $((a,b),P)$ be in $\pi_{i+1} \circ t_i$. Then:

$$\begin{aligned}
&((a,b),P) \in \pi_{i+1} \circ t_i \\
\implies &\exists (c,d) \in r_{i+1} \text{ such that } ((a,b),(c,d)) \in t_i \text{ and } ((c,d),P) \in \pi_{i+1} \\
\implies &\exists (c,d) \in r_{i+1} \text{ such that } (\pi_i(a,b), \pi_{i+1}(c,d)) \in s_i \text{ and } ((c,d),P) \in \pi_{i+1} \\
\implies &(\pi_i(a,b),P) \in s_i \text{ and } ((a,b),\pi_i(a,b)) \in \pi_i \\
\implies &((a,b),P) \in s_i \circ \pi_i.
\end{aligned}$$

So $\pi_{i+1} \circ t_i \subseteq s_i \circ \pi_i$.

From this point forward, since Y_i is a quotient of r_i with the projection π_i serving as the map between factor spaces, the key function $t_i|^{\pi_{i+1}^{-1}\{P\}}$ is equal to $t_i|^P$ since $\pi_{i+1}^{-1}\{P\} = P$. This simplifies our notation.

Definition Let r be a subset of $W \times X$ and Y be a quotient of X with quotient map $\pi : X \to Y$. The statement that r *is compatible with* Y means for each $P \in Y$ and $x \in \mathrm{dom}(r|^P)$, we have $\pi(x) \subseteq \mathrm{dom}(r|^P)$.

Lemma 2.4.1. *For each $i \in \omega$ suppose t_i is function decomposable relative to π_{i+1} and compatible with Y_{i+1}. Then $(\mathbf{r}, \mathbf{Y}, \mathbf{t}, \mathbf{s}, \pi)$ is a commutative generalized limit system.*

Proof. Let $((a,b), P)$ be in $s_i \circ \pi_i$. Then $(\pi_i(a,b), P) \in s_i$. So there is a $((p,q),(u,v)) \in t_i$ such that $\pi_i(p,q) = \pi_i(a,b)$ and $\pi_{i+1}(u,v) = P$. So $(u,v) \in P$, and thus $((p,q),(u,v)) \in t_i|^P$, so $(p,q) \in \mathrm{dom}(t_i|^P)$. So $\pi_i(p,q) \subseteq \mathrm{dom}(t_i|^P)$. So $(a,b) \in \pi_i(a,b) = \pi_i(p,q) \subseteq \mathrm{dom}(t_i|^P)$. So there is a $(f,g) \in r_{i+1}$ such that $((a,b),(f,g)) \in t_i|^P$, so $(f,g) \in P$. Thus $\pi_{i+1}(f,g) = P$. So $((a,b),(f,g)) \in t_i$ and $\pi_{i+1}(f,g) = P$. Thus $((a,b),P) \in \pi_{i+1} \circ t_i$, so $s_i \circ \pi_i \subseteq \pi_{i+1} \circ t_i$. We have from above that $\pi_{i+1} \circ t_i \subseteq s_i \circ \pi_i$, so $s_i \circ \pi_i = \pi_{i+1} \circ t_i$. □

For each $i \in \omega$, let X_i be a compact Hausdorff space, r_i be a reversibly upper semi-continuous subset of $X_i \times X_{i+1}$.

Let Y_0 be the partition of r_0 where each element of r_0 is in a singleton equivalence class. Define $\pi_0 : r_0 \to Y_0$ by $\pi_0(a,b) = \{(a,b)\}$.

For each $i \in \omega$ where $i > 0$, let Y_i be a partition of r_i, with π_i the function that assigns each element of r_i to the partition element containing it in Y_i. Giving r_i the subspace topology from the product $X_i \times X_{i+1}$, assign Y_i the quotient topology inherited from r_i via π_i.

Finally, for each $i \in \omega$ let s_i be $\{(\pi_i(a,b), \pi_{i+1}(c,d)) \mid ((a,b),(c,d)) \in t_i\}$.

Theorem 2.3. *Suppose for each $i \in \omega$, t_i is function decomposable relative to π_{i+1} and compatible with Y_{i+1}, and $(\mathbf{Y}, \mathbf{s})$ is subsequently separable. Then $\varprojlim \mathbf{s} \cong \varprojlim \mathbf{r}$.*

Proof. $\varprojlim \mathbf{r} \cong \varprojlim \mathbf{t}$ by Theorem 2.2, so we intend to show that $\varprojlim \mathbf{s} \cong \varprojlim \mathbf{t}$.

Define $\pi : \varprojlim \mathbf{t} \to \varprojlim \mathbf{s}$ to be the function so that for each $\mathbf{x} \in \varprojlim \mathbf{t}$ and each $i \in \omega$, $\pi(\mathbf{x})_i = \pi_i(x_i)$. Since Y_0 is the discrete partition of r_0 with topology induced by π_0, π_0 is a homeomorphism and thus Y_0 is Hausdorff. By Lemma 2.3.1, $\varprojlim \mathbf{t}$ is compact. Since each

π_i is a projection, it is continuous. By Lemma 2.4.1, $(\mathbf{r}, \mathbf{t}, \mathbf{Y}, \mathbf{s}, \pi)$ is commutative. So by Theorem 2.1, $\varprojlim \mathbf{r} \cong \varprojlim \mathbf{t}$. □

Definition For each $i \in \omega$ let X_i be a topological space and r_i be a subset of $X_i \times X_{i+1}$. The statement that $(\mathbf{X}, \mathbf{r})$ *satisfies condition* θ' means there is a sequence of partitions $\mathbf{Y}$ with each Y_i a partition of X_i with quotient map $\pi_i : X_i \to Y_i$ and for each $i \in \omega$ if $s_i = \{(\pi_i(a), \pi_{i+1}(b)) \mid (a, b) \in r_i\}$ then:

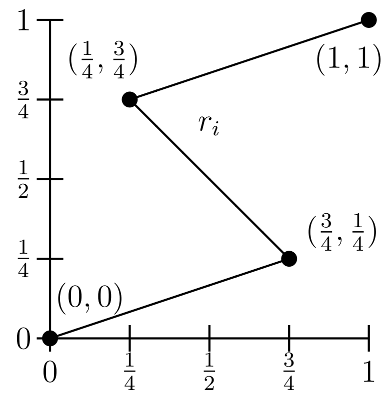
1. Y_0 is the discrete partition of X_0.
2. $(\mathbf{Y}, \mathbf{s})$ is subsequently separable.
3. For each $i \in \omega$, r_i is function decomposable relative to π_{i+1}.
4. For each $i \in \omega$, r_i is compatible with Y_{i+1}.

Condition θ' allows that the factor spaces Y_i under the quotient topology may not be Hausdorff, while the resulting space $\varprojlim \mathbf{s}$ is Hausdorff. We now ask the question, given a limit space $\varprojlim \mathbf{r}$ that induces $\varprojlim \mathbf{t}$, how does one produce partitions Y_i that demonstrate that $(\mathbf{r}, \mathbf{t})$ satisfies condition θ'? Consider the case where each bonding map r_i (viewed as the subspace of $X_i \times X_{i+1}$) is an arc. This is true, for example, when each X_i is the closed interval $[0, 1]$ and r_i^{-1} is a continuous function (as in a traditional inverse limit), but it is also true for many instances of generalized inverse limits.

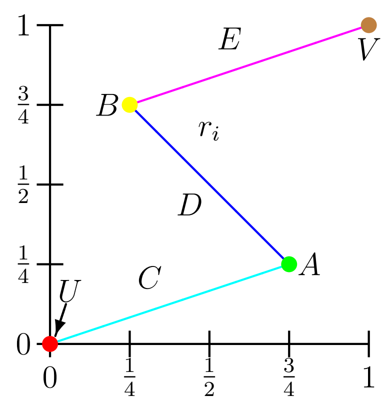
We suggest the following technique: Y_0 shall be the discrete partition of r_0.

For $i > 0$, the arc r_i shall be partitioned into sets that are either singletons or open intervals of the arc. In choosing how to partition the arc, the goal is to partition r_i into as few parts as possible while ensuring that no open interval can be intersected twice by a vertical line in the graph of r_i, and that every element of Y_i related by s_i to a singleton in Y_{i+1} is a singleton (i.e., if the graphs of r_i and r_{i+1}^{-1} are lined up horizontally and a horizontal line passes through a point of r_{i+1}^{-1} that is a singleton in Y_{i+1} (after swapping coordinates) then any point of r_i intersected by the horizontal line is a singleton in Y_i).

Consider the example where each X_i is the interval $[0,1]$ and each r_i has the following graph:

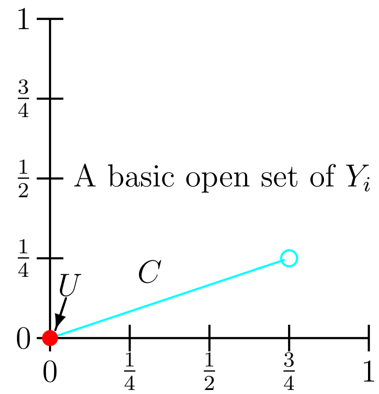


We can achieve the desired result by "breaking" the graph at the points where the graph changes direction:



Each solid dot and open line segment represents a distinct element in the partition. Note each element passes the vertical line test when considered individually.

A partition with the above properties will have a quotient topology characterized by the following basis: Since each r_i is an arc and each element of the partition is a connected subset of the arc, there is a natural ordering of the partition, and the set of all subsets of Y_i of the form $\{Q \mid A < Q < B\}$, $\{Q \mid U \leq Q < B\}$, or $\{Q \mid A < Q \leq V\}$ where A and B are singletons and U and V are the partition elements containing the left and right endpoints of the arc, respectively, is a basis for the topology of Y_i.



Such a partition sequence will demonstrate condition θ':

1. Y_0 is the discrete partition of r_0: By construction.

2. $(\mathbf{Y}, \mathbf{s})$ is subsequently separable: Let P be in Y_i and each of A and B be in $s_i\{P\}$ such that $A < B$. Suppose there is no singleton T of Y_{i+1} with $A < T < B$. Then A and B are singletons and are the endpoints of an open interval G of Y_{i+1}. Since A and B are singletons, and $A, B \in s_i\{P\}$, P is a singleton.

   We can write $A = \{(s,t)\}$, $B = \{(y,z)\}$, and $P = \{(u,v)\}$.

   Since (P,A) and (P,B) are in s_i, which is $\{(\pi_i(a,b), \pi_{i+1}(c,d)) \mid ((a,b),(c,d)) \in t_i\}$, $((p,q),(u,v)) \in t_i$ and $((p,q),(w,z)) \in t_i$, so $q = u$ and $q = w$. So $A = \{(q,v)\}$ and $B = \{(q,z)\}$. Since A, G, and B when unioned form an arc in r_{i+1} and the points associated with A and B lie on the same vertical line, G is an open line segment joining A to B. Thus there must be a vertical line intersecting G twice. This is a contradiction, so there must be a singleton T of Y_{i+1} with $A < T < B$. The sets $\{Q \mid U \leq Q < T\}$ and $\{Q \mid T < Q \leq V\}$ where U and V are the parts containing the left and right endpoints of the arc r_{i+1}, respectively, are disjoint open sets separating A and B.

3. For each $i \in \omega$, t_i is function decomposable relative to π_{i+1}: Let P be in Y_{i+1}, and consider $t_i|^P$: a point (u, v) in r_i is related by $t_i|^P$ to a point (p, q) in P if and only if when r_i is aligned horizontally with r_{i+1}^{-1}, (u, v) and (p, q) belong to the same horizontal line. Thus $t_i|^P$ is a function when no point (u, v) of r_i belongs to the same horizontal line as two points of P in the inverse graph of r_{i+1}.

   A horizontal line in the inverse graph of r_{i+1} corresponds to a vertical line in the graph of r_{i+1}, and it was stipulated that no two points of P may belong to the same vertical line.Thus $t_i|^P$ is a function.

4. For each $i \in \omega$, t_i is compatible with Y_{i+1}: Let (p, q) be in $\mathrm{dom}(t_i|^P)$.Suppose $\pi_i(p, q)$ is not a subset of $\mathrm{dom}(t_i|^P)$, and (u, v) is an element of $\pi_i(p, q)$ that is not in $\mathrm{dom}(t_i|^P)$. So no pair of P has first element v.

   Without loss of generality suppose $v > q$. Let F be the set of all numbers that are the first number in a pair in P. Then F is connected since it is the continuous image of the connected set P. Let l be the least upper bound of F; $q \leq l < v$ since $(p, q) \in \mathrm{dom}(t_i|^P)$ and $(u, v) \notin \mathrm{dom}(t_i|^P)$. Let G be the set of all numbers that are the second number in a pair in $\pi_i(p, q)$. G is connected since it the the continuous image of the connected set $\pi_i(p, q)$. v and q are in G, so $[q, v] \subseteq G$. Thus there is a pair in $\pi_i(p, q)$ with second number l. Let (w, l) be such an element.An endpoint of P must have first coordinate l. Let (l, z) be such an endpoint of P. $\{(l, z)\}$ must be an element of Y_{i+1}. So $\pi_i(w, l)$ is a singleton.

   This is a contradiction, since $q \neq v$, so $(p, q) \neq (u, v)$ and $(p, q), (u, v) \in \pi_i(p, q) = \pi_i(w, l)$.So there can be no element (u, v) of $\pi_i(p, q)$ that is not in $\mathrm{dom}(t_i|^P)$, so $\pi_i(p, q) \subseteq \mathrm{dom}(t_i|^P)$.

This completes the demonstration of the four sub-conditions of condition θ', so condition θ' is satisfied.

2.5 Application to Topological Groups

In the case where each X_i is a group, $\prod_{i \in \omega} X_i$ is a group under the induced operation of coordinate-wise multiplication. Under certain conditions, $\varprojlim \mathbf{r}$ may be a subgroup of the product. One sufficient condition is as follows:

Lemma 2.5.1. *For each $i \in \omega$ let X_i be a group and r_i be a subgroup of $X_i \times X_{i+1}$. Then $\varprojlim \mathbf{r}$ is a subgroup of $\prod_{i \in \omega} X_i$.*

Proof. $\varprojlim \mathbf{r}$ is a subset of the group $\prod_{i \in \omega} X_i$, so we must show that the identity $\mathbf{e}$ of $\prod_{i \in \omega} X_i$ is in $\varprojlim \mathbf{r}$, and $\varprojlim \mathbf{r}$ is closed under inversion and the group operation.

The identity element $\mathbf{e}$ of $\prod_{i \in \omega} X_i$ is defined so that for each $i \in \omega$, e_i is the identity element of X_i. For each $i \in \omega$, r_i is a subgroup of $X_i \times X_{i+1}$, so $(e_i, e_{i+1}) \in r_i$. So $\mathbf{e} \in \varprojlim \mathbf{r}$.

Let $\mathbf{x}$ be in $\varprojlim \mathbf{r}$. The inverse $\mathbf{x}^{-1}$ of $\mathbf{x}$ is defined in $\prod_{i \in \omega} X_i$ so that for each $i \in \omega$, $(\mathbf{x}^{-1})_i = x_i^{-1}$. Let i be in ω. $\mathbf{x} \in \varprojlim \mathbf{r}$, so $(x_i, x_{i+1}) \in r_i$. r_i is a subgroup of $X_i \times X_{i+1}$, so $((\mathbf{x}^{-1})_i, (\mathbf{x}^{-1})_{i+1}) = (x_i^{-1}, x_{i+1}^{-1}) = (x_i, x_{i+1})^{-1} \in r_i$. So $\varprojlim \mathbf{r}$ is closed under inversion.

Let each of $\mathbf{x}$ and $\mathbf{y}$ be in $\varprojlim \mathbf{r}$. Let i be in ω. Since $\mathbf{x} \in \varprojlim \mathbf{r}$, $(x_i, x_{i+1}) \in r_i$. Since $\mathbf{y} \in \varprojlim \mathbf{r}$, $(y_i, y_{i+1}) \in r_i$. Since each of (x_i, x_{i+1}) and (y_i, y_{i+1}) is in r_i, and r_i is a subgroup of $X_i \times X_{i+1}$:

$$((\mathbf{xy})_i, (\mathbf{xy})_{i+1}) = (x_i y_i, x_{i+1} y_{i+1}) = (x_i, x_{i+1})(y_i, y_{i+1}) \in r_i.$$

So $\mathbf{xy} \in \varprojlim \mathbf{r}$.

So $\varprojlim \mathbf{r}$ is a subgroup of $\prod_{i\in\omega} X_i$. $\square$

Corollary. *For each $i \in \omega$ let X_i be a topological group and r_i be a subgroup of $X_i \times X_{i+1}$. Then $\varprojlim \mathbf{r}$ is a topological group.*

Proof. We need only show that the group operation is continuous on $\varprojlim \mathbf{r}$, and this follows directly from the fact that the group operation is continuous on $\prod_{i\in\omega} X_i$. $\square$

We are interested in examining the conditions under which a limit space $\varprojlim \mathbf{r}$ that is a topological group and a homeomorphic finite domain space $\varprojlim \mathbf{s}$ with an induced group structure are isomorphic as topological groups. Of particular interest are instances where the factors Y_i are not even groups, while $\varprojlim \mathbf{s}$ is a topological group.

Definition Let X be a group and Y be a quotient of X. The statement that Y is a *semi-congruent quotient* means that if F, G, H are in Y with $H \cap FG$ nonempty, then $H \subseteq FG$.

Definition Let $(\mathbf{X}, \mathbf{Y}, \mathbf{r}, \mathbf{s}, \pi)$ be a generalized limit system such that for each $i \in \omega$, X_i is a group and Y_i is a partition of X_i with quotient map π_i. The statement that $(\mathbf{X}, \mathbf{Y}, \mathbf{r}, \mathbf{s}, \pi)$ *is operation inducing* means for each $F, G \in \varprojlim \mathbf{s}$, $i \in \omega$, and $H \in Y_i$ with $H \subseteq F_iG_i$, $\pi_{i+1}[F_{i+1}G_{i+1}] \cap (\pi_{i+1} \circ r_i)[H]$ is a singleton.

Lemma 2.5.2. *For each $i \in \omega$ let X_i be a topological group, r_i be a subgroup of $X_i \times X_{i+1}$, and Y_i be a semi-congruent quotient of X_i with quotient map π_i and Y_0 the discrete quotient ($\pi_0(x) = \{x\}$ for all $x \in X_0$), $s_i = \{(\pi_i(x), \pi_{i+1}(y)) \mid (x, y) \in r_i\}$, so that $(\mathbf{X}, \mathbf{Y}, \mathbf{r}, \mathbf{s}, \pi)$ is operation inducing. Then for each $\mathbf{F}, \mathbf{G} \in \varprojlim \mathbf{s}$ there is a unique $\mathbf{H} \in \varprojlim \mathbf{s}$ such that $H_i \subseteq F_iG_i$ for all $i \in \omega$.*

Proof. Let $\mathbf{F}, \mathbf{G}$ be in $\varprojlim \mathbf{s}$.

Claim. *There is an $\mathbf{H} \in \varprojlim \mathbf{s}$ such that $H_i \subseteq F_iG_i$ for all $i \in \omega$.*

Proof. We define such an $\mathbf{H}$ inductively:

$F_0 = \{f_0\}$ and $G_0 = \{g_0\}$ for some $f_0, g_0 \in X_0$. Define $H_0 = \{f_0 g_0\}$. $H_0 = \{f_0 g_0\} \subseteq \{f_0\}\{g_0\} = F_0 G_0$.

Let i be in ω such that for each $j \in \omega$ with $j < i$, H_j is defined so that $H_j \subseteq F_j G_j$ and $(H_{j-1}, H_j) \in s_{j-1}$. We have that $H_{i-1} \subseteq F_{i-1} G_{i-1}$ so $\pi_i[F_i G_i] \cap (\pi_i \circ r_{i-1})[H_{i-1}]$ is a singleton $\{H_i\}$. Also $H_i \in \pi_i[F_i G_i]$, so there is an $f_i \in F_i$ and a $g_i \in G_i$ such that $\pi_i(f_i g_i) = H_i$, i.e., $f_i g_i \in H_i$. Since $f_i g_i \in H_i \cap F_i G_i$, it must be the case that $H_i \subseteq F_i G_i$. Furthermore, $H_i \in (\pi_i \circ r_{i-1})[H_{i-1}] \subseteq (s_{i-1} \circ \pi_{i-1})[H_{i-1}] = s_{i-1}[\pi_{i-1}[H_{i-1}]] = s_{i-1}\{H_{i-1}\}$. So $(H_{i-1}, H_i) \in s_{i-1}$.

By induction $\mathbf{H} \in \varprojlim \mathbf{s}$ and for all $i \in \omega$, $H_i \subseteq F_i G_i$.

The uniqueness of this $\mathbf{H}$ comes from the fact that there is only one partition element satisfying $H_0 \subseteq F_0 G_0$, and given an H_{i-1}, there is only one partition element satisfying $H_i \subseteq F_i G_i$ and $(H_{i-1}, H_i) \in s_i$. $\square$

Definition For each $i \in \omega$ let X_i be a topological group, r_i be a subgroup of $X_i \times X_{i+1}$, and Y_i be a semi-congruent quotient of X_i with quotient map π_i and Y_0 the discrete quotient ($\pi_0(x) = \{x\}$ for all $x \in X_0$), $s_i = \{(\pi_i(x), \pi_{i+1}(y)) \mid (x, y) \in r_i\}$, so that $(\mathbf{X}, \mathbf{Y}, \mathbf{r}, \mathbf{s}, \pi)$ is operation inducing. Then for each $\mathbf{F}, \mathbf{G} \in \varprojlim \mathbf{s}$, define $\mathbf{F} \diamond \mathbf{G}$ to be the unique $\mathbf{H} \in \varinjlim \mathbf{s}$ satisfying $H_i \subseteq F_i G_i$ for all $i \in \omega$.

Corollary. *For each $i \in \omega$ let X_i be a topological group, r_i be a subgroup of $X_i \times X_{i+1}$, and Y_i be a semi-congruent quotient of X_i with $s_i = \{(\pi_i(x), \pi_{i+1}(y)) \mid (x, y) \in r_i\}$, so that $(\mathbf{X}, \mathbf{Y}, \mathbf{r}, \mathbf{s}, \pi)$ is operation inducing. Then $\phi : \varprojlim \mathbf{r} \to \varprojlim \mathbf{s}$ induced by the π_i's has the property that for each $\mathbf{f}, \mathbf{g} \in \varprojlim \mathbf{r}$, $\phi(\mathbf{fg}) = \phi(\mathbf{f}) \diamond \phi(\mathbf{g})$.*

Proof. Let each of $\mathbf{f}, \mathbf{g}$ be in $\varprojlim \mathbf{r}$ and j be in ω. Then:

$$\phi(\mathbf{fg})_j = \pi_j((\mathbf{fg})_j) = \pi_j(f_j g_j) \subseteq \pi_j(f_j)\pi_j(g_j) = \phi(\mathbf{f})_j \phi(\mathbf{g})_j.$$

So $\phi(\mathbf{fg}) \in \varprojlim \mathbf{s}$ and $\phi(\mathbf{fg})_i \subseteq \phi(\mathbf{f})_i \phi(\mathbf{g})_i$ for all $i \in \omega$. Since $\phi(\mathbf{f}) \diamond \phi(\mathbf{g})$ is the unique element of $\varprojlim s_i$ with this property, $\phi(\mathbf{fg}) = \phi(\mathbf{f}) \diamond \phi(\mathbf{g})$. □

Lemma 2.5.3. *For each $i \in \omega$ let X_i be a topological group, r_i be a subgroup of $X_i \times X_{i+1}$, and Y_i be a semi-congruent quotient of X_i with $s_i = \{(\pi_i(x), \pi_{i+1}(y)) \mid (x, y) \in r_i\}$, ϕ be a homeomorphism, and $(\mathbf{X}, \mathbf{Y}, \mathbf{r}, \mathbf{s}, \pi)$ be operation inducing. Then $\varprojlim \mathbf{s}$ is a topological group under the $\diamond$ operation.*

Proof. Let $\mathbf{e}$ be the identity element of $\varprojlim \mathbf{r}$ and each of $\mathbf{F}$, $\mathbf{G}$, and $\mathbf{H}$ be an element of $\varprojlim \mathbf{s}$, and $\mathbf{f} = \phi^{-1}(\mathbf{F})$, $\mathbf{g} = \phi^{-1}(\mathbf{G})$, $\mathbf{h} = \phi^{-1}(\mathbf{H})$.

$\phi(\mathbf{e}) \diamond \mathbf{F} = \phi(\mathbf{e}) \diamond \phi(\mathbf{f}) = \phi(\mathbf{ef}) = \phi(\mathbf{f}) = \mathbf{F}$. (Also $\mathbf{F} \diamond \phi(\mathbf{e}) = \mathbf{F}$ is proved similarly.)

$\mathbf{F} \diamond \phi((\mathbf{f})^{-1}) = \phi(\mathbf{f}) \diamond \phi((\mathbf{f})^{-1}) = \phi(\mathbf{f}(\mathbf{f})^{-1}) = \phi(\mathbf{e})$. (Also $\phi((\mathbf{f})^{-1}) \diamond \mathbf{F} = \phi(\mathbf{e})$ is proved similarly.) Finally,

$$\mathbf{F} \diamond (\mathbf{G} \diamond \mathbf{H}) = \mathbf{F} \diamond (\phi(\mathbf{g}) \diamond \phi(\mathbf{h})) = \phi(\mathbf{f}) \diamond \phi(\mathbf{gh}) = \phi(\mathbf{f}(\mathbf{gh}))$$
$$= \phi((\mathbf{fg})\mathbf{h}) = \phi(\mathbf{fg}) \diamond \phi(\mathbf{h}) = (\phi(\mathbf{f}) \diamond \phi(\mathbf{g})) \diamond \mathbf{H} = (\mathbf{F} \diamond \mathbf{G}) \diamond \mathbf{H}.$$

This establishes that $\varprojlim \mathbf{s}$ is a group under the operation $\diamond$.

Let each of $\mathbf{F}$ and $\mathbf{G}$ be in $\varprojlim \mathbf{s}$ and T be an open set containing $\mathbf{F} \diamond \mathbf{G}$. Let $\mathbf{f} = \phi^{-1}(\mathbf{F})$ and $\mathbf{g} = \phi^{-1}(\mathbf{G})$. Consider $\phi^{-1}[T]$, an open subset of $\varprojlim \mathbf{r}$. Since $\phi(\mathbf{fg}) = \phi(\mathbf{f}) \diamond \phi(\mathbf{g}) = \mathbf{F} \diamond \mathbf{G} \in T$, we have that $\mathbf{fg} \in \phi^{-1}[T]$. $\varprojlim \mathbf{r}$ is a topological group, so there are open sets U and V containing $\mathbf{f}$ and $\mathbf{g}$ respectively so that $UV \subseteq \phi^{-1}[T]$. The sets $\phi[U]$ and $\phi[V]$ are open subsets of $\varprojlim \mathbf{s}$ containing $\mathbf{F}$ and $\mathbf{G}$ respectively.

Let $\mathbf{H}$ be in $\phi[U]$ and $\mathbf{K}$ be in $\phi[V]$, and $\mathbf{h} = \phi^{-1}(\mathbf{H})$, $\mathbf{k} = \phi^{-1}(\mathbf{K})$ (so $\mathbf{h} \in U$ and $\mathbf{k} \in V$). Then:

$$\mathbf{H} \diamond \mathbf{K} = \phi(\mathbf{h}) \diamond \phi(\mathbf{k}) = \phi(\mathbf{hk}) \in \phi[UV] \subseteq \phi[\phi^{-1}[T]] = T.$$

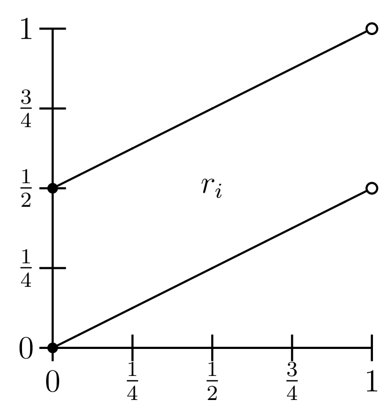
So $\diamond$ is a continuous operation. Thus $\varprojlim \mathbf{s}$ is a topological group. □

Theorem 2.4. *For each $i \in \omega$ let X_i be a topological group, r_i be a subgroup of $X_i \times X_{i+1}$, Y_i be a semi-congruent quotient of X_i with $s_i = \{(\pi_i(x), \pi_{i+1}(y)) \mid (x, y) \in r_i\}$, ϕ be a homeomorphism, and $(\mathbf{X}, \mathbf{Y}, \mathbf{r}, \mathbf{s}, \pi)$ be operation inducing. Then ϕ is a topological group isomorphism.*

Proof. We know that $\varprojlim \mathbf{s}$ is a group with operation $\diamond$ so by the corollary to Lemma 2.5.3, ϕ is a group homomorphism. Since ϕ is a homeomorphism, it is a bijection, and being a bijective homomorphism, ϕ is a group isomorphism.Thus ϕ is a topological group isomorphism. □

Example For the following, we use the realization of the topological group S^1 with $S^1 = [0, 1]$ with 0 and 1 identified, and group operation addition mod 1.

For each $i \in \omega$, let X_i be S^1, n_i be a positive integer greater than 1, r_0 be the identity function on S^1, and for each $i > 0$ let $r_i = \{(x, y) \mid x + m = n_i y \text{ for some } m \in n_i\}$ (shown below for $n_i = 2$).



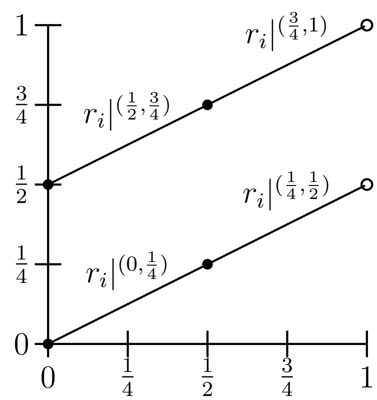
It is easy to verify that $\varprojlim \mathbf{r}$ is homeomorphic to the solenoid with winding sequence $\{n_i\}_{i\in\omega}$.

Let Y_0 be the discrete partition of S^1, k_0 be 2, and for each $i \in \omega$ let k_{i+1} be the greater of $\mathrm{lcm}(n_i, k_i)$ and $2n_i$, $Y_i = \{\{\frac{j}{k_i}\}\}_{j\in k_i} \cup \{(\frac{j}{k_i}, \frac{j+1}{k_i})\}_{j\in k_i}$.

Note that since $k_0 = 2$, each k_i is a multiple of 2 and thus $k_{i+1} \mid n_i k_i$ in both the case where $k_{i+1} = \mathrm{lcm}(k_i, n_i)$ and where $k_{i+1} = 2n_i$.

Since for all $i \in \omega$, r_i^{-1} is a function, for any singleton $\{p\}$, $r_i|^{\{p\}}$ is a singleton and thus a function. As can be seen below, for $i > 0$ and any of the open intervals S in Y_i, $r_i|^S$ is a function as well.

The diagram below shows the situation for $n_i = 2$ and $k_i = 4$.



Each $r_i|^{(\frac{j}{k_i},\frac{j+1}{k_i})} = \{(x, y) \mid x + m = n_i y$ for some $m \in n_i, y \in (\frac{j}{k_i}, \frac{j+1}{k_i})\}$ is a function: Suppose $(x, a), (x, b) \in r_i|^{(\frac{j}{k_i},\frac{j+1}{k_i})}$, so $a, b \in (\frac{j}{k_i}, \frac{j+1}{k_i})$ and $x + m_a = n_i a$ for some $m_a \in n_i$, $x + m_b = n_i b - m_b$ for some $m_b \in n_i$. So,

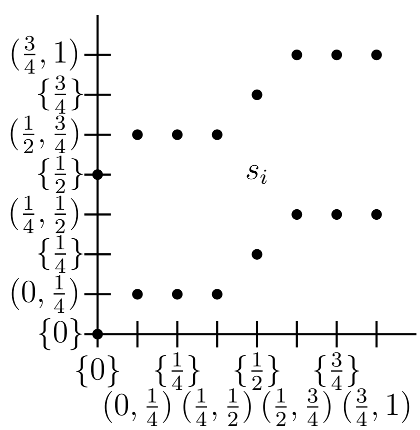
$$\begin{aligned} n_i a - m_a =& x = n_i b - m_b \\ n_i(a - b) =& m_a - m_b. \end{aligned}$$

It must be the case that $n_i > |m_a - m_b|$, which is a contradiction unless $a - b = 0$, so $a = b$. Thus $r_i|^{(\frac{j}{k_i}, \frac{j+1}{k_i})}$ is a function. So for each $i \in \omega$, r_i is function decomposable relative to π_i.

Now we show compatibility. Let x be in the domain of $r_i|^{(\frac{j}{k_{i+1}}, \frac{j+1}{k_{i+1}})}$. Then there is an $m \in n_i$, $y \in (\frac{j}{k_{i+1}}, \frac{j+1}{k_{i+1}})$ so that $x + m = n_i y$. So $x + m \in (\frac{n_i j}{k_{i+1}}, \frac{n_i(j+1)}{k_{i+1}})$ and $x \in (\frac{n_i j - m k_{i+1}}{k_{i+1}}, \frac{n_i j - m k_{i+1} + n_i}{k_{i+1}})$. Let t be in $\pi_i(x) = (\frac{q}{k_i}, \frac{q+1}{k_i})$. Since $k_{i+1} \mid n_i k_i$, there is a c_i such that $k_{i+1} c_i = n_i k_i$. Then $x \in (\frac{c_i n_i j - m c_i k_{i+1}}{n_i k_i}, \frac{c_i n_i j - m c_i k_{i+1} + c_i n_i}{n_i k_i}) = (\frac{c_i n_i j - m n_i k_i}{n_i k_i}, \frac{c_i n_i j - m n_i k_i + c_i n_i}{n_i k_i}) = (\frac{c_i j - m k_i}{k_i}, \frac{c_i j - m k_i + c_i}{k_i})$. Then $q \geq c_i j - m k_i$ and $q + 1 \leq c_i j - m k_i + c_i$, so $t \in (\frac{c_i j - m k_i}{k_i}, \frac{c_i j - m k_i + c_i}{k_i}) \subseteq \operatorname{dom}(r_i|^{(\frac{j}{k_{i+1}}, \frac{j+1}{k_{i+1}})})$ and thus $\pi(x) \subseteq \operatorname{dom}(r_i|^{(\frac{j}{k_{i+1}}, \frac{j+1}{k_{i+1}})})$. So r_i is compatible with Y_i for each $i \in \omega$.

So if we define $s_i = \{(\pi_i(x), \pi_{i+1}(y)) \mid (x, y) \in r_i\}$, then $\pi_{i+1} \circ r_i = s_i \circ \pi_i$.

The diagram below show s_i for $k_i = 4$, and partitions as described.



For $i > 0$, giving Y_i the quotient topology yields a topology with basis consisting of singletons $\{(\frac{j}{k_i}, \frac{j+1}{k_i})\}$ for $j \in n_i$, and triples

$$\{(\frac{j}{k_i}, \frac{j+1}{k_i}), \{\frac{j+1}{k_i}\}, (\frac{j+1}{k_i}, \frac{j+2}{k_i})\}$$

for $j \in n_i$ which consist of singleton sets along with their adjacent open intervals. Since $k_i \geq 2n_{i-1}$, this implies that while Y_i is not Hausdorff, elements of Y_i containing elements of X_i that are distance $1/n_{i-1}$ apart can be separated by open sets.

Since for any $P \in Y_i$ and $A, B \in s_i\{P\}$ with $A \neq B$, A and B contain elements that are distance $1/n_i$ apart, so A and B can be separated by open sets, and thus $(\mathbf{Y}, \mathbf{s})$ is subsequently separable. So $\varprojlim \mathbf{r} \cong \varprojlim \mathbf{s}$ as topological spaces.

Note that in each Y_i with $i > 0$, we cannot define a group operation on Y_i that is compatible with the group operation on X_i. For example for elements $(0, \frac{1}{k_i})$ and $(\frac{1}{k_i}, \frac{2}{k_i})$, $(\frac{1}{k_i}, \frac{3}{k_i}) = (0, \frac{1}{k_i}) + (\frac{1}{k_i}, \frac{2}{k_i})$ contains elements of $(\frac{1}{k_i}, \frac{2}{k_i}), \{\frac{2}{k_i}\}$, and $(\frac{2}{k_i}, \frac{3}{k_i})$. (There is an algebraic notion of a set with a multivalued binary operation that is grouplike called a multigroup [11]. Y_i for $i > 0$ is a multigroup in this case. We don't define a multigroup here since none of the properties are needed except what has already been demonstrated here.)

For $i = 0$, $s_0 = \{(\pi_0(a), \pi_1(b)) \mid (a, b) \in r_0\} = \{(\{a\}, \pi_1(a)) \mid (a, a) \in 1_{S^1}\}$. This is a function.

Let $\mathbf{F}, \mathbf{G}$ be in $\varprojlim \mathbf{s}$. $F_0 = \{u\}$ and $G_0 = \{v\}$ for some $u, v \in S^1$. So if $H \subseteq F_0 + G_0 = \{u\} + \{v\} = \{u + v\}$, then $H = \{u + v\}$. $\pi_1(u + v) \subseteq \pi_1(u) + \pi_1(v)$ so:

$$\begin{aligned}
\pi_1[F_1 + G_1] \cap (\pi_1 \circ r_0)[H] =& \pi_1[s_0(F_0) + s_0(G_0)] \cap (\pi_1 \circ 1_{S^1})\{u + v\} \\
=& \pi_1[s_0(\{u\}) + s_0(\{v\})] \cap \pi_1\{u + v\} \\
=& \pi_1[\pi_1(u) + \pi_1(v)] \cap \{\pi_1(u + v)\} \\
=& \{\pi_1(u + v)\}.
\end{aligned}$$

Clearly $\{\pi_1(u+v)\}$ is a singleton.

For $i > 0$, for any pair $\mathbf{A}, \mathbf{B} \in \varprojlim \mathbf{s}$ the set $A_{i+1} + B_{i+1}$ is a singleton in the case that A_{i+1} and B_{i+1} are singletons, an open interval of length $1/k_{i+1} \leq 1/n_i$ in the case that A_{i+1} is a singleton and B_{i+1} is an open interval or vice versa, or an open interval of length $2/k_{i+1} \leq 2/2n_i = 1/n_i$ in the case that A_{i+1} and B_{i+1} are open intervals. In each of these cases, $\pi_{i+1}[A_{i+1} + B_{i+1}]$ cannot contain a pair of elements distance $1/n_{i+1}$ apart. Thus if it can be shown that for each $\mathbf{F}, \mathbf{G} \in \varprojlim \mathbf{s}$, each $i \in \omega$, and each $H \in Y_i$ such that $H \subseteq F_i + G_i$, $\pi_{i+1}[F_{i+1} + G_{i+1}] \cap (\pi_{i+1} \circ r_i)[H]$ is nonempty, then it is a singleton.

An element $\{\frac{f}{k_i}\}$ of Y_i is s_i-related to an element $\{\frac{t}{k_{i+1}}\}$ if and only if $\frac{f}{k_i}$ is r_i-related to $\frac{t}{k_{i+1}}$, in other words, there is an integer u such that $\frac{f}{k_i} + u = n_i \frac{t}{k_{i+1}}$, or $\frac{f+uk_i}{n_ik_i} = \frac{t}{k_{i+1}}$. Note that since k_{i+1} is a multiple of n_i, there is an integer p_i such that $n_ip_i = k_{i+1}$. Then:

$$\begin{aligned}
& \frac{f+uk_i}{n_ik_i} = \frac{t}{k_{i+1}} \\
\iff & \frac{f+uk_i}{c_ik_{i+1}} = \frac{t}{k_{i+1}} \\
\iff & c_i \mid f + uk_i \\
\iff & c_in_i \mid fn_i + un_ik_i \\
\iff & c_in_i \mid fn_i + uc_ik_{i+1} \\
\iff & c_in_i \mid fn_i + uc_in_ip_i \\
\iff & c_in_i \mid fn_i \\
\iff & c_i \mid f.
\end{aligned}$$

Case 1: $F_i = \{\frac{f}{k_i}\}$, $G_i = \{\frac{g}{k_i}\}$ for $f, g \in k_i$. Then $F_i + G_i = \{\frac{f+g}{k_i}\}$. Let H be in $F_i + G_i$. $H = \{\frac{f+g}{k_i}\}$, so $r_i[H] = \{\frac{f+g+mk_i}{n_ik_i}\}_{m\in\omega}$.

Subcase 1a.: $c_i \mid f$ and $c_i \mid g$. Then:

$$\begin{aligned}
F_{i+1} &= \left\{ \frac{f + uk_i}{n_i k_i} \right\} \text{ for some } u \in \omega, \\
G_{i+1} &= \left\{ \frac{g + vk_i}{n_i k_i} \right\} \text{ for some } v \in \omega, \\
F_{i+1} + G_{i+1} &= \left\{ \frac{f + g + (u+v)k_i}{n_i k_i} \right\}.
\end{aligned}$$

Since $f + g$ is divisible by c_i, $(\pi_{i+1} \circ r_i)[H] = \{\{\frac{f+g+mk_i}{n_i k_i}\} \mid m \in k_{i+1}\}$ which intersects $F_{i+1} + G_{i+1}$.

Subcase 1b.: $c_i \mid f$ and $c_i \nmid g$. Then:

$$\begin{aligned}
F_{i+1} &= \left\{ \frac{f + uk_i}{n_i k_i} \right\} \text{ for some } u \in \omega, \\
G_{i+1} &= \left(\frac{g - g \mod c_i + vk_i}{n_i k_i}, \frac{g - g \mod c_i + vk_i + c_i}{n_i k_i} \right) \text{ for some } v \in \omega, \\
F_{i+1} + G_{i+1} &= \left(\frac{f + uk_i + g - g \mod c_i + vk_i}{n_i k_i}, \frac{f + uk_i + g - g \mod c_i + vk_i + c_i}{n_i k_i} \right) \\
&= \left(\frac{f + g - g \mod c_i + (u+v)k_i}{n_i k_i}, \frac{f + g - g \mod c_i + (u+v)k_i + c_i}{n_i k_i} \right).
\end{aligned}$$

Since $c_i \nmid f + g$, $(\pi_{i+1} \circ r_i)[H] = \{(\frac{f+g-(f+g) \mod c_i + mk_i}{n_i k_i}, \frac{f+g-(f+g) \mod c_i + mk_i + c_i}{n_i k_i}) \mid m \in k_{i+1}\} = \{(\frac{f+g-g \mod c_i + mk_i}{n_i k_i}, \frac{f+g-g \mod c_i + mk_i + c_i}{n_i k_i}) \mid m \in k_{i+1}\}$ which intersects $F_{i+1} + G_{i+1}$.

Subcase 1c.: $n_{i+1} \nmid f$ and $n_{i+1} \mid g$. Similar to above.

Subcase 1d.: $c_i \nmid f$, $c_i \nmid g$, and $c_i \mid f+g$, then $f \mod (c_i) + g \mod c_i = c_i$ and:

$$
\begin{aligned}
F_{i+1} &= \left(\frac{f - f \mod c_i + uk_i}{n_i k_i}, \frac{f - f \mod c_i + uk_i + c_i}{n_i k_i} \right) \text{ for some } u \in \omega, \\
G_{i+1} &= \left(\frac{g - g \mod c_i + vk_i}{n_i k_i}, \frac{g - g \mod c_i + vk_i + c_i}{n_i k_i} \right) \text{ for some } v \in \omega, \\
F_{i+1} + G_{i+1} &= \left(\tfrac{f - f \mod c_i + uk_i + g - g \mod c_i + vk_i}{n_i k_i}, \tfrac{f - f \mod c_i + uk_i + c_i + g - g \mod c_i + vk_i + c_i}{n_i k_i} \right) \\
&= \left(\frac{f + g - c_i + (u+v)k_i}{n_i k_i}, \frac{f + g - c_i + (u+v)k_i + 2c_i}{n_i k_i} \right) \\
&= \left(\frac{f + g + (u+v)k_i - c_i}{n_i k_i}, \frac{f + g + (u+v)k_i + c_i}{n_i k_i} \right).
\end{aligned}
$$

Since $c_i \mid f+g$, $(\pi_{i+1} \circ r_i)[H] = \{\{\frac{f+g+mk_i}{n_i k_i}\} \mid m \in k_{i+1}\}$ which intersects $F_{i+1} + G_{i+1}$ when $m = u + v + 1$.

Subcase 1e.: $c_i \nmid f$, $c_i \nmid g$, and $c_i \nmid f+g$. Then:

$$
\begin{aligned}
F_{i+1} &= \left(\frac{f - f \mod c_i + uk_i}{n_i k_i}, \frac{f - f \mod c_i + uk_i + c_i}{n_i k_i} \right) \text{ for some } u \in \omega, \\
G_{i+1} &= \left(\frac{g - g \mod c_i + vk_i}{n_i k_i}, \frac{g - g \mod c_i + vk_i + c_i}{n_i k_i} \right) \text{ for some } v \in \omega, \\
F_{i+1} + G_{i+1} &= \left(\tfrac{f - f \mod c_i + uk_i + g - g \mod c_i + vk_i}{n_i k_i}, \tfrac{f - f \mod c_i + uk_i + c_i + g - g \mod c_i + vk_i + c_i}{n_i k_i} \right) \\
&= \left(\tfrac{f + g - f \mod c_i - g \mod c_i + (u+v)k_i}{n_i k_i}, \tfrac{f + g - f \mod c_i - g \mod c_i + (u+v)k_i + 2c_i}{n_i k_i} \right).
\end{aligned}
$$

Since $c_i \nmid f+g$, $(\pi_{i+1} \circ r_i)[H] = \{(\frac{f+g-(f+g) \mod c_i + mk_i}{n_i k_i}, \frac{f+g-(f+g) \mod c_i + mk_i + c_i}{n_i k_i}) \mid m \in k_{i+1}\} = \{(\frac{f+g-f \mod c_i - g \mod c_i + mk_i}{n_i k_i}, \frac{f+g-f \mod c_i - g \mod c_i + mk_i + c_i}{n_i k_i}) \mid m \in k_{i+1}\}$ which intersects $F_{i+1} + G_{i+1}$.

Case 2: $F_i = (\frac{f}{k_i}, \frac{f+1}{k_i})$, $G_i = \{\frac{g}{k_i}\}$. $F_i + G_i = (\frac{f+g}{k_i}, \frac{f+g+1}{k_i})$. Let H be in $F_i + G_i$. Then $H = (\frac{f+g}{k_i}, \frac{f+g+1}{k_i})$.

Subcase 2a.: $c_i \mid g$. Then:

$$\begin{aligned}
F_{i+1} =& \left(\tfrac{f-f \mod c_i+uk_i}{n_ik_i}, \tfrac{f-f \mod c_i+uk_i+c_i}{n_ik_i}\right) \text{ for some } u \in \omega, \\
G_{i+1} =& \left\{\frac{g+vk_i}{n_ik_i}\right\} \text{ for some } v \in \omega, \\
F_{i+1}+G_{i+1} =& \left(\frac{f-f \mod c_i+uk_i+g+vk_i}{n_ik_i}, \frac{f-f \mod c_i+uk_i+c_i+g+vk_i}{n_ik_i}\right) \\
=& \left(\frac{f+g-f \mod c_i+(u+v)k_i}{n_ik_i}, \frac{f+g-f \mod c_i+(u+v)k_i+c_i}{n_ik_i}\right).
\end{aligned}$$

We have that $r_i[H] = \cup\{(\frac{f+g-(f+g) \mod c_i+mk_i}{n_ik_i}, \frac{f+g-(f+g) \mod c_i+mk_i+c_i}{n_ik_i}) \mid m \in c_i\}$ and $(\pi_{i+1} \circ r_i)[H] = \{(\frac{f+g-(f+g) \mod c_i+mk_i}{n_ik_i}, \frac{f+g-(f+g) \mod c_i+mk_i+c_i}{n_ik_i}) \mid m \in c_i\} = \{(\frac{f+g-f \mod c_i+mk_i}{n_ik_i}, \frac{f+g-f \mod c_i+mk_i+c_i}{n_ik_i}$ $m \in c_i\}$ which intersects $F_{i+1}+G_{i+1}$.

Subcase 2b. $c_i \nmid g$. Then:

$$\begin{aligned}
F_{i+1} =& \left(\tfrac{f-f \mod c_i+uk_i}{n_ik_i}, \tfrac{f-f \mod c_i+uk_i+c_i}{n_ik_i}\right) \text{ for some } u \in \omega, \\
G_{i+1} =& \left(\frac{g-g \mod c_i+vk_i}{n_ik_i}, \frac{g-g \mod c_i+vk_i+c_i}{n_ik_i}\right) \text{ for some } v \in \omega, \\
F_{i+1}+G_{i+1} =& \left(\tfrac{f-f \mod c_i+uk_i+g-g \mod c_i+vk_i}{n_ik_i}, \tfrac{f-f \mod c_i+uk_i+c_i+g-g \mod c_i+vk_i+c_i}{n_ik_i}\right) \\
=& \left(\tfrac{f+g-f \mod c_i-g \mod c_i+(u+v)k_i}{n_ik_i}, \tfrac{f+g-f \mod c_i-g \mod c_i+(u+v)k_i+2c_i}{n_ik_i}\right).
\end{aligned}$$

We have that $r_i[H] = \cup\{(\frac{f+g-(f+g) \mod c_i+mk_i}{n_ik_i}, \frac{f+g-(f+g) \mod c_i+mk_i+c_i}{n_ik_i}) \mid m \in c_i\}$ and

$$\begin{aligned}
(\pi_{i+1} \circ r_i)[H] =& \{(\tfrac{f+g-(f+g) \mod c_i+mk_i}{n_ik_i}, \tfrac{f+g-(f+g) \mod c_i+mk_i+c_i}{n_ik_i}) \mid m \in c_i\} \\
=& \{(\tfrac{f+g-f \mod c_i-g \mod c_i+mk_i}{n_ik_i}, \tfrac{f+g-f \mod c_i-g \mod c_i+mk_i+c_i}{n_ik_i}) \mid m \in c_i\}
\end{aligned}$$

which intersects $F_{i+1}+G_{i+1}$.

Case 3: $F_i = \{\frac{f}{k_i}\}$, $G_i = (\frac{g}{k_i}, \frac{g+1}{k_i})$. Similar to above.

Case 4: $F_i = (\frac{f}{k_i}, \frac{f+1}{k_i})$, $G_i = (\frac{g}{k_i}, \frac{g+1}{k_i})$. $F_i + G_i = (\frac{f+g}{k_i}, \frac{f+g+2}{k_i})$. Let H be in $F_i + G_i$. Then $H = (\frac{f+g}{k_i}, \frac{f+g+1}{k_i})$, $H = \{\frac{f+g+1}{k_i}\}$, or $H = (\frac{f+g+1}{k_i}, \frac{f+g+2}{k_i})$. Then:

$$\begin{aligned}
F_{i+1} =& (\tfrac{f-f \mod c_i+uk_i}{n_ik_i}, \tfrac{f-f \mod c_i+uk_i+c_i}{n_ik_i}),\\
G_{i+1} =& (\tfrac{g-g \mod c_i+vk_i}{n_ik_i}, \tfrac{g-g \mod c_i+vk_i+c_i}{n_ik_i}),\\
F_{i+1}+G_{i+1} =& (\tfrac{f-f \mod c_i+uk_i+g-g \mod c_i+vk_i}{n_ik_i}, \tfrac{f-f \mod c_i+uk_i+c_i+g-g \mod c_i+vk_i+c_i}{n_ik_i})\\
=& (\tfrac{f+g-f \mod c_i-g \mod c_i+(u+v)k_i}{n_ik_i}, \tfrac{f+g-f \mod c_i-g \mod c_i+(u+v)k_i+2c_i}{n_ik_i}).
\end{aligned}$$

Subcase 4a.: $H = (\frac{f+g}{k_i}, \frac{f+g+1}{k_i})$. Then:

$$\begin{aligned}
r_i[H] =& \cup\{(\tfrac{f+g-(f+g) \mod c_i+mk_i}{n_ik_i}, \tfrac{f+g-(f+g) \mod c_i+mk_i+c_i}{n_ik_i}) \mid m \in c_i\},\\
(\pi_{i+1}\circ r_i)[H] =& \{(\tfrac{f+g-(f+g) \mod c_i+mk_i}{n_ik_i}, \tfrac{f+g-(f+g) \mod c_i+mk_i+c_i}{n_ik_i}) \mid m \in c_i\}\\
=& \{(\tfrac{f+g-f \mod c_i-g \mod c_i+mk_i}{n_ik_i}, \tfrac{f+g-f \mod c_i-g \mod c_i+mk_i+c_i}{n_ik_i}) \mid m \in c_i\}
\end{aligned}$$

which intersects $F_{i+1} + G_{i+1}$.

Subcase 4b.: $H = \{\frac{f+g+1}{k_i}\}$ and $c_i \mid f+g+1$. Then:

$$\begin{aligned}
r_i[H] =& \{\frac{f+g+1+mk_i}{n_ik_i} \mid m \in c_i\},\\
(\pi_{i+1}\circ r_i)[H] =& \{\{\frac{f+g+1+mk_i}{n_ik_i}\} \mid m \in c_i\}\\
=& \{\{\frac{f+g+1-(c_i-1)+c_i-1+mk_i}{n_ik_i}\} \mid m \in c_i\}\\
=& \{\{\frac{f+g-(f+g) \mod c_i+mk_i+c_i}{n_ik_i}\} \mid m \in c_i\}\\
=& \{\{\frac{f+g-f \mod c_i-g \mod c_i+mk_i+c_i}{n_ik_i}\} \mid m \in c_i\}
\end{aligned}$$

which intersects $F_{i+1} + G_{i+1}$.

Subcase 4c.: $H = \{\frac{f+g+1}{k_i}\}$ and $c_i \nmid f + g + 1$. Then:

$$\begin{aligned}
r_i[H] =& \cup \{(\tfrac{f+g+1-(f+g+1) \mod c_i+mk_i}{n_ik_i}, \tfrac{f+g+1-(f+g+1) \mod c_i+mk_i+c_i}{n_ik_i}) \mid m \in c_i\}, \\
(\pi_{i+1} \circ r_i)[H] =& \{(\tfrac{f+g+1-(f+g+1) \mod c_i+mk_i}{n_ik_i}, \tfrac{f+g+1-(f+g+1) \mod c_i+mk_i+c_i}{n_ik_i}) \mid m \in c_i\} \\
=& \{(\tfrac{f+g-f \mod c_i-g \mod c_i+mk_i}{n_ik_i}, \tfrac{f+g-f \mod c_i-g \mod c_i+mk_i+c_i}{n_ik_i}) \mid m \in c_i\}
\end{aligned}$$

which intersects $F_{i+1} + G_{i+1}$.

Subcase 4d.: $H = (\frac{f+g+1}{k_i}, \frac{f+g+2}{k_i})$. Then:

$$\begin{aligned}
r_i[H] =& \cup \{(\tfrac{f+g+1-(f+g+1) \mod c_i+mk_i}{n_ik_i}, \tfrac{f+g+1-(f+g+1) \mod c_i+mk_i+c_i}{n_ik_i}) \mid m \in c_i\}, \\
(\pi_{i+1} \circ r_i)[H] =& \{(\tfrac{f+g+1-(f+g+1) \mod c_i+mk_i}{n_ik_i}, \tfrac{f+g+1-(f+g+1) \mod c_i+mk_i+c_i}{n_ik_i}) \mid m \in c_i\} \\
=& \{(\tfrac{f+g-f \mod c_i-g \mod c_i+mk_i}{n_ik_i}, \tfrac{f+g-f \mod c_i-g \mod c_i+mk_i+c_i}{n_ik_i}) \mid m \in c_i\}
\end{aligned}$$

which intersects $F_{i+1} + G_{i+1}$.

By exhaustion, $\pi_{i+1}[F_{i+1} + G_{i+1}] \cap (\pi_{i+1} \circ r_i)$ is a singleton for each $\mathbf{F}, \mathbf{G} \in \varprojlim \mathbf{s}$, each $i \in \omega$, and each subset H of $F_{i+1} + G_{i+1}$. So $\varprojlim \mathbf{s}$ equipped with the induced operation $\diamond$ is a topological group isomorphic to the solenoid with winding sequence $\{n_i\}_{i\in\omega}$. $\square$

Chapter 3

Categorical Characterization

3.1 Definitions

Definitions from category theory follow [6] and [9].

Definition A *category* $\mathbf{C}$ consists of the following:

- A class called *the objects of* $\mathbf{C}$.
- For each pair x, y of objects, a set $\mathbf{C}(x, y)$, the elements of which are called *morphisms from x to y*. For a morphism $\alpha \in \mathbf{C}(x, y)$, x and y are called the *source* and *target* of α, respectively. Notationally morphisms are treated like functions, although in general they may not be. So $\alpha : x \to y$ means $\alpha \in \mathbf{C}(x, y)$ for objects x, y of $\mathbf{C}$. (In the literature some authors permit $\mathbf{C}(x, y)$ to be a proper class. In this understanding, a category for which $\mathbf{C}(x, y)$ is a set for each pair x, y of objects is called *locally small*.)
- For each triple x, y, z of objects, a map $\mathbf{C}(y, z) \times \mathbf{C}(x, y) \to \mathbf{C}(x, z)$ called *composition* and denoted $(\alpha, \beta) \to \alpha \circ \beta$ with the following properties:
  1. $(\alpha \circ \beta) \circ \gamma = \alpha \circ (\beta \circ \gamma)$ for all trios of morphisms α, β, γ such that the compositions exist.
  2. For each object x of $\mathbf{C}$, there is a morphism $1_x : x \to x$ called *the identity on x* such that $\alpha \circ 1_x = \alpha$ and $1_x \circ \beta = \beta$ for all morphisms α, β for which the compositions exist.

Definition Let each of $\mathbf{C}$ and $\mathbf{D}$ be a category. The statement that $F : \mathbf{C} \to \mathbf{D}$ *is a functor* means that F is a class function with domain all the objects and morphisms of $\mathbf{C}$, mapping objects of $\mathbf{C}$ to objects of $\mathbf{D}$ and morphisms of $\mathbf{C}$ to morphisms of $\mathbf{D}$, having the properties:

1. For each morphism $\alpha : x \to y$ of $\mathbf{C}$, $F(\alpha)$ has source $F(x)$ and target $F(y)$. Sometimes subscript notation is used, e.g. $F_\alpha : F_x \to F_y$.

2. For each object x of $\mathbf{C}$, $F(1_x) = 1_{F(x)}$ (or $F_{1_x} = 1_{F_x}$).

3. For each pair α, β of morphisms of $\mathbf{C}$ for which $\alpha \circ \beta$ is defined, $F(\alpha \circ \beta) = F(\alpha) \circ F(\beta)$ (or $F_{\alpha \circ \beta} = F_\alpha \circ F_\beta$).

Definition Let each of $\mathbf{I}$ and $\mathbf{C}$ be a category, and $F : \mathbf{I} \to \mathbf{C}$ be a functor. The statement that (x, α) *is a cone to* F means for each object i of $\mathbf{I}$, $\alpha_i : x \to F_i$ is a morphism of $\mathbf{C}$ and if $\tau : i \to j$ is a morphism of $\mathbf{I}$ then $F_\tau \circ \alpha_i = \alpha_j$.

$$\begin{tikzcd}
 & x \arrow[dl, "\alpha_i"'] \arrow[dr, "\alpha_j"] & \\
F_i \arrow[rr, "F_\tau"'] & & F_j
\end{tikzcd}$$

Definition Let each of $\mathbf{I}$ and $\mathbf{C}$ be a category ($\mathbf{I}$ can be thought of as an index category), and l be an object of $\mathbf{C}$. The statement that (l, π) *is a limit of* F means (l, π) is a cone to F, and if (x, α) is a cone to F, then there is a unique morphism $\gamma : x \to l$ of $\mathbf{C}$ such that for each object i of $\mathbf{I}$, $\pi_i \circ \gamma = \alpha_i$. In this case (l, π) is also said to be a terminal cone to F.

$$\begin{tikzcd}
x \arrow[rr, dashed, "\exists! \gamma"] \arrow[dr, "\alpha_i"'] & & l \arrow[dl, "\pi_i"] \\
 & F_i &
\end{tikzcd}$$

Definition Define $\mathbf{I}$ to be the category with objects n and $(n, n+1)$ for each $n \in \omega$, and morphisms the symbols $n_* : (n, n+1) \to n$ and $(n+1)^* : (n, n+1) \to n+1$ for each $n \in \omega$, as well as the identity morphism on each object.

$$\begin{tikzcd}
\cdots \arrow[dr, "n^*"] & & (n, n+1) \arrow[dl, "n_*"'] \arrow[dr, "(n+1)^*"] & & \cdots \arrow[dl, "(n+1)_*"'] \\
\cdots & n & & n+1 & \cdots
\end{tikzcd}$$

Definition The category **Top** is the category having topological spaces as objects, and continuous functions as morphisms. Identity and composition are defined as the terminology suggests.

Definition Let each of **C** and **D** be a category and each of F and G be a functor from **C** to **D**. $\eta : F \to G$ is called a *natural transformation from F to G* if for each object x of **C**, $\eta_x : F(x) \to G(x)$ is a morphism, such that for each morphism $\alpha : x \to y$ of **C** the following square commutes:

$$
\begin{tikzcd}
F(x) \arrow[r, "\eta_x"] \arrow[d, "F(\alpha)"'] & G(x) \arrow[d, "G(\alpha)"] \\
F(y) \arrow[r, "\eta_y"'] & G(y)
\end{tikzcd}
$$

That is, $G(\alpha) \circ \eta_x = \eta_y \circ F(\alpha)$.

Definition Define a category $\mathbf{Top}^{\mathbf{I}}$ having objects functors from **I** to **Top** and morphisms natural transformations between functors. In this case a natural transformation $\eta : X \to Y$ will make the diagram below commute.

$$
\begin{tikzcd}
\cdots \arrow[dr, "X_{n^*}"] & & X_{(n,n+1)} \arrow[dl, "X_{n_*}"'] \arrow[dr, "X_{(n+1)^*}"] \arrow[dd, "\eta_{(n,n+1)}"'] & & \cdots \arrow[dl, "X_{(n+1)_*}"'] \\
\cdots & X_n \arrow[dd, "\eta_n"'] & & X_{n+1} \arrow[dd, "\eta_{n+1}"'] & \cdots \\
\cdots \arrow[dr, "Y_{n^*}"] & & Y_{(n,n+1)} \arrow[dl, "Y_{n_*}"'] \arrow[dr, "Y_{(n+1)^*}"] & & \cdots \arrow[dl, "Y_{(n+1)_*}"'] \\
\cdots & Y_n & & Y_{n+1} & \cdots
\end{tikzcd}
$$

For each object X of $\mathbf{Top}^{\mathbf{I}}$, the identity 1_X on X is defined by $(1_X)_i = 1_{X_i}$ for each object i of **I**. The composition $\theta \circ \eta$ is defined by $(\theta \circ \eta)_i = \theta_i \circ \eta_i$ for each object i of **I**. Note that each functor in $\mathbf{Top}^{\mathbf{I}}$ can in a certain sense be thought of as a structural copy of the category **I** that lies within **Top**.

Definition Let $\mathbf{C}$ be a category with object class O, morphism class M, and composition operator $\circ$, and let O' be a subset of O and M' be a subset of M, such that every morphism of M' has source and target in O'. Then $O', M', \circ'$ (where $\circ'$ is $\circ$ restricted to M') is said to be a *subcategory of* $\mathbf{C}$ if it is a category.

3.2 Main Results

Theorem 3.1.

1. *For each functor X from* $\mathbf{I}$ *to* $\mathbf{Top}$, *a limit* (U, π) *of* X *exists and is unique up to homeomorphism.*

2. *If each of* X *and* Y *is a functor from* $\mathbf{I}$ *to* $\mathbf{Top}$, $(L(X), \pi^X)$ *is a limit of* X, *and* $(L(Y), \pi^Y)$ *is a limit of* Y, *then each morphism* $\eta : X \to Y$ *of* $\mathbf{Top}^{\mathbf{I}}$ *induces a continuous function* $L(\eta) : L(X) \to L(Y)$ *unique in having the property that* $\pi_i^Y \circ L(\eta) = \eta_i \circ \pi_i^X$ *for each object* i *of* $\mathbf{I}$.

3. *Let* $\mathbf{D}$ *be a subcategory of* $\mathbf{Top}^{\mathbf{I}}$, $(L(X), \pi^X)$ *be a limit of* X *for each functor* X *of* $\mathbf{D}$, *and* $L(\eta)$ *be as in 2 for each morphism* η *of* $\mathbf{Top}^{\mathbf{I}}$ *in* $\mathbf{D}$. *Then* $L : \mathbf{D} \to \mathbf{Top}$ *is a functor.*

Proof.

To prove (1), we note that $\mathbf{I}$ is what is known as a *small category*, meaning its object and morphism classes are sets. It is well known that a limit of X exists in **Top** for every functor X from a small category (see [9], p. 133) (when a category has this property we say it is *small complete* or *complete*). The canonical construction of such a limit is

$$U = \{p \in \prod_{i \in O(\mathbf{I})} X_i \mid X_\alpha(p_y) = p_z \text{ for every morphism } \alpha : y \to z \text{ of } \mathbf{I}\}$$

(Where $O(\mathbf{I})$ is the object class of $\mathbf{I}$). The topology for this space is the subspace topology inherited from the product topology. (We will later be using a different, homeomorphic construction of the limit of a functor X.) The construction for π is given by defining $\pi_i(p) = p_i$ for each object i of $\mathbf{I}$ and each $p \in U$.

Category theoretical limits are well known to be unique up to isomorphism (homeomorphism in this case) in all contexts (see [9], p. 69).

(2) is well known ([9], p. 114 exercise 3). The outline of the proof is that for each morphism $\eta : X \to Y$ of $\mathbf{Top}^{\mathbf{I}}$, $(L(X), \eta \circ \pi^X)$ is a cone to Y, and since $(L(Y), \pi^Y)$ is a terminal cone to Y, there is a unique morphism $L(\eta) : L(X) \to L(Y)$ having the desired property.

(3) is a slight generalization of a special case of a well known theorem of category theory ([9], p. 114 exercise 3) that says that if a category $\mathbf{C}$ is small complete and $\mathbf{I}$ is small then $L : \mathbf{C}^{\mathbf{I}} \to \mathbf{C}$ is a functor (our modification is to extend this notion to the case where the map is defined on a subcategory of $\mathbf{C}^{\mathbf{I}}$). We give the proof here although it is essentially the proof for the non-generalized version.

Let X be an object of $\mathbf{D}$, i be an object of $\mathbf{I}$. $\pi_i^X \circ 1_{L(X)} = \pi_i^X = 1_{X_i} \circ \pi_i^X = (1_X)_i \circ \pi_i^X$ and $L(1_X)$ is the unique morphism such that $\pi_i^X \circ L(1_X) = (1_X)_i \circ \pi_i^X$, so $L(1_X) = 1_{L(X)}$. Let each of $\eta : X \to Y$ and $\theta : Y \to Z$ be a morphism of $\mathbf{D}$. Then $L(\theta)$ is the unique morphism from $L(Y) \to L(Z)$ such that for each object i of $\mathbf{I}$, $\pi_i^Z \circ L(\theta) = \theta_i \circ \pi_i^Y$. Furthermore $L(\eta)$ is the unique morphism from $L(X) \to L(Y)$ such that for each object i of $\mathbf{I}$, $\pi_i^Y \circ L(\eta) = \eta_i \circ \pi_i^X$. Let i be an object of $\mathbf{I}$. Then:

$$\pi_i^Z \circ L(\theta) \circ L(\eta) = \theta_i \circ \pi_i^Y \circ L(\eta) = \theta_i \circ \eta_i \circ \pi_i^X = (\theta \circ \eta)_i \circ \pi_i^X.$$

$L(\theta \circ \eta)$ is the unique morphism from $L(X) \to L(Z)$ such that for each object x of $\mathbf{I}$, $\pi_x^Z \circ L(\theta \circ \eta) = (\theta \circ \eta)_x \circ \pi_x^X$. We have that $L(\theta) \circ L(\eta)$ has this property, so $L(\theta) \circ L(\eta) = L(\theta \circ \eta)$. So L is a functor. □

Definition Define **GLim** to be the subcategory of $\mathbf{Top}^{\mathbf{I}}$ of which a functor $X : \mathbf{I} \to \mathbf{Top}$ is an object only in case for each $n \in \omega$:

1. $X_{(n,n+1)}$ is a subspace of $X_n \times X_{n+1}$.

2. $X_{n_*} : X_{(n,n+1)} \to X_n$ is the restriction of the projection map onto the left coordinate.

3. $X_{(n+1)^*} : X_{(n,n+1)} \to X_{n+1}$ is the restriction of the projection map onto the right coordinate.

A natural transformation η is a morphism of **GLim** if η is a morphism of $\mathbf{Top^I}$ with source and target belonging to **GLim**.

Note that if X is an object of **GLim** and for each $n \in \omega$ we define $r_n = X_{(n,n+1)}$, then $\{X_n\}_{n\in\omega}, \{r_n\}_{n\in\omega}$ is a generalized inverse sequence. Then $\varprojlim\{X_{(n,n+1)}\}_{n\in\omega} = \varprojlim\{r_n\}_{n\in\omega} = \varprojlim \mathbf{r}$.

Theorem 3.2. *Let X be an object of* **GLim**. *Define π^X such that for each $n \in \omega$, $\pi_n^X : \varprojlim\{X_{(n,n+1)}\}_{n\in\omega} \to X_n$ is defined by $\pi_n^X(p) = p_n$ for each $p \in \varprojlim\{X_{(n,n+1)}\}_{n\in\omega}$, and $\pi_{(n,n+1)}^X : \varprojlim\{X_{(n,n+1)}\}_{n\in\omega} \to X_{(n,n+1)}$ is defined by $\pi_{(n,n+1)}^X(p) = (p_n, p_{n+1})$ for each $p \in \varprojlim\{X_{(n,n+1)}\}_{n\in\omega}$. Then $(\varprojlim\{X_{(n,n+1)}\}_{n\in\omega}, \pi^X)$ is a limit of X.*

Proof.

Claim. $(\varprojlim\{X_{(n,n+1)}\}_{n\in\omega}, \pi^X)$ *is a cone to X.*

Proof. We want to show the following diagram commutes for each $n \in \omega$:

$$\begin{tikzcd}
 & \varprojlim\{X_{(n,n+1)}\}_{n\in\omega} \arrow[d, "\pi^X_{(n,n+1)}"] \arrow[ddl, bend right, "\pi^X_n"'] \arrow[ddr, bend left, "\pi^X_{n+1}"] & \\
 & X_{(n,n+1)} \arrow[dl, "X_{n_*}"'] \arrow[dr, "X_{(n+1)^*}"] & \\
X_n & & X_{n+1}
\end{tikzcd}$$

Let p be in $\varprojlim\{X_{(n,n+1)}\}_{n\in\omega}$. We have that

$$\begin{aligned}
(X_{n_*} \circ \pi^X_{(n,n+1)})(p) =& X_{n_*}(\pi^X_{(n,n+1)}(p)) = X_{n_*}(p_n, p_{n+1}) \\
=& p_n = \pi_n^X(p).
\end{aligned}$$

So $X_{n_*} \circ \pi^X_{(n,n+1)} = \pi^X_n$. Also,

$$
\begin{aligned}
(X_{(n+1)^*} \circ \pi^X_{(n,n+1)})(p) =& X_{(n+1)^*}(\pi^X_{(n,n+1)}(p)) = X_{(n+1)^*}(p_n, p_{n+1}) \\
=& p_{n+1} = \pi^X_{n+1}(p).
\end{aligned}
$$

So $X_{(n+1)^*} \circ \pi^X_{(n,n+1)} = \pi^X_{n+1}$. So $(\varprojlim \{X_{(n,n+1)}\}_{n\in\omega}, \pi^X)$ is a cone to X. $\square$

Claim. $(\varprojlim \{X_{(n,n+1)}\}_{n\in\omega}, \pi^X)$ *is a terminal cone to* X*, that is, if* W *is space with family* α *such that for each object* i *of* $\mathbf{I}$*,* $\alpha_i : W \to X_i$ *is a continuous map, such that*

$$
\begin{tikzcd}
 & W \arrow[d, "\alpha_{(n,n+1)}"] \arrow[ddl, bend right, "\alpha_n"'] \arrow[ddr, bend left, "\alpha_{n+1}"] & \\
 & X_{(n,n+1)} \arrow[dl, "X_{n_*}"'] \arrow[dr, "X_{(n+1)^*}"] & \\
X_n & & X_{n+1}
\end{tikzcd}
$$

commutes for each $n \in \omega$*, then there is a unique* $\gamma : W \to \varprojlim \{X_{(n,n+1)}\}_{n\in\omega}$ *such that* $\pi^X_i \circ \gamma = \alpha_i$ *for each object* i *of* $\mathbf{I}$*.*

Proof. Let W be such a space with α such a family. Define $\gamma : W \to \varprojlim \{X_{(n,n+1)}\}_{n\in\omega}$ such that for each $w \in W$, $n \in \omega$, $\gamma(w)_n = \alpha_n(w)$.

γ is well defined: For all $n \in \omega$,

$$
\begin{aligned}
(\gamma(w)_n, \gamma(w)_{n+1}) =& (\alpha_n(w), \alpha_{n+1}(w)) \\
=& ((X_{n_*} \circ \alpha_{(n,n+1)})(w), (X_{(n+1)^*} \circ \alpha_{(n,n+1)})(w)) \\
=& (X_{n_*}(\alpha_{(n,n+1)}(w)), X_{(n+1)^*}(\alpha_{(n,n+1)}(w))) \\
=& \alpha_{(n,n+1)}(w) \in X_{(n,n+1)}.
\end{aligned}
$$

So $\gamma(w) \in \varprojlim\{X_{(n,n+1)}\}_{n\in\omega}$.

γ is continuous: By the fact that it is the product of continuous maps.

γ has the desired commutative property: Let w be in W. Then:

$$(\pi_n^X \circ \gamma)(w) = \pi_n^X(\gamma(w)) = (\gamma(w))_n = \alpha_n(w).$$

So $\pi_n^X \circ \gamma = \alpha_n$. Furthermore:

$$\begin{aligned}
X_{n_*}((\pi^X_{(n,n+1)} \circ \gamma)(w)) =& X_{n_*}(\pi^X_{(n,n+1)}(\gamma(w))) = X_{n_*}(\gamma(w)_n, \gamma(w)_{n+1}) \\
=& X_{n_*}(\alpha_n(w), \alpha_{n+1}(w)) = \alpha_n(w) \\
=& (X_{n_*} \circ \alpha_{(n,n+1)})(w) \\
=& X_{n_*}(\alpha_{(n,n+1)}(w)), \\
X_{(n+1)^*}((\pi^X_{(n,n+1)} \circ \gamma)(w)) =& X_{(n+1)^*}(\pi^X_{(n,n+1)}(\gamma(w))) = X_{(n+1)^*}(\gamma(w)_n, \gamma(w)_{n+1}) \\
=& X_{(n+1)^*}(\alpha_n(w), \alpha_{n+1}(w)) = \alpha_{n+1}(w) \\
=& (X_{(n+1)^*} \circ \alpha_{(n,n+1)})(w) \\
=& X_{(n+1)^*}(\alpha_{(n,n+1)}(w)).
\end{aligned}$$

Since X_{n_*} and $X_{(n+1)^*}$ are the projections onto the first and second coordinates respectively for a subspace of $X_n \times X_{n+1}$, and $X_{n_*}((\pi^X_{(n,n+1)} \circ \gamma)(w)) = X_{n_*}(\alpha_{(n,n+1)}(w))$ and $X_{(n+1)^*}((\pi^X_{(n,n+1)} \circ \gamma)(w)) = X_{(n+1)^*}(\alpha_{(n,n+1)}(w))$ we have that $(\pi^X_{(n,n+1)} \circ \gamma)(w) = \alpha_{(n,n+1)}(w)$ for each $w \in W$, and hence $\pi^X_{(n,n+1)} \circ \gamma = \alpha_{(n,n+1)}$.

γ has this property uniquely: Let $\gamma' : W \to \varprojlim\{X_{(n,n+1)}\}_{n\in\omega}$ have the property that

$\pi_n^X \circ \gamma' = \alpha_n$ and $\pi_{(n,n+1)}^X \circ \gamma' = \alpha_{(n,n+1)}$ for each $n \in \omega$. Then for all $w \in W$, $n \in \omega$,

$$(\gamma(w))_n = \pi_n^X(\gamma(w)) = (\pi_n^X \circ \gamma)(w) = \alpha_n(w) = (\pi_n^X \circ \gamma')(w) = \pi_n^X(\gamma'(w)) = (\gamma'(w))_n.$$

So $\gamma(w) = \gamma'(w)$ and thus $\gamma = \gamma'$. Thus $(\varprojlim\{X_{(n,n+1)}\}_{n\in\omega}, \pi^X)$ is a terminal cone. □

Thus $(\varprojlim\{X_{(n,n+1)}\}_{n\in\omega}, \pi^X)$ is a limit of X. □

Corollary 3.2.1. *$L(X) = \varprojlim\{X_{(n,n+1)}\}_{n\in\omega}$ and $L(\eta)$ as in part 2 of Theorem 3.1 defines a functor L from* **GLim** *to* **Top**.

Proof. From Theorem 3.2, $(\varprojlim\{X_{(n,n+1)}\}_{n\in\omega}, \pi^X)$ is a limit.

Each morphism $\eta : X \to Y$ of **GLim** is a morphism of $\mathbf{Top}^{\mathbf{I}}$, so part 2 of Theorem 3.1 allows us to define a continuous function $L(\eta) : L(X) \to L(Y)$ uniquely having the property that $\pi_i^Y \circ L(\eta) = \eta_i \circ \pi_i^X$ for each object i in $\mathbf{I}$.

So by part 3 of Theorem 3.1, L is a functor. □

Note that in our construction of $\mathbf{Top}^{\mathbf{I}}$ and **GLim**, we have imposed none of the standard conditions, such as requiring that each factor space be a compact metric space or that each bonding map be upper semi-continuous. Because of the general nature of Theorem 3.1 part 3, we may replace **GLim** with any subcategory of $\mathbf{Top}^{\mathbf{I}}$ we desire and achieve the same result. For example, if we define $\mathbf{GLim}_{CMet,USC}$ to be the subcategory of **GLim** where each functor maps each object n of $\mathbf{I}$ to a compact metric space and each object $(n, n+1)$ of $\mathbf{I}$ to a closed subset of $X_n \times X_{n+1}$ (which is a graph corresponding to an upper semi-continuous set-valued bonding map) then we can similarly construct a functor from this category of inverse sequences to **Top**.

Theorem 3.2 leads to a characterization of generalized inverse limits as "category theoretical" limits of functors in **GLim**, which can be thought of as inverse sequences of the form

$$\begin{tikzcd}
\cdots \arrow[dr, "X_{n^*}"] & & X_{(n,n+1)} \arrow[dl, "X_{n_*}"'] \arrow[dr, "X_{(n+1)^*}"] & & \cdots \arrow[dl, "X_{(n+1)_*}"'] \\
\cdots & X_n & & X_{n+1} & \cdots
\end{tikzcd}$$

with each $X_{(n,n+1)}$ the graph of a set valued map bonding X_n and X_{n+1}, and $X_{n_*}, X_{(n+1)^*}$ the projections of this graph into the factor spaces.

Corollary 3.2.2. *A space U is representable as a generalized inverse limit on factor spaces $\{X_n\}_{n\in\omega}$ if and only if for each $n \in \omega$, there is a subset $X_{(n,n+1)}$ of $X_n \times X_{n+1}$ so that if $X_{n_*} : X_{(n,n+1)} \to X_n$ and $X_{(n+1)^*} : X_{(n,n+1)} \to X_{n+1}$ are as defined earlier, X is a functor of* **GLim** *having U as (the object part of) a category theoretical limit.*

Proof. Let U be a topological space.

($\Longrightarrow$): Suppose U is representable as an inverse limit on factor spaces $\{X_n\}_{n\in\omega}$. Then U is homeomorphic to $\varprojlim\{X_{(n,n+1)}\}_{n\in\omega}$ where $X_{(n,n+1)}$ is a subset of $X_n \times X_{n+1}$. Define X_{n_*} and $X_{(n+1)^*}$ to be the projections onto the left and right coordinates, respectively. Then X is a functor belonging to **GLim** having $\varprojlim\{X_{(n,n+1)}\}_{n\in\omega}$ as a limit by Theorem 3.2. Since U is homeomorphic to $\varprojlim\{X_{(n,n+1)}\}_{n\in\omega}$, U is a also a limit.

($\Longleftarrow$): Suppose for each $n \in \omega$, there is a subset $X_{(n,n+1)}$ of $X_n \times X_{n+1}$ so that if $X_{n_*} : X_{(n,n+1)} \to X_n$ and $X_{(n+1)^*} : X_{(n,n+1)} \to X_{n+1}$ are as defined earlier, X is a functor of **GLim** having U as (the object part of) a category theoretical limit. By Theorem 3.2, $\varprojlim\{X_{(n,n+1)}\}_{n\in\omega}$ is also a limit, and since limits are unique up to homeomorphism, U and $\varprojlim\{X_{(n,n+1)}\}_{n\in\omega}$ are homeomorphic. So U is representable as the generalized limit $\varprojlim\{X_{(n,n+1)}\}_{n\in\omega}$ on factor spaces $\{X_n\}_{n\in\omega}$. □

Another characterization arises as follows. (For the following, for a function α, $\mathrm{im}(\alpha)$ is the image of α.)

Theorem 3.3. *A topological space U is representable as a generalized inverse limit on factor spaces $\{X_n\}_{n\in\omega}$ if and only if there are continuous functions $\{\psi^X_{(n,n+1)} : U \to X_n \times X_{n+1}\}_{n\in\omega}$ such that $\pi^* \circ \psi^X_{(n,n+1)} = \pi_* \circ \psi^X_{(n+1,n+2)}$ for each $n \in \omega$ (where π_*, π^* are the projections onto the left and right coordinates respectively), having the property that if W is a space with continuous functions $\{\alpha_{(n,n+1)} : W \to \mathrm{im}(\psi^X_{(n,n+1)})\}_{n\in\omega}$ such that $\pi^* \circ \alpha_{(n,n+1)} = \pi_* \circ \alpha_{(n+1,n+2)}$ for each $n \in \omega$, then there is a continuous function $\gamma : W \to U$ uniquely having the property that $\psi^X_{(n,n+1)} \circ \gamma = \alpha_{(n,n+1)}$ for each $n \in \omega$.*

$$\begin{tikzcd}
& U \arrow[dl, "\psi^X_{(n,n+1)}"'] \arrow[dr, "\psi^X_{(n+1,n+2)}"] & \\
X_n \times X_{n+1} \arrow[dr, "\pi^*"'] & & X_{n+1} \times X_{n+2} \arrow[dl, "\pi_*"] \\
& X_{n+1} &
\end{tikzcd}$$

Proof. Let U be a topological space and X_i be a topological space for each $i \in \omega$.

($\Longrightarrow$): Suppose U is representable as a generalized inverse limit on factor spaces $\{X_i\}_{i\in\omega}$. Then by Corollary 3.2.2, for each $n \in \omega$, there is a subset $X_{(n,n+1)}$ of $X_n \times X_{n+1}$ so that if $X_{n_*} : X_{(n,n+1)} \to X_n$ and $X_{(n+1)^*} : X_{(n,n+1)} \to X_{n+1}$ are as defined earlier, X is a functor of **GLim** having U as (the object part of) a category theoretical limit. So there are continuous functions $\psi^X_n : U \to X_n$ and $\psi^X_{(n,n+1)} : U \to X_{(n,n+1)}$ for each $n \in \omega$ so that (U, ψ^X) is a category theoretical limit. Let W be a space with continuous functions $\{\alpha_{(n,n+1)} : W \to \mathrm{im}(\psi^X_{(n,n+1)})\}_{n\in\omega}$ such that $\pi^* \circ \alpha_{(n,n+1)} = \pi_* \circ \alpha_{(n+1,n+2)}$ for each $n \in \omega$. Note that for each $n \in \omega$ since $\mathrm{im}(\psi^X_{(n,n+1)}) \subseteq X_{(n,n+1)}$, $\alpha_{(n,n+1)}$ is also a continuous function into $X_{(n,n+1)}$. For each $n \in \omega$ define $\alpha_n = \pi_* \circ \alpha_{n,n+1}$. Then for each $n \in \omega$, note that $X_{n_*} = \pi_*|_{X_{(n,n+1)}}$ and $X_{(n+1)^*} = \pi^*|_{X_{(n,n+1)}}$, so $X_{n_*} \circ \alpha_{(n,n+1)} = \pi_* \circ \alpha_{(n,n+1)} = \alpha_n$ and $X_{(n+1)^*} \circ \alpha_{(n,n+1)} = \pi^* \circ \alpha_{(n,n+1)} = \pi_* \circ \alpha_{(n+1,n+2)} = \alpha_{n+1}$. So (W, α) is a cone to X, and thus since (U, ψ^X) is a category theoretical limit of X, there is a continuous function $\gamma : W \to U$

uniquely having the property that $\psi_n^X \circ \gamma = \alpha_n$ and $\psi_{(n,n+1)}^X \circ \gamma = \alpha_{(n,n+1)}$ for each $n \in \omega$. So the family $\{\psi_{(n,n+1)}^X : U \to X_n \times X_{n+1}\}_{n\in\omega}$ has the desired properties.

($\Longleftarrow$): Suppose there are continuous functions $\{\psi_{(n,n+1)}^X : U \to X_n \times X_{n+1}\}_{n\in\omega}$ such that $\pi^* \circ \psi_{(n,n+1)}^X = \pi_* \circ \psi_{(n+1,n+2)}^X$ for each $n \in \omega$ (where π_*, π^* are the projections onto the left and right coordinates respectively), having the property that if W is a space with continuous functions $\{\alpha_{(n,n+1)} : W \to \operatorname{im}(\psi_{(n,n+1)}^X)\}_{n\in\omega}$ such that $\pi^* \circ \alpha_{(n,n+1)} = \pi_* \circ \alpha_{(n+1,n+2)}$ for each $n \in \omega$, then there is a continuous function $\gamma : W \to U$ uniquely having the property that $\psi_{(n,n+1)}^X \circ \gamma = \alpha_{(n,n+1)}$ for each $n \in \omega$. For each $n \in \omega$, define $\psi_n^X : U \to X_n$ by $\psi_n^X = \pi_* \circ \psi_{(n,n+1)}^X$, define $X_{(n,n+1)} = \operatorname{im}(\psi_{(n,n+1)}^X)$, define $X_{n_*} = \pi_*|_{X_{(n,n+1)}}$, and define $X_{(n+1)^*} = \pi^*|_{X_{(n,n+1)}}$. Note that X is a functor of **GLim**. The pair (U, ψ^X) is a cone to X since $X_{n_*} \circ \psi_{(n,n+1)}^X = \pi_* \circ \psi_{(n,n+1)}^X = \psi_n^X$ and $X_{(n+1)^*} \circ \psi_{(n,n+1)}^X = \pi^* \circ \psi_{(n,n+1)}^X = \pi^* \circ \psi_{(n+1,n+2)}^X = \psi_{n+1}^X$.

Claim. *(U, ψ^X) is a terminal cone to X.*

Proof. Let W be a topological space and α be a family such that for each $n \in \omega$, $\alpha_n : W \to X_n$ and $\alpha_{(n,n+1)} : W \to X_{(n,n+1)}$ is each a continuous function, $X_{n_*} \circ \alpha_{(n,n+1)} = \alpha_n$ and $X_{(n+1)^*} \circ \alpha_{(n,n+1)} = \alpha_{n+1}$. Since $X_{(n,n+1)} = \operatorname{im}(\psi_{(n,n+1)}^X)$, we have a family $\{\alpha_{(n,n+1)} : W \to \operatorname{im}(\psi_{(n,n+1)}^X)\}_{n\in\omega}$ of continuous functions with the desired domain and codomain, and for each $n \in \omega$:

$$\pi^* \circ \alpha_{(n,n+1)} = X_{(n+1)^*} \circ \alpha_{(n,n+1)} = \alpha_{n+1} = X_{(n+1)_*} \circ \alpha_{(n+1,n+2)} = \pi_* \circ \alpha_{(n+1,n+2)}.$$

So by assumption there is a continuous function $\gamma : W \to U$ uniquely having the property that $\psi_{(n,n+1)}^X \circ \gamma = \alpha_{(n,n+1)}$ for each $n \in \omega$. Thus,

$$\psi_n^X \circ \gamma = \pi_* \circ \psi_{(n,n+1)}^X \circ \gamma = \pi_* \circ \alpha_{(n,n+1)} = X_{n_*} \circ \alpha_{(n,n+1)} = \alpha_n.$$

So (U, ψ^X) is a terminal cone to X. $\square$

So (U, ψ^X) is a category theoretical limit of X, so by Corollary 3.2.2, U is representable as a generalized inverse limit on factor spaces $\{X_n\}_{n\in\omega}$. $\square$

3.3 Adjoint Pairs

It is well known that every topological space U can be realized as a traditional inverse limit where all the factor spaces are U, by letting each bonding map be the identity. We intend to show that this idea induces a functor from **Top** to **GLim**, which together with the functor L forms what is called an adjoint pair.

Definition Let each of $\mathbf{C}$ and $\mathbf{D}$ be a category and each of $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{C} \to \mathbf{D}$ be a functor. (F, G) is said to be an *adjoint pair* if for each object x of $\mathbf{D}$ and object y of $\mathbf{C}$, there is a bijection $\eta_{xy} : \mathbf{D}(F(x), y) \to \mathbf{C}(x, G(y))$ such that for each morphism $\alpha : x \to x'$ of $\mathbf{C}$ and each morphism $\beta : y \to y'$ of $\mathbf{D}$, the following diagram commutes:

$$\begin{tikzcd}
\mathbf{D}(F(x'), y) \arrow[r, "\eta_{x'y}"] \arrow[d, "F(\alpha)^*"] & \mathbf{C}(x', G(y)) \arrow[d, "\alpha^*"] \\
\mathbf{D}(F(x), y) \arrow[r, "\eta_{xy}"] \arrow[d, "\beta_*"] & \mathbf{C}(x, G(y)) \arrow[d, "G(\beta)_*"] \\
\mathbf{D}(F(x), y') \arrow[r, "\eta_{xy'}"] & \mathbf{C}(x, G(y'))
\end{tikzcd}$$

Where for each morphism $\gamma : x \to y$ of a category $\mathbf{A}$, $\gamma_* : \mathbf{A}(z, x) \to \mathbf{A}(z, y)$ is left composition by γ, and $\gamma^* : \mathbf{A}(y, z) \to \mathbf{A}(x, z)$ is right composition by γ. Adjoint pairs describe a relationship between functors which is similar to but weaker than the relationship functors have which map between isomorphic or equivalent categories. If (F, G) is an adjoint pair then F is said to be *left adjoint to* G, and G is said to be *right adjoint to* F.

We begin by defining a functor from $\mathbf{Top}^{\mathbf{I}}$ to its subcategory **GLim**.

Theorem 3.4. *Let $I : \mathbf{GLim} \to \mathbf{Top}^{\mathbf{I}}$ be the inclusion functor. For each functor X in $\mathbf{Top}^{\mathbf{I}}$, define ϕ^X so that for each $n \in \omega$, $\phi^X_n = 1_{X_n}$ and $\phi^X_{(n,n+1)}$ is defined so that $\phi^X_{(n,n+1)}(p) = (X_{n_*}(p), X_{(n+1)^*}(p))$ for each $p \in X_{(n,n+1)}$. For each functor X in $\mathbf{Top}^{\mathbf{I}}$ define $P(X)$ so that for each $n \in \omega$, $P(X)_n = X_n$ and $P(X)_{(n,n+1)}$ is the space $\mathrm{im}(\phi^X_{(n,n+1)})$ endowed with* *the subspace topology inherited from* $X_n \times X_{n+1}$*, and for each* $n \in \omega$ *define* $P(X)_{n*}$ *and* $P(X)_{(n+1)^*}$ *to be the projection maps onto the left and right coordinates. Then:*

1. *For each morphism* $\eta : X \to Y$ *of* $\mathbf{Top^I}$ *there is a morphism* $P(\eta) : P(X) \to P(Y)$ *of* **GLim** *uniquely having the property that* $I(P(\eta)) \circ \phi^X = \phi^Y \circ \eta$*.*

2. P *is a functor from* $\mathbf{Top^I}$ *to* **GLim***.*

3. ϕ *is a natural transformation from the identity functor on* $\mathbf{Top^I}$ *to* $I \circ P$*.*

4. *For any functor* X *or morphism* η *of* **GLim**, $\phi^{I(X)} = 1_{I(X)}$*,* $P(I(X)) = X$*, and* $P(I(\eta)) = \eta$*.*

Proof.

To prove (1), let $\eta : X \to Y$ be a morphism of $\mathbf{Top^I}$, and define $P(\eta)$ so that for each $n \in \omega$, $P(\eta)_n = \eta_n$ and $P(\eta)_{(n,n+1)} = (\eta_n \times \eta_{n+1})\Big|_{P(X)_{(n,n+1)}}$.

$P(\eta)$ is well defined: We want to show that $(\eta_n(a), \eta_{n+1}(b))$ is in $P(Y)_{(n,n+1)}$ for each $(a, b) \in P(X)_{(n,n+1)}$. Since $(a, b) \in P(X)_{(n,n+1)}$, $(a, b) \in \mathrm{im}(\phi^X_{(n,n+1)})$, so $(a, b) = \phi^X_{(n,n+1)}(p) = (X_{n_*}(p), X_{(n+1)^*}(p))$ for some $p \in X_{(n,n+1)}$. So $a = X_{n_*}(p)$ and $b = X_{(n+1)^*}(p)$. So we have:

$$
\begin{aligned}
(\eta_n(a), \eta_{n+1}(b)) =& (\eta_n(X_{n_*}(p)), \eta_{n+1}(X_{(n+1)^*}(p))) \\
=& ((\eta_n \circ X_{n_*})(p), (\eta_{n+1} \circ X_{(n+1)^*})(p)) \\
=& ((Y_{n_*} \circ \eta_{(n,n+1)})(p), (Y_{(n+1)^*} \circ \eta_{(n,n+1)})(p)) \\
=& (Y_{n_*}(\eta_{(n,n+1)}(p)), Y_{(n+1)^*}(\eta_{(n,n+1)}(p))) \\
=& \phi^Y_{(n,n+1)}(\eta_{(n,n+1)}(p)) \in \mathrm{im}(\phi^Y_{(n,n+1)}) = P(Y)_{(n,n+1)}.
\end{aligned}
$$

$P(\eta)_{(n,n+1)}$ is continuous since it is constructed from continuous functions using continuity-preserving operations. So $P(\eta)$ is well defined.

Let n be in ω and p be in $X_{(n,n+1)}$. Then:

$$\begin{aligned}
(I(P(\eta)) \circ \phi^X)_{(n,n+1)}(p) =& (I(P(\eta))_{(n,n+1)} \circ \phi^X_{(n,n+1)})(p) \\
=& I(P(\eta))_{(n,n+1)}(\phi^X_{(n,n+1)}(p)) \\
=& I(P(\eta))_{(n,n+1)}(X_{n_*}(p), X_{(n+1)^*}(p)) \\
=& (\eta_n(X_{n_*}(p)), \eta_{n+1}(X_{(n+1)^*}(p))) \\
=& ((\eta_n \circ X_{n_*})(p), (\eta_{n+1} \circ X_{(n+1)^*})(p)) \\
=& ((Y_{n_*} \circ \eta_{(n,n+1)})(p), (Y_{(n+1)^*} \circ \eta_{(n,n+1)})(p)) \\
=& (Y_{n_*}(\eta_{(n,n+1)}(p)), Y_{(n+1)^*}(\eta_{(n,n+1)}(p))) \\
=& \phi^Y_{(n,n+1)}(\eta_{(n,n+1)}(p)) = (\phi^Y_{(n,n+1)} \circ \eta_{(n,n+1)})(p) \\
=& (\phi^Y \circ \eta)_{(n,n+1)}(p).
\end{aligned}$$

So $(I(P(\eta)) \circ \phi^X)_{(n,n+1)} = (\phi^Y \circ \eta)_{(n,n+1)}$. Let p be in X_n. Then:

$$\begin{aligned}
(I(P(\eta)) \circ \phi^X)_n(p) =& (I(P(\eta))_n \circ \phi^X_n)(p) = (\eta_n \circ 1_{X_n})(p) = \eta_n(p) \\
=& (1_{Y_n} \circ \eta_n)(p) = (\phi^Y_n \circ \eta_n)(p) = (\phi^Y \circ \eta)_n(p).
\end{aligned}$$

So $(I(P(\eta)) \circ \phi^X)_n = (\phi^Y \circ \eta)_n$. Thus $I(P(\eta)) \circ \phi^X = \phi^Y \circ \eta$. Let $\gamma : P(X) \to P(Y)$ be a morphism of **GLim** such that $I(\gamma) \circ \phi^X = \phi^Y \circ \eta$. Let n be in ω. Then:

$$\begin{aligned}
I(\gamma)_n =& I(\gamma)_n \circ 1_{X_n} = I(\gamma)_n \circ \phi^X_i = (I(\gamma) \circ \phi^X)_n = (\phi^Y \circ \eta)_n \\
=& (I(P(\eta)) \circ \phi^X)_n = I(P(\eta))_n \circ \phi^X_n \\
=& P(\eta)_n \circ 1_{X_n} = I(P(\eta))_n \circ 1_{P(X)_n} = I(P(\eta))_n \\
\implies \gamma_n =& P(\eta)_n.
\end{aligned}$$

Let (a,b) be in $P(X)_{(n,n+1)}$. Then $(a,b) = \phi^X_{(n,n+1)}(p)$ for some $p \in X_{(n,n+1)}$. So:

$$
\begin{aligned}
I(\gamma)_{(n,n+1)}(a,b) =& I(\gamma)_{(n,n+1)}(\phi^X_{(n,n+1)}(p)) = (I(\gamma)_{(n,n+1)} \circ \phi^X_{(n,n+1)})(p) \\
=& (I(\gamma) \circ \phi^X)_{(n,n+1)}(p) = (\phi^Y \circ \eta)_{(n,n+1)}(p) \\
=& (I(P(\eta)) \circ \phi^X)_{(n,n+1)}(p) = (I(P(\eta))_{(n,n+1)} \circ \phi^X_{(n,n+1)})(p) \\
=& I(P(\eta))_{(n,n+1)}(\phi^X_{(n,n+1)}(p)) = I(P(\eta))_{(n,n+1)}(a,b) \\
\implies \ I(\gamma)_{(n,n+1)} =& I(P(\eta))_{(n,n+1)} \\
\implies \ \gamma_{(n,n+1)} =& P(\eta)_{(n,n+1)}.
\end{aligned}
$$

So $\gamma = P(\eta)$ and $P(\eta)$ has the desired property uniquely.

For (2), let n be in ω, each of X, Y, Z be a functor of $\mathbf{Top}^{\mathbf{I}}$ and each of $\eta : X \to Y$ and $\theta : Y \to Z$ be a morphism of $\mathbf{Top}^{\mathbf{I}}$. Then:

$$
\begin{aligned}
(P(1_X))_n =& 1_{X_n} = 1_{(P(X))_n} = (1_{P(X)})_n, \\
(P(1_X))_{(n,n+1)} =& (1_{X_n} \times 1_{X_{n+1}})\Big|_{(P(X))_{(n,n+1)}} = 1_{(P(X))_{(n,n+1)}} = (1_{P(X)})_{(n,n+1)}, \\
(P(\theta \circ \eta))_n =& (\theta \circ \eta)_n = \theta_n \circ \eta_n = P(\theta)_n \circ P(\eta)_n = (P(\theta) \circ P(\eta))_n \\
P(\theta \circ \eta)_{(n,n+1)} =& ((\theta \circ \eta)_n \times (\theta \circ \eta)_{n+1})\Big|_{(P(X))_{(n,n+1)}} \\
=& ((\theta_n \circ \eta_n) \times (\theta_{n+1} \circ \eta_{n+1}))\Big|_{(P(X))_{(n,n+1)}} \\
=& (\theta_n \times \theta_{n+1}) \circ (\eta_n \circ \eta_{n+1})\Big|_{(P(X))_{(n,n+1)}} \\
=& (\theta_n \times \theta_{n+1})\Big|_{(P(Y))_{(n,n+1)}} \circ (\eta_n \circ \eta_{n+1})\Big|_{(P(X))_{(n,n+1)}} \\
& \text{(Since } \operatorname{im}((\eta_n \circ \eta_{n+1})\Big|_{(P(X))_{(n,n+1)}}) \subseteq (P(Y))_{(n,n+1)}) \\
=& (P(\theta))_{(n,n+1)} \circ (P(\eta))_{(n,n+1)} = (P(\theta) \circ P(\eta))_{(n,n+1)}.
\end{aligned}
$$

So $P(1_X) = 1_{P(X)}$ and $P(\theta \circ \eta) = P(\theta) \circ P(\eta)$. So P is a functor.

(3) follows directly from 1 and 2.

For (4), let X be an object of **GLim** and $\eta : X \to Y$ be a morphism of **GLim**. Then $X_{(n,n+1)} \subseteq X_n \times X_{n+1}$, X_{n_*} is projection onto the left coordinate, and $X_{(n+1)^*}$ is projection onto the right coordinate for all $n \in \omega$ (and thus $I(X)$ has those same properties). We have that $\phi_n^{I(X)} = 1_{(I(X))_n}$ for each $n \in \omega$ and $\phi_{(n,n+1)}^{I(X)}(a,b) = ((I(X))_{n_*}(a,b), I(X)_{(n+1)^*}(a,b)) = (a,b)$ for all $n \in \omega$. So $\phi_{(n,n+1)}^{I(X)} = 1_{(I(X))_{(n,n+1)}}$ and thus $\phi^{I(X)} = 1_{I(X)}$. $(P(I(X)))_n = (I(X))_n = X_n$ for all $n \in \omega$ while $(P(I(X)))_{(n,n+1)} = \mathrm{im}(\phi_{(n,n+1)}^{I(X)}) = \mathrm{im}(1_{(I(X))_{(n,n+1)}}) = (I(X))_{(n,n+1)} = X_{(n,n+1)}$ for all $n \in \omega$. So $P(I(X)) = X$. We have that $(P(I(\eta)))_n = (I(\eta))_n = \eta_n$ for all $n \in \omega$, and for all $n \in \omega$:

$$
\begin{aligned}
&(P(I(\eta)))_{(n,n+1)}(a,b) = ((I(\eta))_n \times (I(\eta))_{n+1})\Big|_{(P(I(X)))_{(n,n+1)}}(a,b) \\
=&((I(\eta))_n(a), (I(\eta))_{n+1}(b)) \\
=&(((I(\eta))_n \circ (I(X))_{n_*})(a,b), ((I(\eta))_{n+1} \circ (I(X))_{(n+1)^*})(a,b)) \\
=&(((I(Y))_{n_*} \circ (I(\eta))_{(n,n+1)})(a,b), ((I(Y))_{(n+1)^*} \circ (I(\eta))_{(n,n+1)})(a,b)) \\
=&(I(\eta))_{(n,n+1)}(a,b) = \eta_{(n,n+1)}(a,b).
\end{aligned}
$$

So $P(I(\eta)) = \eta$. □

Theorem 3.5. *(P, I) is an adjoint pair.*

Proof. For each $\rho \in \mathbf{GLim}(P(X), Y) \to \mathbf{Top}^{\mathbf{I}}(X, I(Y))$ define $\theta_{XY}(\rho) = I(\rho) \circ \phi^X$.

θ_{XY} is an injection: Let each of σ, ρ be in $\mathbf{GLim}(P(X), Y)$ (that is, each of σ, ρ is a morphism between generalized inverse sequences $P(X)$ and Y of **GLim**) with $\theta_{XY}(\sigma) = \theta_{XY}(\rho)$.

For each $n \in \omega$, $\phi_n^X = 1_{X_n}$ so:

$$\begin{aligned}\sigma_n =& \sigma_n \circ 1_{X_n} = \sigma_n \circ \phi_n^X = (\sigma \circ \phi^X)_n = (\theta_{XY}(\sigma))_n \\ =& (\theta_{XY}(\rho))_n = (\rho \circ \phi^X)_n = \rho_n \circ \phi_n^X = \rho_n \circ 1_{X_n} = \rho_n.\end{aligned}$$

Let (a, b) be in $P(X)_{(n,n+1)}$. Then $(a, b) = \phi^X_{(n,n+1)(p)}$ for some $p \in X_{(n,n+1)}$. So:

$$\begin{aligned}\sigma_{(n,n+1)}(a, b) =& \sigma_{(n,n+1)}(\phi^X_{(n,n+1)}(p)) = (\sigma_{(n,n+1)} \circ \phi^X_{(n,n+1)})(p) \\ =& (\sigma \circ \phi^X)_{(n,n+1)}(p) = (\theta_{XY}(\sigma))_{(n,n+1)}(p) \\ =& (\theta_{XY}(\rho))_{(n,n+1)}(p) = (\rho \circ \phi^X)_{(n,n+1)}(p) \\ =& (\rho_{(n,n+1)} \circ \phi^X_{(n,n+1)})(p) = \rho_{(n,n+1)}(\phi^X_{(n,n+1)}(p)) \\ =& \rho_{(n,n+1)}(a, b).\end{aligned}$$

So $\sigma_{(n,n+1)} = \rho_{(n,n+1)}$ and thus $\sigma = \rho$. So θ_{XY} is an injection.

θ_{XY} is a surjection: Let ξ be in $\mathbf{Top^I}(X, I(Y))$. Note $\phi^{I(Y)} = 1_{I(Y)}$ since Y is an object of **GLim**. Then:

$$\theta_{XY}(P(\xi)) = I(P(\xi)) \circ \phi^X = \phi^{I(Y)} \circ \xi = 1_{I(Y)} \circ \xi = \xi.$$

So θ_{XY} is a surjection.

Having shown that θ_{XY} is a bijection for each object X of $\mathbf{Top^I}$ and object Y of **GLim**, it remains to be shown that for a morphism $\alpha : X \to X'$ of $\mathbf{Top^I}$ and a morphism $\beta : Y \to Y'$

of **GLim**, the following diagram commutes:

$$\begin{tikzcd}
\mathbf{GLim}(P(X'),Y) \arrow[r, "\theta_{X'Y}"] \arrow[d, "P(\alpha)^*"] & \mathbf{Top}^{\mathbf{I}}(X',I(Y)) \arrow[d, "\alpha^*"] \\
\mathbf{GLim}(P(X),Y) \arrow[r, "\theta_{XY}"] \arrow[d, "\beta_*"] & \mathbf{Top}^{\mathbf{I}}(X,I(Y)) \arrow[d, "I(\beta)_*"] \\
\mathbf{GLim}(P(X),Y') \arrow[r, "\theta_{XY'}"] & \mathbf{Top}^{\mathbf{I}}(X,I(Y'))
\end{tikzcd}$$

Let σ be in $\mathbf{GLim}(P(X'),Y)$. Then:

$$\begin{aligned}
(\alpha^* \circ \theta_{X'Y})(\sigma) =& \alpha^*(\theta_{X'Y}(\sigma)) = \alpha^*(I(\sigma)\circ\phi^{X'}) = I(\sigma)\circ\phi^{X'}\circ\alpha \\
=& I(\sigma)\circ I(P(\alpha))\circ\phi^X = I(\sigma\circ P(\alpha))\circ\phi^X = I(P(\alpha)^*(\sigma))\circ\phi^X \\
=& \theta_{XY}(P(\alpha^*)(\sigma)) = (\theta_{XY}\circ P(\alpha)^*)(\sigma).
\end{aligned}$$

So $\alpha^*\circ\theta_{X'Y} = \theta_{XY}\circ P(\alpha)^*$ and thus the top square commutes. Let ρ be in $\mathbf{GLim}(P(X),Y)$. Then:

$$\begin{aligned}
(I(\beta)_*\circ\theta_{XY})(\rho) =& I(\beta)_*(\theta_{XY}(\rho)) = I(\beta)_*(I(\rho)\circ\phi^X) = I(\beta)\circ I(\rho)\circ\phi^X \\
=& I(\beta\circ\rho)\circ\phi^X = \theta_{XY'}(\beta\circ\rho) = \theta_{XY'}(\beta_*(\rho)) = (\theta_{XY'}\circ\beta_*)(\rho).
\end{aligned}$$

So $I(\beta)_*\circ\theta_{XY} = \theta_{XY'}\circ\beta_*$ and thus the bottom square commutes. So (P,I) is an adjoint pair. $\square$

Define a functor Δ from **Top** to $\mathbf{Top}^{\mathbf{I}}$ so that for each topological space U, $(\Delta(U))_n = (\Delta(U))_{(n,n+1)} = U$ and $(\Delta(U))_{n*} = (\Delta(U))_{(n+1)^*} = 1_U$ for each $n \in \omega$. It is easy to verify that this is a functor, known as the *diagonal functor*. Then the following is well known ([9], p. 88 table):

Lemma. *Any limit functor from* $\mathbf{Top}^{\mathbf{I}}$ *to* **Top** *as described in Theorem 3.1 is right adjoint to* Δ.

We may then define a functor L' from $\mathbf{Top}^{\mathbf{I}}$ to $\mathbf{Top}$ so that $L'(X) = L(X)$ for every object X of **GLim**, and $L(Y)$ is the canonical limit described in the proof of Theorem 3.1 part 1. Note then that $L = L' \circ I$.

Theorem 3.6. $(P \circ \Delta, L)$ *is an adjoint pair.*

Proof. It is easy to verify that since (Δ, L') is an adjoint pair and (P, I) is an adjoint pair, then $(P \circ \Delta, L' \circ I) = (P \circ \Delta, L)$ is an adjoint pair. Indeed this is true in general for two adjoint pairs that share a category in common. $\square$

Note what $P \circ \Delta$ does to a topological space U. The object $\Delta(U)$ represents the inverse sequence below:

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[dr, "1_U"] & & U \arrow[dl, "1_U"'] \arrow[dr, "1_U"] & & \cdots \arrow[dl, "1_U"'] \\
\cdots & U & & U & \cdots
\end{tikzcd}$$

This is of course *not* a generalized inverse sequence, so

$$(P(\Delta(U)))_{(n,n+1)}(u) = (1_U(u), 1_U(u)) = (u, u) \text{ for all } n \in \omega, u \in U.$$

So $P(\Delta(U))$ is the generalized inverse sequence where each factor space is U and the graph of each bonding map is the diagonal graph, which corresponds to the identity bonding map. This inverse sequence is known to have inverse limit homeomorphic to U. So $P \circ \Delta$ can be thought of as the functor which sends each topological space U to this generalized inverse sequence and each continuous function $\alpha : U \to V$ to $(P \circ \Delta)(\alpha)$ where for each $n \in \omega$, $(P \circ \Delta)(\alpha)_n = \alpha$ and $(P \circ \Delta)(\alpha)_{(n,n+1)}$ is defined by $(P \circ \Delta)(\alpha)_{(n,n+1)}(u) = (\alpha(u), \alpha(u))$ for all $u \in U$.

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