Derivations of subalgebras of the Lie algebra of block upper triangular matrices

by

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Abstract

This dissertation studies the derivations of some subalgebras of the Lie algebra of block upper triangular matrices. Specifically, we study the derivations of the Lie algebra of strictly block upper triangular matrices and the Lie algebra of so-called dominant upper triangular (DUT) ladder matrices, which are block upper triangular matrices that take zero on some preset nonconsecutive diagonal blocks.

The dissertation consists of six chapters. Chapter 1 provides a brief introduction, background information, and some related literatures to the topics to be studied.

In Chapter 2, we introduce the definitions and basic properties of matrices, ladder matrices, and Lie algebras. We also describe some linear transformations between matrix spaces that satisfy certain special properties. These linear transformations will appear in the derivations of Lie algebra to be studied.

Chapter 3 provides an explicit description of the derivations of the Lie algebra $\mathcal{N}$ of strictly block upper triangular matrices over a field $\mathbb{F}$.

In Chapter 4, we completely characterize the derivations of the Lie algebra $M_{\mathcal{L}}$ of dominant upper triangular (DUT) ladder matrices over a filed $\mathbb{F}$ with $\text{char}(\mathbb{F}) \neq 2$. In exploring the results, we obtain some properties of these Lie algebras and their derivations.

Chapter 5 discusses the derivations of the Lie algebra $[M_{\mathcal{L}}, M_{\mathcal{L}}]$ for the so-called strongly dominant upper triangular (SDUT) ladder $\mathcal{L}$ over a field $\mathbb{F}$ with $\text{char}(\mathbb{F}) \neq 2, 3$.

The final chapter provides some potential future research directions on those Lie algebras that we study in this dissertation.

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Chapter 1

Introduction

Let $(\mathfrak{g}, [\ ,\ ])$ be a Lie algebra over a field $\mathbb{F}$ or a ring R. A **derivation** of $\mathfrak{g}$ is an $\mathbb{F}$-linear map (resp. an R -linear map) $f : \mathfrak{g} \to \mathfrak{g}$ that satisfies

$$f([X,Y]) = [f(X),Y] + [X,f(Y)] \qquad \text{for all } X,Y \in \mathfrak{g}.$$

Let $\mathrm{Der}(\mathfrak{g})$ denote the set of all derivations of the Lie algebra $\mathfrak{g}$, which itself forms a Lie algebra with Lie bracket defined by $[f,g] = f \circ g - g \circ f$ for all $f,g \in \mathrm{Der}(\mathfrak{g})$ [17, p.15]. The Lie algebra $\mathrm{Der}(\mathfrak{g})$ is called the **derivation algebra** of $\mathfrak{g}$. The dissertation is a study of the derivations of some subalgebras of the Lie algebra of block upper triangular matrices. The subalgebras we study include the Lie algebra of strictly block upper triangular matrices and the Lie algebra of so-called dominant upper triangular (DUT) ladder matrices, which are block upper triangular matrices that take zero on some preset nonconsecutive diagonal blocks.

The derivations of Lie algebras play an important role in disclosing the structure of Lie algebras. On the other hand, by Ado-Iwasawa theorem, every finite dimensional Lie algebra over a field can be realized as a matrix Lie algebra [1, 12]. In recent years, significant progress has been made in studying the derivations and generalized derivations of Lie algebras (esp. matrix Lie algebras) over a field or a ring. Here is some of the progress: Chen determined the structure of certain generalized derivations of a parabolic subalgebra of the general linear Lie algebra $\mathfrak{gl}(n,\mathbb{F})$ over a field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) \neq 2$ and $|\mathbb{F}| > n \geq 3$ [6]. Brice described the derivation algebras of the parabolic subalgebras of a reductive Lie algebra over an algebraically closed field of characteristic zero and over the real field, and proved the zero-product determined property of such derivation algebras [4]. Let R be a commutative ring with identity. Cheung characterized proper Lie derivations and gave sufficient conditions for any Lie derivation of a triangular algebra over R to be proper [7]. Du and Wang investigated the Lie derivations of 2×2 block generalized matrix algebras over R [9]. Wang, Ou, and Yu described the derivations of intermediate Lie algebras between the Lie algebra of diagonal matrices and that of upper triangular matrices in $\mathfrak{gl}(n, R)$ [28]. Wang and Yu characterized all the derivations of parabolic subalgebras of $\mathfrak{gl}(n, R)$ [27]. Ou, Wang, and Yao described the derivations of the Lie algebra of strictly upper triangular matrices in $\mathfrak{gl}(n, R)$ [23]. Ji, Yang, and Chen studied the biderivations of the algebra of strictly upper triangular matrices in $\mathfrak{gl}(n, R)$ [15]. More recently, Benkovič described the Lie derivations and Lie triple derivations of upper triangular matrix algebras over a unital algebra [3]. There are some other results on the Lie triple derivations of certain matrix Lie algebras, such as the algebra $\mathfrak{gl}(n, R)$ [18], the algebra of upper triangular matrices of $\mathfrak{gl}(n, R)$ [2], the parabolic subalgebras of $\mathfrak{gl}(n, R)$ [20], and the algebra of strictly upper triangular matrices of $\mathfrak{gl}(n, R)$ [29].

Let $\mathbb{F}$ be a field, let $M_{m,n}$ be the set of all $m \times n$ matrices over $\mathbb{F}$, and put $M_n := M_{n,n}$. Let $\mathcal{N}$ denote the set of all strictly block upper triangular matrices in M_n relative to a given partition. Then $\mathcal{N}$ can be viewed as a Lie subalgebra of M_n with the standard Lie bracket $[X, Y] = XY - YX$. In the first part of the dissertation, we study the derivations of the Lie algebra $\mathcal{N}$ over $\mathbb{F}$ (Theorems 3.2 and 3.14). The motivation for this work comes from Ou, Wang and Yao's work on the derivations of the Lie algebra of strictly upper triangular matrices in $\mathfrak{gl}(n, R)$, where R is a commutative ring with identity [23]. The results that we obtain on the derivations of $\mathcal{N}$ could be viewed as extensions of Ou, Wang and Yao's results in the special case $R = \mathbb{F}$.

The derivation algebras $\mathrm{Der}(\mathcal{N})$ for all possible $\mathcal{N}$ constitute an important family of the derivation algebras of nilpotent Lie algebras. First of all, each $\mathcal{N}$ is a direct sum of root spaces in the root space decomposition of $\mathfrak{sl}(n, \mathbb{F})$ or $\mathfrak{gl}(n, \mathbb{F})$, so that $\mathcal{N}$ and $\mathrm{Der}(\mathcal{N})$

have elegant graded structures relative to the roots. Secondly, a derivation $f \in \mathrm{Der}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ always maps the center $\mathrm{Z}(\mathfrak{g})$ into itself. There is an induced Lie algebra homomorphism $\phi : \mathrm{Der}(\mathfrak{g}) \to \mathrm{Der}(\mathfrak{g}/\mathrm{Z}(\mathfrak{g}))$, $f \mapsto \bar{f}$, defined by

$$\bar{f}(a + \mathrm{Z}(\mathfrak{g})) := f(a) + \mathrm{Z}(\mathfrak{g}) \quad \text{for } f \in \mathrm{Der}(\mathfrak{g}),\ a \in \mathfrak{g}.$$

The $\mathrm{Ker}\,\phi$ consists of all $f \in \mathrm{End}(\mathfrak{g})$ such that $\mathrm{Ker}\, f \supseteq [\mathfrak{g}, \mathfrak{g}]$ and $\mathrm{Im}\, f \subseteq \mathrm{Z}(\mathfrak{g})$. Moreover, $\mathrm{Der}(\mathfrak{g})/\mathrm{Ker}\,\phi$ is isomorphic to a subalgebra of $\mathrm{Der}(\mathfrak{g}/\mathrm{Z}(\mathfrak{g}))$. If $\mathfrak{g}$ is nilpotent, then $\mathfrak{g}/\mathrm{Z}(\mathfrak{g}) \simeq \mathrm{ad}\,\mathfrak{g}$ is isomorphic to a subalgebra of the specific Lie algebra $\mathcal{N}$ of strictly upper triangular matrices in $\mathrm{End}(\mathfrak{g})$ [10, Engel's Theorem]. Therefore, knowledge of $\mathrm{Der}(\mathcal{N})$ would be helpful to explore the derivations of an arbitrary nilpotent Lie algebra.

The notion of ladder matrix, introduced by Brice and Huang in [5], generalizes the notion of block upper triangular matrix. A ladder matrix is, roughly speaking, a matrix that has zero entries outside of a ladder shape region determined by a set $\mathcal{L}$ of matrix indices, called a "ladder." When a given ladder $\mathcal{L}$ is "upper triangular," the set $M_{\mathcal{L}}$ of all ladder matrices corresponding to $\mathcal{L}$ is a Lie algebra with respect to the standard Lie bracket.

Classical examples of ladder matrix Lie algebras include the Lie algebras of block upper triangular matrices and of strictly block upper triangular matrices, $M_{p,q}$ embedded in the upper right corner of M_n (when $p \le n$ and $q \le n$), and M_n itself. Nevertheless, not much work has been done on ladder matrix Lie algebras in general. A non-classical example of a ladder matrix Lie algebra arises from the nilpotent Lie algebra $\mathfrak{g}$ constructed by Dixmier and Lister [8], to disprove the converse of a statement of Jacobson [13]. The corresponding derivation algebra $\mathrm{Der}(\mathfrak{g})$ is well-embedded in a special nilpotent ladder matrix Lie algebra. Let T_n denote the space of upper triangular matrices in M_n. In [14], it is shown without a formal introduction that any subset $M_{\mathcal{L}}$ of T_n consisting of ladder matrices is a subspace invariant under the triangular matrix similarity.

In the second part of the dissertation, we define two special types of ladders, namely "dominant upper triangular" (DUT) and "strongly dominant upper triangular" (SDUT). We give an explicit description of the derivations of the Lie algebra $M_{\mathcal{L}}$ of DUT ladder matrices over a field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) \neq 2$ (Theorem 4.4), and the derivations of the Lie algebra $[M_{\mathcal{L}}, M_{\mathcal{L}}]$ associated with an SDUT ladder $\mathcal{L}$ over a field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) \neq 2, 3$ (Theorem 5.8).

In general, a derivation of a Lie algebra stabilizes each subalgebra appearing in the derived series of the Lie algebra. Moreover, the derived series of a non-solvable Lie algebra of upper triangular ladder matrices terminates at the Lie algebra $[M_{\mathcal{L}}, M_{\mathcal{L}}]$ for a certain SDUT ladder $\mathcal{L}$. Therefore, knowledge of the derivations of the Lie algebras $[M_{\mathcal{L}}, M_{\mathcal{L}}]$ associated with SDUT ladders $\mathcal{L}$ are useful in finding the derivations of non-solvable Lie algebras of upper triangular ladder matrices.

The organization of this dissertation is as follows. In Chapter 2, we introduce ladder matrices and basic Lie theory, and determine some linear maps between matrix spaces that satisfy certain special properties. In Chapter 3, we completely characterize the derivations of the Lie algebra $\mathcal{N}$ of strictly block upper triangular matrices in M_n over a field $\mathbb{F}$. In Chapter 4, we characterize the derivations of the Lie algebra $M_{\mathcal{L}}$ of DUT ladder matrices in M_n over a field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) \neq 2$, and give some examples and applications. In Chapter 5, we discuss the derivations of the Lie algebra $[M_{\mathcal{L}}, M_{\mathcal{L}}]$ associated with an SDUT ladder $\mathcal{L}$ over a field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) \neq 2, 3$. Finally, in Chapter 6, we give some future research directions.

Chapter 2

Preliminaries

In this chapter, we introduce background information and notation needed throughout the dissertation. We also describe some linear transformations between matrix spaces that satisfy some special properties.

2.1 Matrices

Let m and n be positive integers and let $\mathbb{F}$ be a field. The set of all $m \times n$ matrices over $\mathbb{F}$ is denoted by $M_{m,n}(\mathbb{F})$, and $M_{n,n}(\mathbb{F})$ is abbreviated to $M_n(\mathbb{F})$. If $\mathbb{F}$ is known, $M_n(\mathbb{F})$ is further abbreviated to M_n and $M_{m,n}(\mathbb{F})$ to $M_{m,n}$. Matrices are usually denoted by capital letters. A **submatrix** of a matrix $A \in M_{m,n}$ is a matrix that can be obtained by deleting some rows and columns of A.

Let $[n] := \{1, 2, \cdots, n\}$. An **ordered partition** of $[n]$ is a sequence $(n_1, n_2, \cdots, n_t)$ such that $t, n_1, \cdots, n_t \in \mathbb{Z}^+$ and $\sum_{i=1}^{t} n_i = n$. If $(n_1, n_2, \cdots, n_t)$ is an ordered partition of $[n]$, then a matrix in M_n can be partitioned into a $t \times t$ **block matrix** (or **partitioned matrix**) such that the (i, j) **block** has the size $n_i \times n_j$. Explicitly, a matrix $A \in M_n$ can be viewed in the following **block matrix form** according to the given ordered partition $(n_1, n_2, \cdots, n_t)$ of $[n]$:

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1t} \\ A_{21} & A_{22} & \cdots & A_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ A_{t1} & A_{t2} & \cdots & A_{tt} \end{bmatrix}, \qquad A_{ij} \in M_{n_i,n_j}, \quad i, j \in [t].$$

A matrix $A \in M_n$ of the form

$$A = \begin{bmatrix} A_{11} & * & \cdots & * \\ 0 & A_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & A_{tt} \end{bmatrix}$$

in which $A_{ii} \in M_{n_i}$, $i \in [t]$, with $\sum_{i=1}^{t} n_i = n$ and $*$ denotes any entry, is called **block upper triangular**. Similarly, we can define **block lower triangular**, **strictly block upper triangular**, and **strictly block lower triangular**.

2.2 Ladders and Ladder matrices

Fix a field $\mathbb{F}$. Let M_n denote the set of all $n \times n$ matrices over $\mathbb{F}$. Let $E_{ij} \in M_n$ denote the matrix with the only non-zero entry 1 in the (i, j) position. Recall that $[n] := \{1, 2, \cdots, n\}$.

Definition 2.1. *A subset $\{(i_1, j_1), \cdots, (i_s, j_s)\}$ of the set $[n] \times [n]$ is called a* **ladder of size** n *if*

$$i_1 < i_2 < \cdots < i_s \qquad \text{and} \qquad j_1 < j_2 < \cdots < j_s.$$

Given a ladder $\mathcal{L} = \{(i_1, j_1), \cdots, (i_s, j_s)\}$ of size n, a matrix $A = (a_{ij}) \in M_n$ is called an **$\mathcal{L}$-ladder matrix** *if*

$$a_{ij} \neq 0 \;\Rightarrow\; \text{there exists } \ell \in [s] \text{ such that } i \leq i_\ell \text{ and } j_\ell \leq j.$$

The set of all $\mathcal{L}$-ladder matrices is denoted by $M_{\mathcal{L}}$.

Explicitly, $M_{\mathcal{L}}$ consists of matrices in M_n that have nonzero entries only in the upper right direction of some (i_ℓ, j_ℓ) in $\mathcal{L}$.

Definition 2.2. *A ladder $\mathcal{L} = \{(i_1, j_1), \cdots, (i_s, j_s)\}$ of size n is called*

- **upper triangular:** *if* $i_\ell < j_{\ell+1}$ *for* $\ell \in [s-1]$*, equivalently all "inner corner entry positions"* $(i_\ell, j_{\ell+1})$ $(\ell \in [s-1])$ *of matrices in* $M_{\mathcal{L}}$ *are located on the strictly upper triangular part;*

- **dominant upper triangular (DUT):** *if* $j_\ell \le i_\ell < j_{\ell+1}$ *for* $\ell \in [s-1]$*, equivalently* $\mathcal{L}$ *is upper triangular and all "outer corner entry positions"* (i_ℓ, j_ℓ) $(\ell \in [s])$ *of matrices in* $M_{\mathcal{L}}$ *are located on the lower triangular part;*

- **strongly dominant upper triangular (SDUT):** *if* $j_\ell < i_\ell < j_{\ell+1}$ *for* $\ell \in [s-1]$*, equivalently* $\mathcal{L}$ *is upper triangular and all "outer corner entry positions"* (i_ℓ, j_ℓ) $(\ell \in [s])$ *of matrices in* $M_{\mathcal{L}}$ *are located on the strictly lower triangular part.*

When $\mathcal{L}$ *is upper triangular, a matrix in* $M_{\mathcal{L}}$ *is called an* **upper triangular ladder matrix***. Similarly for the others.*

In [5], Brice and Huang proved that if $\mathcal{L}$ is an upper triangular ladder of size n, then $M_{\mathcal{L}}$ with matrix product is a subalgebra of M_n. Naturally, $M_{\mathcal{L}}$ with respect to the standard Lie bracket $[X, Y] = XY - YX$ is a Lie subalgebra of M_n (aka. $\mathfrak{gl}(n, \mathbb{F})$), in which we call $M_{\mathcal{L}}$ a Lie algebra of upper triangular ladder matrices.

Given a ladder $\mathcal{L} = \{(i_1, j_1), \cdots, (i_s, j_s)\}$ of size n, the matrices in $M_{\mathcal{L}}$ may be viewed as block matrices by dividing the rows and columns after those indexed by the set

$$\big(\{i_1, i_2, \cdots, i_s\} \cup \{j_1 - 1, j_2 - 1, \cdots, j_s - 1\}\big) \setminus \{0, n\}. \tag{2.1}$$

Suppose matrices in M_n are partitioned into $t \times t$ block matrix form according to (2.1). We define the **block index set** of matrices in $M_{\mathcal{L}}$ as

$$\Omega(\mathcal{L}) := \{(i, j) \in [t] \times [t] : \text{ the } (i, j) \text{ block of a matrix of } M_{\mathcal{L}} \text{ is nonzero}\}.$$

We call an element of $\Omega(\mathcal{L})$ a **block index**. The block index set $\Omega(\mathcal{L})$ collects the positions of possibly nonzero blocks of matrices in $M_{\mathcal{L}}$. Denote by $M_{\mathcal{B}}$ the set of all block upper triangular matrices in M_n with the same block matrix form as $M_{\mathcal{L}}$. The set $M_{\mathcal{B}}$ is itself a set of ladder matrices for a special DUT ladder $\mathcal{B}$. Obviously, $\Omega(\mathcal{B}) = \{(i,j) : 1 \le i \le j \le t\}$.

Example 2.3. *Consider the ladder $\mathcal{L} = \{(1,1),(4,3),(5,5)\}$ of size 7. By definition 2.2, $\mathcal{L}$ is a DUT ladder. The matrix form in $M_{\mathcal{L}}$ is given in Figure 2.1(a). Figure 2.1(b) indicates the block matrix form in $M_{\mathcal{L}}$ obtained by dividing the rows and columns after those indexed by (2.1):*

$$\big(\{1,4,5\} \cup \{1-1, 3-1, 5-1\}\big) \setminus \{0,7\} = \{1,2,4,5\}.$$

The matrices in $M_{\mathcal{L}}$ are clearly conformal to this block matrix form, in the sense that if a matrix in $M_{\mathcal{L}}$ has a nonzero entry in the (i,j) block then every matrix whose nonzero entries are located in the (i,j) block is in $M_{\mathcal{L}}$. The block index set $\Omega(\mathcal{L})$ of matrices of $M_{\mathcal{L}}$ is

$$\Omega(\mathcal{L}) \;=\; \{(i,j) : 1 \le i \le j \le 5\} \setminus \{(2,2),(5,5)\}.$$

$$\begin{array}{ccccccc}
* & * & * & * & * & * & * \\
0 & 0 & * & * & * & * & * \\
0 & 0 & * & * & * & * & * \\
0 & 0 & * & * & * & * & * \\
0 & 0 & 0 & 0 & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}$$

(a) matrix form of $M_{\mathcal{L}}$

$$\left[\begin{array}{c|c|cc|c|cc}
* & * & * & * & * & * & * \\ \hline
0 & 0 & * & * & * & * & * \\ \hline
0 & 0 & * & * & * & * & * \\
0 & 0 & * & * & * & * & * \\ \hline
0 & 0 & 0 & 0 & * & * & * \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]$$

(b) block matrix form of $M_{\mathcal{L}}$

Figure 2.1: Ladder $\mathcal{L} = \{(1,1),(4,3),(5,5)\}$ of size 7

Finally, $M_{\mathcal{B}}$ is the set of block upper triangular matrices according to the partition in Figure 2.1(b), and $\Omega(\mathcal{B}) = \{(i,j) : 1 \le i \le j \le 5\}$.

The different kinds of ladder $\mathcal{L}$ could be easily distinguished by the block matrix form of $M_{\mathcal{L}}$ and the corresponding $M_{\mathcal{B}}$:

- $\mathcal{L}$ is upper triangular if and only if $M_{\mathcal{L}} \subseteq M_{\mathcal{B}}$;

- $\mathcal{L}$ is DUT if and only if for every $(i,j) \in \Omega(\mathcal{L})$ there exists an integer k such that $(k,k) \in \Omega(\mathcal{L})$ and $i \leq k \leq j$;
- $\mathcal{L}$ is SDUT if and only if $\mathcal{L}$ is DUT, and every nonzero diagonal block of a matrix in $M_{\mathcal{L}}$ has size greater than 1.

The next theorem completely characterizes DUT ladders in terms of the block matrix form of matrices in $M_{\mathcal{L}}$ obtained by (2.1).

Theorem 2.4. *1. Let $\mathcal{L}$ be a ladder of size n, and the matrices in $M_{\mathcal{L}}$ have a $t \times t$ block matrix form obtained by* (2.1)*. Then $\mathcal{L}$ is DUT if and only if*

$$\Omega(\mathcal{L}) = \{(i,j) : 1 \leq i \leq j \leq t\} \setminus \{(i,i) \mid i \in S\}$$

for a certain subset $S \subseteq [t]$ that consists of some non-consecutive integers. In particular, if $\mathcal{L}$ is DUT, then $(i,j) \in \Omega(\mathcal{L})$ for every $i,j \in [t]$ with $i < j$.

2. Equivalently, a ladder $\mathcal{L}$ of size n is DUT if and only if $M_{\mathcal{L}}$ consists of block upper triangular matrices of M_n that have zero submatrices on some preset non-consecutive diagonal blocks, according to a given ordered partition of $[n]$.

Proof. It suffices to prove the first statement. Let $\mathcal{L} = \{(i_1,j_1), \cdots, (i_s,j_s)\}$ be a ladder of size n. Let $\{k_1 \cdots, k_{t-1}\}$ be the corresponding set given by (2.1), which determines an ordered partition $(k_1, k_2 - k_1, \cdots, k_{t-1} - k_{t-2}, n - k_{t-1})$ of $[n]$, and the corresponding $t \times t$ block matrix form of matrices in M_n. The set $M_{\mathcal{B}}$ of block upper triangular matrices is a set of ladder matrices for the ladder

$$\mathcal{B} = \{(k_1, 1), (k_2, k_1 + 1), \cdots, (k_{t-1}, k_{t-2} + 1), (n, k_{t-1} + 1)\}.$$

Clearly, $\Omega(\mathcal{B}) = \{(i,j) \mid 1 \leq i \leq j \leq t\}$.

Suppose $\Omega(\mathcal{L}) = \Omega(\mathcal{B}) \setminus \{(i,i) \mid i \in S\}$ where S is a subset of $[t]$ that consists of some non-consecutive integers. Then $M_{\mathcal{L}} \subseteq M_{\mathcal{B}}$. On the other hand, each pair $(i_\ell, j_\ell) \in \mathcal{L}$

is the position of the most lower-left entry of the (p, q) block of matrices in $M_{\mathcal{L}}$ for some $(p, q) \in \Omega(\mathcal{L})$. If $p < q$, then any of $(p, p), (p+1, p+1), \cdots, (q, q)$ is not in $\Omega(\mathcal{L})$, which contradicts the assumption of $\Omega(\mathcal{L})$. Therefore, $p = q$. Then $\mathcal{L} \subseteq \mathcal{B}$. Since $\mathcal{B}$ is DUT, it is clear from the definition that $\mathcal{L}$ is also DUT.

Now assume that $\mathcal{L}$ is DUT (one may refer to Example 2.3 for the following argument). Then $\Omega(\mathcal{L}) \subseteq \{(i, j) : 1 \leq i \leq j \leq t\}$, and for every $(i, j) \in \Omega(\mathcal{L})$ there exists an integer p such that $(p, p) \in \Omega(\mathcal{L})$ and $i \leq p \leq j$. Hence every $(i_\ell, j_\ell) \in \mathcal{L}$ is the position of the most lower-left entry of a diagonal (p, p) block of matrices for some $(p, p) \in \Omega(\mathcal{L})$. So $(i_\ell, j_\ell) = (k_p, k_{p-1} + 1)$ (set $k_0 := 0$ and $k_t := n$). If there exists $i \in [t-1]$ such that neither (i, i) nor $(i+1, i+1)$ is in $\Omega(\mathcal{L})$, then $(i, i+1)$ is not in $\Omega(\mathcal{L})$. Then k_i cannot be the row position (resp. $k_i + 1$ cannot be the column position) of the most lower-left entry of a diagonal (p, p) block for any $(p, p) \in \Omega(\mathcal{L})$, which means that k_i is not in the set (2.1). This is a contradiction. Similarly, if there exists $(i, j) \in [t] \times [t]$ such that $i < j$ and $(i, j) \notin \Omega(\mathcal{L})$, then none of $(i, i), (i+1, i+1), \cdots, (j, j)$ is in $\Omega(\mathcal{L})$, which leads to the same contradiction. Therefore, we must have $\Omega(\mathcal{L}) = \{(i, j) : 1 \leq i \leq j \leq t\} \setminus \{(i, i) \mid i \in S\}$ where S is a subset of $[t]$ that consists of some non-consecutive integers. □

A direct consequence of Theorem 2.4 is the counting of sets of DUT ladder matrices.

Corollary 2.5. *Let* $(F_t)_{t=1}^{\infty} = (1, 1, 2, 3, 5, \cdots)$ *be the Fibonacci sequence.*

1. *Given a* $t \times t$ *block matrix form in* M_n*, the number of* $M_{\mathcal{L}}$ *corresponding to a DUT ladder* $\mathcal{L}$ *associated with this block form equals* F_{t+2}*.*

2. *Given* $n \in \mathbb{Z}^+$*, the number of* $M_{\mathcal{L}}$ *such that* $\mathcal{L}$ *is a DUT ladder and* $M_{\mathcal{L}} \subseteq M_n$ *equals* F_{2n+1}*.*

Proof. 1. Let b_t denote the number of $M_{\mathcal{L}}$ corresponding to a DUT ladder $\mathcal{L}$ associated with the given $t \times t$ block matrix form in M_n. Clearly $b_1 = 2 = F_3$ and $b_2 = 3 = F_4$. It suffices to prove that the sequence (b_t) satisfies the same recursive formula as (F_{t+2})

does, that is,

$$b_t = b_{t-1} + b_{t-2}. \tag{2.2}$$

By Theorem 2.4, b_t equals the number of ways to choose non-consecutive diagonal blocks in a given $t \times t$ block form. If the first diagonal block is chosen, then the second one should be skipped, and there are b_{t-2} ways to choose the remaining diagonal blocks; if the first diagonal block is not chosen, then there are b_{t-1} ways to choose the remaining diagonal blocks. Therefore, (2.2) is true.

2. Given $t \in [n]$, there are $\binom{n-1}{t-1}$ ways to partition matrices of M_n into a $t \times t$ block form; each block form corresponds to F_{t+2} sets $M_{\mathcal{L}}$ of DUT ladder matrices. Put $r_1 := \frac{1+\sqrt{5}}{2}$ and $r_2 := \frac{1-\sqrt{5}}{2}$ and note that r_1 and r_2 are the roots of $x^2 - x - 1 = 0$. The Binet's Fibonacci number formula says that

$$F_t = \frac{1}{\sqrt{5}} r_1^t - \frac{1}{\sqrt{5}} r_2^t.$$

Therefore, the number of $M_{\mathcal{L}}$ such that $\mathcal{L}$ is a DUT and $M_{\mathcal{L}} \subseteq M_n$

$$\begin{aligned} &= \sum_{t=1}^{n} \binom{n-1}{t-1} F_{t+2} = \sum_{t=1}^{n} \binom{n-1}{t-1} \left(\frac{1}{\sqrt{5}} r_1^{t+2} - \frac{1}{\sqrt{5}} r_2^{t+2} \right) \\ &= \frac{1}{\sqrt{5}} \left[r_1^3 (1+r_1)^{n-1} - r_2^3 (1+r_2)^{n-1} \right] = \frac{1}{\sqrt{5}} \left[r_1^3 (r_1^2)^{n-1} - r_2^3 (r_2^2)^{n-1} \right] = F_{2n+1}. \square \end{aligned}$$

2.3 Lie algebra

A vector space $\mathfrak{g}$ over a field $\mathbb{F}$, with a product $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, denoted by $(X, Y) \mapsto [X, Y]$ and called the **Lie bracket** of X and Y, is called a **Lie algebra** over $\mathbb{F}$ [10, p.1] if the following axioms are satisfied:

1. The Lie bracket is bilinear.

2. $[X, X] = 0$ for all $X \in \mathfrak{g}$.

3. The Jacobi identity $[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0$ holds for all $X,Y,Z \in \mathfrak{g}$.

A classical example of a Lie algebra is the general linear algebra $\mathfrak{gl}(V)$ consisting of all linear operators on a vector space V with the Lie bracket defined by

$$[X,Y] = XY - YX \quad \text{for all} \quad X,Y \in \mathfrak{gl}(V).$$

The set M_n of $n \times n$ matrices over $\mathbb{F}$ can be viewed as a Lie algebra, denoted by $\mathfrak{gl}(n,\mathbb{F})$, with the Lie bracket defined by

$$[X,Y] = XY - YX \quad \text{for all} \quad X,Y \in M_n.$$

Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras over a field $\mathbb{F}$. A linear transformation $\varphi : \mathfrak{g} \to \mathfrak{h}$ is called a **Lie algebra homomorphism** if

$$\varphi[X,Y] \quad = \quad [\varphi(X),\varphi(Y)] \quad \text{for all} \quad X,Y \in \mathfrak{g}.$$

By the bilinearity of the Lie bracket and the Jacobi identity, the linear transformation $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ defined by

$$\mathrm{ad}\, X(Y) := [X,Y] \qquad \text{for all } X,Y \in \mathfrak{g}$$

is a Lie algebra homomorphism and therefore a representation of $\mathfrak{g}$, called the **adjoint representation** of $\mathfrak{g}$. The adjoint representation is important in the study of Lie algebras.

The Lie algebra $\mathfrak{g}$ is said to be **abelian** if $[X,Y]=0$ for all $X,Y \in \mathfrak{g}$. A subspace $\mathfrak{s}$ of $\mathfrak{g}$ is called a **subalgebra** or **Lie subalgebra** if $[X,Y] \in \mathfrak{s}$ for all $X,Y \in \mathfrak{s}$; it is called an **ideal** if $[X,Y] \in \mathfrak{s}$ for all $X \in \mathfrak{g}$ and $Y \in \mathfrak{s}$.

For later use we mention a couple of notions, analogous to those which arise in group theory. Let $\mathfrak{s}$ be a subalgebra of $\mathfrak{g}$. The **normalizer** $\mathrm{N}(\mathfrak{s})$ of $\mathfrak{s}$ is defined by

$$\mathrm{N}(\mathfrak{s}) = \{X \in \mathfrak{g} : [X, Y] \in \mathfrak{s} \quad \text{for all} \quad Y \in \mathfrak{s}\}.$$

The **centralizer** $\mathrm{Z}(\mathfrak{s})$ of $\mathfrak{s}$ is defined by

$$\mathrm{Z}(\mathfrak{s}) = \{X \in \mathfrak{g} : [X, Y] = 0 \quad \text{for all} \quad Y \in \mathfrak{s}\}.$$

Both $\mathrm{N}(\mathfrak{s})$ and $\mathrm{Z}(\mathfrak{s})$ are subalgebra of $\mathfrak{g}$.

It is natural to study a Lie algebra $\mathfrak{g}$ via its ideals. Define a sequence of ideals of $\mathfrak{g}$ (the **derived series**) by

$$\mathfrak{g}^{(0)} = \mathfrak{g},\ \ \mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}],\ \ \mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}], \cdots, \mathfrak{g}^{(k)} = [\mathfrak{g}^{(k-1)}, \mathfrak{g}^{(k-1)}].$$

We call $\mathfrak{g}$ **solvable** if $\mathfrak{g}^{(k)} = 0$ for some k [10, p.10]. For example, the Lie algebra of upper triangular matrices in M_n is solvable.

Define a sequence of ideals of a Lie algebra $\mathfrak{g}$ (the **lower central series**) by

$$\mathfrak{g}^{0} = \mathfrak{g},\ \ \mathfrak{g}^{1} = [\mathfrak{g}, \mathfrak{g}],\ \ \mathfrak{g}^{2} = [\mathfrak{g}, \mathfrak{g}^{1}], \qquad \mathfrak{g}^{k} = [\mathfrak{g}, \mathfrak{g}^{k-1}].$$

$\mathfrak{g}$ is called **nilpotent** if $\mathfrak{g}^{k} = 0$ for some k [10, p.11]. For example, the Lie algebra of strictly upper triangular matrices and the Lie algebra of strictly block upper triangular matrices in M_n are nilpotent. It is easy to check that $\mathfrak{g}^{(k)} \subseteq \mathfrak{g}^{k}$ for all k, and it follows that nilpotent Lie algebras are solvable.

2.4 Derivations of Lie algebras

Recall that a derivation of Lie algebra $\mathfrak{g}$ over a field $\mathbb{F}$ is an $\mathbb{F}$-linear map $f : \mathfrak{g} \to \mathfrak{g}$ that satisfies

$$f([X,Y]) = [f(X),Y] + [X, f(Y)] \quad \text{for all } X, Y \in \mathfrak{g}.$$

The collection $\mathrm{Der}(\mathfrak{g})$ of all derivations of $\mathfrak{g}$ is a Lie algebra with the Lie bracket $[f,g] = f \circ g - g \circ f$ for all $f, g \in \mathrm{Der}(\mathfrak{g})$. Certain derivations of $\mathfrak{g}$ arise quite naturally, as follows. If $X \in \mathfrak{g}$, $Y \mapsto [X,Y]$ is an endomorphism of $\mathfrak{g}$, which we denote $\mathrm{ad}\, X$. Then, $\mathrm{ad}\, X \in \mathrm{Der}(\mathfrak{g})$, because of the Jocabi identity. Derivations of this form are called **inner**, all others **outer**. The collection $\mathrm{ad}(\mathfrak{g})$ of all inner derivations of $\mathfrak{g}$ is an ideal of $\mathrm{Der}(\mathfrak{g})$ [10, p.8].

2.5 Some linear transformations between matrix spaces

The purpose of this section is to describe some linear transformations between matrix spaces that satisfy certain special properties. These results will be useful in the study of derivations of matrix Lie algebras. Recall that $[n] := \{1, 2, \cdots, n\}$. Let $E_{pq}^{(mn)} \in M_{m,n}$ denote the matrix with the only non-zero entry 1 in the (p,q) position.

Lemma 2.6. *If linear transformations $\phi : M_{m,p} \to M_{m,q}$ and $\varphi : M_{n,p} \to M_{n,q}$ satisfy that*

$$\phi(AB) = A\varphi(B) \qquad \textit{for all} \quad A \in M_{m,n}, \quad B \in M_{n,p},$$

then there is $X \in M_{p,q}$ such that $\phi(C) = CX$ for $C \in M_{m,p}$ and $\varphi(D) = DX$ for $D \in M_{n,p}$.

Proof. For any $j \in [n]$ and $B \in M_{n,p}$,

$$\phi(E_{1j}^{(mn)} B) = E_{1j}^{(mn)} \varphi(B).$$

All such $E_{1j}^{(mn)}B$ span the first row space of $M_{m,p}$. So ϕ sends the first row of $M_{m,p}$ to the first row of $M_{m,q}$. There exists a unique $X \in M_{p,q}$ such that

$$E_{1j}^{(mn)}\varphi(B) = \phi(E_{1j}^{(mn)}B) = E_{1j}^{(mn)}BX, \qquad \text{for all } \ j \in [n], \ B \in M_{n,p}.$$

Therefore, $\varphi(B) = BX$. Then $\phi(AB) = A\varphi(B) = ABX$ for any $A \in M_{m,n}$ and $B \in M_{n,p}$. All such AB span $M_{m,p}$. So $\phi(C) = CX$ for all $C \in M_{m,p}$. □

Lemma 2.7. *If linear transformations $\phi : M_{m,p} \to M_{n,p}$ and $\varphi : M_{m,q} \to M_{n,q}$ satisfy that*

$$\phi(BA) = \varphi(B)A \qquad \textit{for all} \quad A \in M_{q,p}, \ B \in M_{m,q},$$

then there is $X \in M_{n,m}$ such that $\phi(C) = XC$ for $C \in M_{m,p}$ and $\varphi(D) = XD$ for $D \in M_{m,q}$.

Proof. The proof (omitted) is similar to that of Lemma 2.6. □

Lemma 2.8. *Suppose $\mathbb{F}$ is an arbitrary field. If $X \in M_m$ and $Y \in M_n$ satisfy that $XA = AY$ for all $A \in M_{m,n}$, then $X = \lambda I_m$ and $Y = \lambda I_n$ for certain $\lambda \in \mathbb{F}$.*

Proof. For any $(i, j) \in [m] \times [n]$,

$$XE_{ij}^{(mn)} = E_{ij}^{(mn)}Y.$$

Comparing the $(i, j)^{th}$ entry, we get $x_{ii} = y_{jj}$. Comparing the $(p, j)^{th}$ entry for $p \neq i$, we get $x_{pi} = 0$. Comparing the $(i, q)^{th}$ entry for $q \neq j$, we get $0 = y_{jq}$. Therefore, $X = \lambda I_m$ and $Y = \lambda I_n$ for some $\lambda \in \mathbb{F}$. □

Lemma 2.9. *If linear transformations $\phi : M_{m,p} \to M_{m,q}$ and $\varphi : M_{q,n} \to M_{p,n}$ satisfy that*

$$\phi(A)B = A\varphi(B) \qquad \textit{for all} \quad A \in M_{m,p}, \quad B \in M_{q,n},$$

then there is $X \in M_{p,q}$ such that $\phi(C) = CX$ for $C \in M_{m,p}$ and $\varphi(D) = XD$ for $D \in M_{q,n}$.

Proof. For any $j \in [p]$ and any $E_{kl}^{(qn)} \in M_{q,n}$,

$$\phi(E_{1j}^{(mp)})E_{kl}^{(qn)} = E_{1j}^{(mp)}\varphi(E_{kl}^{(qn)}),$$

which shows that the only possibly nonzero row of $\phi(E_{1j}^{(mp)})$ is the first row. So ϕ maps the first row of $M_{m,p}$ to the first row of $M_{m,q}$. There exists a unique $X \in M_{p,q}$ such that

$$E_{1j}^{(mp)}\varphi(E_{kl}^{(qn)}) = \phi(E_{1j}^{(mp)})E_{kl}^{(qn)} = E_{1j}^{(mp)}XE_{kl}^{(qn)}, \quad \text{for all } j \in [p], \quad E_{kl}^{(qn)} \in M_{q,n}.$$

Therefore, $\varphi(E_{kl}^{(qn)}) = XE_{kl}^{(qn)}$ for all $E_{kl}^{(qn)} \in M_{q,n}$. So $\varphi(B) = XB$ for $B \in M_{q,n}$. Then $\phi(A)B = AXB$ for any $A \in M_{m,p}$ and $B \in M_{q,n}$. Hence $\phi(A) = AX$ for all $A \in M_{m,p}$. □

Lemma 2.10. *If linear transformations $f : M_{p,r} \to M_{p,r}$, $g : M_{p,q} \to M_{p,q}$, and $h : M_{q,r} \to M_{q,r}$ satisfy that*

$$f(AB) = g(A)B + Ah(B) \quad \textit{for all} \quad A \in M_{p,q}, \quad B \in M_{q,r}, \tag{2.3}$$

then there exist $X \in M_p, Y \in M_r, Z \in M_q$ such that

$$\begin{aligned} f(C) &= XC + CY \qquad \textit{for} \ \ C \in M_{p,r}, & (2.4)\\ g(A) &= XA + AZ \qquad \textit{for} \ \ A \in M_{p,q}, & (2.5)\\ h(B) &= BY - ZB \qquad \textit{for} \ \ B \in M_{q,r}. & (2.6) \end{aligned}$$

Proof. For any $n \in [p]$, $j, k \in [q]$, $m \in [r]$, $E_{nj}^{(pq)} \in M_{p,q}$ and $E_{km}^{(qr)} \in M_{q,r}$,

$$f(E_{nj}^{(pq)}E_{km}^{(qr)}) = g(E_{nj}^{(pq)})E_{km}^{(qr)} + E_{nj}^{(pq)}h(E_{km}^{(qr)}). \tag{2.7}$$

We further discuss (2.7) in two cases:

1. $j \neq k$: the left side of (2.7) is zero and

$$g(E_{nj}^{(pq)})E_{km}^{(qr)} = -E_{nj}^{(pq)}h(E_{km}^{(qr)}). \tag{2.8}$$

2. $j = k$: the left side of (2.7) is $f(E_{nm}^{(pr)})$, and according to (2.7), the only possibly nonzero entries of $f(E_{nm}^{(pr)})$ are

$$\begin{aligned} f(E_{nm}^{(pr)})_{im} &= g(E_{nk}^{(pq)})_{ik} \qquad \text{for all } \; i \in [p],\; i \neq n; && (2.9)\\ f(E_{nm}^{(pr)})_{n\ell} &= h(E_{km}^{(qr)})_{k\ell} \qquad \text{for all } \; \ell \in [r],\; \ell \neq m; && (2.10)\\ f(E_{nm}^{(pr)})_{nm} &= g(E_{nk}^{(pq)})_{nk} + h(E_{km}^{(qr)})_{km}. && (2.11) \end{aligned}$$

Next we define a linear transformation $f' : M_{p,r} \to M_{p,r}$ such that property (2.3) still holds. For $C \in M_{p,r}$, let

$$f'(C) := \left[\sum_{i,j \in [p]} f(E_{j1}^{(pr)})_{i1} E_{ij}^{(pp)}\right] C + C \left[\sum_{k,\ell \in [r]} f(E_{1k}^{(pr)})_{1\ell} E_{k\ell}^{(rr)}\right] - f(E_{11}^{(pr)})_{11} C. \tag{2.12}$$

Then for any $n \in [p]$, $m \in [r]$ and $E_{nm}^{(pr)} \in M_{p,r}$,

$$f'(E_{nm}^{(pr)}) = \sum_{i \in [p]} f(E_{n1}^{(pr)})_{i1} E_{im}^{(pr)} + \sum_{\ell \in [r]} f(E_{1m}^{(pr)})_{1\ell} E_{n\ell}^{(pr)} - f(E_{11}^{(pr)})_{11} E_{nm}^{(pr)},$$

which implies that the only possibly nonzero entries of $f'(E_{nm}^{(pr)})$ are

$$\begin{aligned} f'(E_{nm}^{(pr)})_{im} &= f(E_{n1}^{(pr)})_{i1} = f(E_{nm}^{(pr)})_{im} \qquad \text{for } i \in [p],\; i \neq n, && (2.13)\\ f'(E_{nm}^{(pr)})_{n\ell} &= f(E_{1m}^{(pr)})_{1\ell} = f(E_{nm}^{(pr)})_{n\ell} \qquad \text{for } \ell \in [r],\; \ell \neq m, && (2.14)\\ f'(E_{nm}^{(pr)})_{nm} &= f(E_{n1}^{(pr)})_{n1} + f(E_{1m}^{(pr)})_{1m} - f(E_{11}^{(pr)})_{11} = f(E_{nm}^{(pr)})_{nm}, && (2.15) \end{aligned}$$

where the last equality in (2.13), (2.14) and (2.15) is by (2.9), (2.10) and (2.11) respectively. Therefore, $f' = f$ on each $E_{nm}^{(pr)} \in M_{p,r}$ and thus on the whole $M_{p,r}$. Denote

$$X := \sum_{i,j\in[p]} f(E_{j1}^{(pr)})_{i1} E_{ij}^{(pp)} - f(E_{11}^{(pr)})_{11} I_p, \qquad Y := \sum_{k,\ell\in[r]} f(E_{1k}^{(pr)})_{1\ell} E_{k\ell}^{(rr)}. \tag{2.16}$$

We get $f(C) = f'(C) = XC + CY$ for $C \in M_{p,r}$. So (2.4) is done. Now for $A \in M_{p,q}$ and $B \in M_{q,r}$, by (2.3),

$$g(A)B + Ah(B) = f(AB) = XAB + ABY \Longrightarrow (g(A) - XA)B = A(BY - h(B)).$$

Applying Lemma 2.9 to $\phi : M_{p,q} \to M_{p,q}$ defined by $\phi(A) = g(A) - XA$ and $\varphi : M_{q,r} \to M_{q,r}$ defined by $\varphi(B) = BY - h(B)$, we will find $Z \in M_q$ such that

$$\begin{aligned} g(A) - XA &= \phi(A) = AZ \qquad \text{for } A \in M_{p,q}, \\ BY - h(B) &= \varphi(B) = ZB \qquad \text{for } B \in M_{q,r}, \end{aligned}$$

which imply (2.5) and (2.6). □

Chapter 3

Derivations of the Lie algebra of strictly block upper triangular matrices

In this chapter, we explicitly describe the derivations of the Lie algebra $\mathcal{N}$ of strictly block upper triangular matrices in M_n over a field $\mathbb{F}$. In the rest of this chapter, we fix a $t \times t$ block matrix form of matrices in M_n corresponding to a given ordered partition $(n_1, n_2, \cdots, n_t)$ of $[n]$. We first introduce some notations.

Definition 3.1. *1. Let $M_{\mathcal{B}}$ denote the set of all block upper triangular matrices in M_n.*

2. Let $\mathcal{M}_{ij}$ denote the set of all submatrices in the (i,j) block of matrices in M_n. The (i,j) block of a matrix $A \in M_n$ is denoted by A_{ij} or $(A)_{ij}$. If $A \in M_n$ is not given, A_{ij} may refer to an arbitrary matrix in $\mathcal{M}_{ij}$.

3. Let $\mathcal{M}^{ij}$ denote the set of matrices in M_n that take zero outside of the (i,j) block. For a matrix $B \in \mathcal{M}_{ij}$, let B^{ij} denote the embedding of B into $\mathcal{M}^{ij}$ by placing B on the (i,j) block. If $B \in \mathcal{M}_{ij}$ is not given, B^{ij} may refer to an arbitrary matrix in $\mathcal{M}^{ij}$.

A notation of double index, say $\mathcal{M}_{ij}$, may be written as $\mathcal{M}_{i,j}$ (resp. $\mathcal{M}^{ij}$ as $\mathcal{M}^{i,j}$) for clarity purpose.

3.1 Derivations of $\mathcal{N}$ for $\mathrm{char}(\mathbb{F}) \neq 2$

In this section, we give an explicit description of $\mathrm{Der}(\mathcal{N})$ for the Lie algebra $\mathcal{N}$ over $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) \neq 2$. The Lie algebra $M_{\mathcal{B}}$ is the normalizer of $\mathcal{N}$ in M_n by direct computation. For any $X \in M_{\mathcal{B}}$, the adjoint action

$$\mathrm{ad}\, X : \mathcal{N} \to \mathcal{N}, \qquad Y \mapsto [X, Y],$$

is a derivation of $\mathcal{N}$. Now we state our main theorem of this section.

Theorem 3.2. *Suppose* $\mathrm{char}(\mathbb{F}) \neq 2$. *Then every derivation f of the Lie algebra $\mathcal{N}$ can be (not uniquely) written as*

$$f \quad = \quad \mathrm{ad}\, X + \varphi_{1t} + \phi_{2t}^{12} + \phi_{1,t-1}^{t-1,t}, \tag{3.1}$$

where the summand components are described below:

1. *$X \in M_{\mathcal{B}}$ is a block upper triangular matrix in M_n.*

2. *$\varphi_{1t} \in \mathrm{End}(\mathcal{N})$ satisfies that $\mathrm{Ker}\,\varphi_{1t}$ contains $[\mathcal{N}, \mathcal{N}] = \bigoplus_{1<i+1<j\leq t} \mathcal{M}^{ij}$, and $\mathrm{Im}\,\varphi_{1t}$ is contained in the center $\mathrm{Z}(\mathcal{N}) = \mathcal{M}^{1t}$ of $\mathcal{N}$.*

3. *$\phi_{2t}^{12} \equiv 0$ unless the $(1,2)$ block of a matrix of $\mathcal{N}$ has only one row, in which $\phi_{2t}^{12} \in \mathrm{Der}(\mathcal{N})$ such that $\phi_{2t}^{12}(\mathcal{M}^{12}) \subseteq \mathcal{M}^{2t}$ and $\phi_{2t}^{12}(\mathcal{M}^{ij}) = 0$ for the other $\mathcal{M}^{ij} \subseteq \mathcal{N}$; the explicit form of ϕ_{2t}^{12} is given in Lemma 3.6;*

4. *$\phi_{1,t-1}^{t-1,t} \equiv 0$ unless the $(t-1,t)$ block of a matrix of $\mathcal{N}$ has only one column, in which $\phi_{1,t-1}^{t-1,t} \in \mathrm{Der}(\mathcal{N})$ such that $\phi_{1,t-1}^{t-1,t}(\mathcal{M}^{t-1,t}) \subseteq \mathcal{M}^{1,t-1}$ and $\phi_{1,t-1}^{t-1,t}(\mathcal{M}^{ij}) = 0$ for the other $\mathcal{M}^{ij} \subseteq \mathcal{N}$; the explicit form of $\phi_{1,t-1}^{t-1,t}$ is given in Lemma 3.7.*

The cases $t = 1$ and $t = 2$ are trivial. So we assume $t \geq 3$ in the following discussion. Before proving the Theorem 3.2, we present several auxiliary results on the images $f(\mathcal{M}^{ij})$ for $f \in \mathrm{Der}(\mathcal{N})$ and $\mathcal{M}^{ij} \subseteq \mathcal{N}$. We do not assume $\mathrm{char}(\mathbb{F}) \neq 2$ in Lemmas 3.1 – 3.10. The next lemma concerns the range of f on superdiagonal blocks of matrices of $\mathcal{N}$ except for $\mathcal{M}^{12}$ and $\mathcal{M}^{t-1,t}$.

Lemma 3.3. *For $f \in \mathrm{Der}(\mathcal{N})$ and $1 < k < t-1$,*

$$f(\mathcal{M}^{k,k+1}) \subseteq \sum_{p=1}^{k-1} \mathcal{M}^{p,k+1} + \sum_{q=k+1}^{t} \mathcal{M}^{kq} + \mathrm{Z}(\mathcal{N}). \tag{3.2}$$

In other words, the image $f(\mathcal{M}^{k,k+1})$ is located on the k-th block row and the $(k+1)$-th block column of matrices of $\mathcal{N}$ as well as in the center $\mathrm{Z}(\mathcal{N}) = \mathcal{M}^{1t}$.

Proof. Given any $A^{k,k+1} \in \mathcal{M}^{k,k+1}$, it suffices to prove that $f(A^{k,k+1})_{ij} = 0$ for $i < j$, $i \neq k$, $j \neq k+1$, and $(i,j) \neq (1,t)$. Either $i > 1$ or $j < t$. Without loss of generality, suppose $j < t$ (similarly for $i > 1$). Then for any $A^{jt} \in \mathcal{M}^{jt}$, the (i,t) block of $f([A^{k,k+1}, A^{jt}])$ is

$$0 = f([A^{k,k+1}, A^{jt}])_{it} = [f(A^{k,k+1}), A^{jt}]_{it} + [A^{k,k+1}, f(A^{jt})]_{it} = f(A^{k,k+1})_{ij}(A^{jt})_{jt}$$

where the last equality is by the assumptions on i, j. Therefore $f(A^{k,k+1})_{ij} = 0$. □

Now consider the range of f on $\mathcal{M}^{12}$ and $\mathcal{M}^{t,t-1}$ for $f \in \mathrm{Der}(\mathcal{N})$. The case $\mathrm{char}(\mathbb{F}) \neq 2$ would be simpler in the following lemma.

Lemma 3.4. *Let* $f \in \mathrm{Der}(\mathcal{N})$. *Then*

$$f(\mathcal{M}^{12}) \subseteq \sum_{q=2}^{t} \mathcal{M}^{1q} + \mathcal{M}^{2t} + \mathcal{M}^{3t}, \tag{3.3}$$

$$f(\mathcal{M}^{t-1,t}) \subseteq \sum_{p=1}^{t-1} \mathcal{M}^{pt} + \mathcal{M}^{1,t-1} + \mathcal{M}^{1,t-2}. \tag{3.4}$$

Furthermore, when $\mathrm{char}(\mathbb{F}) \neq 2$, *the* $(3,t)$ *block of matrices in* $f(\mathcal{M}^{12})$ *and the* $(1,t-2)$ *block of matrices in* $f(\mathcal{M}^{t-1,t})$ *are zero.*

Proof. The case $t = 3$ is obviously true. We now assume that $t \geq 4$.

To get (3.3), we prove that $f(A^{12})_{ij} = 0$ for any $A^{12} \in \mathcal{M}^{12}$, $1 < i < j$, and $(i,j) \notin \{(2,t),(3,t)\}$. Either $i > 3$ or $j < t$.

1. Suppose $j < t$. Then for any $A^{jt} \in \mathcal{M}^{jt}$,

$$0 = f([A^{12}, A^{jt}])_{it} = [f(A^{12}), A^{jt}]_{it} + [A^{12}, f(A^{jt})]_{it}.$$

   Therefore, $0 = f(A^{12})_{ij}(A^{jt})_{jt}$, and thus $f(A^{12})_{ij} = 0$.

2. Suppose $3 < i$. Then for any $A^{3i} \in \mathcal{M}^{3i}$,

$$0 = f([A^{12}, A^{3i}])_{3j} = [f(A^{12}), A^{3i}]_{3j} + [A^{12}, f(A^{3i})]_{3j}.$$

Therefore, $0 = -(A^{3i})_{3i} f(A^{12})_{ij}$, which implies that $f(A^{12})_{ij} = 0$.

Overall, (3.3) is done.

Next, when $\mathrm{char}(\mathbb{F}) \neq 2$, we show that $f(A^{12})_{3t} = 0$. For any $A^{23} \in \mathcal{M}^{23}$,

$$\begin{aligned} 0 &= f([A^{12}, [A^{12}, A^{23}]])_{1t} \\ &= [f(A^{12}), [A^{12}, A^{23}]]_{1t} + [A^{12}, [f(A^{12}), A^{23}]]_{1t} + [A^{12}, [A^{12}, f(A^{23})]]_{1t} \\ &= -2(A^{12})_{12}(A^{23})_{23} f(A^{12})_{3t}. \end{aligned}$$

Since $\mathrm{char}(\mathbb{F}) \neq 2$, $0 = (A^{12})_{12}(A^{23})_{23} f(A^{12})_{3t}$. Given A^{12}, the matrix $(A^{12})_{12}(A^{23})_{23}$ for any $A^{23} \in \mathcal{M}^{23}$ could be any matrix in $\mathcal{M}_{13}$ with rank no more than $\operatorname{rank} A^{12}$. Therefore $f(A^{12})_{3t} = 0$.

The proofs of (3.4) and $f(\mathcal{M}^{t-1,t})_{1,t-2} = 0$ when $\mathrm{char}(\mathbb{F}) \neq 2$ are similar. $\square$

Definition 3.5. *Given $i, j \in [t]$, $p \in [n_i]$, $q \in [n_j]$, let $E^{ij}_{pq} \in M_n$ denote the matrix with the only nonzero entry 1 in the (p, q) position of the (i, j) block. Clearly $E^{ij}_{pq} \in \mathcal{M}^{ij}$. The matrix E^{ij}_{pq} is called the (p, q) standard matrix of $\mathcal{M}^{ij}$.*

The next two lemmas explicitly describe the $(2, t)$ block of matrices in $f(\mathcal{M}^{12})$ and the $(1, t-1)$ block of matrices in $f(\mathcal{M}^{t-1,t})$ for $f \in \mathrm{Der}(\mathcal{N})$.

Lemma 3.6. *For $f \in \mathrm{Der}(\mathcal{N})$, the image $f(\mathcal{M}^{12})_{2t}$ has the following properties, according to the row size n_1 of the $(1, 2)$ block.*

1. *If $n_1 \geq 2$, then $f(\mathcal{M}^{12})_{2t} = 0$.*

2. *If $n_1 = 1$, then $\{E^{12}_{1i} \mid i \in [n_2]\}$ is a basis of $\mathcal{M}^{12}$; the i-th row of $f(E^{12}_{1j})_{2t}$ is equal to the j-th row of $f(E^{12}_{1i})_{2t}$ for any $i, j \in [n_2]$.*

Conversely, any $f \in \mathrm{End}(\mathcal{N})$ *that satisfies* $f(\mathcal{M}^{ij}) = 0$ *for* $\mathcal{M}^{ij} \subseteq \mathcal{N}$ *with* $(i, j) \neq (1, 2)$, $f(\mathcal{M}^{12}) \subseteq \mathcal{M}^{2t}$, *and the above hypothesis, is in* $\mathrm{Der}(\mathcal{N})$.

Proof. Let $f \in \mathrm{Der}(\mathcal{N})$.

1. When $n_1 \geq 2$, it suffices to prove that $f(E^{12}_{ij})_{2t} = 0$ for any $i \in [n_1], j \in [n_2]$. Since $n_1 \geq 2$, we can choose $r \in [n_1] - \{i\}$. Then for any $s \in [n_2]$, $E^{12}_{rs} \in \mathcal{M}^{12}$ and

$$\begin{aligned} 0 = f([E^{12}_{rs}, E^{12}_{ij}])_{1t} &= [f(E^{12}_{rs}), E^{12}_{ij}]_{1t} + [E^{12}_{rs}, f(E^{12}_{ij})]_{1t} \\ &= -(E^{12}_{ij})_{12} f(E^{12}_{rs})_{2t} + (E^{12}_{rs})_{12} f(E^{12}_{ij})_{2t}. \end{aligned}$$

   Therefore, $(E^{12}_{rs})_{12} f(E^{12}_{ij})_{2t} = (E^{12}_{ij})_{12} f(E^{12}_{rs})_{2t}$. Comparing the r-th rows on both sides, we see that the s-th row of $f(E^{12}_{ij})_{2t}$ is zero. Since $s \in [n_2]$ is arbitrary, we have $f(E^{12}_{ij})_{2t} = 0$.

2. Suppose $n_1 = 1$. The case $n_2 = 1$ is trivial. Now we assume that $n_2 \geq 2$. For any $j \in [n_2]$, we can choose $i \in [n_2] - \{j\}$. Then

$$\begin{aligned} 0 = f([E^{12}_{1j}, E^{12}_{1i}])_{1t} &= [f(E^{12}_{1j}), E^{12}_{1i}]_{1t} + [E^{12}_{1j}, f(E^{12}_{1i})]_{1t} \\ &= -(E^{12}_{1i})_{12} f(E^{12}_{1j})_{2t} + (E^{12}_{1j})_{12} f(E^{12}_{1i})_{2t}. \end{aligned}$$

   Therefore,

$$(E^{12}_{1i})_{12} f(E^{12}_{1j})_{2t} = (E^{12}_{1j})_{12} f(E^{12}_{1i})_{2t}.$$

   Comparing the first rows, we see that the i-th row of $f(E^{12}_{1j})_{2t}$ is equal to the j-th row of $f(E^{12}_{1i})_{2t}$ for $i \neq j$.

The last statement is easy to verify. □

Lemma 3.7. *For* $f \in \mathrm{Der}(\mathcal{N})$*, the image* $f(\mathcal{M}^{t-1,t})_{1,t-1}$ *satisfies the following properties, according to the column size* n_t *of the* $(t-1, t)$ *block.*

1. *If* $n_t \geq 2$, *then* $f(\mathcal{M}^{t-1,t})_{1,t-1} = 0$.

2. *If* $n_t = 1$, *then* $\{E_{i1}^{t-1,t} \mid i \in [n_{t-1}]\}$ *is a basis of* $\mathcal{M}^{t-1,t}$; *the* i*-th column of* $f(E_{j1}^{t-1,t})_{1,t-1}$ *is equal to the* j*-th column of* $f(E_{i1}^{t-1,t})_{1,t-1}$ *for any* $i, j \in [n_{t-1}]$.

Conversely, any $f \in \mathrm{End}(\mathcal{N})$ *that satisfies* $f(\mathcal{M}^{ij}) = 0$ *for* $\mathcal{M}^{ij} \subseteq \mathcal{N}$ *and* $(i,j) \neq (t-1,t)$, $f(\mathcal{M}^{t-1,t}) \subseteq \mathcal{M}^{1,t-1}$, *and the above hypothesis, is in* $\mathrm{Der}(\mathcal{N})$.

Proof. The proof (omitted) is similar to that of Lemma 3.6. □

Next we consider the range of f on the other blocks of matrices in $\mathcal{N}$.

Lemma 3.8. *For* $f \in \mathrm{Der}(\mathcal{N})$, $i, j \in [t]$ *and* $j > i+1$, *the image* $f(\mathcal{M}^{ij})$ *satisfies that:*

1. *If* $\mathrm{char}(\mathbb{F}) \neq 2$, *then*

$$f(\mathcal{M}^{ij}) \subseteq \sum_{p=1}^{i-1} \mathcal{M}^{pj} + \sum_{q=j}^{t} \mathcal{M}^{iq}. \tag{3.5}$$

2. *If* $\mathrm{char}(\mathbb{F}) = 2$, *then* (3.5) *still holds for* $(i,j) \notin \{(1,3),(t-2,t)\}$, *and*

$$f(\mathcal{M}^{13}) \subseteq \sum_{q=3}^{t} \mathcal{M}^{1q} + \mathcal{M}^{2t}, \tag{3.6}$$

$$f(\mathcal{M}^{t-2,t}) \subseteq \sum_{p=1}^{t-2} \mathcal{M}^{pt} + \mathcal{M}^{1,t-1}. \tag{3.7}$$

Proof. First assume $\mathrm{char}(\mathbb{F}) \neq 2$. Let $j = i+k$, $k \geq 2$. We prove (3.5) by induction on k.

1. $k = 2$: $\mathcal{M}^{i,i+2} = \mathcal{M}^{i,i+1}\mathcal{M}^{i+1,i+2} = [\mathcal{M}^{i,i+1}, \mathcal{M}^{i+1,i+2}]$. For $A^{i,i+1} \in \mathcal{M}^{i,i+1}$ and $A^{i+1,i+2} \in \mathcal{M}^{i+1,i+2}$,

$$\begin{aligned} f([A^{i,i+1}, A^{i+1,i+2}]) &= [f(A^{i,i+1}), A^{i+1,i+2}] + [A^{i,i+1}, f(A^{i+1,i+2})] \\ &\in \mathcal{M}^{i,i+2} + \sum_{p=1}^{i-1} \mathcal{M}^{p,i+2} + \sum_{q=i+3}^{t} \mathcal{M}^{i,q} \end{aligned} \tag{3.8}$$

where the last relation is by Lemmas 3.3 and 3.4. Thus $k = 2$ is done.

2. $k=\ell>2$: Suppose (3.5) holds for all $k<\ell$ where $\ell>2$ is given. Now $\mathcal{M}^{i,i+\ell} = \mathcal{M}^{i,i+2}\mathcal{M}^{i+2,i+\ell} = [\mathcal{M}^{i,i+2},\mathcal{M}^{i+2,i+\ell}]$. For any $A^{i,i+2}\in\mathcal{M}^{i,i+2}$ and $A^{i+2,i+\ell}\in\mathcal{M}^{i+2,i+\ell}$,

$$\begin{aligned} f([A^{i,i+2},A^{i+2,i+\ell}]) &= [f(A^{i,i+2}),A^{i+2,i+\ell}]+[A^{i,i+2},f(A^{i+2,i+\ell})] \\ &\in \mathcal{M}^{i,i+\ell}+\sum_{p=1}^{i-1}\mathcal{M}^{p,i+\ell}+\sum_{q=i+\ell+1}^{t}\mathcal{M}^{i,q} \end{aligned} \tag{3.9}$$

where the last relation is by induction hypothesis, the case $k=2$, and Lemmas 3.3 and 3.4. So (3.5) is true for $k=\ell$.

3. Overall, (3.5) is true for all k.

Now consider the case $\mathrm{char}(\mathbb{F})=2$. We get the same relation (3.8) except for $i=1$ and $i=t-2$, according to Lemma 3.4. For $i=1$, by Lemmas 3.3 and 3.4,

$$f([A^{12},A^{23}])=[A^{12},f(A^{23})]+[f(A^{12}),A^{23}]\in\sum_{q=3}^{t}\mathcal{M}^{1q}+[\mathcal{M}^{2t}+\mathcal{M}^{3t},A^{23}]\subseteq\sum_{q=3}^{t}\mathcal{M}^{1q}+\mathcal{M}^{2t}.$$

We get (3.6). Similarly, we can get (3.7). The relation (3.9) is unaffected by (3.6) and (3.7) when $i=1$ or $(i,\ell)=(t-4,4)$. So the induction can be proceeded for $\mathrm{char}(\mathbb{F})=2$. □

Now the range of $f\in\mathrm{Der}(\mathcal{N})$ on $\mathcal{M}^{ij}\subseteq\mathcal{N}$ is limited. The next lemma explicitly describes the f-images of each $\mathcal{M}^{ij}\subseteq\mathcal{N}$ in almost all nonzero blocks. It essentially implies that the f-images on these blocks are the same as the images of the adjoint action of a block upper triangular matrix. Denote the index set

$$\Omega := \{(p,q)\in[t]\times[t]\mid p<q\}\setminus\{(1,t-1),(1,t),(2,t)\}. \tag{3.10}$$

Lemma 3.9. *Let $f\in\mathrm{Der}(\mathcal{N})$. Then for any $(p,q)\in\Omega$, there exists $X_{pq}\in\mathcal{M}_{pq}$ such that*

$$f(A^{ip})_{iq} = -(A^{ip})_{ip}X_{pq}\quad \textit{for all}\ \ A^{ip}\in\mathcal{M}^{ip}\subseteq\mathcal{N}, \tag{3.11}$$

$$f(A^{qj})_{pj} = X_{pq}(A^{qj})_{qj}\quad \textit{for all}\ \ A^{qj}\in\mathcal{M}^{qj}\subseteq\mathcal{N}. \tag{3.12}$$

Proof. Given $p < q$ in $[t]$, we prove (3.11) and (3.12) in the following steps:

1. We prove (3.12) for $(q, j) = (t-1, t)$. Then $1 < p < t-1$. For $A^{t-1,t} \in \mathcal{M}^{t-1,t}$ and $A^{1p} \in \mathcal{M}^{1p}$,

$$\begin{aligned} 0 &= f([A^{1p}, A^{t-1,t}])_{1t} = [f(A^{1p}), A^{t-1,t}]_{1t} + [A^{1p}, f(A^{t-1,t})]_{1t} \\ &= f(A^{1p})_{1,t-1}(A^{t-1,t})_{t-1,t} + (A^{1p})_{1p} f(A^{t-1,t})_{pt}. \end{aligned}$$

   Therefore,

$$-f(A^{1p})_{1,t-1}(A^{t-1,t})_{t-1,t} = (A^{1p})_{1p} f(A^{t-1,t})_{pt}.$$

   Applying Lemma 2.9 to $\phi : \mathcal{M}_{1p} \to \mathcal{M}_{1,t-1}$ defined by $\phi(C) = -f(C^{1p})_{1,t-1}$ and $\varphi : \mathcal{M}_{t-1,t} \to \mathcal{M}_{pt}$ defined by $\varphi(D) = f(D^{t-1,t})_{pt}$, we will find $X_{p,t-1} \in \mathcal{M}_{p,t-1}$ such that $f(A^{t-1,t})_{pt} = X_{p,t-1}(A^{t-1,t})_{t-1,t}$ for all $A^{t-1,t} \in \mathcal{M}^{t-1,t}$.

2. Similarly, we can prove (3.11) for $(i, p) = (1, 2)$ via Lemma 2.9. In other words, for $2 < q < t$, there is $-Y_{2q} \in \mathcal{M}_{2q}$ such that $f(A^{12})_{1q} = -(A^{12})_{12} Y_{2q}$ for all $A^{12} \in \mathcal{M}^{12}$.

3. Now we prove (3.12) for $(q, j) \neq (t-1, t)$. Then $q < t-1$. Given any $j' > j$ in $[t]$, we have $\mathcal{M}^{qj'} = \mathcal{M}^{qj}\mathcal{M}^{jj'} = [\mathcal{M}^{qj}, \mathcal{M}^{jj'}]$. Then for $A^{qj} \in \mathcal{M}^{qj}$ and $A^{jj'} \in \mathcal{M}^{jj'}$,

$$f(A^{qj}A^{jj'})_{pj'} = f([A^{qj}, A^{jj'}])_{pj'} = [f(A^{qj}), A^{jj'}]_{pj'} + [A^{qj}, f(A^{jj'})]_{pj'} = f(A^{qj})_{pj}(A^{jj'})_{jj'}.$$

   Applying Lemma 2.7 to $\phi : \mathcal{M}_{qj'} \to \mathcal{M}_{pj'}$ defined by $\phi(C) = f(C^{qj'})_{pj'}$ and $\varphi : \mathcal{M}_{qj} \to \mathcal{M}_{pj}$ defined by $\varphi(D) = f(D^{qj})_{pj}$, we will find $X_{pq} \in \mathcal{M}_{pq}$ such that $f(A^{qj})_{pj} = X_{pq}(A^{qj})_{qj}$ for all $A^{qj} \in \mathcal{M}^{qj}$ and $(q, j) \neq (t-1, t)$.

4. Similarly, we can prove (3.11) for $(i, p) \neq (1, 2)$ via Lemma 2.6. In other words, there exists $-Y_{pq} \in \mathcal{M}_{pq}$ such that $f(A^{ip})_{iq} = -(A^{ip})_{ip} Y_{pq}$ for all $A^{ip} \in \mathcal{M}^{ip}$ and $(i, p) \neq (1, 2)$.

5. Finally, for any $A^{ip} \in \mathcal{M}^{ip}, A^{qj} \in \mathcal{M}^{qj}$, we have $i < p < q < j$, $[A^{ip}, A^{qj}] = 0$, so that

$$\begin{aligned} 0 &= f([A^{ip}, A^{qj}])_{ij} = [f(A^{ip}), A^{qj}]_{ij} + [A^{ip}, f(A^{qj})]_{ij} \\ &= f(A^{ip})_{iq}(A^{qj})_{qj} + (A^{ip})_{ip} f(A^{qj})_{pj} \\ &= -(A^{ip})_{ip} Y_{pq} (A^{qj})_{qj} + (A^{ip})_{ip} X_{pq} (A^{qj})_{qj}. \end{aligned}$$

Therefore, $X_{pq} = Y_{pq}$. The equalities (3.11) and (3.12) are proved. □

The next lemma concerns the derivations with image in the center of $\mathcal{N}$.

Lemma 3.10. *Suppose $f \in \operatorname{End}(\mathcal{N})$ satisfies that*

$$f(\mathcal{N}) \subseteq \mathrm{Z}(\mathcal{N}) = \mathcal{M}^{1t}, \qquad \operatorname{Ker} f \supseteq [\mathcal{N}, \mathcal{N}] = \sum_{i,j \in [t], i+1<j} \mathcal{M}^{ij}.$$

Then $f \in \operatorname{Der}(\mathcal{N})$.

Proof. The f satisfying the above conditions also satisfies the derivation property:

$$f([\mathcal{N}, \mathcal{N}]) = 0 = [f(\mathcal{N}), \mathcal{N}] + [\mathcal{N}, f(\mathcal{N})].$$ □

Now we are ready to prove Theorem 3.2.

Proof of Theorem 3.2. By Lemma 3.9, for $(p, q) \in \Omega$ we can find a matrix $X_{pq} \in \mathcal{M}_{pq}$ that satisfies (3.11) and (3.12). Let $X^{pq} := (X_{pq})^{pq} \in \mathcal{M}^{pq}$, and let

$$X_0 := \sum_{(p,q) \in \Omega} X^{pq} \ \in \ \mathcal{N}, \qquad f_0 := f - \operatorname{ad} X_0 \ \in \ \operatorname{Der}(\mathcal{N}). \tag{3.13}$$

The equalities (3.11) and (3.12) imply that

$$f_0(\mathcal{M}^{ip})_{iq} = 0 \quad \text{for all} \ \ \mathcal{M}^{ip} \subseteq \mathcal{N}, \qquad f_0(\mathcal{M}^{qj})_{pj} = 0 \quad \text{for all} \ \ \mathcal{M}^{qj} \subseteq \mathcal{N}. \tag{3.14}$$

By (3.2), (3.5), and Lemma 3.4 for $\mathrm{char}(\mathbb{F}) \neq 2$, for any $\mathcal{M}^{ij} \subseteq \mathcal{N}$, the only possibly nonzero blocks of matrices of $f_0(\mathcal{M}^{ij})$ are the (i,j) block and the following:

1. the $(1,t)$ block when $j = i+1$, and
2. the $(2,t)$ block when $(i,j) = (1,2)$, or
3. the $(1,t-1)$ block when $(i,j) = (t-1,t)$.

Define $\phi_{2t}^{12}, \phi_{1,t-1}^{t-1,t} \in \mathrm{End}(\mathcal{N})$ such that for $A \in \mathcal{N}$,

$$\phi_{2t}^{12}(A) := f_0(A^{12})^{2t} = f(A^{12})^{2t}, \tag{3.15}$$

$$\phi_{1,t-1}^{t-1,t}(A) := f_0(A^{t-1,t})^{1,t-1} = f(A^{t-1,t})^{1,t-1}. \tag{3.16}$$

Then Lemmas 3.6 and 3.7 show that $\phi_{2t}^{12}, \phi_{1,t-1}^{t-1,t} \in \mathrm{Der}(\mathcal{N})$. We get a derivation

$$f_1 := f_0 - \phi_{2t}^{12} - \phi_{1,t-1}^{t-1,t} = f - \mathrm{ad}\, X_0 - \phi_{2t}^{12} - \phi_{1,t-1}^{t-1,t}. \tag{3.17}$$

Define $\varphi_{1t} \in \mathrm{End}(\mathcal{N})$ such that for $A \in \mathcal{N}$,

$$\varphi_{1t}(A) := \sum_{i=1}^{t-1} f_1(A^{i,i+1})^{1t} = \sum_{i=1}^{t-1} f(A^{i,i+1})^{1t}. \tag{3.18}$$

Then Lemma 3.10 implies that $\varphi_{1t} \in \mathrm{Der}(\mathcal{N})$. We get a new derivation

$$f_2 := f_1 - \varphi_{1t} = f - \mathrm{ad}\, X_0 - \phi_{2t}^{12} - \phi_{1,t-1}^{t-1,t} - \varphi_{1t} \tag{3.19}$$

where $f_2(\mathcal{M}^{ij}) \subseteq \mathcal{M}^{ij}$.

To get (3.1), it suffices to prove the following claim regarding f_2: there exist $X^{ii} \in \mathcal{M}^{ii}$ for $i \in [t]$ such that for each $k \in [t-1]$, the derivation

$$f_2^{(k)} := f_2 - \sum_{i=1}^{k+1} \mathrm{ad}\, X^{ii}$$

satisfies that $f_2^{(k)}(\mathcal{M}^{pq}) = 0$ for $1 \le p < q \le k+1$. The proof is done by induction on k:

1. $k = 1$: For any $A^{12} \in \mathcal{M}^{12}$ and $A^{23} \in \mathcal{M}^{23}$, we have

$$\begin{aligned} f_2(A^{12}A^{23})_{13} &= f_2([A^{12}, A^{23}])_{13} = [f_2(A^{12}), A^{23}]_{13} + [A^{12}, f_2(A^{23})]_{13} \\ &= f_2(A^{12})_{12}(A^{23})_{23} + (A^{12})_{12} f_2(A^{23})_{23}. \end{aligned} \tag{3.20}$$

   By (2.5) in Lemma 2.10, there exist $X^{11} \in \mathcal{M}^{11}$ and $Y^{22} \in \mathcal{M}^{22}$ such that

$$f_2(A^{12})_{12} = (X^{11}A^{12} + A^{12}Y^{22})_{12}.$$

   Define $X^{22} := -Y^{22} \in \mathcal{M}^{22}$. Then

$$f_2(A^{12})_{12} = (X^{11}A^{12} - A^{12}X^{22})_{12}.$$

   Let $f_2^{(1)} := f_2 - \operatorname{ad} X^{11} - \operatorname{ad} X^{22}$. Then $f_2^{(1)}(\mathcal{M}^{12}) = 0$. The claim holds for $k = 1$.

2. $k = 2$: Applying (3.20) to $f_2^{(1)}$:

$$f_2^{(1)}(A^{12}A^{23})_{13} = f_2^{(1)}(A^{12})_{12}(A^{23})_{23} + (A^{12})_{12} f_2^{(1)}(A^{23})_{23} = (A^{12})_{12} f_2^{(1)}(A^{23})_{23}.$$

   By Lemma 2.6, there exists $Y^{33} \in \mathcal{M}^{33}$ such that $f_2^{(1)}(A^{13})_{13} = (A^{13}Y^{33})_{13}$ and $f_2^{(1)}(A^{23})_{23} = (A^{23}Y^{33})_{23}$. Define $X^{33} := -Y^{33} \in \mathcal{M}^{33}$. Then $f_2^{(1)}(A^{13})_{13} = (-A^{13}X^{33})_{13}$ and $f_2^{(1)}(A^{23})_{23} = (-A^{23}X^{33})_{23}$. Let $f_2^{(2)} := f_2^{(1)} - \operatorname{ad} X^{33}$. Then $f_2^{(2)}(\mathcal{M}^{pq}) = 0$ for $1 \le p < q \le 3$. So $k = 2$ is done.

3. $k = \ell > 2$: Suppose the claim holds for all $k < \ell$ where $\ell > 2$ is given. In other words, there exist $X^{ii} \in \mathcal{M}^{ii}$ for all $i \in [\ell]$ such that $f_2^{(\ell-1)} := f_2 - \sum_{i=1}^{\ell} \operatorname{ad} X^{ii}$ satisfies that $f_2^{(\ell-1)}(\mathcal{M}^{pq}) = 0$ for $1 \le p < q \le \ell$. Similar to (3.20), for any $p \in [\ell - 1]$, $A^{p,\ell} \in \mathcal{M}^{p,\ell}$,

$A^{\ell,\ell+1} \in \mathcal{M}^{\ell,\ell+1}$,

$$\begin{aligned} f_2^{(\ell-1)}(A^{p,\ell}A^{\ell,\ell+1})_{p,\ell+1} &= f_2^{(\ell-1)}(A^{p,\ell})_{p,\ell}(A^{\ell,\ell+1})_{\ell,\ell+1} + (A^{p,\ell})_{p,\ell}f_2^{(\ell-1)}(A^{\ell,\ell+1})_{\ell,\ell+1} \\ &= (A^{p,\ell})_{p,\ell}f_2^{(\ell-1)}(A^{\ell,\ell+1})_{\ell,\ell+1}. \end{aligned}$$

By Lemma 2.6, there exists $Y^{\ell+1,\ell+1} \in \mathcal{M}^{\ell+1,\ell+1}$ such that

$$f_2^{(\ell-1)}(A^{p,\ell+1})_{p,\ell+1} = (A^{p,\ell+1}Y^{\ell+1,\ell+1})_{p,\ell+1} \qquad \text{for all } \ p \in [\ell].$$

Define $X^{\ell+1,\ell+1} := -Y^{\ell+1,\ell+1} \in \mathcal{M}^{\ell+1,\ell+1}$. Then

$$f_2^{(\ell-1)}(A^{p,\ell+1})_{p,\ell+1} = (-A^{p,\ell+1}X^{\ell+1,\ell+1})_{p,\ell+1} \qquad \text{for all } \ p \in [\ell].$$

Let $f_2^{(\ell)} := f_2^{(\ell-1)} - \operatorname{ad} X^{\ell+1,\ell+1}$. Then $f_2^{(\ell)}(\mathcal{M}^{p,\ell+1}) = 0$ for $p \in [\ell]$. So $k = \ell$ is proved.

Overall, the claim is completely proved; in particular, $f_2^{(t-1)}(\mathcal{N}) = 0$. Let $X := X_0 + \sum_{i=1}^{t} X^{ii}$, then we get (3.1). □

3.2 Derivations of $\mathcal{N}$ for $\operatorname{char}(\mathbb{F}) = 2$

When $\operatorname{char}(\mathbb{F}) = 2$, $\operatorname{Der}(\mathcal{N})$ is not completely described by Theorem 3.2. In fact, Lemmas 3.4, 3.8, and [23, Section 2(D)] motivate us to construct the following example.

Example 3.11. *Suppose* $\operatorname{char}(\mathbb{F}) = 2$. *Let* $\mathcal{N}$ *consist of strictly upper triangular matrices in* M_4*. So* $\mathcal{N}$ *has a basis* $\mathcal{B} := \{E_{12}, E_{13}, E_{14}, E_{23}, E_{24}, E_{34}\}$*, where* E_{ij} *denotes the matrix in* M_4 *that has the only nonzero entry 1 in the* (i,j) *position. Define* $f \in \operatorname{End}(\mathcal{N})$ *by* $f(E_{12}) := -E_{34}$*,* $f(E_{13}) := E_{24}$*, and* $f(E) := 0$ *for all other matrices* $E \in \mathcal{B}$*. We prove that*

$$f([E,E']) = [f(E),E'] + [E,f(E')] \tag{3.21}$$

for any $E, E' \in \mathcal{B}$, *so that* $f \in \mathrm{Der}(\mathcal{N})$. *There are only two cases where one side of* (3.21) *is possibly nonzero:*

1. $\{E, E'\} = \{E_{12}, E_{23}\}$, *where* $f([E, E']) = f(E_{13}) = E_{24}$, $[f(E), E'] + [E, f(E')] = E_{24}$.

2. $\{E, E'\} = \{E_{12}, E_{13}\}$, *where* $f([E, E']) = 0$, $[f(E), E'] + [E, f(E')] = 2E_{14} = 0$.

Therefore, $f \in \mathrm{Der}(\mathcal{N})$. *However,* f *can not be written as in* (3.1).

In Section 3.1, Lemmas 3.4 and 3.8 have special statements for $\mathrm{char}(\mathbb{F}) = 2$, while Lemmas 3.3, 3.6, 3.7, 3.9, 3.10 remain unchanged. The following two lemmas completely describe the images of a derivation on additional nonzero blocks when $\mathrm{char}(\mathbb{F}) = 2$.

Lemma 3.12. *For* $f \in \mathrm{Der}(\mathcal{N})$, *the images* $f(\mathcal{M}^{12})_{3t}$ *and* $f(\mathcal{M}^{13})_{2t}$ *satisfy the following properties, according to the row size* n_1 *of the first block row.*

1. *If* $n_1 \geq 2$, *then*

$$f(\mathcal{M}^{12})_{3t} = 0, \qquad f(\mathcal{M}^{13})_{2t} = 0. \tag{3.22}$$

2. *If* $n_1 = 1$, *then* $\mathcal{M}^{12}$ *has a basis* $\{E_{1i}^{12} \mid i \in [n_2]\}$ *and* $\mathcal{M}^{13}$ *has a basis* $\{E_{1j}^{13} \mid j \in [n_3]\}$; *the* i*-th row of* $f(E_{1j}^{13})_{2t}$ *is equal to the* j*-th row of* $f(E_{1i}^{12})_{3t}$ *for any* $i \in [n_2]$ *and* $j \in [n_3]$.

Conversely, any $f \in \mathrm{End}(\mathcal{N})$ *that satisfies* $f(\mathcal{M}^{ij}) = 0$ *for* $\mathcal{M}^{ij} \subseteq \mathcal{N}$ *and* $(i, j) \notin \{(1, 2), (1, 3)\}$, $f(\mathcal{M}^{12}) \subseteq \mathcal{M}^{3t}$, $f(\mathcal{M}^{13}) \subseteq \mathcal{M}^{2t}$, *and the above hypothesis, is in* $\mathrm{Der}(\mathcal{N})$.

Proof. Suppose $f \in \mathrm{Der}(\mathcal{N})$. Given $E_{ij}^{12} \in \mathcal{M}^{12}$ and $E_{rs}^{13} \in \mathcal{M}^{13}$,

$$0 = f([E_{ij}^{12}, E_{rs}^{13}])_{1t} = [E_{ij}^{12}, f(E_{rs}^{13})]_{1t} + [f(E_{ij}^{12}), E_{rs}^{13}]_{1t} = (E_{ij}^{12})_{12} f(E_{rs}^{13})_{2t} - (E_{rs}^{13})_{13} f(E_{ij}^{12})_{3t}.$$

Therefore,

$$(E_{ij}^{12})_{12} f(E_{rs}^{13})_{2t} = (E_{rs}^{13})_{13} f(E_{ij}^{12})_{3t}. \tag{3.23}$$

1. If $n_1 \geq 2$, then for a fixed $E_{ij}^{12} \in \mathcal{M}^{12}$, we can choose $r \in [n_1] \setminus \{i\}$. Comparing the r-th rows in the equality (3.23), we see that the s-th row of $f(E_{ij}^{12})_{3t}$ is zero. Then $f(E_{ij}^{12})_{3t} = 0$ since s is arbitrary. Similarly, $f(E_{rs}^{13})_{2t} = 0$. We get (3.22).

2. If $n_1 = 1$, then $i = r = 1$ in (3.23), which implies that the j-th row of $f(E_{1s}^{13})_{2t}$ is equal to the s-th row of $f(E_{1j}^{12})_{3t}$.

Conversely, suppose $f \in \mathrm{End}(\mathcal{N})$ satisfies that $f(\mathcal{M}^{ij}) = 0$ for $\mathcal{M}^{ij} \subseteq \mathcal{N}$ and $(i,j) \notin \{(1,2),(1,3)\}$, $f(\mathcal{M}^{12}) \subseteq \mathcal{M}^{3t}$, $f(\mathcal{M}^{13}) \subseteq \mathcal{M}^{2t}$, and the hypothesis in Lemma 3.12 (1) or (2). When $n_1 \geq 2$, $f \equiv 0$; when $n_1 = 1$, f satisfies (3.23) for $i = r = 1$ and all $j \in [n_2]$, $s \in [n_3]$. In both cases, f satisfies the derivation property and thus $f \in \mathrm{Der}(\mathcal{N})$. □

Lemma 3.13. *For $f \in \mathrm{Der}(\mathcal{N})$, the images $f(\mathcal{M}^{t-1,t})_{1,t-2}$ and $f(\mathcal{M}^{t-2,t})_{1,t-1}$ satisfy the following properties, according to the column size n_t of the last block column.*

1. *If $n_t \geq 2$, then*

$$f(\mathcal{M}^{t-1,t})_{1,t-2} = 0, \qquad f(\mathcal{M}^{t-2,t})_{1,t-1} = 0. \tag{3.24}$$

2. *If $n_1 = 1$, then $\mathcal{M}^{t-1,t}$ has a basis $\{E_{i1}^{t-1,t} \mid i \in [n_{t-1}]\}$ and $\mathcal{M}^{t-2,t}$ has a basis $\{E_{j1}^{t-2,t} \mid j \in [n_{t-2}]\}$; the i-th column of $f(E_{j1}^{t-2,t})_{1,t-1}$ is equal to the j-th column of $f(E_{i1}^{t-1,t})_{1,t-2}$ for any $i \in [n_{t-1}]$ and $j \in [n_{t-2}]$.*

Conversely, any $f \in \mathrm{End}(\mathcal{N})$ that satisfies $f(\mathcal{M}^{ij}) = 0$ for $\mathcal{M}^{ij} \subseteq \mathcal{N}$ and $(i,j) \notin \{(t-1,t),(t-2,t)\}$, $f(\mathcal{M}^{t-1,t}) \subseteq \mathcal{M}^{1,t-2}$, $f(\mathcal{M}^{t-2,t}) \subseteq \mathcal{M}^{1,t-1}$, and the above hypothesis, is in $\mathrm{Der}(\mathcal{N})$.

Proof. The proof (omitted) is similar to that of Lemma 3.12. □

Now we are able to describe $\mathrm{Der}(\mathcal{N})$ for the case $\mathrm{char}(\mathbb{F}) = 2$.

Theorem 3.14. *When* $\mathrm{char}(\mathbb{F}) = 2$*, every derivation* f *of the Lie algebra* $\mathcal{N}$ *can be (not uniquely) written as*

$$f \quad = \quad \mathrm{ad}\, X + \varphi_{1t} + \phi_{2t}^{12} + \phi_{1,t-1}^{t-1,t} + \psi_{3t;2t}^{12;13} + \psi_{1,t-2;1,t-1}^{t-1,t;t-2,t}, \tag{3.25}$$

where the summand components X*,* φ_{1t}*,* ϕ_{2t}^{12}*,* $\phi_{1,t-1}^{t-1,t}$ *are described in Theorem 3.2, and* $\psi_{3t;2t}^{12;13}$ *and* $\psi_{1,t-2;1,t-1}^{t-1,t;t-2,t}$ *are determined as follow:*

1. $\psi_{3t;2t}^{12;13} \equiv 0$ *unless the first block row of matrices of* $\mathcal{N}$ *has only one row, in which* $\psi_{3t;2t}^{12;13} \in \mathrm{Der}(\mathcal{N})$ *maps* $\mathcal{M}^{12}$ *to* $\mathcal{M}^{3t}$*,* $\mathcal{M}^{13}$ *to* $\mathcal{M}^{2t}$*, and the other* $\mathcal{M}^{ij} \subseteq \mathcal{N}$ *to 0; the explicit form of* $\psi_{3t;2t}^{12;13}$ *is given in Lemma 3.12;*

2. $\psi_{1,t-2;1,t-1}^{t-1,t;t-2,t} \equiv 0$ *unless the last block column of matrices of* $\mathcal{N}$ *has only one column, in which* $\psi_{1,t-2;1,t-1}^{t-1,t;t-2,t} \in \mathrm{Der}(\mathcal{N})$ *maps* $\mathcal{M}^{t-1,t}$ *to* $\mathcal{M}^{1,t-2}$*,* $\mathcal{M}^{t-2,t}$ *to* $\mathcal{M}^{1,t-1}$*, and the other* $\mathcal{M}^{ij} \subseteq \mathcal{N}$ *to 0; the explicit form of* $\psi_{1,t-2;1,t-1}^{t-1,t;t-2,t}$ *is given in Lemma 3.13.*

Proof. Given $f \in \mathrm{Der}(\mathcal{N})$, we can proceed the proof of Theorem 3.2 up to (3.14). Then we define $\psi_{3t;2t}^{12;13}, \psi_{1,t-2;1,t-1}^{t-1,t;t-2,t} \in \mathrm{End}(\mathcal{N})$ such that for $A \in \mathcal{N}$,

$$\begin{aligned} \psi_{3t;2t}^{12;13}(A) &:= f_0(A^{12})^{3t} + f_0(A^{13})^{2t} = f(A^{12})^{3t} + f(A^{13})^{2t}, \\ \psi_{1,t-2;1,t-1}^{t-1,t;t-2,t}(A) &:= f_0(A^{t-1,t})^{1,t-2} + f_0(A^{t-2,t})^{1,t-1} = f(A^{t-1,t})^{1,t-2} + f(A^{t-2,t})^{1,t-1}. \end{aligned}$$

Both linear maps are derivations by Lemmas 3.12 and 3.13. Subtracting them from f, we can continue the remaining proof of Theorem 3.2. □

Chapter 4

Derivations of the Lie algebra of dominant upper triangular ladder matrices

In this chapter, we explicitly characterize the derivations of the Lie algebra $M_{\mathcal{L}}$ of ladder matrices in M_n associated with a dominant upper triangular (DUT) ladder $\mathcal{L}$ over a field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) \neq 2$, and provide some consequent results. Recall that $\mathcal{L}$ is a DUT ladder of size n if and only if $M_{\mathcal{L}}$ consists of block upper triangular matrices in M_n that have zero submatrices on some preset non-consecutive diagonal blocks, corresponding to a given ordered partition of $[n]$ (Theorem 2.4). In the rest of this chapter, we fix the $t \times t$ block matrix form of matrices in M_n determined by $\mathcal{L}$ through (2.1).

4.1 Properties of the Lie algebra of DUT ladder matrices

We describe some properties of the Lie algebra of DUT ladders matrices in this section. We adapt the notations $M_{\mathcal{B}}$, $\mathcal{M}_{ij}$, and $\mathcal{M}^{ij}$ in Definition 3.1 here. Recall that $\Omega(\mathcal{L})$ and $\Omega(\mathcal{B})$ denote the block index set of matrices of $M_{\mathcal{L}}$ and $M_{\mathcal{B}}$, respectively. We now introduce some new notations.

Definition 4.1. *1. In $\mathcal{M}_{kk}$, let I_{kk} denote the identity matrix, $\mathfrak{sl}_{kk}$ the set of traceless matrices, respectively.*

2. Let $I^{kk} \in \mathcal{M}^{kk}$ denote the matrix with the submatrix I_{kk} in the (k,k) block.

3. Let $\mathfrak{sl}^{kk}$ denote the set of traceless matrices of $\mathcal{M}^{kk}$.

A notation of double index, say $\mathcal{M}_{ij}$, may be written as $\mathcal{M}_{i,j}$ (resp. $\mathcal{M}^{ij}$ as $\mathcal{M}^{i,j}$) for clarity purpose, as in Chapter 3.

The *normalizer* $\mathrm{N}(M_{\mathcal{L}})$ and the *centralizer* $\mathrm{Z}(M_{\mathcal{L}})$ of subalgebra $M_{\mathcal{L}}$ in M_n are:

$$\begin{aligned} \mathrm{N}(M_{\mathcal{L}}) &= \{A \in M_n : [A, B] \in M_{\mathcal{L}} \text{ for all } B \in M_{\mathcal{L}}\}, \\ \mathrm{Z}(M_{\mathcal{L}}) &= \{A \in M_n : [A, B] = 0 \text{ for all } B \in M_{\mathcal{L}}\}. \end{aligned}$$

They are explicitly described by the following two lemmas.

Lemma 4.2. *If $\mathcal{L}$ is a DUT ladder of size n, then $\mathrm{N}(M_{\mathcal{L}}) = M_{\mathcal{B}}$, the algebra of block upper triangular matrices in M_n.*

Proof. We first show that $\mathrm{N}(M_{\mathcal{L}}) \subseteq M_{\mathcal{B}}$. Suppose on the contrary, there is $A \in \mathrm{N}(M_{\mathcal{L}})$ such that the (i, j) block $A_{ij} \neq 0$ for some $1 \leq j < i \leq t$. There are two cases:

1. $i > j + 1$: We have $\mathcal{M}^{j,j+1} \subseteq M_{\mathcal{L}}$ by Theorem 2.4. So $[A, B^{j,j+1}] \in M_{\mathcal{L}}$ for $B^{j,j+1} \in \mathcal{M}^{j,j+1}$. However, its $(i, j+1)$ block is

$$([A, B^{j,j+1}])_{i,j+1} = [A_{ij}, (B^{j,j+1})_{j,j+1}] = A_{ij}(B^{j,j+1})_{j,j+1} \neq \{0\},$$

   which contradicts to the DUT assumption of $\mathcal{L}$.

2. $i = j + 1$: By Theorem 2.4, either $\mathcal{M}^{jj} \subseteq M_{\mathcal{L}}$ or $\mathcal{M}^{j+1,j+1} \subseteq M_{\mathcal{L}}$. Without loss of generality, suppose $\mathcal{M}^{jj} \subseteq M_{\mathcal{L}}$. Then $[A, B^{jj}] \in M_{\mathcal{L}}$ for $B^{jj} \in \mathcal{M}^{jj}$. However, its (i, j) block is

$$([A, B^{jj}])_{ij} = A_{ij}(B^{jj})_{jj} \neq \{0\},$$

   which contradicts the DUT assumption of $\mathcal{L}$.

Therefore, $A \in M_{\mathcal{B}}$ and thus $\mathrm{N}(M_{\mathcal{L}}) \subseteq M_{\mathcal{B}}$.

For any $(i, j) \in [t] \times [t]$ with $i \leq j$, the possibly nonzero blocks of matrices in $[\mathcal{M}^{ij}, M_{\mathcal{L}}]$ are those (i, q) blocks with $q \geq j$ and (p, j) blocks with $p \leq i$, all of which belong to $M_{\mathcal{L}}$. Hence $M_{\mathcal{B}} \subseteq \mathrm{N}(M_{\mathcal{L}})$. $\square$

Lemma 4.3. *Let $\mathcal{L}$ be a DUT ladder of size n, $\Omega(\mathcal{L})$ the block index set of matrices of $M_{\mathcal{L}}$, and I_n the identity matrix in M_n.*

1. *If both $(1,1)$ and (t,t) are not in $\Omega(\mathcal{L})$, then $\mathrm{Z}(M_{\mathcal{L}}) = \mathbb{F}I_n + \mathcal{M}^{1t}$.*

2. *Otherwise, $\mathrm{Z}(M_{\mathcal{L}}) = \mathbb{F}I_n$.*

Proof. Clearly $\mathrm{Z}(M_{\mathcal{L}}) \subseteq \mathrm{N}(M_{\mathcal{L}})$. The possibly nonzero blocks of any $A \in \mathrm{Z}(M_{\mathcal{L}})$ are A_{ij} for some $1 \leq i \leq j \leq t$. If $A_{ij} \neq 0$ and $2 \leq i < j$, then $\mathcal{M}^{i-1,i} \subseteq M_{\mathcal{L}}$, and we can find $B^{i-1,i} \in \mathcal{M}^{i-1,i}$ such that

$$([B^{i-1,i}, A])_{i-1,j} = [(B^{i-1,i})_{i-1,i}, A_{i,j}] \neq 0,$$

which contradicts to the assumption $A \in \mathrm{Z}(M_{\mathcal{L}})$. Thus $A_{ij} = 0$ for all $2 \leq i < j \leq t$. Similarly, $A_{ij} = 0$ for all $1 \leq i < j \leq t-1$. So the only possibly nonzero blocks of $A \in \mathrm{Z}(M_{\mathcal{L}})$ are A_{1t} and A_{ii} for $i \in [t]$.

If $(1,1) \in \Omega(\mathcal{L})$, then $0 = [I^{11}, A]_{1t} = [I_{11}, A_{1t}] = A_{1t}$. Similarly, $(t,t) \in \Omega(\mathcal{L})$ implies that $A_{1t} = 0$. If neither $(1,1)$ nor (t,t) is in $\Omega(\mathcal{L})$, then $\mathcal{M}^{1t} \subseteq \mathrm{Z}(M_{\mathcal{L}})$ by direct computation.

Now for any $i, j \in [t]$ with $i < j$ and $B^{ij} \in \mathcal{M}^{ij} \subseteq M_{\mathcal{L}}$,

$$0 = ([A, B^{ij}])_{ij} = A_{ii}(B^{ij})_{ij} - (B^{ij})_{ij}A_{jj}.$$

Applying lemma 2.8, we find $\lambda \in \mathbb{F}$ such that $A_{ii} = \lambda I_{ii}$ and $A_{jj} = \lambda I_{jj}$.

In summary, $\mathrm{Z}(M_{\mathcal{L}})$ is described by the statements 1 and 2. □

4.2 Derivations of the Lie algebra $M_{\mathcal{L}}$ of DUT ladder matrices

We introduce the main theorem of this chapter here and provide some consequent results. Note that the adjoint representation $\mathrm{ad} : M_n \to \mathrm{Der}(M_n)$ defined by $\mathrm{ad}\, A(B) = [A, B]$

induces a Lie algebra homomorphism

$$\mathrm{ad}(\cdot)|_{M_{\mathcal{L}}} : \mathrm{N}(M_{\mathcal{L}})/\mathrm{Z}(M_{\mathcal{L}}) \to \mathrm{Der}(M_{\mathcal{L}}), \quad X \mapsto \mathrm{ad}(X)|_{M_{\mathcal{L}}},$$

which will be used in the next theorem. Denote by $\mathrm{ad}(\mathfrak{s})|_{M_{\mathcal{L}}}$ the set of all $\mathrm{ad}(X)|_{M_{\mathcal{L}}}$ for $X \in \mathfrak{s}$ and $\mathfrak{s} \subseteq \mathrm{N}(M_{\mathcal{L}})/\mathrm{Z}(M_{\mathcal{L}})$.

Theorem 4.4. (*Main theorem*) *Suppose* $\mathrm{char}(\mathbb{F}) \neq 2$*. Let* $\mathcal{L}$ *be a DUT ladder of size* n *with the corresponding* $t \times t$ *block matrix form of matrices in* M_n *determined by* (2.1)*. Then the derivation algebra* $\mathrm{Der}(M_{\mathcal{L}})$ *can be decomposed as follow:*

$$\begin{aligned} \mathrm{Der}(M_{\mathcal{L}}) &= \mathrm{ad}(\mathrm{N}(M_{\mathcal{L}})/\mathrm{Z}(M_{\mathcal{L}}))|_{M_{\mathcal{L}}} \ltimes \mathcal{D} & (4.1)\\ &= (\mathrm{ad}\,(M_{\mathcal{L}}) \oplus \mathcal{D}) \rtimes \left(\bigoplus_{(k,k)\in\Omega(\mathcal{B})\setminus\Omega(\mathcal{L})} \mathrm{ad}\left(\mathcal{M}^{kk}\right)|_{M_{\mathcal{L}}} \right) & (4.2) \end{aligned}$$

where the normalizer $\mathrm{N}(M_{\mathcal{L}})$ *and the centralizer* $\mathrm{Z}(M_{\mathcal{L}})$ *are described by Lemmas 4.2 and 4.3, respectively, and*

$$\mathcal{D} := \{\phi \in \mathrm{End}(M_{\mathcal{L}}) : \mathrm{Ker}\,\phi \supseteq [M_{\mathcal{L}}, M_{\mathcal{L}}],\ \mathrm{Im}\,\phi \subseteq \mathrm{Z}(M_{\mathcal{L}}) \cap M_{\mathcal{L}}\}. \tag{4.3}$$

Moreover, both $\mathrm{ad}(M_{\mathcal{L}})$ *and* $\mathcal{D}$ *are ideals of* $\mathrm{Der}(M_{\mathcal{L}})$*.*

Explicitly, we have the following cases:

1. *If* $\Omega(\mathcal{L}) = \Omega(\mathcal{B})$*, i.e.* $M_{\mathcal{L}}$ *is the Lie algebra of block upper triangular matrices in* M_n*, then every* $f \in \mathrm{Der}(M_{\mathcal{L}})$ *corresponds to an* $X \in M_{\mathcal{B}}/\mathbb{F}I_n$ *and* $c_1, \cdots, c_t \in \mathbb{F}$*, such that*

$$f(A) = \mathrm{ad}\,X(A) + \left(\sum_{(k,k)\in\Omega(\mathcal{L})} c_k \mathrm{tr}(A_{kk}) \right) I_n \qquad \textit{for} \quad A \in M_{\mathcal{L}}. \tag{4.4}$$

2. *If $\Omega(\mathcal{L}) \neq \Omega(\mathcal{B})$ but at least one of $(1,1)$ and (t,t) is in $\Omega(\mathcal{L})$, then every $f \in \mathrm{Der}(M_{\mathcal{L}})$ corresponds to an $X \in M_{\mathcal{B}}/\mathbb{F}I_n$, such that*

$$f(A) = \mathrm{ad}\, X(A) \qquad \textit{for} \quad A \in M_{\mathcal{L}}. \tag{4.5}$$

3. *If both $(1,1)$ and (t,t) are not in $\Omega(\mathcal{L})$, then every $f \in \mathrm{Der}(M_{\mathcal{L}})$ corresponds to an $X \in M_{\mathcal{B}}/(\mathbb{F}I_n + \mathcal{M}^{1t})$ and $Y^{1tk} \in \mathcal{M}^{1t}$ for each $(k,k) \in \Omega(\mathcal{L})$, such that*

$$f(A) = \mathrm{ad}\, X(A) + \sum_{(k,k)\in\Omega(\mathcal{L})} \mathrm{tr}(A_{kk})Y^{1tk} \qquad \textit{for} \quad A \in M_{\mathcal{L}}. \tag{4.6}$$

A detailed proof of Theorem 4.4 will be given in Section 4.3. The special case $M_{\mathcal{L}} = M_{\mathcal{B}}$ is included in a paper of Dengyin Wang and Qiu Yu [27, Theorem 4.1]. Moreover, Daniel Brice has obtained a formula similar to (4.1) for the derivation algebra of the parabolic subalgebra of a reductive Lie algebra over a $\mathbb{C}$-like fields or over $\mathbb{R}$ [4].

Example 4.5. *Theorem 4.4 is not true when $\mathrm{char}(\mathbb{F}) = 2$. Consider $M_{\mathcal{L}} = M_2$ with the basis $\mathcal{B} = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ where E_{ij} denotes the matrix in M_2 that has the only nonzero entry 1 in the (i,j) position. Define $f \in \mathrm{End}(M_{\mathcal{L}})$ by $f(E_{12}) = E_{21}$ and $f(E_{ij}) = 0$ for $(i,j) = (1,1), (2,1), (2,2)$. It is straightforward to verify that*

$$f([E,E']) = [f(E), E'] + [E, f(E')] \tag{4.7}$$

for any $E, E' \in \mathcal{B}$, since there are only two cases that either side of (4.7) is nonzero: $\{E,E'\} = \{E_{11}, E_{12}\}$ or $\{E_{12}, E_{22}\}$. Therefore $f \in \mathrm{Der}(M_{\mathcal{L}})$. However, f is not an element of $\mathrm{ad}(\mathrm{N}(M_{\mathcal{L}})/\mathrm{Z}(M_{\mathcal{L}}))|_{M_{\mathcal{L}}} \ltimes \mathcal{D}$ in (4.1).

Corollary 4.6. *Suppose $\mathrm{char}(\mathbb{F}) \neq 2$. Let $\mathcal{L}$ be a DUT ladder of size n. If $\Omega(\mathcal{L}) \neq \Omega(\mathcal{B})$, then every $f \in Der(M_{\mathcal{L}})$ maps $\mathcal{M}^{ij} \subseteq M_{\mathcal{L}}$ for $(i,j) \in \Omega(\mathcal{L})$ to a sum of $\mathcal{M}^{pq} \subseteq M_{\mathcal{L}}$ for some $(p,q) \in \Omega(\mathcal{L})$ such that $p \leq i \leq j \leq q$.*

Proof. The corollary is a direct consequence of Theorem 4.4(2) and (3). □

In general, Corollary 4.6 may not be true if $\mathcal{L}$ is not a DUT ladder which can be seen via the following example.

Example 4.7. *Suppose $\mathbb{F}$ is an arbitrary field. Let $\mathcal{L} = \{(1,2),(3,4)\}$ be a ladder of size 5. Then $\mathcal{L}$ is not DUT ladder. The Lie algebra $M_{\mathcal{L}}$ has the following block form:*

$$\left(\begin{array}{c|cc|cc} 0 & a_{12} & a_{13} & a_{14} & a_{15} \\ \hline 0 & 0 & 0 & a_{24} & a_{25} \\ 0 & 0 & 0 & a_{34} & a_{35} \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right), \quad a_{ij} \in \mathbb{F}.$$

So $M_{\mathcal{L}}$ has a basis $\mathcal{B} = \{E_{12}, E_{13}, E_{14}, E_{15}, E_{24}, E_{25}, E_{34}, E_{35}\}$ where E_{ij} denote the matrix in M_5 that has the only nonzero entry 1 in the (i,j) position. Given $a, b \in \mathbb{F}$, define $f \in$ $\mathrm{End}(M_{\mathcal{L}})$ by

$$f(E_{12}) := \left(\begin{array}{c|cc|cc} 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & b \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right), \quad f(E_{13}) := \left(\begin{array}{c|cc|cc} 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

and $f(E) = 0$ for all other matrices E in the basis $\mathcal{B}$. We prove that

$$f([E,E']) = [f(E),E'] + [E,f(E')] \quad \textit{for all} \quad E,E' \in \mathcal{B}, \tag{4.8}$$

so that f is a derivation of $M_{\mathcal{L}}$. On one hand, $[E,E'] \in \mathrm{span}\{E_{14}, E_{15}\}$ and thus $f([E,E']) =$ 0; on the other hand, in (4.8), $[f(E),E'] \neq 0$ or $[E,f(E')] \neq 0$ only when $\{E,E'\} =$

$\{E_{12}, E_{13}\}$, *for which the equality* (4.8) *is easily verified. Therefore,* $f \in \mathrm{Der}(M_{\mathcal{L}})$. *However,* f *maps* $\mathcal{M}^{12}$ *into* $\mathcal{M}^{23}$.

An important family of ladders is that of "1-step" ladders $\mathcal{L} = \{(i,j)\}$ of size n. Many 1-step ladders are DUT. The derivations of the Lie algebras $M_{\mathcal{L}}$ associated with these ladders over $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) \neq 2$ are explicitly characterized in the following example by using Theorem 4.4.

Example 4.8. *Suppose* $\mathrm{char}(\mathbb{F}) \neq 2$. *Let* $\mathcal{L} = \{(i,j)\}$ *be a* 1*-step ladder of size* n.

1. *If* $i < j$, *then* $M_{\mathcal{L}}$ *is abelian and it is straightforward to check that every endomorphism of* $M_{\mathcal{L}}$ *is a derivation.*

2. *If* $i = n$ *or* $j = 1$, *then there are three cases:*

   (a) *If* $i = n$ *and* $j = 1$, *then* $M_{\mathcal{L}} = M_n$, *and* $\mathrm{Der}(M_{\mathcal{L}}) = \mathrm{Der}(M_n)$.

   (b) *If* $i \neq n$ *and* $j = 1$, *then we can consider*

$$M_{\mathcal{L}} = \left\{ \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix} \in M_n : A_{11} \in \mathcal{M}_{11},\ A_{12} \in \mathcal{M}_{12} \right\}, \textit{and}$$

$$\mathrm{Der}(M_{\mathcal{L}}) = \mathrm{ad} \left\{ \begin{bmatrix} X_{11} & X_{12} \\ 0 & X_{22} \end{bmatrix} : X_{11} \in \mathcal{M}_{11},\ X_{12} \in \mathcal{M}_{12},\ X_{22} \in \mathcal{M}_{22} \right\} \Bigg|_{M_{\mathcal{L}}}.$$

   (c) *If* $i = n$ *and* $j \neq 1$, *then we can consider*

$$M_{\mathcal{L}} = \left\{ \begin{bmatrix} 0 & A_{12} \\ 0 & A_{22} \end{bmatrix} \in M_n : A_{12} \in \mathcal{M}_{12},\ A_{22} \in \mathcal{M}_{22} \right\}, \textit{and}$$

$$\mathrm{Der}(M_{\mathcal{L}}) = \mathrm{ad} \left\{ \begin{bmatrix} X_{11} & X_{12} \\ 0 & X_{22} \end{bmatrix} : X_{11} \in \mathcal{M}_{11},\ X_{12} \in \mathcal{M}_{12},\ X_{22} \in \mathcal{M}_{22} \right\} \Bigg|_{M_{\mathcal{L}}}.$$

3. If $n > i \geq j > 1$, then we can consider

$$M_{\mathcal{L}} = \left\{ \begin{bmatrix} 0 & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & 0 \end{bmatrix} \in M_n : A_{ij} \in \mathcal{M}_{ij} \right\}, \textit{and}$$

$$\mathrm{Der}(M_{\mathcal{L}}) = \mathrm{ad}\left\{ \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ 0 & X_{22} & X_{23} \\ 0 & 0 & X_{33} \end{bmatrix} \right\}\Bigg|_{M_{\mathcal{L}}} \ltimes \left\{ f_Y : f_Y(A) = \mathrm{tr}(A_{22}) \begin{bmatrix} 0 & 0 & Y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}$$

where $X_{ij} \in \mathcal{M}_{ij}$ and $Y \in \mathcal{M}_{13}$.

Note that the cases (2) and (3) are direct consequence of Theorem 4.4.

4.3 Proof of Theorem 4.4

We give a proof Theorem 4.4 here. In the rest of this section, we assume that $\mathbb{F}$ is a field with $\mathrm{char}(\mathbb{F}) \neq 2$, and $\mathcal{L}$ is a DUT ladder of size n. Recall that E^{ij}_{pq} denote the (p, q) standard matrix in $\mathcal{M}^{ij}$ (Definition 3.5).

Before proving the Theorem 4.4, we first present several results on the images $f(\mathcal{M}^{ij})$ for $f \in \mathrm{Der}(M_{\mathcal{L}})$ and $\mathcal{M}^{ij} \subseteq M_{\mathcal{L}}$. The next lemma concerns the f-image of some special matrices on the diagonal blocks of matrices of $M_{\mathcal{L}}$.

Lemma 4.9. *For $f \in \mathrm{Der}(M_{\mathcal{L}})$, the f-images of $I^{kk}, E^{kk}_{\ell,\ell} \in \mathcal{M}^{kk}$ satisfy that*

$$f(I^{kk}),\ f(E^{kk}_{\ell,\ell}) \in \sum_{i=1}^{k-1} \mathcal{M}^{ik} + \sum_{j=k+1}^{t} \mathcal{M}^{kj} + (\mathrm{Z}(M_{\mathcal{L}}) \cap M_{\mathcal{L}}) \tag{4.9}$$

where (by Lemma 4.3)

$$Z(M_{\mathcal{L}}) \cap M_{\mathcal{L}} = \begin{cases} \mathbb{F} I_n & \text{if } \Omega(\mathcal{L}) = \Omega(\mathcal{B}); \\ \mathcal{M}^{1t} & \text{if } (1,1) \notin \Omega(\mathcal{L}) \text{ and } (t,t) \notin \Omega(\mathcal{L}); \\ 0 & \text{otherwise.} \end{cases} \tag{4.10}$$

Proof. We prove (4.9) for $f(I^{kk})$ here, and the case of $f(E^{kk}_{\ell\ell})$ is similar.

1. First we investigate $f(I^{kk})_{jj}$. When $k < j$,

$$\begin{aligned} f(A^{kj})_{kj} &= f([I^{kk}, A^{kj}])_{kj} = [f(I^{kk}), A^{kj}]_{kj} + [I^{kk}, f(A^{kj})]_{kj} \\ &= f(I^{kk})_{kk}(A^{kj})_{kj} - (A^{kj})_{kj} f(I^{kk})_{jj} + f(A^{kj})_{kj} \end{aligned}$$

   Therefore

$$f(I^{kk})_{kk}(A^{kj})_{kj} = (A^{kj})_{kj} f(I^{kk})_{jj} \qquad \text{for} \quad A^{kj} \in \mathcal{M}^{kj}.$$

   Lemma 2.8 implies that $f(I^{kk})_{kk} = \lambda I_{kk}$ and $f(I^{kk})_{jj} = \lambda I_{jj}$ for a $\lambda \in \mathbb{F}$. The same equation holds for $k > j$. In the situation $\Omega(\mathcal{L}) \neq \Omega(\mathcal{B})$, there exists $(p,p) \notin \Omega(\mathcal{L})$, which forces $f(I^{kk})_{pp} = 0$ and thus $f(I^{kk})_{jj} = 0$ for all $j \in [t]$.

2. Next we prove that $f(I^{kk})_{ij} = 0$ for $i < j$, $i \neq k$, $j \neq k$, and $(i,j) \neq (1,t)$. Either $i > 1$ or $j < t$. Without loss of generality, suppose $j < t$ (similarly for $i > 1$). Then

$$f([I^{kk}, A^{jt}])_{it} = [f(I^{kk}), A^{jt}]_{it} + [I^{kk}, f(A^{jt})]_{it}. \tag{4.11}$$

   (a) If $k \neq t$, then (4.11) becomes $0 = f(I^{kk})_{ij}(A^{jt})_{jt}$ for any $A^{jt} \in \mathcal{M}^{jt}$. So $f(I^{kk})_{ij} = 0$.

   (b) If $k = t$, then (4.11) becomes

$$-f(A^{jt})_{it} = f(I^{kk})_{ij}(A^{jt})_{jt} - f(A^{jt})_{it}.$$

Again we get $0 = f(I^{kk})_{ij}(A^{jt})_{jt}$ and thus $f(I^{kk})_{ij} = 0$.

3. Finally, if $(1,1) \in \Omega(\mathcal{L})$ or $(t,t) \in \Omega(\mathcal{L})$, say $(1,1) \in \Omega(\mathcal{L})$, then for any $(k,k) \in \Omega(\mathcal{L})$ and $k \notin \{1,t\}$,

$$0 = f([I^{11}, I^{kk}])_{1t} = [f(I^{11}), I^{kk}]_{1t} + [I^{11}, f(I^{kk})]_{1t} = f(I^{kk})_{1t}.$$

Lemma 4.3 implies (4.10). Therefore, (4.9) is proved. □

For $(p,q) \in \Omega(\mathcal{L})$, we have

$$\mathcal{M}^{pq} \cap [M_{\mathcal{L}}, M_{\mathcal{L}}] = \begin{cases} \mathfrak{sl}^{pp}, & \text{if } p = q; \\ \mathcal{M}^{pq}, & \text{if } p < q. \end{cases}$$

Next we investigate the image of $f \in \mathrm{Der}(M_{\mathcal{L}})$ on each block in $[M_{\mathcal{L}}, M_{\mathcal{L}}]$.

Lemma 4.10. *For $f \in \mathrm{Der}(M_{\mathcal{L}})$, $(p,q) \in \Omega(\mathcal{L})$, and $A^{pq} \in \mathcal{M}^{pq} \cap [M_{\mathcal{L}}, M_{\mathcal{L}}]$,*

$$f(A^{pq}) \in \mathcal{M}^{pq} + \sum_{i=1}^{p-1} \mathcal{M}^{iq} + \sum_{j=q+1}^{t} \mathcal{M}^{pj}. \tag{4.12}$$

Proof. There are two cases for $(p,q) \in \Omega(\mathcal{L})$:

1. $p = q$: Then $\mathcal{M}^{pq} \cap [M_{\mathcal{L}}, M_{\mathcal{L}}] = \mathfrak{sl}^{pp} = [\mathfrak{sl}^{pp}, \mathfrak{sl}^{pp}]$. For $B^{pp}, C^{pp} \in \mathfrak{sl}^{pp}$,

$$f([B^{pp}, C^{pp}]) = [f(B^{pp}), C^{pp}] + [B^{pp}, f(C^{pp})]. \tag{4.13}$$

Since $f(B^{pp})$ and $f(C^{pp})$ are block upper triangular matrices, the nonzero (i,j) blocks of the right side of (4.13) satisfy that $p = i \leq j$ or $i \leq j = p$. Thus (4.12) holds in this case.

2. $p < q$: Then $\mathcal{M}^{pq} \cap [M_{\mathcal{L}}, M_{\mathcal{L}}] = \mathcal{M}^{pq}$. Let $q = p + k$ and we prove (4.12) by induction on k. For better display, we also use $\{\cdot\}^{ij}$ here to denote the embedding of $\mathcal{M}_{ij}$ to $\mathcal{M}^{ij} \subseteq M_n$.

   (a) $k = 1$: By Theorem 2.4, at least one of (p, p) and $(p+1, p+1)$ is in $\Omega(\mathcal{L})$. Without lost of generality, suppose $(p, p) \in \Omega(\mathcal{L})$. Then for $A^{p,p+1} \in \mathcal{M}^{p,p+1}$,

$$\begin{aligned} f(A^{p,p+1}) &= f([I^{pp}, A^{p,p+1}]) \\ &= [f(I^{pp}), A^{p,p+1}] + [I^{pp}, f(A^{p,p+1})] \\ &= \sum_{i=1}^{p-1} \left\{ f(I^{pp})_{ip} (A^{p,p+1})_{p,p+1} \right\}^{i,p+1} + \sum_{j=p+1}^{t} \left\{ f(A^{p,p+1})_{pj} \right\}^{pj} - \sum_{i=1}^{p-1} \left\{ f(A^{p,p+1})_{ip} \right\}^{ip} \end{aligned}$$

   where the last equality is given by Lemma 4.9. Therefore,

$$f(A^{p,p+1}) + \sum_{i=1}^{p-1} \left\{ f(A^{p,p+1})_{ip} \right\}^{ip} = \sum_{i=1}^{p-1} \left\{ f(I^{pp})_{ip} A_{p,p+1} \right\}^{i,p+1} + \sum_{j=p+1}^{t} \left\{ f(A^{p,p+1})_{pj} \right\}^{pj}$$

   On one hand, as $\mathrm{char}(\mathbb{F}) \neq 2$, the nonzero blocks on the left side of the above equality are those of $f(A^{p,p+1})$; on the other hand, the right side of this equality has nonzero (i, j) blocks only for $1 \leq i \leq p-1 < p+1 = j$ or $i = p < p+1 \leq j \leq t$. So $k = 1$ is done.

   (b) $k = \ell$: Suppose the statement is true for all $k < \ell$ where $\ell \geq 2$. Now $\mathcal{M}^{p,p+\ell} = [\mathcal{M}^{p,p+1}, \mathcal{M}^{p+1,p+\ell}]$, and

$$f([B^{p,p+1}, C^{p+1,p+\ell}]) = [f(B^{p,p+1}), C^{p+1,p+\ell}] + [B^{p,p+1}, f(C^{p+1,p+\ell})]$$

   By induction hypothesis, $f(B^{p,p+1})$ has nonzero blocks only on the p block row and the $(p+1)$ block column in the upper right direction of $(p, p+1)$ block, so that $[f(B^{p,p+1}), C^{p+1,p+\ell}]$ has nonzero blocks only on the $(p + \ell)$ block column above the $(p, p+\ell)$ block and on the $(p, p+\ell)$ block. Similarly, $[B^{p,p+1}, f(C^{p+1,p+\ell})]$ has nonzero blocks only on the p block row to the right of the $(p, p+\ell)$ block and on the $(p, p+\ell)$ block. So (4.12) is true for $k = \ell$.

(c) Overall, (4.12) is verified for all the cases. □

Now we are ready to prove Theorem 4.4. The basic idea is to explore what remain in $\mathrm{Der}(M_{\mathcal{L}})$ after factoring out $\mathrm{ad}\,(\mathrm{N}(M_{\mathcal{L}}))\,|_{M_{\mathcal{L}}} = \mathrm{ad}\,(M_{\mathcal{B}})\,|_{M_{\mathcal{L}}}$. Given $X \in M_{\mathcal{B}}$, $A \in M_{\mathcal{L}}$,

$$\mathrm{ad}\, X(A) = \sum_{(p,q)\in\Omega(\mathcal{B})} \sum_{(i,j)\in\Omega(\mathcal{L})} [X^{pq}, A^{ij}].$$

A summand $[X^{pq}, A^{ij}]$ is nonzero only if $i = q$ or $p = j$. In other words, $\mathrm{ad}\, X^{pq}$ has nonzero action only on the q block row or the p block column of A. It motivates us to investigate the relationship of $f(A^{ip})$ and $f(A^{qj})$ for given $f \in \mathrm{Der}(M_{\mathcal{L}})$ and $1 \le p \le q \le t$.

Proof of Theorem 4.4.

1. If $\Omega(\mathcal{L}) = \Omega(\mathcal{B})$ i.e. $M_{\mathcal{L}}$ is the Lie algebra of block upper triangular matrices of M_n, by [27, Theorem 4.1] and the assumption $\mathrm{char}(\mathbb{F}) \neq 2$, every $f \in \mathrm{Der}(M_{\mathcal{L}})$ corresponds to $X \in M_{\mathcal{L}}$ and $\mu \in {M_{\mathcal{L}}}^*$ such that

$$f(A) = \mathrm{ad}\, X(A) + \mu(A) I_n.$$

Then $\mu([M_{\mathcal{L}}, M_{\mathcal{L}}]) = 0$ by derivation property. All $\mathcal{M}^{ij}$ with $i < j$ are in $[M_{\mathcal{L}}, M_{\mathcal{L}}]$. So $\mu(A) = \sum_{k\in[t]} \mu(A^{kk})$. Recall that the (p, q) standard matrix in $\mathcal{M}^{ij}$ is denoted by E^{ij}_{pq}. Given $k \in [t]$, we have $A^{kk} - \mathrm{tr}(A^{kk})E^{kk}_{11} \in [M_{\mathcal{L}}, M_{\mathcal{L}}]$ so that

$$\mu\left(A^{kk}\right) = \mathrm{tr}(A^{kk})\mu\left(E^{kk}_{11}\right).$$

Denote $c_k = \mu\left(E_{11}^{[kk]}\right)$. Then

$$f(A) = \operatorname{ad} X(A) + \left(\sum_{k\in[t]} c_k \operatorname{tr}(A^{kk})\right) I_n.$$

This is (4.4). The formulae (4.1) and (4.2) for $\Omega(\mathcal{L}) = \Omega(\mathcal{B})$ immediately follow.

2. In the remaining of the proof, we assume $\Omega(\mathcal{L}) \neq \Omega(\mathcal{B})$, so that matrices of $M_{\mathcal{L}}$ have at least one zero submatrix on diagonal blocks.

   Suppose $(k,k) \in \Omega(\mathcal{L})$. For any $A, B \in \mathcal{M}_{kk}$, $A^{kk}, B^{kk} \in \mathcal{M}^{kk}$ and

$$f([A^{kk}, B^{kk}])_{kk} = [f(A^{kk})_{kk}, (B^{kk})_{kk}] + [A^{kk}, f(B^{kk})_{kk}].$$

   So $f(\cdot^{kk})_{kk} : \mathcal{M}_{kk} \to \mathcal{M}_{kk}$, $(A \mapsto A^{kk} \mapsto f(A^{kk})_{kk})$, is a derivation of $\mathcal{M}_{kk}$. Since $\operatorname{char}(\mathbb{F}) \neq 2$, according to [27, Corollary 5.1] [1], there is $X_{kk} \in \mathcal{M}_{kk}$ and $\lambda_k \in \mathbb{F}$ such that

$$f(A^{kk})_{kk} = [X_{kk}, (A^{kk})_{kk}] + \lambda_k \operatorname{tr}(A^{kk}) I_{kk} \qquad \text{for} \quad A^{kk} \in \mathcal{M}^{kk}.$$

   We prove that $\lambda_k = 0$ for all k. Recall that E_{pq}^{ij} denotes the (p,q) standard matrix in $\mathcal{M}^{ij}$. If we set up $A^{kk} = E_{11}^{kk}$. On one hand, the $(1,1)$ entry of

$$f(E_{11}^{kk})_{kk} = [X_{kk}, (E_{11}^{kk})_{kk}] + \lambda_k I_{kk}$$

   equals λ_k. On the other hand, for any $\ell \in [t]$ with $\ell > k$,

$$\begin{aligned} f(E_{11}^{k\ell})_{k\ell} &= f([E_{11}^{kk}, E_{11}^{k\ell}])_{k\ell} = [f(E_{11}^{kk}), E_{11}^{k\ell}]_{k\ell} + [E_{11}^{kk}, f(E_{11}^{k\ell})]_{k\ell} \\ &= f(E_{11}^{kk})_{kk}(E_{11}^{k\ell})_{k\ell} - (E_{11}^{k\ell})_{kl} f(E_{11}^{kk})_{\ell\ell} + (E_{11}^{kk})_{kk} f(E_{11}^{k\ell})_{k\ell}. \end{aligned}$$

[1] $\operatorname{Der}(\mathfrak{gl}(m,\mathbb{F}))$ has additional elements when $\operatorname{char}(\mathbb{F}) = 2$ and $m = 2$ [27, Corollary 5.1].

Therefore,

$$f(E_{11}^{kk})_{kk}(E_{11}^{k\ell})_{k\ell} = (I_{kk} - (E_{11}^{kk})_{kk})f(E_{11}^{k\ell})_{k\ell} + (E_{11}^{k\ell})_{k\ell}f(E_{11}^{kk})_{\ell\ell}.$$

Comparing the $(1,1)$ entry of both sides, we see that the $(1,1)$ entries of $f(E_{11}^{kk})_{kk}$ and $f(E_{11}^{kk})_{\ell\ell}$ are equal. The same result holds for $\ell < k$. By assumption $\Omega(\mathcal{L}) \neq \Omega(\mathcal{B})$, there exists $(\ell,\ell) \notin \Omega(\mathcal{L})$, where $f(E_{11}^{kk})_{\ell\ell} = 0$. Hence $\lambda_k = 0$. Overall, for any $(k,k) \in \Omega(\mathcal{L})$, there exists $X_{kk} \in \mathcal{M}_{kk}$ such that

$$f(A^{kk})_{kk} = [X_{kk}, (A^{kk})_{kk}] \qquad \text{for all } \; A \in \mathcal{M}_{kk}.$$

3. Given $p, q \in [t]$ and $p < q$, we claim that there exists $X_{pq} \in \mathcal{M}_{pq}$ such that

$$f(A^{ip})_{iq} = \operatorname{ad} X_{pq}(A^{ip})_{ip}, \qquad \text{for any } \; (i,p) \in \Omega(\mathcal{L}), \quad \text{and} \tag{4.14}$$

$$f(A^{qj})_{pj} = \operatorname{ad} X_{pq}(A^{qj})_{qj}, \qquad \text{for any } \; (q,j) \in \Omega(\mathcal{L}). \tag{4.15}$$

There are several situations:

(a) Suppose $(q,j) = (t,t) \in \Omega(\mathcal{L})$. For any $A, B \in \mathcal{M}_{tt}$, $A^{tt}, B^{tt} \in \mathcal{M}^{tt}$ and

$$f([A^{tt}, B^{tt}])_{pt} = [f(A^{tt}), B^{tt}]_{pt} + [A^{tt}, f(B^{tt})]_{pt} = f(A^{tt})_{pt}(B^{tt})_{tt} - f(B^{tt})_{pt}A^{tt}_{tt}.$$

Recall that I^{kk} denote the matrix of $\mathcal{M}^{kk}$ with the identity matrix I_{kk} in the (k,k) block. Set $B^{tt} = I^{tt}$. Then $f(A^{tt})_{pt} = f(I^{tt})_{pt}(A^{tt})_{tt}$ for $A^{tt} \in \mathcal{M}^{tt}$. Denote $X_{pt} := f(I^{tt})_{pt} \in \mathcal{M}_{pt}$. We have $f(A^{tt})_{pt} = X_{pt}(A^{tt})_{tt}$ and so

$$f(A^{tt})_{pt} = \operatorname{ad} X_{pt}(A^{tt})_{tt} \qquad \text{for all} \quad A = (A^{tt})_{tt} \in \mathcal{M}_{tt}.$$

(b) Suppose $(i,p) = (1,1) \in \Omega(\mathcal{L})$. Similarly, let $Y_{1q} := -f(I^{11})_{1q} \in \mathcal{M}_{1q}$ then

$$f(A^{11})_{1q} = -(A^{11})_{11}Y_{1q} = \operatorname{ad} Y_{1q}(A^{11})_{11} \qquad \text{for all} \quad A = (A^{11})_{11} \in \mathcal{M}_{11}.$$

(c) Suppose $(q,j) \in \Omega(\mathcal{L}) \setminus \{(t,t)\}$. Either $q < t$ or $j < t$. Without loss of generality, suppose $j < t$. Let $j' := j+1$. Then $(j,j'), (q,j'), (p,j), (p,j') \in \Omega(\mathcal{L})$, and $\mathcal{M}^{qj'} = \mathcal{M}^{qj}\mathcal{M}^{jj'} = [\mathcal{M}^{qj}, \mathcal{M}^{jj'}]$. For $A \in \mathcal{M}_{qj}$, $A^{qj} \in \mathcal{M}^{qj}$; $B \in \mathcal{M}_{jj'}$, $B^{jj'} \in \mathcal{M}^{jj'}$, and

$$f(A^{qj}B^{jj'})_{pj'} = f([A^{qj}, B^{jj'}])_{pj'} = [f(A^{qj}), B^{jj'}]_{pj'} + [A^{qj}, f(B^{jj'})]_{pj'} = f(A^{qj})_{pj}(B^{jj'})_{jj'}.$$

Applying Lemma 2.7 to $\phi : \mathcal{M}_{qj'} \to \mathcal{M}_{pj'}$ defined by $\phi(C) := f(C^{qj'})_{pj'}$ and $\varphi : \mathcal{M}_{qj} \to \mathcal{M}_{pj}$ defined by $\varphi(D) := f(D^{qj})_{pj}$, we can find $X_{pq} \in \mathcal{M}_{pq}$ such that $f(A^{qj})_{pj} = X_{pq}(A^{qj})_{qj}$ for $A = (A^{qj})_{qj} \in \mathcal{M}_{qj}$, and $f(F^{qj'})_{pj'} = X_{pq}(F^{qj'})_{qj'}$ for $F = (F^{qj'})_{qj'} \in \mathcal{M}_{qj'}$. In particular, X_{pq} is independent of j. So

$$f(A^{qj})_{pj} = \operatorname{ad} X_{pq}(A^{qj})_{qj} \quad \text{for} \quad A = (A^{qj})_{qj} \in \mathcal{M}_{qj}.$$

(d) Suppose $(i,p) \in \Omega(\mathcal{L}) \setminus \{(1,1)\}$. Either $i > 1$ or $p > 1$. Without loss of generality, suppose $i > 1$ (similarly for $p > 1$). Let $i' := i-1$. Then $(i',i), (i',p), (i,q), (i',q) \in \Omega(\mathcal{L})$, and $\mathcal{M}^{i'p} = \mathcal{M}^{i'i}\mathcal{M}^{ip} = [\mathcal{M}^{i'i}, \mathcal{M}^{ip}]$. For $B \in \mathcal{M}_{i'i}$, $B^{i'i} \in \mathcal{M}^{i'i}$; $A \in \mathcal{M}_{ip}$, $A^{ip} \in \mathcal{M}^{ip}$, and

$$f(B^{i'i}A^{ip})_{i'q} = f([B^{i'i}, A^{ip}])_{i'q} = [f(B^{i'i}), A^{ip}]_{i'q} + [B^{i'i}, f(A^{ip})]_{i'q} = (B^{i'i})_{i'i}f(A^{ip})_{iq}.$$

Applying Lemma 2.6 to $\phi : \mathcal{M}_{i'p} \to \mathcal{M}_{i'q}$ defined by $\phi(C) := f(C^{i'p})_{i'q}$ and $\varphi : \mathcal{M}_{ip} \to \mathcal{M}_{iq}$ defined by $\varphi(D) := f(D^{ip})_{iq}$, we can find $Z_{pq} \in \mathcal{M}_{pq}$ such that $f(A^{ip})_{iq} = (A^{ip})_{ip}Z_{pq}$ for $A = (A^{ip})_{ip} \in \mathcal{M}^{ip}$, and $f(F^{i'p})_{i'q} = (F^{i'p})_{i'p}Z_{pq}$

for $F = (F^{i'p})_{i'p} \in \mathcal{M}_{i'p}$. Define $Y_{pq} := -Z_{pq} \in \mathcal{M}_{pq}$. In particular, Y_{pq} is independent of i. So

$$f(A^{ip})_{iq} = \operatorname{ad} Y_{pq}(A^{ip})_{ip}, \quad \text{for} \quad A = (A^{ip})_{ip} \in \mathcal{M}_{ip}.$$

(e) Given any $(i,p),(q,j) \in \Omega(\mathcal{L})$, we have $[A^{ip}, A^{qj}] = 0$, so that

$$\begin{aligned} 0 &= f([A^{ip}, A^{qj}])_{ij} = [f(A^{ip}), A^{qj}]_{ij} + [A^{ip}, f(A^{qj})]_{ij} \\ &= f(A^{ip})_{iq}(A^{qj})_{qj} + (A^{ip})_{ip} f(A^{qj})_{pj} \\ &= -(A^{ip})_{ip} Y_{pq}(A^{qj})_{qj} + (A^{ip})_{ip} X_{pq}(A^{qj})_{qj}. \end{aligned}$$

Therefore, $X_{pq} = Y_{pq}$.

Overall, we successfully find $X_{pq} \in \mathcal{M}_{pq}$ that satisfies (4.14) and (4.15). Let $X^{pq} = (X_{pq})^{pq} \in \mathcal{M}^{pq}$.

4. From 2 and 3, we can construct a matrix in $M_{\mathcal{L}}$:

$$X_0 := \sum_{(k,k)\in\Omega(\mathcal{L})} X^{kk} + \sum_{1\le p<q\le t} X^{pq}.$$

Define the derivation

$$f_1 := f - \operatorname{ad} X_0. \tag{4.16}$$

Then for any $(k,k) \in \Omega(\mathcal{L})$, $1 \le p < q \le t$, and $(i,p),(q,j) \in \Omega(\mathcal{L})$, we have

$$f_1(\mathcal{M}^{kk})_{kk} = 0, \qquad f_1(\mathcal{M}^{ip})_{iq} = 0, \qquad f_1(\mathcal{M}^{qj})_{pj} = 0.$$

By Lemmas 4.9 and 4.10, f_1 belong to the following set:

$$\begin{aligned} D_0 := \{g \in \mathrm{Der}(M_{\mathcal{L}}) \quad | \quad & g(\mathcal{M}^{kk}) \in \mathrm{Z}(M_{\mathcal{L}}) \cap M_{\mathcal{L}} \text{ for } (k,k) \in \Omega(\mathcal{L}), \\ & g(\mathcal{M}^{pq}) \subseteq \mathcal{M}^{pq} \text{ for } 1 \leq p < q \leq t\}. \end{aligned} \tag{4.17}$$

It remains to describe the subalgebra D_0 of $\mathrm{Der}(M_{\mathcal{L}})$.

5. Given $f' \in \mathrm{Der}(M_{\mathcal{L}})$, $p, q \in [t]$ with $p < q$, and $k \in [t]$ with $p \leq k \leq q$, Lemmas 4.9 and 4.10 imply that

$$\begin{aligned} f'(A^{pk}A^{kq})_{pq} &= f'([A^{pk}, A^{kq}])_{pq} = [f'(A^{pk}), A^{kq}]_{pq} + [A^{pk}, f'(A^{kq})]_{pq} \\ &= f'(A^{pk})_{pk}(A^{kq})_{kq} + (A^{pk})_{pk} f'(A^{kq})_{kq}. \end{aligned} \tag{4.18}$$

This formula will be frequently used in the following computations.

6. We prove the following claim regarding f_1 defined in (4.16): there exist $Y^{ii} \in \mathcal{M}^{ii}$ for $i \in [t]$, such that for each $k \in [t]$, the derivation $f_1^{(k)} := \left(f_1 - \sum_{i=1}^{k} \mathrm{ad}\, Y^{ii}\right)\Big|_{M_{\mathcal{L}}}$ has the images

$$\begin{cases} f_1^{(k)}(\mathcal{M}^{qq}) = f_1(\mathcal{M}^{qq}), & \text{for } (q,q) \in \Omega(\mathcal{L}),\ q \leq k; \\ f_1^{(k)}(\mathcal{M}^{pq}) = 0, & \text{for } (p,q) \in \Omega(\mathcal{L}),\ 1 \leq p < q \leq k. \end{cases} \tag{4.19}$$

Moreover, $Y^{ii} \in \mathbb{F}I^{ii}$ whenever $(i,i) \in \Omega(\mathcal{L})$.

The proof is proceeded by induction on k:

(a) $k = 1$ and 2: There are two subcases:

- If $(1,1) \in \Omega(\mathcal{L})$, we let $Y^{11} = 0 \in \mathcal{M}^{11}$ so that $f_1^{(1)} = f_1$. By (4.18),

$$f_1^{(1)}(A^{11}A^{12})_{12} = f_1^{(1)}(A^{11})_{11}(A^{12})_{12} + (A^{11})_{11} f_1^{(1)}(A^{12})_{12} = (A^{11})_{11} f_1^{(1)}(A^{12})_{12}.$$

By Lemma 2.6, there exists $Z^{22} \in \mathcal{M}^{22}$, such that $f_1^{(1)}(A^{12})_{12} = (A^{12}Z^{22})_{12}$. Define $Y^{22} = -Z^{22}$. Then $f_1^{(1)}(A^{12})_{12} = -(A^{12}Y^{22})_{12}$. Let $f_1^{(2)} = f_1^{(1)} - \operatorname{ad} Y^{22}$. Then $f_1^{(2)}(A^{12}) = 0$. If furthermore $(2,2) \in \Omega(\mathcal{L})$, then by (4.18),

$$0 = f_1^{(2)}(A^{12}A^{22})_{12} = f_1^{(2)}(A^{12})_{12}(A^{22})_{22} + (A^{12})_{12}f_1^{(2)}(A^{22})_{22} = (A^{12})_{12}f_1^{(2)}(A^{22})_{22}.$$

Thus

$$0 = f_1^{(2)}(A^{22})_{22} = f_1(A^{22})_{22} - ([Y^{22}, A^{22}])_{22} = -([Y^{22}, A^{22}])_{22}.$$

So $Y^{22} \in \mathbb{F}I^{22}$ and $f_1^{(2)}(A^{22}) = f_1(A^{22})$. The claim holds for $k = 1, 2$.

- If $(1,1) \notin \Omega(\mathcal{L})$, then $(2,2) \in \Omega(\mathcal{L})$ by Theorem 2.4. By (4.18),

$$f_1(A^{12}A^{22})_{12} = f_1(A^{12})_{12}(A^{22})_{22} + (A^{12})_{12}f_1(A^{22})_{22} = f_1(A^{12})_{12}(A^{22})_{22}.$$

By Lemma 2.7, there exists $Y^{11} \in \mathcal{M}^{11}$ such that $f_1(A^{12})_{12} = (Y^{11}A^{12})_{12}$. Let $Y^{22} = 0 \in \mathcal{M}^{22}$, $f_1^{(1)} = f_1 - \operatorname{ad} Y^{11}$, and $f_1^{(2)} = f_1^{(1)} - \operatorname{ad} Y^{22}$. Then the claim holds for $k = 1, 2$.

(b) $k = \ell > 2$: Suppose the claim holds for $k = \ell - 1 \geq 2$. So there exist $Y^{11} \in \mathcal{M}^{11}, \cdots, Y^{\ell-1,\ell-1} \in \mathcal{M}^{\ell-1,\ell-1}$, such that $f_1^{(\ell-1)} := f_1 - \sum_{i=1}^{\ell-1} \operatorname{ad} Y^{ii}$ satisfies (4.19) for $k = \ell - 1$. Clearly $f_1^{(\ell-1)} \in D_0$. For any $p \in [\ell - 2]$, by (4.18),

$$\begin{aligned} f_1^{(\ell-1)}(A^{p,\ell-1}A^{\ell-1,\ell})_{p\ell} &= f_1^{(\ell-1)}(A^{p,\ell-1})_{p,\ell-1}(A^{\ell-1,\ell})_{\ell-1,\ell} + (A^{p,\ell-1})_{p,\ell-1}f_1^{(\ell-1)}(A^{\ell-1,\ell})_{\ell-1,\ell} \\ &= (A^{p,\ell-1})_{p,\ell-1}f_1^{(\ell-1)}(A^{\ell-1,\ell})_{\ell-1,\ell}. \end{aligned}$$

By Lemma 2.6, there exists $Z^{\ell\ell} \in \mathcal{M}^{\ell\ell}$, such that

$$f_1^{(\ell-1)}(A^{p\ell})_{p\ell} = (A^{p\ell}Z^{\ell\ell})_{p\ell} \qquad \text{for all} \quad p \in [\ell - 1].$$

Define $Y^{\ell\ell} = -Z^{\ell\ell}$. Let $f_1^{(\ell)} := f_1^{(\ell-1)} - \operatorname{ad} Y^{\ell\ell}$. Then $f_1^{(\ell)}(A^{p\ell}) = 0$ for $p \in [\ell - 1]$. In the case $(\ell, \ell) \in \Omega(\mathcal{L})$, by (4.18),

$$\begin{aligned} 0 = f_1^{(\ell)}(A^{\ell-1,\ell}A^{\ell\ell})_{\ell-1,\ell} &= f_1^{(\ell)}(A^{\ell-1,\ell})_{\ell-1,\ell}(A^{\ell\ell})_{\ell\ell} + (A^{\ell-1,\ell})_{\ell-1,\ell} f_1^{(\ell)}(A^{\ell\ell})_{\ell\ell} \\ &= (A^{\ell-1,\ell})_{\ell-1,\ell} f_1^{(\ell)}(A^{\ell\ell})_{\ell\ell}. \end{aligned}$$

So

$$0 = f_1^{(\ell)}(A^{\ell\ell})_{\ell\ell} = \left(f_1 - \sum_{i=1}^{\ell} \operatorname{ad} Y^{ii} \right) (A^{\ell\ell})_{\ell\ell} = -([Y^{\ell\ell}, A^{\ell\ell}])_{\ell\ell}.$$

Thus $Y^{\ell\ell} \in \mathbb{F}I^{\ell\ell}$ and $f_1^{(\ell)}(A^{\ell\ell}) = f_1(A^{\ell\ell})$. The claim is proved for $k = \ell$.

(c) Overall, the claim holds for every $k \in [t]$.

7. The derivation $f_1^{(t)} = f_1 - \sum_{i=1}^{t} \operatorname{ad} Y^{ii}$ sends each $\mathcal{M}^{kk}$ for $(k,k) \in \Omega(\mathcal{L})$ to $\mathrm{Z}(M_{\mathcal{L}}) \cap M_{\mathcal{L}}$, and $\mathcal{M}^{pq}$ for $1 \leq p < q \leq t$ to 0. For any $A, B \in M_{\mathcal{L}}$,

$$f_1^{(t)}([A,B]) = [f_1^{(t)}(A), B] + [A, f_1^{(t)}(B)] = 0.$$

Therefore, $f_1^{(t)} \in \mathcal{D}$ for $\mathcal{D}$ defined in (4.3). Every $\phi \in \mathcal{D}$ satisfies $\phi([A,B]) = 0 = [\phi(A), B] + [A, \phi(B)]$ for $A, B \in M_{\mathcal{L}}$. Thus $\mathcal{D} \subseteq \operatorname{Der} M_{\mathcal{L}}$. So far we have

$$\operatorname{Der}(M_{\mathcal{L}}) = (\operatorname{ad} M_{\mathcal{B}})\,|_{M_{\mathcal{L}}} + \mathcal{D}.$$

If $(1,1) \in \Omega(\mathcal{L})$ or $(t,t) \in \Omega(\mathcal{L})$, then $\mathrm{Z}(M_{\mathcal{L}}) \cap M_{\mathcal{L}} = 0$ implies that $\mathcal{D} = 0$. We get (4.5).

If neither $(1,1)$ nor (t,t) is in $\Omega(\mathcal{L})$, then $\mathrm{Z}(M_{\mathcal{L}}) \cap M_{\mathcal{L}} = \mathcal{M}^{1t}$. The set $\{E_{11}^{kk} \mid (k,k) \in \Omega(\mathcal{L})\}$ spans a subalgebra complement to $[M_{\mathcal{L}}, M_{\mathcal{L}}]$ in $M_{\mathcal{L}}$. By a similar calculation as in 1, one can show that for any $\phi \in \mathcal{D}$ and $A \in M_{\mathcal{L}}$,

$$\phi(A) = \sum_{(k,k)\in\Omega(\mathcal{L})} \phi(A^{kk}) = \sum_{(k,k)\in\Omega(\mathcal{L})} \mathrm{tr}(A^{kk})\phi(E_{11}^{kk}).$$

Denote $Y^{1tk} := \phi(E_{11}^{kk}) \in \mathcal{M}^{1t}$ for $(k,k) \in \Omega(\mathcal{L})$. We get (4.6).

In all the cases, the equations (4.5) and (4.6) as well as (4.4) imply (4.1) and (4.2) by a easy computation. So Theorem 4.4 is completely proved. □

Chapter 5

Derivations of the Lie algebra $[M_{\mathcal{L}}, M_{\mathcal{L}}]$ associated with a strongly dominant upper triangular ladder $\mathcal{L}$

Recall that a ladder $\mathcal{L} = \{(i_1, j_1), \cdots, (i_s, j_s)\}$ of size n is called strongly dominant upper triangular (SDUT) if $j_\ell < i_\ell < j_{\ell+1}$ for $\ell \in [s-1]$ (Definition 2.2). In this chapter, we give an explicit description of the derivations of the Lie algebra $[M_{\mathcal{L}}, M_{\mathcal{L}}]$ for an SDUT ladder $\mathcal{L}$ of size n over a field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) \neq 2, 3$. The Lie algebra $[M_{\mathcal{L}}, M_{\mathcal{L}}]$ consists of matrices of $M_{\mathcal{L}}$ that have a submatrix of zero trace on every diagonal block. In the rest of this chapter, we fix the $t \times t$ block matrix form of matrices in M_n determined by $\mathcal{L}$ through (2.1). Let us make the following notation.

Definition 5.1. *Given an upper triangular ladder $\mathcal{L}$ of size n, let $M^0_{\mathcal{L}}$ denote the Lie subalgebra of $M_{\mathcal{L}}$ consisting of matrices that have zero trace on every diagonal block.*

Any derivation of a Lie algebra $\mathfrak{g}$ preserves derived series of $\mathfrak{g}$. Given an upper triangular ladder $\mathcal{L}$, the derived series of $M_{\mathcal{L}}$ is

$$M_{\mathcal{L}} = M^{(0)}_{\mathcal{L}} \trianglerighteq M^{(1)}_{\mathcal{L}} \trianglerighteq M^{(2)}_{\mathcal{L}} \trianglerighteq \cdots, \qquad M^{(k)}_{\mathcal{L}} := [M^{(k-1)}_{\mathcal{L}}, M^{(k-1)}_{\mathcal{L}}].$$

The following observations are straightforward in the view point of block matrix form:

1. When $k \geq 1$, each $M^{(k)}_{\mathcal{L}} = M^0_{\mathcal{L}_k}$ for some upper triangular ladder $\mathcal{L}_k$ contained in $\mathcal{L}$.
2. The Lie algebra $M_{\mathcal{L}}$ is non-solvable if and only if its derived series terminates at a nonzero $M^0_{\mathcal{L}_*}$, where $\mathcal{L}_*$ is the maximal SDUT ladder contained in $\mathcal{L}$. Precisely,

$$\mathcal{L}_* = \{(i_\ell, j_\ell) \in \mathcal{L} \mid i_\ell > j_\ell\}.$$

Example 5.2. *In M_8, the forms of $M_{\mathcal{L}}$, $M_{\mathcal{L}*}$, and $M^0_{\mathcal{L}*}$ associate with an upper triangular ladder $\mathcal{L} = \{(2,1),(3,3),(4,4),(7,6)\}$ of size 8 are illustrated below. In particular, we see that $\mathcal{L}_* = \{(2,1),(7,6)\}$ is an SDUT of size 8.*

$M_{\mathcal{L}}$	$M_{\mathcal{L}*}$	$M^0_{\mathcal{L}*}$
$\begin{pmatrix} * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ & & * & * & * & * & * & * \\ & & & * & * & * & * & * \\ & & & & 0 & * & * & * \\ & & & & & * & * & * \\ & & & & & * & * & * \\ & & & & & & & * \end{pmatrix}$	$\begin{pmatrix} * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ & & 0 & 0 & 0 & * & * & * \\ & & & 0 & 0 & * & * & * \\ & & & & 0 & * & * & * \\ & & & & & * & * & * \\ & & & & & * & * & * \\ & & & & & & & 0 \end{pmatrix}$	$\begin{pmatrix} a & * & * & * & * & * & * & * \\ * & -a & * & * & * & * & * & * \\ & & 0 & 0 & 0 & * & * & * \\ & & & 0 & 0 & * & * & * \\ & & & & 0 & * & * & * \\ & & & & & b & * & * \\ & & & & & * & -b & * \\ & & & & & & & 0 \end{pmatrix}$

The above observations indicate that the structure of $\mathrm{Der}(M^0_{\mathcal{L}})$ associated with an SDUT ladder $\mathcal{L}$ will be useful in finding the derivations of non-solvable Lie algebras of upper triangular ladder matrices.

5.1 Some linear transformations between matrix spaces $\mathfrak{sl}_n$ and $M_{m,n}$

In this section, we describe some linear transformations between matrix spaces $\mathfrak{sl}_n$ and $M_{m,n}$ that satisfy some special properties. Let $E^{(mn)}_{pq} \in M_{m,n}$ denote the the matrix with the only nonzero entry 1 in the (p,q) position. We first give two lemmas similar to Lemmas 2.6 and 2.7.

Lemma 5.3. *Suppose $n \geq 2$. If linear transformations $\phi : M_{m,n} \to M_{m,q}$ and $\varphi : \mathfrak{sl}_n \to M_{n,q}$ satisfy that*

$$\phi(AB) = A\varphi(B) \qquad \text{for all} \quad A \in M_{m,n}, \quad B \in \mathfrak{sl}_n, \tag{5.1}$$

then there is $X \in M_{n,q}$ such that $\phi(C) = CX$ for $C \in M_{m,n}$ and $\varphi(D) = DX$ for $D \in \mathfrak{sl}_n$.

Lemma 5.3 is very similar to a special case ($p = n$) of Lemma 2.6, except that the domain of φ is $\mathfrak{sl}_n$ instead of $M_{n,n} = M_n$. The proof of Lemma 5.3 (omitted) is totally parallel to that of Lemma 2.6, using the key fact that $\{E_{1j}^{(mn)}B \mid j \in [n], B \in \mathfrak{sl}_n\}$ still spans the first row space of $M_{m,n}$. Similarly, we have the following lemma.

Lemma 5.4. *Suppose $n \geq 2$. If linear transformations $\phi : M_{n,q} \to M_{m,q}$ and $\varphi : \mathfrak{sl}_n \to M_{m,n}$ satisfy that*

$$\phi(BA) = \varphi(B)A \qquad \text{for all} \quad A \in M_{n,q},\ B \in \mathfrak{sl}_n, \tag{5.2}$$

then there is $X \in M_{m,n}$ such that $\phi(C) = XC$ for $C \in M_{n,q}$ and $\varphi(D) = XD$ for $D \in \mathfrak{sl}_n$.

Next we give two lemmas related to the Lie bracket operation.

Lemma 5.5. *Suppose* $\operatorname{char}(\mathbb{F}) \neq 2, 3$. *If a linear transformation $\phi : \mathfrak{sl}_n \to M_{n,m}$ satisfies that*

$$\phi(AB - BA) = A\phi(B) - B\phi(A), \quad \text{for all } A, B \in \mathfrak{sl}_n, \tag{5.3}$$

then there is $X \in M_{n,m}$ such that $\phi(C) = CX$ for $C \in \mathfrak{sl}_n$.

Proof. The case $n = 1$ is obviously true. We now assume that $n \geq 2$. Let $\{E_{ij} \mid i, j \in [n]\}$ be the standard basis of M_n. Then $\mathfrak{sl}_n$ has the standard basis $\{E_{ij} \mid i, j \in [n], i \neq j\} \cup \{H_i \mid i \in [n-1]\}$, where $H_i := E_{ii} - E_{i+1,i+1}$. We have $M_n = \mathfrak{sl}_n \oplus \mathbb{F}E_{11}$.

First we prove that the only possibly nonzero row of $\phi(E_{ij})$ ($i \neq j$) is the i-th row, and the only possibly nonzero rows of $\phi(H_i) = \phi(E_{ii} - E_{i+1,i+1})$ ($i \in [n-1]$) are the i-th and the $(i+1)$-th rows.

Suppose $i, j \in [n]$ with $i < j$. Denote $E := E_{ij}$, $F := E_{ji}$, and $H := E_{ii} - E_{jj}$. Then

$$2\phi(E) = \phi([H, E]) = H\phi(E) - E\phi(H) \quad \Longrightarrow \quad (2I_n - H)\phi(E) = -E\phi(H).$$

When $\operatorname{char}(\mathbb{F}) \neq 2, 3$, the matrix $2I_n - H = \operatorname{diag}(2, 2, \cdots, \underset{i}{1}, \cdots, \underset{j}{3}, \cdots, 2)$ is invertible and diagonal. The matrix $(2I_n - H)^{-1}$ is again diagonal with 1 as the i-th diagonal entry. So we have

$$\phi(E) = -(2I_n - H)^{-1} E_{ij} \phi(H) = -E_{ij}\phi(H).$$

In particular, $\phi(E_{ij}) = \phi(E)$ has zeros outside of the i-th row. Similar argument works for E_{ji}.

For $H_i = E_{ii} - E_{i+1,i+1}$, we have

$$\phi(H_i) = \phi([E_{i,i+1}, E_{i+1,i}]) = E_{i,i+1}\phi(E_{i+1,i}) - E_{i+1,i}\phi(E_{i,i+1}).$$

Therefore, $\phi(H_i)$ has zeros outside of the i-th and the $(i+1)$-th rows.

Next we extend the map ϕ from the domain $\mathfrak{sl}_n$ to the domain M_n such that property (5.3) still hold in M_n. Define the linear transformation $\phi^+ : M_n \to M_{n,m}$ as follow:

$$\begin{cases} \phi^+(A) = \phi(A), & \text{for } A \in \mathfrak{sl}_n; \\ \phi^+(E_{11}) = E_{12}\phi(E_{21}). \end{cases}$$

Then ϕ^+ is an extension of ϕ from $\mathfrak{sl}_n$ to M_n. To verify (5.3)-like property for ϕ^+ in M_n, it suffices to prove the following equality for all A in the standard basis of $\mathfrak{sl}_n$:

$$\phi^+(E_{11}A - AE_{11}) = E_{11}\phi^+(A) - A\phi^+(E_{11}) = E_{11}\phi(A) - AE_{12}\phi(E_{21}). \tag{5.4}$$

1. $A = E_{1j}$, $1 \neq j \in [n]$: the left side of (5.4) is $\phi^+(E_{1j}) = \phi(E_{1j})$. The right side of (5.4) is $E_{11}\phi(E_{1j})$. Both sides are clearly equal since $\phi(E_{1j})$ has zero entries outside of the first row.

2. $A = E_{i1}$, $1 \neq i \in [n]$: the proof is similar.

3. $A = E_{ij}$, $i, j \in [n] \setminus \{1\}$, $i \neq j$: both sides of (5.4) are zero.

4. $A = H_1 = E_{11} - E_{22}$: the left side of (5.4) is zero. The right side of (5.4) is

$$E_{11}\phi(H_1) - H_1E_{12}\phi(E_{21}) = E_{11}\phi(H_1) - E_{12}\phi(E_{21}).$$

We have

$$-2\phi(E_{21}) = \phi([H_1, E_{21}]) = H_1\phi(E_{21}) - E_{21}\phi(H_1) = -\phi(E_{21}) - E_{21}\phi(H_1),$$

where the last equality holds since $\phi(E_{21})$ has zeros outside of the second row. Therefore, $\phi(E_{21}) = E_{21}\phi(H_1)$, and the right side of (5.4) is

$$E_{11}\phi(H_1) - E_{12}\phi(E_{21}) = E_{11}\phi(H_1) - E_{12}E_{21}\phi(H_1) = 0.$$

So both sides are equal.

5. $A = H_i$, $i \in [n-1] \setminus \{1\}$: Both sides of (5.4) are clearly zero.

Overall, (5.4) is proved. We have

$$\phi^+(AB - BA) = A\phi^+(B) - B\phi^+(A), \quad \text{for all } A, B \in M_n. \tag{5.5}$$

Finally, let $B = I_n$ in (5.5), then

$$0 = A\phi^+(I_n) - I_n\phi^+(A) \quad \Rightarrow \quad \phi^+(A) = A\phi^+(I_n).$$

Setting $X := \phi^+(I_n)$, we get $\phi(A) = AX$ for all $A \in \mathfrak{sl}_n$. □

Similarly, we have the following result.

Lemma 5.6. *Suppose* $\mathrm{char}(\mathbb{F}) \neq 2, 3$. *If a linear transformation* $\phi : \mathfrak{sl}_n \to M_{m,n}$ *satisfies that*

$$\phi(AB - BA) = \phi(A)B - \phi(B)A, \quad \textit{for all } A, B \in \mathfrak{sl}_n, \tag{5.6}$$

then there is $X \in M_{m,n}$ *such that* $\phi(C) = XC$ *for* $C \in \mathfrak{sl}_n$.

The statements of Lemmas 5.5 and 5.6 also hold when $\mathrm{char}(\mathbb{F}) = 2$, but the proofs should be adjusted slightly. We will not need the case $\mathrm{char}(\mathbb{F}) = 2$ here. The following counterexample shows that Lemma 5.5 is not true when $\mathrm{char}(\mathbb{F}) = 3$. Likewise for Lemma 5.6.

Example 5.7. *Suppose* $\mathrm{char}(\mathbb{F}) = 3$. *In* M_2, *let* $H := E_{11} - E_{22}$, *and* $\phi : \mathfrak{sl}_2 \to M_2$ *the linear map given by*

$$\phi(E_{12}) := E_{21}, \qquad \phi(E_{21}) := 0, \qquad \phi(H) := 0.$$

Then ϕ *satisfies* (5.3) *since*

$$\begin{aligned} \phi([H, E_{12}]) &= 2\phi(E_{12}) = 2E_{21} = -E_{21} = H\phi(E_{12}) - E_{12}\phi(H), \\ \phi([H, E_{21}]) &= -2\phi(E_{21}) = 0 = H\phi(E_{21}) - E_{21}\phi(H), \\ \phi([E_{12}, E_{21}]) &= \phi(H) = 0 = E_{12}\phi(E_{21}) - E_{21}\phi(E_{12}). \end{aligned}$$

However, there is no $X \in M_2$ *such that* $\phi(E_{12}) = E_{21} = E_{12}X$.

5.2 Derivations of the Lie algebra $M^0_{\mathcal{L}}$ associated an SDUT ladder $\mathcal{L}$

In this section, we state the theorem about the derivations of the Lie algebra $M^0_{\mathcal{L}}$ associated with an SDUT ladder $\mathcal{L}$ over a field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) \neq 2, 3$. We recall that the $M_{\mathcal{B}}$ denote Lie algebra of block upper triangular matrices in M_n.

Theorem 5.8. *Suppose* $\mathrm{char}(\mathbb{F}) \neq 2, 3$. *Let* $\mathcal{L}$ *be an SDUT ladder of size* n. *Then every derivation* $f \in \mathrm{Der}(M^0_{\mathcal{L}})$ *can be extended to a derivation* $f^+ \in \mathrm{Der}(M_{\mathcal{L}})$ *such that* $f^+|_{M^0_{\mathcal{L}}} = f$. *In particular, there exists a block upper triangular matrix* $X \in M_{\mathcal{B}}$ *such that*

$$f(B) = \mathrm{ad}\, X(B) = [X, B], \qquad \text{for all } \; B \in M^0_{\mathcal{L}}. \tag{5.7}$$

We can write

$$\mathrm{Der}(M^0_{\mathcal{L}}) = \mathrm{ad}(\mathrm{N}(M_{\mathcal{L}})/\mathrm{Z}(M_{\mathcal{L}}))|_{M^0_{\mathcal{L}}}. \tag{5.8}$$

The proof of Theorem 5.8 will be given in next section, after we present several auxiliary lemmas and their proofs.

Example 5.9. *When* $\mathrm{char}(\mathbb{F}) = 2$ *or* 3, *we show by counterexamples that* $\mathrm{Der}(M^0_{\mathcal{L}})$ *is not in the form of* (5.8).

- $\mathrm{char}(\mathbb{F}) = 2$*: Let* $M_{\mathcal{L}} = M_2$, *so that* $M^0_{\mathcal{L}} = \mathfrak{sl}_2$. *Let* f *be the derivation of* M_2 *given in Example 4.5, that is,* $f(E_{12}) = E_{21}$, *and* $f(E_{ij}) = 0$ *for* $(i, j) \in \{(1, 1), (2, 2), (2, 1)\}$. *Then* $f|_{\mathfrak{sl}_2}$ *is a derivation of* $\mathfrak{sl}_2$. *However, there is no* $X \in M_{\mathcal{B}} = M_2$ *such that* $f|_{\mathfrak{sl}_2}(E_{12}) = [X, E_{12}]$.

- $\mathrm{char}(\mathbb{F}) = 3$*: Let* $\mathcal{L} = \{(2, 1)\}$ *be a ladder in* M_4. *Then* $M^0_{\mathcal{L}}$ *consists of matrices in* M_4 *that takes the following forms:*

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & -a_{11} & a_{23} & a_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad a_{ij} \in \mathbb{F}.$$

So $M^0_{\mathcal{L}}$ *has a basis* $\mathcal{B} = \{E_{11} - E_{22}, E_{12}, E_{13}, E_{14}, E_{21}, E_{23}, E_{24}\}$. *Define* $f \in \mathrm{End}(M^0_{\mathcal{L}})$ *by* $f(E_{12}) := E_{24}$, *and* $f(E) = 0$ *for all other matrices* E *in the basis* $\mathcal{B}$. *We prove that*

$$f([E, E']) = [f(E), E'] + [E, f(E')] \tag{5.9}$$

for any distinct $E, E' \in \mathcal{B}$, *so that* $f \in \mathrm{Der}(M^0_{\mathcal{L}})$. *The only case that the left side or the right side of* (5.9) *is nonzero is* $\{E, E'\} = \{E_{11} - E_{22}, E_{12}\}$, *in which*

$$f([E, E']) = 2f(E_{12}) = 2E_{24}, \qquad [f(E), E'] + [E, f(E')] = -E_{24}.$$

Since $\mathrm{char}(\mathbb{F}) = 3$*, the equality* (5.9) *holds for this case. Therefore,* (5.9) *holds for all* $\{E, E'\} \subseteq \mathcal{B}$*, and* $f \in \mathrm{Der}(M^0_{\mathcal{L}})$*. However, there is no matrix* $X \in M_4$*, such that* $f(E_{12}) = [X, E_{12}]$.

5.3 Proof of Theorem 5.8

The main goal of this section is to prove Theorem 5.8. We adapt the notations $M_{\mathcal{B}}$, $\mathcal{M}_{ij}$, $\mathcal{M}^{ij}$, $\mathfrak{sl}_{kk}$ and $\mathfrak{sl}^{kk}$ in Definitions 3.1 and 4.1 here. Recall that $\Omega(\mathcal{L})$ and $\Omega(\mathcal{B})$ denote the block index set of matrices of $M_{\mathcal{L}}$ and $M_{\mathcal{B}}$, respectively.

We first present several results on the images $f(\mathfrak{sl}^{kk})$, $f(\mathcal{M}^{ij})$ for $f \in Der(M^0_{\mathcal{L}})$ and $\mathfrak{sl}^{kk}, \mathcal{M}^{ij} \subseteq M^0_{\mathcal{L}}$.

Lemma 5.10. *Suppose* $\mathrm{char}(\mathbb{F}) \neq 2$*. Then for any* $f \in \mathrm{Der}(M^0_{\mathcal{L}})$*:*

$$f(\mathfrak{sl}^{kk}) \subseteq \mathfrak{sl}^{kk} + \sum_{i=1}^{k-1} \mathcal{M}^{ik} + \sum_{j=k+1}^{t} \mathcal{M}^{kj}, \qquad \textit{for} \quad (k,k) \in \Omega(\mathcal{L}); \tag{5.10}$$

$$f(\mathcal{M}^{pq}) \subseteq \mathcal{M}^{pq} + \sum_{i=1}^{p-1} \mathcal{M}^{iq} + \sum_{j=q+1}^{t} \mathcal{M}^{pj}, \qquad \textit{for} \quad 1 \leq p < q \leq t. \tag{5.11}$$

The proof below is similar to that of Lemma 4.10, with some slight adjustments.

Proof. Given $(k,k) \in \Omega(\mathcal{L})$, we have $[\mathfrak{sl}^{kk}, \mathfrak{sl}^{kk}] = \mathfrak{sl}^{kk}$ in $M^0_{\mathcal{L}}$. For $A^{kk}, B^{kk} \in \mathfrak{sl}^{kk}$,

$$f([A^{kk}, B^{kk}]) = [f(A^{kk}), B^{kk}] + [A^{kk}, f(B^{kk})] \ \in \ \mathfrak{sl}^{kk} + \sum_{i=1}^{k-1} \mathcal{M}^{ik} + \sum_{j=k+1}^{t} \mathcal{M}^{kj}.$$

So (5.10) is done.

Given $1 \leq p < q \leq t$, we prove (5.11) by induction on $\ell := q - p$:

1. $\ell = 1$: Here $(p,q) = (p, p+1) \in \Omega(\mathcal{L})$. By Theorem 2.4, at least one of (p,p) and $(p+1, p+1)$ is in $\Omega(\mathcal{L})$. Without loss of generality, suppose $(p,p) \in \Omega(\mathcal{L})$. Let $\{\cdot\}^{ij}$

also denote the embedding of $\mathcal{M}_{ij}$ to $\mathcal{M}^{ij} \subseteq M_n$. For $A^{pp} \in \mathfrak{sl}^{pp}$, $A^{p,p+1} \in \mathcal{M}^{p,p+1}$,

$$\begin{aligned} f(A^{pp}A^{p,p+1}) &= f([A^{pp}, A^{p,p+1}]) = [f(A^{pp}), A^{p,p+1}] + [A^{pp}, f(A^{p,p+1})] \\ &\in \mathcal{M}^{p,p+1} + \sum_{i=1}^{p-1} \mathcal{M}^{i,p+1} + \sum_{j=p+2}^{t} \mathcal{M}^{pj} \\ &\quad - \sum_{i=1}^{p-1} \left\{ f(A^{p,p+1})_{ip}(A^{pp})_{pp} \right\}^{ip} + \left\{ [(A^{pp})_{pp}, f(A^{p,p+1})_{pp}] \right\}^{pp}. \end{aligned} \tag{5.12}$$

Recall that $E_{kj}^{p,p+1}$ denote the (k,j) standard matrix in $\mathcal{M}^{p,p+1}$. To get (5.11) for $q-p=1$, it remains to prove that $f(E_{kj}^{p,p+1})_{ip} = 0$ for any given standard matrix $E_{kj}^{p,p+1}$ in $\mathcal{M}^{p,p+1}$ and $i \in [p]$. There are two cases:

- $i \in [p-1]$: (5.12) shows that for $A^{pp} \in \mathfrak{sl}^{pp}$ and $A^{p,p+1} \in \mathcal{M}^{p,p+1}$,

$$f(A^{pp}A^{p,p+1})_{ip} = -f(A^{p,p+1})_{ip}(A^{pp})_{pp}. \tag{5.13}$$

  Since $\mathcal{L}$ is SDUT ladder, the size of the submatrix in the (p,p) block of matrices in $\mathfrak{sl}^{pp}$ is $m \times m$ such that $m \geq 2$. So we can choose $s \in [m] \setminus \{k\}$. Then

$$f(E_{kj}^{p,p+1})_{ip} = f(E_{ks}^{pp}E_{sj}^{p,p+1})_{ip} = -f(E_{sj}^{p,p+1})_{ip}(E_{ks}^{pp})_{pp}. \tag{5.14}$$

  However, we also have

$$\begin{aligned} 0 &= f([E_{kj}^{p,p+1}, E_{sj}^{p,p+1}])_{i,p+1} \\ &= [f(E_{kj}^{p,p+1}), E_{sj}^{p,p+1}]_{i,p+1} + [E_{kj}^{p,p+1}, f(E_{sj}^{p,p+1})]_{i,p+1} \\ &= f(E_{kj}^{p,p+1})_{ip}(E_{sj}^{p,p+1})_{p,p+1} - f(E_{sj}^{p,p+1})_{ip}(E_{kj}^{p,p+1})_{p,p+1} \\ &= -f(E_{sj}^{p,p+1})_{ip}(E_{ks}^{pp}E_{sj}^{p,p+1})_{p,p+1} - f(E_{sj}^{p,p+1})_{ip}(E_{kj}^{p,p+1})_{p,p+1} \qquad \text{(by (5.14))} \\ &= -2f(E_{sj}^{p,p+1})_{ip}(E_{kj}^{p,p+1})_{p,p+1}. \end{aligned}$$

Since char($\mathbb{F}$) $\neq 2$, the k-th column of $f(E_{sj}^{p,p+1})_{ip}$ must be zero. Then (5.14) shows that $f(E_{kj}^{p,p+1})_{ip} = -f(E_{sj}^{p,p+1})_{ip}E_{ks}^{pp} = 0$.

- $i = p$: (5.12) shows that for $A^{pp} \in \mathfrak{sl}^{pp}$ and $A^{p,p+1} \in \mathcal{M}^{p,p+1}$,

$$f(A^{pp}A^{p,p+1})_{pp} = [(A^{pp})_{pp}, f(A^{p,p+1})_{pp}] = (A^{pp})_{pp}f(A^{p,p+1})_{pp} - f(A^{p,p+1})_{pp}(A^{pp})_{pp}.$$

In particular, for $r \in [m] \setminus \{k\}$, we have $E_{kr}^{pp} \in \mathfrak{sl}^{pp}$ and

$$f(E_{kj}^{p,p+1})_{pp} = f(E_{kr}^{pp}E_{rj}^{p,p+1})_{pp} = (E_{kr}^{pp})_{pp}f(E_{rj}^{p,p+1})_{pp} - f(E_{rj}^{p,p+1})_{pp}(E_{kr}^{pp})_{pp}. \quad (5.15)$$

Denote

$$A = \Big[a_{ij}\Big]_{m\times m} := f(E_{kj}^{p,p+1})_{pp}.$$

(5.15) implies that all nonzero entries of A are located in the k-th row and the r-th column. If $m \geq 3$, we can replace r by any $s \in [m] \setminus \{k, r\}$ in (5.15) to show that all nonzero entries of A are located in the k-th row. In both $m = 2$ and $m \geq 3$ cases, we have

$$A = (E_{kk}^{pp})_{pp}A + a_{rr}(E_{rr}^{pp})_{pp}. \quad (5.16)$$

Applying (5.15) twice, we get

$$\begin{aligned} A &= \big[(E_{kr}^{pp})_{pp}, f(E_{rj}^{p,p+1})_{pp}\big] = \big[(E_{kr}^{pp})_{pp}, \big[(E_{rk}^{pp})_{pp}, f(E_{kj}^{p,p+1})_{pp}\big]\big] \\ &= (E_{kk}^{pp})_{pp}A - (E_{kr}^{pp})_{pp}A(E_{rk}^{pp})_{pp} - (E_{rk}^{pp})_{pp}A(E_{kr}^{pp})_{pp} + A(E_{rr}^{pp})_{pp} \\ &= (A - a_{rr}(E_{rr}^{pp})_{pp}) - (E_{kr}^{pp})_{pp}\{(E_{kk}^{pp})_{pp}A + a_{rr}(E_{rr}^{pp})_{pp}\}(E_{rk}^{pp})_{pp} \\ &\quad -(E_{rk}^{pp})_{pp}A(E_{kr}^{pp})_{pp} + A(E_{rr}^{pp})_{pp} \quad \text{(by (5.16))} \\ &= A - a_{rr}\{(E_{rr}^{pp})_{pp} + (E_{kk}^{pp})_{pp}\} - (E_{rk}^{pp})_{pp}A(E_{kr}^{pp})_{pp} + A(E_{rr}^{pp})_{pp}. \end{aligned}$$

Therefore,

$$a_{rr}\{(E^{pp}_{rr})_{pp} + (E^{pp}_{kk})_{pp}\} + (E^{pp}_{rk})_{pp} A (E^{pp}_{kr})_{pp} = A(E^{pp}_{rr})_{pp}.$$

Comparing the (k,k) (resp. (r,r), (k,r)) entry, we get $a_{rr} = 0$ (resp. $a_{kk} = 0$, $a_{kr} = 0$). Since $r \in [m] \setminus \{k\}$ is arbitrary, we have $f(E^{p,p+1}_{kj})_{pp} = 0$.

The case $\ell = 1$ is done.

2. Suppose (5.11) is true for all $\ell < k$. Now for any $(p, p+k) \in \Omega(\mathcal{L})$, we have $[\mathcal{M}^{p,p+1}, \mathcal{M}^{p+1,p+k}] = \mathcal{M}^{p,p+k}$ in $M^0_{\mathcal{L}}$, and by induction hypothesis,

$$\begin{aligned} f(A^{p,p+1}A^{p+1,p+k}) &= f([A^{p,p+1}, A^{p+1,p+k}]) = [f(A^{p,p+1}), A^{p+1,p+k}] + [A^{p,p+1}, f(A^{p+1,p+k})] \\ &\in \mathcal{M}^{p,p+k} + \sum_{i=1}^{p-1} \mathcal{M}^{i,p+k} + \sum_{j=p+k+1}^{t} \mathcal{M}^{pj}. \end{aligned}$$

Therefore, (5.11) is true for $\ell = k$.

3. Overall, (5.11) is proved for all $(p,q) \in \Omega(\mathcal{L})$ with $p < q$. □

Lemma 5.11. *Suppose* $\mathrm{char}(\mathbb{F}) \neq 2, 3$. *Let* $f \in \mathrm{Der}(M^0_{\mathcal{L}})$. *Then for any* $1 \le p < q \le t$, *there exists* $X_{pq} \in \mathcal{M}_{pq}$ *such that*

$$f(A^{ip})_{iq} = -(A^{ip})_{ip} X_{pq}, \quad \text{for all } (i,p) \in \Omega(\mathcal{L}) \text{ and } A^{ip} \in \mathcal{M}^{ip} \cap M^0_{\mathcal{L}}, \tag{5.17}$$

$$f(A^{qj})_{pj} = X_{pq}(A^{qj})_{qj}, \quad \text{for all } (q,j) \in \Omega(\mathcal{L}) \text{ and } A^{qj} \in \mathcal{M}^{qj} \cap M^0_{\mathcal{L}}. \tag{5.18}$$

The proof below is similar to that of (4.14) and (4.15) inside the proof of Theorem 4.4.

Proof. Given $p < q$ in $[t]$, we consider the following four situations:

1. Suppose $(q,j) = (t,t) \in \Omega(\mathcal{L})$. For any $A^{tt}, B^{tt} \in \mathfrak{sl}^{tt}$,

$$f([A^{tt}, B^{tt}])_{pt} = [f(A^{tt}), B^{tt}]_{pt} + [A^{tt}, f(B^{tt})]_{pt} = f(A^{tt})_{pt}(B^{tt})_{tt} - f(B^{tt})_{pt}(A^{tt})_{tt}.$$

Applying Lemma 5.6 to the map $\phi : \mathfrak{sl}_{tt} \to \mathcal{M}_{pt}$ defined by $\phi(C) = f(C^{tt})_{pt}$, we can find $X_{pt} \in \mathcal{M}_{pt}$ such that $f(A^{tt})_{pt} = X_{pt}(A^{tt})_{tt}$ for $A^{tt} \in \mathfrak{sl}^{tt}$.

2. Similarly, when $(i,p) = (1,1)$, there exists $Y_{1q} \in \mathcal{M}_{1q}$ such that $f(A^{11})_{1q} = -(A^{11})_{11}Y_{1q}$ for $A^{11} \in \mathfrak{sl}^{11}$.

3. Suppose $(q,j) \in \Omega(\mathcal{L})$, $(q,j) \neq (t,t)$. Then $q < t$. Given any $j < j'$ in $[t]$, we have $(j,j'), (q,j'), (p,j), (p,j') \in \Omega(\mathcal{L})$, and $\mathcal{M}^{qj'} = \mathcal{M}^{qj}\mathcal{M}^{jj'} = [\mathcal{M}^{qj}, \mathcal{M}^{jj'}]$.

   - If $q = j$, then for $A^{qj} \in \mathfrak{sl}^{qq}$ and $A^{jj'} \in \mathcal{M}^{jj'}$,

     $$f(A^{qj}A^{jj'})_{pj'} = f([A^{qj}, A^{jj'}])_{pj'} = [f(A^{qj}), A^{jj'}]_{pj'} + [A^{qj}, f(A^{jj'})]_{pj'} = f(A^{qj})_{pj}A_{jj'}.$$

     Applying Lemma 5.4 to the map $\phi : \mathcal{M}_{qj'} \to \mathcal{M}_{pj'}$ defined by $\phi(C) = f(C^{qj})_{pj'}$, and $\varphi : \mathfrak{sl}_{qq} \to \mathcal{M}_{pq}$ defined by $\varphi(D) = f(D^{qq})_{pj}$, there exists $X_{pq} \in \mathcal{M}_{pq}$ such that $f(A^{qj})_{pj} = X_{pq}(A^{qj})_{qj}$ for $A^{qj} \in \mathfrak{sl}^{qq}$, and $f(A^{qj'})_{pj'} = X_{pq}(A^{qj'})_{qj'}$ for any $j' > j$ in $[t]$ and any $A^{qj'} \in \mathcal{M}^{qj'}$.

   - If $q < j$, then for $A^{qj} \in \mathcal{M}^{qj}$ and $A^{jj'} \in \mathcal{M}^{jj'}$, we still have

     $$f(A^{qj}A^{jj'})_{pj'} = f([A^{qj}, A^{jj'}])_{pj'} = [f(A^{qj}), A^{jj'}]_{pj'} + [A^{qj}, f(A^{jj'})]_{pj'} = f(A^{qj})_{pj}(A^{jj'})_{jj'}.$$

     Applying Lemma 2.7, there exists a (unique) $X_{pq} \in \mathcal{M}_{pq}$ such that $f(A^{qj})_{pj} = X_{pq}(A^{qj})_{qj}$ for all $j > q$ in $[t]$ and $A^{qj} \in \mathcal{M}^{qj}$.

4. Suppose $(i,p) \in \Omega(\mathcal{L})$ and $(i,p) \neq (1,1)$. Similar to the proceeding argument, there exists $-Y_{pq} \in \mathcal{M}_{pq}$ such that $f(A^{ip})_{iq} = (A^{ip})_{ip}Y_{pq}$ for $(i,p) \in \Omega(\mathcal{L})$ and $A^{ip} \in \mathcal{M}^{ip}$.

5. For any $(i,p),(q,j) \in \Omega(\mathcal{L})$, we have $[A^{ip}, A^{qj}] = 0$. So

$$\begin{aligned} 0 &= f([A^{ip}, A^{qj}])_{ij} = [f(A^{ip}), A^{qj}]_{ij} + [A^{ip}, f(A^{qj})]_{ij} \\ &= f(A^{ip})_{iq}(A^{qj})_{qj} + (A^{ip})_{ip} f(A^{qj})_{pj} \\ &= -(A^{ip})_{ip} Y_{pq} (A^{qj})_{qj} + (A^{ip})_{ip} X_{pq} (A^{qj})_{ip}. \end{aligned}$$

Therefore, $X_{pq} = Y_{pq}$. □

Now we are ready to prove Theorem 5.8.

Proof of Theorem 5.8. Recall that $E^{kk}_{\ell\ell}$ denote the (ℓ,ℓ) standard matrix in $\mathcal{M}^{kk}$. We have the Lie subalgebra decomposition

$$M_{\mathcal{L}} = \operatorname{span}\{E^{kk}_{11} \mid (k,k) \in \Omega(\mathcal{L})\} \ltimes M^0_{\mathcal{L}}.$$

Given $f \in \operatorname{Der}(M^0_{\mathcal{L}})$, we define $f^+(A) := f(A)$ for $A \in M^0_{\mathcal{L}}$. The next step is to define $f^+(E^{kk}_{11})$ for each $(k,k) \in \Omega(\mathcal{L})$ appropriately so that $f^+ \in \operatorname{Der}(M_{\mathcal{L}})$. We will let

$$f^+(E^{kk}_{11}) \ \in \ \mathfrak{sl}^{kk} + \sum_{i=1}^{k-1} \mathcal{M}^{ik} + \sum_{j=k+1}^{t} \mathcal{M}^{kj}$$

and define the nonzero blocks of $f^+(E^{kk}_{11})$ as follow.

1. The (k,k) block: it is easy to see that $f(\cdot^{kk})_{kk} : \mathfrak{sl}_{kk} \to \mathfrak{sl}_{kk}$, $(A \mapsto A^{kk} \mapsto f(A^{kk})_{kk})$, is a derivation of $\mathfrak{sl}_{kk}$. Since $\operatorname{char}(\mathbb{F}) \neq 2$, there exists $X_{kk} \in \mathfrak{sl}_{kk}$ such that $f(A^{kk})_{kk} = [X_{kk}, (A^{kk})_{kk}]$ for $A^{kk} \in \mathfrak{sl}^{kk}$. Define

$$f^+(E^{kk}_{11})_{kk} := [X_{kk}, (E^{kk}_{11})_{kk}]. \tag{5.19}$$

2. The (i,k) block, $i<k$: by Lemma 5.11, there exists $X_{ik}\in\mathcal{M}_{ik}$ such that $f(A^{kj})_{ij}=X_{ik}(A^{kj})_{kj}$ for any $(k,j)\in\Omega(\mathcal{L})$. Define

$$f^+(E_{11}^{kk})_{ik}:=X_{ik}(E_{11}^{kk})_{kk}\quad\text{for all } i\in[k-1]. \tag{5.20}$$

3. The (k,j) block, $k<j$: by Lemma 5.11, there exists $X_{kj}\in\mathcal{M}_{kj}$ such that for all $(i,k)\in\Omega(\mathcal{L})$ we have $f(A^{ik})_{ij}=-(A^{ik})_{ik}X_{kj}$. Define

$$f^+(E_{11}^{kk})_{kj}:=-(E_{11}^{kk})_{kk}X_{kj}\quad\text{for all } k<j\le t. \tag{5.21}$$

The above process uniquely defines a linear map $f^+\in\mathrm{End}(\mathrm{M}_{\mathcal{L}})$ such that $f^+|_{M_{\mathcal{L}}^0}=f$. Next we verify that $f^+\in\mathrm{Der}(\mathrm{M}_{\mathcal{L}})$. It suffices to prove that for every $(i,j)\in\Omega(\mathcal{L})$,

$$f^+([E_{11}^{kk},A^{ij}])=[f^+(E_{11}^{kk}),A^{ij}]+[(E_{11}^{kk})_{kk},f^+(A^{ij})]\quad\text{for all } A^{ij}\in\mathcal{M}^{ij}\cap M_{\mathcal{L}}^0. \tag{5.22}$$

Denote

$$X^k:=X^{kk}+\sum_{i=1}^{k-1}X^{ik}+\sum_{j=k+1}^{t}X^{kj} \tag{5.23}$$

where $X^{kk}:=(X_{kk})^{kk}\in\mathfrak{sl}^{kk}$, $X^{ik}:=(X_{ik})^{ik}\in\mathcal{M}^{ik}$, and $X^{kj}:=(X_{kj})^{kj}\in\mathcal{M}^{kj}$. Then (5.19), (5.20), and (5.21) imply that $f^+(E_{11}^{kk})=[X^k,E_{11}^{kk}]$. So (5.22) is equivalent to

$$f([E_{11}^{kk},A^{ij}])=[[X^k,E_{11}^{kk}],A^{ij}]+[E_{11}^{kk},f(A^{ij})]\quad\text{for all } A^{ij}\in\mathcal{M}^{ij}\cap M_{\mathcal{L}}^0. \tag{5.24}$$

We will prove (5.24) for each block $(i,j)\in\Omega(\mathcal{L})$:

1. $(k,k)\in\Omega(\mathcal{L})$: the matrices X^{kk}, X^{ik} $(i<k)$, and X^{kj} $(k<j)$ satisfy that

$$f(A^{kk})=[X^k,A^{kk}]\quad\text{for all } A^{kk}\in\mathfrak{sl}^{kk},$$

   where X^k is given by (5.23). Therefore, (5.24) is true for $(i,j)=(k,k)\in\Omega(\mathcal{L})$.

2. (k,j), $k<j\le t$: when $(i,j)=(k,j)$, we have

$$[E_{11}^{kk},A^{kj}]=E_{11}^{kk}A^{kj}=E_{12}^{kk}E_{21}^{kk}A^{kj}=[E_{12}^{kk},[E_{21}^{kk},A^{kj}]].$$

So (5.24) is equivalent to the following equalities:

$$\begin{aligned}
&f([E_{12}^{kk},[E_{21}^{kk},A^{kj}]])=[[X^k,E_{11}^{kk}],A^{kj}]+[E_{11}^{kk},f(A^{kj})]\\
\iff\ &f(E_{12}^{kk})E_{21}^{kk}A^{kj}+E_{12}^{kk}f(E_{21}^{kk})A^{kj}+E_{12}^{kk}E_{21}^{kk}f(A^{kj})=[X^k,E_{11}^{kk}]A^{kj}+E_{11}^{kk}f(A^{kj})\\
\iff\ &f(E_{12}^{kk})E_{21}^{kk}A^{kj}+E_{12}^{kk}f(E_{21}^{kk})A^{kj}=[X^k,E_{11}^{kk}]A^{kj}\quad(\text{for all } A^{kj}\in\mathcal{M}^{kj})
\end{aligned}$$

$$\begin{aligned}
\iff\ &f(E_{12}^{kk})E_{21}^{kk}+E_{12}^{kk}f(E_{21}^{kk})=[X^k,E_{11}^{kk}]\\
\iff\ &[X^k,E_{12}^{kk}]E_{21}^{kk}+E_{12}^{kk}[X^k,E_{21}^{kk}]=[X^k,E_{11}^{kk}].
\end{aligned}$$

The last equality is obviously true.

3. (i,k), $1\le i<k$: similarly, we can prove (5.24) for the case $(i,j)=(i,k)$.

4. $(i,j)\in\Omega(\mathcal{L})$, $i\ne k$, $j\ne k$: the left side of (5.24) is zero. We investigate the right side of (5.24) in three cases:

   (a) $i\le j<k$: the only possibly nonzero block in the right side of (5.24) is the (i,k) block, which is

$$\begin{aligned}
[[X^k,E_{11}^{kk}],A^{ij}]_{ik}+[E_{11}^{kk},f(A^{ij})]_{ik}&=-(A^{ij})_{ij}[X^k,E_{11}^{kk}]_{jk}-f(A^{ij})_{ik}(E_{11}^{kk})_{kk}\\
&=-(A^{ij})_{ij}[X^k,E_{11}^{kk}]_{jk}+(A^{ij})_{ij}X_{jk}(E_{11}^{kk})_{kk}\ (\text{by Lemma 5.11})\\
&=-(A^{ij})_{ik}(X^{jk})_{jk}(E_{11}^{kk})_{kk}+(A^{ij})_{ij}X_{jk}(E_{11}^{kk})_{kk}\quad(\text{by (5.23)})\\
&=0.
\end{aligned}$$

   So (5.24) is done for this case.

(b) $k < i \leq j$: similarly, we can prove (5.24) for this case.

(c) $i < k < j$: the right side of (5.24) is

$$[[X^k, E_{11}^{kk}], A^{ij}] + [E_{11}^{kk}, f(A^{ij})] = 0 + 0 = 0.$$

So (5.24) holds.

Overall, we have proved (5.24). Therefore, $f^+ \in \mathrm{Der}(M_{\mathcal{L}})$ and $f^+|_{M_{\mathcal{L}}^0} = f$. By Theorem 4.4, there is $X \in M_{\mathcal{B}}$ such that $f(B) = [X, B]$ for all $B \in M_{\mathcal{L}}^0$. □

Chapter 6

Future research topics

In this chapter, we discuss some potential future research directions on those matrix Lie algebras that we have studied in this dissertation.

Other than the Lie algebras of block upper triangular matrices and strictly block upper triangular matrices, not much work has been done on the Lie algebra $M_{\mathcal{L}'}$ of upper triangular ladder matrices. Many interesting research directions arise on this topic, for examples, the description of $\mathrm{Der}(M_{\mathcal{L}'})$ in terms of Levi-decomposition and root space decomposition of $M_{\mathcal{L}'}$, the study of Lie triple derivations of $M_{\mathcal{L}'}$, extensions of these results to semisimple or reductive Lie algebra, etc.

For most upper triangular ladders $\mathcal{L}'$, the complete description of $\mathrm{Der}(M_{\mathcal{L}'})$ remains an open problem. The explicit description of the derivations of the Lie algebra $M_{\mathcal{L}}$ for an DUT ladder matrices has been given over a filed $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) \neq 2$ (Theorem 4.4). However, we have not investigated the $\mathrm{char}(\mathbb{F}) = 2$ case. Example 4.5 suggests that $\mathrm{Der}(M_{\mathcal{L}})$ has some additional elements when $\mathrm{char}(\mathbb{F}) = 2$. In the future, I would like to continue my investigation on $\mathrm{Der}(M_{\mathcal{L}})$ when $\mathrm{char}(\mathbb{F}) = 2$, as well as the study of $\mathrm{Der}(M_{\mathcal{L}'})$ for other upper triangular ladders $\mathcal{L}'$.

A **Lie triple derivation** (or simply **triple derivation**) of Lie algebra $\mathfrak{g}$ is an $\mathbb{F}$-linear map $f : \mathfrak{g} \to \mathfrak{g}$ that satisfies

$$f([X,[Y,Z]]) = [f(X),[Y,Z]] + [X,[f(Y),Z]] + [X,[Y,f(Z)]]$$

for all $X, Y, Z \in \mathfrak{g}$.

The triple derivations of Lie algebras are apparently a generalization of its derivations. It is easy to show that every derivation of Lie algebra is a triple derivation. However, the converse statement is in general not true, which can be seen via the following example.

Example 6.1. *Let* $\mathfrak{g}$ *be a Lie algebra of strictly upper triangular matrices in* M_3*. So* $\mathfrak{g}$ *has a basis* $\mathcal{B} := \{E_{12}, E_{13}, E_{23}\}$*, where* E_{ij} *denotes the standard matrix in* M_3 *that has the only nonzero entry 1 in the* (i, j) *position. Define* $f \in \mathrm{End}(\mathfrak{g})$ *by* $f(E_{23}) := E_{23}$*, and* $f(E) := 0$ *for all other matrices* $E \in \mathcal{B}$*. Since* $[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] = 0$*, it is straightforward to check that* f *is a triple derivation of* $\mathfrak{g}$*. On the other hand,* $f([E_{12}, E_{23}]) = f(E_{13}) = 0$ *and* $[f(E_{12}), E_{23}] + [E_{12}, f(E_{23})] = [E_{12}, E_{23}] = E_{13}$*. So* f *is not a derivation of* $\mathfrak{g}$*.*

The triple derivations of matrix Lie algebras have been extensively studied over a ring R [2, 18, 20, 29]. In [29], Wang and Li explicitly described the triple derivations of the Lie algebra of strictly upper triangular matrices over a commutative ring R with identity. However, no work has been done on the triple derivations of the Lie algebra of strictly block upper triangular matrices. At this point, we want to ask the following question.

Question 6.2. *Can we extend Wang and Li's result in [29] on triple derivations of the Lie algebra of strictly upper triangular matrices to the Lie algebra of strictly block upper triangular matrices?*

As we mentioned earlier, many interesting questions on the Lie algebra of upper triangular ladder matrices remain to be addressed. The triple derivation is one of them but it seems a lot of works are needed to be done to have a complete description of them. In the next step, I would like to study the triple derivation of the Lie algebra $M_{\mathcal{L}}$ of DUT ladder matrices. This Lie algebra satisfies the following Lie bracket relation

$$[M_{\mathcal{L}}, M_{\mathcal{L}}] = [M_{\mathcal{L}}, [M_{\mathcal{L}}, M_{\mathcal{L}}]]$$

by a direct computation. This suggests the following question.

Question 6.3. *Does the Lie algebra $M_{\mathcal{L}}$ of DUT ladder matrices have any triple derivations other than derivations?*

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