Generalization of Ky Fan-Amir-Moéz-Horn-Mirsky's result on the eigenvalues and real singular values of a matrix

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Generalization of Ky Fan-Amir-Moéz-Horn-Mirsky's result on the eigenvalues and real singular values of a matrix

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Generalization of Ky Fan-Amir-Moéz-Horn-Mirsky's result on the eigenvalues and real singular values of a matrix

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Wen Yan, son of Changyou Yan and Pingju Wu, was born in Xupu, Hunan, P. R. China, on January 15, 1973. He attended the public school of Xupu, Hunan, P.R.China. After graduating from Xupu High School, he entered Xiangtan University in September, 1990, from which he received his B.S. (Math) in June, 1994. He started his graduate study in the Department of Mathematics of Xiangtan University in August, 1994 and received his M.S. (Math) in January, 1997. He continued his graduate study in the Department of Mathematics of Auburn University in August, 2000.

DISSERTATION ABSTRACT

GENERALIZATION OF KY FAN-AMIR-MOÉZ-HORN-MIRSKY'S RESULT ON THE EIGENVALUES AND REAL SINGULAR VALUES OF A MATRIX

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Ky Fan's result states that the real part of the eigenvalues of an $n \times n$ complex matrix A is majorized by the eigenvalues of the Hermitian part of A. The converse was established by Amir-Moéz and Horn, and Mirsky, independently. We extend the results in the context of complex semisimple Lie algebras. Inequalities associated with the classical complex Lie algebras are given. The real case is also discussed.

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Chapter 1

Introduction

Let $\mathbb{C}_{n\times n}$ be the space of $n \times n$ complex matrices. Each $A \in \mathbb{C}_{n\times n}$ has the Hermitian decomposition

$$A = \frac{1}{2}(A - A^*) + \frac{1}{2}(A + A^*), \tag{1.1}$$

where $*$ denotes the complex conjugate transpose. Clearly the matrix $A_1 := \frac{1}{2}(A - A^*)$ is skew-Hermitian, i.e., $A_1^* = -A_1$ (called the *skew-Hermitian part* of A) and $A_2 := \frac{1}{2}(A + A^*)$ is Hermitian, i.e., $A_2^* = A_2$ (called the *Hermitian part* of A).

There are three important sets of scalars associated with A, known as the *eigenvalues*, denoted by $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$, the *real singular values*, denoted by $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ and the *imaginary singular values* of A, denoted by $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n$. An eigenvalue λ of A is a scalar such that there exists a nonzero vector $z \in \mathbb{C}^n$ such that

$$Az = \lambda z.$$

The real singular values of A are the eigenvalues of the Hermitian part $\frac{1}{2}(A + A^*)$ of A. The imaginary singular values of A are the eigenvalues of the Hermitian matrix $\frac{1}{2i}(A - A^*)$.

There is a nice result of Ky Fan [7], [23, p.239] relating the eigenvalues of A and the real singular values of A. We need the following important notion called *majorization* [23] in order to state the result of Ky Fan. Majorization has a lot of applications in different branches of mathematics [3, 16, 23].

Definition 1.1 Let $a, b \in \mathbb{R}^n$. We say that a is *majorized* by b, denoted by $a \prec b$, if

$$\begin{aligned} \sum_{i=1}^{k} a_{[i]} &\leq \sum_{i=1}^{k} b_{[i]}, \qquad k = 1, \ldots, n-1, \\ \sum_{i=1}^{n} a_{[i]} &= \sum_{i=1}^{n} b_{[i]}, \end{aligned}$$

where $a_{[1]} \geq a_{[2]} \geq \cdots \geq a_{[n]}$ and $b_{[1]} \geq b_{[2]} \geq \cdots \geq b_{[n]}$ are the rearrangements of the entries of a and b, respectively, in nonincreasing order.

Theorem 1.2 (Ky Fan) Given $A \in \mathbb{C}_{n\times n}$, the real parts of the eigenvalues $(\lambda_1, \ldots, \lambda_n)^T \in \mathbb{C}^n$ of A is majorized by the real singular values $(\alpha_1, \ldots, \alpha_n)^T \in \mathbb{R}^n$ of A, i.e., $\operatorname{Re}\lambda \prec \alpha$.

The converse was established by Amir-Moéz and Horn [1], and independently by Mirsky [24]. It was later rediscovered by Sherman and Thompson [29]. The study can be traced back to some old results of Bendixson [2], Hirsch [13], and Bromwich [4]. Also see [23, p.237-239]. We state the result in the following theorem.

Theorem 1.3 (Amir-Moéz-Horn and Mirsky) If $\lambda \in \mathbb{C}^n$ and $\alpha \in \mathbb{R}^n$ such that $\operatorname{Re}\lambda \prec \alpha$, then there exists $A \in \mathbb{C}_{n\times n}$ such that λ's are the eigenvalues of A and α's are the real singular values of A.

Similar results hold for the imaginary part of the eigenvalues of A and the imaginary singular values of A.

The following is the result of Sherman and Thompson [29].

Theorem 1.4 (Sherman and Thompson) If H is a given Hermitian matrix with eigenvalues $\beta \in \mathbb{R}^n$ and if $\alpha \in \mathbb{R}^n$ satisfies $\alpha \prec \beta$, then there exists a skew Hermitian matrix K such that α is the real part of the eigenvalues of $K + H$.

Remark 1.5 The result of Sherman and Thompson is indeed equivalent to Amir-Moéz-Horn-Mirsky's result. We now establish their equivalence.

We will use the fact that the eigenvalues, real singular values, and the imaginary singular values of any $A \in \mathbb{C}_{n\times n}$ are invariant under unitary similarity.

Theorem 1.3 $\Longrightarrow$ Theorem 1.4: Given $\alpha \prec \beta$, where β is the eigenvalues of a Hermitian matrix H, by Amir-Moéz-Horn-Mirsky's result (Theorem 1.3), there is a matrix A_1 such that α is the real part of the eigenvalues of A_1 and β is the eigenvalues of the Hermitian part of A_1. Since H and $\frac{A_1+A_1^*}{2}$ are both Hermitian with the same eigenvalues β's, by the spectral decomposition for Hermitian matrices [16, p.171], they are both unitarily similar to the diagonal matrix diag β, hence unitarily similar to each other. Let U be the unitary matrix such that $H = U^{-1}\frac{A_1+A_1^*}{2}U$. Let $K := U^{-1}\frac{A_1-A_1^*}{2}U$. Then K is skew Hermitian, and $A := K + H = U^{-1}A_1U$. By the fact that eigenvalues and real singular values are invariant under unitary similarity, A and A_1 have the same eigenvalues and real singular values. Therefore K is the required skew Hermitian matrix. This proves that Amir-Moéz-Horn-Mirsky's result implies the result of Sherman and Thompson.

Theorem 1.4 $\Longrightarrow$ Theorem 1.3: Conversely, suppose $\operatorname{Re}\lambda \prec \beta$ for $\lambda \in \mathbb{C}^n$ and $\beta \in \mathbb{R}^n$. Let $H = \operatorname{diag}\beta$. By Sherman and Thompson's result there is a skew Hermitian matrix K such that $\operatorname{Re}\lambda$ is the real part of the eigenvalues μ's of $A_1 := K + H$ and the real singular values of A_1 are β's. By Schur triangularization theorem (Theorem 2.1), there exists a unitary matrix U such that $B := U^{-1}A_1U$ is upper triangular and $\mu = \operatorname{diag} B$. Then $A := B + i\operatorname{diag}(\operatorname{Im}\lambda - \operatorname{Im}\mu)$ has eigenvalues λ's and the Hermitian part of A is $\frac{B+B^*}{2} = U^{-1}\frac{A_1+A_1^*}{2}U$. Thus A and A_1 have the same real singular values β's. Therefore A has eigenvalues λ's and real singular values β's. So the result of Sherman and Thompson implies Amir-Moéz-Horn-Mirsky's result.

How would Ky Fan-Amir-Moéz-Horn-Mirsky's result be if we restrict our attention to complex skew symmetric matrices? If $A \in \mathbb{C}_{n\times n}$ is skew symmetric, then its eigenvalues occur in pair but opposite in sign, since A and $A^T = -A$ have the same characteristic polynomial. We will see in Chapter 6 that majorization remains to be the key, except the statements are stronger and we will separately consider the even and odd cases. Similarly we consider the symplectic case as well in Chapter 5.

We will have semisimple Lie algebra as a unified framework and develop our main result in Chapter 4. The following is the motivation for our study. A translation of A, that is, $A + \xi I$ for some $\xi \in \mathbb{C}$, would translate the eigenvalues by ξ and the real singular values by $\operatorname{Re}\xi$. Thus it is sufficient to consider those $A \in \mathbb{C}_{n\times n}$ such that $\operatorname{tr} A = 0$ in Ky Fan-Amir-Moéz-Horn-Mirsky's result. Recall that

$$\mathfrak{sl}(n, \mathbb{C}) := \{A \in \mathbb{C}_{n\times n} : \operatorname{tr} A = 0\}$$

is the Lie algebra of the *special linear group*

$$\mathrm{SL}(n, \mathbb{C}) := \{A \in \mathbb{C}_{n\times n} : \det A = 1\}.$$

The *special unitary group*

$$\mathrm{SU}(n) := \{U \in \mathbb{C}_{n\times n} : U^*U = I_n,\ \det U = 1\}$$

is a maximal compact subgroup of $\mathrm{SL}(n, \mathbb{C})$. The diagonal matrices in $\mathfrak{sl}(n, \mathbb{C})$ form a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{sl}(n, \mathbb{C})$, and those in $\mathfrak{h}$ with purely imaginary diagonal entries form a Cartan subalgebra $\mathfrak{t}$ of the Lie algebra

$$\mathfrak{su}(n) := \{A \in \mathfrak{sl}(n, \mathbb{C}) : A + A^* = 0\}$$

of $SU(n)$. As a real $\mathrm{SU}(n)$-module, $\mathfrak{sl}(n, \mathbb{C})$ is just the direct sum of two copies of the adjoint module $\mathfrak{su}(n)$:

$$\mathfrak{sl}(n, \mathbb{C}) = \mathfrak{su}(n) \oplus i\mathfrak{su}(n)$$

which in our case is essentially the well known Hermitian decomposition of a complex matrix (1.1). By the Schur triangularization theorem (See Theorem 2.1) for complex matrices, the eigenvalues of a matrix $A \in \mathfrak{sl}(n, \mathbb{C})$ may be viewed as the image of an element $Y \in \mathrm{AdSU}(n)(A) \cap \mathfrak{b}$ under the orthogonal projection

$\rho : \mathfrak{sl}(n, \mathbb{C}) \to \mathfrak{h}$ with respect to the inner product

$$\langle X, Y \rangle = \mathrm{Re}\,\mathrm{tr}\, XY^*, \qquad X, Y \in \mathfrak{sl}(n, \mathbb{C}),$$

where $\mathrm{SU}(n)$ acts on $\mathfrak{sl}(n, \mathbb{C})$ via the the adjoint representation, $\mathrm{AdSU}(n)(A)$ is the orbit of A under the action of $\mathrm{SU}(n)$, and $\mathfrak{b}$ is the Borel subalgebra of $\mathfrak{sl}(n, \mathbb{C})$ consisting of $n \times n$ upper triangular matrices. Thus taking the real part of the eigenvalues of A amounts to sending Y via the orthogonal projection

$$\pi : \mathfrak{sl}(n, \mathbb{C}) \to i\mathfrak{t}$$

with respect to $\langle \cdot, \cdot \rangle$. The majorization relation $\alpha \prec \beta$ is equivalent to $\alpha \in \mathrm{conv}\, S_n\beta$ for $\alpha, \beta \in \mathbb{R}^n$ by Theorem 2.7, where $\mathrm{conv}\, S_n\beta$ denotes the convex hull of the orbit of β under the action of S_n. It is a direct result of Theorem 2.3 (Hardy-Littlewood-Polyá) and Theorem 2.6 (Birkhoff). So the result of Ky Fan (Theorem 1.2) may be stated as

$$\pi(\mathrm{AdSU}(n)(X + Z) \cap \mathfrak{b}) \subset \mathrm{conv}\, S_n Z, \tag{1.2}$$

where $Z \in i\mathfrak{t}$, $X \in \mathfrak{su}(n)$ and S_n is the full symmetric group on $\{1, \ldots, n\}$. The result of Amir-Moéz-Horn and Mirsky (in the version of Sherman and Thompson) may be written as

$$\mathrm{conv}\, S_n Z \subset \cup_{X \in \mathfrak{su}(n)} \pi(\mathrm{AdSU}(n)(X + Z) \cap \mathfrak{b}). \tag{1.3}$$

Combining (1.2) and (1.3) we have

$$\cup_{X \in \mathfrak{su}(n)} \pi(\mathrm{AdSU}(n)(X + Z) \cap \mathfrak{b}) = \mathrm{conv}\, S_n Z. \tag{1.4}$$

We will extend (1.4) in the context of finite dimensional complex semisimple Lie algebras (Theorem 4.6), as one of our main results in this dissertation. Notice that (1.4) may be stated as

$$\pi((\mathfrak{su}(n) + \mathrm{AdSU}(n)(Z)) \cap \mathfrak{b}) = \mathrm{conv}\, S_n Z.$$

In particular, for each $U \in \mathfrak{sl}(n, \mathbb{C})$,

$$\pi(\mathrm{AdSU}(n)(U) \cap \mathfrak{b}) \subset \mathrm{conv}\, S_n z,$$

where $Z \in \mathrm{AdSU}(n)(\frac{1}{2}(U - \theta U)) \cap i\mathfrak{t}$, where

$$\theta(X) = -X^*, \qquad X \in \mathfrak{sl}(n, \mathbb{C}).$$

Before we prove the extension of (1.4) in Theorem 4.6 in Chapter 4, we give a detailed proof of Ky Fan-Amir-Moéz -Horn-Mirsky's result in Chapter 2. We then introduce some preliminary materials of complex semisimple Lie algebras in Chapter 3 to pave the road for Chapter 4. In Chapter 4, we use Kostant's result [22, Theorem 8.2] to show that (1.4) remains true for all complex semisimple Lie algebras. The interesting inequalities corresponding to the classical Lie algebras, alike majorization, are discussed in Chapter 5 and Chapter 6. In Chapter 7, we discuss the case for the real Lie algebra $\mathfrak{sl}(n, \mathbb{R})$ and obtain a result for $\mathfrak{sl}(n, \mathbb{R})$ which is similar to Theorem 4.6. In particular we consider $\mathfrak{su}(1, 1)$, a real form of $\mathfrak{sl}(2, \mathbb{C})$. In Chapter 8, we consider some inequalities relating the eigenvalues, the real and imaginary singular values for $\mathfrak{sl}(2, \mathbb{C})$ and $\mathfrak{sl}(2, \mathbb{R})$.

Chapter 2

The proof of the result of Ky Fan-Amir-Moéz-Horn-Mirsky

In this chapter we review the proof of the result of Ky Fan, Amir-Moéz and Horn, and Mirsky. We shall then point out the key elements of the proof, which will be essential for the extension in Chapter 4. The proof makes use of the well known Schur's triangularization theorem [16, p.79], a result of Schur [30] and a result of A. Horn [15] on the diagonal and the eigenvalues of a Hermitian matrix.

We denote by $\mathrm{U}(n) = \{U \in \mathbb{C}_{n\times n} : U^*U = I_n\}$ the unitary group.

Theorem 2.1 (Schur's triangularization theorem) Given $A \in \mathbb{C}_{n\times n}$ with eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ in any prescribed order, there is a unitary matrix $U \in \mathrm{U}(n)$ such that UAU^{-1} is upper triangular and $\mathrm{diag}\,(UAU^{-1}) = \lambda$, where $\lambda = (\lambda_1, \ldots, \lambda_n)^T$.

Remark 2.2 Indeed $\mathrm{U}(n)$ in Theorem 2.1 can be replaced by the special unitary group $\mathrm{SU}(n)$.

We will use the following result of Hardy, Littlewood and Polyá [11, p.49] to establish Schur's result (Theorem 2.4). A nonnegative matrix D is called a *doubly stochastic* if the row sums and column sums of D are 1.

Theorem 2.3 (Hardy-Littlewood-Polyá) Let $\alpha, \beta \in \mathbb{R}^n$. Then $\alpha \prec \beta$ if and only if there exists a $n \times n$ doubly stochastic matrix D such that $\alpha = D\beta$.

Clearly the diagonal entries of a Hermitian matrix are real. Schur [30], [16, p.193] obtained the following nice result relating the diagonal and the eigenvalues of a Hermitian matrix $A \in \mathbb{C}_{n\times n}$.

Theorem 2.4 (Schur) Let $A \in \mathbb{C}_{n\times n}$ be a Hermitian matrix. The diagonal $d = (d_1, \ldots, d_n)^T \in \mathbb{R}^n$ of A is majorized by the vector $\lambda = (\lambda_1, \ldots, \lambda_n)^T \in \mathbb{R}^n$ of the eigenvalues of A, i.e., $d \prec \lambda$.

Proof: Let $A = (a_{ij})$. By the spectral decomposition [16, p.171] there exists $U \in \mathrm{U}(n)$ such that $A = U(\operatorname{diag}\lambda)\, U^{-1}$. By direct computation,

$$a_{ij} = \sum_{s=1}^{n} u_{is}\bar{u}_{js}\lambda_s.$$

Thus $d = D\lambda$, where $D = (d_{ij}) = (|u_{ij}|^2)$ is called an *orthostochastic* matrix. An orthostochastic matrix is clearly a doubly stochastic matrix. By Theorem 2.3 $d \prec \lambda$. $\square$

The converse of Theorem 2.4 was obtained by A. Horn [15]. The original proof is a long and intricate argument. The following simple proof was first obtained by Chan and Li [5] and later rediscovered by Zha and Zhang [35].

Theorem 2.5 (A. Horn) If $d, \lambda \in \mathbb{R}^n$ and $d \prec \lambda$, then there exists a real symmetric matrix $A \in \mathbb{R}_{n\times n}$ such that $\operatorname{diag} A = d$ and λ's are the eigenvalues of A.

Proof: Suppose $d \prec \lambda$. Since permutation similarity would not change the eigenvalues of a matrix but permute the diagonal entries, we may assume that $d_1 \geq d_2 \geq \cdots \geq d_n$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. We use induction on n. When $n = 2$, the real symmetric matrix

$$A = \begin{pmatrix} d_1 & \xi \\ \xi & d_2 \end{pmatrix}$$

has the desired property if we choose $\xi = \frac{\sqrt{2}}{2}(\lambda_1^2 + \lambda_2^2 - d_1^2 - d_2^2)^{1/2}$. Suppose the statement is true when $n = m$. Let $n = m + 1$. Suppose $2 \le j \le m + 1$ be the largest index such that $\lambda_{j-1} \ge d_1 \ge \lambda_j$. Clearly $\lambda_1 \ge \max\{d_1, \lambda_1 + \lambda_j - d_1\} \ge \min\{d_1, \lambda_1 + \lambda_j - d_1\} \ge \lambda_j$. Then there exists a 2×2 orthogonal matrix U_1 such that

$$U_1 \begin{pmatrix} \lambda_1 & \\ & \lambda_j \end{pmatrix} U_1^{-1} = \begin{pmatrix} d_1 & * \\ * & \lambda_1 + \lambda_j - d_1 \end{pmatrix}.$$

Set $U_2 := U_1 \oplus I_{m-1}$. Then

$$\begin{aligned} A_1 &:= U_2 \operatorname{diag}(\lambda_1, \lambda_j, \lambda_2, \ldots, \lambda_{j-1}, \lambda_{j+1}, \ldots, \lambda_{m+1}) U_2^{-1} \\ &= \begin{pmatrix} d_1 & * \\ * & \lambda_1 + \lambda_j - d_1 \end{pmatrix} \oplus \operatorname{diag}(\lambda_2, \ldots, \lambda_{j-1}, \lambda_{j+1}, \ldots, \lambda_{m+1}). \end{aligned}$$

We are going to show that

$$(d_2, \ldots, d_n) \prec (\lambda_1 + \lambda_j - d_1, \lambda_2, \ldots, \lambda_{j-1}, \lambda_{j+1}, \ldots, \lambda_{m+1}). \qquad (2.1)$$

To proceed, notice that $\lambda_{j-1} \ge d_1 \ge d_2$,

$$d_2 \le \max\{\lambda_1 + \lambda_j - d_1, \lambda_2, \ldots, \lambda_{j-1}, \lambda_{j+1}, \ldots, \lambda_{m+1}\}.$$

Moreover

$$\sum_{i=2}^{k} d_i \leq (k-1)d_1 \leq \sum_{i=2}^{k} \lambda_i, \quad k = 2, \ldots, j-1,$$
$$\sum_{i=2}^{k} d_i = \sum_{i=1}^{k} d_i - d_1 \leq \sum_{i=1}^{k} \lambda_i - d_1 = (\lambda_1 + \lambda_j - d_1) + \sum_{i=2, i\neq j}^{k} \lambda_i, \qquad k = j, \ldots, m,$$
$$\sum_{i=2}^{m+1} d_i = \sum_{i=1}^{m+1} d_i - d_1 = \sum_{i=1}^{m+1} \lambda_i - d_1 = (\lambda_1 + \lambda_j - d_1) + \sum_{i=2, i\neq j}^{m+1} \lambda_i.$$

Hence (2.1) is established. By the inductive hypothesis, there exists an $m \times m$ orthogonal matrix U_3 such that

$$U_3 \operatorname{diag}(\lambda_1 + \lambda_j - d_1, \lambda_2, \ldots, \lambda_{j-1}, \lambda_{j+1}, \ldots, \lambda_{m+1}) U_3^{-1}$$

has diagonal $(d_2, \ldots, d_{m+1})$. Then $A := U_4 A_1 U_4^{-1}$ has diagonal d and eigenvalues λ, where $U_4 := 1 \oplus U_3$. □

It is straight forward to show that the set of $n \times n$ doubly stochastic matrices Ω_n is a convex set in $\mathbb{R}_{n\times n}$. Birkhoff [16, p.527] showed that it is the convex hull of the permutation matrices.

Theorem 2.6 (Birkhoff) A matrix $D \in \mathbb{R}_{n\times n}$ is a doubly stochastic matrix if and only if it is a convex combination of permutation matrices, i.e., there are permutation matrices $P_1, \ldots, P_m$ and nonnegative scalars $\alpha_1, \ldots, \alpha_m \in \mathbb{R}$ such that $\alpha_1 + \ldots + \alpha_m = 1$ and $D = \alpha_1 P_1 + \ldots + \alpha_m P_m$.

Combining and rewriting Theorem 2.4 and Theorem 2.5, we have the following statement.

Theorem 2.7 (Schur and A.Horn) Let $\lambda \in \mathbb{R}^n$. Then

$$\begin{aligned} & \operatorname{diag}\{U(\operatorname{diag}(\lambda_1, \ldots, \lambda_n))\, U^{-1} : U \in \mathrm{O}(n)\} \\ = \;& \operatorname{diag}\{U(\operatorname{diag}(\lambda_1, \ldots, \lambda_n))\, U^{-1} : U \in \mathrm{U}(n)\} \\ = \;& \{d \in \mathbb{R}^n : d \prec \lambda\} \\ = \;& \operatorname{conv} S_n \lambda, \end{aligned}$$

where $\mathrm{U}(n)$ is the unitary group, $\mathrm{O}(n)$ is the orthogonal group, S_n is the full symmetric group on $\{1, \ldots, n\}$ and conv denotes the convex hull of the underlying set in $\mathbb{R}^n$.

Proof: Since $\mathrm{O}(n) \subset \mathrm{U}(n)$ and in view of Theorem 2.4,

$$\begin{aligned} & \operatorname{diag}\{U(\operatorname{diag}(\lambda_1, \ldots, \lambda_n))\, U^{-1} : U \in \mathrm{O}(n)\} \\ \subset \;& \operatorname{diag}\{U(\operatorname{diag}(\lambda_1, \ldots, \lambda_n))\, U^{-1} : U \in \mathrm{U}(n)\} \\ \subset \;& \{d \in \mathbb{R}^n : d \prec \lambda\}. \end{aligned}$$

By Theorem 2.5, $\{d \in \mathbb{R}^n : d \prec \lambda\} \subset \operatorname{diag}\{U(\operatorname{diag}(\lambda_1, \ldots, \lambda_n))\, U^{-1} : U \in \mathrm{O}(n)\}$. So we established the first two set equalities. Theorem 2.3 asserts that

$$\{d \in \mathbb{R}^n : d \prec \lambda\} = \Omega_n \lambda,$$

where Ω_n is the set of $n \times n$ doubly stochastic matrices. By Theorem 2.6

$$\{d \in \mathbb{R}^n : d \prec \lambda\} = \operatorname{conv} S_n \lambda$$

follows immediately. $\square$

Remark 2.8 We may replace $\mathrm{O}(n)$ by $\mathrm{SO}(n)$, the special orthogonal group and $\mathrm{U}(n)$ by $\mathrm{SU}(n)$, respectively, in Theorem 2.7.

We will make use of Theorem 2.1 and Theorem 2.4 to prove the result of Ky Fan, namely Theorem 1.2. Then we use Theorem 2.5 to prove Amir-Moéz-Horn-Mirsky's result, namely Theorem 1.3 which is the converse of Theorem 1.2. We first combine Theorem 1.2 and Theorem 1.3 together in the following statement.

Theorem 2.9 (Ky Fan-Amir-Moéz-Horn-Mirsky) Let $A \in \mathbb{C}_{n\times n}$ with eigenvalues $\lambda = (\lambda_1, \ldots, \lambda_n)^T$ and real singular values $\alpha = (\alpha_1, \ldots, \alpha_n)^T$. Then $\operatorname{Re}\lambda \prec \alpha$. Conversely, if $\lambda \in \mathbb{C}^n$, $\alpha \in \mathbb{R}^n$ such that $\operatorname{Re}\lambda \prec \alpha$, then there exists $A \in \mathbb{C}_{n\times n}$ with eigenvalues λ's and real singular values α's.

Proof: (Ky Fan) Suppose that A has eigenvalues $\lambda_1, \ldots, \lambda_n$. By Theorem 2.1 there exists a unitary matrix $U \in \mathrm{U}(n)$ such that

$$Y := UAU^{-1} = \begin{pmatrix} \lambda_1 & * & * \\ & \ddots & * \\ & & \lambda_n \end{pmatrix}$$

is upper triangular. Now $A = U^{-1}YU$ and

$$\frac{A + A^*}{2} = U^{-1}(\frac{Y + Y^*}{2})U,$$

thus A and Y have the same eigenvalues and real singular values. Now $\frac{1}{2}(Y + Y^*)$ is Hermitian and has diagonal entries $\operatorname{Re}\lambda$'s and eigenvalues α's. By Theorem 2.4, $\operatorname{Re}\lambda \prec \alpha$.

(Amir-Moéz-Horn-Mirsky) Conversely, suppose $\lambda \in \mathbb{C}^n$ and $\alpha \in \mathbb{R}^n$ are given such that $\operatorname{Re}\lambda \prec \alpha$. By Theorem 2.5, there is a Hermitian matrix $H = (h_{ij}) \in \mathbb{C}_{n\times n}$ with eigenvalues α's and diagonal entries $\operatorname{Re}\lambda$'s. The upper triangular matrix

$$A := \begin{pmatrix} \lambda_1 & 2h_{12} & \dots & 2h_{1n} \\ & \lambda_2 & \dots & 2h_{2n} \\ & & \ddots & \vdots \\ & & & \lambda_n \end{pmatrix} \in C_{n\times n}$$

has eigenvalues λ's and real singular values α's since $\frac{1}{2}(A + A^*) = H$, of which the eigenvalues are α's. This completes the proof. □

The key point in the proof of Ky Fan's result is to obtain an upper triangular matrix Y which is similar to the original matrix A under unitary similarity. Since eigenvalues and real singular values are invariant under unitary similarity, A and Y have the same eigenvalues and real singular values. So the real part of the eigenvalues of A are the real part of the diagonal of Y. Application of Schur's result on $\frac{(Y+Y^*)}{2}$ then finishes the proof. The key element in the proof of Amir-Moéz-Horn-Mirsky's result is Theorem 2.5.

We may view taking the eigenvalues of A (the real part, respectively) as a projection from $\mathbb{C}_{n\times n}$ to $\mathbb{C}^n$ ($\mathbb{R}^n$, respectively), after turning A into an upper triangular matrix via Schur's triangularization theorem. With this view in mind we rewrite Theorem 2.9 in the following form.

Theorem 2.10 Given $A \in \mathbb{C}_{n\times n}$. Let $W_n \subset \mathbb{C}_{n\times n}$ be the space of upper triangular matrices. Then

$$\operatorname{Re}\operatorname{diag}\{\{UAU^{-1} : U \in U(n)\} \cap W_n\} = \{d \in \mathbb{R}^n : d \prec \alpha\} = \operatorname{conv} S_n\alpha, \tag{2.2}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)^T$ is the vector of eigenvalues of $\frac{1}{2}(A + A^*)$.

Proof: The last equality is in Theorem 2.7. The first equality is another form of Theorem 2.9. $\square$

The above idea will be extended in Chapter 4 for the complex semisimple Lie algebras.

Chapter 3

Preliminaries

In this chapter we introduce some notations and basic concepts of complex semisimple Lie algebras. Most of the material can be found in any standard Lie theory textbook such as [12, 18, 20]. In particular we will review the details of the root space decomposition of classical semisimple Lie algebras.

In this chapter all Lie algebras are finite dimensional over $\mathbb{C}$, unless specifically noted.

Definition 3.1 [20, p.2] A finite dimensional vector space $\mathfrak{g}$ over $\mathbb{C}$ is called a *complex Lie algebra* if there is a product $[X, Y]$ for $X, Y \in \mathfrak{g}$ that is linear in each variable and satisfies

(a) $[X, X] = 0$ for all $X \in \mathfrak{g}$ (and hence $[X, Y] = -[Y, X]$) and

(b) the *Jacobi identity*

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0. \tag{3.1}$$

The *real* Lie algebra is defined analogously by changing the base field to $\mathbb{R}$ in the above definition.

A Lie *subalgebra* $\mathfrak{h}$ of $\mathfrak{g}$ is a subspace satisfying $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$. An *ideal* $\mathfrak{h}$ in $\mathfrak{g}$ is a subspace satisfying $[\mathfrak{h}, \mathfrak{g}] \subseteq \mathfrak{h}$. It is automatically a Lie subalgebra. The Lie algebra $\mathfrak{g}$ is said to be *abelian* if $[\mathfrak{g}, \mathfrak{g}] = 0$.

Definition 3.2 [20, p.8-9] Let $\mathfrak{g}$ be a finite dimensional Lie algebra. We define recursively

$$\mathfrak{g}^0 = \mathfrak{g}, \quad \mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}], \quad \mathfrak{g}^{j+1} = [\mathfrak{g}^j, \mathfrak{g}^j];$$

$$\mathfrak{g}_0 = \mathfrak{g}, \quad \mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}], \quad \mathfrak{g}_{j+1} = [\mathfrak{g}, \mathfrak{g}_j].$$

The sequence of ideals $\mathfrak{g}^0 \supset \mathfrak{g}^1 \supset \mathfrak{g}^2 \cdots$ is called the *derived series* of $\mathfrak{g}$ and the sequence $\mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \mathfrak{g}_2 \cdots$ is called the descending central series of $\mathfrak{g}$. We say that $\mathfrak{g}$ is *solvable* if $\mathfrak{g}^j = 0$ for some j and $\mathfrak{g}$ *nilpotent* if $\mathfrak{g}_j = 0$ for some j.

Example 3.3 [20, p.3] Let $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ denote the associative algebra of all $n \times n$ matrices with complex entries. We can turn $\mathfrak{g}$ into a Lie algebra by introducing the product

$$[X, Y] = XY - YX, \qquad X, Y \in \mathfrak{g}.$$

The set of all $n \times n$ upper triangular matrices $\mathfrak{u}$ is a solvable subalgebra in $\mathfrak{g}$.

A Lie algebra $\mathfrak{g}$ is *simple* if $\mathfrak{g}$ is nonabelian and has no proper ideals. Hence simple Lie algebras are at least 3-dimensional. A Lie algebra $\mathfrak{g}$ is *semisimple* if $\mathfrak{g}$ has no nonzero solvable ideals. Every simple Lie algebra is semisimple.

For any complex Lie algebra $\mathfrak{g}$ we get a linear map

$$\mathrm{ad} : \mathfrak{g} \to \operatorname{End} \mathfrak{g} \qquad (\operatorname{ad} X)(Y) = [X, Y], \quad X, Y \in \mathfrak{g}.$$

This linear map is called the *adjoint representation* of $\mathfrak{g}$, one of the most important maps in the theory of Lie algebras. If X and Y are in $\mathfrak{g}$, then $\operatorname{ad} X \circ \operatorname{ad} Y$ is a linear transformation from $\mathfrak{g}$ to itself, and it is meaningful to define

$$B(X, Y) = \operatorname{tr}\,(\operatorname{ad} X \circ \operatorname{ad} Y).$$

Then $B(\cdot, \cdot)$ is a symmetric bilinear form on $\mathfrak{g}$ known as the *Killing form* of $\mathfrak{g}$.

Killing form is a very useful tool in the theory of Lie algebras. Cartan [20, p.25] obtained two useful criteria for semisimplicity and solvability, respectively, of a Lie algebra $\mathfrak{g}$ by considering its Killing form.

Theorem 3.4 (Cartan's Criterion for Semisimplicity) The complex Lie algebra $\mathfrak{g}$ is semisimple if and only if the Killing form for $\mathfrak{g}$ is nondegenerate, i.e., if $X \in \mathfrak{g}$ and $B(X, Y) = 0$ for all $Y \in \mathfrak{g}$, then $X = 0$.

Theorem 3.5 (Cartan's Criterion for Solvability) The complex Lie algebra $\mathfrak{g}$ is solvable if and only if its Killing form satisfies $B(X, Y) = 0$ for all $X \in \mathfrak{g}$ and $Y \in [\mathfrak{g}, \mathfrak{g}]$.

Let us consider the root space decomposition of a complex Lie algebra. From now on we assume that $\mathfrak{g}$ is a complex semisimple Lie algebra unless specified.

Definition 3.6 A Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is called a *Cartan subalgebra* [12, p.130] if $\mathfrak{h}$ is maximal abelian and for each $H \in \mathfrak{h}$, the endomorphism $\operatorname{ad}_{\mathfrak{g}} H$ is semisimple, i.e., diagonalizable.

It is known that any finite dimensional complex Lie algebra has a Cartan subalgebra. The importance of the Cartan subalgebras for unraveling the structure of $\mathfrak{g}$ lies in the following fundamental fact.

Theorem 3.7 [18, p.84] The Cartan subalgebras of a complex semisimple Lie algebra $\mathfrak{g}$ are all conjugate under the adjoint group $\mathrm{Int}\,\mathfrak{g}$, where $\mathrm{Int}\,\mathfrak{g} \subset \mathrm{GL}(\mathfrak{g})$ is the analytic subgroup of $\mathrm{GL}(\mathfrak{g})$ with Lie subalgebra $\mathrm{ad}\,\mathfrak{g} \subset \mathrm{End}\,(\mathfrak{g})$.

Since all the Cartan subalgebras of $\mathfrak{g}$ are conjugate, there is no harm in choosing one, say, $\mathfrak{h}$. Since $\mathfrak{h}$ is abelian and the underlying field $\mathbb{C}$ is algebraically closed, the adjoint representation $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}\,(\mathfrak{g})$, restricted to $\mathfrak{h}$, splits $\mathfrak{g}$ up as a direct sum of one-dimensional subspaces. In other words, if $\alpha \in \mathfrak{h}^*$ and if we set

$$\mathfrak{g}^\alpha := \{X \in \mathfrak{g} : [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{h}\},$$

then $\mathfrak{g}$ is a direct sum of the $\mathfrak{g}^\alpha$. Since $\mathfrak{g}$ is finite dimensional, only finitely many of the $\mathfrak{g}^\alpha$ are nonzero. If $\alpha \neq 0$ and $\mathfrak{g}^\alpha \neq 0$, then α is called a *root* of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Members of $\mathfrak{g}^\alpha$ are called *root vectors* for the root α. Let Δ denote the set of all roots, a finite subset of $\mathfrak{h}^*$. We now have the well known *root space decomposition* of $\mathfrak{g}$ with respect to $\mathfrak{h}$.

Proposition 3.8 [20, p.88] Let $\mathfrak{g}$ be a complex semisimple Lie algebra and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Then

$$\mathfrak{g} = \mathfrak{g}^0 \dot{+} \dot{\sum}_{\alpha\in\Delta} \mathfrak{g}^\alpha. \tag{3.2}$$

It satisfies

(a) $\mathfrak{h} = \mathfrak{g}^0$,

(b) $[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] = \mathfrak{g}^{\alpha+\beta}$ (with $\mathfrak{g}^{\alpha+\beta}$ understood to be 0 if $\alpha + \beta$ is not a root),

(c) $B|_{\mathfrak{h}\times\mathfrak{h}}$ is nondegenerate; consequently there is an vector space isomorphism $\tau : \mathfrak{h}^* \longrightarrow \mathfrak{h}$ such that $H_\alpha := \tau(\alpha)$, $\alpha \in \mathfrak{h}^*$ satisfies $\alpha(H) = B(H, H_\alpha)$ for all $H \in \mathfrak{h}$,

(d) Δ spans $\mathfrak{h}^*$, the dual space of $\mathfrak{h}$,

(e) $\dim \mathfrak{g}^\alpha = 1$ for all $\alpha \in \Delta$.

The real span $V = \sum_{\alpha\in\Delta} \mathbb{R}\alpha \subset \mathfrak{h}^*$ is of dimension $\dim_{\mathbb{C}}\mathfrak{h}$. Since the Killing form $B(\cdot,\cdot)$ remains nondegenerate in the restriction to $\mathfrak{h}$, hence defines an isomorphism $\alpha \mapsto H_\alpha$ (as in Proposition 3.8(c) via Riesz's representation theorem) of $\mathfrak{h}^*$ onto $\mathfrak{h}$, and a bilinear form on the dual space $\mathfrak{h}^*$ of $\mathfrak{h}$ by transporting via $B(\cdot,\cdot)$ in the following fashion:

$$\langle \varphi, \psi \rangle = B(H_\varphi, H_\psi) = \varphi(H_\psi) = \psi(H_\varphi), \quad \varphi, \psi \in \mathfrak{h}^*, \tag{3.3}$$

where H_φ and H_ψ are defined in Proposition 3.8(c).

It turns out that the restriction of the bilinear form on V, $\langle \cdot,\cdot \rangle|_{V\times V}$, is real and positive definite, i.e., a real inner product so that V acquires the structure of an Euclidean space.

Theorem 3.9 [20, p.101] Let Δ be the root system of $(\mathfrak{g}, \mathfrak{h})$. Let $V = \sum_{\alpha\in\Delta} \mathbb{R}\alpha \subset \mathfrak{h}^*$ and $\mathfrak{h}_0 = \sum_{\alpha\in\Delta} \mathbb{R}H_\alpha \subset \mathfrak{h}$. Then

1. The real space V is an Euclidean space with the inner product $\langle\cdot,\cdot\rangle|_{V\times V}$, the restriction of the bilinear form defined by (3.3) to $V\times V$. Moreover, the members of V are exactly those linear functionals in $\mathfrak{h}^*$ that are real on $\mathfrak{h}_0$ and the restriction of the operation of those linear functionals from $\mathfrak{h}$ to $\mathfrak{h}_0$ is an $\mathbb{R}$ isomorphism of V onto $\mathfrak{h}_0^*$. In particular $V|_{\mathfrak{h}_0}=\mathfrak{h}_0^*$.

2. $\mathfrak{h}^*=V\oplus iV$.

3. $\mathfrak{h}=\mathfrak{h}_0\oplus i\mathfrak{h}_0$.

Theorem 3.10 [20, p.99-103] Let Δ be the root system of $(\mathfrak{g},\mathfrak{h})$. Then

1. Δ spans $\mathfrak{h}_0^*$,

2. the orthogonal transformations $s_\alpha(\varphi):=\varphi-\frac{2\langle\varphi,\alpha\rangle}{\langle\alpha,\alpha\rangle}\alpha$, for all $\varphi\in V$ where $\alpha\in\Delta$, carry Δ onto itself,

3. $\frac{2\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle}$ is an integer for any $\alpha,\beta\in\Delta$.

We now introduce a notion of positivity in V so that for any nonzero $\varphi\in V$ so that

1. exactly one of φ and $-\varphi$ is positive,

2. the sum of positive elements is positive and any positive multiple of a positive element is positive.

One way to define positivity is by means of a lexicographic ordering. Fix a basis $\{\varphi_1,\ldots,\varphi_n\}$ of V, define positivity as follows: We say that $\varphi>0$ if there exists an index k such that $\langle\varphi,\varphi_i\rangle=0$ for $1\le i\le k-1$, and $\langle\varphi,\varphi_k\rangle>0$. We say that a root α is *simple* if $\alpha > 0$ and if α does not decompose as $\alpha = \beta_1 + \beta_2$ with β_1 and β_2 both positive roots. Then there are n linearly independent simple roots $\alpha_1, \ldots, \alpha_n$ such that if a root β is written as $\beta = x_1\alpha_1 + \cdots + x_n\alpha_n$, then all the x_j's are integers with the same sign (if 0 is allowed to be positive or negative). Denote by Δ^+ the set of of all positive roots which would uniquely determine a set $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ of simple roots and Π is called a *simple system*.

We know from Proposition 3.10 that $s_\alpha : \mathfrak{h}_0^* \to \mathfrak{h}_0^*$ defined by

$$s_\alpha(\beta) = \beta - \frac{2\langle \beta, \alpha\rangle}{\langle \alpha, \alpha\rangle}\alpha, \quad \beta \in \mathfrak{h}_0^*,$$

is a reflection of the space $\mathfrak{h}_0^*$ that fixes Δ. The set $\{s_\alpha : \alpha \in \Delta\}$ generates a finite reflection group [19], denoted by $W(\mathfrak{g}, \mathfrak{h})$ or simply W, called the *Weyl group* of $(\mathfrak{g}, \mathfrak{h})$. It is clearly a subgroup of $\mathrm{O}(\mathfrak{h}_0^*)$, the orthogonal group of $\mathfrak{h}_0^*$. The Weyl group W can be generated by a smaller set of generators known as the *simple reflections* $\{s_\alpha : \alpha \in \Pi\}$. Notice that the Weyl group W can be viewed as a subgroup of $O(\mathfrak{h}_0)$ since we analogously have the reflections on $\mathfrak{h}_0$:

$$s_H(L) = L - \frac{B(H, L)}{B(H, H)}H, \qquad H, L \in \mathfrak{h}_0,$$

and it is clear that

$$\tau \circ s_\alpha(\beta) = s_{\tau(\alpha)}(\tau(\beta)), \qquad \alpha, \beta \in \mathfrak{h}_0^*.$$

In other words

$$\tau \circ s_\alpha \circ \tau^{-1} = s_{\tau(\alpha)},$$

and thus s_α is identified with $s_{\tau(\alpha)}$, where τ is given in Proposition 3.8 (c).

Let $\alpha \in \Delta$ and P^*_α be the hyperplane in $\mathfrak{h}^*_0$ defined by

$$\langle \alpha, \xi \rangle = 0, \quad \xi \in \mathfrak{h}^*_0.$$

Let

$$C = \mathfrak{h}^*_0 - \bigcup_{\alpha \in \Delta} P^*_\alpha,$$

i.e., C is the complement of the union of all P^*_α ($\alpha \in \Delta$). A component of C is said to be a *Weyl chamber* of $\mathfrak{h}^*_0$ with respect to Δ. The Weyl chamber

$$C_0 = \{\xi \in \mathfrak{h}^*_0 : (\alpha_i, \xi) > 0,\ i = 1, \ldots, n\}$$

is called the *fundamental Weyl chamber*. The choice of one among Δ^+, Π and C_0 determines the others.

Let us consider some standard models of the classical simple Lie algebras $\mathfrak{a}_n$, $\mathfrak{b}_n$, $\mathfrak{c}_n$ and $\mathfrak{d}_n$. We will look at their root space decompositions and Weyl groups. Let S_n be the full symmetric group on $\{1, \ldots, n\}$ and E_{ij} be the square matrix of appropriate size with (i, j)th entry 1 and 0 elsewhere.

Example 3.11 [12, p.186] [20, p.80] A model for the simple Lie algebra $\mathfrak{a}_n$ ($n \geq 1$) is $\mathfrak{g} := \mathfrak{sl}(n+1, \mathbb{C})$, the algebra of all $(n+1) \times (n+1)$ complex matrices with trace 0. Let $\mathfrak{h}$ be the set of all diagonal matrices in $\mathfrak{g}$. Then $\mathfrak{h}$ is a Cartan subalgebra of

$\mathfrak{g}$. Define $e_j \in \mathfrak{h}^*$ for $j = 1, \ldots, n+1$ by

$$e_j \begin{pmatrix} h_1 & & \\ & \ddots & \\ & & h_{n+1} \end{pmatrix} = h_j.$$

Direct matrix computation yields

$$(\operatorname{ad} H)E_{jk} = [H, E_{jk}] = (e_j(H) - e_k(H))E_{jk} \quad \text{for all } H \in \mathfrak{h}.$$

Thus E_{jk} is a simultaneous eigenvector for all $\operatorname{ad} H$, $H \in \mathfrak{h}$. The root system of $(\mathfrak{g}, \mathfrak{h})$ is

$$\Delta = \{\pm(e_j - e_k) : 1 \leq j < k \leq n\}.$$

The root space decomposition is

$$\mathfrak{g} = \mathfrak{h} \dot{+} \dot{\sum}_{e_j - e_k \in \Delta} \mathfrak{g}^{e_j - e_k},$$

where

$$\mathfrak{g}^{e_j - e_k} = \mathbb{C}E_{jk}.$$

The simple roots are

$$\Pi = \{e_j - e_{j+1} : j = 1, \ldots, n\}.$$

Notice that $e_j - e_k$ can be written as

$$e_j - e_k = (e_j - e_{j+1}) + \ldots + (e_k - e_{k+1}) \quad \text{if } j < k,$$

and

$$e_j - e_k = -(e_k - e_{k+1}) - \ldots - (e_{j-1} - e_j) \quad \text{if } j > k.$$

The Killing form of (complex) $\mathfrak{g}$ is [12, p.187]

$$B(X, Y) = 2(n+1)\,\mathrm{tr}\, XY, \qquad X, Y \in \mathfrak{g}.$$

For any $\alpha = e_j - e_k \in \Delta$, $H_{e_j - e_k} = \frac{1}{2(n+1)}(E_{jj} - E_{kk})$ (see Proposition 3.8 (c)). So $\mathfrak{h}_0$ is the space of real diagonal matrices which is identified with the hyperplane

$$\{(h_1, \ldots, h_{n+1})^T \in \mathbb{R}^{n+1} : \sum_{j=1}^{n+1} h_j = 0\}$$

in $\mathbb{R}^{n+1}$ naturally:

$$\mathfrak{h}_0 \ni \mathrm{diag}\,(h_1, \ldots, h_{n+1}) \mapsto (h_1, \ldots, h_{n+1})^T \in \mathbb{R}^{n+1}.$$

If $\alpha = e_j - e_{j+1} \in \Pi$, the reflection s_α (identified with s_{H_α}) acts on $H = (h_1, \ldots, h_{n+1})^T \in \mathfrak{h}_0$ by

$$\begin{aligned} s_\alpha(H) &= H - \frac{2B(H_\alpha, H)}{B(H_\alpha, H_\alpha)} H_\alpha \\ &= H - (h_j - h_{j+1}) H_\alpha \\ &= \mathrm{diag}\,(h_1, \ldots, h_{j-1}, h_{j+1}, h_j, h_{j+2}, \ldots, h_{n+1}). \end{aligned}$$

So the action of $s_{e_j - e_{j+1}}$ on $H = \mathrm{diag}\,(h_1, \ldots, h_{n+1})$ is to switch the jth and $(j+1)$st entries. Thus the Weyl group is the full symmetric group S_{n+1} on the set

$\{1, \ldots, n\}$. With the identification W acts on $\mathfrak{h}_0$ by

$$(h_1, \ldots, h_{n+1})^T \mapsto (h_{\sigma(1)}, \ldots, h_{\sigma(n+1)})^T, \quad \sigma \in S_{n+1}.$$

Example 3.12 [20, p.83] A model of the simple Lie algebra $\mathfrak{b}_n$ $(n \geq 1)$ is $\mathfrak{g} := \mathfrak{so}(2n+1, \mathbb{C})$, the set of all $(2n+1) \times (2n+1)$ complex skew symmetric matrices. The subalgebra

$$\mathfrak{h} = \left\{ H = \begin{pmatrix} 0 & ih_1 \\ -ih_1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & ih_n \\ -ih_n & 0 \end{pmatrix} \oplus (0) : h_1, \ldots, h_n \in \mathbb{C} \right\}.$$

is a Cartan subalgebra of $\mathfrak{g}$. Let

$$e_j(\text{above } H) = h_j, \quad 1 \leq j \leq n.$$

The root system of $(\mathfrak{g}, \mathfrak{h})$ is

$$\Delta = \{\pm e_j \pm e_k : 1 \leq j \neq k \leq n\} \cup \{\pm e_k : 1 \leq k \leq n\}.$$

The root space decomposition is

$$\mathfrak{g} = \mathfrak{h} \dot{+} \dot{\sum}_{\alpha \in \Delta} \mathfrak{g}^\alpha, \qquad \mathfrak{g}^\alpha = \mathbb{C} E_\alpha,$$

and with E_α as defined below. To define E_α, first let $j < k$ and let $\alpha = \pm e_j \pm e_k$. Then E_α is 0 except in the sixteen entries corresponding to the jth and kth pairs of indices, i.e.,

$$E_\alpha \quad = \quad \begin{array}{c} \begin{array}{cc} j & k \end{array} \\ \left(\begin{array}{cc} 0 & X_\alpha \\ -X_\alpha^T & 0 \end{array}\right) \end{array} \quad \begin{array}{c} \\ j \\ k \end{array}$$

with

$$X_{e_j-e_k} = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}, \qquad X_{e_j+e_k} = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix},$$

$$X_{-e_j+e_k} = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}, \qquad X_{-e_j-e_k} = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}.$$

To define E_α for $\alpha = \pm e_l$, write

$$E_\alpha \quad = \quad \begin{array}{c} \begin{array}{cc} \text{pair} & \text{entry} \\ l & 2n+1 \end{array} \\ \left(\begin{array}{cc} 0 & X_\alpha \\ -X_\alpha^T & 0 \end{array}\right) \end{array}$$

with 0's elsewhere and with

$$X_{e_l} = \begin{pmatrix} 1 \\ -i \end{pmatrix}, \qquad X_{-e_l} = \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

The simple roots are

$$\Pi = \{e_j - e_{j+1} : 1 \leq j \leq n-1\} \cup \{e_n\}.$$

The Killing form of (complex) $\mathfrak{g}$ is [12, p.189]

$$B(X,Y) = (2n-1)\,\mathrm{tr}\, XY, \qquad X, Y \in \mathfrak{g}.$$

Notice that $\mathfrak{h}_0 = \{H \in \mathfrak{h} : h_1, \dots, h_n \in \mathbb{R}\}$. If we identify $\mathfrak{h}_0$ with $\mathbb{R}^n$ in the natural way,

$$\mathfrak{h}_0 \ni \begin{pmatrix} 0 & ih_1 \\ -ih_1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & ih_n \\ -ih_n & 0 \end{pmatrix} \oplus (0) \mapsto (h_1, \dots, h_n)^T \in \mathbb{R}^n,$$

then

$$s_{e_j - e_{j+1}}(h_1, \dots, h_n) = (h_1, \dots, h_{j-1}, h_{j+1}, h_j, h_{j+2}, \dots, h_n),$$

i.e., switching the j and the $(j+1)$st entries, and

$$s_{e_n}(h_1, \dots, h_n) = (h_1, \dots, h_{n-1}, -h_n).$$

Thus the Weyl group W of $(\mathfrak{g}, \mathfrak{h})$ acts on $\mathfrak{h}_0$ by

$$(h_1, \dots, h_n)^T \mapsto (\pm h_{\sigma(1)}, \dots, \pm h_{\sigma(n)})^T, \quad \sigma \in S_n.$$

Example 3.13 [20, p.85] The simple Lie algebra $\mathfrak{c}_n$ $(n \geq 1)$ may be realized as $\mathfrak{g} := \mathfrak{sp}(n, \mathbb{C}) = \mathfrak{sp}(n) \oplus i\mathfrak{sp}(n)$, where is the set of $2n \times 2n$ complex matrices of the following form:

$$\mathfrak{sp}(n, \mathbb{C}) = \{X \in \mathfrak{sl}(2n, \mathbb{C}) : X^T J + JX = 0\},$$

where $J = J_{n,n}$ is the $2n \times 2n$ matrix

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

So [12, p.447]

$$\mathfrak{sp}(n, \mathbb{C}) = \{ \begin{pmatrix} A_1 & A_2 \\ A_3 & -A_1^T \end{pmatrix} : A_2, A_3 \in \mathbb{C}_{n \times n} \text{ complex symmetric}, \ A_1 \in \mathbb{C}_{n \times n} \}$$

and

$$\mathfrak{sp}(n) = \{ \begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix} : A, B \in \mathbb{C}_{n \times n}, A^* = -A, \ B^T = B \}.$$

Now

$$\mathfrak{h} := \{ \operatorname{diag}(h_1, \ldots, h_n, -h_1, \ldots, -h_n) : h_1, \ldots, h_n \in \mathbb{C} \}$$

is a Cartan subalgebra of $\mathfrak{g}$. Let $e_j \in \mathfrak{h}^*$ be

$$e_j(\operatorname{diag}(h_1, \ldots, h_n, -h_1, \ldots, -h_n)) = h_j, \quad 1 \leq j \leq n.$$

Then the root system of $(\mathfrak{g}, \mathfrak{h})$ is

$$\Delta = \{ \pm e_j \pm e_k : 1 \leq j \neq k \leq n \} \cup \{ \pm 2e_k : 1 \leq k \leq n \}.$$

The corresponding root spaces are

$$\begin{aligned}
\mathfrak{g}^{e_j - e_k} &= \mathbb{C}(E_{j,k} - E_{k+n,j+n}), & \mathfrak{g}^{2e_l} &= \mathbb{C}(E_{l,l+n}),\\
\mathfrak{g}^{e_j + e_k} &= \mathbb{C}(E_{j,k+n} + E_{k,j+n}), & \mathfrak{g}^{-2e_l} &= \mathbb{C}(E_{l+n,l}),\\
\mathfrak{g}^{-e_j - e_k} &= \mathbb{C}(E_{j+n,k} + E_{k+n,j}),
\end{aligned}$$

where $1 \leq j \neq k \leq n$ and $1 \leq l \leq n$. The simple roots are

$$\Pi = \{e_j - e_{j+1} : 1 \leq j \leq n-1\} \cup \{2e_n\}.$$

The Killing form of (complex) $\mathfrak{g}$ is [12, p.190]

$$B(X, Y) = (2n+2)\,\mathrm{tr}\, XY, \qquad X, Y \in \mathfrak{g}.$$

Notice that $\mathfrak{h}_0 = \{H \in \mathfrak{h} : h_1, \ldots, h_n \in \mathbb{R}\}$. If we identify $\mathfrak{h}_0$ with $\mathbb{R}^n$ in the natural way,

$$\mathfrak{h}_0 \ni \mathrm{diag}\,(h_1, \ldots, h_n, -h_1, \ldots, -h_n) \mapsto (h_1, \ldots, h_n)^T \in \mathbb{R}^n,$$

then Weyl group W of $(\mathfrak{g}, \mathfrak{h})$ acts on $\mathfrak{h}_0$ by

$$(h_1, \ldots, h_n)^T \mapsto (\pm h_{\sigma(1)}, \ldots, \pm h_{\sigma(n)})^T, \quad \sigma \in S_n.$$

Example 3.14 [20, p.85] The simple Lie algebra $\mathfrak{d}_n$ $(n \geq 3)$ may be realized as $\mathfrak{g} := \mathfrak{so}(2n, \mathbb{C}) = \mathfrak{so}(2n) + i\mathfrak{so}(2n)$, the algebra of $2n \times 2n$ complex skew symmetric matrices. The subalgebra

$$\mathfrak{h} = \left\{ H = \begin{pmatrix} 0 & h_1 \\ -h_1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & h_n \\ -h_n & 0 \end{pmatrix} : h_1, \ldots, h_n \in \mathbb{C} \right\},$$

is a Cartan subalgebra of $\mathfrak{g}$. The root system of $(\mathfrak{g}, \mathfrak{h})$ is

$$\Delta = \{\pm e_j \pm e_k : 1 \leq j < k \leq n\},$$

where

$$e_j(\text{above } H) = h_j, \quad 1 \leq j \leq n.$$

The corresponding root spaces $\mathfrak{g}^{\pm e_j \pm e_k}$, $\pm e_j \pm e_k \in \Delta$ are similar to those defined for $\mathfrak{b}_n$ in Example 3.12. The simple roots are

$$\Pi = \{e_j - e_{j+1} : 1 \leq j \leq n-1\} \cup \{e_{n-1} + e_n\}.$$

The Killing form of $\mathfrak{g}$ is [12, p.188]

$$B(X,Y) = (2n-2) \operatorname{tr} XY, \qquad X, Y \in \mathfrak{g}.$$

Notice that $\mathfrak{h}_0 = \{H \in \mathfrak{h} : h_1, \ldots, h_n \in \mathbb{R}\}$. If we identify $\mathfrak{h}_0$ with $\mathbb{R}^n$ similar to Example 3.12, i.e.,

$$\mathfrak{h}_0 \ni \begin{pmatrix} 0 & ih_1 \\ -ih_1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & ih_n \\ -ih_n & 0 \end{pmatrix} \mapsto (h_1, \ldots, h_n)^T \in \mathbb{R}^n,$$

then

$$s_{e_j-e_{j+1}}(h_1,\ldots,h_n)=(h_1,\ldots,h_{j-1},h_{j+1},h_j,h_{j+2},\ldots,h_n),$$

i.e., switching the jth and the $(j+1)$st entries, and

$$s_{e_{n-1}+e_n}(h_1,\ldots,h_n)=(h_1,\ldots,-h_n,-h_{n-1}).$$

Thus the The Weyl group W of $(\mathfrak{g},\mathfrak{h})$ acts on $\mathfrak{h}_0$ by

$$(h_1,\ldots,h_n)^T\mapsto(\pm h_{\sigma(1)},\ldots,\pm h_{\sigma(n)})^T,\quad \sigma\in S_n,$$

where the number of negative signs is even.

Chapter 4

The complex semisimple case

In this chapter we assume that $\mathfrak{g}$ is a complex semisimple Lie algebra and use the notations in the previous chapter.

A *complex Lie group* is a Lie group G possessing a complex analytic structure such that multiplication and inversion are holomorphic. For such a group the complex structure induces a multiplication-by-i mapping in the Lie algebra $\mathfrak{g}$ such that $\mathfrak{g}$ becomes a Lie algebra over $\mathbb{C}$ [20, p.55].

A semisimple Lie group G has a complexification $G^{\mathbb{C}}$ if $G^{\mathbb{C}}$ is a complex connected Lie group such that G is Lie subgroup of $G^{\mathbb{C}}$ and the Lie algebra of $G^{\mathbb{C}}$ is the complexification of the Lie algebra of G [20, p.404]. Not every semisimple Lie group has a complexification. Even if G has a complexification, the complexification is not necessarily unique up to isomorphism. But if G is compact, $G^{\mathbb{C}}$ exists and is unique [20, p.375].

Let K be a real compact connected semisimple Lie group, G its complexification, and let $\mathfrak{k}$ and $\mathfrak{g}$ be their respective Lie algebras. Thus $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$. We fix a maximal torus T of K and denote its Lie algebra by $\mathfrak{t}$. Then $\mathfrak{h} = \mathfrak{t} \oplus i\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$ (now $i\mathfrak{t}$ is $\mathfrak{h}_0$ in Chapter 3). Let the root space decomposition of $\mathfrak{g}$ be

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}^{\alpha},$$

where Δ is the root system of $(\mathfrak{g}, \mathfrak{h})$ and $\mathfrak{g}^\alpha$ is the root space of the root $\alpha \in \Delta$. Fix a simple system Π for Δ. The set of positive roots (with respect to Π) is denoted by Δ^+. The Weyl group of $(\mathfrak{g}, \mathfrak{h})$ will be denoted by W. A subalgebra of $\mathfrak{g}$ is called a *Borel subalgebra* of $\mathfrak{g}$ if it is a maximal solvable subalgebra of $\mathfrak{g}$. We introduce the maximal nilpotent subalgebras $\mathfrak{n}$ and $\mathfrak{n}^-$ of $\mathfrak{g}$:

$$\mathfrak{n} = \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}^\alpha, \quad \mathfrak{n}^- = \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}^{-\alpha}.$$

Then

$$\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n} \tag{4.1}$$

is a (standard) Borel subalgebra of $\mathfrak{g}$. Let B be the corresponding Borel subgroup of G.

An *automorphism* of $\mathfrak{g}$ is an invertible linear map $L \in \mathrm{GL}(\mathfrak{g})$ that respects bracket

$$[L(X), L(Y)] = L[X, Y] \qquad \text{for all} \quad X, Y \in \mathfrak{g}.$$

Denote by $\mathrm{Aut}\,\mathfrak{g}$ the group of automorphisms of $\mathfrak{g}$. The adjoint group $\mathrm{Int}\,(\mathfrak{g})$ (see the definition in Theorem 3.7) is a normal subgroup of $\mathrm{Aut}\,\mathfrak{g}$. The adjoint group $\mathrm{Int}\,(\mathfrak{g})$ is generated by $e^{\mathrm{ad}\,X}$ $(= \mathrm{Ad}(e^X)$ [12, p.128]), where $X \in \mathfrak{g}$. Its elements are called *inner automorphisms*. Since the complexification G of K is connected, $\mathrm{Ad}(G) = \mathrm{Int}\,(\mathfrak{g})$ [12, p.129] and is the identity component of $\mathrm{Aut}\,(\mathfrak{g})$ [12, p.132].

We have the following facts about the Borel subalgebras of a complex Lie algebra $\mathfrak{g}$.

Theorem 4.1 [18, p.84] The Borel subalgebras of a complex semisimple Lie algebra $\mathfrak{g}$ are all conjugate under $\mathrm{Int}\,\mathfrak{g}$.

The following is a recent generalization of the Schur triangularization theorem (Theorem 2.1) in the context of complex semisimple Lie algebras by Djoković and Tam [6].

Proposition 4.2 (Djoković and Tam) Let $\mathfrak{g}$ be a complex semisimple Lie algebra.

1. The Borel subalgebras of $\mathfrak{g}$ are all conjugate under $\mathrm{Ad}K$.
2. Let $\mathfrak{b}$ be any Borel subalgebra of $\mathfrak{g}$. Then $\mathrm{Ad}K(X)$ intersects $\mathfrak{b}$ for each $X \in \mathfrak{g}$.

Proof: (1) Let $\mathfrak{b}'$ be any Borel subalgebra and let $\mathfrak{b}$ the standard Borel algebra given in (4.1). By Theorem 4.1 all Borel subalgebras are conjugate under $\mathrm{Int}\,\mathfrak{g} = \mathrm{Ad}G$. So there is $g \in G$ such that $\mathfrak{b}' = \mathrm{Ad}(g)\mathfrak{b}$. The global Iwasawa decomposition [12, p.275] states that $G = KAN$ (G is viewed as a real group), where K, A, and N denote the analytic subgroups of G with Lie algebras $\mathfrak{k}$, $i\mathfrak{t}$, and $\mathfrak{n}$, respectively. Thus $G = KB$, where $B \supset AN$ is the analytic subgroup of G with Lie algebra $\mathfrak{b}$. Therefore there exist $k \in K$ and $b \in B$ such that $g = kb$. So

$$\mathfrak{b}' = \mathrm{Ad}(g)\mathfrak{b} = \mathrm{Ad}(k)\mathrm{Ad}(b)\mathfrak{b} = \mathrm{Ad}(k)\mathfrak{b}.$$

(2) Let $\mathfrak{b}$ be any Borel subalgebra of $\mathfrak{g}$. For any $X \in \mathfrak{g}$, X is contained in some Borel subalgebra $\mathfrak{b}'$ of $\mathfrak{g}$. Thus $\mathrm{Ad}(k)X \in \mathrm{Ad}(k)\mathfrak{b}' = \mathfrak{b}$ by the first part for some $k \in K$. □

Let θ be the *Cartan involution* of $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$, i.e., θ is identity on $\mathfrak{k}$ and negative identity on $i\mathfrak{k}$. Then θ is semilinear, i.e., $\theta(\lambda X + \mu Y) = \overline{\lambda} X + \overline{\mu} Y,\ X, Y \in \mathfrak{g}$, respects the bracket $\theta[X, Y] = [\theta X, \theta Y]$ and is an involution, i.e., $\theta^2 = 1$. Moreover $B_\theta(X, Y) := 2\mathrm{Re}\, B(X, \theta Y)$ is an inner product on $\mathfrak{g}$ and $\mathfrak{k}$ and $i\mathfrak{k}$ are orthogonal with respect to $B_\theta(\cdot, \cdot)$.

Example 4.3 Consider $\mathfrak{g} := \mathfrak{sl}(n, \mathbb{C})$. The Hermitian decomposition $\mathfrak{sl}(n, \mathbb{C}) = \mathfrak{su}(n) + i\mathfrak{su}(n)$ suggests that $\mathfrak{k} = \mathfrak{su}(n)$ and the corresponding Cartan involution is $\theta(X) = -X^*,\ X \in \mathfrak{g}$. Moreover $\theta(E_{ij}) = -E_{ji},\ i \neq j$ and $B_\theta(X, Y) = -B(X, \theta(Y)) = 2(n+1)\mathrm{tr}\, XY^*$.

Proposition 4.4 $\theta(\mathfrak{h}) = \mathfrak{h}$ and $\theta(\mathfrak{g}^\alpha) = \mathfrak{g}^{-\alpha}$ for all $\alpha \in \Delta$.

Proof: Clearly $\theta(i\mathfrak{t}) = i\mathfrak{t}$ and $\theta\mathfrak{t} = \mathfrak{t}$ so that $\theta\mathfrak{h} = \mathfrak{h}$. Let $X \in \mathfrak{g}^\alpha$ and $H := iH_1 + H_2 \in \mathfrak{t} \dot{+} i\mathfrak{t},\ H_1, H_2 \in i\mathfrak{t}$. Then $[H, \theta X] = \theta[\theta H, X] = \theta[\theta(iH_1 + H_2), X] = \theta[\theta iH_1, X] + \theta[\theta H_2, X] = \theta[iH_1, X] + \theta[-H_2, X] = \theta i\alpha(H_1)X - \alpha(H_2)\theta X = -\alpha(H)\theta X$ since α takes real values on $i\mathfrak{t}$. $\Box$

We will use the following result of Kostant [22] to prove Theorem 4.6.

Theorem 4.5 (Kostant) Use the same notations of K, $\mathfrak{t}$ and W as above. Let $\pi : \mathfrak{k} \to \mathfrak{t}$ be the orthogonal projection with respect to the Killing form. If $Z \in \mathfrak{t}$, then $\pi(\mathrm{Ad}K(Z)) = \mathrm{conv}\, WZ$,

For any complex semisimple Lie algebra $\mathfrak{g}$ whose connected Lie group is G, it is known that [20, p.302] that $\mathfrak{g}$ always has a compact real form $\mathfrak{k}$, i.e., $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$. Let K be a connected subgroup of G corresponding to $\mathfrak{k}$.

The following is an extension of Ky Fan-Amir-Moéz-Horn-Mirsky's result in the context of complex semisimple Lie algebras.

Theorem 4.6 Let $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$ be a complex semisimple Lie algebra, where $\mathfrak{k}$ is the Lie algebra of a semisimple Lie group K. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{k}$. Let $\pi : \mathfrak{g} \to i\mathfrak{t}$ be the orthogonal projection with respect to the Killing form of the realification $\mathfrak{g}^{\mathbb{R}}$ of $\mathfrak{g}$. If $Z \in i\mathfrak{t}$, then

$$\cup_{X\in\mathfrak{k}} \pi(\mathrm{Ad}K(X+Z) \cap \mathfrak{b}) = \mathrm{conv}\, WZ, \tag{4.2}$$

where $\mathfrak{b}$ is given in (4.1) and W is the Weyl group of $(\mathfrak{g}, \mathfrak{h})$. Equivalently

$$\pi((\mathfrak{k} + \mathrm{Ad}K(Z)) \cap \mathfrak{b}) = \mathrm{conv}\, WZ. \tag{4.3}$$

In particular, for each $U \in \mathfrak{g}$, $\pi(\mathrm{Ad}K(U) \cap \mathfrak{b}) \subset \mathrm{conv}\, WZ$, where $Z \in \mathrm{Ad}K(\frac{1}{2}(U - \theta U)) \cap i\mathfrak{t}$.

Proof: Notice that

$$\begin{aligned}
& \cup_{X\in\mathfrak{k}} \pi(\mathrm{Ad}K(X+Z) \cap \mathfrak{b}) \\
= \;& \pi(\cup_{X\in\mathfrak{k}} \mathrm{Ad}K(X+Z) \cap \mathfrak{b}) \\
= \;& \pi([\cup_{X\in\mathfrak{k}} \cup_{k\in K} (\mathrm{Ad}k(X) + \mathrm{Ad}k(Z))] \cap \mathfrak{b}) \\
= \;& \pi([\cup_{k\in K} \cup_{X\in\mathfrak{k}} (\mathrm{Ad}k(X) + \mathrm{Ad}k(Z))] \cap \mathfrak{b})) \\
= \;& \pi(\cup_{k\in K} (\mathfrak{k} + \mathrm{Ad}k(Z)) \cap \mathfrak{b})) \quad \text{since } \mathrm{Ad}_G(k)|_{\mathfrak{k}} = \mathrm{Ad}_K(k) \in \mathrm{Aut}\,(\mathfrak{k}) \\
= \;& \pi((\mathfrak{k} + \mathrm{Ad}K(Z)) \cap \mathfrak{b})
\end{aligned}$$

and Proposition 4.2 ensures that $\pi((\mathfrak{k} + \mathrm{Ad}K(Z)) \cap \mathfrak{b})$ is nonempty.

We now give a proof of (4.2). Let $Y \in \mathfrak{k} + \mathrm{Ad}K(Z)$. Then $Y = X + \mathrm{Ad}k(Z)$ for some $X \in \mathfrak{k}$ and $k \in K$. Since $\mathfrak{k} \perp i\mathfrak{t}$ under $B_\theta(\cdot, \cdot)$,

$$\pi(Y) = \pi(X + \mathrm{Ad}k(Z)) = \pi(\mathrm{Ad}k(Z)).$$

By Theorem 4.5 $\pi(\mathrm{Ad}K(Z)) = \mathrm{conv}\, WZ$. Thus $\pi(Y) \in \mathrm{conv}\, WZ$ and

$$\pi((\mathfrak{k} + \mathrm{Ad}K(Z)) \cap \mathfrak{b}) \subset \pi(\mathfrak{k} + \mathrm{Ad}K(Z)) \subset \mathrm{conv}\, WZ. \tag{4.4}$$

Conversely, let $\beta \in \mathrm{conv}\, WZ$. By Theorem 4.5 again, there exists $Y \in \mathrm{Ad}K(Z)$ such that $\pi(Y) = \beta$. Recall the root space decomposition from Proposition 3.8

$$\mathfrak{g} = \mathfrak{h} \dot{+} \dot{\sum}_{\alpha \in \Delta^+} (\mathfrak{g}^\alpha \oplus \mathfrak{g}^{-\alpha}).$$

The direct sum $\mathfrak{g}^\alpha \oplus \mathfrak{g}^{-\alpha}$ is not orthogonal. Write

$$Y = Y_0 + \sum_{\alpha \in \Delta^+} (Y_\alpha + Y_{-\alpha}),$$

where $Y_0 \in \mathfrak{h}$, $Y_\alpha \in \mathfrak{g}^\alpha$ and $Y_{-\alpha} \in \mathfrak{g}^{-\alpha}$. Since $Y \in i\mathfrak{k}$, $i\mathfrak{k}$ is the -1 eigenspace of θ, we have

$$-Y_0 + \sum_{\alpha \in \Delta^+} (-Y_\alpha - Y_{-\alpha}) = -Y = \theta Y = \theta Y_0 + \sum_{\alpha \in \Delta^+} (\theta Y_\alpha + \theta Y_{-\alpha}).$$

Since the sums are direct and $\theta\mathfrak{g}^\alpha = \mathfrak{g}^{-\alpha}$ ($\alpha \in \Delta$), it follows that $Y_0 \in \mathfrak{h} \cap i\mathfrak{k}$, i.e., $Y_0 \in i\mathfrak{t}$. Moreover $Y_{-\alpha} = -\theta Y_\alpha$ for all $\alpha \in \Delta$. Then

$$Y = Y_0 + \sum\nolimits_{\alpha\in\Delta^+}(Y_\alpha - \theta Y_\alpha),$$

and $Y_0 = \pi(Y) = \beta$. Set

$$X := \sum\nolimits_{\alpha\in\Delta^+}(Y_\alpha + \theta Y_\alpha) \in \mathfrak{k}.$$

Then

$$X + Y = Y_0 + 2\sum\nolimits_{\alpha\in\Delta^+} Y_\alpha \in (X + \mathrm{Ad}K(Z)) \cap \mathfrak{b}.$$

Clearly $\pi(X+Y) = \pi(Y) = \beta$. This proves

$$\pi((\mathfrak{k} + \mathrm{Ad}K(Z)) \cap \mathfrak{b}) \supset \mathrm{conv}\, WZ. \tag{4.5}$$

Combining (4.4) and (4.5) we conclude

$$\pi((\mathfrak{k} + \mathrm{Ad}K(Z)) \cap \mathfrak{b}) = \mathrm{conv}\, WZ.$$

For $U \in \mathfrak{g}$, $\mathrm{Ad}K(U) \cap \mathfrak{b}$ is nonempty by Proposition 4.2. We decompose $U = \frac{1}{2}(U + \theta U) + \frac{1}{2}(U - \theta U)$. Clearly

$$\begin{aligned}\pi(\mathrm{Ad}K(U) \cap \mathfrak{b}) &= \pi(\mathrm{Ad}K(\frac{1}{2}(U+\theta U) + \frac{1}{2}(U - \theta U)) \cap \mathfrak{b}) \\ &\subset \cup_{X\in\mathfrak{k}}\pi(\mathrm{Ad}K(X + \frac{1}{2}(U - \theta U)) \cap \mathfrak{b})\end{aligned}$$

$$= \operatorname{conv} WZ,$$

where $Z \in \operatorname{Ad}K(\frac{1}{2}(U - \theta U)) \cap i\mathfrak{t}$. $\Box$

Remark 4.7 When $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, the theorem is simply Ky Fan-Amir-Moéz-Horn-Mirsky's result with an appropriate translation.

Remark 4.8 Let $N(T)$ denote the normalizer of $\mathfrak{t}$ in K with respect to the adjoint action of K, i.e., $N(T) = \{k \in K : \operatorname{Ad}(k)\,\mathfrak{t} = \mathfrak{t}\}$. Consider the group homomorphism $\nu : N(T) \to \mathrm{O}(\mathfrak{t})$, $n \mapsto (\operatorname{Ad} n)|_{\mathfrak{t}}$, from $N(T)$ into $\mathrm{O}(\mathfrak{t})$ which denotes the group of orthogonal linear transformations of the real space $\mathfrak{t}$. The kernel of ν is T and W is the image of ν. So ν defines a group isomorphism between the group $N(T)/T$ onto the Weyl group W of $(\mathfrak{g}, \mathfrak{h})$ if we identify $i\mathfrak{h}_0$ and $\mathfrak{t}$. Knapp calls $N(T)/T$ the analytically defined Weyl group and W the algebraically defined Weyl group [20, p.207].

Remark 4.9 The statement of Theorem 4.6 remains true when the Cartan subspace $i\mathfrak{k}$ is replaced by $\mathfrak{k}$. But we need to change the projection π to $\pi_1 : \mathfrak{g} \to \mathfrak{t}$ and assume that $Z \in \mathfrak{t}$. This becomes the generalization of the Ky Fan-Amir-Moéz-Horn-Mirsky's result about the imaginary part of the eigenvalues and imaginary singular values.

Remark 4.10 Notice that $\mathfrak{g}$ is an inner product space equipped with the natural inner product $\langle X, Y\rangle := B_\theta(X, Y) = -B(X, \theta Y)$, and $\mathfrak{k}$ and $i\mathfrak{k}$ are orthogonal since $[\mathfrak{k}, i\mathfrak{k}] \subset i\mathfrak{k}$, $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$ and $[i\mathfrak{k}, i\mathfrak{k}] \subset \mathfrak{k}$. Thus

$$\|X\|^2 = \|\frac{1}{2}(X + \theta X)\|^2 + \|\frac{1}{2}(X - \theta X)\|^2.$$

When $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, it simply asserts that the square of the Frobenius norm of X is the sum of squares of the real and imaginary singular values [1, Theorem 5].

The complex classical Lie algebras $\mathfrak{a}_n$ $(n \geq 1)$, $\mathfrak{b}_n$ $(n \geq 1)$, $\mathfrak{c}_n$ $(n \geq 1)$ and $\mathfrak{d}_n$ ($n \geq 2$ and $\mathfrak{d}_1$ is not semisimple) are semisimple. Indeed they are simple except $\mathfrak{d}_2$. So Theorem 4.6 holds for them. The following

$$\mathfrak{a}_1 \approx \mathfrak{b}_1 \approx \mathfrak{c}_1, \quad \mathfrak{b}_2 \approx \mathfrak{c}_2, , \quad \mathfrak{a}_3 \approx \mathfrak{d}_3, \quad \mathfrak{d}_2 \approx \mathfrak{a}_1 \oplus \mathfrak{a}_1.$$

are the only isomorphisms [12, p.465] which hold between the complex classical Lie algebras. All occur among low dimensional algebras. We now draw the pictures of the convex hull $\operatorname{conv} W\beta$ for $\beta \in i\mathfrak{t}$ for several low dimensional cases.

Example 4.11 (1) $\mathfrak{a}_1$: For $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, $\mathfrak{h} = \left\{ \begin{pmatrix} h_1 & 0 \\ 0 & -h_1 \end{pmatrix} : h_1 \in \mathbb{C} \right\}$ and the only simple root is $\alpha_1 = e_1 - e_2 \in \mathfrak{h}^*$ (See the notations in Example 3.11), where

$$\alpha_1 \begin{pmatrix} h_1 & 0 \\ 0 & -h_1 \end{pmatrix} = 2h_1.$$

The Weyl group of $(\mathfrak{g}, \mathfrak{h})$ is $W = \{1, s_{\alpha_1}\}$, where $s_{\alpha_1}(\alpha_1) = -\alpha_1$ and $s_{\alpha_1}(-\alpha_1) = \alpha_1$. Thus W acts on $\mathfrak{h}_0 = i\mathfrak{t} = \left\{ \begin{pmatrix} h_1 & 0 \\ 0 & -h_1 \end{pmatrix} : h_1 \in \mathbb{R} \right\}$:

$$\begin{pmatrix} h_1 & 0 \\ 0 & -h_1 \end{pmatrix} \mapsto \begin{pmatrix} h_1 & 0 \\ 0 & -h_1 \end{pmatrix}, \text{ or } \begin{pmatrix} -h_1 & 0 \\ 0 & h_1 \end{pmatrix}.$$

So if $\beta = \operatorname{diag}(\beta_1, -\beta_1) \in \mathfrak{h}_0$, then

$$\operatorname{conv} W\beta = \left\{ \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix} : -\beta_1 \leq x \leq \beta_1 \right\},$$

which is identified with the line segment $[-\beta_1, \beta_1]$.

(2) $\mathfrak{a}_2$: For $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$, the simple roots are $\alpha_1 = e_1 - e_2$, $\alpha_2 = e_2 - e_3$ (See the notations in Example 3.11). The Weyl group is generated by the reflections $\{s_{\alpha_1}, s_{\alpha_2}\}$ with respective to the hyperplanes perpendicular to α_1 and α_2, respectively. If we identify $\mathfrak{h}_0 = i\mathfrak{t}$ with the hyperplane $H_0 := \{x \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\}$ in $\mathbb{R}^3$. The Weyl group W of $(\mathfrak{g}, \mathfrak{h})$ acts on $\mathfrak{h}_0$ by

$$(h_1, h_2, h_3)^T \mapsto (h_{\sigma(1)}, h_{\sigma(2)}, h_{\sigma(3)})^T, \quad \sigma \in S_3,$$

where S_3 is the full symmetric group on the set $\{1, 2, 3\}$. For any $\beta \in i\mathfrak{t}$, $W\beta$ consists of six points on the hyperplane H_0, hence $\operatorname{conv} W\beta$ is a hexagon including its interior. The shaded region is the intersection of $\operatorname{conv} W\beta$ and the (closed) fundamental Weyl chamber.

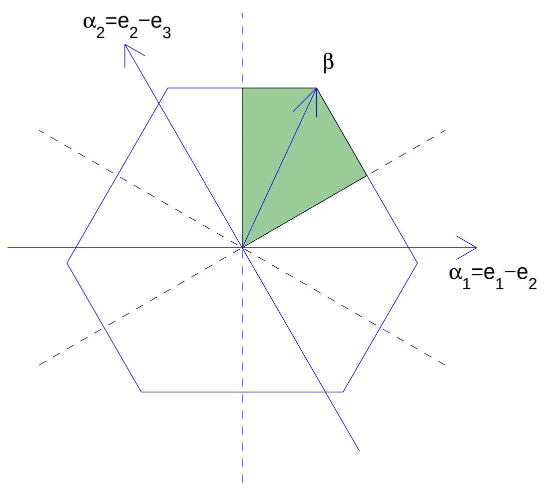

Figure 4.1: The Convex Hull $\operatorname{conv} W\beta$ For $\mathfrak{a}_2$

(3) $\mathfrak{b}_2$: For $\mathfrak{g} = \mathfrak{so}(5, \mathbb{C})$, the simple roots are $\alpha_1 = e_1 - e_2$, $\alpha_2 = e_2$ (See Example 3.12). The Weyl group is generated by the reflections $\{s_{\alpha_1}, s_{\alpha_2}\}$ and if we identify $\mathfrak{h}_0 = i\mathfrak{t}$ with $\mathbb{R}^2$ in the natural way, then the Weyl group acts on $i\mathfrak{t}$ by

$$(h_1, h_2)^T \mapsto (\pm h_{\sigma(1)}, \pm h_{\sigma(2)})^T, \quad \sigma \in S_2.$$

For any $\beta \in i\mathfrak{t}$, the convex hull $\operatorname{conv} W\beta$ is an octagon including its interior. The shaded region is the intersection of $\operatorname{conv} W\beta$ and the (closed) fundamental Weyl chamber.

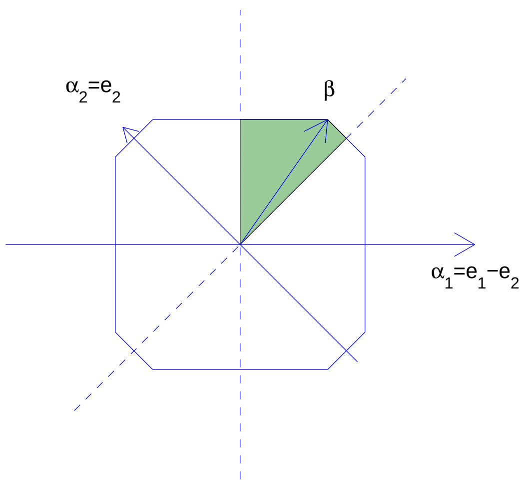

Figure 4.2: The Convex Hull $\operatorname{conv} W\beta$ For $\mathfrak{b}_2$

(4) $\mathfrak{c}_2$: Similar to $\mathfrak{so}(5, \mathbb{C})$, the simple roots of $\mathfrak{g} = \mathfrak{sp}(2, \mathbb{C})$ are $\alpha_1 = e_1 - e_2$, $\alpha_2 = 2e_2$ (See Example 3.13). The graph of the convex hull is identical with the one of $\mathfrak{g} = \mathfrak{so}(5, \mathbb{C})$ since they have the same Weyl group.

(5) $\mathfrak{d}_2$: For $\mathfrak{g} = \mathfrak{so}(4, \mathbb{C})$, the simple roots are $\alpha_1 = e_1 - e_2$, $\alpha_2 = e_1 + e_2$ (See Example 3.14). The Weyl group is generated by the reflections $\{s_{\alpha_1}, s_{\alpha_2}\}$

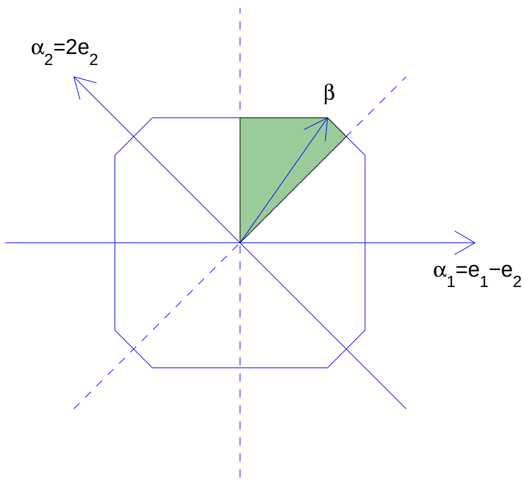

Figure 4.3: The Convex Hull $\operatorname{conv} W\beta$ For $\mathfrak{c}_2$

and acts on $\mathfrak{h}_0 = i\mathfrak{t}$ by

$$(h_1, h_2)^T \mapsto \pm(h_{\sigma(1)}, h_{\sigma(2)})^T, \quad \sigma \in \ S_2,$$

if we identify $i\mathfrak{t}$ with $\mathbb{R}^2$. The graph of $\operatorname{conv} W\beta$ for $\beta \in i\mathfrak{t}$ is a rectangle including its interior. The shaded region is the intersection of $\operatorname{conv} W\beta$ and the (closed) fundamental Weyl chamber.

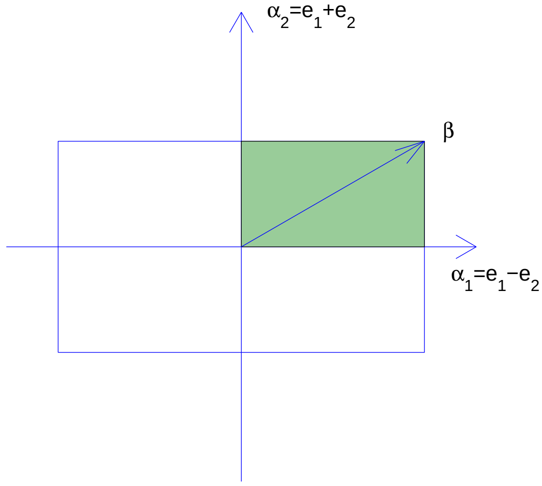

Figure 4.4: The Convex Hull $\operatorname{conv} W\beta$ For $\mathfrak{o}_2$

Chapter 5

The inequalities associated with $\mathfrak{a}_n$ and $\mathfrak{c}_n$

In Chapter 4 we obtained the extension of Ky Fan-Amir-Moéz-Horn-Mirsky's result in the context of complex semisimple Lie algebras. In this chapter and the next we will analyze the result for the classical Lie algebras $\mathfrak{a}_n$, $\mathfrak{b}_n$, $\mathfrak{c}_n$ and $\mathfrak{d}_n$, which are realized in matrix models. We obtain some interesting inequalities which are similar to majorization. In this chapter, we use the same notations as we did in the previous chapters. Let K be a real semisimple compact connected Lie group whose complexification is G. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K, respectively. Let $\mathfrak{h} = \mathfrak{t} \oplus i\mathfrak{t}$ be the Cartan subalgebra of $\mathfrak{g}$, where $\mathfrak{t}$ is the Cartan subalgebra of $\mathfrak{k}$. Let $\mathfrak{b}$ be the (standard) Borel subalgebra of $\mathfrak{g}$:

$$\mathfrak{b} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta^+} \mathfrak{g}^\alpha,$$

where Δ^+ is the set of positive roots in the root system Δ of $(\mathfrak{g}, \mathfrak{h})$. Denote by $\Pi = \{\alpha_j,\ j = 1, \dots, n\}$ the set of simple roots and set $V = \sum_{i=1}^n \mathbb{R}\alpha_i$. Let $\rho : \mathfrak{g} \to \mathfrak{h}$ and $\pi : \mathfrak{g} \to i\mathfrak{t}$ be the orthogonal projections with respect to the Killing form. In this chapter, we use $(\cdot, \cdot)$ to denote the Killing form $B(\cdot, \cdot)$ of $\mathfrak{g}$ restricted to $\mathfrak{h}_0 = i\mathfrak{t}$, and identify $(i\mathfrak{t})^*$ with $i\mathfrak{t}$ under the identification defined by Proposition 3.8 (c) $(\tau : \alpha \mapsto H_\alpha)$. Under this identification, the (closed) fundamental Weyl chamber of $\mathfrak{g}$ is

$$(i\mathfrak{t})_+ = \{H \in i\mathfrak{t} : \alpha_j(H) \geq 0,\ j = 1, \ldots, n\}.$$

The vectors $\frac{2\alpha_i}{(\alpha_i, \alpha_i)}$, $i = 1, \ldots, n$ again form a basis of V. Let $\{\lambda_1, \ldots, \lambda_n\}$ be the dual basis:

$$\frac{2(\lambda_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij}, \qquad i, j = 1, \ldots, n.$$

They are called the *fundamental dominant weights* [18, p.67].

Proposition 5.1 Let $\mathfrak{g}$ be a complex semisimple Lie algebra. The (closed) fundamental Weyl chamber $(i\mathfrak{t})_+$ for $\mathfrak{g}$ is the cone C generated by the fundamental dominant weights $\lambda_j,\ j = 1, \ldots, n$, i.e., $C = \{\sum_{j=1}^n a_j \lambda_j : a_j \geq 0, j = 1, \ldots, n\}$.

Proof: Denote by

$$\langle \delta, \gamma \rangle := \frac{2(\delta, \gamma)}{(\gamma, \gamma)}.$$

Let $\alpha \in (i\mathfrak{t})_+$. Thus $(\alpha, \alpha_j) \geq 0$ for $j = 1, \ldots, n$. We have

$$\alpha = \sum_{j=1}^n \langle \alpha, \alpha_j \rangle \lambda_j,$$

hence $\alpha \in C$. This proves $(i\mathfrak{t})_+ \subset C$. Conversely, let $\alpha \in C$. Then $\alpha = \sum_{j=1}^n a_j \lambda_j$ with $a_j \geq 0,\ j = 1, \ldots, n$. Thus $\langle \alpha, \alpha_j \rangle = a_j \geq 0$ and hence $(\alpha, \alpha_j) \geq 0$ for $j = 1, \ldots, n$. This proves $\alpha \in (i\mathfrak{t})_+$. So $C \subset (i\mathfrak{t})_+$. Therefore $(i\mathfrak{t})_+ = C$. $\square$

The *dual cone* $\mathrm{dual}_{i\mathfrak{t}}(i\mathfrak{t})_+ \subset i\mathfrak{t}$ of the fundamental Weyl chamber $(i\mathfrak{t})_+$ is defined as

$$\mathrm{dual}_{i\mathfrak{t}}(i\mathfrak{t})_+ := \{X \in i\mathfrak{t} : (X, Y) \geq 0, \text{ for all } Y \in (i\mathfrak{t})_+\},$$

which may be written as

$$\{X \in i\mathfrak{t} : (X, \lambda_j) \geq 0, \ j = 1, \ldots, n\}.$$

Kostant [22, Lemma 3.3] proved the following result which is very useful to derive inequalities that completely describe $\mathrm{conv}\, WZ$.

Lemma 5.2 (Kostant)

1. Let $Z \in (i\mathfrak{t})_+$. For any $w \in W$, $Z - wZ \in \mathrm{dual}_{i\mathfrak{t}}(i\mathfrak{t})_+$.

2. Let $Y, Z \in (i\mathfrak{t})_+$, then $Y \in \mathrm{conv}\, WZ$ if and only if $Z - Y \in \mathrm{dual}_{i\mathfrak{t}}(i\mathfrak{t})_+$, where W is the Weyl group of $(\mathfrak{g}, \mathfrak{h})$.

Proof: (1) If $Z - w_0 Z \notin \mathrm{dual}_{i\mathfrak{t}}(i\mathfrak{t})_+$ for some $id \neq w_0 \in W$, then there would exist $X \in (i\mathfrak{t})_+$ such that $(Z - w_0 Z, X) < 0$. Let us fix this X. Because W is finite, there exist a $id \neq w \in W$ so that $(Z - wZ, X) = (Z, X) - (wZ, X)$ is minimal, or equivalently (wZ, X) is maximal. Since W acts simply transitively on the Weyl chambers, $wZ \notin (i\mathfrak{t})_+$ and there exists a simple root $\alpha \in \Pi$ such that $\alpha(wZ) < 0$. Let H_α be the be the corresponding element of α in $i\mathfrak{t}$. Then we have

$$(s_\alpha(wZ), X) \quad = \quad (wZ - \frac{\alpha(wZ)}{|\alpha|^2} H_\alpha, X)$$

$$
\begin{aligned}
&= (wZ, X) - (\frac{\alpha(wZ)}{(H_\alpha, H_\alpha)} H_\alpha, X) \\
&= (wZ, X) - \frac{\alpha(wZ)}{(H_\alpha, H_\alpha)} \alpha(X) \\
&> (wZ, X)
\end{aligned}
$$

since $\alpha(X) > 0$ and $\alpha(wZ) < 0$. This contradicts the maximality of (wZ, X).

(2) If $Y, Z \in (i\mathfrak{t})_+$ and $Y \in \operatorname{conv} WZ$, then there exist $a_w \geq 0$, $w \in W$ such that

$$
Y = \sum_{w \in W} a_w wZ, \quad \sum_{w \in W} a_w = 1.
$$

Thus

$$
\begin{aligned}
Z - Y &= \sum_{w \in W} a_w Z - \sum_{w \in W} a_w wZ \\
&= \sum_{w \in W} a_w (Z - wZ).
\end{aligned}
$$

Since $Z - wZ \in \operatorname{dual}_{i\mathfrak{t}}(i\mathfrak{t})_+$ by the first part and $a_w \geq 0$ for all $w \in W$, $Z - Y \in \operatorname{dual}_{i\mathfrak{t}}(i\mathfrak{t})_+$.

Conversely, let $Y, Z \in (i\mathfrak{t})_+$ and suppose that $Y \notin \operatorname{conv} WZ$ Then Y and $\operatorname{conv} WZ$ lie on different sides of some hyperplane in $i\mathfrak{t}$. Then there exists $X \in i\mathfrak{t}$ such that $(Y, X) > (wZ, X)$ for all $w \in W$. Since $W \in \mathrm{O}(i\mathfrak{t})$, $(Z, w^{-1}X) = (wZ, X) < (Y, X)$. Choose $w \in W$ such that $w^{-1}X \in (i\mathfrak{t})_+$. Since $Z, w^{-1}X \in (i\mathfrak{t})_+$, we have $(Z - w^{-1}Z, w^{-1}X) \geq 0$ by the first part, or equivalently,

$$
(Z, w^{-1}X) \geq (w^{-1}Z, w^{-1}X) = (Z, X).
$$

Thus $(Z - Y, X) < 0$, i.e., $Z - Y \notin \mathrm{dual}_{i\mathfrak{t}}(i\mathfrak{t})_+$. □

The following picture illustrate the geometric meaning of Lemma 5.2 for $\mathfrak{sl}(3,\mathbb{C})$ in which $C := \mathrm{dual}_{i\mathfrak{t}}(i\mathfrak{t})_+$ is the dual cone generated by α_1 and α_2.

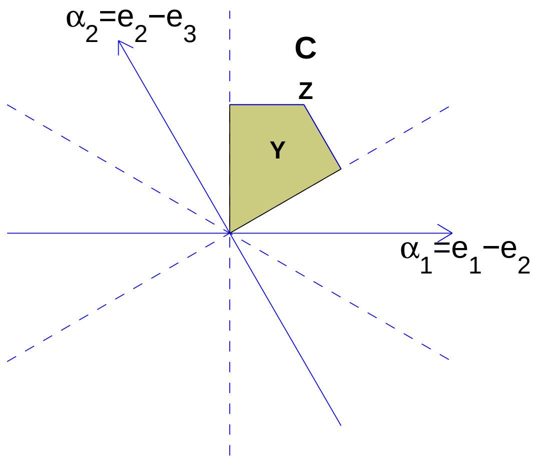

Figure 5.1: $Z - Y$ in $C := \mathrm{dual}_{i\mathfrak{t}}(i\mathfrak{t})_+$ for $\mathfrak{sl}(3,\mathbb{C})$

The shaded region is the intersection of the fundamental Weyl chamber $(i\mathfrak{t})_+$ and the backward cone, $-C + Z$, centered at Z.

Lemma 5.3 Given $Y, Z \in i\mathfrak{t}$, $Y \in \mathrm{conv}\, WZ$ if and only if $Y_+ - Z_+ \in \mathrm{dual}_{i\mathfrak{t}}(i\mathfrak{t})_+$, where W is the Weyl group of $(\mathfrak{g}, \mathfrak{h})$, where Y_+ denotes the element in the singleton set $WY \cap (i\mathfrak{t})_+$.

Proof: The Weyl group W acts transitively on Weyl chambers, i.e., $WY \cap (i\mathfrak{t})_+$ is nonempty [18, p.51] for each $Y \in (i\mathfrak{t})_+$, which is a singleton set since W is a group.

Clearly each $\omega \in W$ fixes $\mathrm{conv}\, WZ$, i.e., $\omega(\mathrm{conv}\, WZ) = \mathrm{conv}\, WZ$ and hence $WY \subset \mathrm{conv}\, WZ$ if and only if $Y \in \mathrm{conv}\, WZ$. Thus $Y \in \mathrm{conv}\, WZ$ if and only if $Y_+ \subset \operatorname{conv} WZ_+$. By Lemma 5.2, $Y \in \operatorname{conv} WZ$ if and only if $Y_+ - Z_+ \in \operatorname{dual}_{i\mathfrak{t}}(i\mathfrak{t})_+$. □

Example 5.4 Referring to Example 3.11 concerning the simple Lie algebra $\mathfrak{a}_n$ ($n \geq 1$), we still use the model $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{C})$. Let $K = \mathrm{SU}(n+1)$. Then $\mathfrak{k} = \mathfrak{su}(n+1)$ and $\theta(X) = -X^*$, $X \in \mathfrak{g}$. Denote $\mathfrak{h}$ by the set of all diagonal matrices in $\mathfrak{g}$, which is a Cartan subalgebra. The root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$ is

$$\mathfrak{g} = \mathfrak{h} \dot{+} \dot{\sum}_{\alpha \in \Delta^+} \mathfrak{g}^{\alpha} \oplus \mathfrak{g}^{-\alpha},$$

where $\Delta = \{\pm(e_j - e_k) : 1 \leq j < k \leq n+1\}$ and $\mathfrak{g}^{e_j - e_k} = \mathbb{C}E_{jk}$. Identify $\mathfrak{h}_0 = i\mathfrak{t}$ with the hyperplane $\{x \in \mathbb{R}^{n+1} : x_1 + \ldots + x_{n+1} = 0\}$ in $\mathbb{R}^{n+1}$:

$$\operatorname{diag}(h_1, \ldots, h_{n+1}) \mapsto (h_1, \ldots, h_{n+1})^T.$$

The Weyl group W of $(\mathfrak{g}, \mathfrak{h})$ acts on $i\mathfrak{t}$ by

$$(h_1, \ldots, h_{n+1})^T \mapsto (h_{\sigma(1)}, \ldots, h_{\sigma(n+1)})^T, \quad \sigma \in S_{n+1}.$$

The positive roots in Δ are

$$e_j - e_k, \; 1 \leq j < k \leq n+1,$$

and the simple roots are

$$\alpha_j = e_j - e_{j+1}, \quad j = 1, \ldots, n.$$

The fundamental dominant weights are [18, p.69] [20, p.289]

$$\lambda_k = \frac{n-k+1}{n+1}\sum_{j=1}^{k} e_j - \frac{k}{n+1}\sum_{j=k+1}^{n+1} e_j, \quad k = 1, \ldots, n.$$

The (closed) fundamental Weyl chamber $i\mathfrak{t}_+$ is

$$\{(h_1, \ldots, h_{n+1})^T \in i\mathfrak{t} : h_1 \geq \ldots \geq h_{n+1}\}.$$

The dual cone of $(i\mathfrak{t})_+$ in $i\mathfrak{t}$ is

$$\mathrm{dual}_{\,i\mathfrak{t}} i\mathfrak{t}_+ = \{(x_1, ..., x_{n+1})^T \in i\mathfrak{t} : \sum_{j=1}^{k} x_j \geq 0, k = 1, ..., n, \text{ and } \sum_{j=1}^{n+1} x_j = 0\}.$$

Since $W = S_{n+1}$, $\alpha_+ = (\alpha_{[1]}, \ldots, \alpha_{[n+1]}) \in (i\mathfrak{t})_+$, i.e., rearrangement of $\alpha \in i\mathfrak{t}$ in nonincreasing order. By Lemma 5.3, given $\alpha, \beta \in i\mathfrak{t}$, $\alpha \in \mathrm{conv}\, W\beta$ if and only if $\beta_+ - \alpha_+ \in \mathrm{dual}_{\,i\mathfrak{t}}(i\mathfrak{t})_+$, i.e.,

$$\sum_{j=1}^{k} \alpha_{[j]} \leq \sum_{j=1}^{k} \beta_{[j]}, \quad k = 1, \ldots, n, \tag{5.1}$$

$$\sum_{j=1}^{n+1} \alpha_{[j]} = \sum_{j=1}^{n+1} \beta_{[j]} = 0, \tag{5.2}$$

In other words, $\alpha \prec \beta$, i.e., majorization.

Theorem 4.6 holds for the simple $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{C})$, $n \geq 1$. Let us consider the inequalities associated with $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{C})$. It is easy to see that the projections $\rho : \mathfrak{b} \longrightarrow \mathfrak{h}$ and $\pi : \mathfrak{b} \longrightarrow i\mathfrak{t}$ amount to taking the eigenvalues and the real part of the eigenvalues, respectively, of the matrices in $\mathfrak{b}$. We know that $\mathrm{Ad}K(Y) \cap \mathfrak{b} \subset i\mathfrak{t}$ for any $Y \in i\mathfrak{k}$, hence $\mathrm{Ad}K(Y) \cap \mathfrak{b}$ are the eigenvalues of $Y \in i\mathfrak{k}$ (essentially spectral theorem on Hermitian matrices). For any $X \in \mathfrak{sl}(n+1, \mathbb{C})$, if $\alpha = (\alpha_1, \ldots, \alpha_{n+1})^T \in \pi(\mathrm{Ad}K(X) \cap \mathfrak{b})$, and $\beta = (\beta_1, ..., \beta_{n+1})^T$ is the eigenvalues of $\frac{1}{2}(X + X^*) = \frac{1}{2}(X - \theta(X))(\in i\mathfrak{k})$, then by Theorem 4.6 we know that $\alpha \in W\beta$. By (5.1) and (5.2) we have

$$\begin{aligned} \sum_{j=1}^{k} \alpha_{[j]} &\leq \sum_{j=1}^{k} \beta_{[j]}, \quad k = 1, ..., n, \\ \sum_{j=1}^{n+1} \alpha_{[j]} &= \sum_{j=1}^{n+1} \beta_{[j]} = 0, \end{aligned}$$

where $(\alpha_{[1]}, \ldots, \alpha_{[n+1]})^T, (\beta_{[1]}, \ldots, \beta_{[n+1]})^T \in (i\mathfrak{t})_+$ are the rearrangements of the entries of α and β, respectively, in nonincreasing order. Notice that the rearrangement of α, say, is simply to map α to its representative in the fundamental Weyl $(i\mathfrak{t})_+$ via the Weyl group action. The result is essentially Ky Fan. If we change the projection π to $\pi_1 : \mathfrak{g} \to \mathfrak{t}$, by similar argument, we get the result of Ky Fan's result about the imaginary part of eigenvalues and imaginary singular values. Conversely, if $\alpha \prec \beta$, where $\alpha, \beta \in \mathbb{R}^{n+1}$, i.e., (5.1) and (5.2) hold for α and β, then Theorem 4.6 on $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{C})$ asserts that there exists $X \in \mathfrak{g}$ such that the real part of the eigenvalues of X is majorized by the real singular values of X, i.e., the result of Amir-Moeź-Horn and Mirsky up to a translation. Therefore Ky Fan-Amir-Moeź-Horn-Mirsky's result is a special case of Theorem 4.6.

Example 5.5 [20, p.85] Consider the simple complex Lie algebra $\mathfrak{c}_n$, which is realized as $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{C}) = \mathfrak{sp}(n) \oplus i\mathfrak{sp}(n)$ and $\theta(X) = -X^*$. Recall that

$$\mathfrak{sp}(n, \mathbb{C}) = \{X \in \mathfrak{sl}(2n, \mathbb{C}) : X^T J + JX = 0\},$$

where $J = J_{n,n}$ is the $2n \times 2n$ matrix

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Recall

$$\mathfrak{sp}(n, \mathbb{C}) = \{\begin{pmatrix} A_1 & A_2 \\ A_3 & -A_1^T \end{pmatrix} : A_2, A_3 \in \mathbb{C}_{n\times n} \text{ complex symmetric, } A_1 \in \mathbb{C}_{n\times n}\}.$$

If $\lambda \in \mathbb{C}$ is an eigenvalue of $A = \begin{pmatrix} A_1 & A_2 \\ A_3 & -A_1^T \end{pmatrix} \in \mathfrak{sp}(n, \mathbb{C})$ with eigenvector

$$x = \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{C}^{2n},$$

i.e., $Ax = \lambda x$, then

$$\begin{pmatrix} -v \\ u \end{pmatrix} \in \mathbb{C}^{2n}$$

is also an eigenvector of A corresponding to the eigenvalue $-\lambda$. Thus the eigenvalues of $A \in \mathfrak{sp}(n, \mathbb{C})$ occur in pair but opposite in sign. So do the real singular values of A. We will see that again when we discuss the projection $\rho : \mathfrak{b} \to \mathfrak{h}$.

The symplectic group $K = \mathrm{Sp}(n)$ consists of the matrices of the form

$$\begin{pmatrix} U & -\overline{V} \\ V & \overline{U} \end{pmatrix} \in \mathrm{U}(2n),$$

and $\mathfrak{k} = \mathfrak{sp}(n)$. Thus $\theta(X) := -X^* \in \mathfrak{g}$.

The group $G = \mathrm{Sp}(n, \mathbb{C}) = \{g \in \mathrm{GL}(2n, \mathbb{C}) : g^T J g = J\}$, i.e., the group of matrices g that preserves the bilinear form

$$\langle x, y\rangle := x_1 y_{n+1} + x_2 y_{n+2} + \cdots + x_n y_{2n}, \qquad x, y \in \mathbb{C}^n.$$

Matrices in $\mathrm{Sp}(n, \mathbb{C})$ are called *symplectic matrices*. Indeed $\mathrm{Sp}(n, \mathbb{C}) \subset \mathrm{SL}(2n, \mathbb{C})$ by considering the Pfaffian (See Remark 6.6). As we did in Example 3.13, let

$$\mathfrak{h} = \{\mathrm{diag}\,(h_1, \ldots, h_n, -h_1, \ldots, -h_n) : h_1, \ldots, h_n \in \mathbb{C}\}.$$

Let us identify $\mathfrak{h}_0 = i\mathfrak{t}$ with $\mathbb{R}^n$ in the natural way:

$$\mathrm{diag}\,(h_1, \ldots, h_n, -h_1, \ldots, -h_n) \mapsto (h_1, \ldots, h_n)^T.$$

The positive roots are

$$\{e_j \pm e_k : 1 \le j < k \le n\} \cup \{2e_l : 1 \le l \le n\}.$$

The Weyl group W of $(\mathfrak{g}, \mathfrak{h})$ acts on $i\mathfrak{t}$:

$$(h_1, \dots, h_n)^T \mapsto (\pm h_{\sigma(1)}, \dots, \pm h_{\sigma(n)})^T, \quad \sigma \in S_n.$$

The simple roots are

$$\alpha_j = e_j - e_{j+1},\ j = 1, \dots, n-1, \quad \alpha_n = 2e_n,$$

and the fundamental dominant weights [18, p.67] [20, p.289] are

$$\lambda_k = \sum_{j=1}^{k} e_j, \quad k = 1, \dots, n.$$

The (closed) fundamental Weyl chamber $(i\mathfrak{t})_+$ is

$$(i\mathfrak{t})_+ = \{(h_1, \dots, h_n)^T : h_1 \geq \cdots \geq h_n \geq 0\}.$$

The dual cone of $(i\mathfrak{t})_+$ in $i\mathfrak{t}$ is

$$\operatorname{dual}_{i\mathfrak{t}}(i\mathfrak{t})_+ = \{(h_1, \dots, h_n)^T \in i\mathfrak{t} : \sum_{j=1}^{k} h_j \geq 0,\ k = 1, \dots, n\}.$$

The condition that $\beta - \alpha \in \operatorname{dual}_{i\mathfrak{t}}(i\mathfrak{t})_+$ is equivalent to

$$\sum_{j=1}^{k} \alpha_j \leq \sum_{j=1}^{k} \beta_j, \quad k = 1, \dots, n. \tag{5.3}$$

Because of the Weyl group action, if $\alpha = (\alpha_1, \ldots, \alpha_n) \in i\mathfrak{t}$, then

$$\alpha_+ = (|\alpha|_{[1]}, \ldots, |\alpha|_{[n]}) \in (i\mathfrak{t})_+,$$

where $|\alpha| = (|\alpha_1|, \ldots, |\alpha_n|)$. In other words, α_+ is the rearrangement of the absolute values of α's in nonincreasing order. By Lemma 5.3, if $\alpha, \beta \in i\mathfrak{t}$, then $\alpha \in \operatorname{conv} W\beta$ if and only if

$$\sum_{j=1}^{k} |\alpha|_{[j]} \leq \sum_{j=1}^{k} |\beta|_{[j]}, \quad k = 1, \ldots, n.$$

We remark that the orthogonal projection $\rho : \mathfrak{b} \to \mathfrak{h}$ with respective to the Killing form of $\mathfrak{g}$ amounts to taking eigenvalues and $\pi : \mathfrak{b} \to \mathfrak{h}$ is equivalent to taking the real part of the eigenvalues, since

$$\mathfrak{b} = \left\{ \begin{pmatrix} A_1 & A_2 \\ 0 & -A_1^T \end{pmatrix}, \ A_1 \text{ is upper triangular}, {A_2}^T = A_2 \right\}.$$

and $\mathfrak{h}$ consists of diagonal matrices.

Definition 5.6 Let $a, b \in \mathbb{R}^n$. We say that a is *weakly majorized* by b, denoted by $a \prec_w b$, if

$$\sum_{i=1}^{k} a_{[i]} \leq \sum_{i=1}^{k} b_{[i]}, \qquad k = 1, \ldots, n,$$

where $a_{[1]} \geq a_{[2]} \geq \cdots \geq a_{[n]}$ and $b_{[1]} \geq b_{[2]} \geq \cdots \geq b_{[n]}$ are the rearrangements of the entries of a and b, respectively, in nonincreasing order.

Thus we have the following result which basically asserts that majorization plays the same role $\mathfrak{sp}(n, \mathbb{C})$ as in $\mathfrak{sl}(n, \mathbb{C})$.

Proposition 5.7 The n largest nonnegative real parts of the eigenvalues of an $A \in \mathfrak{sp}(n, \mathbb{C})$ are weakly majorized by the n largest nonnegative real singular values of A. Conversely given two nonnegative n-tuples $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$, if $\alpha \prec_w \beta$, then there exists an $A \in \mathfrak{sp}(n, \mathbb{C})$ such that $\pm\alpha_1, \ldots, \pm\alpha_n$ are the real parts of the eigenvalues of A and $\pm\beta_1, \ldots, \pm\beta_n$ are the real singular values of A.

Chapter 6

The inequalities associated with $\mathfrak{b}_n$ and $\mathfrak{d}_n$

As we already noticed in Chapter 1, eigenvalues of a skew symmetric matrix occur in pair, opposite in sign, since A and $A^T = -A$ have the same characteristic polynomial. We now proceed to investigate the relation between the real parts of eigenvalues of A and the real singular values of A.

Example 6.1 [18, p.3] In Example 3.12, we used the model $\mathfrak{so}(2n+1, \mathbb{C})$ for $\mathfrak{b}_n$. In the model $\mathfrak{g} = \mathfrak{so}(2n+1, \mathbb{C})$, $G = \mathrm{SO}(2n+1, \mathbb{C})$, $K = \mathrm{SO}(2n+1)$, $\mathfrak{k} = \mathfrak{so}(2n+1)$, $\theta(X) = -X^*$, $X \in \mathfrak{g}$. Unlike $\mathfrak{sl}(n, \mathbb{C})$ and $\mathfrak{sp}(n, \mathbb{C})$, the form of the Borel subalgebra $\mathfrak{b}$ does not make it transparent that $\rho(\mathrm{Ad}(K)X \cap \mathfrak{b})$ and $\pi(\mathrm{Ad}(K)X \cap \mathfrak{b})$ amount taking the eigenvalues and the real parts of the eigenvalues of X, respectively. In order to see that we switch to another model.

Notice that

$$G = \mathrm{SO}(2n+1, \mathbb{C}) := \{g \in \mathrm{SL}(2n+1, \mathbb{C}) : g^T g = I_{2n+1}\},$$

is the group of matrices preserving the symmetric bilinear form [20, p.70]

$$\langle x, y \rangle := x_1 y_1 + \cdots + x_{2n+1} y_{2n+1}, \qquad x \in \mathbb{C}^{2n+1}.$$

If we change the quadratic form to

$$\langle x, y \rangle := x_1 y_1 + x_2 y_{n+1} + \cdots + x_n y_{2n+1}, \qquad x \in \mathbb{C}^{2n+1},$$

then the group becomes

$$\tilde{G} := \{g \in \mathrm{SL}(2n+1, \mathbb{C}) : g^T J g = J\},$$

where

$$J := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{pmatrix}.$$

The two groups G and $\tilde{G}$ are isomorphic via the isomorphisms

$$i_S(S) : \tilde{G} \to G, \qquad i_S(g) := SgS^{-1}, \tag{6.1}$$

where

$$S^T S = J. \tag{6.2}$$

Such $S \in \mathrm{GL}(2n+1, \mathbb{C})$ exists by Takagi's factorization [16, p.204-205], for example,

$$S = (1) \oplus \frac{e^{-i\pi/4}}{\sqrt{2}} \begin{pmatrix} iI_n & I_n \\ I_n & iI_n \end{pmatrix}. \tag{6.3}$$

It may be view as the restriction of the automorphism $i_S \in \mathrm{GL}(2n+1, \mathbb{C})$ defined by $i_S(g) := SgS^{-1}$, $g \in \mathrm{GL}(2n+1, \mathbb{C})$. Obviously $S \in \mathrm{GL}(2n+1, \mathbb{C})$ but not in $G := \mathrm{SO}(2n+1)$ nor $\tilde{G}$. By matrix differentiation, the Lie algebra $\tilde{\mathfrak{g}}$ is the set of matrices A such that $JA = -A^T J$. Direct computation [18, p.3] leads to the explicit form of $\tilde{\mathfrak{g}}$:

$$\tilde{\mathfrak{g}} = \left\{ \begin{pmatrix} 0 & -b^T & -a^T \\ a & A_1 & A_2 \\ b & A_3 & -A_1^T \end{pmatrix} : A_1, A_2, A_3 \in \mathbb{C}_{n\times n},\ A_2 = -A_2^T, A_3 = -A_3^T, a, b \in \mathbb{C}^n \right\},$$

which can also be deduced from the Lie algebra isomorphism of $\tilde{\mathfrak{g}}$ onto $\mathfrak{g}$:

$$\mathrm{Ad}(S) : \tilde{\mathfrak{g}} \to \mathfrak{g}, \tag{6.4}$$

in which $\mathrm{Ad}(S)$ (abuse of notation) is identified with the restriction of $\mathrm{Ad}(S)$: $\mathfrak{gl}(2n+1, \mathbb{C}) \to \mathfrak{gl}(2n+1, \mathbb{C})$ onto $\tilde{\mathfrak{g}}$. We will see that the change of model enables us to see that the projection $\tilde{\rho} : \tilde{\mathfrak{b}} \to \tilde{\mathfrak{h}}$ amounts to taking eigenvalues, where

$$\tilde{\mathfrak{h}} := \{\mathrm{diag}\,(0, h_1, \ldots, h_n, -h_1, \ldots, -h_n) : h_1, \ldots, h_n \in \mathbb{C}\},$$

a Cartan subalgebra of $\tilde{\mathfrak{g}}$. To describe $\tilde{\mathfrak{b}}$, consider the root space decomposition of $\tilde{\mathfrak{g}}$ with respect to $\tilde{\mathfrak{h}}$:

$$\tilde{\mathfrak{g}} = \tilde{\mathfrak{h}} \dot{+} \dot{\sum}_{\alpha \in \Delta^+} \tilde{\mathfrak{g}}^{\alpha} \oplus \tilde{\mathfrak{g}}^{-\alpha},$$

where $\tilde{\Delta}$ is the set of all roots of $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}})$. Notice that

$$\tilde{\mathfrak{b}} := \tilde{\mathfrak{h}} \dot{+} \dot{\sum}_{\alpha \in \tilde{\Delta}^+} \tilde{\mathfrak{g}}^{\alpha}$$

is a Borel subalgebra, where the positive roots in $\tilde{\Delta}^+$ are

$$\{\tilde{e}_j - \tilde{e}_k : 1 \le j < k \le n\} \cup \{e_l : 1 \le l \le n\},$$

where

$$\tilde{e}_j(\mathrm{diag}\,(0, h_1, \ldots, h_n, -h_1, \ldots, -h_n)) = h_j, \qquad j = 1, 2, \ldots$$

The root spaces are [34, p.7]

$$\tilde{\mathfrak{g}}^{\tilde{e}_j - \tilde{e}_k} = \mathbb{C}\begin{pmatrix} 0 & & \\ & E_{jk} & \\ & & -E_{kj} \end{pmatrix}, \qquad \tilde{\mathfrak{g}}^{-\tilde{e}_j + \tilde{e}_k} = \mathbb{C}\begin{pmatrix} 0 & & \\ & E_{kj} & \\ & & -E_{jk} \end{pmatrix},$$

$$\tilde{\mathfrak{g}}^{\tilde{e}_j + \tilde{e}_k} = \mathbb{C}\begin{pmatrix} 0 & & \\ & 0 & E_{jk} - E_{kj} \\ & & 0 \end{pmatrix}, \qquad \tilde{\mathfrak{g}}^{-\tilde{e}_j - \tilde{e}_k} = \mathbb{C}\begin{pmatrix} 0 & & \\ & 0 & \\ & -E_{jk} + E_{kj} & 0 \end{pmatrix},$$

where $1 \le j < k \le n$, and

$$\tilde{\mathfrak{g}}^{\tilde{e}_l} = \mathbb{C}\begin{pmatrix} 0 & 0 & {\epsilon_l}^T \\ -\epsilon_l & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{\mathfrak{g}}^{-\tilde{e}_l} = \mathbb{C}\begin{pmatrix} 0 & -\epsilon_l^T & 0 \\ 0 & 0 & 0 \\ \epsilon_l & 0 & 0 \end{pmatrix},$$

where $\{\epsilon_l : 1 \le l \le n\}$ is the standard basis of $\mathbb{R}^n$. So

$$\tilde{\mathfrak{b}} = \left\{ \begin{pmatrix} 0 & 0 & -u^T \\ u & A_1 & A_2 \\ 0 & 0 & -A_1^T \end{pmatrix} : A_1, A_2 \in \mathbb{C}_{n\times n},\ A_1 \text{ upper triangular},\ A_2 = -A_2^T,\ u \in \mathbb{C}^n \right\}.$$

Clearly the diagonal elements of each matrix in $\tilde{\mathfrak{b}}$ are its eigenvalues in which the nonzero ones appear in pair but of opposite signs. Thus the projection $\tilde{\rho} : \tilde{\mathfrak{b}} \to \tilde{\mathfrak{h}}$ amounts to taking the eigenvalues of the elements in $\tilde{\mathfrak{b}}$. Decompose

$$\tilde{\mathfrak{h}} = \tilde{\mathfrak{t}} \oplus i\tilde{\mathfrak{t}},$$

where

$$i\tilde{\mathfrak{t}} = \{\mathrm{diag}\,(0, h_1, \ldots, h_n, -h_1, \ldots, -h_n) : h_1, \ldots, h_n \in \mathbb{R}\}$$

and we identify $i\tilde{\mathfrak{t}}$ with $\mathbb{R}^n$ in the natural way that

$$\mathrm{diag}\,(0, h_1, \ldots, h_n, -h_1, \ldots, -h_n) \mapsto (h_1, \ldots, h_n)^T,$$

then the Weyl group W of $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}})$ acts on $i\tilde{\mathfrak{t}}$ by

$$(h_1, \ldots, h_n)^T \mapsto (\pm h_{\sigma(1)}, \ldots, \pm h_{\sigma(n)})^T, \quad \sigma \in S_n.$$

The simple roots are

$$\alpha_j = e_j - e_{j+1},\ j = 1, \ldots, n-1, \quad \alpha_n = e_n.$$

The fundamental dominant weights [18, p.69] [20, p.289] are

$$\lambda_k = \sum_{j=1}^{k} e_j, \quad k = 1, \ldots, n-1, \quad \lambda_n = \frac{1}{2}\sum_{j=1}^{n} e_j.$$

The (closed) fundamental Weyl Chamber $(i\tilde{\mathfrak{t}})_+$ is identified as

$$(i\tilde{\mathfrak{t}})_+ = \{(h_1, \ldots, h_n)^T \in i\tilde{\mathfrak{t}} : h_1 \geq \ldots \geq h_n \geq 0\}.$$

The dual cone of $(i\tilde{\mathfrak{t}})_+$ in $i\tilde{\mathfrak{t}}$ is identified as

$$\operatorname{dual}_{i\tilde{\mathfrak{t}}}(i\tilde{\mathfrak{t}})_+ = \{(h_1, \ldots, h_n)^T \in i\tilde{\mathfrak{t}} : \sum_{k=1}^{j} h_k \geq 0,\ j = 1, \ldots, n\}.$$

By Lemma 5.2, given $\alpha, \beta \in (i\tilde{\mathfrak{t}})_+$, i.e., α and β are identified as nonnegative vectors in $\mathbb{R}^n$, the condition $\alpha \in \operatorname{conv} W\beta$ is equivalent to $\beta - \alpha \in \operatorname{dual}_{i\tilde{\mathfrak{t}}}(i\tilde{\mathfrak{t}})_+$, i.e.,

$$\sum_{j=1}^{k} \alpha_j \leq \sum_{j=1}^{k} \beta_j, \quad k = 1, \ldots, n, \tag{6.5}$$

i.e., $\alpha \prec_w \beta$.

Lemma 6.2 Let $\sigma : \mathfrak{g}_1 \to \mathfrak{g}_2$ be a Lie algebra isomorphism of $\mathfrak{g}_1$ onto $\mathfrak{g}_2$. If $\mathfrak{g}_1$ is semisimple, then $\mathfrak{g}_2$ is semisimple. Moreover,

1. if $\mathfrak{h}_1$ is a Cartan subalgebra in $\mathfrak{g}_1$, then $\sigma(\mathfrak{h}_1)$ is a Cartan subalgebra in $\mathfrak{g}_2$,

2. if $\mathfrak{b}_1$ is a Borel subalgebra in $\mathfrak{g}_1$, then $\sigma(\mathfrak{b}_1)$ is a Borel subalgebra in $\mathfrak{g}_2$.

Proof: Since $\operatorname{ad}_{\mathfrak{g}_2}(\sigma X) = \sigma \circ \operatorname{ad}_{\mathfrak{g}_1} X \circ \sigma^{-1}$, $X \in \mathfrak{g}_1$ and $\operatorname{tr} AB = \operatorname{tr} BA$, we have [12, p.131]

$$B_{\mathfrak{g}_2}(\sigma(X), \sigma(Y)) = B_{\mathfrak{g}_1}(X, Y), \qquad X, Y \in \mathfrak{g}_2.$$

By Cartan's criterion of semisimplicity (Theorem 3.4), $\mathfrak{g}_2$ is semisimple.

(1) Clearly $\sigma(\mathfrak{h}_1)$ is a subalgebra of $\mathfrak{g}_2$. We need to prove that $\sigma(\mathfrak{h}_1)$ is maximal abelian and $\operatorname{ad}\mathfrak{h}_2$ diagonalizable. First $\mathfrak{h}_2$ is abelian since isomorphisms respect bracket:

$$[\sigma(\mathfrak{h}_1), \sigma(\mathfrak{h}_1)]_{\mathfrak{g}_2} = [\mathfrak{h}_1, \mathfrak{h}_1]_{\mathfrak{g}_1} = 0.$$

If $H_2 \in \mathfrak{g}_2$ such that $[H_2, \sigma(\mathfrak{h}_1)]_{\mathfrak{g}_2} = 0$, then

$$0 = [H_2, \sigma(\mathfrak{h}_1)]_{\mathfrak{g}_2} = \sigma[\sigma^{-1}(H_2), \mathfrak{h}_1]_{\mathfrak{g}_1}.$$

Since $\mathfrak{h}_1$ is maximal abelian, $\sigma^{-1}(H_2) \in \mathfrak{h}_1$, i.e., $H_2 \in \sigma(\mathfrak{h}_1)$. So $\sigma(\mathfrak{h}_1)$ is maximal abelian. Since $\operatorname{ad}\mathfrak{h}_1$ is diagonalizable, $\sigma : \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$ is an isomorphism of $\mathfrak{g}_1$ onto $\mathfrak{g}_2$, and

$$\operatorname{ad}(\sigma(\mathfrak{h}_1)) = \sigma \circ \operatorname{ad}\mathfrak{h}_1 \circ \sigma^{-1},$$

$\operatorname{ad}(\sigma(\mathfrak{h}_1))$ is diagonalizable. Therefore $\mathfrak{h}_2 = \sigma(\mathfrak{h}_1)$ is a Cartan subalgebra of $\mathfrak{g}_2$.

(2) Let $\mathfrak{b}_2 := \sigma(\mathfrak{b}_1)$. Since $\mathfrak{b}_1$ is a Borel subalgebra, the kth derived subalgebra ${\mathfrak{b}_1}^k = 0$ for some positive integer k. By induction the jth derived algebra of $\sigma(\mathfrak{b}_1)$ is

$$[\sigma(\mathfrak{b}_1)]^j = \sigma(\mathfrak{b}_1^j), \qquad j = 1, 2, \ldots$$

Thus $\mathfrak{b}_2^k = [\sigma(\mathfrak{b}_1)]^k = \sigma(\mathfrak{b}_1^k) = 0$, i.e., $\mathfrak{b}_2 = \sigma(\mathfrak{b}_1)$ is solvable. If $\mathfrak{b}_2' \supset \mathfrak{b}_2$ is a maximal solvable subalgebra of $\mathfrak{g}_2$, then $\sigma^{-1}(\mathfrak{b}_2') \supset \mathfrak{b}_1$ is solvable subalgebra of $\mathfrak{g}_1$. Thus $\sigma^{-1}(\mathfrak{b}_2') = \mathfrak{b}_1$. Hence $\mathfrak{b}_2' = \sigma(\mathfrak{b}_1) = \mathfrak{b}_2$, i.e., $\mathfrak{b}_2 = \sigma(\mathfrak{b}_1)$ is a Borel subalgebra of $\mathfrak{g}_2$. $\square$

Lemma 6.3 Let $\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{h}$ be given in Example 3.12. There exists a matrix similarity $\sigma : \tilde{\mathfrak{g}} \longrightarrow \mathfrak{g}$ such that

1. $\sigma(\tilde{\mathfrak{h}}) = \mathfrak{h}$ and $\sigma(\tilde{\mathfrak{b}}) = \mathfrak{b}$,

2. $\sigma(\tilde{\mathfrak{h}}^\perp) = \mathfrak{h}^\perp$,

3. $\rho \circ \sigma = \sigma \circ \tilde{\rho}$, where $\rho : \mathfrak{g} \to \mathfrak{h}$ and $\tilde{\rho} : \tilde{\mathfrak{g}} \to \tilde{\mathfrak{h}}$ are the projections,

4. $\rho|_{\mathfrak{b}} : \mathfrak{b} \to \mathfrak{h}$ amounts to take eigenvalues of $X \in \mathfrak{b}$.

Proof: (1) Recall from (6.4) that $\mathrm{Ad}(S) : \tilde{\mathfrak{g}} \to \mathfrak{g}$ defined by $\mathrm{Ad}(S)X = SXS^{-1}$, $X \in \tilde{\mathfrak{g}}$. By Lemma 6.2 $\mathrm{Ad}(S)\tilde{\mathfrak{h}}$ is a Cartan subalgebra of $\mathfrak{g}$ and $\mathrm{Ad}(S)\tilde{\mathfrak{b}}$ is a Borel subalgebra of $\mathfrak{g}$ containing $\mathrm{Ad}(S)\tilde{\mathfrak{h}}$. By Theorem 4.1, there exists $\xi \in \mathrm{Ad}(\mathfrak{g})$ such that $\xi(\mathrm{Ad}(S)\tilde{\mathfrak{b}}) = \mathfrak{b}$. Now $\xi(\mathrm{Ad}(S)\tilde{\mathfrak{h}})$ and $\mathfrak{h}$ are both Cartan subalgebras of the solvable algebra $\mathfrak{b}$, so [18, Theorem 16.2] there exists $\tau' \in \mathrm{Int}\,(\mathfrak{b})$ for which $\tau' \circ \xi(\mathrm{Ad}(S)\tilde{\mathfrak{h}}) = \mathfrak{h}$. But τ' is the restriction to $\mathfrak{b}$ of some $\tau \in \mathrm{Int}\,\mathfrak{g}$ [18, p.84] so that $\tau \circ \xi(\mathrm{Ad}(S)\tilde{\mathfrak{h}}) = \mathfrak{h}$. Then $\sigma := \tau \circ \xi \circ \mathrm{Ad}(S) : \tilde{\mathfrak{g}} \to \mathfrak{g}$ is a matrix similarity (thus a Lie algebra isomorphism from $\tilde{\mathfrak{g}}$ onto $\mathfrak{g}$) satisfying

$$\sigma(\tilde{\mathfrak{g}}) = \mathfrak{g}, \qquad \sigma(\tilde{\mathfrak{h}}) = \mathfrak{h}, \qquad \sigma(\tilde{\mathfrak{b}}) = \mathfrak{b}. \tag{6.6}$$

(2) Since $\mathrm{ad}_{\mathfrak{g}}(\sigma\tilde{X}) = \sigma \circ \mathrm{ad}_{\tilde{\mathfrak{g}}}X \circ \sigma^{-1}$ and $\mathrm{tr}\, AB = \mathrm{tr}\, BA$, we have [12, p.131]

$$B_{\mathfrak{g}}(\sigma(\tilde{X}), \sigma(\tilde{Y})) = B_{\tilde{\mathfrak{g}}}(\tilde{X}, \tilde{Y}),$$

where $B_{\mathfrak{g}}(\cdot,\cdot)$ and $B_{\tilde{\mathfrak{g}}}(\cdot,\cdot)$ are the respective Killing forms of $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$. Let $\tilde{X} \in \tilde{\mathfrak{h}}^\perp$. For each $Y \in \mathfrak{h}$, there exists $\tilde{Y} \in \tilde{\mathfrak{h}}$ such that $\sigma(\tilde{Y}) = Y$. Thus

$$B_{\mathfrak{g}}(\sigma(\tilde{X}), Y) = B_{\mathfrak{g}}(\sigma(\tilde{X}), \sigma(\tilde{Y})) = B_{\tilde{\mathfrak{g}}}(\tilde{X}, \tilde{Y}) = 0.$$

So $\sigma(X_1) \in \mathfrak{h}^{\perp}$. This proves that $\sigma(\tilde{\mathfrak{h}}^{\perp}) \subset \mathfrak{h}^{\perp}$. For dimension reason, $\sigma(\tilde{\mathfrak{h}}^{\perp}) = \mathfrak{h}^{\perp}$.

(3) Each $\tilde{X} \in \tilde{\mathfrak{g}}$ can be decomposed as $\tilde{X} = \tilde{X}_{\tilde{\mathfrak{h}}} + \tilde{X}_{\tilde{\mathfrak{h}}^{\perp}}$, where $\tilde{X}_{\tilde{\mathfrak{h}}} \in \tilde{\mathfrak{h}}$ and $\tilde{X}_{\tilde{\mathfrak{h}}^{\perp}} \in \tilde{\mathfrak{h}}^{\perp}$. So

$$\rho \circ \sigma(\tilde{X}) = \rho \circ \sigma(\tilde{X}_{\tilde{\mathfrak{h}}} + \tilde{X}_{\tilde{\mathfrak{h}}^{\perp}}) = \rho(\sigma(\tilde{X}_{\tilde{\mathfrak{h}}})) = \sigma(\tilde{X}_{\tilde{\mathfrak{h}}}) = \sigma(\tilde{\rho}(\tilde{X}_{\tilde{\mathfrak{h}}} + \tilde{X}_{\tilde{\mathfrak{h}}^{\perp}})) = \sigma \circ \tilde{\rho}(\tilde{X}).$$

Thus $\rho \circ \sigma = \sigma \circ \tilde{\rho}$.

(4) We have the following commuting diagram.

$$\begin{tikzcd}
\tilde{\mathfrak{b}} \arrow[r, "\sigma"] \arrow[d, "\tilde{\rho}"'] & \mathfrak{b} \arrow[d, "\rho"] \\
\tilde{\mathfrak{h}} \arrow[r, "\sigma"] & \mathfrak{h}
\end{tikzcd}$$

The projection $\tilde{\rho}|_{\tilde{\mathfrak{b}}} : \tilde{\mathfrak{b}} \to \tilde{\mathfrak{h}}$ amounts to taking eigenvalues of $\tilde{X} \in \tilde{\mathfrak{g}}$ and σ is a matrix similarity which of course preserves eigenvalues. For each $X \in \mathfrak{b}$, there exist $\tilde{X} \in \mathfrak{b}'$ such that $\sigma(\tilde{X}) = X$. So $\rho(X) = \rho \circ \sigma(\tilde{X}) = \sigma \circ \tilde{\rho}(\tilde{X})$. Since $\rho(X) \in \mathfrak{h}$ and $\tilde{\rho}(\tilde{X}) \in \tilde{\mathfrak{h}}$ is a diagonal matrix having eigenvalues as the diagonal elements. So $\rho : \mathfrak{b} \to \mathfrak{h}$ amounts to taking eigenvalues because of the form of $\mathfrak{h}$. □

Proposition 6.4 With the notations in Example 3.12 and given $X \in \mathfrak{so}(2n+1, \mathbb{C})$, $\rho(\mathrm{Ad}(K)X \cap \mathfrak{b})$ and $\pi(\mathrm{Ad}(K)X \cap \mathfrak{b})$ amount taking the eigenvalues and the real parts of the eigenvalues of X, respectively.

Proof: Let $\sigma : \tilde{\mathfrak{g}} \to \mathfrak{g}$ be in Lemma 6.3. Let $g \in \mathrm{GL}(2n+1, \mathbb{C})$ such that $\sigma(\tilde{X}) = g\tilde{X}g^{-1} = \mathrm{Ad}(g)\tilde{X}$, $X \in \tilde{\mathfrak{g}}$ since the groups under discussion are matrix groups. Let $\tilde{K} := g^{-1}Kg = i_{g^{-1}}(K)$. For each $X \in \mathfrak{g}$, set $\tilde{X} = \mathrm{Ad}(g^{-1})X \in \tilde{\mathfrak{g}}$.

Then

$$
\begin{aligned}
\mathfrak{h} \ni \rho(\mathrm{Ad}(K)X \cap \mathfrak{b}) &= \sigma \circ \tilde{\rho} \circ \sigma^{-1}(\mathrm{Ad}(g\tilde{K}g^{-1})\mathrm{Ad}(g)\tilde{X} \cap \mathrm{Ad}(g)\tilde{\mathfrak{b}}) \\
&= \mathrm{Ad}(g) \circ \tilde{\rho} \circ \mathrm{Ad}(g^{-1})(\mathrm{Ad}(g)\mathrm{Ad}(\tilde{K})\tilde{X} \cap \mathrm{Ad}(g)\tilde{\mathfrak{b}}) \\
&= g\tilde{\rho}(\mathrm{Ad}(\tilde{K})\tilde{X} \cap \tilde{\mathfrak{b}})g^{-1}
\end{aligned}
$$

Now $\mathrm{Ad}(\tilde{K})\tilde{X} \cap \tilde{\mathfrak{b}}$ yields the eigenvalues of $\tilde{X} = g^{-1}Xg$, i.e., the eigenvalues of X. We conclude that $\rho(\mathrm{Ad}(K)X \cap \mathfrak{b})$ yields the eigenvalues of X because of the form of $\mathfrak{h}$. The argument for π is similar. □

So we have the following result for $\mathfrak{so}(2n+1, \mathbb{C})$ by using Theorem 4.6.

Proposition 6.5 The n nonnegative real parts of the eigenvalues of a $(2n+1) \times (2n+1)$ complex skew symmetric matrix A are weakly majorized by the n nonnegative real singular values of A. Conversely given two nonnegative n-tuples $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$, if $\alpha \prec_w \beta$, then there exists a $(2n+1) \times (2n+1)$ skew symmetric matrix A such that $\pm\alpha_1, \dots, \pm\alpha_n, 0$ are the real parts of the eigenvalues of A and $\pm\beta_1, \dots, \pm\beta_n, 0$ are the real singular values of A.

Remark 6.6 To facilitate further discussion we introduce a notion known as Pfaffian [10, Appendix D] of a complex skew symmetric matrix. Let $X = (x_{ij})$ be a complex skew-symmetric matrix, i.e., $X^T = -X$. If $X \in \mathfrak{so}(2n+1, \mathbb{C})$, then

$$\det(X) = \det(-X^T) = (-1)^n \det X = 0.$$

On the other hand, if $X \in \mathfrak{so}(2n,\mathbb{C})$, then its determinant is a perfect square:

$$\det X = \mathrm{Pf}\,(X)^2,$$

where

$$\mathrm{Pf}\,(X) := \sum_{\sigma \in S_{2n}} \mathrm{sgn}(\sigma) x_{\sigma(1)\sigma(2)} \cdot \ldots \cdot x_{\sigma(2n-1)\sigma(2n)}$$

such that $\sigma(2r-1) < \sigma(2r)$ for $1 \leq r \leq n$, and $\sigma(2r-1) \leq \sigma(2r+1)$ for $1 \leq r \leq n-1$. There are $(2n-1) \cdot (2n-3) \cdot \ldots \cdot 3 \cdot 1$ terms in this sum. Equivalently,

$$\mathrm{Pf}\,(X) = \frac{1}{2^n n!} \sum \mathrm{sgn}(\sigma) x_{\sigma(1)\sigma(2)} \cdot \ldots \cdot x_{\sigma(2n-1)\sigma(2n)}.$$

Example 6.7

$$\mathrm{Pf} \begin{pmatrix} 0 & x_{12} \\ -x_{12} & 0 \end{pmatrix} = x_{12},$$

$$\mathrm{Pf} \begin{pmatrix} 0 & x_{12} & x_{13} & x_{14} \\ -x_{12} & 0 & x_{23} & x_{24} \\ -x_{13} & -x_{23} & 0 & x_{34} \\ -x_{14} & -x_{24} & -x_{34} & 0 \end{pmatrix} = x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23},$$

and

$$\mathrm{Pf} \left[\begin{pmatrix} 0 & x_1 \\ -x_1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & x_n \\ -x_n & 0 \end{pmatrix} \right] = x_1 \cdots x_n.$$

The following are some properties of Pfaffian.

Proposition 6.8 For any $A \in \mathfrak{so}(2n,\mathbb{C})$.

1. $\mathrm{Pf}\,(A)^2 = \det\,(A)$.

2. $\mathrm{Pf}\,(BAB^T) = \det\,(B)\mathrm{Pf}\,(A), \quad B \in \mathbb{C}_{2n\times 2n}$.

3. $\mathrm{Pf}\,(\lambda A) = \lambda^n \mathrm{Pf}\,(A), \quad \lambda \in \mathbb{C}$.

4. $\mathrm{Pf}\,(A^T) = (-1)^n \mathrm{Pf}\,(A)$

5. $\mathrm{Pf}\,(A_1 \oplus A_2) = \mathrm{Pf}\,(A_1)\mathrm{Pf}\,(A_2)$.

6. For any $M \in \mathbb{C}_{n\times n}$,

$$\mathrm{Pf}\begin{pmatrix} 0 & M \\ -M^T & 0 \end{pmatrix} = (-1)^{n(n-1)/2}\det M.$$

It follows from Proposition 6.8 (2) that the determinant of any symplectic matrix A, i.e., $A^TJA = J$, is 1:

$$\mathrm{Pf}\,(J) = \mathrm{Pf}\,(A^TJA) = \det\,(A)\mathrm{Pf}\,(J).$$

If $X \in \mathfrak{so}(2n, \mathbb{C})$ and $A \in \mathbb{C}_{2n\times 2n}$, then $AXA^T \in \mathfrak{so}(2n, \mathbb{C})$ and

$$\mathrm{Pf}\,(AXA^T) = (\det A) \cdot \mathrm{Pf}\,(X).$$

For any $Y \in \mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$ and any $k \in \mathrm{SO}(2n)$, then $k^{-1} = k^T$, $\det k = 1$ and hence $\mathrm{Ad}k(Y) = kYk^{-1} = kYk^T$, we have

$$\mathrm{Pf}\,(Y) = \mathrm{Pf}\,(\mathrm{Ad}k(Y)). \tag{6.7}$$

So the Pfaffian is an invariant polynomial of a skew-symmetric matrix and is invariant under under special orthogonal similarity. It is important in the theory of characteristic classes. In particular, it can be used to define the Euler class of a Riemannian manifold which is used in the generalized Gauss-Bonnet theorem.

Example 6.9 [20, p.85] Similar to the Lie algebra $\tilde{\mathfrak{g}}$ in Example 6.1, we choose another model $\tilde{\mathfrak{g}}$ for Lie algebra $\mathfrak{d}_n$, which is equivalent to the model $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$ in Example 3.14. Let

$$\tilde{\mathfrak{g}} = \left\{ \begin{pmatrix} A_1 & A_2 \\ A_3 & -A_1^T \end{pmatrix} : A_1, A_2, A_3 \in \mathbb{C}_{n\times n},\ A_2^T = -A_2, A_3^T = -A_3 \right\}.$$

Then

$$\tilde{\mathfrak{h}} = \{\operatorname{diag}(h_1, \dots, h_n, -h_1, \dots, -h_n) : h_j \in \mathbb{C}, j = 1, \dots, n\}$$

is a Cartan subalgebra. The positive roots are

$$\{e_j \pm e_k : 1 \leq j < k \leq n\}.$$

The root spaces are

$$\tilde{\mathfrak{g}}^{e_j - e_k} = \begin{pmatrix} E_{jk} & 0 \\ 0 & -E_{kj} \end{pmatrix}, \qquad \tilde{\mathfrak{g}}^{-e_j + e_k} = \begin{pmatrix} E_{kj} & 0 \\ 0 & -E_{jk} \end{pmatrix},$$

$$\tilde{\mathfrak{g}}^{e_j + e_k} = \begin{pmatrix} 0 & E_{jk} - E_{kj} \\ 0 & 0 \end{pmatrix}, \quad \tilde{\mathfrak{g}}^{-e_j - e_k} = \begin{pmatrix} 0 & 0 \\ -E_{jk} + E_{kj} & 0 \end{pmatrix},$$

where $1 \leq j < k \leq n$. Identify $\tilde{\mathfrak{h}}_0 = i\tilde{\mathfrak{t}}$ with $\mathbb{R}^n$ in the natural way. The Weyl group W of $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}})$ acts on $\tilde{\mathfrak{h}}_0$:

$$(h_1, \ldots, h_n)^T \mapsto (\pm h_{\sigma(1)}, \ldots, \pm h_{\sigma(n)})^T, \ \sigma \in S_n,$$

where the number of negative signs is even. The simple roots are

$$\alpha_j = e_j - e_{j+1}, \ j = 1, \ldots, n-1, \ \alpha_n = e_{n-1} + e_n,$$

and the fundamental dominant weights [18, p.69] [20, p.289] are

$$\lambda_j = \sum_{j=1}^{k} e_j, \ k = 1, \ldots, n-2, \quad \lambda_{n-1} = \frac{1}{2}(\sum_{j=1}^{n-1} e_j - e_n), \ \lambda_n = \frac{1}{2}\sum_{j=1}^{n} e_j.$$

The (closed) fundamental Weyl chamber $(i\tilde{\mathfrak{t}})_+$ is

$$(i\tilde{\mathfrak{t}})_+ = \{(h_1, \ldots, h_n)^T : h_1 \geq \ldots \geq h_{n-1} \geq |h_n|\}.$$

The dual cone of $(i\tilde{\mathfrak{t}})_+$ in $i\tilde{\mathfrak{t}}$ is

$$\operatorname{dual}_{i\tilde{\mathfrak{t}}}(i\tilde{\mathfrak{t}})_+ = \{(h_1, \ldots, h_n)^T \in i\tilde{\mathfrak{t}} : \sum_{j=1}^{k} h_j \geq 0, \ k = 1, \ldots, n-1, \ \sum_{j=1}^{n-1} h_j - h_n \geq 0\}.$$

The condition that $\beta - \alpha \in \operatorname{dual}_{i\tilde{\mathfrak{t}}}(i\tilde{\mathfrak{t}})_+$ amounts to the following inequalities.

$$\sum_{j=1}^{k} \alpha_j \quad \leq \quad \sum_{j=1}^{k} \beta_j, \quad k = 1, \ldots, n-2,$$

$$\begin{aligned}\sum_{j=1}^{n-1}\alpha_j - \alpha_n &\le \sum_{j=1}^{n-1}\beta_j - \beta_n\\ \sum_{j=1}^{n}\alpha_j &\le \sum_{j=1}^{n}\beta_j\end{aligned}$$

The inequalities are equivalent to

$$\sum_{j=1}^{k}\alpha_j \le \sum_{j=1}^{k}\beta_j, \quad k = 1, \ldots, n, \tag{6.8}$$

$$\sum_{j=1}^{n-1}\alpha_j - \alpha_n \le \sum_{j=1}^{n-1}\beta_j - \beta_n. \tag{6.9}$$

When $\alpha, \beta \in (i\mathfrak{t})_+$, i.e., $\alpha_1 \ge \cdots \ge \alpha_{n-1} \ge |\alpha_n|$ and $\beta_1 \ge \cdots \ge \beta_{n-1} \ge |\beta_n|$, (6.8) and (6.9) are equivalent to

$$\begin{aligned}\alpha \prec_w \beta, \quad i.e., \quad \sum_{j=1}^{k}\alpha_j &\le \sum_{j=1}^{k}\beta_j, \quad k = 1, \ldots, n,\\ \sum_{j=1}^{n-1}\alpha_j - \alpha_n &\le \sum_{j=1}^{n-1}\beta_j - \beta_n\end{aligned}$$

The above condition is clearly stronger than weakly majorization.

Similar to the simple Lie algebra $\mathfrak{b}_n$, there is an matrix similarity σ between the two models $\tilde{\mathfrak{g}}$ in Example 6.9 and $\mathfrak{so}(2n, \mathbb{C})$ of $\mathfrak{d}_n$ which preserves eigenvalues. Moreover we have the following commuting diagram.

$$\begin{tikzcd} \tilde{\mathfrak{b}} \arrow[r, "\sigma"] \arrow[d, "\tilde{\rho}"'] & \mathfrak{b} \arrow[d, "\rho"] \\ \tilde{\mathfrak{h}} \arrow[r, "\sigma"] & \mathfrak{h} \end{tikzcd}$$

Proposition 6.10 With the notations in Example 3.14 and given $X \in \mathfrak{so}(2n, \mathbb{C})$, $\rho(\mathrm{Ad}(K)X \cap \mathfrak{b})$ and $\pi(\mathrm{Ad}(K)X \cap \mathfrak{b})$ amount taking the eigenvalues and the real parts of the eigenvalues of X, respectively.

Let $Y \in i\mathfrak{k} = i\mathfrak{sp}(n)$ have eigenvalues $\pm\beta_1, \ldots, \pm\beta_n$ such that $\beta_1, \ldots, \beta_n$ are nonnegative. There exists $k \in \mathrm{SO}(2n)$ [16, p.107] such that

$$\mathrm{Ad}k(Y) = \begin{pmatrix} 0 & i\beta_1 \\ -i\beta_1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & i\beta_{n-1} \\ -i\beta_{n-1} & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & \delta i\beta_n \\ -\delta i\beta_n & 0 \end{pmatrix} \in i\mathfrak{t},$$

where $\delta = \pm 1$. So by (6.7) and Example 6.7

$$\mathrm{Pf}\,(Y) = \delta i^n \beta_1 \cdots \beta_{n-1}\beta_n,$$

and $\delta = \mathrm{sign}[(-i)^n \mathrm{Pf}\,(Y)]$ is uniquely determined by Y.

Proposition 6.11 Let $A \in \mathfrak{so}(2n, \mathbb{C})$ and let $\pm\beta_1, \ldots, \pm\beta_n$ be the real singular values of A with $\beta_j \geq 0, j = 1, \ldots, n$. Suppose

$$\begin{pmatrix} 0 & i\alpha_1 \\ -i\alpha_1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & i\alpha_{n-1} \\ -i\alpha_{n-1} & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & i\alpha_n \\ -i\alpha_n & 0 \end{pmatrix} \in \pi(\mathrm{Ad}K(A) \cap \mathfrak{b}),$$

so that $\pm\alpha_1, \ldots, \pm\alpha_n$ are the real part of the eigenvalues of A. Set $|\alpha| = (|\alpha_1|, \ldots, |\alpha_n|)$. Then

$$\begin{aligned} \sum_{j=1}^{k} |\alpha|_{[j]} &\leq \sum_{j=1}^{k} \beta_{[j]}, \quad k = 1, \ldots, n-2, \\ \sum_{j=1}^{n-1} |\alpha|_{[j]} + \mathrm{sign}\,(\alpha_1 \cdots \alpha_n)|\alpha|_{[n]} &\leq \sum_{j=1}^{n-1} \beta_{[j]} + [\mathrm{sign}\,\{(-i)^n \mathrm{Pf}\,(\frac{1}{2}(A + A^*))\}]\,\beta_{[n]}, \end{aligned}$$

$$\sum_{j=1}^{n-1}|\alpha|_{[j]}-\mathrm{sign}\,(\alpha_1\cdots\alpha_n)|\alpha|_{[n]} \;\le\; \sum_{j=1}^{n-1}\beta_{[j]}-[\mathrm{sign}\,\{(-i)^n\mathrm{Pf}\,(\frac{1}{2}(A+A^*))\}]\,\beta_{[n]},$$

(the sign of zero may be taken 1 or -1). Conversely, suppose $(\alpha_1,\ldots,\alpha_n)^T,(\beta_1,\ldots,\beta_n)^T\in\mathbb{R}^n$ satisfying the inequalities,

$$\begin{aligned}
\sum_{j=1}^{k}|\alpha|_{[j]} &\le \sum_{j=1}^{k}|\beta|_{[j]},\quad k=1,\ldots,n-2,\\
\sum_{j=1}^{n-1}|\alpha|_{[j]}+\mathrm{sign}(\alpha_1\cdots\alpha_n)|\alpha|_{[n]} &\le \sum_{j=1}^{n-1}|\beta|_{[j]}+\mathrm{sign}\,(\beta_1\cdots\beta_n)\,|\beta|_{[n]},\\
\sum_{j=1}^{n-1}|\alpha|_{[j]}-\mathrm{sign}(\alpha_1\cdots\alpha_n)|\alpha|_{[n]} &\le \sum_{j=1}^{n-1}|\beta|_{[j]}-\mathrm{sign}\,(\beta_1\cdots\beta_n)\,|\beta|_{[n]},
\end{aligned}$$

or equivalently

$$\begin{aligned}
\sum_{j=1}^{k}|\alpha|_{[j]} &\le \sum_{j=1}^{k}|\beta|_{[j]},\quad k=1,\ldots,n-2,\\
\sum_{j=1}^{n-1}|\alpha|_{[j]}+|\alpha|_{[n]} &\le \sum_{j=1}^{n-1}|\beta|_{[j]}+\mathrm{sign}\,(\Pi_{j=1}^{n}(\alpha_j\beta_j))\,|\beta|_{[n]},\\
\sum_{j=1}^{n-1}|\alpha|_{[j]}-|\alpha|_{[n]} &\le \sum_{j=1}^{n-1}|\beta|_{[j]}-\mathrm{sign}\,(\Pi_{j=1}^{n}(\alpha_j\beta_j))\,|\beta|_{[n]},
\end{aligned}$$

where $|\alpha|_{[j]}$ and $|\beta|_{[j]},\ j=1,\ldots,n$, are the rearrangements of the entries of $|\alpha|$ and $|\beta|$, respectively, in nonincreasing order. Then we can find $A\in\mathfrak{so}(2n,\mathbb{C})$ such that $\pm\alpha$'s are the real part of the eigenvalues of A, $\pm\beta$'s are the real singular values of A and

$$\mathrm{sign}\,[(-i)^n\mathrm{Pf}\,(\frac{1}{2}(A+A^*))]=\mathrm{sign}\,(\beta_1\cdots\beta_n).$$

Proof: Since

$$Y := \begin{pmatrix} 0 & i\alpha_1 \\ -i\alpha_1 & 0 \end{pmatrix} \oplus \cdots \begin{pmatrix} 0 & i\alpha_{n-1} \\ -i\alpha_{n-1} & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & i\alpha_n \\ -i\alpha_n & 0 \end{pmatrix} \in \pi(\mathrm{Ad}K(A) \cap \mathfrak{b}),$$

the element $Y_+ \in (i\mathfrak{t})_+ \cap WY$ is of the form

$$Y_+ = \begin{pmatrix} 0 & i|\alpha|_{[1]} \\ -i|\alpha|_{[1]} & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & i|\alpha|_{[n-1]} \\ -i|\alpha|_{[n-1]} & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & \delta i|\alpha|_{[n]} \\ -\delta i|\alpha|_{[n]} & 0 \end{pmatrix},$$

where $\delta = \mathrm{sign}\,(\alpha_1 \cdots \alpha_n)$. Similarly, there exists $Z_+ \in \mathrm{Ad}K(\frac{1}{2}(A + A^*) \cap (i\mathfrak{t})_+$ such that

$$Z_+ = \begin{pmatrix} 0 & i|\beta|_{[1]} \\ -i|\beta|_{[1]} & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & i|\beta|_{[n-1]} \\ -i|\beta|_{[n-1]} & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & \delta' i|\beta|_{[n]} \\ -\delta' i|\beta|_{[n]} & 0 \end{pmatrix},$$

where $\delta' = \mathrm{sign}\,\{\mathrm{Pf}\,((-i)^n \frac{1}{2}(A + A^*))\}$. Now $Y_+ \in \mathrm{conv}\, WZ_+$ by Theorem 4.6 and Lemma 5.3. So $Z_+ - Y_+ \in \mathrm{dual}_{\,i\mathfrak{t}}(i\mathfrak{t})_+$ by Lemma 5.2. Under the identification $(|\alpha|_{[1]}, \ldots, |\alpha|_{[n-1]}, \delta|\alpha|_{[n]})$ and $(|\beta|_{[1]}, \ldots, |\beta|_{[n-1]}, \delta'|\beta|_{[n]})$ satisfy (6.8) and (6.9), i.e.,

$$\begin{aligned} (|\alpha|_{[1]}, \ldots, |\alpha|_{[n-1]}, \delta|\alpha|_{[n]})^T \quad &\prec_w \quad (|\beta|_{[1]}, \ldots, |\beta|_{[n-1]}, \delta'\beta_{[n]})^T, \\ \sum_{j=1}^{n-1} |\alpha|_{[j]} - \delta|\alpha|_{[n]} \quad &\leq \quad \sum_{j=1}^{n-1} \beta_{[j]} - \delta'|\beta|_{[n]}, \end{aligned}$$

which are equivalent to the first set of inequalities.

Conversely, if $\alpha, \beta \in \mathbb{R}^n$ satisfy the second set of inequalities, then

$$(|\alpha|_{[1]}, \ldots, |\alpha|_{[n-1]}, \mathrm{sign}\,(\alpha_1 \cdots \alpha_n)|\alpha|_{[n]})^T \quad \prec_w \quad (|\beta|_{[1]}, \ldots, |\beta|_{[n-1]}, \mathrm{sign}\,(\beta_1 \cdots \beta_n)\beta_{[n]})^T,$$

$$\sum_{j=1}^{n-1} |\alpha|_{[j]} - \delta|\alpha|_{[n]} \quad \leq \quad \sum_{j=1}^{n-1} \beta_{[j]} - \delta'|\beta|_{[n]}.$$

Thus

$$(|\alpha|_{[1]}, \dots, |\alpha|_{[n-1]}, \mathrm{sign}\,(\alpha_1 \cdots \alpha_n)|\alpha|_{[n]})^T \in \mathrm{conv}\, W(|\beta|_{[1]}, \dots, |\beta|_{[n-1]}, \mathrm{sign}\,(\beta_1 \cdots \beta_n)\beta_{[n]})^T.$$

Then by Theorem 4.6 and Proposition 6.12 there exists $A \in \mathfrak{so}(2n, \mathbb{C})$ such that the real parts of the eigenvalues are $\pm\alpha_1, \dots, \pm\alpha_n$ and the real singular values are $\pm\beta_1, \dots, \pm\beta_n$, and by the invariance of the Pfaffian under adjoint action of $K = \mathrm{SO}(2n)$ we have

$$\mathrm{sign}\,[(-i)^n \mathrm{Pf}\,(\frac{1}{2}(A + A^*))] = \mathrm{sign}\,(\beta_1 \cdots \beta_n).$$

Notice that $\mathrm{sign}\,(ab) = \mathrm{sign}\,(a)\mathrm{sign}\,(b)$ for any real numbers a and b. If $\delta_1 = 1$, the second set of inequalities is exactly the third set of inequalities. If $\delta_1 = -1$, the last two inequalities are identical to the last two inequalities in the second set. Thus the last set of three inequalities is equivalent to the second set. Hence if the last three equations are true, then there exists $A \in \mathfrak{so}(2n)$ satisfying the required conditions. $\square$

Proposition 6.12 Let $A \in \mathfrak{so}(2n, \mathbb{C})$ and let $\alpha_1, \dots, \alpha_n$ be the largest n nonnegative real part of the eigenvalues of A and let $\beta_1, \dots, \beta_n$ be the largest n nonnegative real singular values of A. Then either

1. $(\alpha_1, \dots, \alpha_{n-1}, \alpha_n)^T \in \mathrm{conv}\, W(\beta_1, \dots, \beta_n)^T$, or

2. $(\alpha_1, \ldots, \alpha_{n-1}, -\alpha_n)^T \in \operatorname{conv} W(\beta_1, \ldots, \beta_n)^T$.

Conversely if $\alpha, \beta \in \mathbb{R}^n$ have the above relationship, then there exists $A \in \mathfrak{so}(2n, \mathbb{C})$ such that $\pm\alpha_1, \ldots, \pm\alpha_n$ are the real parts of the eigenvalues of A and $\pm\beta_1, \ldots, \pm\beta_n$ are the real singular values of A.

Proof: Since $\alpha_1, \ldots, \alpha_n$ are the largest n nonnegative real part of the eigenvalues of A and $\beta_1, \ldots, \beta_n$ are the largest n real singular values of A, by considering the action of the Weyl group on A, either $(\alpha_1, \ldots, \alpha_n)^T$ or $(\alpha_1, \ldots, \alpha_{n-1}, -\alpha_n)^T$ (but usually not both) is in $\pi(\mathrm{Ad}K(A) \cap \mathfrak{b})$ under the identification, where $K = \mathrm{SO}(2n)$. Similarly either $(\beta_1, \ldots, \beta_n)^T$ or $(\beta_1, \ldots, \beta_{n-1}, -\beta_n)^T$ is in $\mathrm{Ad}K(\frac{1}{2}(A + A^*)) \cap i\mathfrak{t}$. By Theorem 4.6 we have the following four possibilities

$$(\alpha_1, \ldots, \alpha_{n-1}, \delta_1\alpha_n)^T \in \operatorname{conv} W(\beta_1, \ldots, \beta_{n-1}, \delta_2\beta_n)^T,$$

where $\delta_1 = \pm 1$ and $\delta_2 = \pm 1$. But

$$(\alpha_1, \ldots, \alpha_{n-1}, \delta_1\alpha_n)^T \in \operatorname{conv} W(\beta_1, \ldots, \beta_{n-1}, -\beta_n)^T$$

is the same as

$$(\alpha_1, \ldots, \alpha_{n-1}, -\delta_1\alpha_n)^T \in \operatorname{conv} W(\beta_1, \ldots, \beta_{n-1}, \beta_n)^T.$$

Thus either

1. $(\alpha_1, \ldots, \alpha_{n-1}, \alpha_n)^T \in \operatorname{conv} W(\beta_1, \ldots, \beta_n)^T$, or

2. $(\alpha_1, \ldots, \alpha_{n-1}, -\alpha_n)^T \in \operatorname{conv} W(\beta_1, \ldots, \beta_n)^T$.

Conversely, if either of the conditions is true, then Theorem 4.6 and Proposition 6.12 guarantee the existence of the required A. □

Remark 6.13 Proposition 6.5 is no longer true for $\mathfrak{o}_n$. We can see this clearly from the following example for $n = 2$. Let $\alpha = (1/2, 0)^T$ and $\beta = (1, 1)^T$. Obviously $\alpha \prec_w \beta$. By Proposition 6.12, there is no $A \in \mathfrak{so}(2, \mathbb{C})$ with the real part of the eigenvalues $\pm\alpha$'s and real singular values $\pm\beta$'s. See the following figure: $L_1 = \operatorname{conv} W(1, 1)^T$, $L_2 = \operatorname{conv} W(1, -1)^T$, and the square with vertices $(1, 1), (1, -1), (-1, 1), (-1, -1)$ (the shaded area) is the set of all the vectors, in particular $\alpha = (1/2, 0)^T$, weakly majorized by $\beta = (1, 1)^T$. But the union of L_1 and L_2 is a "cross" which is clearly not convex.

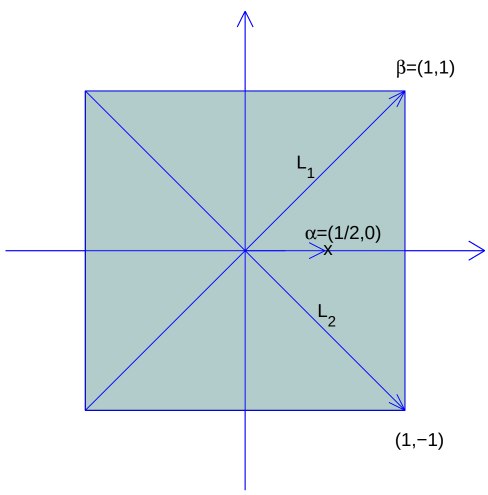

Figure 6.1: The Union of the Convex Hulls: $L_1 \cup L_2$

Chapter 7

The real semisimple case

The proof [1] given by Amir-Moéz and Horn for the converse of Ky Fan's result also works for $\mathfrak{sl}(n,\mathbb{R})$, a normal real form of $\mathfrak{sl}(n,\mathbb{C})$. However the study would be intricate for real semisimple Lie algebras. Theorem 4.6 concerns a complex semisimple Lie algebra $\mathfrak{g}$, a Cartan subalgebra $\mathfrak{h}$ and a Borel subalgebra $\mathfrak{b}$. In the complex semisimple case $\mathfrak{g}$, all maximal solvable subalgebras in $\mathfrak{g}$ are conjugate via the adjoint group $\mathrm{Ad}G$ of $\mathfrak{g}$ [18, Section 16.4] (it is also true for Cartan subalgebras). The Borel subalgebra $\mathfrak{b}$ in Section 2 and 3 is the "standard" one with respect to the chosen Cartan subalgebra $\mathfrak{h} = \mathfrak{t} \oplus i\mathfrak{t}$ and the basis Π for the root system Δ.

One may consider the real semisimple case. From now on $\mathfrak{g}$ denotes a real semisimple Lie algebra with Killing form $B(\cdot,\cdot)$. A subalgebra $\mathfrak{h}$ is called a Cartan subalgebra of $\mathfrak{g}$ if the complexification $\mathfrak{h}^{\mathbb{C}}$ of $\mathfrak{h}$ is a Cartan subalgebra of the complexification $\mathfrak{g}^{\mathbb{C}}$ of $\mathfrak{g}$ [20, p.318]. It is well known [21] [31, p.397] that Cartan subalgebras of a real semisimple Lie algebra are not conjugate in general (unless $\mathfrak{g}$ is compact). But there are only finitely many conjugacy classes of Cartan subalgebras [31, p.395].

Example 7.1 Let $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$ and let

$$\mathfrak{a} = \mathbb{R}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathfrak{b} = \mathbb{R}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then $\mathfrak{a}$ and $\mathfrak{b}$ are Cartan subalgebras of $\mathfrak{g}$. They cannot be conjugate in $\mathfrak{g}$ since $e^{\mathfrak{a}} = \{\mathrm{diag}\,((e^{\lambda}, e^{-\lambda}) : \lambda \in \mathbb{R}\}$ is not compact, but $e^{\mathfrak{b}} = \mathrm{SO}(2)$ is compact.

Borel subalgebras in a complex Lie algebra are (complex) maximal solvable algebras and there is only one conjugacy class since all Borel subalgebras are conjugate. However, in the real case, there are different conjugacy classes of maximal solvable subalgebras [25, 26, 27]. The conjugacy classes of maximal solvable subalgebras may be obtained via the non-conjugate Cartan subalgebras [25] and the procedure will be outlined.

A decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

into a direct sum is called a *Cartan decomposition* if

1. the map $\theta : X + Y \mapsto X - Y$ $(X \in \mathfrak{k}, Y \in \mathfrak{p})$ is an automorphism of $\mathfrak{g}$, i.e., $\theta \in \mathrm{Aut}\,(\mathfrak{g})$.

2. The bilinear form $B_\theta(X, Y) = -B(X, \theta Y)$ is positive definite on $\mathfrak{g}$.

Since $\theta^2 = 1$, B_θ is a symmetric bilinear form. The first condition amounts to the following:

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}.$$

So $\mathfrak{k}$ and $\mathfrak{p}$ are orthogonal to each other under $B(\cdot, \cdot)$ and under $B_\theta(\cdot, \cdot)$. Since $B_\theta(\cdot, \cdot)$ is positive definite, $B(\cdot, \cdot)$ is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$. The subspace $\mathfrak{p} \subset \mathfrak{g}$ is called a *Cartan subspace* of $\mathfrak{g}$ and $\mathfrak{k}$ is a subalgebra of $\mathfrak{g}$.

An involutory automorphism $\theta \in \mathrm{Aut}\,(\mathfrak{g})$ such that the symmetric bilinear form $B_\theta(X, Y) = -B(X, \theta Y)$ is positive definite is called a *Cartan involution*. A

Cartan involution determines a Cartan decomposition of $\mathfrak{g}$ and vice versa. The importance of the Cartan decomposition is that it is unique up to conjugacy, i.e., if $\mathfrak{g} = \mathfrak{k}' \oplus \mathfrak{p}'$ is another Cartan decomposition, there exists $\varphi \in \operatorname{Int}(\mathfrak{g})$ such that $\mathfrak{k}' = \varphi(\mathfrak{k})$ and $\mathfrak{p}' = \varphi(\mathfrak{p})$ [20, p.301].

For a complex semisimple Lie algebra $\mathfrak{u}$, the only Cartan involutions of its realification $\mathfrak{u}_{\mathbb{R}}$ are the conjugations with respect to the compact real forms of $\mathfrak{u}$ [20, p.302].

Let $\mathfrak{g} = \mathfrak{k} \dot{+} \mathfrak{p}$ be the Cartan decomposition associated with the Cartan involution θ, that is, $\mathfrak{k}$ is the $+1$ eigenspace of θ and $\mathfrak{p}$ is the -1 eigenspace of θ. Since the adjoint of $\operatorname{ad} X$ is [20, p.304]

$$(\operatorname{ad} X)^* = -\operatorname{ad} \theta X, \qquad X \in \mathfrak{g},$$

$\operatorname{ad} X$ is represented by a symmetric matrix if $X \in \mathfrak{p}$, and by a skew symmetric matrix if $X \in \mathfrak{k}$. Fix a maximal abelian subspace $\mathfrak{a}_{\mathfrak{p}}$ in $\mathfrak{p}$. For any $H \in \mathfrak{a}_{\mathfrak{p}} \subset \mathfrak{p}$,

$$(\operatorname{ad} H)^* = \operatorname{ad} H,$$

and hence $\operatorname{ad} \mathfrak{a}_{\mathfrak{p}}$ is a commuting family of self-adjoint transformations of $\mathfrak{g}$. So [16, p.103] $\mathfrak{g}$ is the orthogonal direct sum of the subspaces

$$\mathfrak{g}^\alpha = \{X \in \mathfrak{g} : (\operatorname{ad} H)X = \alpha(H)X \text{ for all } H \in \mathfrak{a}_{\mathfrak{p}}\}, \qquad \alpha \in \mathfrak{a}_{\mathfrak{p}}^*$$

If $\alpha \neq 0$ and $\mathfrak{g}^\alpha \neq 0$, we call α a *restricted root* of $(\mathfrak{g}, \mathfrak{a}_{\mathfrak{p}})$. The set of all restricted roots is denoted by Σ. Any nonzero $\mathfrak{g}^\alpha$ is called a *restricted root space*, and each member of $\mathfrak{g}^\alpha$ is called a *restricted root vector* for the restricted root α. The decomposition of $\mathfrak{g}$,

$$\mathfrak{g} = \mathfrak{g}^0 \dot{+} \dot{\sum}_{\alpha\in\Sigma^+} (\mathfrak{g}^\alpha \oplus \mathfrak{g}^{-\alpha}), \tag{7.1}$$

is called the *restricted root space decomposition* of $\mathfrak{g}$ relative to $\mathfrak{a}_\mathfrak{p}$ [20, p.313], where Σ^+ is the set of restricted positive roots (with respect to a fixed base Π of Σ) of the root system Σ of $(\mathfrak{g}, \mathfrak{a}_\mathfrak{p})$. Unlike the root system for a complex semisimple Lie algebra, the restricted root system of $(\mathfrak{g}, \mathfrak{a}_\mathfrak{p})$ need not be *reduced*, i.e., it may happen that $\alpha \in \Sigma$ and $2\alpha \in \Sigma$. Moreover $\dim \mathfrak{g}^\alpha$ may be larger than 1 and usually $\mathfrak{g}^0$ is bigger than $\mathfrak{a}_\mathfrak{p}$. We also have the orthogonal sum [20, p.313]

$$\mathfrak{g}^0 = \mathfrak{a}_\mathfrak{p} \dot{+} \mathfrak{m},$$

where $\mathfrak{m} = Z_\mathfrak{k}(\mathfrak{a}_\mathfrak{p})$, the centralizer of $\mathfrak{a}_\mathfrak{p}$ in $\mathfrak{k}$. Hence

$$\mathfrak{k} \cap \mathfrak{g}^0 = \mathfrak{m}, \qquad \mathfrak{p} \cap \mathfrak{g}^0 = \mathfrak{a}_\mathfrak{p}.$$

The root system Σ does not of itself determine $\mathfrak{g}$ up to isomorphism. The Weyl group of $(\mathfrak{g}, \mathfrak{a}_\mathfrak{p})$ which may be defined as the normalizer of $\mathfrak{a}_\mathfrak{p}$ in K modulo the centralizer of $\mathfrak{a}_\mathfrak{p}$ in K, will be denoted by W.

Since $\mathrm{Ad}(K)$ preserves the Killing form and $\mathfrak{k}$, $\mathrm{Ad}(K)\mathfrak{a}_\mathfrak{p} \subset \mathfrak{p}$. The following is the original Kostant's convexity theorem [22] and Theorem 4.5 is a particular case.

Theorem 7.2 (Kostant) Let $Z \in \mathfrak{a}_\mathfrak{p}$, then $\pi(\mathrm{Ad}(K)Z) = \mathrm{conv}\, WZ$ where $\pi : \mathfrak{p} \to \mathfrak{a}_\mathfrak{p}$ is the orthogonal projection with respect to the Killing form.

Since θ is -1 on $\mathfrak{p}$ and thus on $\mathfrak{a}_\mathfrak{p}$, and $[H, \theta X] = \theta[\theta H, X] = -\theta[H, X] = -\alpha(H)\theta X$ for all $X \in \mathfrak{g}^\alpha$, $H \in \mathfrak{a}_\mathfrak{p}$, we have [20, 313]

Proposition 7.3 $\theta(\mathfrak{a}_\mathfrak{p}) = \mathfrak{a}_\mathfrak{p}$ and $\theta(\mathfrak{g}^\alpha) = \mathfrak{g}^{-\alpha}$ for all $\alpha \in \Sigma$.

Let

$$\mathfrak{n} = \sum_{\alpha \in \Sigma^+} \mathfrak{g}^\alpha$$

which is a nilpotent subalgebra of $\mathfrak{g}$. Let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}_\mathfrak{p}$. Then $\mathfrak{a}$ is a Cartan subalgebra. We have $\mathfrak{a}_\mathfrak{p} = \mathfrak{a} \cap \mathfrak{p}$ and if we put $\mathfrak{a}_\mathfrak{k} := \mathfrak{a} \cap \mathfrak{k}$, then $\mathfrak{a} = \mathfrak{a}_\mathfrak{k} \oplus \mathfrak{a}_\mathfrak{p}$. Let

$$\mathfrak{b} := \mathfrak{a} \dot{+} \mathfrak{n} = \mathfrak{a} \dot{+} \dot{\sum}_{\alpha \in \Sigma^+} \mathfrak{g}^\alpha$$

which is a maximal solvable subalgebra of $\mathfrak{g}$.

Theorem 7.4 Let $\mathfrak{g} = \mathfrak{k} \dot{+} \mathfrak{p}$ be a Cartan decomposition associated with the Cartan involution θ. Let $\mathfrak{a}_\mathfrak{p}$ be a maximal abelian subspace of $\mathfrak{p}$. Let $\mathfrak{g} = \mathfrak{g}^0 \dot{+} \dot{\sum}_{\alpha \in \Sigma^+} (\mathfrak{g}^\alpha \oplus \mathfrak{g}^{-\alpha})$ be the restricted root space decomposition of the real semisimple Lie algebra $\mathfrak{g}$ with respect to $\mathfrak{a}_\mathfrak{p}$ and set $\mathfrak{b} = \mathfrak{a} \dot{+} \dot{\sum}_{\alpha \in \Sigma^+} \mathfrak{g}^\alpha$. Let $\pi : \mathfrak{g} \to \mathfrak{a}_\mathfrak{p}$ be the orthogonal projection with respect to $B_\theta(\cdot, \cdot)$. Then for each $\beta \in \mathfrak{a}_\mathfrak{p}$,

$$\pi((\mathfrak{k} + \mathrm{Ad}K(\beta)) \cap \mathfrak{b}) = \operatorname{conv} W\beta.$$

Proof: Notice that

$$\pi((\mathfrak{k} + \mathrm{Ad}K(\beta)) \cap \mathfrak{b})) \subset \pi(\mathfrak{k} + \mathrm{Ad}K(\beta)) = \pi(\mathrm{Ad}K(\beta)) = \operatorname{conv} W\beta$$

by Theorem 7.2.

Suppose $\gamma \in \operatorname{conv} W\beta$, where $\gamma, \beta \in \mathfrak{a}_{\mathfrak{p}}$. By Theorem 7.2 again, there exists $Y \in \operatorname{Ad}K(\beta)$ such that $\pi(Y) = \gamma$. Let $Y = Y_0 + \sum_{\alpha\in\Sigma^+}(Y_\alpha + Y_{-\alpha})$, where $Y_0 \in \mathfrak{a}_{\mathfrak{p}} = \mathfrak{g}^0 \cap \mathfrak{p}$, $Y_\alpha \in \mathfrak{g}^\alpha$, $Y_{-\alpha} \in \mathfrak{g}^{-\alpha}$ for $\alpha \in \Sigma^+$. Since $Y \in \mathfrak{p}$, $\mathfrak{p}$ is the -1 eigenspace of θ, and $\theta\mathfrak{g}^\alpha = \mathfrak{g}^{-\alpha}$ by Proposition 7.3, we have

$$-Y_0 + \sum_{\alpha\in\Sigma^+}(-Y_\alpha - Y_{-\alpha}) = -Y = \theta(Y) = \theta Y_0 + \sum_{\alpha\in\Sigma^+}(\theta Y_\alpha + \theta Y_{-\alpha}).$$

Since the sums are direct, $Y_{-\alpha} = -\theta Y_\alpha$. Then $Y = Y_0 + \sum_{\alpha\in\Sigma^+}(Y_\alpha - \theta Y_\alpha)$, and $Y_0 = \pi(Y) = \gamma$. Similar to the proof of Theorem 4.6, set $X := Y_0 + 2\sum_{\alpha\in\Sigma^+} Y_\alpha \in \mathfrak{b}$. Clearly $\pi(X) = Y_0 = \gamma$, $\frac{1}{2}(X - \theta X) = Y$, and $\beta \in \operatorname{Ad}K(Y) \cap \mathfrak{a}_{\mathfrak{p}}$. $\square$

Remark 7.5 Theorem 7.4 provides Amir-Moéz-Horn-Mirsky's type result for the real semisimple Lie algebras. The algebra $\mathfrak{b} := \mathfrak{a}\dot{+}\sum_{\alpha\in\Sigma^+}\mathfrak{g}^\alpha$ will be called the standard maximal solvable subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}_{\mathfrak{p}}$. For example, when $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$, $\mathfrak{a} = \mathfrak{a}_{\mathfrak{p}}$ may be chosen as the space of real diagonal matrices and $\mathfrak{a}_{\mathfrak{k}} = 0$.

Unlike the complex case, given $X \in \mathfrak{g}$, the adjoint orbit $\operatorname{Ad}K(X)$ may not intersect $\mathfrak{b}$; in this case Ky-Fan's type result would be trivial. For example, consider $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$, $\mathfrak{b}$ the algebra of upper triangular matrices, $X = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ $(b \neq 0)$ whose eigenvalues are in complex conjugate pair. In general, in $\mathfrak{sl}(n, \mathbb{R})$ one would encounter the same problem and the following result in matrix theory is well known [17, p.82].

Proposition 7.6 For any $X \in \mathfrak{sl}(n,\mathbb{R})$, there exists $k \in \mathrm{SO}(n)$ such that kXk^{-1} is of block upper triangular form where the (main diagonal) blocks are either 1×1 or 2×2:

$$\begin{pmatrix} A_1 & & & * \\ & A_2 & & \\ & & \ddots & \\ & & & A_s \end{pmatrix},$$

with zero trace, where $A_k = \begin{pmatrix} a_k & b_k \\ -b_k & a_k \end{pmatrix}$, or $A_k = (c_k)$, $a_k, b_k, c_k \in \mathbb{R}$, $k = 1, \ldots, s$.

Indeed the above forms in Proposition 7.6 are associated with the so called standard maximal solvable subalgebras of $\mathfrak{sl}(n,\mathbb{R})$. Since each conjugacy class is uniquely determined by $(\deg A_1, \deg A_2, \ldots, \deg A_s)$ where $A_k = 1$ or 2, $k = 1, \ldots, s$. There are

$$N_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

(the Fibonacci number defined by $N_n = N_{n-1} + N_{n-2}$, $N_1 = 1$, $N_2 = 2$) conjugacy classes of maximal solvable subalgebras [26, p.1032] of $\mathfrak{sl}(n,\mathbb{R})$.

Motivated by Proposition 4.2 and the case $\mathfrak{sl}(n,\mathbb{R})$ in Proposition 7.6, we now ask whether for any element $X \in \mathfrak{g}$, $\mathrm{Ad}K(X)$ intersects *some* standard maximal solvable subalgebra $\mathfrak{s}$.

To be specific, we introduce some notions. Fix a Cartan decomposition of the real semisimple Lie algebra $\mathfrak{g} = \mathfrak{k} \dot{+} \mathfrak{p}$ and let θ be the associated Cartan involution. Fix a maximal abelian subspace $\mathfrak{a}_\mathfrak{p}$ in $\mathfrak{p}$.

A Cartan subalgebra $\mathfrak{c}$ is called a *standard Cartan subalgebra* (relative to θ and $\mathfrak{a}_{\mathfrak{p}}$) [28, p.405]. If $\mathfrak{c} = \mathfrak{c}_{\mathfrak{k}} \dot{+} \mathfrak{c}_{\mathfrak{p}}$, where $\mathfrak{a}_{\mathfrak{k}} \subset \mathfrak{c}_{\mathfrak{k}} := \mathfrak{c} \cap \mathfrak{k}$ and $\mathfrak{c}_{\mathfrak{p}} := \mathfrak{c} \cap \mathfrak{p} \subset \mathfrak{a}_{\mathfrak{p}}$. In this case $\mathfrak{c}_{\mathfrak{k}}$ is called the *toral part*, and $\mathfrak{c}_{\mathfrak{p}}$ is called the *vector part* [31, p.379].

Sugiura [31, Theorem 2, Theorem 3] proved that

Theorem 7.7 (Sugiura) Let $\mathfrak{g}$ be a real semisimple Lie algebra.

1. Every Cartan subalgebra of $\mathfrak{g}$ is conjugate to a standard Cartan subalgebra via $\mathrm{Int}\,(\mathfrak{g})$.

2. Two standard Cartan subalgebras are conjugate via $\mathrm{Int}\,(\mathfrak{g})$ if and only if their vector parts are conjugate under the Weyl group W of $(\mathfrak{g}, \mathfrak{a}_{\mathfrak{p}})$.

Therefore it is sufficient to consider standard Cartan subalgebras in order to find the conjugacy classes of Cartan subalgebras. Rothschild [28, p.405] showed that two standard Cartan subalgebras are conjugate if and only if their toral parts are conjugate.

The idea of (1) in Theorem 7.7 is as follow: If $\mathfrak{c}$ is any Cartan subalgebra of $\mathfrak{g}$, there exists a conjugate of $\mathfrak{c}$ which is θ-stable, i.e., $\mathfrak{c} = \mathfrak{c}_{\mathfrak{k}} \dot{+} \mathfrak{c}_{\mathfrak{p}}$ such that $\mathfrak{c}_{\mathfrak{k}} = \mathfrak{c} \cap \mathfrak{k}$ and $\mathfrak{c}_{\mathfrak{p}} = \mathfrak{c} \cap \mathfrak{p}$. The vector part $\mathfrak{c}_{\mathfrak{p}}$ is an abelian subalgebra of $\mathfrak{p}$ and hence is contained in some maximal abelian subalgebra in $\mathfrak{p}$. By conjugating $\mathfrak{c}$ via K, we may arrange that $\mathfrak{c}_{\mathfrak{p}} \subset \mathfrak{a}_{\mathfrak{p}}$, and then by conjugating again, leaving $\mathfrak{c}_{\mathfrak{p}}$ fixed, we can also arrange that $\mathfrak{a}_{\mathfrak{k}} \subset \mathfrak{c}_{\mathfrak{k}}$. One has [12, p.259]

Proposition 7.8 (Sugiura) Any maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ that contains $\mathfrak{a}_{\mathfrak{p}}$ is a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{a}_{\mathfrak{p}} = \mathfrak{a} \cap \mathfrak{p}$ and $\mathfrak{a}_{\mathfrak{k}} = \mathfrak{a} \cap \mathfrak{k}$.

A result of Mostow [25, Theorem 4.1] asserts that each maximal solvable subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ contains a Cartan subalgebra $\mathfrak{c}$, for example, compact Cartan subalgebra of $\mathfrak{g}$ is maximal solvable [25, Lemma 4.1]. If $\mathfrak{c}$ is a standard Cartan subalgebra, such $\mathfrak{s}$ is called a *standard maximal solvable subalgebra* (with respect to $\mathfrak{a}_\mathfrak{p}$ and θ). Each G-conjugate of $\mathfrak{s}$ is still a maximal solvable subalgebra, due to Cartan's criterion of solvability [20, Proposition 1.43], i.e., Theorem 3.5 remains true for real case, and the adjoint action of G respects the bracket and preserves the Killing form. Thus each conjugacy class of maximal solvable subalgebras under the adjoint action of G contains a standard maximal solvable subalgebra $\mathfrak{s}$ by Theorem 7.7.

Let $\mathfrak{c} = \mathfrak{c}_\mathfrak{k} \dot{+} \mathfrak{c}_\mathfrak{p}$ be a standard Cartan subalgebra of a real semisimple Lie algebra $\mathfrak{g}$. Since $\mathfrak{c}_\mathfrak{p} \subset \mathfrak{a}_\mathfrak{p}$, $\operatorname{ad} \mathfrak{c}_\mathfrak{p}$ is a family of commuting self adjoint linear transformations of $\mathfrak{g}$. So $\mathfrak{g}$ has a root space decomposition with respect to $\mathfrak{c}_\mathfrak{p}$:

$$\mathfrak{g} = \mathfrak{g}^0 \dot{+} \sum_{\alpha \in R} \mathfrak{g}^\alpha,$$

where $\mathfrak{g}^\alpha := \{X \in \mathfrak{g} : [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{c}_\mathfrak{p}\}$ and $R \subset {\mathfrak{c}_\mathfrak{p}}^*$ is the set of roots which do not vanish identically on $\mathfrak{c}_\mathfrak{p}$. An element $H \in \mathfrak{c}_\mathfrak{p}$ is called $\mathfrak{c}_\mathfrak{p}$-*singular* if there exists $\alpha \in R$ such that $\alpha(H) = 0$, otherwise H is called $\mathfrak{c}_\mathfrak{p}$-*general*. A connected component in the set of $\mathfrak{c}_\mathfrak{p}$-general elements of $\mathfrak{c}_\mathfrak{p}$ is called a $\mathfrak{c}_\mathfrak{p}$-*chamber*. For any $H \in \mathfrak{c}_\mathfrak{p}$, $\alpha \in \Sigma$, set

$$\mathfrak{g}^\alpha(H) := \{X \in \mathfrak{g} : (\operatorname{ad} H - \alpha(H))X = 0\}.$$

Let C be a $\mathfrak{c}_\mathfrak{p}$-chamber in $\mathfrak{c}_\mathfrak{p}$. Then $\mathfrak{g}^\alpha(H)$ is independent of the choice of H in C and thus may be written as $\mathfrak{g}^\alpha(C)$. Set

$$\mathfrak{g}^+(H) := \sum_{\alpha>0} \mathfrak{g}^\alpha(H) = \sum_{\alpha>0} \mathfrak{g}^\alpha(C)$$

which is also independent of the choice of H in C so it may be denoted by $\mathfrak{g}^+(C)$. The following is Mostow's result [23, Theorem 4.1].

Theorem 7.9 (Mostow) Let $\mathfrak{g}$ be a real semisimple Lie algebra.

1. Any maximal solvable subalgebra of $\mathfrak{g}$ contains a Cartan subalgebra. Hence it is conjugate via $\operatorname{Int}(\mathfrak{g})$ to a standard maximal solvable subalgebra.

2. Any maximal solvable subalgebra of $\mathfrak{g}$ containing a standard Cartan subalgebra $\mathfrak{c} = \mathfrak{c}_\mathfrak{k} \dot{+} \mathfrak{c}_\mathfrak{p}$ is of the form $\mathfrak{c} + \mathfrak{g}^+(C)$ for some $\mathfrak{c}_\mathfrak{p}$-chamber C.

Let $N := \{g \in G : \mathrm{Ad}(g)\mathfrak{c}_\mathfrak{p} = \mathfrak{c}_\mathfrak{p}\}$ and $N_K := \{k \in K : \mathrm{Ad}(k)\mathfrak{c} = \mathfrak{c}\}$. Then the $\mathfrak{c}_\mathfrak{p}$-restrictions of N and N_K coincide [23, p.515]. Notice that $\mathfrak{g}^+(C_1)$ and $\mathfrak{g}^+(C_2)$ are conjugate for two different $\mathfrak{c}_\mathfrak{p}$-chambers C_1 and C_2 if and only if the chambers are conjugate under the $\mathfrak{c}_\mathfrak{p}$-restriction of N_K.

Example 7.10 For the special case $\mathfrak{c}_\mathfrak{p} = \mathfrak{a}_\mathfrak{p}$, the $\mathfrak{c}_\mathfrak{p}$-chambers are the Weyl chambers which are permuted transitively by the Weyl group. So all standard maximal solvable subalgebras containing the standard Cartan subalgebra $\mathfrak{a} = \mathfrak{a}_\mathfrak{k} \dot{+} \mathfrak{a}_\mathfrak{p}$ are conjugate. One may pick $\mathfrak{b} = (\mathfrak{a}_\mathfrak{k} \dot{+} \mathfrak{a}_\mathfrak{p}) \dot{+} \sum_{\alpha>0} \mathfrak{g}^\alpha$ in Theorem 7.4.

To list all non-conjugate standard maximal solvable subalgebras of $\mathfrak{g}$, we first find the standard Cartan subalgebras $\mathfrak{c} = \mathfrak{c}_{\mathfrak{k}} \dot{+} \mathfrak{c}_{\mathfrak{p}}$. Then find the non-conjugate $\mathfrak{c}_{\mathfrak{p}}$-chambers of $\mathfrak{c}$, which yields all the standard maximal solvable subalgebras of $\mathfrak{g}$ containing $\mathfrak{c}$ via Theorem 7.9.

Example 7.11 When $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{R})$, there are two conjugacy classes of Cartan subalgebras [20] represented by the standard Cartan subalgebras:

$$\mathfrak{c}_1 = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & -a-b \end{pmatrix} : a, b \in \mathbb{R} \right\}, \qquad \mathfrak{c}_2 = \left\{ \begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & -2a \end{pmatrix} : a, b \in \mathbb{R} \right\}.$$

However there are three conjugacy classes of maximal solvable subalgebras [26, p.1032] represented by the standard maximal solvable subalgebras:

$$\mathfrak{s}_1 := \left\{ \begin{pmatrix} a & c & e \\ 0 & b & d \\ 0 & 0 & -a-b \end{pmatrix} : a, b, c, d, e \in \mathbb{R} \right\},$$

$$\mathfrak{s}_2 := \left\{ \begin{pmatrix} a & b & c \\ -b & a & d \\ 0 & 0 & -2a \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\},$$

$$\mathfrak{s}_3 := \left\{ \begin{pmatrix} -2a & c & d \\ 0 & a & b \\ 0 & -b & a \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}.$$

Notice that $\mathfrak{s}_2$ and $\mathfrak{s}_3$ contain conjugate standard Cartan subalgebras corresponding to $\mathfrak{c}_2$.

We now address the question whether for any $X \in \mathfrak{g}$, $\mathrm{Ad}K(X) \cap \mathfrak{s} \neq \phi$ for some standard maximal solvable subalgebra $\mathfrak{s}$ of $\mathfrak{g}$.

Proposition 7.12 If $\mathfrak{g}$ is a compact semisimple Lie algebra, then for any $X \in \mathfrak{g}$, $\mathrm{Ad}K(X) \cap \mathfrak{s} \neq \phi$ for some standard maximal solvable subalgebra $\mathfrak{s}$ of $\mathfrak{g}$.

Proof: When $\mathfrak{g}$ is compact we have $\mathfrak{g} = \mathfrak{k}$, $\mathfrak{p} = 0$, $\mathfrak{a}_{\mathfrak{p}} = 0$ and $\mathfrak{a}_{\mathfrak{k}} = \mathfrak{t}$ which is a Cartan subalgebra. By [25, Lemma 4.1] $\mathfrak{t}$ is a maximal solvable subalgebra of $\mathfrak{k}$. Since maximal tori are conjugate in compact Lie group [20, p.202], $\mathrm{Ad}K(X) \cap \mathfrak{t} \neq \phi$ for any $X \in \mathfrak{k}$. $\square$

In this case Theorem 7.4 is reduced to Theorem 5.2.

Proposition 7.13 In general it is not true that given arbitrary $X \in \mathfrak{g}$, $\mathrm{Ad}K(X)$ intersects $\mathfrak{s}$ for some standard maximal solvable subalgebra $\mathfrak{s}$ of $\mathfrak{g}$.

Proof: Consider the real simple Lie algebra $\mathfrak{g} = \mathfrak{su}_{1,1}$. We consider the group:

$$\mathrm{SU}(1,1) = \left\{ \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} : |\alpha|^2 - |\beta|^2 = 1 \right\},$$

whose Lie algebra is a real form of $\mathfrak{sl}(2,\mathbb{C})$:

$$\mathfrak{su}_{1,1} = \left\{ \begin{pmatrix} ia & c \\ \overline{c} & -ia \end{pmatrix} : a \in \mathbb{R}, c \in \mathbb{C} \right\},$$

$$K = \left\{ \mathrm{diag}\,(e^{i\theta}, e^{-i\theta}) : \theta \in \mathbb{R} \right\},$$

$$\mathfrak{k} = \left\{ \begin{pmatrix} ia & 0 \\ 0 & -ia \end{pmatrix} : a \in \mathbb{R} \right\},$$

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & c \\ \overline{c} & 0 \end{pmatrix} : c \in \mathbb{C} \right\},$$

$$\mathfrak{a}_\mathfrak{p} = \left\{ \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix} : b \in \mathbb{R} \right\}.$$

There are two non-conjugate standard Cartan subalgebras: $\mathfrak{k}$ and $\mathfrak{a}_\mathfrak{p}$ [31, p.401]. They are also the two [27, p.518] standard maximal solvable subalgebras of $\mathfrak{su}_{1,1}$. Since $\mathfrak{k}$ is a compact Cartan subalgebra of $\mathfrak{su}_{1,1}$, it is maximal solvable [25, Lemma 4.1]. We know that all maximal solvable subalgebras containing $\mathfrak{a}_\mathfrak{p}$ are conjugate. Let us consider the root space decomposition of $\mathfrak{g} = \mathfrak{su}(1,1)$ with respect to $\mathfrak{a}_\mathfrak{p}$. The roots are $R = \{\alpha, -\alpha\}$ [20, p.314], where

$$\alpha \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix} = 2b, \quad b \in \mathbb{R}, \qquad \mathfrak{g}^\alpha = \mathbb{R} \begin{pmatrix} -i & i \\ -i & i \end{pmatrix}, \quad \mathfrak{g}^{-\alpha} = \mathbb{R} \begin{pmatrix} -i & -i \\ i & i \end{pmatrix}.$$

Let

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(indeed any $\mathfrak{c}_p$-general element H in $\mathfrak{c}_\mathfrak{p}$ works). By Theorem 7.9,

$$\mathfrak{s} = \mathfrak{a}_\mathfrak{p} + \mathfrak{g}^\alpha(H) = \mathbb{R} \begin{pmatrix} -ia & ia+b \\ -ia+b & ia \end{pmatrix}, \quad a, b \in \mathbb{R}.$$

is the only standard maximal solvable subalgebra containing $\mathfrak{a}_\mathfrak{p}$. Let

$$X = \begin{pmatrix} -i & \epsilon \\ \epsilon & i \end{pmatrix}, \quad 0 < \epsilon < 1.$$

Clearly $\mathfrak{k} \cap \mathrm{Ad}K(X) = \phi$. Now $\mathfrak{s} \cap \mathrm{Ad}K(X) = \phi$, otherwise let

$$\begin{aligned} \mathrm{Ad}\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} X &= \begin{pmatrix} -i & \epsilon e^{-i2\theta} \\ \epsilon e^{i2\theta} & i \end{pmatrix} \\ &= \begin{pmatrix} -ia & ia+b \\ -ia+b & ia \end{pmatrix} \in \mathfrak{s}, \quad \text{for some } a, b, \theta \in \mathbb{R}. \end{aligned}$$

Thus $a = 1$, and $\epsilon e^{-i2\theta} = ia + b = i + b$. Then $\epsilon = |i + b| \geq 1$, a contradiction. $\square$

Proposition 7.14 Let $\mathfrak{c} = \mathfrak{c}_\mathfrak{k} \dot{+} \mathfrak{c}_\mathfrak{p}$ be a standard Cartan subalgebra of $\mathfrak{g}$. Let $\pi_{\mathfrak{c}_\mathfrak{p}} : \mathfrak{g} \to \mathfrak{c}_\mathfrak{p}$ be the orthogonal projection onto $\mathfrak{c}_\mathfrak{p} \subset \mathfrak{a}_\mathfrak{p}$. Then for any $\beta \in \mathfrak{c}_\mathfrak{p}$,

$$\mathrm{conv}\, W\beta \cap \mathfrak{c}_\mathfrak{p} = \pi_{\mathfrak{c}_\mathfrak{p}}(\mathrm{Ad}K(\beta)) = \pi_{\mathfrak{c}_\mathfrak{p}}(\mathrm{conv}\, W\beta),$$

Proof: The second equality follows immediately from Theorem 7.2. If $\gamma \in \mathrm{conv}\, W\beta \cap \mathfrak{c}_\mathfrak{p}$, by Theorem 7.2 there is $k \in K$ such that $\gamma = \pi_{\mathfrak{a}_\mathfrak{p}}(\mathrm{Ad}(k)\beta)$. Now $\gamma \in \mathfrak{c}_\mathfrak{p}$ implies $\gamma = \pi_{\mathfrak{c}_\mathfrak{p}}(\gamma) = \pi_{\mathfrak{c}_\mathfrak{p}}(\pi_{\mathfrak{a}_\mathfrak{p}}(\mathrm{Ad}(k)\beta)) = \pi_{\mathfrak{c}_\mathfrak{p}}(\mathrm{Ad}(k)\beta)$. So we have

$$\mathrm{conv}\, W\beta \cap \mathfrak{c}_\mathfrak{p} \subset \pi_{\mathfrak{c}_\mathfrak{p}}(\mathrm{Ad}K(\beta)).$$

To establish the converse inclusion, we will show that $\operatorname{conv} W\beta$ is symmetric with respect to $\mathfrak{c}_\mathfrak{p}$, i.e., if $\gamma \in \operatorname{conv} W\beta$ such that $\gamma = \gamma_1 + \gamma_2$, where $\gamma_1 \in \mathfrak{c}_\mathfrak{p}$ and $\gamma_2 \in \mathfrak{c}_p^\perp \cap \mathfrak{a}_\mathfrak{p}$, then $\gamma_1 - \gamma_2 \in \operatorname{conv} W\beta$.

Let $\mathfrak{a}$ be the maximal abelian subalgebra containing $\mathfrak{a}_\mathfrak{p}$. By Proposition 7.8 $\mathfrak{a} = \mathfrak{a}_\mathfrak{k} + \mathfrak{a}_\mathfrak{p}$ is a Cartan subalgebra of $\mathfrak{g}$, where $\mathfrak{a}_\mathfrak{k} = \mathfrak{a} \cap \mathfrak{k}$ and $\mathfrak{a}_\mathfrak{p} = \mathfrak{a} \cap \mathfrak{p}$. Let $\mathfrak{h} := \mathfrak{a}^\mathbb{C}$ and $\mathfrak{g}^\mathbb{C}$ be the complexifications of $\mathfrak{a}$ and $\mathfrak{g}$ respectively. The root space decomposition of $\mathfrak{g}^\mathbb{C}$ with respect to $\mathfrak{h}$ is

$$\mathfrak{g}^\mathbb{C} = \mathfrak{h} + \sum_{\alpha \in \Delta} (\mathfrak{g}^\mathbb{C})^\alpha,$$

where $\Delta \subset \mathfrak{h}_0^*$ is the root system of $(\mathfrak{g}^\mathbb{C}, \mathfrak{h})$ and $\mathfrak{h}_0 := \mathfrak{a}_\mathfrak{p} + i\mathfrak{a}_\mathfrak{k}$ is spanned by H_α, $\alpha \in \Delta$ (see Proposition 3.8). Let $\rho : \mathfrak{h}_0^* \to \mathfrak{a}_\mathfrak{p}^*$ be the restriction to $\mathfrak{a}_\mathfrak{p}$. Then for each $\alpha \in \Delta$, either $\rho(\alpha)$ is zero or is an element of Σ (that is why Σ is called the restricted root system of $(\mathfrak{g}, \mathfrak{a}_\mathfrak{p})$) [12, p.260-263].

Since $\mathfrak{c}_\mathfrak{p}$ is the vector part of some standard Cartan subalgebra $\mathfrak{c}$, by [31, Theorem 5], there exist ℓ roots $\alpha_1, \ldots, \alpha_\ell \in \Delta$ such that

1. $\alpha_i \pm \alpha_j \notin \Delta$, $1 \le i, j \le \ell$ and $\alpha_i \pm \alpha_j \neq 0$ if $i \neq j$.

2. $\mathfrak{c}_\mathfrak{p}^\perp \cap \mathfrak{a}_\mathfrak{p} = \sum_{i=1}^\ell \mathbb{R} H_{\alpha_i}$.

In particular $H_{\alpha_i} \in \mathfrak{c}_\mathfrak{p}^\perp \cap \mathfrak{a}_\mathfrak{p} \subset \mathfrak{a}_\mathfrak{p}$ and thus $\rho(\alpha_1), \ldots, \rho(\alpha_\ell)$ are in Σ. By [12, p.457] $\alpha_1, \ldots, \alpha_\ell$ are orthogonal.

Extend $\{H_{\alpha_1}, \ldots, H_{\alpha_\ell}\}$ to an orthogonal basis $\{H_{\alpha_1}, \ldots, H_{\alpha_\ell}, X_1, \ldots, X_m\}$ of $\mathfrak{a}_\mathfrak{p}$. To show that $\operatorname{conv} W\beta$ is symmetric about $\mathfrak{c}_\mathfrak{p}$, it suffices to show the symmetry for $W\beta$. Let $\lambda = \lambda_1 + \lambda_2 \in W\beta$, where $\lambda_1 \in \mathfrak{c}_\mathfrak{p}$ and $\lambda_2 \in \mathfrak{c}_\mathfrak{p}^\perp \cap \mathfrak{a}_\mathfrak{p}$. Then $\lambda_1 =$

$\sum_{i=1}^m \eta_i X_i$ and $\lambda_2 = \sum_{i=1}^\ell \xi_i H_{\alpha_i}$. Then apply the reflection $s := s_{\rho(\alpha_1)} \cdots s_{\rho(\alpha_\ell)} \in W$ on $\lambda \in \operatorname{conv} W\beta \subset \mathfrak{a}_\mathfrak{p}$:

$$s\lambda = s_{\rho(\alpha_1)} \cdots s_{\rho(\alpha_\ell)}\lambda_1 + s_{\rho(\alpha_1)} \cdots s_{\rho(\alpha_\ell)}\lambda_2 = s_{\alpha_1} \cdots s_{\alpha_\ell}\lambda_1 - s_{\alpha_1} \cdots s_{\alpha_\ell}\lambda_2 = \lambda_1 - \lambda_2,$$

to have the desired symmetry. Now if $\gamma \in \operatorname{conv} W\beta$ with $\gamma = \gamma_1 + \gamma_2$, $\gamma_1 \in \mathfrak{c}_\mathfrak{p}$ and $\gamma_2 \in \mathfrak{c}_\mathfrak{p}^\perp \cap \mathfrak{a}_\mathfrak{p}$, then $\pi_{\mathfrak{c}_\mathfrak{p}}(\gamma) = \gamma_1$. But $\gamma_1 - \gamma_2 \in \operatorname{conv} W\beta$ by the symmetry of $\operatorname{conv} W\beta$ with respect to $\mathfrak{c}_\mathfrak{p}$. So $\gamma_1 \in \operatorname{conv} W\beta$ and thus $\gamma_1 \in \operatorname{conv} W\beta \cap \mathfrak{c}_\mathfrak{p}$. □

Suppose $\mathfrak{s}$ is a standard maximal solvable subalgebra containing the standard Cartan subalgebra $\mathfrak{c}$. Since $\mathfrak{c}_\mathfrak{p} \subset \mathfrak{a}_\mathfrak{p} \subset \mathfrak{p}$, $\mathfrak{c}_\mathfrak{p} \perp \mathfrak{k}$ so that

$$\pi_{\mathfrak{c}_\mathfrak{p}}((\mathfrak{k} + \mathrm{Ad}K(\beta)) \cap \mathfrak{s}) \subset \pi_{\mathfrak{c}_\mathfrak{p}}(\mathfrak{k} + \mathrm{Ad}K(\beta)) = \pi_{\mathfrak{c}_\mathfrak{p}}(\mathrm{Ad}K(\beta)) = \operatorname{conv} W\beta \cap \mathfrak{c}_\mathfrak{p}.$$

Due to Theorem 7.4 and Proposition 7.12, one may ask whether the set equality holds. The following example shows that the set inclusion is strict, even for some normal real Lie algebras. (A real semisimple Lie algebra $\mathfrak{g}$ is called *normal* [31, p.392] if $\mathfrak{g}$ has a Cartan subalgebra whose toral part is zero).

Example 7.15 Consider $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{R})$ in Example 7.15. Choose the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \dot{+} \mathfrak{p}$, where $\mathfrak{k}$ is the set of all real skew symmetric matrices and $\mathfrak{p}$ is the set of all real symmetric matrices in $\mathfrak{g}$. Let $\mathfrak{a}_\mathfrak{p} = \{\mathrm{diag}\,(a, b, -a-b) : a, b \in \mathbb{R}\}$. Consider the Cartan subalgebra $\mathfrak{c} = \mathfrak{c}_\mathfrak{k} \dot{+} \mathfrak{c}_\mathfrak{p}$ of $\mathfrak{g}$ corresponding to

$$\mathfrak{s}_2 = \left\{ \begin{pmatrix} a & b & c \\ -b & a & d \\ 0 & 0 & -2a \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\},$$

such that

$$\mathfrak{c}_{\mathfrak{p}} = \{\operatorname{diag}(a, a, -2a) : a \in \mathbb{R}\}.$$

Let $\beta = \operatorname{diag}(1, 1, -2) \in \mathfrak{c}_{\mathfrak{p}}$. Then $W\beta = S_3\beta = \{H_1, H_2, H_3\}$ where

$$H_1 = \operatorname{diag}(1, 1, -2), \quad H_2 = \operatorname{diag}(1, -2, 1), \quad H_3 = \operatorname{diag}(-2, 1, 1).$$

Then

$$H = \frac{1}{2}(H_2 + H_3) = \operatorname{diag}(-\frac{1}{2}, -\frac{1}{2}, 1) \in \operatorname{conv} W\beta \cap \mathfrak{c}_{\mathfrak{p}}.$$

We now claim $H \notin \pi_{\mathfrak{c}_{\mathfrak{p}}}\{(\mathfrak{k} + \operatorname{Ad}K(\beta)) \cap \mathfrak{s}_2\}$. Otherwise, let $H = \pi_{\mathfrak{c}_{\mathfrak{p}}}(Y + \operatorname{Ad}k(\beta))$ for some $Y \in \mathfrak{k}$ and $k \in K$ and $Y + \operatorname{Ad}k(\beta) \in \mathfrak{s}_2$. Since $\operatorname{diag} Y = 0$, the diagonal of $\operatorname{Ad}k(\beta)$ must be

$$\operatorname{diag}(\operatorname{Ad}k(\beta)) = \operatorname{diag}(-\frac{1}{2}, -\frac{1}{2}, 1).$$

That is to say, there exists

$$k = \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} \in \mathrm{SO}(3)$$

such that

$$\begin{aligned} \operatorname{diag}(k(\operatorname{diag}\beta)k^T) &= \operatorname{diag}(u_1^2 + u_2^2 - 2u_3^2, v_1^2 + v_2^2 - 2v_3^2, w_1^2 + w_2^2 - 2w_3^2) \\ &= (-\frac{1}{2}, -\frac{1}{2}, 1). \end{aligned}$$

So

$$
\begin{aligned}
u_1^2 + u_2^2 - 2u_3^2 &= -1/2, \\
v_1^2 + v_2^2 - 2v_3^2 &= -1/2, \\
w_1^2 + w_2^2 - 2w_3^2 &= 1.
\end{aligned}
$$

Because $k \in \mathrm{SO}(3)$, we have

$$
\begin{aligned}
u_1^2 + u_2^2 + u_3^2 &= 1, \\
v_1^2 + v_2^2 + v_3^2 &= 1, \\
w_1^2 + w_2^2 + w_3^2 &= 1.
\end{aligned}
$$

So

$$
u_3 = \pm\frac{\sqrt{2}}{2}, \quad v_3 = \pm\frac{\sqrt{2}}{2}, \quad w_3 = 0.
$$

Let $w_1 = \cos\gamma$, $w_2 = \sin\gamma$. Because u and v are perpendicular to w, we can assume that

$$
u = (u_1, u_2, u_3) = (\delta_1\frac{\sqrt{2}}{2}\sin\gamma, -\delta_1\frac{\sqrt{2}}{2}\cos\gamma, \delta_2\frac{\sqrt{2}}{2}),
$$

where δ_1, $\delta_2 = \pm 1$. Similarly, we can assume that

$$
v = (v_1, v_2, v_3) = (\epsilon_1\frac{\sqrt{2}}{2}\sin\gamma, -\epsilon_1\frac{\sqrt{2}}{2}\cos\gamma, \epsilon_2\frac{\sqrt{2}}{2}),
$$

where ϵ_1, $\epsilon_2 = \pm 1$. By direct computation, the $(1,2)$ and $(2,1)$ entries of $\mathrm{Ad}k(\beta)$ are equal to $\frac{1}{2}\delta_1\epsilon_1 - \delta_2\epsilon_2 \neq 0$. Thus no $X \in \mathfrak{k}$ (skew symmetric matrices) can make

$X + \mathrm{Ad}k(\beta)$ an element of $\mathfrak{s}_2$. Therefore we cannot find $k \in K$ and $X \in \mathfrak{k}$ such that $X + \mathrm{Ad}k(\beta) \in \mathfrak{s}_2$ and

$$\pi_{\mathfrak{c}_{\mathfrak{p}}}(X + \mathrm{Ad}k(\beta)) = \mathrm{diag}\,(-\frac{1}{2}, -\frac{1}{2}, 1) \in \mathrm{conv}\, W\beta \cap \mathfrak{c}_{\mathfrak{p}}.$$

One can get the same conclusion on $\mathfrak{s}_2$ similarly, but not on $\mathfrak{s}_1$ due to Theorem 7.4.

Remark 7.16 In Example 7.15, $\mathfrak{sl}(3, \mathbb{R})$ is the simplest nontrivial example. Consider $\mathfrak{g} := \mathfrak{sl}(2, \mathbb{R}) = \mathfrak{k} + \mathfrak{p}$ where $\mathfrak{k}$ is the set of all real skew symmetric matrices and $\mathfrak{p}$ is the set of all real symmetric matrices in $\mathfrak{g}$. Let $\mathfrak{a}_{\mathfrak{p}} = \{\mathrm{diag}\,(a, -a) : a \in \mathbb{R}\}$. Now $\mathfrak{g}$ has two standard Cartan subalgebras

$$\mathfrak{c}_1 := \mathfrak{a}_{\mathfrak{p}} = \Big\{ \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} : a \in \mathbb{R} \Big\}, \qquad \mathfrak{c}_2 := \mathfrak{so}(2) = \Big\{ \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} : b \in \mathbb{R} \Big\},$$

and the corresponding standard maximal solvable subalgebras are

$$\mathfrak{s}_1 := \Big\{ \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix} : a, b \in \mathbb{R} \Big\}, \qquad \mathfrak{s}_2 := \mathfrak{so}(2) = \Big\{ \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} : b \in \mathbb{R} \Big\},$$

respectively. Now Theorem 7.4 implies

$$\pi_{\mathfrak{c}_{\mathfrak{p}}}((\mathfrak{k} + \mathrm{Ad}K(\beta)) \cap \mathfrak{s}_1) = \mathrm{conv}\, W\beta \cap (\mathfrak{c}_1)_{\mathfrak{p}}, \qquad \beta \in (\mathfrak{c}_1)_{\mathfrak{p}}.$$

Since $(\mathfrak{c}_2)_{\mathfrak{p}} = 0$, the statement

$$\pi_{\mathfrak{c}_{\mathfrak{p}}}((\mathfrak{k} + \mathrm{Ad}K(\beta)) \cap \mathfrak{s}_2) = \mathrm{conv}\, W\beta \cap (\mathfrak{c}_2)_{\mathfrak{p}}, \qquad \beta \in (\mathfrak{c}_2)_{\mathfrak{p}}$$

is then trivial.

We conclude this chapter by asking whether $\mathrm{Ad}(K)X \cap \mathfrak{s} \neq \phi$ for some standard maximal solvable subalgebra $\mathfrak{s}$ of $\mathfrak{g}$, if $\mathfrak{g}$ is a normal real semisimple Lie algebra. The question is motivated by Proposition 7.6.

Chapter 8

The eigenvalues and the real and imaginary singular values for $\mathfrak{sl}(2, \mathbb{C})$ and $\mathfrak{sl}(2, \mathbb{R})$

Ky Fan-Amir-Moéz-Horn-Mirsky's result asserts that the real part of the eigenvalues of a matrix is majorized by the real singular values, and conversely if there exist $\lambda \in \mathbb{C}$, $\beta \in \mathbb{R}^n$ such that $\beta \prec \operatorname{Re} \lambda$, then there is a matrix with eigenvalues λ's and real singular values β's. A similar result for the imaginary part of the eigenvalues and the imaginary singular values is also given. Then we may ask the following question: what is the necessary and sufficient condition on the given vectors $\lambda \in \mathbb{C}$, $\alpha, \beta \in \mathbb{R}^n$ so that a matrix $A \in \mathbb{C}_{n\times n}$ exists with eigenvalues λ's, real singular values α's and imaginary singular values β's? Condition stronger than majorization is expected. The following is the simplest case and it shows that the norm condition in Remark 4.10 is not sufficient.

Proposition 8.1 Let $\alpha, \beta \in \mathbb{R}$ and $a + ib \in \mathbb{C}$. Then there exists $A \in \mathfrak{sl}(2, \mathbb{C})$ whose eigenvalues, real singular values, and imaginary singular values are $\pm(a+ib)$, $\pm\alpha$, and $\pm\beta$, respectively, if and only if $(-a, a) \prec (-\alpha, \alpha)$, $(-b, b) \prec (-\beta, \beta)$, and $\beta^2 - b^2 = \alpha^2 - a^2$.

Proof: Let $A \in \mathfrak{sl}(2, \mathbb{C})$ whose eigenvalues, real singular values, and imaginary singular values are $\pm(a + ib)$, $\pm\alpha$, and $\pm\beta$, respectively. After an appropriate unitary similarity, we may assume that A is in upper triangular form:

$$A = \begin{pmatrix} a+ib & c \\ 0 & -a-ib \end{pmatrix}.$$

Then

$$(A+A^*)/2 = \begin{pmatrix} a & c/2 \\ \bar{c}/2 & -a \end{pmatrix}, \qquad (A-A^*)/2i = \begin{pmatrix} b & c/2i \\ -\bar{c}/2i & -b \end{pmatrix}.$$

The eigenvalues of the matrices are $\pm(a+ib)$, $\pm\alpha = \pm(a^2+\frac{1}{4}|c|^2)^{1/2}$, and $\pm\beta = \pm(b^2+\frac{1}{4}|c|^2)^{1/2}$. So $(-a,a) \prec (-\alpha,\alpha)$, $(-b,b) \prec (-\beta,\beta)$ and $\beta^2 - b^2 = \alpha^2 - a^2 = \frac{1}{4}|c|^2$. Conversely, if the conditions are satisfied, the above triangular matrix A (thus not unique) is the required one with $\alpha^2 - a^2 = \frac{1}{4}|c|^2$, $\square$

The following is the corresponding result for the real case.

Proposition 8.2 Let $\alpha, \beta \in \mathbb{R}$ and $a+ib \in \mathbb{C}$. Then there exists $A \in \mathfrak{sl}(2,\mathbb{R})$ whose eigenvalues, real singular values, and imaginary singular values are $\pm(a+ib)$, $\pm\alpha$, and $\pm\beta$, respectively, if and only if (1) $b=0$, $(-a,a) \prec (-\alpha,\alpha)$, and $\beta^2 = \alpha^2 - a^2$, or (2) $a = \alpha = 0$, $b = \pm\beta$.

Proof: If the eigenvalues of $A \in \mathfrak{sl}(2,\mathbb{R})$ are complex, they must be conjugate to each other, that is, $\pm ib$, $b \in \mathbb{R}$. Otherwise, they must be of the form $\pm a$, $a \in \mathbb{R}$. By Proposition 8.1, or by observing each $A \in \mathfrak{sl}(2,\mathbb{R})$ is (special) orthogonally similar to one of the forms:

$$(a) \begin{pmatrix} a & c \\ 0 & -a \end{pmatrix}, \quad a, c \in \mathbb{R}, \qquad (b) \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}, \quad b \in \mathbb{R},$$

accordingly, we have the necessary conditions. Conversely, (1) let A be in the form (a) whose eigenvalues, real singular values, and imaginary singular values are $\pm a$, $\pm\alpha = \pm(a^2 + \frac{1}{4}c^2)^{1/2}$, and $\pm\beta = \pm(\frac{1}{4}c^2)^{1/2}$. Thus set $c = \pm 2|\beta|$; (2) let A be in the form (b) and it is obvious. $\square$

But for the case $n \geq 3$, the problem becomes much more complicated. The problem is similar to the recently settled Horn's problem on the eigenvalues of sum of Hermitian matrices [8, 9]. Further research is needed for a clear understanding.

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