A Computational Study of Modern Approaches to Risk-Averse Stochastic Optimization Using Financial Portfolio Allocation Model

by

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A thesis submitted to the Graduate Faculty of
Auburn University
in partial fulfillment of the
requirements for the Degree of
Master of Science

Auburn, Alabama
Aug 6, 2016

Keywords: Stochastic programming, Risk, Optimization

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Abstract

We study some novel approaches to risk-averse stochastic optimization. Our goal is to numerically evaluate whether these methods result in an improved decision making under conditions of uncertainty. The methodology used relies on the financial portfolio optimization model used as a testing framework. We track the behavior of trading strategies made based on Conditional-Value-at-Risk (CVaR), Higher Moment Coherent Risk (HMCR) measure and Log-Exponential Convex Risk (LogExpCR) measures, three of the approaches recently proposed to deal with risk in stochastic operations research problems. We use historical data from S&P 100 assets during the period from 2006 to 2015, which includes the Global Financial Crisis. In our analysis we have observed that more advanced HMCR and LogExpCR measures result in better performance compared to CVaR portfolios, especially in the case of heavy-tailed distributions of the uncertainties. While this is in accordance with most of the previous findings presented in the literature this work represents an attempt at a more comprehensive comparative study of risk measures. We have also observed some behaviors that go against general expectations, and hence require additional attention in the future research.

Acknowledgments

Foremost, I would like to express my sincere gratitude to my advisor Prof. Aleksandr Vinel for the continuous support of my research. His guidance helped me in all the time of research and writing of this thesis. I could not have imagined that I finish my degree without him.

Besides my advisor, I would like to thank the rest of my thesis committee: Prof. Jorge Valenzuela and Prof. Jeffrey S. Smith, for their insightful comments, and hard questions.

I would like to thank my country and the Republic of Korea Army who gave me a chance to have a wonderful experience. I will pay them back by devotion and royalty.

Thanks also go to my fellow graduate student, Kim and Charlie. Special thanks go to beautiful and thoughtful Ray family. Without your support, this result should not be here.

Finally, I would like to acknowledge with gratitude, the support and love of my family. My lovely wife and son, my parent and parents in law, and my brother. They all kept me going, and give me so much love as much as I cannot express.

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Chapter 1

INTRODUCTION

Risk, understood as a potential for losses, is inevitably present whenever decisions have to be made under conditions of uncertainty. Intuitively, as long as the decision maker cannot accurately predict the future, the decision making process is risky. Furthermore, anticipation of risk is not enough. Indeed, although we can expect the risks, quantifying them is required to measure and manage the performance of uncertain operation. For that reason, mainly in the area of the operations research and the finance, efforts on designing tools for quantifying and modeling risk have been made. While various risk measures have been presented, the study of reliability and accuracy of such tools should be considered as a separate research question.

Therefore, to generate a coherent and reliable result, it will be important to develop better risk measures and to supplement their weaknesses based on the assessment of existent risk measures. The ultimate goal here is to identify certain properties and characteristics of the problems under consideration that would favor one risk measurement approach over another, thus letting the decision maker choose the one best suited for the particular model.

1.1 Background

Historically, the Mean-Variance model(MV) pioneered the area of quantifying the risk. Yet, very quickly it was understood that it does not have certain properties that would be required from a practical risk measure. For example, variance equally penalizes positive and negative outcomes that deviate from the average, which is clearly not ideal. Another widely used approach is known as the Value at Risk (VaR), which is equivalent to so-called chance constraints. Still, there exists a large body of literature documenting issues and limitations associated with this approach.

Recently, a number of approaches have been proposed in the literature, that are aimed at identifying both rigorous ways of defining measures of risk and particular candidates for practically useful measures. Conditional Value at Risk (CVaR), proposed by Rockafellar and Uryasev (2000), is suggested as more reliable and alternative risk measure than the VaR. It has been shown to possess most of the important properties required from a measure of risk, and is slowly becoming a de-facto standard in stochastic optimization. Further, Krokhmal (2007) and Vinel and Krokhmal (2015) proposed a novel approach to generate so-called coherent risk measures more easily. Higher Moment Coherent Risk measure (HMCR) is one of the results following this advance. Furthermore, the Convolution Representation is presented for generating suitable risk measures for math programming. By adding utility function to the Convolution Representing, the Log-Exponential Convex Risk (LogExpCR) measure was defined.

Although these coherent and suitable for optimizing risk measures are presented, finding pros and cons of the risk measure and comparing each other still remain to study. To observe the performance of risk measure, Rockafellar and Uryasev(2000) suggested the portfolio optimization problem. Since the portfolio is selected based on random variables which have rate of returns with probability and the result can explain numerically, it is a suitable tool for comparison. Not only in this case, HMCR and LogExpCR also use the portfolio optimization for the verification and comparing with existent risk measures.

These former studies lead us to examine three coherent and convex risk measures stated above and compare them. To compare the performance of risk measure, we will also use the portfolio optimization problem suggested by many researchers.

1.2 Problem Statement

The purpose of our research is as follows. First, we will examine historical risk measure development footstep. Next, we will choose risk measures which can present a coherent and reasonable result. Since our research is focused on making decisions based on stochastic programming, mathematical suitability will be also considered.

Secondly, we will observe how the risk measures quantify and expect the future risk, and how the quantified risk reduces the worst case in the future. To satisfy our purpose, the decision that yields the most minimized risk should be made under the uncertainty. After making a decision, the decision should be realized in the future. According to the decision procedure, we could observe how the risk measure control risk.

Finally, we will evaluate the observing data objectively further, and we intend to find the characteristic of risk measures and analyze their behavior under various conditions.

Note that partially due to the relative novelty of the considered approaches, to the best of our knowledge, there has not been a rigorous study of risk measures in the literature. Hence, the aim of this research is to provide a comprehensive analysis comparing some recent developments using a practical real-world application area and data.

1.3 Research Methodology

As we noted above, we need a suitable stochastic problem that makes a decision under risk distribution and the problem is able to quantify risk. In this study, we will employ financial portfolio optimization model for the following reasons. First, the financial market has always been a major driving force behind research in risk-averse stochastic operations research. While other application areas constantly arise, financial optimization remains a standard test model for novel results in the field. Second, there exists an abundance of publicly available real-life historical data on the performance of financial assets. There are thousands of stocks traded daily throughout the world and information on real-time prices can be easily accessed. Finally, we expect that so-call heavy-tailed behavior of the uncertainty plays a major role in the relative performance of risk measures, and financial data is known to exhibit such behavior. Moreover, by adjusting the time frame of the assets' returns we can change the "heavy-tailness" of the uncertainty, and hence, control for it in our study.

In this thesis, the portfolio optimization problem is defined as a minimizing risk with portfolio return constrained by the threshold. Additionally, Risk function of the problem will take the CVaR, HMCR, LogExpCR or VaR respectively. The portfolio optimization problem will be solved by the following procedure. 1,000 rate of return random variable sets are generated by the historical stock market data from the past to one day before the rebalancing date. By using this scenario set, the portfolio optimization problem will be solved. Then, the portfolio return will be realized according to the optimal solution over the next few days. We will repeat this procedure a hundred times. This procedure will be coded in C^{++} software, and solved by CPLEX (a commercially available solver).

The S&P 100 historical stock market data from January 2006 to December 2015 will be used in our research. However, for the research purpose, we will select 50 stocks from S&P 100. This is done in order to limit the number of decision variables, and thus reduce the computational effort. Additionally, we select for the more heavy-tailed distributions according to their kurtosis. To illustrate different risk conditions, we will conduct an experiment changing various parameter values within the same historical data set.

The average portfolio returns and the Sharpe ratio will be used for evaluating the performance of the portfolios. Average portfolio return is the primary measure for an investor, since it directly corresponds to the gain from trading. Since the Sharpe ratio adjusts the expected portfolio return by the standard deviation of return, the Sharpe ratio is well known standard measure evaluating the relative risk of a portfolio.

Not only are we comparing risk measures by the performance of the generating portfolio, but we also intend to find stock selecting tendency by tracking the optimal solution and solution stock's kurtosis and skewness.

We would like to emphasize here that our goal here is to study the novel tools proposed in the operations research literature, and not to use the presented portfolio optimization problem for making a good invest model or obtaining an economical result.

1.4 Research Plan

This research proceeds as follows, Chapter 2 describes the literature review of the general risk measures and the portfolio optimization. Chapter 3 constructs the portfolio optimization problem implemented by three types of risk measures. Chapter 4 analyzes the result of the portfolio problem model from Chapter 3. Here we separate the analysis within and between the historical data set. Furthermore, we examine the tendency of selecting stocks of each risk measure. Chapter 5 summarizes and presents conclusions.

Chapter 2

LITERATURE REVIEW

Since Markowitz set up portfolio's risk as the variance of return in his modern optimization problem, many other risk measures have been researched. Thus, we will review studies that focused on the characteristic of well-known risk measurements and coherency. Then, we will review portfolio optimizing with risk measures being used in our research. In our literature review, we follow "Modeling and Optimization of Risk" review paper presented in Krokhmal et al. (2011).

2.1 Review of Risk Measures

As mentioned above, Markowitz (1952, 1959) presented the foundations of the modern theory of risk management. In his mean-variance (MV) model, he set the risk as the variance of the portfolio $\sigma^2(X(x,\omega))$ under scenario ω and wanted to minimize it under the condition that expected portfolio return exceeds a predefined threshold r_0.

$$\min_{x\in s}\{\sigma^2\left(X\left(x,\omega\right)\right)|E\left[X\left(x,\omega\right)\right]\}\geq\ r_0 \tag{1}$$

Intuitively, given equal expected return an investor is assumed to prefer the outcomes with less variance, i.e., with the more sure realization of the expected reward. However, the variance is clearly not ideal for this purpose. To begin with, the decision maker should not penalize a deviation from the average if this deviation is positive, i.e., brings in a better reward, which is not the case in MV model.

Hence, Markowitz (1987) redeemed this theoretical weakness by replacing the variance with the lower standard semi-deviation and these types of risk measures were called as the Downside Risk Measures. The Downside Risk Measures are also seen in more recent works by Ogryczak and Ruszczyński (1999, 2001, 2002).

One of the most popular risk measurement, also a downside risk measurement, is the Value at Risk JP Morgan (1994); Jorion (1997); Duffie and Pan (1997). The Value at Risk (VaR) is denoted as $VaR_{\alpha}(X)$ and this notation means that the value of X is the highest loss with probability α. If we assume X is \$10,000 and α is 95%, it means that we have less than 5% of probability of losing over \$10,000. Mathematically, $VaR_{\alpha}(X)$ is defined as

$$VaR_{\alpha}(X) = \sup \{ z \mid P\{ X \leq z \} < \alpha \} \tag{2}$$

However, because of non-convexity, minimization of risk using the VaR is not efficient. Moreover, lack of convexity results in counter-intuitive behavior in some application leading to sub-optimal performance. Hence, while the definition of VaR is easily explained and is widely used in practice, it is also generally agreed that it is not methodologically sound approach to risk-averse stochastic optimization.

While a number of alternative candidates to downside measures of risk have been proposed in the literature, many have been shown to lack in one aspect or another. Artzner et al. (1999) proposed a standard for constructing "good" risk measures. Artzner et al. (1999) and Delbaen (2002) presented four axioms for being a good risk measure and called risk measures satisfying these four requirements as *coherent measures of risk*. The coherent measure of risk is defined as mapping $\rho : \mathbf{X} \mapsto \mathbb{R}$ that satisfies the following four axioms:

(A1) monotonicity: $X \leq 0 \Rightarrow \rho \leq 0$ for all $X \in \mathbf{X}$,

(A2) sub-additivity: $\rho(X+Y) \leq \rho(X) + \rho(Y)$ for all $X, Y \in \mathbf{X}$,

(A3) positive homogeneity: $\rho(\lambda X) = \lambda \rho(X)$ for all $X \in \mathbf{X}, \lambda > 0$

(A4) translation invariance : $T\rho(X+a)=\rho(X)+a$ for all $X\in\mathbf{X}, a\in\mathbb{R}$.

The Conditional Value at Risk(CVaR), Higher Moment Coherent Risk Measures(HMCR), and Log-Exponential Convex Measures of Risk(LogExpCR) that we will discuss on our thesis satisfy all four axioms.

Rockafellar and Uryasev (2000, 2002) designed Conditional Value at Risk (CVaR) measure to remedy VaR, mainly non-convexity. They defined CVaR as the average of loss that exceeds $VaR_\alpha(X)$ in a confidence level α:

$$CVaR_\alpha(X) = -E\{X|X < -VaR_\alpha(X)\} \tag{3}$$

They also proved that the CVaR optimization problem can draw optimal value by this function,

$$CVaR_\alpha(X) = \min_{\eta\in\mathbb{R}} \eta + (1-\alpha)^{-1}E(X-\eta)_+ \tag{4}$$

and this result helps to implement coherent risk measurement to the stochastic optimization problem.

To generate coherent risk measures more easily, Krokhmal (2007) proposed the stochastic optimization problem which can draw a coherent measure of risk as the optimal value.

$$\rho(X) = \inf_{\eta\in\mathbb{R}} \{\eta + \phi(X+\eta)\} \tag{5}$$

Krokhmal (2007) also introduced the family of Higher Moment Coherent Risk measures (HMCR) by using equation (5). HMCR designed to accumulate exceeding loss by the use of higher moment. Krokhmal defined HMCR with certain value of p($\geq$ 1) and confidence level of α:

$$HMCR_{p,\alpha}(X) = \min_{\eta\in\mathbb{R}} \eta + (1-\alpha)^{-1}\|(X-\eta)_+\|_p \tag{6}$$

Another advanced method that can present risk measure is the convolution representation, further, by summing up utility function, the Certainty Equivalent Measures are presented.

$$\rho(X) = \min_{\eta} \eta + (1-\alpha)^{-1} \nu^{-1} E\nu([X-\eta]_+) \tag{7}$$

More recently, Vinel and Krokhmal (2015) presented Log Exponential Convex Risk Measure (LogExpCR).

$$\rho(X) = \min_{\eta} \eta + (1-\alpha)^{-1} \log_{\lambda} E\lambda_+^{[X-\eta]} \tag{8}$$

Since LogExpCR implements the deutility function which has a rational preference, this risk measure gives a higher penalty to extreme loss than a moderate one.

As seen above, recent risk measure developments have a tendency to focus on the tail of the risk distribution. The reason for this tendency is that tail parts of the heavy-tailed risk distribution have the probability as much as we might ignore it. However, losses or damages that came from the probability of the tail parts can be much more tremendous than our expectation. Kousky and Cooke (2009) presented evidence that the notion of fat (or heavy) tails plays a crucial role in evaluating risk. These properties can make risk measures neglect catastrophes even if much more risk can be made when they happen together. For this reason, researchers design risk measures to emphasize the tails of the respective distribution.

2.2 Portfolio Optimization

In order to assess the performance of risk measurement, the portfolio optimization has been widely used by many researchers. The portfolio optimization problem can be solved by either maximizing profit under constraining risk measure or the minimizing risk measure under the constraining profit level.

Rockafellar and Uryasev (2000) show that the portfolio optimization problem under the CVaR measure constraints can be solved by linear programming, which can minimize the CVaR. Furthermore, they solved the portfolio optimization with historical data under the CVaR constraints and compared this result with the MV model of Markowitz. They concluded that the difference between them was not significant but this result was drawn by close to normal historical data.

Krokhmal (2007) compared the HMCR (p = 3) with the MV, CVaR and SMCR (p = 2) by using the portfolio optimization problem. He tried to remedy shortcomings of the previous research by picking a high kurtosis historical data set. The experiment was conducted under various conditions by changing the threshold returns and the risk measure's alpha level. In the result, he concluded HMCR and SMCR were well performed compared to the Mean-Variance and CVaR.

Vinel and Krokhmal (2015) conducted the portfolio optimization problem analysis with LogExpCR and CVaR measures and compared its portfolio. This paper used three types of the historical data set that are corresponding to 2-day, 10-day, and 1-month return. They stated that these changes were the key in the comparison and LogExpCR over performed in more heavy-tailed distribution.

A few other case studies have been performed in the literature evaluating the performance of various risk measures in portfolio optimization model, to name a few, Rockafellar et al. (2006b,a); Pastor and Stambaugh (2000); ORTOBELLI et al. (2005); Rachev et al. (2009); Allen et al. (2016) among others. For example, Allen et al. (2016) emphasized the influence of the global financial crisis and showed that in some cases even a naive "equal distribution" approach can yield best results.

2.3 Summary

Historically, there have been many attempts at quantifying and controlling risk. Although Markowitz presented a pioneering method to measuring the risk, it had an obvious weakness. Many follow-ups, which mitigate some of the drawbacks of previous measures, also have certain issues. This led the researchers to propose axioms of relatively good risk measures and the risk measure satisfying all four axioms defined as the coherent measurement of risk. Since CVaR, SMCR, and LogExpCR satisfy axioms of coherent measures of risk, we will choose these three measures for comparing them. Furthermore, they also have convexity which is the important property in the stochastic programming area.

For comparing risk measures, we will use the portfolio optimization problem as conducted many times in previous studies. From the previous research, we have found important points of research method. First, heavy-tailed distribution of historical data set can lead to the significant difference between risk measures. Second, according to the change of parameter, risk measures behaved differently. Finally changing scenario set could have an effect on managing extreme losses. Thus we will use high kurtosis historical data with different types of return calculation and change parameters for comparison.

Chapter 3

RESEARCH MODEL DESCRIPTION

In this research, we will examine the general portfolio optimization problem. Then, we will show the implementation of the Conditional Value at Risk(CVaR) measure, the Higher Moment Coherent Risk Measure(HMCR), and Log-Exponential Convex Risk(LogExpCR) measure. Furthermore, we will also present the historical data set and its statistical properties. Finally, we will explain the realization procedure of the decision that is made by the past scenarios expected value.

3.1 General Portfolio Optimization Problem

The portfolio optimization problem is, basically, the stochastic programming problem that seeks the portfolio that has the highest expected return while the risk is minimized. The decision variables of the problem are the asset proportion of the portfolio and values are defined without any future realizations. The random variables are the historical rate of return set. In this problem, the random variables are distributed according to a set of discrete scenarios for each asset. The general formulation considered here is following Krokhmal (2007):

$$\min_{x} \quad \rho(-r^\top x) \tag{1}$$

$$\text{s.t.} \quad -I^\top x = 1 \tag{2}$$

$$E(r^\top x_i) \geq \mathbf{R} \tag{3}$$

$$x_i \geq 0. \tag{4}$$

where $x = (x_1, \ldots, x_n)^\top$ is the vector of stock proportion in the portfolio, $r = (r_1, \ldots, r_n)^\top$ is the random vector of assets' returns, and $I = (1, \ldots, 1)^\top$. In the real life, a balancing portfolio has many considerations. However, we will only constrain budget and return value, because the purpose of this research is not the development of a practical financial tool.

Loss function

First of all, since this problem is designed to find minimum risk, we need loss function that will be decided by decision variables and random variables. According to risk function, this loss will be quantified and compared. In this problem, reward function can be recognized intuitively as the return of portfolio and also, in the same sense, loss function will be negative of it., $X(x, \omega) = -r^\top x_i$.

Objective function

According to Rockafellar and Uryasev (2000), if the reward function is concave and risk measure is convex, the maximizing reward under risk constraints and the minimizing risk under reward constraints generate the same efficient frontier. Based on this study, although the portfolio optimization problem is made for maximizing profit, we can define our objective function as minimize risk function to satisfy our purpose. Furthermore, our risk function $\rho(x)$ will implement the CVaR, HMCR, and LogExpCR.

Budget Constraints

In constraints (2), sum of x_i is limited by one and this means that stocks are selected under the limited budget.

Value Constraints

Since we minimize our risk measures under required reward, constraints should be imposed on the minimum required level. Therefore, constraints (3) defined the predetermined return $\mathbf{R}$ as the minimum requirement for the expected return of portfolio. Furthermore, in our study $\mathbf{R}$ is calculated as the highest average return of stocks according to the scenario data set multiplied by a predetermined parameter r_0, $\mathbf{R} = r_0 \mathrm{max}_i\{E_\omega r_i(\omega)\}$.

Generating historical data set

Each random return r_i of stock i has 1000 discrete scenario realizations according to

$$r_{ij} = \frac{Price_{ij+d} - Price_{ij}}{Price_{ij}}, \qquad j = 1, \ldots, 1000 \tag{5}$$

and we assumed all scenarios have equal probability, $p(\omega) = 1/1000$. The prices are understood as the close price of the selected assets.

3.2 Conditional Value at Risk Model

Conditional Value at Risk(CVaR) is defined as the average of loss that exceeds $VaR_\alpha(X)$ in a confidence level α. If we assumed $f(x,y)$ as loss function with decision vector x and random vector y, the probability $\Psi(x,\eta)$ where $f(x,y)$ is not exceeding η is

$$\Psi(x,\eta) = \int_{f(x,y)\leq\eta} p(y)dy \tag{6}$$

and $\Psi(x,\eta)$ becomes the cumulative distribution function as a function of η for fixed x.

According to $\Psi(x,\eta)$ distribution, we can define VaR(x) with confidence level α,

$$VaR_\alpha(X) = \min\{\eta \in \mathbb{R} : \psi(x,\eta) \geq \alpha\} \tag{7}$$

and CVaR(x) with the confidence level α,

$$CVaR_\alpha(X) = (1-\alpha)^{-1} \int_{f(x,y)\geq VaR_\alpha(X)} f(x,y)p(y)dy \tag{8}$$

However, we need to obtain the value of VaR for calculating CVaR value.

Alternatively, CVaR can be calculated by the optimization problem,

$$CVaR_\alpha(X) = \min_{\eta \in \mathbb{R}} \phi(x, \eta) = \min_{\eta \in \mathbb{R}} \eta + (1-\alpha)^{-1} E(f(x, y) - \eta)^+ \tag{9}$$

where $[x]^+ = max\{x, 0\}$.

For implementing CVaR into the portfolio problem, it can be shown that

$$\min_{(x,\eta) \in X \times \mathbb{R}} \phi(x, \eta) = \min_{x \in X} CVaR_\alpha(X) \tag{10}$$

Although we can find our objective function as $\min_{\eta \in \mathbb{R}} \eta + (1-\alpha)^{-1} E(f(x, y) - \eta)^+$, we still need to implement $E(f(x, y) - \eta)^+$ function to linear programming. We can rewrite the objective function as below.

$$\begin{aligned} & \eta + (1-\alpha)^{-1} E(f(x, y) - \eta)^+ \\ = \; & \eta + (1-\alpha)^{-1} p(y)(f(x, y) - \eta)^+ \\ = \; & \eta + (1-\alpha)^{-1} \sum_{j=1}^{m} p_j max\{(f(x, y) - \eta), 0\}. \end{aligned} \tag{11}$$

We can further linearize objective function by adding constraints

$$f(x, y) - \eta \leq z_j, \qquad z_j \geq 0, \qquad j = 1, \ldots, 1000 \tag{12}$$

where $z_j = max\{(f(x, y) - \eta), 0\}$.

Therefore our portfolio problem with CVaR risk measure can be defined as,

$$\min \quad \eta + (1-\alpha)^{-1} \sum_{j=1}^{m} p_j z_j \tag{13}$$

$$\text{s.t.} \quad z_j \geq -\sum_{j=1}^{m} \sum_{i=1}^{n} r_{ij} x_i - \eta, \qquad z_j \geq 0, \qquad j = 1, \ldots, m \tag{14}$$

$$\sum_{i=1}^{n} x_i = 1, \qquad \frac{1}{m} \sum_{j=1}^{m} \sum_{i=1}^{n} r_{ij} x_i \geq R, \qquad x_i \geq 0, \tag{15}$$

which is a linear programming problem.

3.3 Higher Moment Coherent Risk Measure Model

Higher Moment Coherent Risk(HMCR) family is derived from an earlier study which claimed that if function ϕ satisfies monotonicity, sub-additivity, positive homogeneity and $\phi(\eta) > \eta$, then the optimal value of the $\rho(X) = \inf_{\eta \in \mathbb{R}} \eta + \phi(X - \eta)$ is a coherent risk measure.

For generating HMCR, function $\phi(X)$ is defined as $(1-\alpha)^{-1} \|(X-\eta)^+\|_p$, where $\|(X)^+\|_p = (E|X|^+)^{1/p}$ and this satisfies the conditions stated above (Krokhmal, 2007). Therefore, family of HMCR can be defined as,

$$HMCR_{p,\alpha}(X) = \min_{\eta \in \mathbb{R}} \eta + (1-\alpha)^{-1} \|(X-\eta)^+\|_p \tag{16}$$

Since the first moment is $E|X|$ and the case where p is 1 results in $\min_{\eta \in \mathbb{R}} \eta + (1 - \alpha)^{-1} E(f(x,y) - \eta)^+$, the special case where p=1 is the same as CVaR.

In our research, we will use Second Moment Coherent Risk (SMCR) measure, i.e., HMCR with $p = 2$. Similarly to the previous case,

$$
\begin{aligned}
& \eta + (1-\alpha)^{-1} \|(f(x,y)-\eta)^{+}\|_2 \\
= \; & \eta + (1-\alpha)^{-1} [E\{(f(x,y)-\eta)^2\}^2]^{+1/2} \\
= \; & \eta + (1-\alpha)^{-1} [Emax\{f(x,y)-\eta)^2, 0\}]^{1/2} \\
= \; & \eta + (1-\alpha)^{-1} p(y)^{1/2} max\{f(x,y)-\eta)^2, 0\}.
\end{aligned} \tag{17}
$$

Further,

$$
\begin{gathered}
t \geq (w_1^2 + \cdots + w_j^2)^{1/2} \\
w_j \geq -\sum_{j=1}^{m} \sum_{i=1}^{n} r_{ij} x_i - \eta, \qquad w_j \geq 0, \qquad j = 1, \ldots, m.
\end{gathered} \tag{18}
$$

Finally, we can derive our portfolio optimization problem with SMCR as

$$
\begin{aligned}
\min \quad & \eta + (1-\alpha)^{-1} \frac{1}{m^{1/2}} t && (19) \\
\text{s.t.} \quad & t \geq (w_1^2 + \cdots + w_j^2)^{1/2} && (20) \\
& w_j \geq -\sum_{j=1}^{m} \sum_{i=1}^{n} r_{ij} x_i - \eta, \qquad w_j \geq 0, \qquad j = 1, \ldots, m && (21) \\
& \sum_{i=1}^{n} x_i = 1, \qquad \frac{1}{m} \sum_{j=1}^{m} \sum_{i=1}^{n} r_{ij} x_i \geq R, \qquad x_i \geq 0. && (22)
\end{aligned}
$$

This is an instance of the so-called second-order cone programming problem and there exists an extensive body of literature discussing such problems. For more details on the computational aspects of the general HMCR optimization problem, see Krokhmal (2007), Vinel and Krokhmal (2014a), Morenko et al. (2013) and Vinel and Krokhmal (2014b).

3.4 Log Exponential Convex Risk Measure Model

The certainty equivalent risk measure is the theoretical base of the Log-Exponential Convex Measures of Risk(LogExpCR). With the one-sided deutility function $v(t)$, convolution-based measure of risk, $\rho(X) = \inf_\eta \eta + \phi(X - \eta)$ can derive class of certainty equivalent risk measure:

$$\rho(X) = \min_\eta \eta + (1-\alpha)^{-1} \nu^{-1} E\nu([X-\eta]^+) \tag{23}$$

Log-Exponential Convex Measures of Risk(LogExpCR) is defined by taking exponential one-sided deutility function $\nu(t) = -1 + \lambda^{[t]^+}$:

$$\rho_\alpha^{(\lambda)}(X) = \min_\eta \eta + (1-\alpha)^{-1} \log_\lambda E\lambda_+^{[X-\eta]} \tag{24}$$

Furthermore, since exponential deutility function is designed to give a higher value to the more extreme point, the LogExpCR measure puts additional emphasis on the tail of the distribution on the random vector.

Therefore, our portfolio problem with the LogExpCR is defined as,

$$\min \eta + (1-\alpha)^{-1} \log_\lambda E\lambda_+^{[X-\eta]} \tag{25}$$

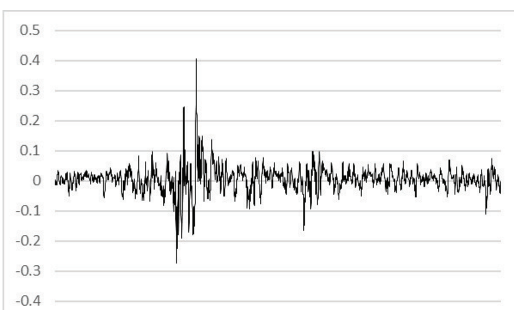
$$\text{s.t.} \sum_{i=1}^n x_i = 1, \qquad \frac{1}{m}\sum_{j=1}^m \sum_{i=1}^n r_{ij} x_i \geq R, \qquad x_i \geq 0. \tag{26}$$

This model can be classified as a convex optimization problem. It can be difficult to solve in general, so in our work we employ an approximation procedure described in Vinel and Krokhmal (2016, 2015).

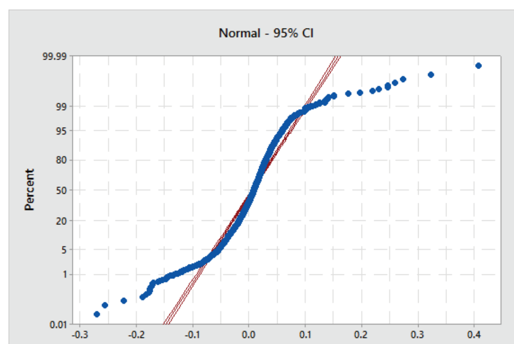
3.5 Dataset Description

S&P 100 stocks from January 2006 to December 2015 were used to generate the rate of return scenario set. Since the CVaR, HMCR and LogExpCR are suitable to quantify risk by implementing heavy-tailed loss distribution, we select 50 stocks from S&P 100 list according to their kurtosis.

Generally, kurtosis greater than 3 indicates a fatter tail than the normal distribution. For research purpose, we select 40 heavy-tailed stocks (kurtosis ≥ 6) and 10 normal distributed stocks (kurtosis ≈ 3) based on 10-day term return calculation. By computing kurtosis of scenario set we can also get the average kurtosis of 9.40 with 56.86 and 3.51 being the maximum and minimum. Thus we can conclude this scenario set is not normally distributed. Figure 3.1 illustrates the actual movement of the rate of return and the probability plot for a sample asset.



(a) Movement of a stock



(b) Probability plot

Figure 3.1: The graphical description of scenario set

3.6 Portfolio rebalancing

Our basic portfolio optimization problem has a ten-day rebalancing period. First the problem is solved based on m historical scenarios and optimal portfolio is obtained. Next, for the next ten time periods, the value of the portfolio is tracked. After that, the historical scenario set is updated with the newly realized information and a new optimal portfolio is determined. The procedure is repeated until the end of the planning horizon. Figure 3.2 illustrates this process. Note that at any moment, the decisions are made based on historical data, while the quality of the outcomes is judged based on the future outcomes.

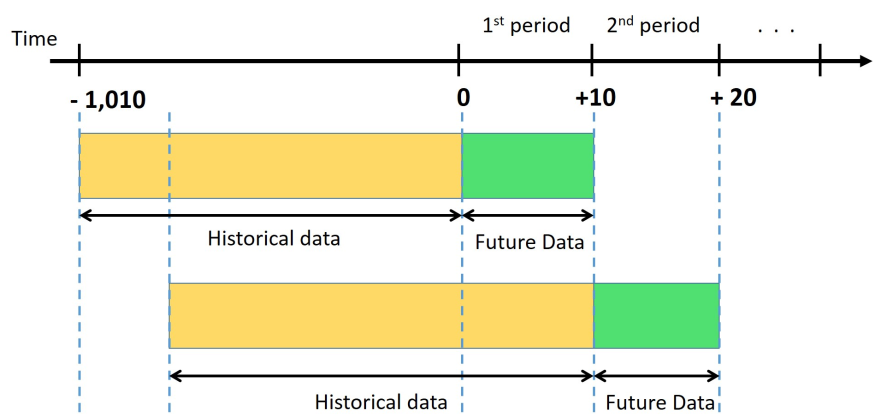


Figure 3.2: Timeline of the portfolio rebalancing

Chapter 4

ANALYSIS RESULT

As mentioned above, we find an optimal portfolio based on historical stock market data set. After finding optimal portfolio composition, we realized portfolio during a 10-day period with stock's own "future" rate of return. Furthermore, we repeated this process over 100 times. For the research purpose, within the same historical data set, we changed r_0 and confidence level α of risk measures. We also use 3 types of historical data for the purpose of finding different behavior between various data set.

To compare risk measures, we will consider the average portfolio return and the Sharpe ratio. As the portfolio is realized a hundred periods, we can observe a hundred portfolio returns. Thus we can average those portfolio returns. However, if we observe only the return of portfolio, analysis can have a weakness to ignore the high deviation of return value. Since high deviation can be highly undesirable, we need a different tool that implements both expected return and deviation. The Sharpe ratio could be suitable assessment tool for comparison of generated portfolios because it is the ratio of the excess return and the standard deviation. It is calculated as follows

$$Sharpe\,ratio = \frac{E[Return_{portfolio} - Retunrn_{benchmark}]}{\sqrt{var[Return_{portfolio} - Retunrn_{benchmark}]}} \tag{1}$$

We defined the benchmark return as the average return of 50 stocks which is used in the problem. As per the definition, having higher ratio represents a better portfolio.

4.1 Research Hypotheses

Hypothesis 1.

In the previous study Krokhmal (2007) and Vinel and Krokhmal (2015), it was observed that HMCR and LogExpCR can generate portfolios that are more profitable than CVaR in most cases. Therefore, we expect that the portfolio returns based on HMCR and LogExpCR will be higher than CVaR for some of the parameter settings.

Hypothesis 2.

We expect that as r_0 increases, the differences in performance between the risk measures will be reduced.

Hypothesis 3.

We expect that HMCR and LogExpCR measures will achieve better Sharpe ratios compared to CVaR.

Hypothesis 4.

Since the HMCR and the LogExpCR model are designed to emphasize the tail part of loss distribution more so than CVaR, we expect that HMCR and LogExpCR will show better performances in the heavy-tailed distribution compared to the normally distributed cases.

Hypothesis 5.

We expect that the difference in performance between the measures will be mirrored in the composition of the optimal portfolios found. Moreover, we expect that HMCR and LogExpCR will produce more diverse portfolios that is composed of more stocks.

4.2 Discussion of Result Within the Same Rebalancing Parameter Value

In the first set of experiments, we have considered various values for parameters r_0, and α. First, the rate of return threshold for selected portfolio parameter 'R', which is calculated by r_0, $\mathbf{R} = r_0 \mathrm{max}_i\{E_\omega r_i(\omega)\}$, is defined by setting r_0 as 1, 10, 30, 50, and 80%. Secondly, the Confidence level of each risk measurement is selected as

$\alpha = 99, 90, 80$ and 70%. Furthermore, historical data is fixed as the 10-day rate of return historical data.

Obtained results are summarized in the table 4.1 and behavior of each risk measurements are presented in the figure from 4.1 to 4.9. In the table 4.1, we exhibit average returns of the portfolio and sharp ratios under various parameter r_0 and α.

		Exp. return			Sharpe ratio		
α	r0	CVaR	LogExp	HMCR	CVaR	LogExp	HMCR
99%	0.01	0.00494	0.00830	0.00831	-0.10395	0.03690	0.03713
	0.1	0.00493	0.00830	0.00831	-0.10424	0.03693	0.03713
	0.3	0.00746	0.00876	0.00885	0.01105	0.05692	0.06019
	0.5	0.01083	0.01209	0.01204	0.12661	0.14748	0.14545
	0.8	0.01462	0.01481	0.01475	0.16619	0.17128	0.17014
90%	0.01	0.00607	0.00824	0.00831	-0.05280	0.03509	0.03723
	0.1	0.00607	0.00824	0.00831	-0.05280	0.03509	0.03723
	0.3	0.00694	0.00876	0.00885	-0.01368	0.05688	0.06027
	0.5	0.00946	0.01191	0.01204	0.09899	0.14275	0.14558
	0.8	0.01341	0.01486	0.01474	0.15661	0.17234	0.17014
80%	0.01	0.00574	0.00823	0.00831	-0.07081	0.03491	0.03723
	0.1	0.00574	0.00823	0.00831	-0.07081	0.03491	0.03724
	0.3	0.00675	0.00875	0.00885	-0.02356	0.05632	0.06023
	0.5	0.00900	0.01194	0.01204	0.08057	0.14357	0.14565
	0.8	0.01313	0.01487	0.01474	0.15223	0.17260	0.17013
70%	0.01	0.00567	0.00823	0.00831	-0.07414	0.03486	0.03721
	0.1	0.00567	0.00823	0.00831	-0.07414	0.03486	0.03720
	0.3	0.00655	0.00874	0.00885	-0.03436	0.05612	0.06024
	0.5	0.00865	0.01197	0.01204	0.06499	0.14452	0.14563
	0.8	0.01321	0.01487	0.01474	0.15896	0.17252	0.17011

Table 4.1: The table of average portfolio return and the Sharpe ratio under various condition

Figures from 4.1 to 4.5 present a different performance according to changing r_0.

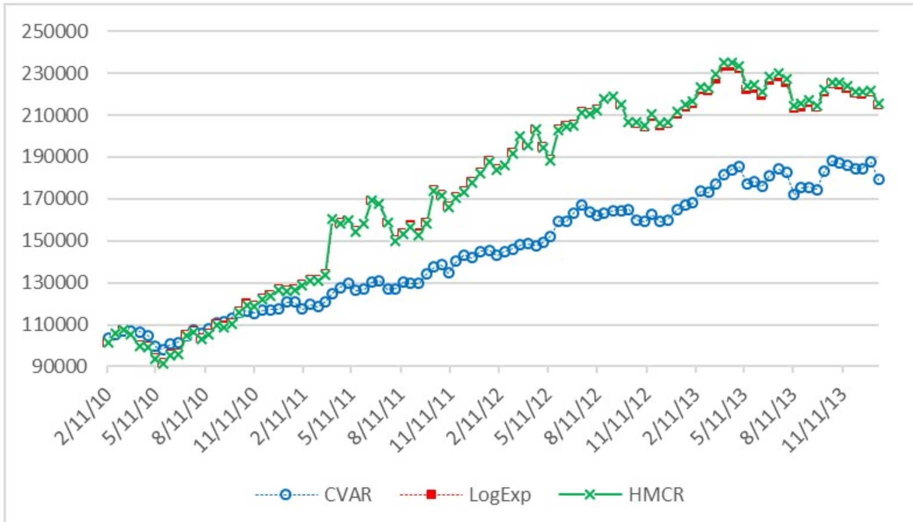


Figure 4.1: The behavior of portfolio value under $r_0 = 1\%$, $\alpha = 90\%$

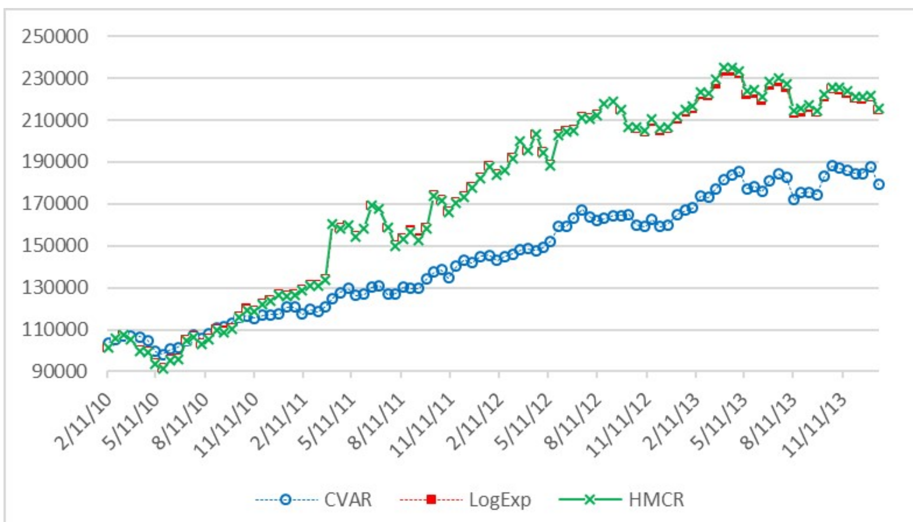


Figure 4.2: The behavior of portfolio value under $r_0 = 10\%$, $\alpha = 90\%$

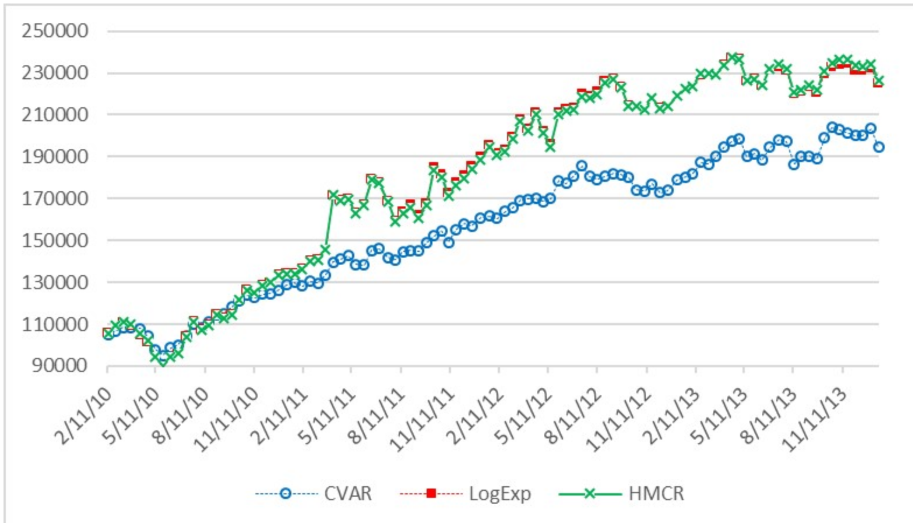


Figure 4.3: The behavior of portfolio value under $r_0 = 30\%$, $\alpha = 90\%$

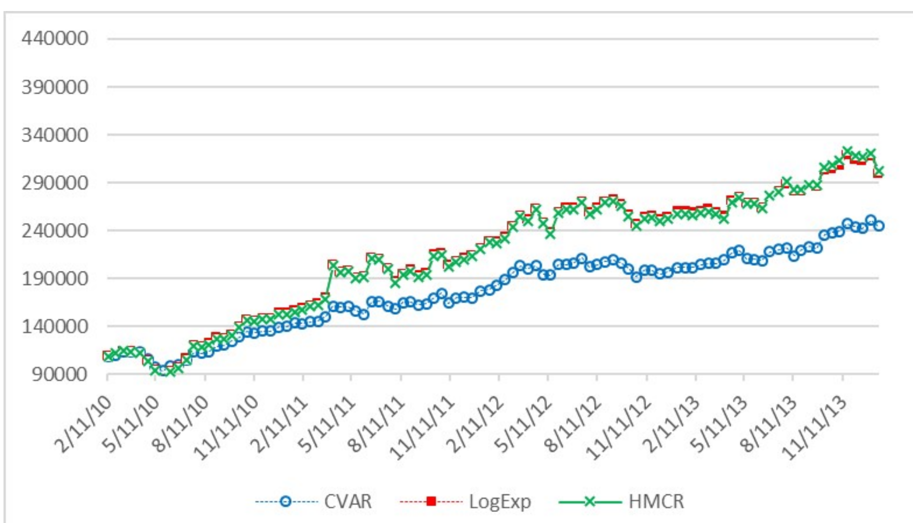


Figure 4.4: The behavior of portfolio value under $r_0 = 50\%$, $\alpha = 90\%$

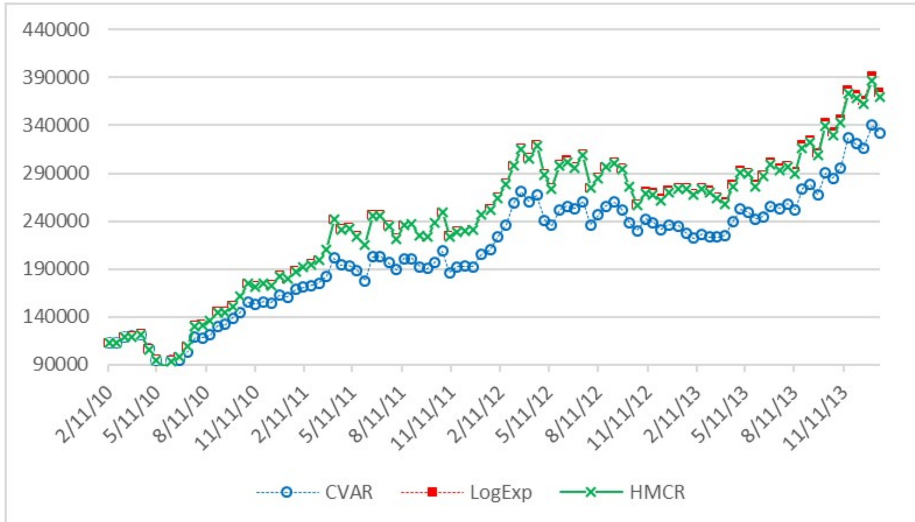


Figure 4.5: The behavior of portfolio value under $r_0 = 80\%$, $\alpha = 90\%$

Figures from 4.6 to 4.9 present a different performance according to changing α.

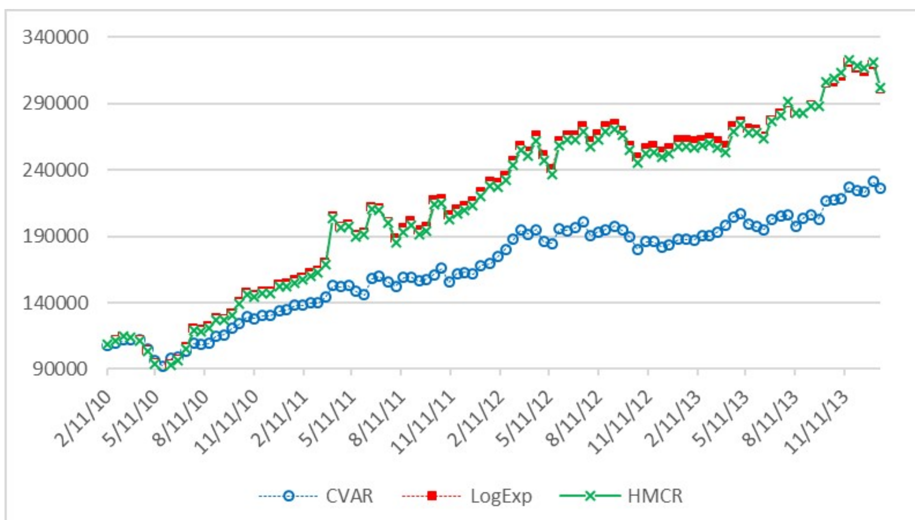


Figure 4.6: The behavior of portfolio value under $r_0 = 50\%$, $\alpha = 70\%$

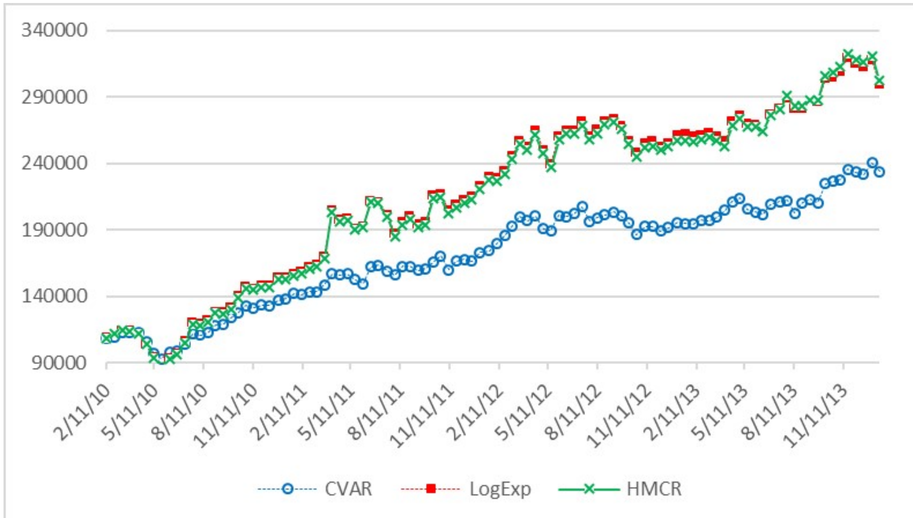


Figure 4.7: The behavior of portfolio value under $r_0 = 50\%$, $\alpha = 80\%$

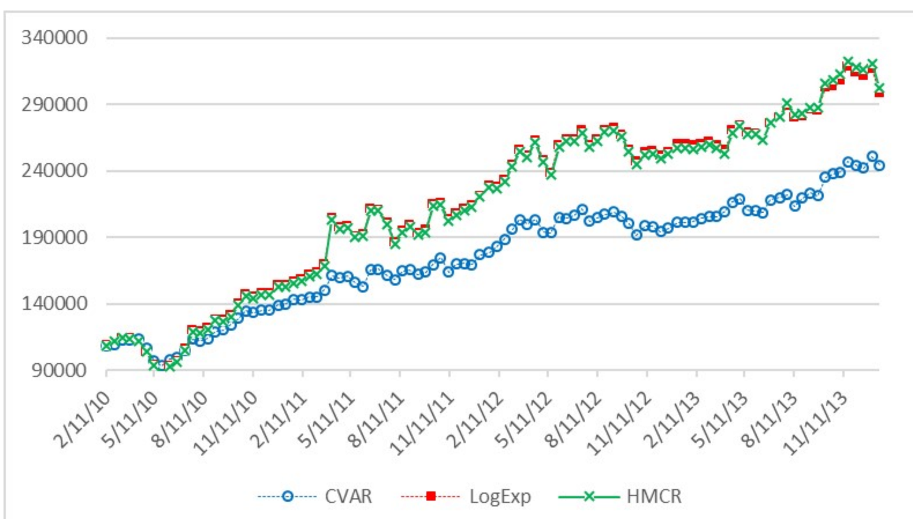


Figure 4.8: The behavior of portfolio value under $r_0 = 50\%$, $\alpha = 90\%$

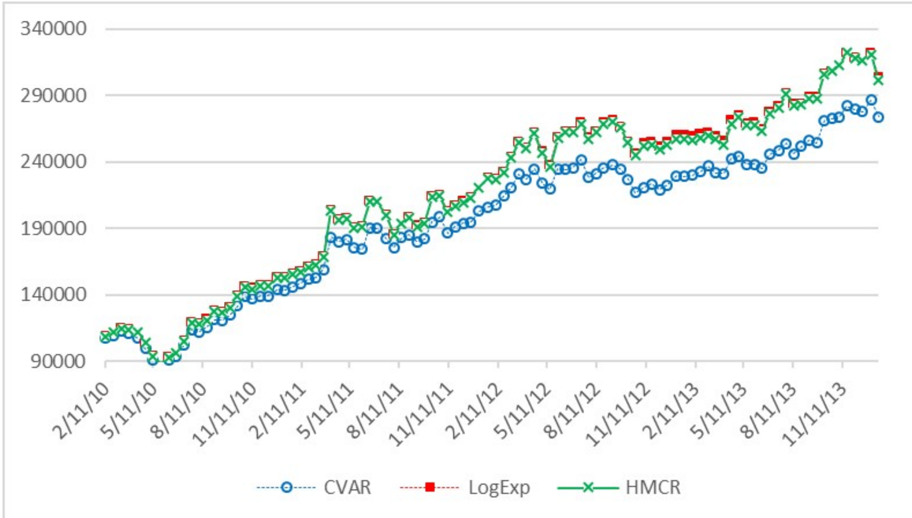


Figure 4.9: The behavior of portfolio value under $r_0 = 50\%$, $\alpha = 99\%$

4.2.1 Changing Parameter r_0

As we expected in Hypothesis 1, generally, both LogExpCR and HMCR portfolios exhibit a higher average returns compared to CVaR. Furthermore, we recognize that the average return of the HMCR is slightly higher than the LogExpCR in the lower threshold situation but in the higher threshold, the opposite result was observed. However, since the difference between them is not significant, we can conclude LogExpCR and HMCR have a similar performance in all situations. This might imply that the two risk measures generated a similar portfolio, and further they quantify risk as a similar value in this loss distribution.

All three portfolios have higher average returns at higher values of r_0. This is due to the fact that every risk measure is forced to compose the portfolio based on the stocks capable of achieving the target return value. Moreover, the difference between the approaches diminishes as the value of r_0 grows.(Hypothesis 2)

Lastly, since the Sharpe ratios for the portfolios generated with HMCR and LogExpCR are larger than for the ones due to CVaR in all cases, which indicates that the use of more advanced risk measures does lead to a reduced riskiness of the portfolios (Hypothesis 3).

4.2.2 Changing Parameter α

Observe that CVaR portfolio shows more noticeable changes compared to LogExpCR and HMCR portfolios when varying the value of α. In all three cases this parameter regulates the "cut-off point", i.e., the part of the distribution that should be considered risky. In case of CVaR this relationship is straightforward as evident from the definition. For HMCR and LogExpCR it is harder to analyze. The fact that the portfolios in this case are less affected by the changes in parameter α may indicate that the range of α used here is insufficient to adequately illustrate the whole variability of these approaches. Hence, a more thorough analysis could be performed here in the future.

4.3 Discussion of Result Between The Different Rebalancing Parameter Values

By varying the rebalancing period duration we can observe the effect that the heaviness of the tails of the loss distributions has on the performance of the risk measures. Indeed, it is well-documented that short-term price of an asset is usually much more noisy than long-term. Similarly, relatively high swings in price in either direction are more likely if the price is tracked on a daily basis, compared to say, monthly changes. Hence, we expect that the one-day return will be more volatile than the 20-day return.

Obtained results are summarized in the table 4.2 and 4.3. The table 4.2 shows the average returns of the portfolios, and the table 4.3 gives the Sharpe ratios.

		1-day			10-day			20-day		
α	r0	CVaR	LogExp	HMCR	CVaR	LogExp	HMCR	CVaR	LogExp	HMCR
99%	0.01	0.00483	0.0061	0.00613	0.00494	0.0083	0.00831	0.00548	0.00594	0.00231
	0.1	0.00483	0.0061	0.00613	0.00493	0.0083	0.00831	0.00548	0.00594	0.00231
	0.3	0.00525	0.00649	0.00653	0.00746	0.00876	0.00885	0.00745	0.00909	0.00322
	0.5	0.00837	0.00776	0.00772	0.01083	0.01209	0.01204	0.00956	0.01078	0.00415
	0.8	0.01238	0.01438	0.01448	0.01462	0.01481	0.01475	0.01449	0.01347	0.00689
90%	0.01	0.00455	0.006	0.00613	0.00607	0.00824	0.00831	0.00509	0.00587	0.00231
	0.1	0.00457	0.006	0.00613	0.00607	0.00824	0.00831	0.00509	0.00587	0.00231
	0.3	0.00538	0.00646	0.00653	0.00694	0.00876	0.00885	0.00676	0.00906	0.00322
	0.5	0.00803	0.00795	0.00773	0.00946	0.01191	0.01204	0.00948	0.0109	0.00415
	0.8	0.01325	0.01429	0.01448	0.01341	0.01486	0.01474	0.01245	0.01347	0.00689
80%	0.01	0.00435	0.006	0.00613	0.00574	0.00823	0.00831	0.00558	0.00588	0.00231
	0.1	0.0044	0.006	0.00613	0.00574	0.00823	0.00831	0.00558	0.00588	0.00231
	0.3	0.0056	0.00645	0.00653	0.00675	0.00875	0.00885	0.00648	0.00906	0.00322
	0.5	0.00827	0.00793	0.00772	0.009	0.01194	0.01204	0.00872	0.01091	0.00415
	0.8	0.01307	0.01429	0.01448	0.01313	0.01487	0.01474	0.01287	0.01347	0.00689
70%	0.01	0.00461	0.006	0.00613	0.00567	0.00823	0.00831	0.00539	0.00588	0.00231
	0.1	0.00465	0.006	0.00613	0.00567	0.00823	0.00831	0.00539	0.00588	0.00231
	0.3	0.00588	0.00645	0.00653	0.00655	0.00874	0.00885	0.00634	0.00907	0.00322
	0.5	0.00861	0.00792	0.00772	0.00865	0.01197	0.01204	0.00846	0.01092	0.00415
	0.8	0.01302	0.01429	0.01448	0.01321	0.01487	0.01474	0.013	0.01347	0.00689

Table 4.2: The table of average portfolio return under between different historical data set

α	r0	1-day CVaR	1-day LogExp	1-day HMCR	10-day CVaR	10-day LogExp	10-day HMCR	20-day CVaR	20-day LogExp	20-day HMCR
99%	0.01	-0.1047	-0.051	-0.0499	-0.104	0.0369	0.0371	-0.0774	-0.058	-0.1934
	0.1	-0.1047	-0.051	-0.0499	-0.1042	0.0369	0.0371	-0.0774	-0.058	-0.1934
	0.3	-0.0965	-0.0333	-0.0317	0.0111	0.0569	0.0602	0.0083	0.0721	-0.1647
	0.5	0.0504	0.0241	0.0222	0.1266	0.1475	0.1455	0.0738	0.1194	-0.1151
	0.8	0.1366	0.1741	0.1759	0.1662	0.1713	0.1701	0.1515	0.1323	-0.0089
90%	0.01	-0.1131	-0.0551	-0.0497	-0.0528	0.0351	0.0372	-0.0927	-0.0606	-0.1932
	0.1	-0.1122	-0.0551	-0.0497	-0.0528	0.0351	0.0372	-0.0927	-0.0606	-0.1932
	0.3	-0.0874	-0.0353	-0.0314	-0.0137	0.0569	0.0603	-0.0213	0.0707	-0.1647
	0.5	0.037	0.0326	0.0225	0.099	0.1428	0.1456	0.0911	0.1233	-0.115
	0.8	0.1568	0.1721	0.176	0.1566	0.1723	0.1701	0.132	0.1322	-0.0088
80%	0.01	-0.1202	-0.0552	-0.0498	-0.0708	0.0349	0.0372	-0.0742	-0.0605	-0.1933
	0.1	-0.1186	-0.0552	-0.0497	-0.0708	0.0349	0.0372	-0.0742	-0.0605	-0.1934
	0.3	-0.0794	-0.0355	-0.0314	-0.0236	0.0563	0.0602	-0.0366	0.071	-0.1647
	0.5	0.0483	0.0318	0.0224	0.0806	0.1436	0.1456	0.0645	0.1237	-0.1151
	0.8	0.1567	0.1721	0.176	0.1522	0.1726	0.1701	0.1463	0.1323	-0.0088
70%	0.01	-0.1114	-0.0552	-0.0498	-0.0741	0.0349	0.0372	-0.0871	-0.0604	-0.1934
	0.1	-0.1101	-0.0552	-0.0499	-0.0741	0.0349	0.0372	-0.0871	-0.0603	-0.1934
	0.3	-0.0673	-0.0357	-0.0314	-0.0344	0.0561	0.0602	-0.0453	0.0711	-0.1647
	0.5	0.0658	0.0312	0.0223	0.065	0.1445	0.1456	0.0555	0.1241	-0.1151
	0.8	0.1557	0.172	0.176	0.159	0.1725	0.1701	0.1528	0.1323	-0.0088

Table 4.3: The table of the Sharpe ratio under between different historical data set

From figure 4.10 to figure 4.12 show the behavior of each risk measure.

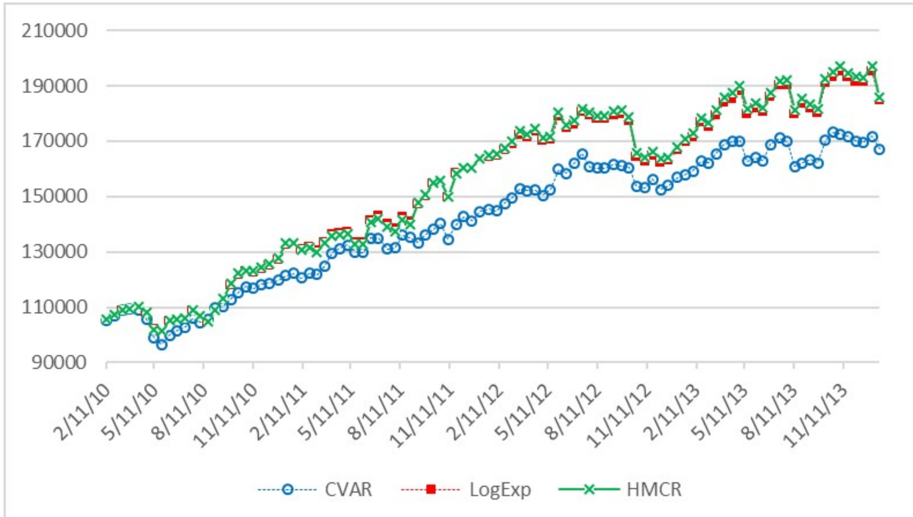


Figure 4.10: The behavior of portfolio value using one-day historical data under $r_0 = 30\%$, $\alpha = 90\%$

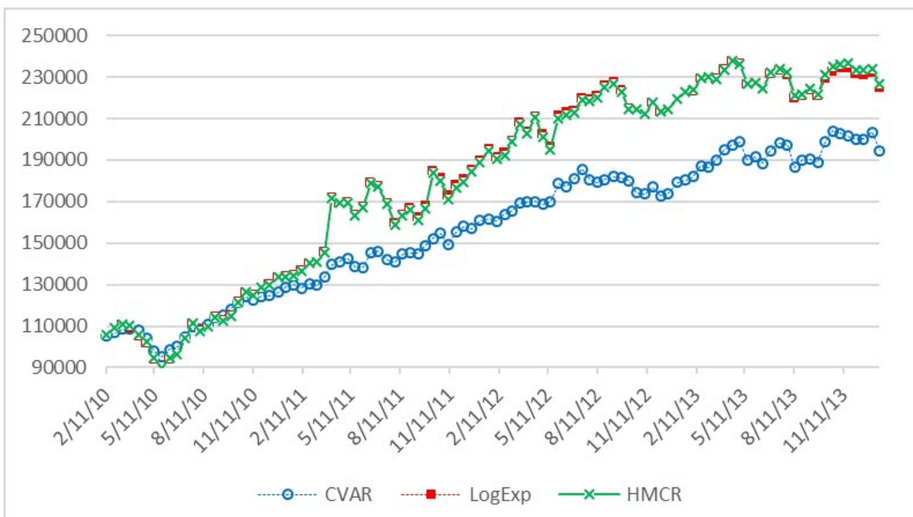


Figure 4.11: The behavior of portfolio value using 10-day(two weeks) historical data under $r_0 = 30\%$, $\alpha = 90\%$

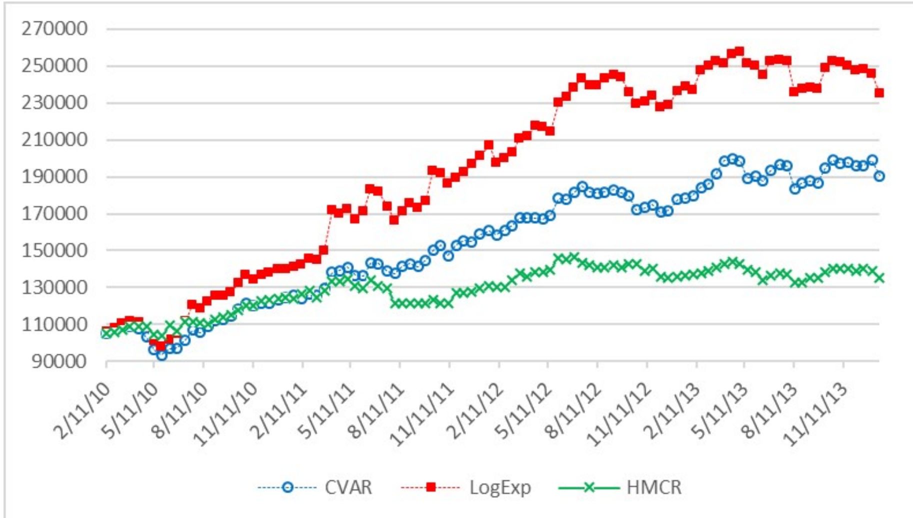


Figure 4.12: The behavior of portfolio value using 20-day(one month) historical data under $r_0 = 30\%$, $\alpha = 90\%$

First of all, we expect that the short-term historical rate of returns set is distributed more closely to heavy tailed while the long-term historical rate of returns set is close to the normal distribution. Therefore, one-day return will be the short-term historical data set and the 20-days return will be the long-term historical rate of return.

However, although we expected (Hypothesis 3) the LogExpCR and the HMCR portfolios show better average returns and Sharpe ratios compared to CVaR in the one-day and 10-day return data, we could not observe a significant difference between short-term rate of return scenario set and long-term rate of return scenario set. Therefore, we can not conclude that the LogExpCR and the HMCR measures risk more effectively than the CVaR in the heavy-tailed distribution.

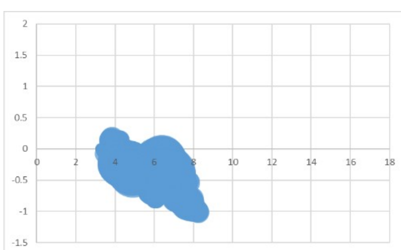
However, in the 20-day return scenario, we observed an unexpected result. The HMCR exhibits the lowest average portfolio return and Sharp ratio. The difference in Sharpe ratios between the CVaR and the LogExp is not regular. At this point, this observation remains unexplained and we intend to study it more closely in the future.

4.4 Discussion of The Optimal Portfolio Composition

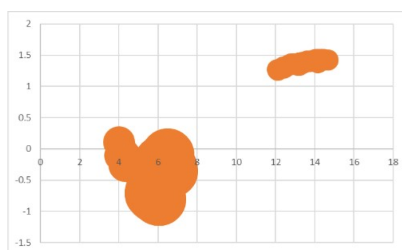
We have focused on kurtosis and skewness of the selected stocks to find whether the distributions were heavy tailed or normal. For this research, we arranged the optimal solution data from the portfolio problem with the 10-day return over 50 periods and calculated the kurtosis and skewness by using the 1,000 rate of return scenario set.

To present tendencies of risk measures intuitively, we used bubble plots where the bubbles represent the selected stocks. The bubbles are scattered on the plain which has X axis as the kurtosis and Y axis as skewness. Furthermore, bubble sizes are determined by the proportion in the optimal portfolio.

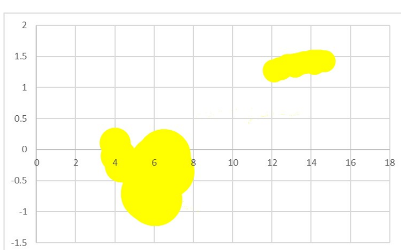
Obtained plots are presented in the figure 4.13, 4.14 and 4.15 for $r_0 = 10$, 50 and 80%, respectively. In the plot, each color presents selection due to each of the risk measure.



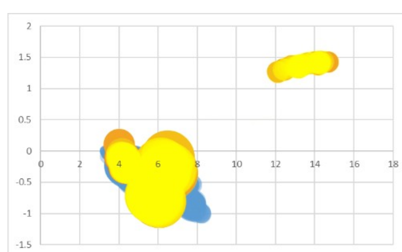
(a) CVaR



(b) LogExp

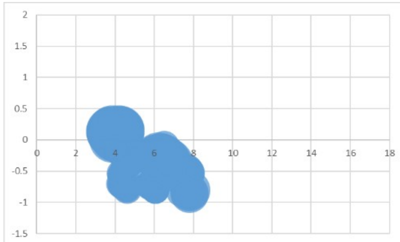


(c) HMCR

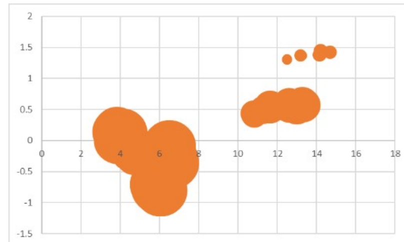


(d) All together

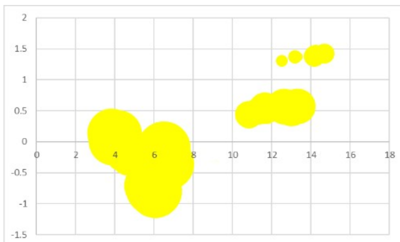
Figure 4.13: The bubble plot of stock composition under $r_0 = 10\%$, $\alpha = 90\%$



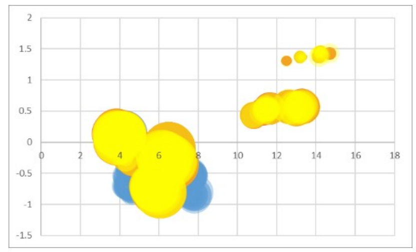
(a) CVaR



(b) LogExp

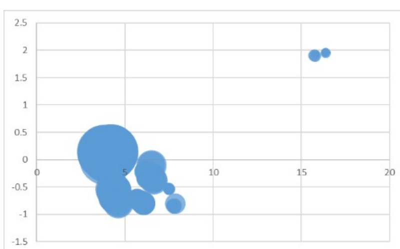


(c) HMCR

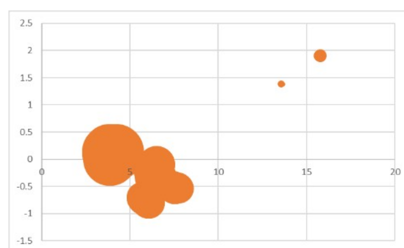


(d) All together

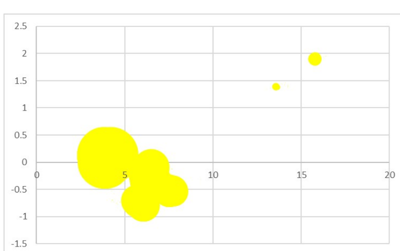
Figure 4.14: The bubble plot of stock composition under $r_0 = 50\%$, $\alpha = 90\%$



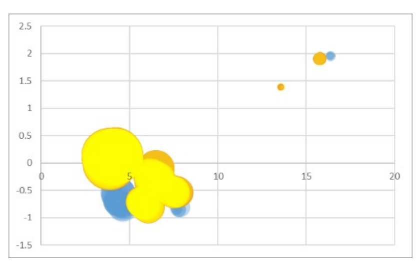
(a) CVaR



(b) LogExp



(c) HMCR



(d) All together

Figure 4.15: The bubble plot of stock composition under $r_0 = 80\%$, $\alpha = 90\%$

We assumed that the stock selecting tendency of the risk measure will be related to the tail of the distribution. Since the kurtosis greater than three means that the distribution is more heavy tailed, the stock on the right side indicates that the stock has heavy tailed scenario sets.

According to the plot, the bubbles are located mostly on the left side. Therefore, we can conclude the risk measure prefer to select comparatively normally distributed stocks.

Observe that the assets selected by HMCR and LogExpCR do not differ much, and are more diverse compared to CVaR. In other words, we can conclude that higher values of Sharpe ratio achieved by HMCR and LogExpCR are accompanied by a higher level of diversification, which is in accordance with the general assumption that diversification leads to better performance.(Hypothesis 5)

A weakness of this approach is the fact that kurtosis and skewness of stocks were not varied evenly. Hence, if we can generate multivariate random variables based on selected mean, kurtosis, and skewness, this research could derive more reliable results.

Chapter 5

SUMMARY AND CONCLUSION

In this research, we have studied Conditional Value at Risk(CVaR), Higher Moment Coherent Risk(HMCR), and Log Exponential Convex Risk(LogExpCR) measures. These risk measures have a number of common characteristics: the axioms of coherency (or convexity)are satisfied, they are convex functions that are well-suited for stochastic programming, and risk is quantified based on the tail of the distribution. However, the difference between them is the way that the tails are handled. Both HMCR and LogExpCR involve a nonlinear approach, which complicates the definition, yet promises a less risky performance due to a better penalization of excessive losses.

We have used a financial portfolio optimization model in order to evaluate the properties of these approaches. This model lets us perform computational experiments with real-life data and is flexible enough to allow for variation of parameters and different uncertainty types. In order to consider various risk condition, we set five different values of the parameter r_0, four different confidence levels within the same scenario set, and we also generated three different scenario sets based on rebalancing periods. The average of the portfolio returns and the Sharpe ratios are used to compare between risk measures.

Chapter 2 introduced the previous work on the risk measure development. We surveyed the relevant literature and justified the choice of the of three measures identified above.

Chapter 3 constructed the portfolio optimization models which is defined as the minimizing risk which will take the CVaR, HMCR, and LogExpCR respectively, with portfolio return constrained by the threshold. We also presented the implementation of each risk measures by proceeding some mathematical procedures. Furthermore, we examined our scenario set whether expected value of the whole scenario set distributed as heavy tailed or normally, then shows how the portfolio is rebalanced.

Chapter 4 analyzed the result derived from the model described in Chapter 3. Before the analysis, we presented our hypotheses based on the definition of each risk measure. The result based on these two values was as following. First of all, both the HMCR and LogExpCR exhibit similar performances in most cases but performed better than CVaR. Secondly, parameter changes were significantly affected the optimal portfolios selected. Furthermore, changing the scenario set also yielded different behaviors of risk measures, with more heavy-tailed distributions resulting in the better relative performance of both nonlinear approaches.

For this research, our contribution is that first, we presented the CVaR, HMCR and LogExpCR comparison that is not studied yet in various situations. We compared them not only in terms of portfolio performance but also based on the tendency of the optimal portfolio composition. In addition, we re-verified that the portfolio optimizing problem is suitable to compare between risk measures in the sense that we can change risk conditions by changing constraints or parameters and also can observe behaviors of risk measure in the manipulated scenario set.

The following future research directions are anticipated. First, we have observed an unexpected discrepancy between HMCR and LogExpCR portfolios in the case of light-tailed distribution of returns. A careful analysis of this phenomenon could yield additional insights into the relative performance of the risk measures. Secondly, a similar analysis could be performed with other decision making models. While portfolio optimization is widely used as a testing model for risk-averse stochastic optimization, other interesting models include natural disaster related decisions, insurance premium evaluation, military applications, etc. Finally, a computational study involving even more approaches to uncertainty quantification could be performed.

Bibliography

Allen, D. E., McAleer, M., Powell, R. J., and Singh, A. K. (2016) "Down-Side Risk Metrics as Portfolio Diversification Strategies across the Global Financial Crisis," *Journal of Risk and Financial Management*, **9** (2), 6.

Artzner, P., Delbaen, F., Eber, J.-M., and Heath, D. (1999) "Coherent measures of risk," *Math. Finance*, **9** (3), 203–228.

Delbaen, F. (2002) "Coherent risk measures on general probability spaces," in: K. Sandmann and P. J. Schönbucher (Eds.) "Advances in Finance and Stochastics: Essays in Honour of Dieter Sondermann," 1–37, Springer.

Duffie, D. and Pan, J. (1997) "An Overview of Value-at-Risk," *Journal of Derivatives*, **4**, 7–49.

Jorion, P. (1997) *Value at Risk: The New Benchmark for Controlling Market Risk*, McGraw-Hill.

JP Morgan (1994) *Riskmetrics*, JP Morgan, New York.

Kousky, C. and Cooke, R. M. (2009) "The unholy trinity: fat tails, tail dependence, and micro-correlations," *Resources for the Future Discussion Paper*, 09–36.

Krokhmal, P., Zabarankin, M., and Uryasev, S. (2011) "Modeling and optimization of risk," *Surveys in Operations Research and Management Science*, **16** (2), 49 – 66.

Krokhmal, P. A. (2007) "Higher moment coherent risk measures," *Quant. Finance*, **7** (4), 373–387.

Markowitz, H. M. (1952) "Portfolio Selection," *Journal of Finance*, **7** (1), 77–91.

Markowitz, H. M. (1959) *Portfolio Selection*, Wiley and Sons, New York, 1st edition.

Markowitz, H. M. (1987) *Mean-variance analysis in portfolio choice and capital markets*, Basil Blackwell, Oxford.

Morenko, Y., Vinel, A., Yu, Z., and Krokhmal, P. (2013) "On p-cone linear discrimination," *European J. Oper. Res.*.

Ogryczak, W. and Ruszczyński, A. (1999) "From stochastic dominance to mean-risk models: Semideviations as risk measures," *European Journal of Operational Research*, **116**, 33–50.

Ogryczak, W. and Ruszczyński, A. (2001) "On consistency of stochastic dominance and mean-semideviation models," *Mathematical Programming*, **89**, 217–232.

Ogryczak, W. and Ruszczyński, A. (2002) "Dual stochastic dominance and related mean-risk models," *SIAM Journal on Optimization*, **13** (1), 60–78.

ORTOBELLI, S., RACHEV, S. T., STOYANOV, S., FABOZZI, F. J., and BIGLOVA, A. (2005) "THE PROPER USE OF RISK MEASURES IN PORTFOLIO THEORY," *International Journal of Theoretical and Applied Finance*, **08** (08), 1107–1133.

Pastor, L. and Stambaugh, R. F. (2000) "Comparing asset pricing models: an investment perspective," *Journal of Financial Economics*, **56** (3), 335 – 381.

Rachev, S. T., Martin, R. D., Racheva, B., and Stoyanov, S. (2009) "Stable ETL optimal portfolios and extreme risk management," in: "Risk Assessment," 235–262, Springer.

Rockafellar, R. T. and Uryasev, S. (2000) "Optimization of Conditional Value-at-Risk," *Journal of Risk*, **2**, 21–41.

Rockafellar, R. T. and Uryasev, S. (2002) "Conditional Value-at-Risk for General Loss Distributions," *Journal of Banking and Finance*, **26** (7), 1443–1471.

Rockafellar, R. T., Uryasev, S., and Zabarankin, M. (2006a) "Master funds in portfolio analysis with general deviation measures," *Journal of Banking & Finance*, **30** (2), 743 – 778, risk Management and Optimization in Finance.

Rockafellar, T. R., Uryasev, S., and Zabarankin, M. (2006b) "Optimality conditions in portfolio analysis with general deviation measures," *Mathematical Programming*, **108** (2), 515–540.

Vinel, A. and Krokhmal, P. (2014a) "On valid inequalities for mixed integer p-order cone programming," *J. Optim. Theory Appl.*, **160** (2), 439–456.

Vinel, A. and Krokhmal, P. (2016) "Mixed integer programming with a class of nonlinear convex constraints," *Discrete Optimization*, accepted for publication.

Vinel, A. and Krokhmal, P. A. (2014b) "Polyhedral approximations in p-order cone programming," *Optim. Methods Softw.*, **29** (6), 1210–1237.

Vinel, A. and Krokhmal, P. A. (2015) "Certainty equivalent measures of risk," *Annals of Operations Research*, 1–21.